An Introduction to Fuzzy Logic

Part II

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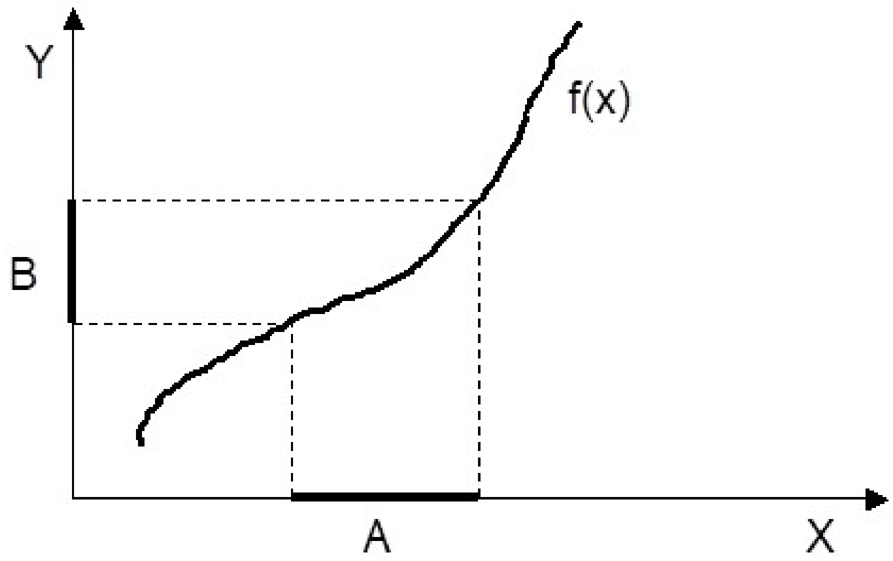
ITALY

Extension of a crisp function

- A crisp function $f: X \to Y$ can be extended to $f: P(X) \to P(Y)$

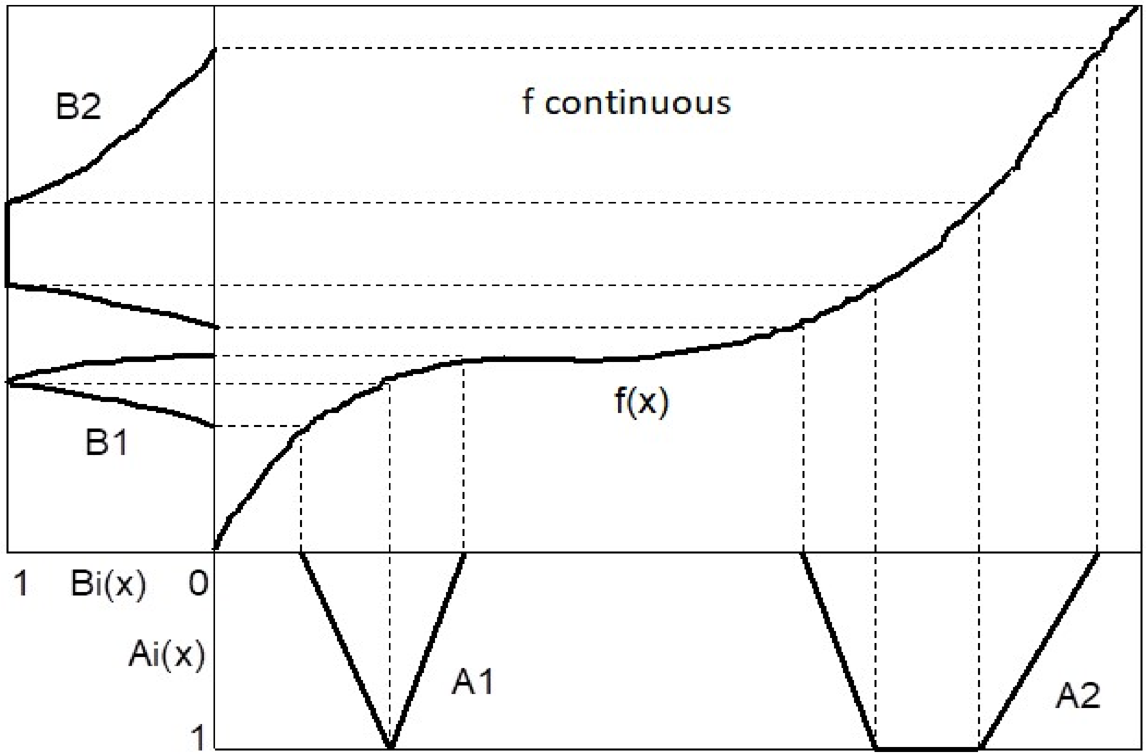
$$\forall A \in P(X): f(A) = \{ y \mid y = f(x), x \in A \}$$



Extension principle

$f: X \rightarrow Y$ induces the following function:

$$f: F(X) \rightarrow F(Y) \qquad [f(A)](y) = \sup_{x \mid y = f(x)} A(x) \quad \forall A \in F(X)$$



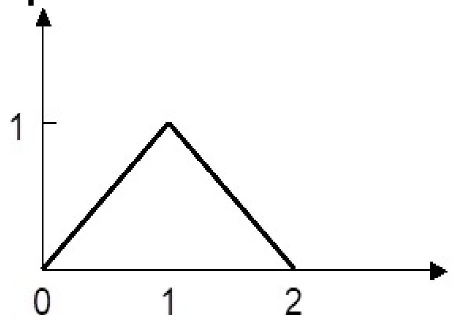
In general, $f: P(X_1 \times X_2 \times \ldots \times X_n) \rightarrow P(Y)$. Therefore,

$$B = f(A_1, A_2, \ldots, A_n)$$

$$\mu_B(y) = \sup_{y = f(x_1, x_2, \ldots, x_n)} \left\{ \min\left[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n) \right] \right\}$$

Fuzzy numbers

A *fuzzy number* is a fuzzy set defined on $\Re$ with a normal and convex membership function.



$$\tilde{1} = \left\{ \frac{.2}{0} + \frac{1}{1} + \frac{.2}{2} \right\}$$

Let * be one of the 4 arithmetic operations +, -, x, /. Let I and J be 2 fuzzy numbers:

$$(I * J)(z) = \sup_{z = x*y} \min[I(x), J(y)]$$

$$\tilde{1}+\tilde{1}=\left(\frac{.2}{0}+\frac{1}{1}+\frac{.2}{2}\right)+\left(\frac{.2}{0}+\frac{1}{1}+\frac{.2}{2}\right)=$$

$$\frac{\min(0.2,0.2)}{0}+\frac{\max[\min(0.2,1),\min(1,0.2)]}{1}$$

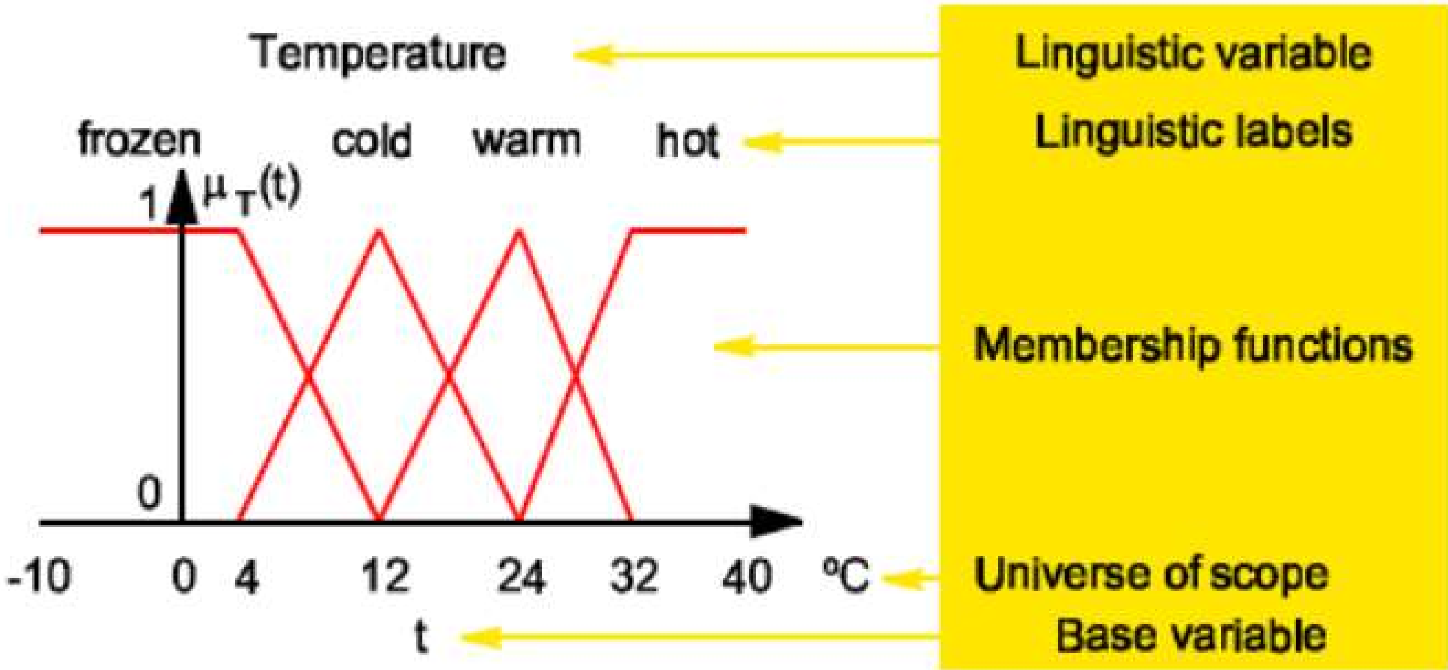
$$+\frac{\max[\min(0.2,0.2),\min(1,1),\min(0.2,0.2)]}{2}$$

$$+\frac{\max[\min(1,0.2),\min(0.2,1)]}{3}+\frac{\min(0.2,0.2)}{4}$$

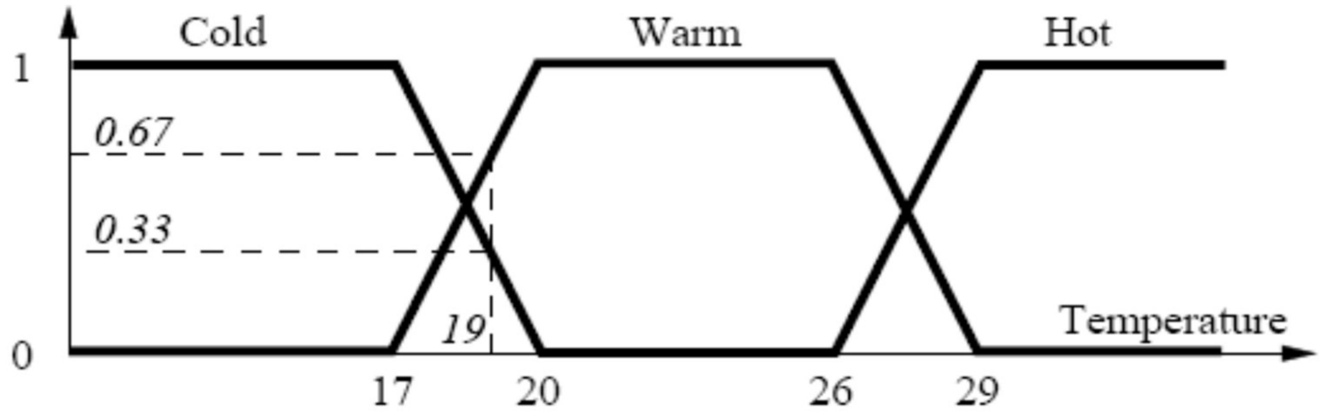
$$=\frac{.2}{0}+\frac{.2}{1}+\frac{1}{2}+\frac{.2}{3}+\frac{.2}{4}=\tilde{2}$$

Linguistic variables

- A *linguistic variable* is a variable whose values are fuzzy sets (*fuzzy numbers*).
- It is defined in terms of a *base variable*, which is a variable in the classical sense (i.e., its values are real numbers).

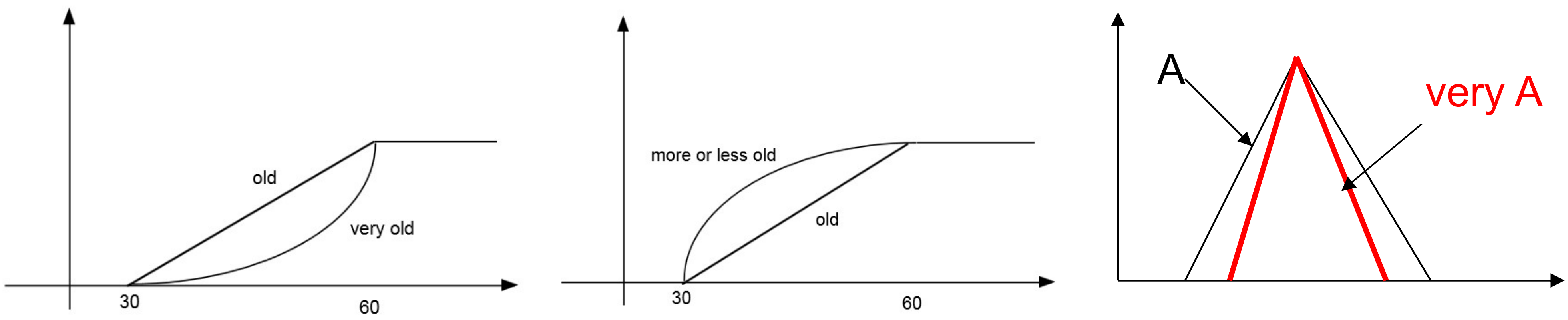


An alternative definition



Linguistic hedges

- *Linguistic hedges* are special linguistic terms that modify other linguistic terms. Examples of hedges are *very*, *more or less*, *fairly*, *extremely*.
- Any linguistic hedge can be interpreted as a unary operation on the unit interval [0,1].
- The hedge *very* is often interpreted as $h(a)=a^2$, while the hedge *more or less* is interpreted as $h(a)=\sqrt{a}$. Other interpretations are possible. Examples:



Fuzzy logic

- Fuzzy logic is an extension of classical logic as it uses fuzzy sets "*to compute with words*".
- 'Logic' stands for a set of principles of reasoning and *rules of inference*. In particular, rules of inference allow conclusions to be drawn from assertions that are known or assumed to be true.

Modus ponens

- One of the most frequently used rules of inference in classical logic is *modus ponens*:

  P

  P → Q (read '*P implies Q*', which is equivalent to '*if P then Q*').

  ―――――

  Q

  or $$(p \wedge (p \rightarrow q)) \rightarrow q$$

  where *P* and *Q* are propositions, which can take only one of two possible values: 'completely true' or 'completely false'. No intermediate truth value is possible.

- A proposition is any statement such as 'it's sunny today' or 'the President of the United States is a woman'.

- In words, *modus ponens* means:
- If 1) P is known to be true, and 2) we assume that $P \rightarrow Q$ is true, then 3) Q must be true.

- For example,

The tomato is red

If the tomato is red, then it is ripe

The tomato is ripe

- But, again, <u>we need a logic with shades of gray</u>.
- Basically, *modus ponens* is too rigid and unable to model any reasoning process. In fact, a conclusion can be drawn only if the known fact *P* coincides exactly with the antecedent of the implication $P \rightarrow Q$.
- In other words, given the rule 'If the tomato is red, then it is ripe', if the tomato is in fact 'almost red', we would like to conclude that it is 'almost ripe'.
- In general, given the rule 'if x is high, then y is low', if x is indeed 'very high', we would like to conclude that y is 'very low':

  x is very high

  if x is high, then y is low

  ---

  y is very low

➔ But *modus ponens* <u>does not work</u> in this case.

Generalized modus ponens

- *Generalized modus ponens* is often presented as follows:

$$\frac{\begin{array}{l} P' \\ P \rightarrow Q \end{array}}{Q'}$$

or $$(p' \wedge (p \rightarrow q)) \rightarrow q'$$

- It is based on *modus ponens*, but the premise *P′* is slightly different from *P* and therefore the conclusion Q′ is slightly different from *Q*. Here, *P′*, *P* → *Q* and *Q′* are *fuzzy propositions*.

Fuzzy propositions

A *fuzzy proposition* can take one of two forms:

- *X is A* (e.g., 'the speed is high')

  where *X* is a linguistic variable (such as 'temperature', 'speed', etc.) and *A* is a fuzzy set, which represents the value of *X* (such as 'high', 'medium', 'low', etc.). For each value *x* of the variable *X*, the *degree of truth* (or *truth value*) of a proposition *p* : *X is A* is given by $T(p) = A(x)$.

- *If X is A then Y is B*

  where *X* and *Y* are linguistic variables, and *A* and *B* are fuzzy sets (e.g., 'if the washing machine is half full, then the washing time is short').

  This second form is called a *conditional fuzzy proposition* (or *fuzzy rule*) and is equivalent to an implication:

  $$X \text{ is } A \rightarrow Y \text{ is } B$$

  It is a fuzzy relation (also called an *implication relation*).

Approximate reasoning

- Generalized modus ponens can now be presented in the form:

Fact:	**X is A′**		**A′**
Rule:	**If X is A then Y is B**	or (for short)	**A → B**
Conclusion:	**Y is B′**		**B′**

- The fuzzy set *B′* is induced by proposition *X is A′* <u>combined</u> with the fuzzy rule *If X is A then Y is B*. This is the essence of *approximate reasoning*.

- We can compute B' as the composition of two relations.
- The first relation is A', which in fact can be considered as a unary fuzzy relation on U, U being the universe on which the fuzzy set A' is defined.
- The second relation is the implication relation, which implements the rule *If X is A then Y is B*. This relation is defined between U and V, where U and V are, respectively, the universes on which the fuzzy sets A and B are defined.
- Note that A' and A are defined on the same universe U; similarly, B' and B are defined on the same universe V.
- In practice, we have two fuzzy relations, namely A' and $A \rightarrow B$, which can be combined to produce B'.

Fuzzy implication

- A fuzzy implication I is a function $I: [0,1]\times[0,1] \rightarrow [0,1]$ which for each possible truth value x and y, respectively, of fuzzy proposition p and fuzzy proposition q, defines the truth value, $I(x,y)$, of the conditional proposition "if p then q".
- This function <u>should</u> be an extension of the classical implication $p \rightarrow q$ from $\{0,1\}$ to $[0,1]$.

- In classical logic, I can be defined in several distinct forms. While these forms are equivalent in classical logic, their extensions to fuzzy logic <u>are not</u> equivalent and result in distinct classes of fuzzy implications.
- Two ways to define I in classical logic are, e.g., expressed by the following formulae:

$$I(a,b)=\neg[a\wedge(\neg b)]=\neg a\vee b$$

$$I(a,b)=(a\wedge b)\vee\neg a$$

- When we extend these formulae to fuzzy logic, we interpret the disjunction, conjunction and negation as a fuzzy union (*t*-conorm), a fuzzy intersection (*t*-norm) and a fuzzy complement, respectively. This results in defining *I* in fuzzy logic, e.g., by the formulae

- Kleene-Dienes implication $I(a,b)=\max(1-a,b)$

- Lukasiewicz implication $I(a,b)=\min(1,1-a+b)$

- Zadeh implication $I(a,b)=\max\left(\min(a,b),1-a\right)$

- Mamdani implication $I(a,b)=\min(a,b)$

- Larsen implication $I(a,b)=a\cdot b$

Compositional rule of inference

$$B' = A' \circ (A \rightarrow B)$$

$$B'(y) = \sup_{x \in U} \min[A'(x), I(A(x), B(y))]$$

In general, we have

$$B'(y) = \sup_{x \in U} T[A'(x), I(A(x), B(y))]$$

where T is a t-norm.

Example

Suppose that the fact *A′* is a fuzzy singleton (with support x_0).

The approximate consequence B′ is computed (with Mamdani implication) as

$$B'(y) = \sup_x \min\{A'(x), \min\{A(x), B(y)\}\} \quad \text{for all } y$$

Observing that A′(x) = 0, $\forall x \neq x_0$, the supremum turns into a simple minimum

$$B'(y) = \min\{A'(x_0), A(x_0), B(y)\} = \min\{1, A(x_0), B(y)\} = \min\{A(x_0), B(y)\}$$

Similarly, in the case of Larsen implication.