PECSN, 02/07/2019

Exercise 1

There are two bushes. The length (in cm) of the leaves of bush A is a RV whose PDF is:

$$f(x)=\begin{cases} -\dfrac{k}{256}\cdot x^2+\dfrac{k}{32}\cdot x & 0\le x\le 8 \\ 0 & otherwise \end{cases}$$

The length of the leaves of bush B is a RV whose PDF is:

$$g(x)=\begin{cases} \dfrac{1}{10} & 0\le x\le 10 \\ 0 & otherwise \end{cases}$$

1) Find k
2) Compute the mean and variance of $f(\ )$ and $g(\ )$

Assume now that we pluck a leaf from one of the bushes at random.

3) What is the probability that it is from bush B, given that it is more than 4cm long?

Assume now that we take a sample of 10 leaves from another bush (which is neither A nor B). The length of the leaves is: 5.4, 6.3, 5.8, 7.2, 4.9, 9.2, 7.0, 7.3, 6.9, 10.4

4) Compute the 95% confidence interval for the sample mean
5) Assume that every leaf is
   a. 5cm longer than before.
   b. Double as long as before

   What about the sample mean and the confidence interval?

Exercise 2

Consider a system having K *gates*, through which job requests may arrive. Through each gate, job requests arrive exponentially at a rate λ. Arrivals processes at different gates are independent. When there are $j < K$ requests in the system, $j+1$ gates are open, whereas the others are closed. If there are K or more requests, all the K gates are open. The system has K identical servers, with a serving rate equal to μ.

1) Model the above system as a queueing system and draw the transition-rate diagram.
2) Compute the steady-state probabilities and state the stability condition
3) Compute the state that a random observe is more likely to observe and the mean number of jobs in the system
4) Compute the throughput of the system. Compare it to an M/M/1's and discuss the result.
5) Compute the probabilities observed by an arriving job request
6) Compute the mean response time, the mean waiting time and the mean number of queued jobs.

Exercise 1 - Solution

1) *k* can be computed based on the normalization condition:

$$\int_0^8 \left(-\frac{k}{256}\cdot x^2 + \frac{k}{32}\cdot x\right)dx =$$
$$\frac{k}{32}\cdot\int_0^8 \left(-\frac{1}{8}\cdot x^2 + x\right)dx =$$
$$\frac{k}{64}\cdot\left[-\frac{1}{12}\cdot x^3 + x^2\right]_0^8 =$$
$$\frac{k}{64}\cdot\left[-\frac{512}{12}+64\right] =$$
$$\frac{k}{64}\cdot\left[\frac{-128+192}{3}\right] =$$
$$\frac{k}{3}$$

Since, by normalization, the integral must be equal to 1, it is $k = 3$

2) $g(\ )$ is a uniform RV, hence its mean is $\mu_g = \frac{1}{b-a} = 5$, and its variance is $\sigma^2 = \frac{(b-a)^2}{12} = \frac{25}{3}$.

On the other hand, the mean and variance of $f(\ )$ are computed by solving these integrals:

$$\mu_f = \int_0^8 x\cdot 3\cdot\left(-\frac{1}{256}\cdot x^2 + \frac{1}{32}\cdot x\right)dx =$$
$$= \frac{3}{32}\cdot\int_0^8 \left(-\frac{1}{8}\cdot x^3 + x^2\right)dx$$
$$= \frac{3}{32}\cdot\left[-\frac{1}{32}\cdot x^4 + \frac{1}{3}\cdot x^3\right]_0^8$$
$$= \frac{\cancel{3}}{32}\cdot\left[\frac{-3\cdot 4096+32\cdot 512}{\cancel{3}\cdot 32}\right]$$
$$= \frac{1}{32}\cdot\frac{4096}{32} = 4$$

$$\overline{X_f^2} = \int_0^8 x^2 \cdot 3 \cdot \left(-\frac{1}{256} \cdot x^2 + \frac{1}{32} \cdot x \right) dx =$$

$$= \frac{3}{32} \cdot \int_0^8 \left(-\frac{1}{8} \cdot x^4 + x^3 \right) dx$$

$$= \frac{3}{32} \cdot \left[-\frac{1}{40} \cdot x^5 + \frac{1}{4} \cdot x^4 \right]_0^8$$

$$= \frac{3}{128} \cdot \left[\frac{-32768 + 40960}{10} \right]$$

$$= \frac{3}{128} \cdot \frac{8192}{10} = 3 \cdot \frac{64}{10} = \frac{96}{5}$$

Hence, $\sigma_f^2 = \frac{96}{5} - (4)^2 = \frac{96 - 16 \cdot 5}{5} = \frac{16}{5}$

3) Call L the event “the leaf is more than 4 cm”. By Bayes’ Theorem, we have:

$$P(B|L) = \frac{P(L|B) \cdot P(B)}{P(L|B) \cdot P(B) + P(L|A) \cdot P(A)}.$$

However, $P(B) = P(A) = 1/2$, since the choice is “at random”. Furthermore, $P(L|B) = (10-4)/10 = 0.6$, and:

$$P(L|A) = \frac{3}{32} \cdot \int_4^8 \left(-\frac{1}{8} \cdot x^2 + x \right) dx$$

$$= \frac{3}{64} \cdot \left[-\frac{1}{12} \cdot x^3 + x^2 \right]_4^8$$

$$= \frac{3}{64} \cdot \left[\left(-\frac{512}{12} + 64 \right) - \left(-\frac{64}{12} + 16 \right) \right]$$

$$= \frac{3}{64} \cdot \left[48 - \frac{112}{3} \right]$$

$$= \frac{3}{64} \cdot \frac{32}{3} = \frac{1}{2}$$

Hence:

$$P(B|L) = \frac{P(L|B)}{P(L|B) + P(L|A)} = \frac{0.6}{0.6 + 0.5} = 0.545$$

4) The sample mean is

$$\bar{X} = \frac{1}{n} \cdot \sum_i x_i = 7.04.$$

The sample variance is:

$$S^2 = \frac{1}{n-1} \cdot \sum_i \left(x_i - \bar{X} \right)^2 = \frac{25.424}{9} = 2.824.$$

From the tabulated Student-s t function, I need $t_{\alpha/2,n-1} = t_{0.025,9} = 2.262$. The semi-width of the confidence interval centered around $\bar{X}$ is therefore:

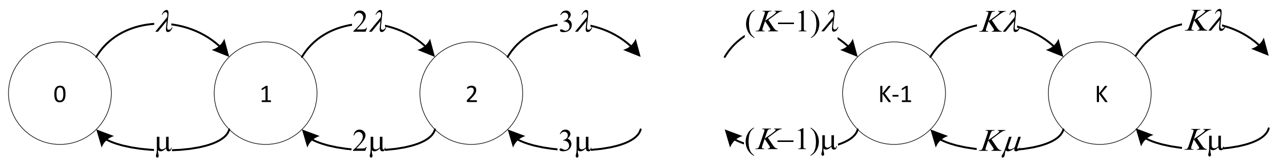
$$w = \frac{S}{\sqrt{n}} \cdot t_{\alpha/2,n-1} = \sqrt{\frac{2.824}{9}} \cdot 2.262 = 1.27.$$

The confidence interval is thus $I=[5.77;8.31]$

5) if each leaf is 5 cm longer, the mean will be 5cm larger and the variance will stay the same, hence the confidence interval will have the same width, i.e. $I_a = I+5=[10.77;13.31]$. If, on the other hand, the length is doubled, the mean will be doubled as well, and the standard deviation will be double as much. Hence, we will have $I_b = 2I=[11.54;16.62]$.

Exercise 2 – solution

The TR diagram is the following:



From the above (transitions are nearest-neighbor) one straightforwardly obtains $p_j = \left(\frac{\lambda}{\mu}\right)^j \cdot p_0$. Call $\rho = \lambda/\mu$, then the SS probabilities and the stability condition are the same as an M/M/1's, i.e. $p_j = (1-\rho)\cdot\rho^j$, as long as $\rho < 1$.
The state that a random observer is more likely to observe is therefore state 0. The mean number of jobs in the system is given by the Kleinrock function $E[N] = \frac{\rho}{1-\rho}$.

The throughput is

$$\gamma = \sum_{j=1}^{+\infty} j\cdot\mu_j = \sum_{j=1}^{K} j\cdot\mu\cdot(1-\rho)\cdot\rho^j + \sum_{j=K+1}^{+\infty} K\cdot\mu\cdot(1-\rho)\cdot\rho^j$$
$$= \mu\cdot(1-\rho)\sum_{j=1}^{K} j\cdot\rho^j + K\cdot\mu\cdot(1-\rho)\sum_{j=K+1}^{+\infty}\rho^j$$

$$= \mu\cdot(1-\rho)\rho\frac{1-(K+1)\rho^K + K\rho^{K+1}}{(1-\rho)^2} + K\cdot\mu\cdot(1-\rho)\frac{\rho^{K+1}}{1-\rho}$$
$$= \lambda\cdot\frac{1-(K+1)\rho^K + K\rho^{K+1}}{1-\rho} + K\cdot\mu\cdot\rho^{K+1}$$
$$= \lambda\cdot\left[\frac{1-\rho^K - K\rho^K(1-\rho)}{1-\rho} + K\cdot\rho^K\right]$$
$$= \lambda\cdot\frac{1-\rho^K}{1-\rho}$$

When K=1, this system is an M/M/1, and from the above expression we have $\gamma = \lambda$. When K>1, the throughput is *larger* than an M/M/1's, since the average arrival rate is larger than λ. Note that we always have $\gamma = \bar{\lambda}$.

The probabilities observed by an arriving job are $r_j = \frac{\lambda_j}{\bar{\lambda}}\cdot p_j$. Therefore we have:

$$r_j = \begin{cases} (j+1) \cdot \dfrac{1-\rho}{1-\rho^K}(1-\rho) \cdot \rho^j & j < K \\ K \cdot \dfrac{1-\rho}{1-\rho^K}(1-\rho) \cdot \rho^j & j \geq K \end{cases}$$

The mean response time, by Little's Law, is:

$$E[R] = \frac{E[N]}{\gamma} = \frac{\rho}{1-\rho} \cdot \frac{1}{\lambda} \cdot \frac{1-\rho}{1-\rho^K} = \frac{1}{\mu} \cdot \frac{1}{1-\rho^K}$$

Moreover, we get:

$$E[W] = E[R] - E[t_s] = \frac{1}{\mu} \cdot \frac{\rho^K}{1-\rho^K}$$

And:

$$E[Nq] = \gamma \cdot E[W] = \frac{1}{\mu} \cdot \frac{\rho^K}{1-\rho^K} \cdot \lambda \cdot \frac{1-\rho^K}{1-\rho} = \frac{\rho^{K+1}}{1-\rho}$$