An Introduction to Fuzzy Logic
Part I

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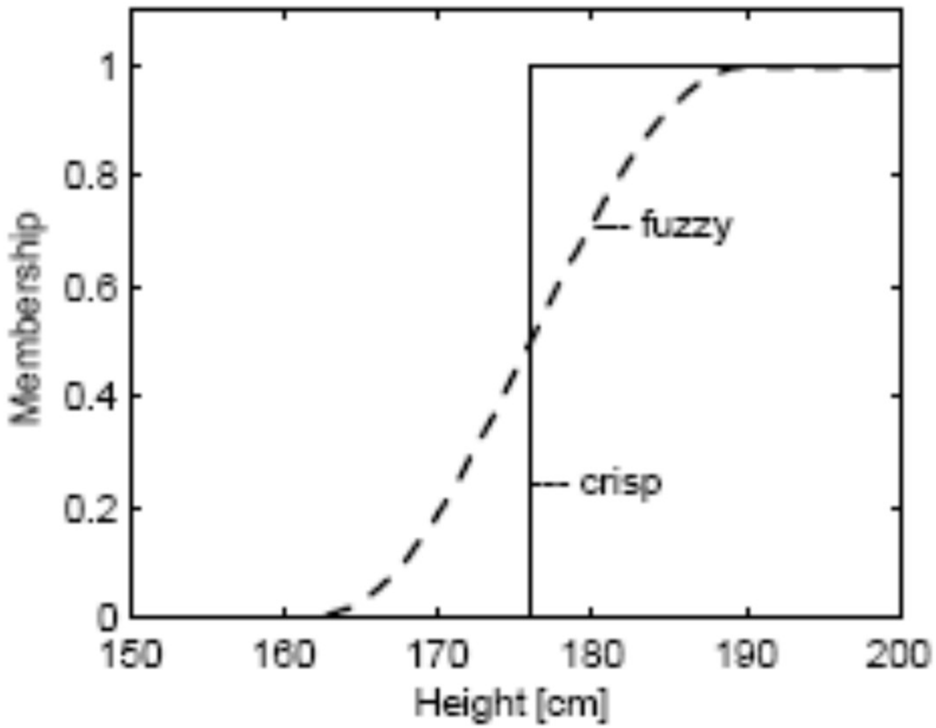
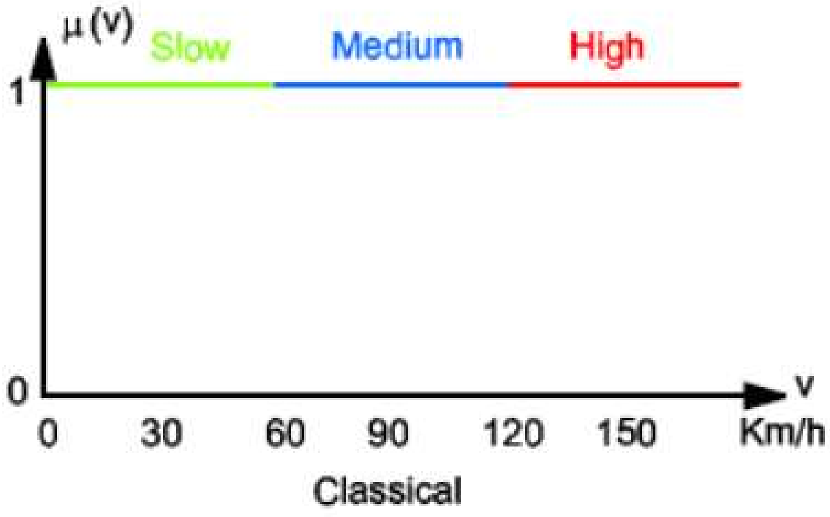
Fuzzy sets

- *Classical* (or *crisp*) set: a given element either belongs or does not belong to the set.
- *Fuzzy* set: each element can belong to the set with any degree between 0 and 1.
- Each fuzzy set is defined on a set of elements that form the *universe of discourse*, and the degree to which an element belongs to the fuzzy set is given by a *membership function*.
- Examples: the set of high temperatures, the set of low speeds, the set of nice days, etc.

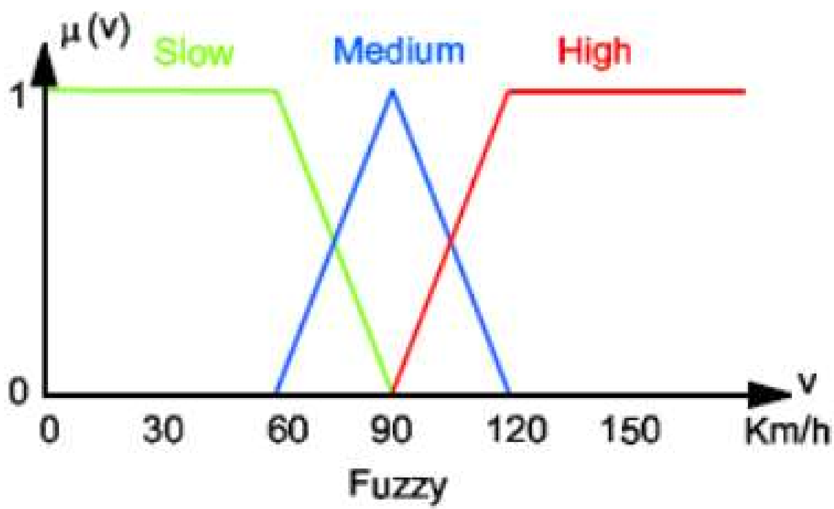
- Example: set of tall men



- Another example: speed



Representation of speed with classical sets



Representation of speed with fuzzy sets

More formally:

- A fuzzy set *A* is characterized by its *membership function*

$$\mu_A : X \rightarrow [0,1]$$

  where *X* is the *universe of discourse*, or *universe of scope* (continuous or discrete).
- The value $\mu_A(x)$, or simply *A*(*x*), is the membership degree of *x* to fuzzy set *A*:

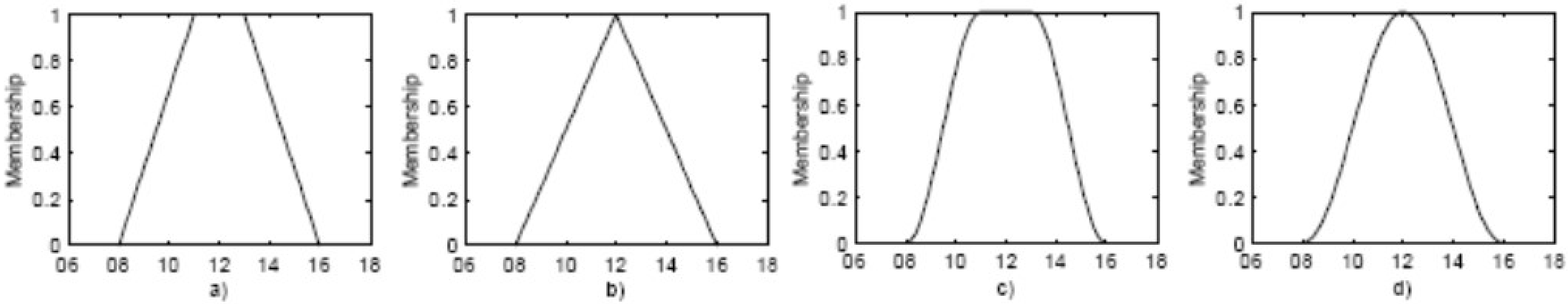
$$A = \left\{ \left(x, \mu_A(x) \right) \mid x \in X \right\}$$

X discrete $\quad A = \left\{ \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots \right\} = \left\{ \sum_i \frac{\mu_A(x_i)}{x_i} \right\} \quad A = \sum_X \mu_A(x)/x$

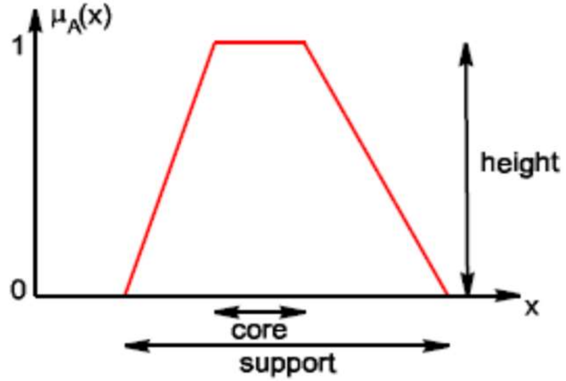
X continuous $\quad A = \left\{ \int \frac{\mu_A(x)}{x} \right\} \quad A = \int_X \mu_A(x)/x$

Membership function

- Typical membership functions are *bell-shaped* membership functions, *triangular* membership functions, and *trapezoidal* membership functions.
- For example, suppose you want to define the concept “around noon”:



Properties of fuzzy sets



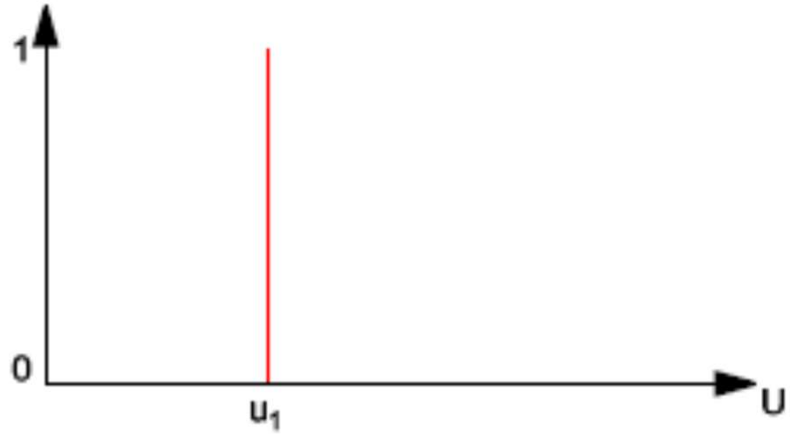
$$\text{height}(A) = \sup_{x \in X} \mu_A(x)$$

$$\text{core}(A) = \{x \in X \mid \mu_A(x) = 1\}$$

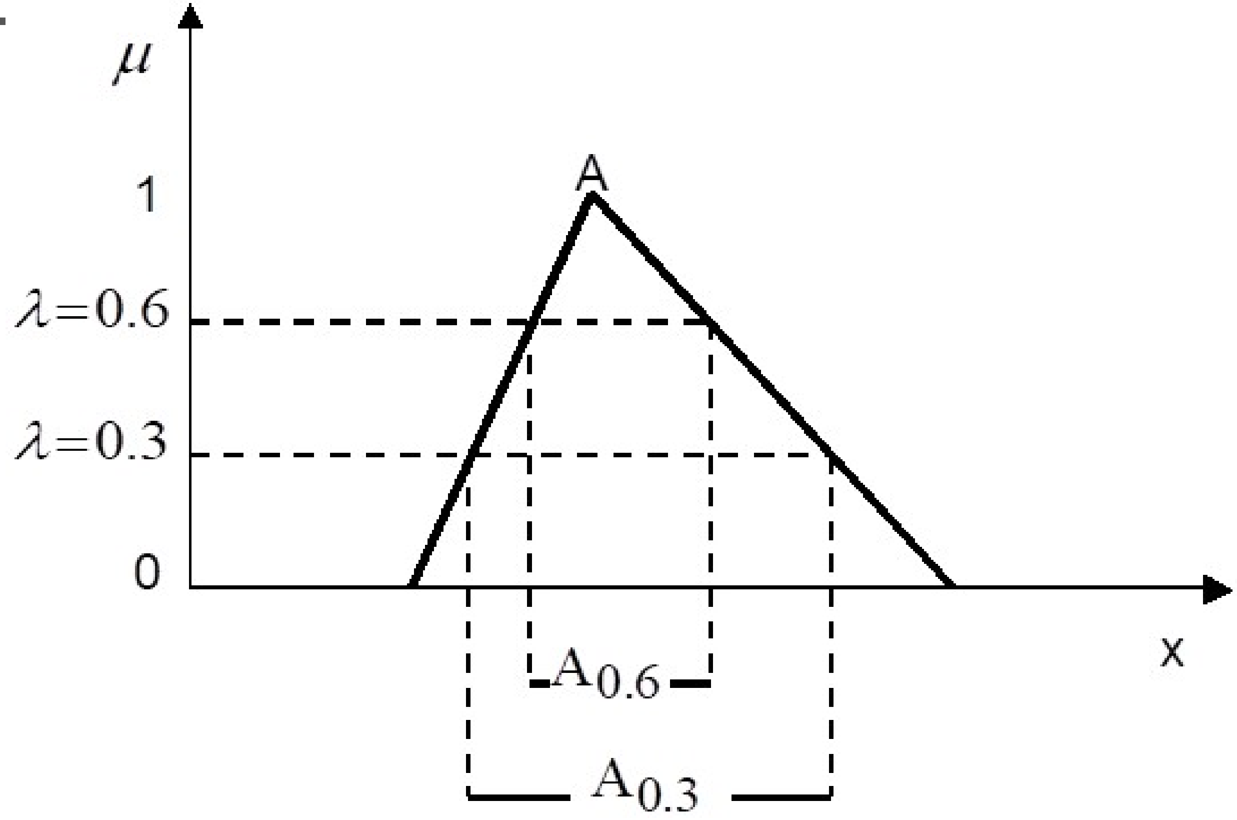
$$\text{support}(A) = \{x \in X \mid \mu_A(x) > 0\}$$

Fuzzy singleton

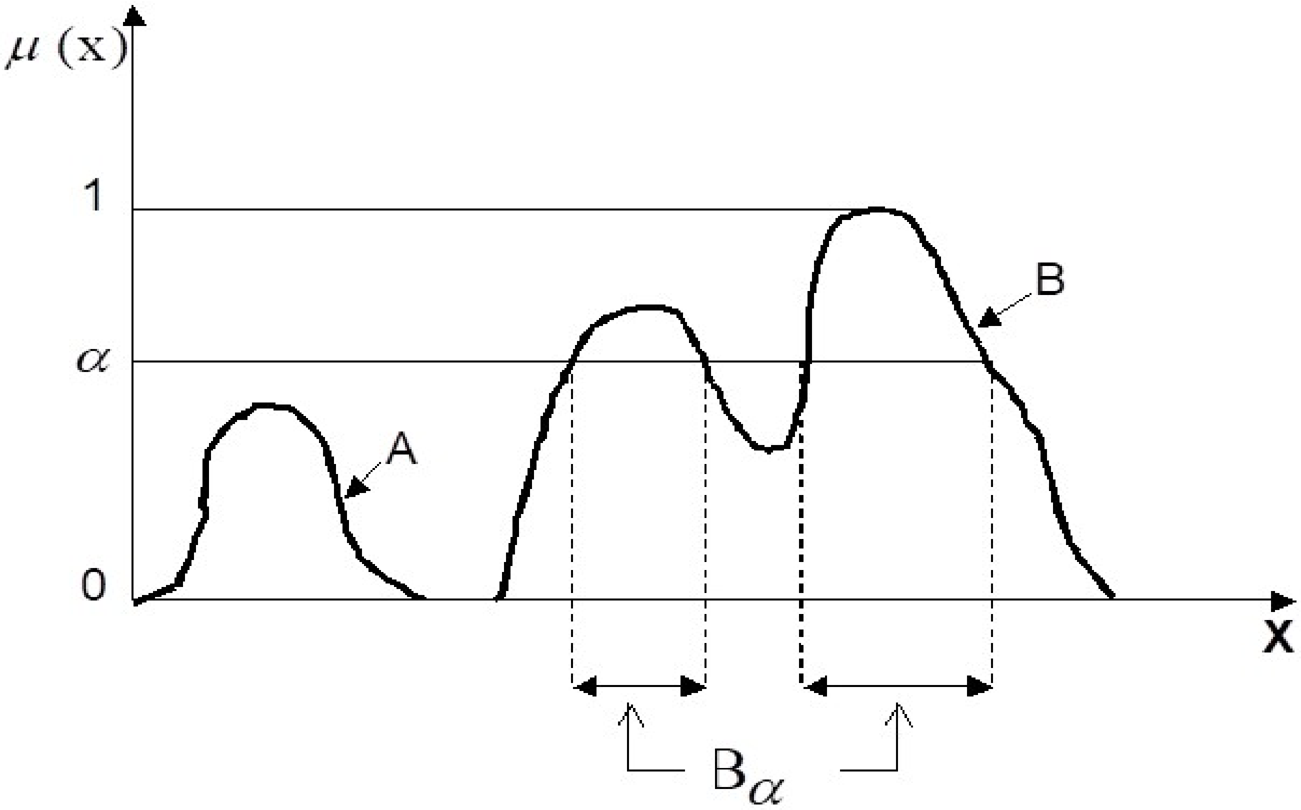
- A fuzzy *singleton* is a fuzzy set whose support is a single point.



Alpha-cut (or lambda-cut)



- The *α-cut* (or *λ-cut*) is the crisp set $A_\alpha = \{x | \mu_A(x) \geq \alpha\}$.
- *Convex* fuzzy set: all α-cuts, $\alpha \in (0,1]$, are convex sets.



A: subnormal convex fuzzy set

B: normal non-convex fuzzy set

Operations on fuzzy sets

- Classical *intersection*, *union* and *complement* can be extended to fuzzy set theory.
- Functions that qualify as *fuzzy intersections* and *fuzzy unions* are *triangular norms* (*t-norms*) and *triangular conorms* (*t-conorms* or *s-norms*), respectively.

$$(A \cap B)(x) = T[A(x), B(x)]$$

$$(A \cup B)(x) = S[A(x), B(x)]$$

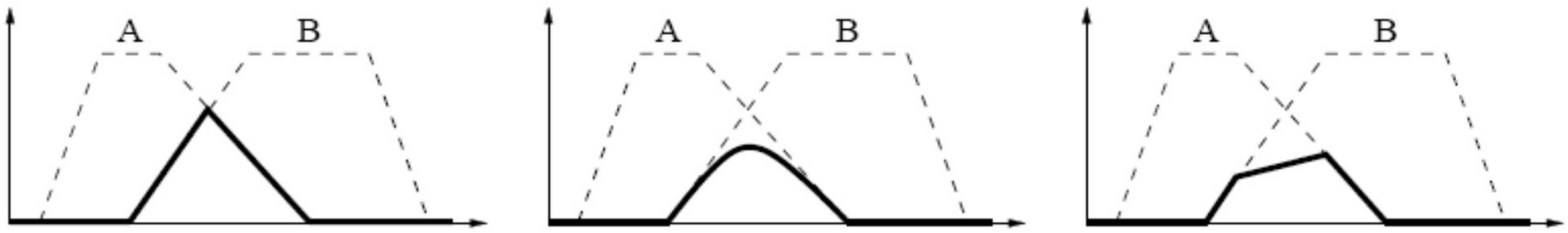
T-norm

- A *triangular norm* (or *t-norm*) is a function T: [0, 1] × [0, 1] → [0, 1] that satisfies the following properties:
  - Commutativity: T(*a*, *b*) = T(*b*, *a*)
  - Monotonicity: T(*a*, *b*) ≤ T(*c*, *d*) if *a* ≤ *c* and *b* ≤ *d*
  - Associativity: T(*a*, T(*b*, *c*)) = T(T(*a*, *b*), *c*)
  - identity element: T(*a*, 1) = *a*

Fundamental t-norms

- **minimum** $T(a,b)=\min(a,b)$
- **product** $T(a,b)=a\cdot b$
- **bounded product** $T(a,b)=\max(0,\ a+b-1)$

$$
\begin{aligned}
\text{minimum} &: \quad \min(\mu_A(x), \mu_B(x)) \\
\text{product} &: \quad \mu_A(x) \cdot \mu_B(x) \\
\text{bounded product} &: \quad \max(0, \mu_A(x) + \mu_B(x) - 1)
\end{aligned}
$$



minimum product bounded product

T-conorm

- *Triangular conorms* (*T-conorms* or *S-norms*) are dual to t-norms. Given a t-norm *T*, the complementary conorm *S*

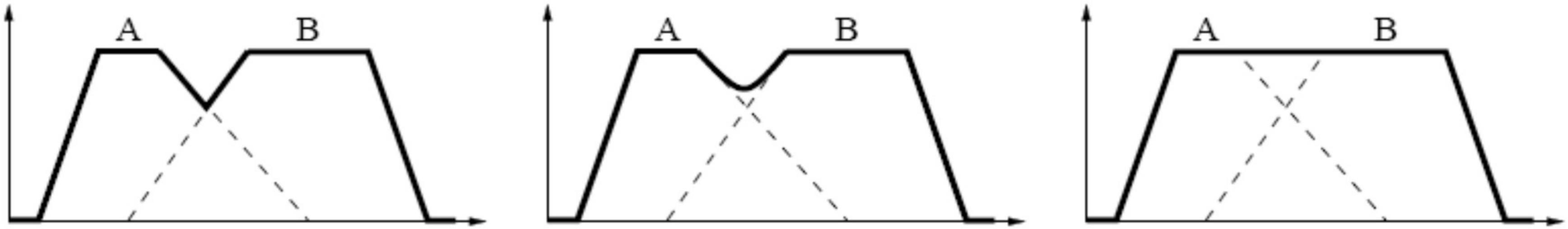
$$S(a,b)=1-T(1-a,1-b)$$

- A t-conorm satisfies the following properties:
  - Commutativity: S(*a*, *b*) = S(*b*, *a*)
  - Monotonicity: S(*a*, *b*) ≤ S(*c*, *d*) if *a* ≤ *c* and *b* ≤ *d*
  - Associativity: S(*a*, S(*b*, *c*)) = S(S(*a*, *b*), *c*)
  - identity element: S(*a*, 0) = *a*

Fundamental t-conorms
(dual t-conorms of the fundamental t-norms)

- **maximum** $T(a,b)=\max(a,b)$
- **probabilistic sum** $T(a,b)=a+b-a\,b$
- **bounded sum** $T(a,b)=\min(1,a+b)$

$$\begin{aligned} \text{maximum} &: \max(\mu_A(x), \mu_B(x)) \\ \text{probabilistic sum} &: \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \\ \text{bounded sum} &: \min(1, \mu_A(x) + \mu_B(x)) \end{aligned}$$



maximum   probabilistic sum   bounded sum

Fuzzy complement

$$\overline{A}(x) = C\big(A(x)\big)$$

where $C:[0,1] \rightarrow [0,1]$ satisfies the following requirements:

- C(0) = 1 and C(1) = 0 (boundary conditions)
- C is monotonic decreasing
- C is a continuous function
- C is involutive, i.e., C(C(x)) = x

Alternative definitions of fuzzy complement

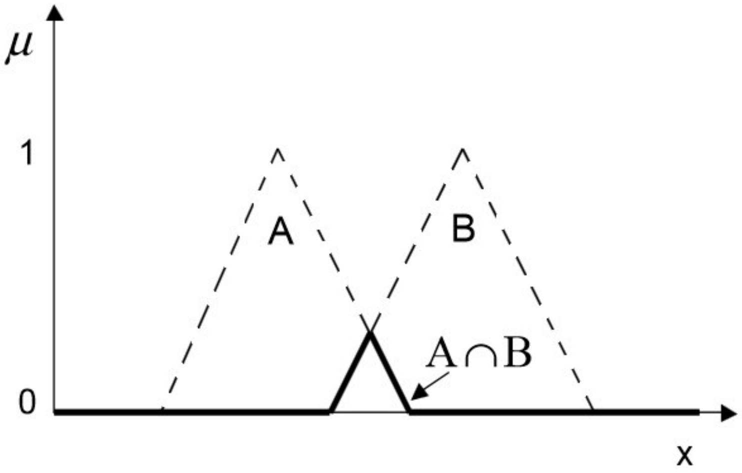
- standard complement $\bar{A}(x) = 1 - A(x)$
- round complement $\bar{A}(x) = \sqrt{1 - [A(x)]^2}$
- Yager $\bar{A}(x) = \left(1 - [A(x)]^p\right)^{1/p}, \ p \in (0, \infty)$
- Sugeno $\bar{A}(x) = \dfrac{1 - x}{1 + sx}, \ s \in (-1, \infty)$

Standard fuzzy operations

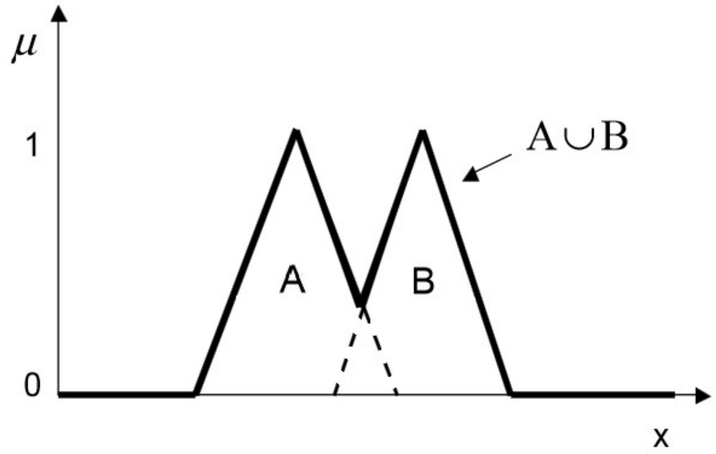
$$\left(A \cap B\right)(x) = \min\{A(x), B(x)\}$$

$$\left(A \cup B\right)(x) = \max\{A(x), B(x)\}$$

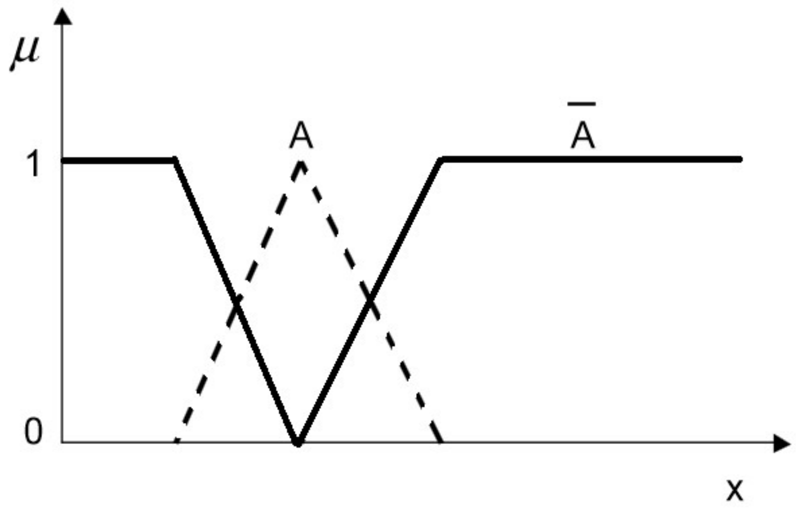
$$\overline{A}(x) = 1 - A(x)$$



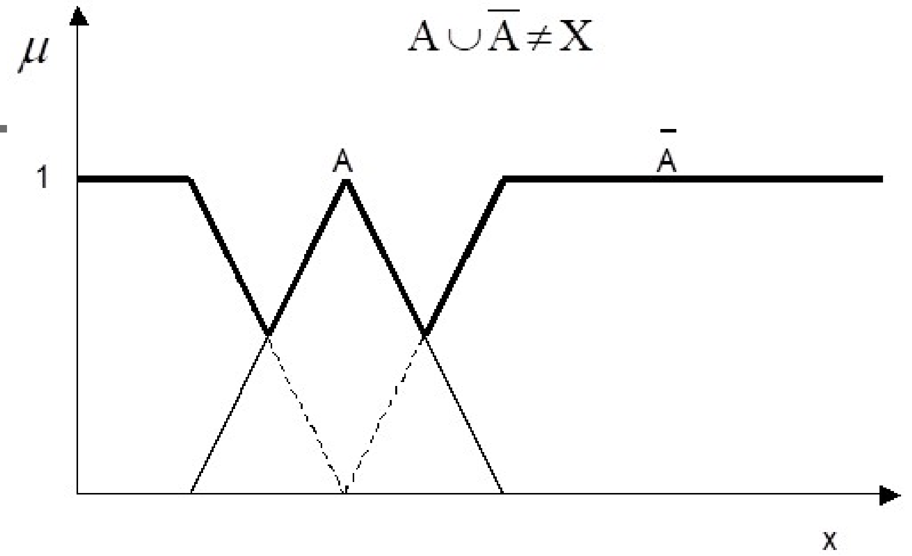
μ
1
0
A
B
A∩B
x



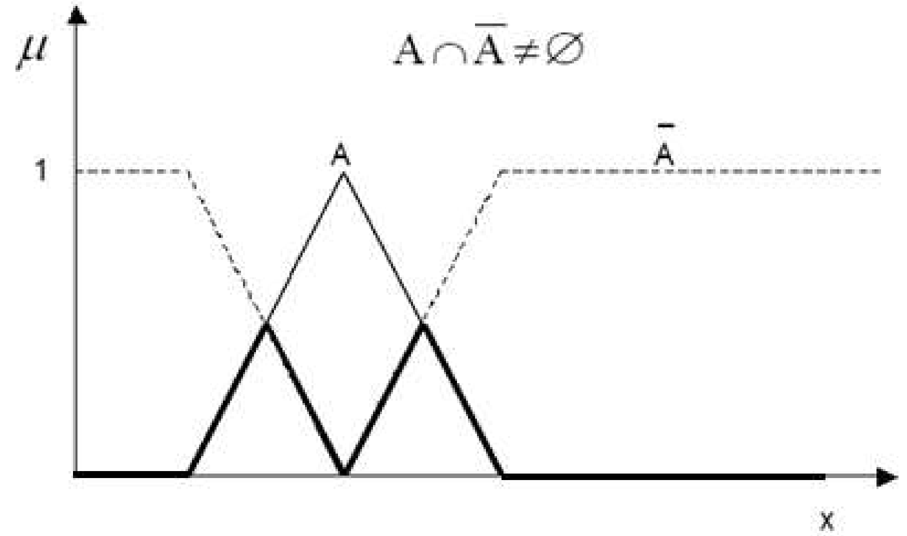
μ
1
0
A∪B
A
B
x



μ
1
0
A
$\bar{A}$
x

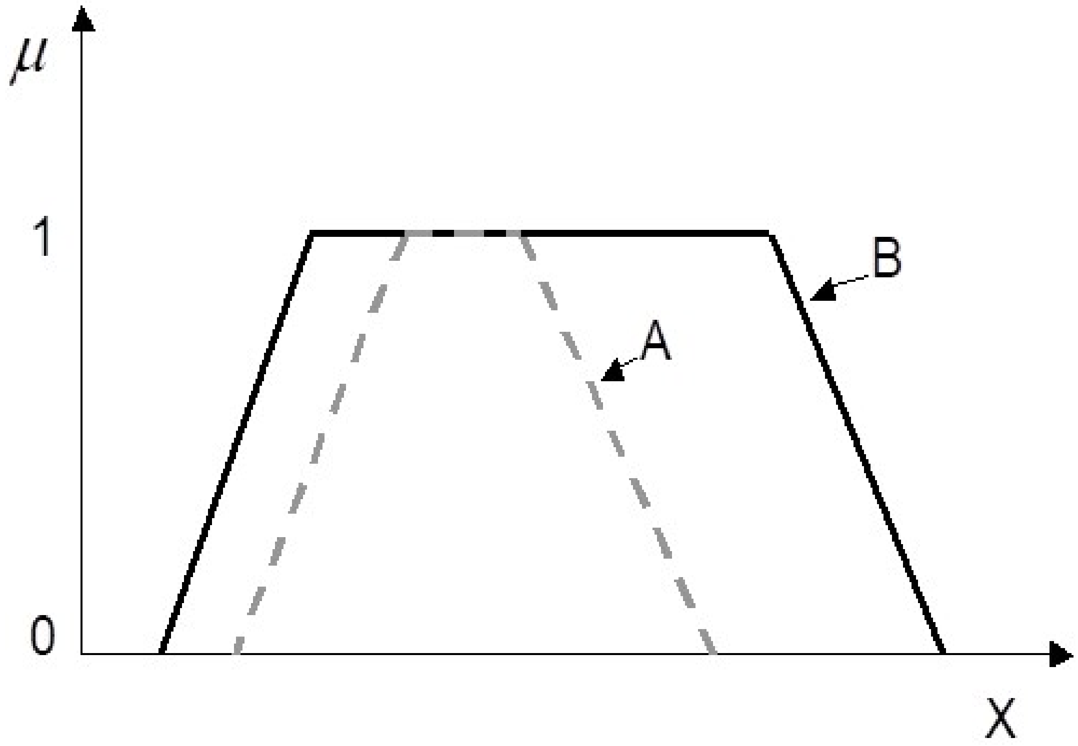


μ
$A \cup \bar{A} \neq X$
1
A
$\bar{A}$
x



μ
$A \cap \bar{A} \neq \varnothing$
1
A
$\bar{A}$
x

Inclusion of fuzzy sets



$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$$

Example 1

A four-person family wants to buy a house that is confortable and large:

- given the universe U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} (number of bedrooms) we define the fuzzy sets

  Comfortable = [0.2 0.5 0.8 1 0.7 0.3 0 0 0 0]

  Large = [0 0 0.2 0.4 0.6 0.8 1 1 1 1]

- The *intersection* of *Comfortable* and *Large* is

min(Comfortable, Large) = [0 0 0.2 0.4 0.6 0.3 0 0 0 0]

- If the goal were to satisfy at least one criterion, we would perform the *union* of *Comfortable* and *Large*

  max(Comfortable, Large) =

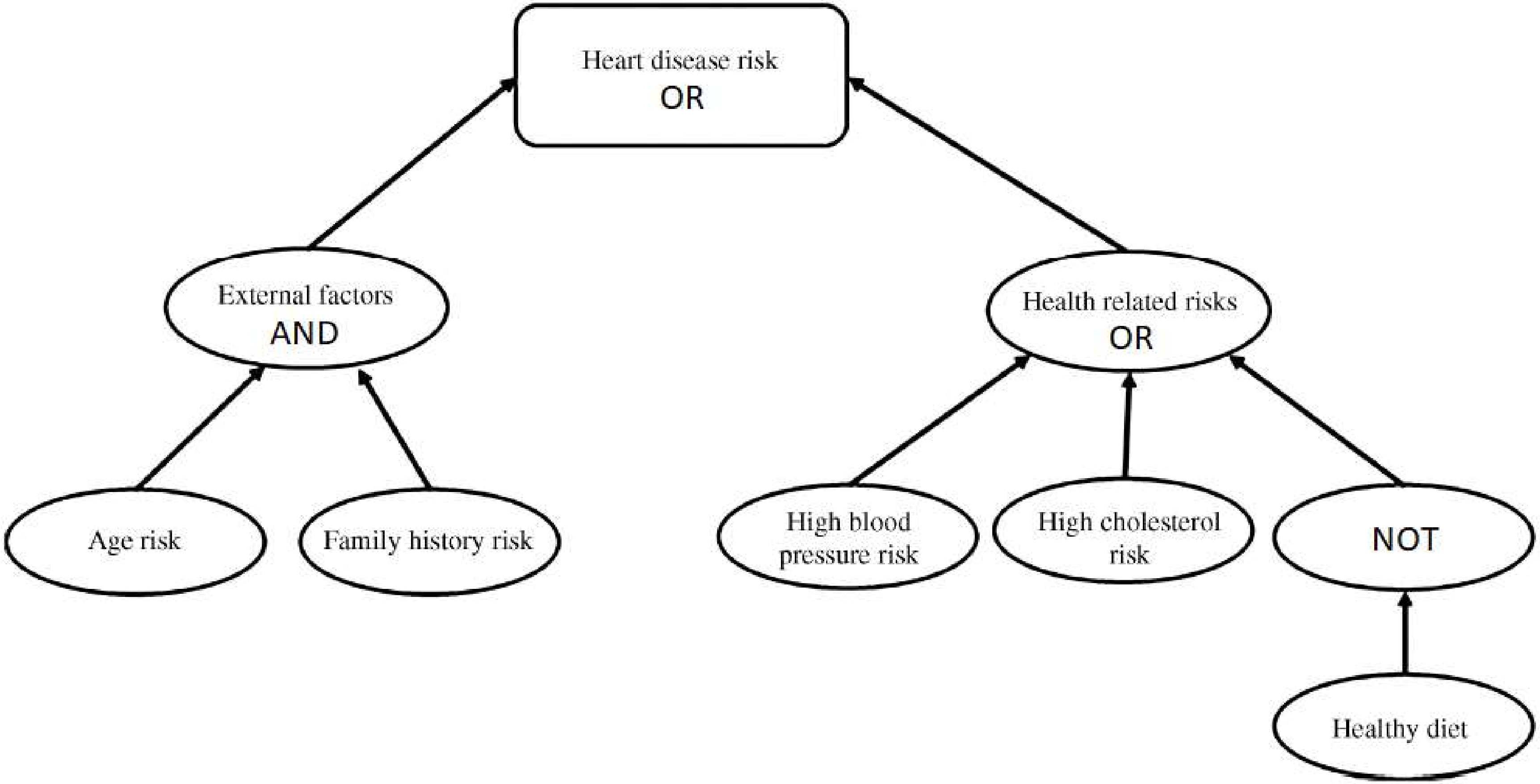
  [0.2 0.5 0.8 1 0.7 0.8 1 1 1 1]

- If the children move away from the family within a year or two, parents may choose to buy a house that is *Comfortable* and *Not Large*, or

$$\min(\text{Comfortable}, 1 - \text{Large}) =$$

$$[\ 0.2 \quad 0.5 \quad 0.8 \quad 0.6 \quad 0.4 \quad 0.2 \quad 0 \quad 0 \quad 0 \quad 0\ ]$$

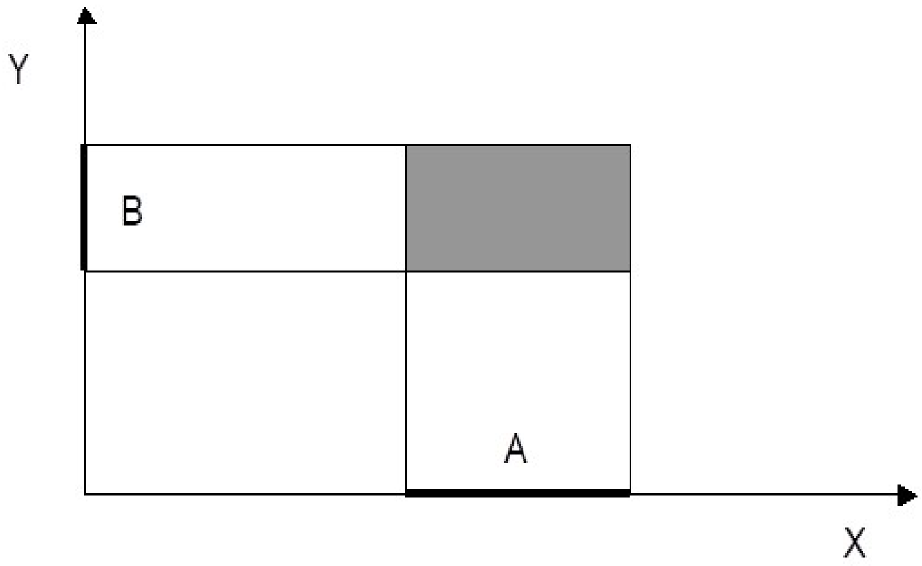
Example 2



Heart disease risk
OR
External factors
AND
Health related risks
OR
Age risk
Family history risk
High blood pressure risk
High cholesterol risk
NOT
Healthy diet

Crisp relations

Cartesian product of crisp sets A and B:



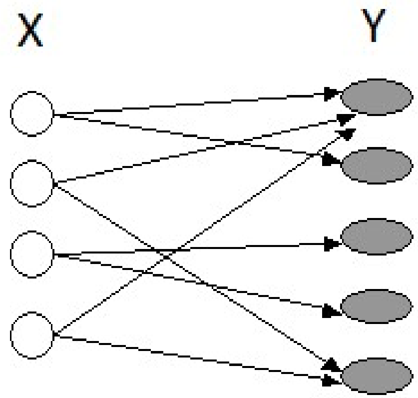
$$A \times B = \{(x, y) | x \in A, y \in B\}$$

A *crisp relation* is a subset of the Cartesian product.

- characteristic function

$$R(x,y)=\begin{cases}1 \text{ if } (x,y)\in R\\ 0 \text{ if } (x,y)\notin R\end{cases}$$

- diagram



- matrix

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ R = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

X = {1,2,3}   Y = {a,b,c}

Composition of crisp relations

$$R : X \rightarrow Y, \quad S : Y \rightarrow Z$$

max-min composition $W = R \circ S : X \rightarrow Z$

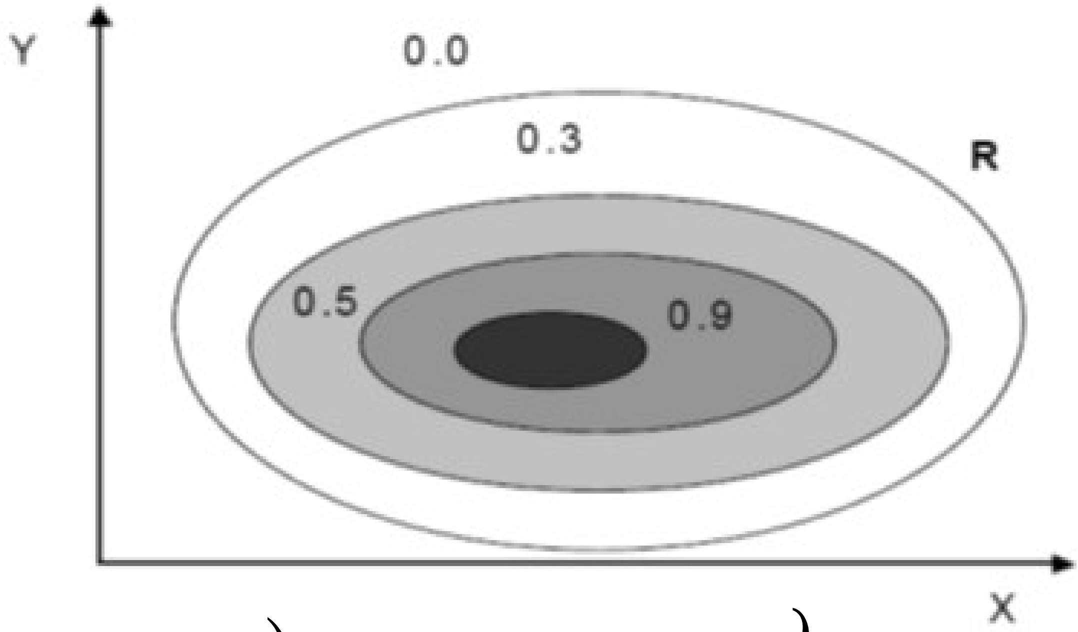
$$\chi_W(x,z) = \bigvee_{y \in Y} \left(\chi_R(x,y) \wedge \chi_S(y,z) \right)$$

max-product (or *max-dot*) composition $W = R \circ S : X \rightarrow Z$

$$\chi_W(x,z) = \bigvee_{y \in Y} \left(\chi_R(x,y) \bullet \chi_S(y,z) \right)$$

Fuzzy relations

- A *fuzzy relation* is a fuzzy set defined on the Cartesian product



$$R : X \times Y \to [0,1]$$

$$R(X,Y) = \{((x,y), \mu_R(x,y)) \mid (x,y) \in X \times Y\}$$

$$R \cup \overline{R} \neq E \qquad E = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R \cap \overline{R} \neq O \qquad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- relation 'very far'

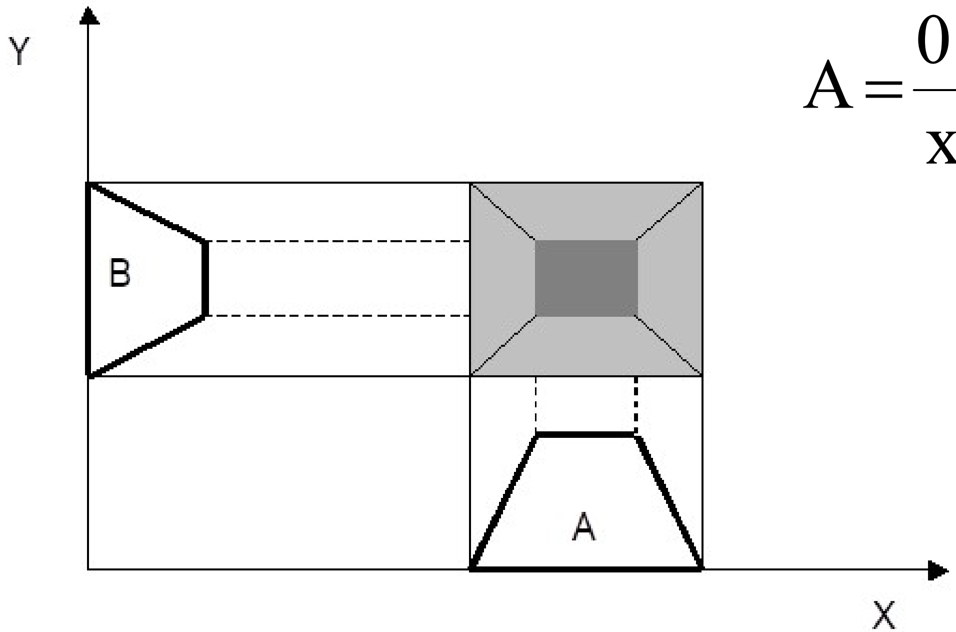
	New York	Paris
Bejing	1	.9
New York	0	.6
Rome	.7	.3

$$R(X,Y) = \{1/\text{NewYork, Bejing} + 0/\text{NewYork, NewYork} + \ldots\}$$

Cartesian product of fuzzy sets

$$A : X \to [0,1],\ B : Y \to [0,1]$$ $A \times B$ is a fuzzy relation:

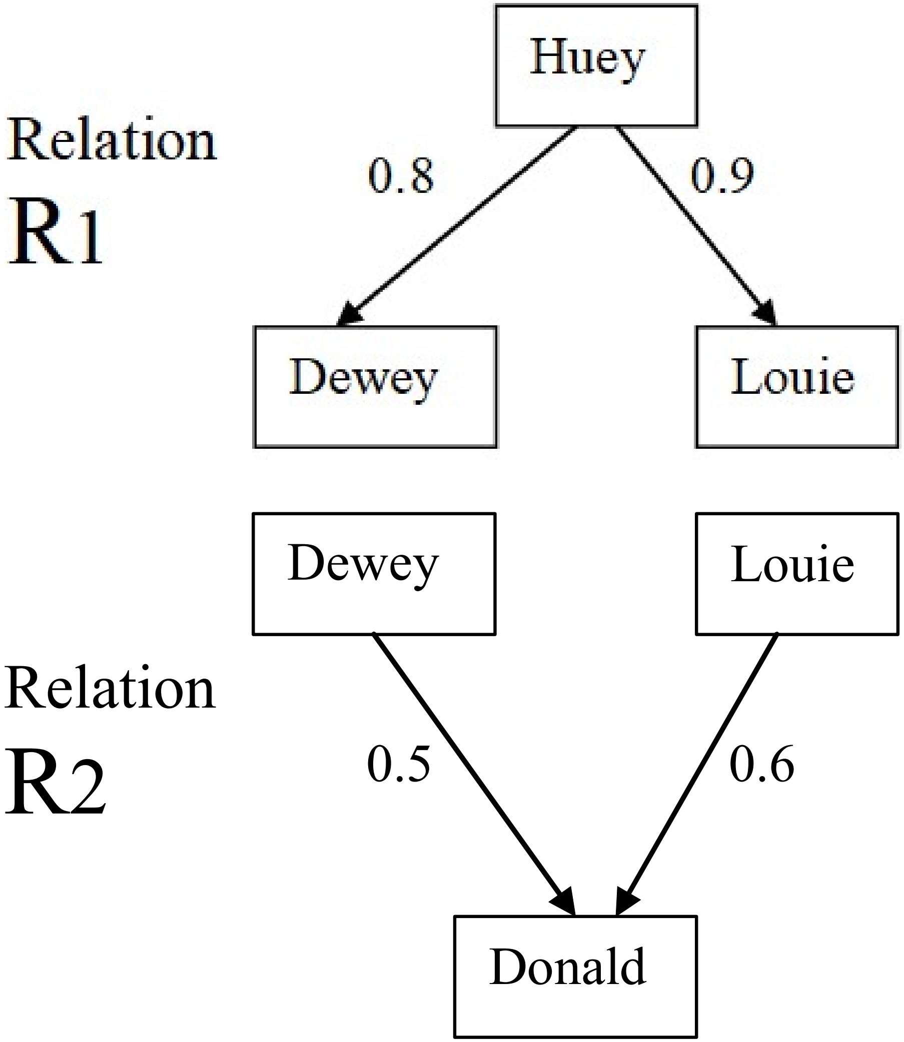
$$A \times B : X \times Y \to [0,1] \quad (x, y) \to \min(A(x), B(y))$$

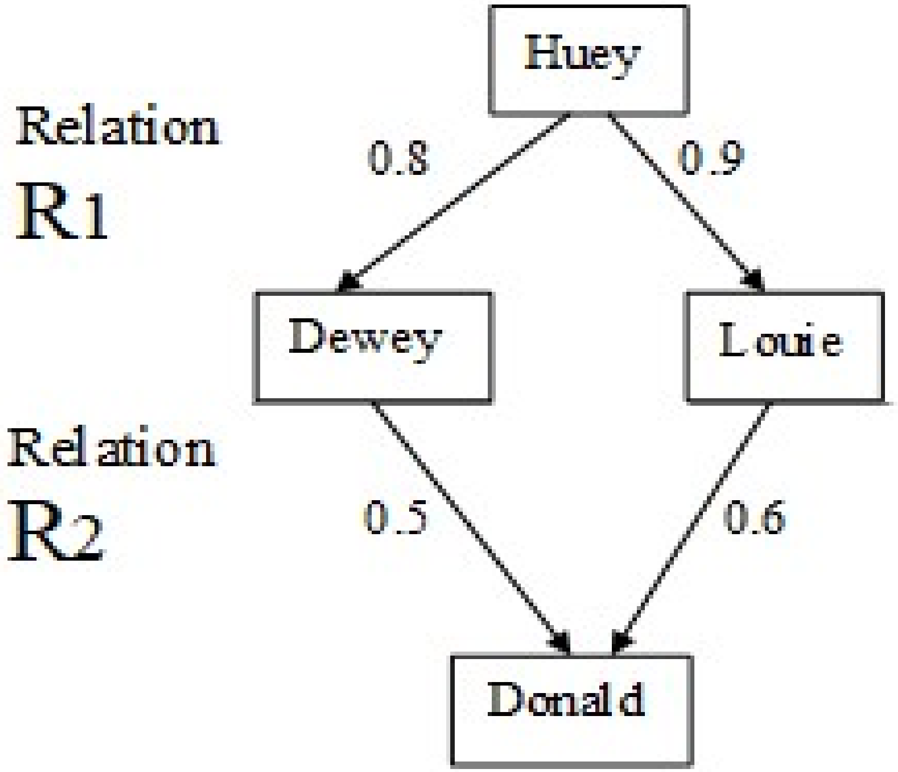


$$A = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \qquad B = \frac{0.3}{y_1} + \frac{0.9}{y_2}$$

$$A \times B = R = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.2 & 0.2 \\ x_2 & 0.3 & 0.5 \\ x_3 & 0.3 & 0.9 \end{array}$$

Example


Relation
R1
Huey
0.8
0.9
Dewey
Louie
Relation
R2
Dewey
Louie
0.5
0.6
Donald


Relation
R1
Relation
R2
Huey
0.8
0.9
Dewey
Louie
0.5
0.6
Donald

Standard composition

- Consider two binary fuzzy relations $R(X,Y)$ and $S(Y,Z)$ with a common set Y Their *standard composition* produces a binary relation $R \circ S$ on $X \times Z$:

$$(R \circ S)(x,z) = \max_{y \in Y} \min\{R(x,y), S(y,z)\}$$

Example

$$\begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.3 & 0.8 \\ 0.6 & 0.9 \end{bmatrix} \\ & R \end{array} \circ \begin{array}{c c} & \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.5 & 0.9 \\ 0.4 & 1 \end{bmatrix} \\ & S \end{array} = \begin{array}{c c} & \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.4 & 0.8 \\ 0.5 & 0.9 \end{bmatrix} \\ & R \circ S \end{array}$$

Sup-T composition of fuzzy relations

- The *sup-T composition* of binary fuzzy relations, where *T* refers to a *t*-norm, generalizes the standard max-min composition:

$$\left(R \circ S\right)(x,z) = \sup_{y \in Y} T\left\{R(x,y), S(y,z)\right\}$$

- The composition of fuzzy relations plays a crucial role in the study of approximate reasoning, as we will see shortly.