Communication systems

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ELECTRONICS AND COMMUNICATIONS SYSTEMS

COMPUTER ENGINEERING

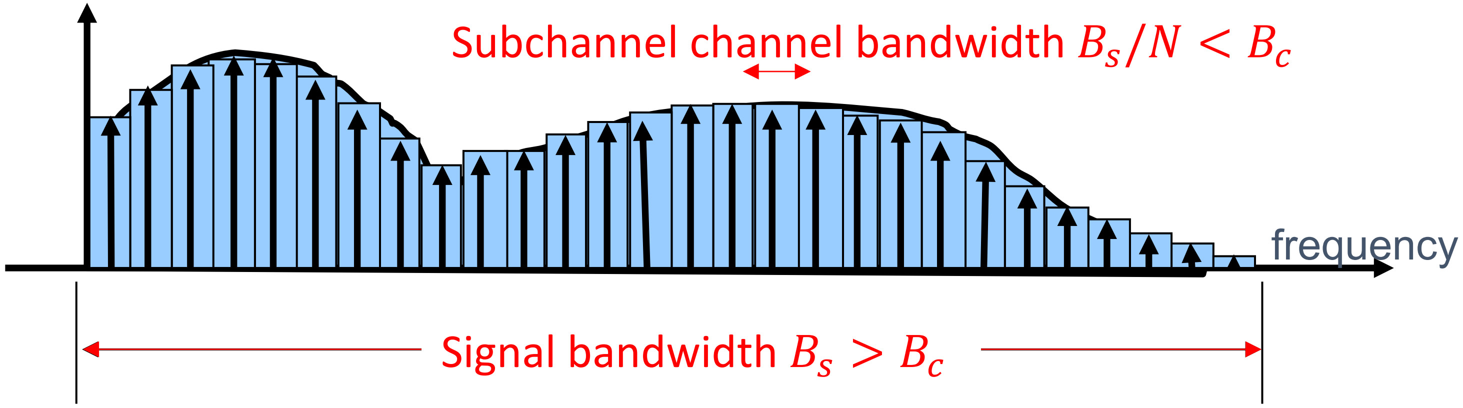
Multi-carrier signals

Multi-carrier technology

- Multi-carrier modulations are at the base of the physical layer for both LTE and 5G.
- Main reasons for the success of multicarrier modulations:
  - Robustness versus multipath fading: as the data rates increase, multipath becomes the major problem for single carrier transmissions.
  - Spectrally efficient.
  - Flexible resource allocation: OFDMA exploits channel frequency diversity by dynamically assigning the radio resources to the users.

OFDM technology

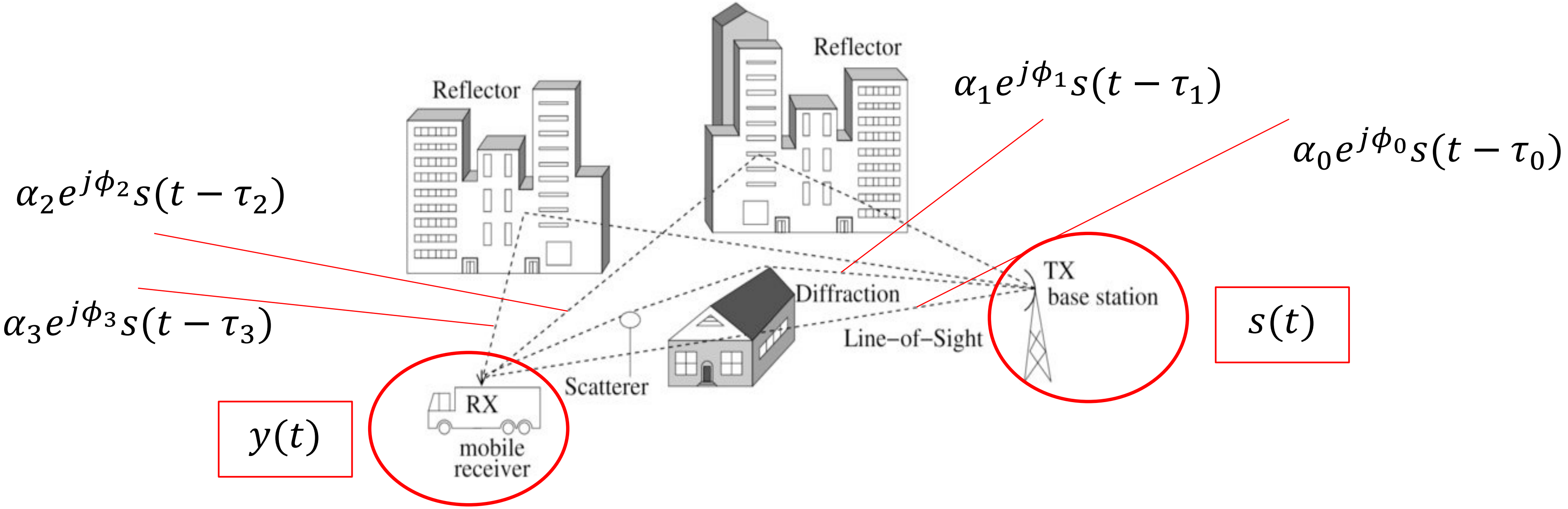
- The multi-carrier concept is that instead of transmitting a single-carrier signal occupying a bandwidth B_s, the data stream is organized in N parallel multicarrier streams each occupying a bandwidth B_s/N.
- Provided that the system is accurately dimensioned, each sub-carrier can be approximated as flat fading.
- The channel is *frequency selective* for $B_s > B_c$, but is flat for each subcarrier of bandwidth $B_s/N < B_c$



The frequency-selective multipath channel

- Assuming there are N_c resolvable paths, the multipath channel takes this form

$$h(t) = A_{LS} \sum_{\ell=0}^{N_c-1} \alpha_\ell e^{j\phi_\ell} \delta(t - \tau_\ell)$$



Channel as a tapped delay line

- When a signal with symbol time T propagates through the channel $h(t)$, the channel impulse response can be resampled at intervals multiple di *T* and the *equivalent channel impulse* response is

$$h_{eq}(t) = \sum_{\ell=0}^{L-1} h(\ell)\delta(t - \ell T)$$

- Even if *L* might be different from N_c, the channel characteristics do not change, and the complex envelope of the signal received through the multipath channel is

$$y(t) = \sum_{m=0}^{N_c-1} \alpha_m e^{j\phi_m} s(t - \tau_m) = \sum_{\ell=0}^{L-1} h(\ell) s(t - \ell T)$$

OFDM signal model (1)

- Because of the multipath, at the receiver the signal is corrupted by inter-symbol interference
$$y(k) = h(0)s(k) + \sum_{\ell=1}^{L-1} h(\ell)s(k-\ell)$$
- Let's consider the transmission of a single block $\boldsymbol{s} = [s(0), s(1), \dots, s(N-1)]$ of N samples.
- After passing through the channel, the received samples are
$$\begin{aligned} y(k) &= \sum_{\ell=0}^{L-1} h(\ell)s(k-\ell) \\ &= h(0)s(k) + \cdots + h(L-1)s(k-L+1) \end{aligned}$$

OFDM signal model (2)

- Since we are considering a single block of N samples, the elements of $\boldsymbol{s}$ are not defined for negative indices.
- Accordingly, the values of the samples $s(-1), s(-2), \ldots, s(L-1)$ is 0 and the received samples are

$$
\begin{aligned}
y(0) &= h(0)s(0) \\
y(1) &= h(0)s(1) + h(1)s(0) \\
&\vdots \\
y(N-1) &= h(0)s(N-1) + h(1)s(N-2) + \cdots + h(L-1)s(N-L)
\end{aligned}
$$

OFDM signal model: matrix notation

- In matrix notation the block of received samples $\mathbf{y}$ can be represented as

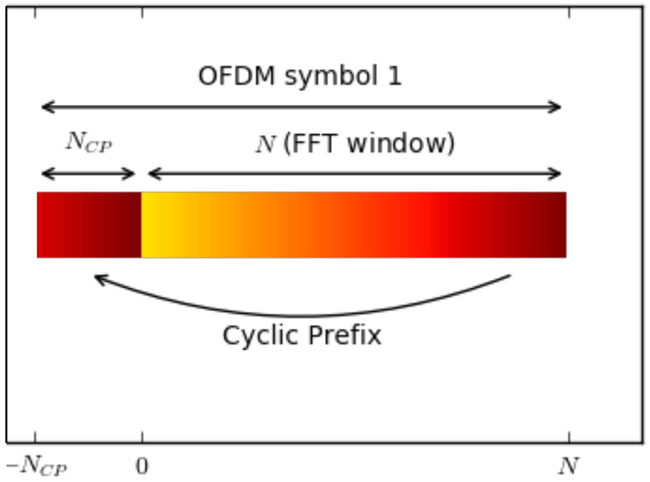
$$\mathbf{y} = \mathcal{H}\mathbf{s}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ h(1) & h(0) & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & h(1) & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ h(L-1) & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & h(L-1) & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & h(L-1) & \ddots & \ddots & h(1) & h(0) \end{bmatrix} \begin{bmatrix} s(0) \\ s(1) \\ \vdots \\ \vdots \\ s(N-1) \end{bmatrix}$$

The elements along any diagonal of the $N \times N$ matrix $\mathcal{H}$ are all equal and $\mathcal{H}$ is called a *Toeplitz* matrix.

OFDM signal model: cyclic extension (1)

- By copying the last $N_{CP} > L$ samples of $\boldsymbol{s}$ and adding them at the beginning of the block, the block assumes a *circular* structure, i.e. the first N_{CP} and last N_{CP} samples are equal, $\bar{\boldsymbol{s}} = [s(N - N_{CP} - 1), \dots, s(N-1), s(0), \dots, s(N-1)]$.



- Keeping the same indexing, the samples with negative indexes take the values $$\bar{s}(-1) = s(N-1), \bar{s}(-2) = s(N-2), \dots, \bar{s}(-N_{CP}) = s(N - N_{CP})$$

OFDM signal model: cyclic extension (2)

- After the cyclic extension, the received signal becomes

$$y(k) = \sum_{\ell=0}^{L-1} h(\ell)\bar{s}(k-\ell)$$

$$y(0) = h(0)\bar{s}(0) + h(1)\bar{s}(-1) + \cdots + h(L-1)\bar{s}(-L+1)$$

$$\Downarrow$$

$$y(0) = h(0)s(0) + h(1)s(N-1) + \cdots + h(L-1)s(N-L+1)$$

$$y(1) = h(0)\bar{s}(1) + h(1)\bar{s}(0) + \cdots + h(L-1)\bar{s}(-L+2)$$

$$\Downarrow$$

$$y(1) = h(0)s(1) + h(1)s(0) + \cdots + h(L-1)s(N-L+2)$$

OFDM signal model: matrix notation

- In matrix notation, the N-dimensional received vector $\mathbf{y}$ can be represented as

$$\mathbf{y} = \bar{\mathcal{H}}\mathbf{s}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & \ddots & \ddots & \ddots & h(3) & h(2) & h(1) \\ h(1) & h(0) & \ddots & \ddots & \ddots & \ddots & h(3) & h(2) \\ \vdots & h(1) & \ddots & \ddots & \ddots & \ddots & \ddots & h(3) \\ h(L-1) & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & h(L-1) & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & h(0) & 0 \\ 0 & \ddots & 0 & h(L-1) & \ddots & \ddots & h(1) & h(0) \end{bmatrix} \begin{bmatrix} s(0) \\ s(1) \\ \vdots \\ \vdots \\ s(N-1) \end{bmatrix}$$

- The N columns of the $N \times N$ matrix $\bar{\mathcal{H}}$ are obtained by a cyclic shift one of each other and the matrix is called *circulant.*
- There is a loss of *power* and *spectral* efficiency: a vector of length $N + N_{CP}$ samples is transmitted for a length-N data vector.

OFDM signal model: channel matrix

- The interesting property of circulant matrices is that they can be diagonalized as

$$\bar{\mathcal{H}} = \mathbf{F}^H \mathbf{H} \mathbf{F}$$

where $\mathbf{F}$ is the normalized Fourier transform matrix, i.e.

$$[\mathbf{F}]_{k,n} = \frac{1}{\sqrt{N}} e^{-\frac{j2\pi kn}{N}}$$

and $\mathbf{H}$ is a diagonal matrix where the n-th element along the diagonal is

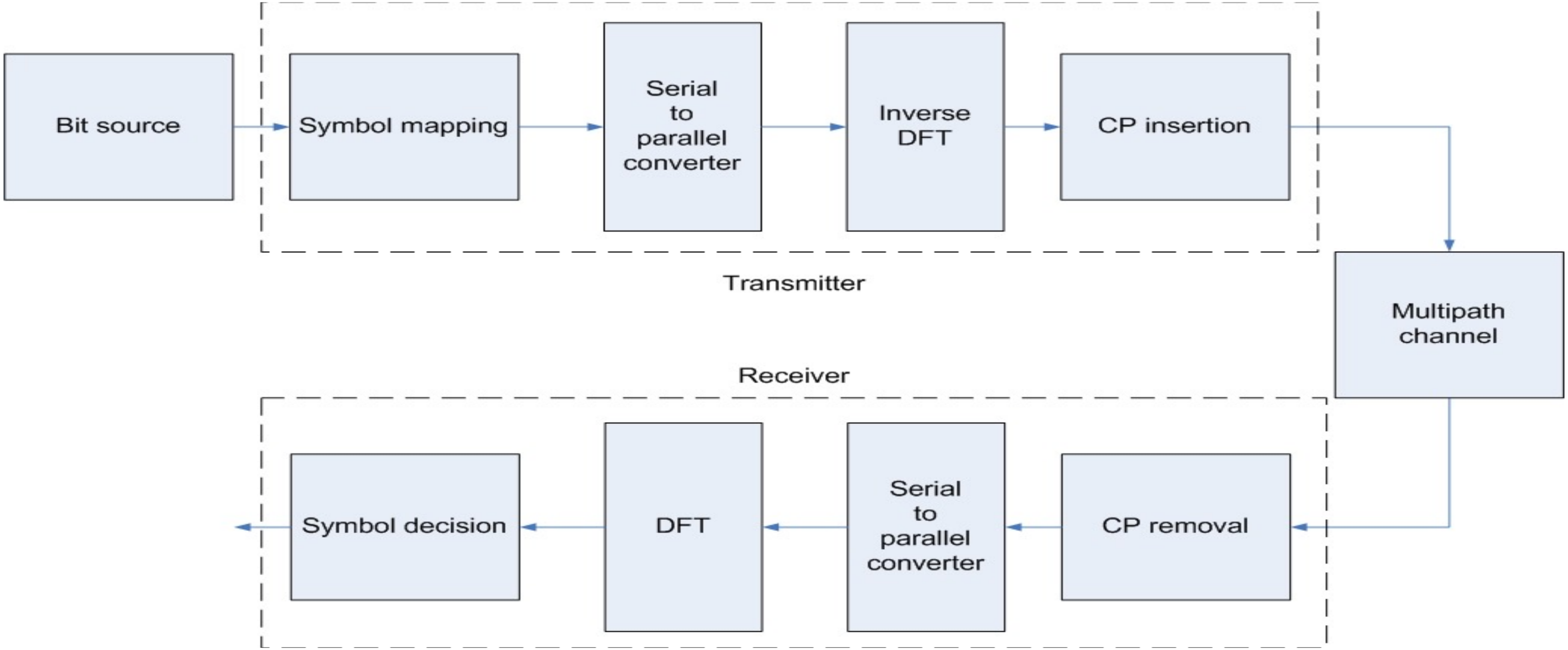
$$[\mathbf{H}]_{n,n} = H(n) = \sum_{\ell=0}^{L-1} h(\ell) e^{-\frac{j2\pi \ell n}{N}}$$

- The matrix $\mathbf{F}$ is unitary, i.e., $\mathbf{F}^H \mathbf{F} = \mathbf{F}\mathbf{F}^{\mathrm{H}} = \mathbf{I}_N$.

OFDM signal model: frequency-domain signal

- If we define $\mathbf{Y} = \mathbf{F}\mathbf{y}$, $\mathbf{S} = \mathbf{F}\mathbf{s}$, the FFT of $\mathbf{y}$ yields
$$\mathbf{Y} = \mathbf{F}\mathbf{y} = \mathbf{F}\overline{\mathcal{H}}\mathbf{s} = \mathbf{F}\mathbf{F}^H\mathbf{H}\mathbf{F}\mathbf{s} = \mathbf{H}\mathbf{S}$$
- Since $\mathbf{H}$ is diagonal, the signal received on subcarrier n depends *exclusively* on the signal transmitted on subcarrier n.
- There is no ISI in the frequency domain!!!
$$Y(n) = H(n)S(n)$$

OFDM baseband transceiver



OFDM baseband transceiver

1. In the serial-to-parallel block, a block of N consecutive data symbols are collected in the vector $\mathbf{S} = [S(0), S(1), \dots, S(N-1)]$.
2. The IDFT block converts $\mathbf{S}$ into a 'time-domain' vector
$$\mathbf{s} = \mathbf{F}^H \mathbf{S}$$
3. A N_{CP}–long cyclic prefix is inserted to create the new time-domain vector of length $N + N_{CP}$
$$\bar{\mathbf{s}} = [s(N - N_{CP} - 1), \dots, s(N-1), s(0), \dots, s(N-1)]$$

OFDM baseband transceiver

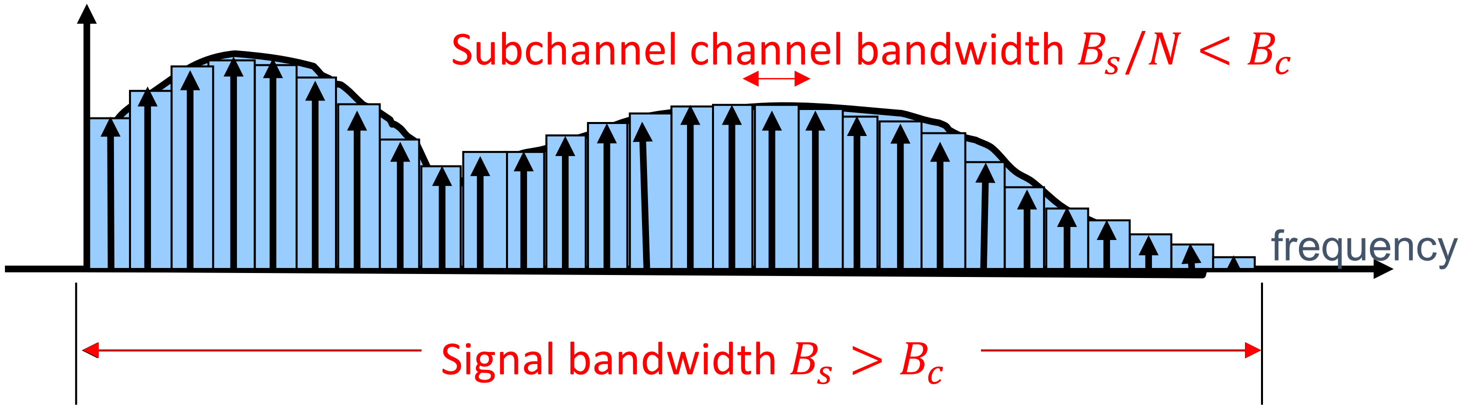
4. The signal propagates through the wireless channel with impulse response $\mathbf{h} = [h(0), h(1), \ldots, h(L-1)]$
$$y(k) = \sum_{\ell=0}^{L-1} h(\ell)\bar{s}(k-\ell)$$
5. At the receiver, the samples corresponding to the CP, which do not carry any information, are discarded and the remaining samples are frequency converted $\mathbf{Y} = \mathbf{F}\underline{\mathbf{y}}$, yielding
$$\mathbf{Y} = \mathbf{F}\bar{\mathcal{H}}\mathbf{s} = \mathbf{F}(\mathbf{F}^H\mathbf{H}\mathbf{F})\mathbf{s}$$

OFDM on multipath channel

- The overall OFDM signal bandwidth is B_s.
- The sampling duration is $T_s = 1/B_s$.
- The OFDM symbol duration is $T_{OFDM} = T_s(N + N_{CP})$
- The bandwidth for each subcarrier is $\Delta f = B_s/N$
- By accurately choosing N we have

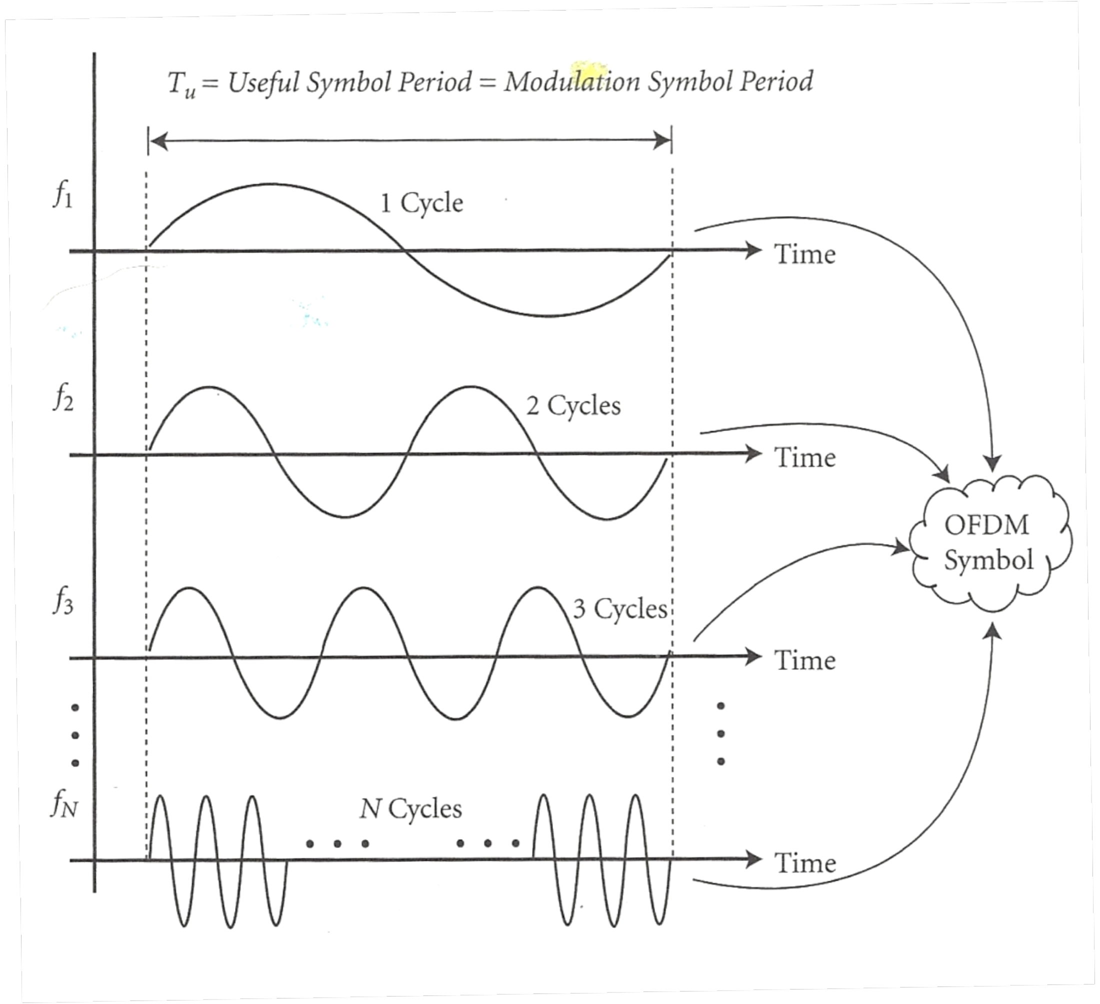
$$T_s < \sigma_\tau \ll T_{OFDM}, B_s > B_c \gg \Delta f$$

- On each subcarrier the channel is *flat*!!



OFDM interpretation

- The signal in the time domain is
$$s(k) = \sum_{n=0}^{N-1} S(n) e^{\frac{j2\pi nk}{N}}, k = 0, \dots, N-1$$
- The k-th sample corresponding to the n-th subcarrier is
$$S(n) e^{\frac{j2\pi nk}{N}} = S(n) e^{j2\pi n \frac{B}{N} kT}$$
$$= S(n) e^{j2\pi n \Delta f kT} = S(n) e^{j2\pi n \Delta f t}\Big|_{t=kT}$$
- Each frequency symbol $S(n)$ is multiplied by a complex exponential oscillating at frequency $n\Delta f$ for a duration of N samples plus the CP.

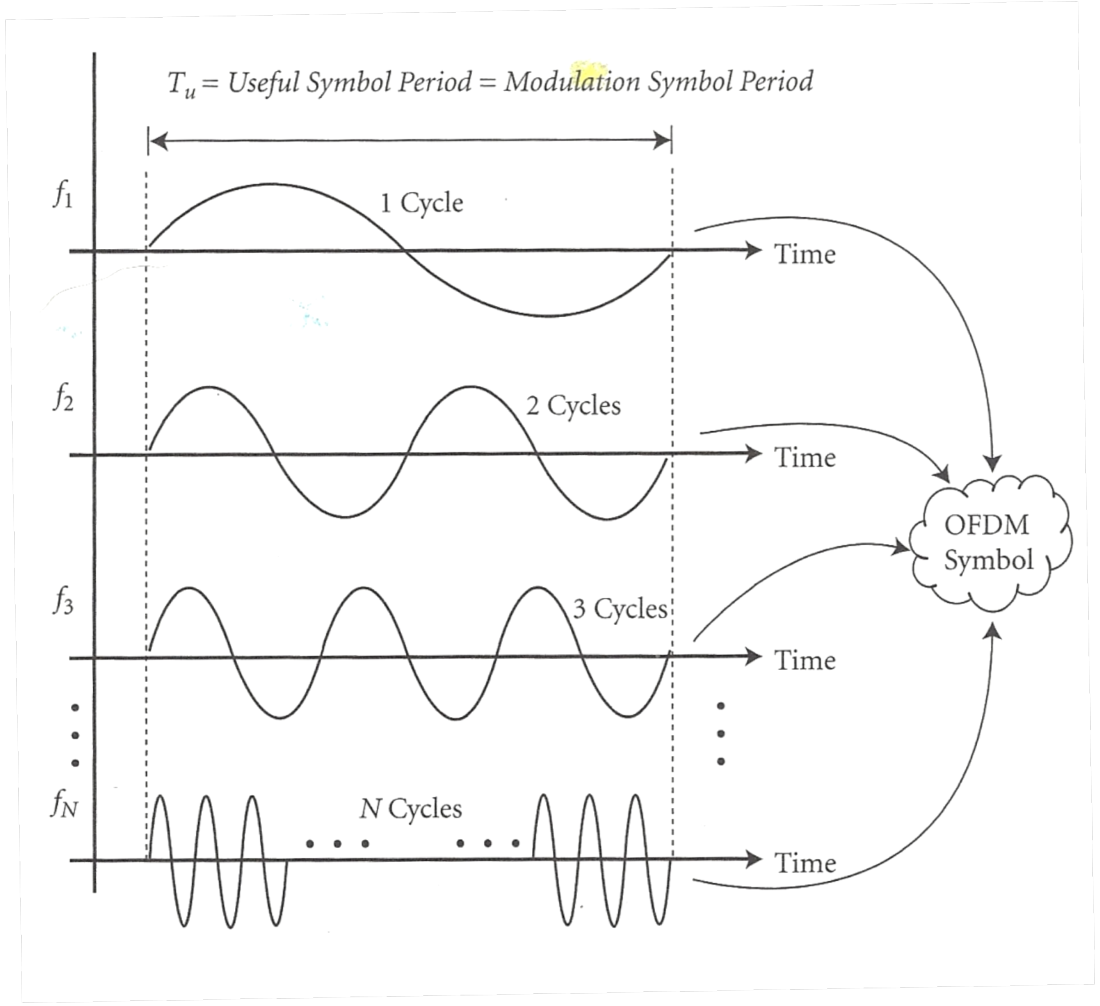


OFDM interpretation

- The symbol duration T is a multiple integer of the period of all subcarriers.
- Since the input signal to the channel is periodic also the output is periodic.
- Thus, the signal received on the n-th subcarrier is

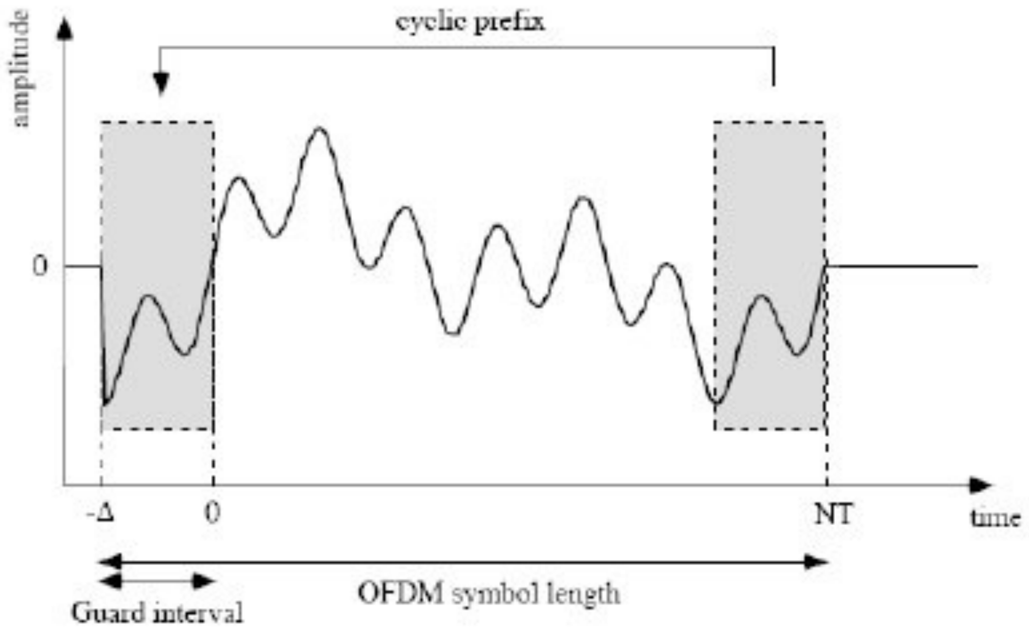
$$y_n(t) = \sum_{\ell=0}^{L-1} h(\ell) s_n(t - \ell \mathrm{T})$$

$$= S(n) \sum_{\ell=0}^{L-1} h(\ell) e^{-j2\pi n \Delta f (t - \ell T)}$$



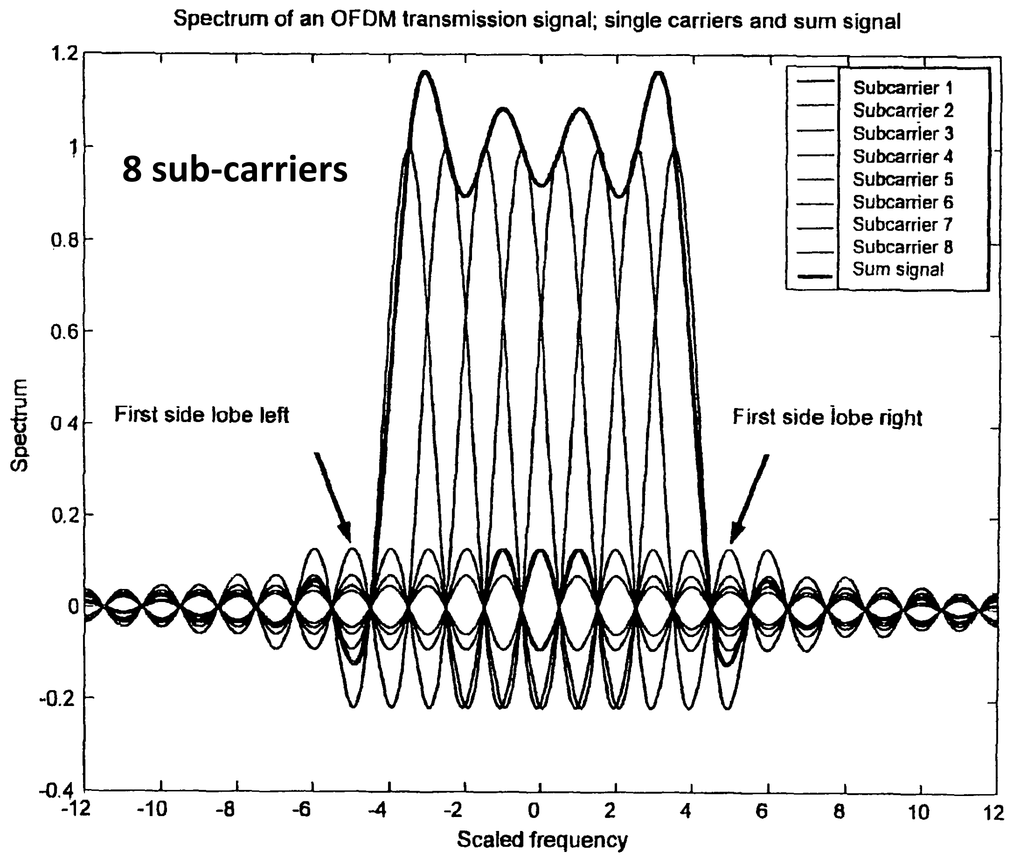
OFDM interpretation

- During an OFDM block the received signal on each subcarrier depends only on the symbol transmitted on that subcarrier.
- The CP insertion exploits this periodicity to render the channel *flat* for each subcarrier.



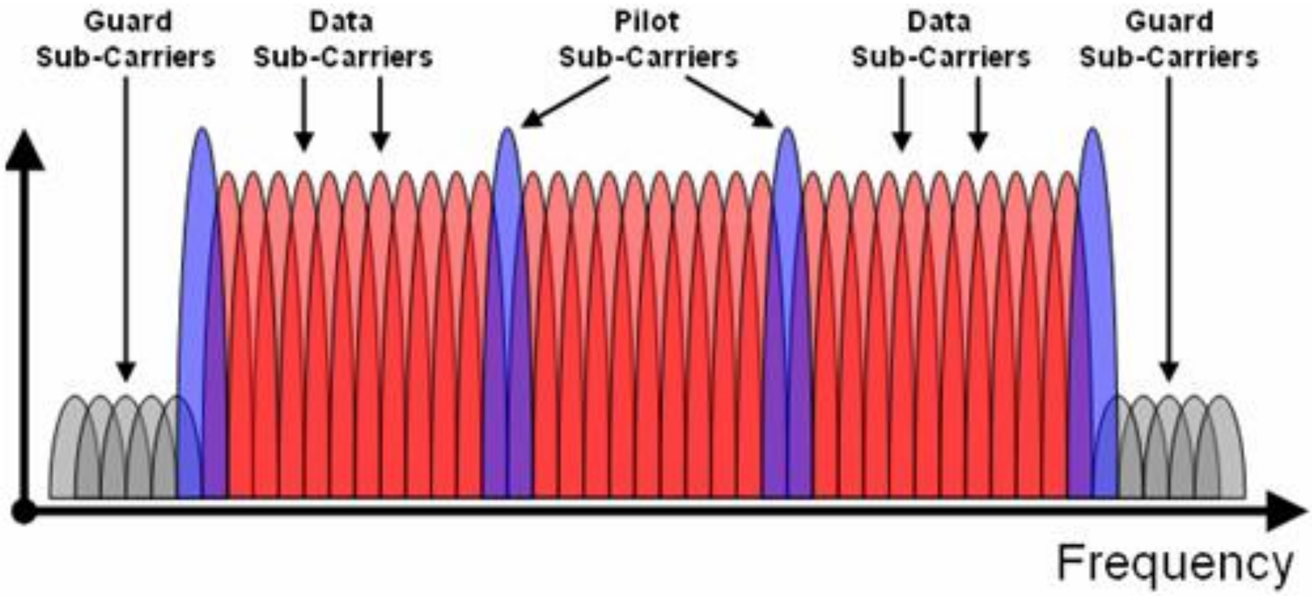
OFDM frequency orthogonality

- The symbol transmitted on a subcarrier is fixed for the duration of an OFDM block.
- This is equivalent to multiply the complex exponential by a 'rect' function for a duration of NT seconds.
- The power spectral density of the OFDM signal is the sum of N 'sinc' functions, one for each subcarrier.
- All the sinc functions are orthogonal by construction and they do not interfere with each other.



OFDM example: WiFi – IEEE 802.11a/g/n/ac

- A WiFi transmission occupies a bandwidth $B = 20$ MHz, which is divided in $N = 64$ sub-carriers spaced $\Delta f = 312.5$ kHz.
  - 802.11a/g use 48 subcarriers for data, 4 for pilot, and 12 as null subcarriers.
  - 802.11n/ac use 52 subcarriers for data, 4 for pilot, and 8 as null.



OFDM example: WiFi – IEEE 802.11a/g/n/ac

- The OFDM block is composed by $N = 64$ and $N_{CP} = 16$ samples.
- The duration of each sample is $T = \frac{1}{B} = \frac{1}{20 \cdot 10e^6} = 50$ ns and the duration of a block is $T_{OFDM} = (64 + 16) \cdot 50 = 4\ \mu$s.
- In general, the delay spread of an indoor channel is $\sigma_\tau < 500$ ns, so that the channel is indeed *flat*

$$T_{OFDM} \gg \sigma_\tau$$

- Assuming that the maximum indoor mobility is $v = 3$ m/s, the Maximum Doppler shift is $f_d = \frac{5 \cdot 10^\wedge 9 \cdot 3}{3 \cdot 10^\wedge 8 \cdot} = 50$ Hz $\Longrightarrow T_c = \frac{1}{2 \cdot 50} = 0.01$ s and the channel is *slow*

$$T_{OFDM} \ll T_c$$

OFDM example: WiFi – IEEE 802.11a/g/n/ac

- Each subcarrier carries a new symbol every $T_{OFDM} = 4\ \mu$s.
- The symbol rate per subcarrier is $\frac{1}{T_{OFDM}} = 0.25 \cdot 10^6$ sym/s.
- There are 48 subcarriers dedicated to data transmissions and the overall symbol rate is $48 \cdot 0.25 \cdot 10^6 = 12 \cdot 10^6$ sym/s.
- Loss of (spectral and energy) efficiency due to the CP insertion is
$$\eta_{CP} = \frac{N_{CP}}{N} = \frac{16}{80} = 20\%$$
- Additional loss of spectral efficiency due to guard subcarriers
$$\eta_{GS} = \frac{16}{64} = 25\%$$

Error rate for OFDM systems

- Considering the presence of noise, the output of the FFT is
$$R(n) = Y(n) + N(n) = H(n)S(n) + N(n)$$
where $N(n) = \mathbf{Fn}$ and the vector $\mathbf{n}$ collects the received noise samples in time, $\mathbf{n} = [n(0), n(1), \dots, n(N-1)]$.
- Due to the properties of the unitary matrix $\mathbf{F}$, the statistics of $N(n)$ are equivalent to the statistics of the noise samples $n(k)$
$$n(k) \in \mathcal{N}(0, \sigma^2) \Leftrightarrow N(n) \in \mathcal{N}(0, \sigma^2)$$
- The decision variable is
$$X(n) = \frac{R(n)}{H(n)} = S(n) + \frac{N(n)}{H(n)}$$

OFDM error probability

- The decision variable on the n-th subcarrier is

$$X(n) = \frac{R(n)}{H(n)} = S(n) + \frac{N(n)}{H(n)} = S(n) + N'(n)$$

  where $S(n)$ is an information symbol and the noise is

$$N'(n) \in \mathcal{N}\left(0, \frac{\sigma^2}{|H(n)|^2}\right)$$

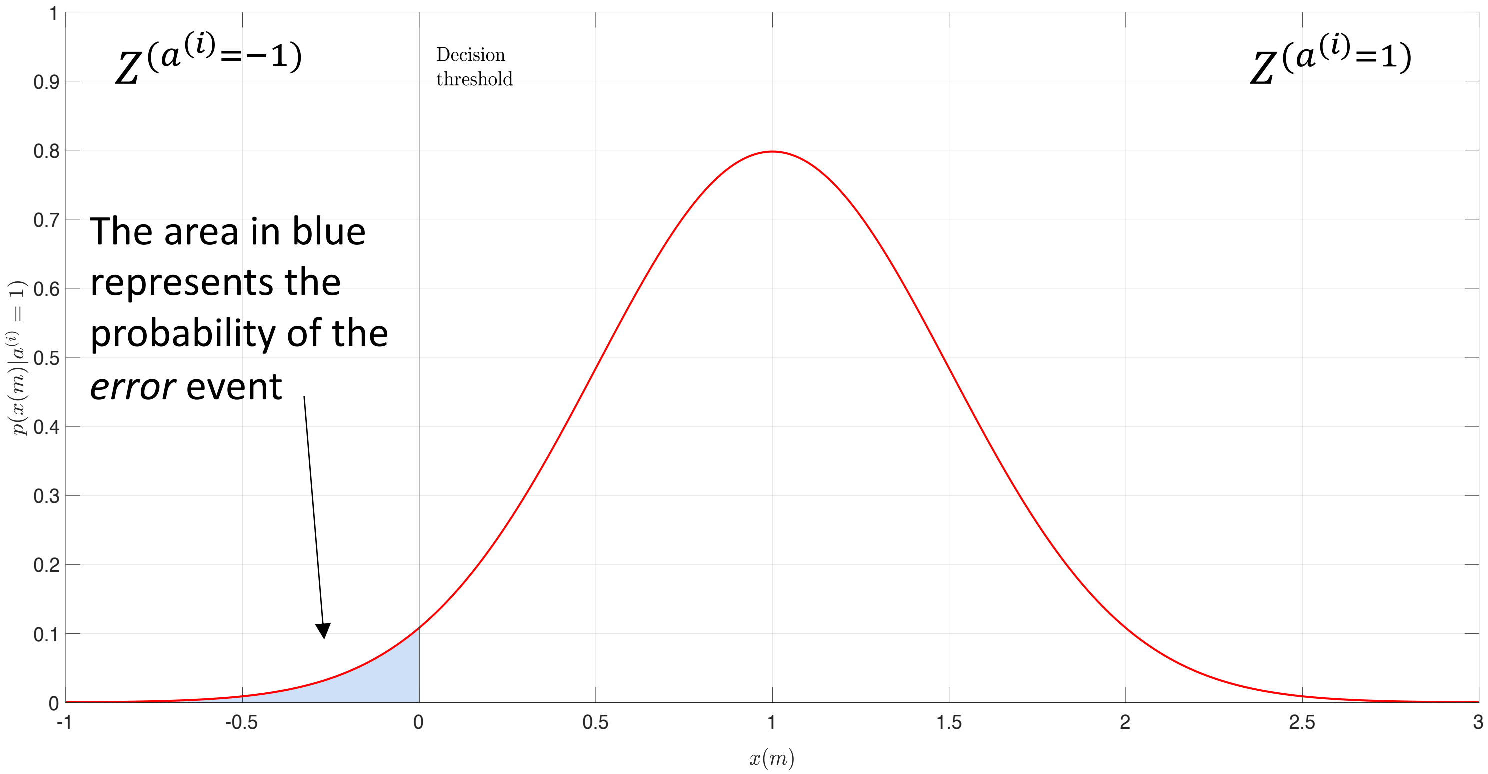
- Hence, the symbol error probability will be

$$P(e|H(n)) = Q\left(\sqrt{\frac{|H(n)|^2}{\sigma^2}}\right) = Q\left(\sqrt{\frac{E_s|H(n)|^2}{AN_0}}\right)$$

PAM error probability ($M = 2$)

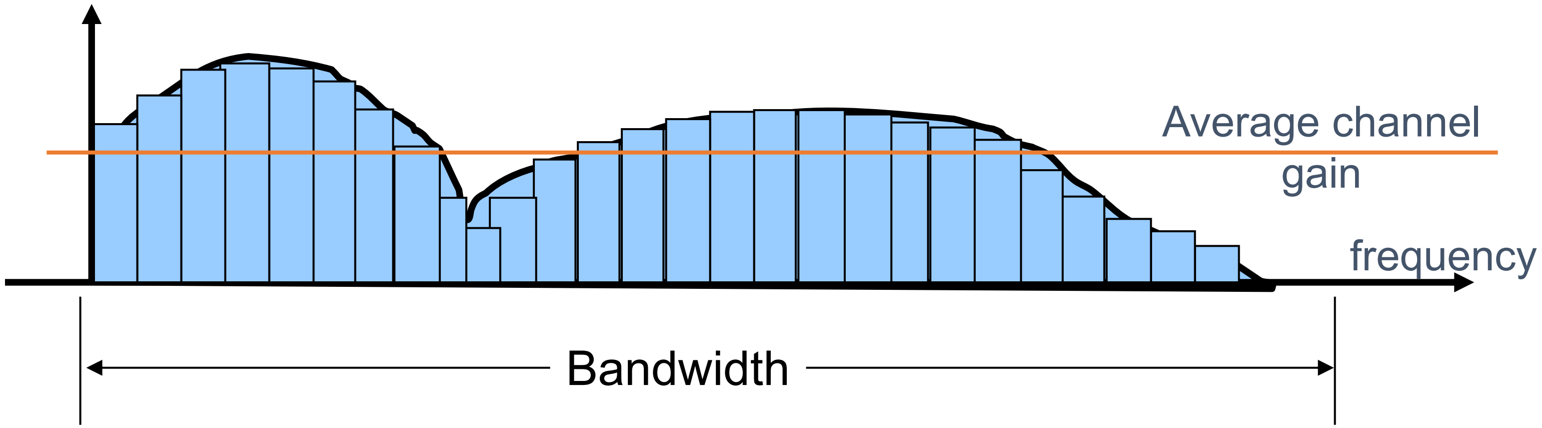
- To compute $P\left(e\middle|a^{(i)}\right)$ we assume that $x(m) = a^{(i)} + n(m)$ and the probability of error is

$$P\left(e\middle|a^{(i)}\right) = Pr\left\{x(m) \notin Z^{(i)}\middle|a_m = a^{(i)}\right\} = Q\left(\frac{d\left(a^{(i)},0\right)}{\sigma}\right) = Q\left(\frac{1}{\sigma}\right)$$



Single-user OFDM

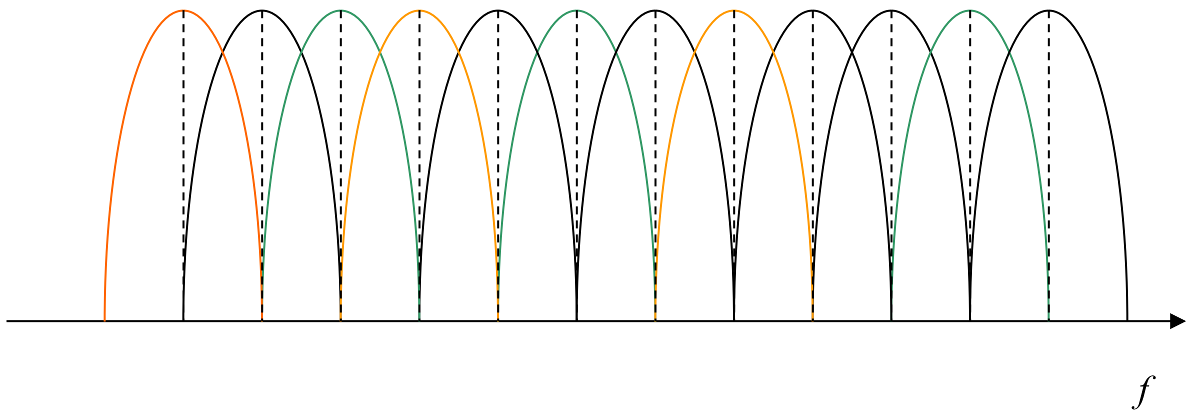
- The wireless channel is divided in N parallel subchannels, each with a different gain $H(n)$.



- By adapting transmission parameters to the different channel gains allows to exploit the channel *frequency diversity.*

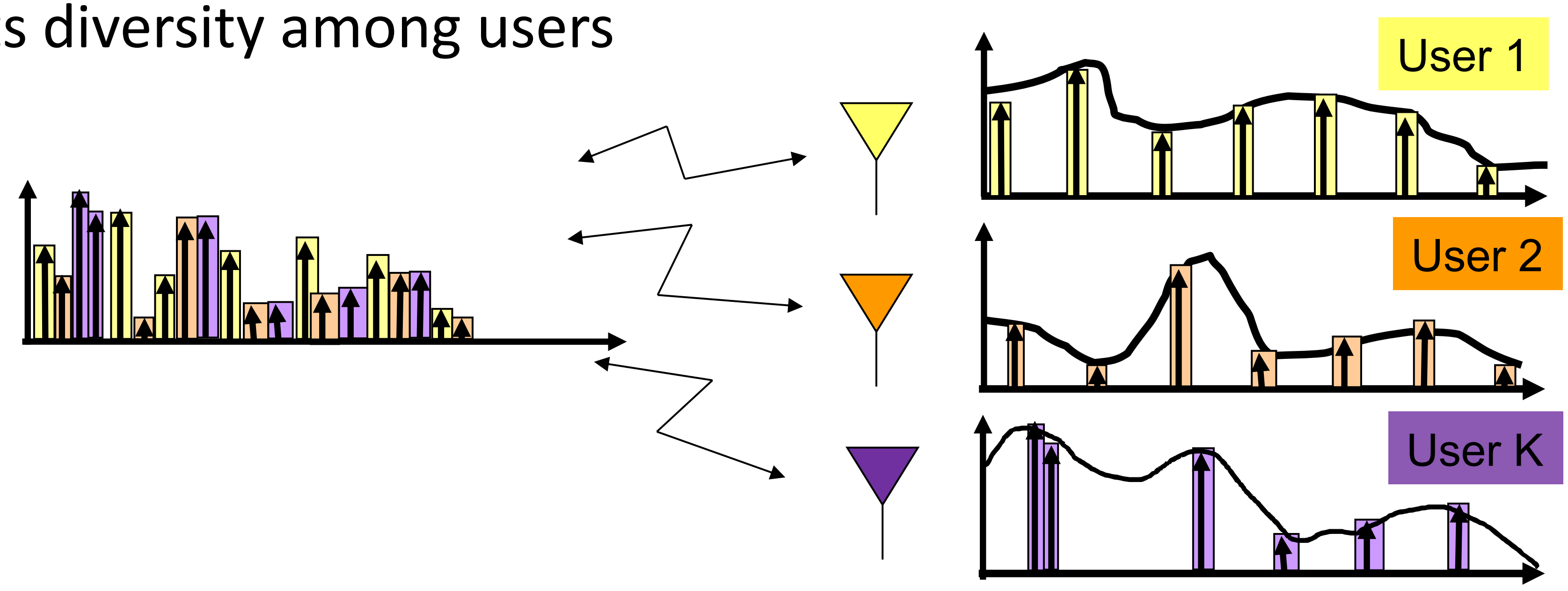
Multi-user OFDM: OFDMA

- OFDM technology can be employed to enforce a multiple access technique: OFDMA
  - Assigns an orthogonal subset of the *N* available sub-carriers to different users



Multiuser OFDM

- Users transmit on different subcarriers at the same time
- The fading gain on each subchannel is independent from user to user
- Adaptive resource allocation is designed to assign to each user its best subchannels according different optimality criterion.
- Exploits diversity among users



Channel capacity

- We assume that the maximum achievable rate is measured as the *Shannon channel capacity*
- Let $P(n)$ be the power of the signal transmitted on sub-channel n, the maximum achievable rate is

$$C = \log_2\left(1 + \frac{P(n)|H(n)|^2}{\sigma^2}\right)$$