PECSN, 23/7/19

Exercise 1

Consider the following function:

$$F(x) = \begin{cases} 0 & x \leq -5 \\ \dfrac{\alpha \cdot x + 5}{\beta + |x|} & x > -5 \end{cases}$$

Where α, β are *positive* constants.

1) Determine under what conditions $F(x)$ is a CDF.

Assume from now on that we are in the above conditions.

2) Compute the PDF of RV X, whose CDF is $F(x)$.
3) Determine under what further conditions $E[X]$ is finite.
4) Assuming $\beta = 5$, compute $E[X]$.

Exercise 2

Consider a distributed system where a *sender* and a *receiver* exchange request/response transactions. The sender is faster than the receiver, hence the system has a flow control window that limits the number of outstanding requests to *K*. The sender is in asymptotic conditions, i.e., it generates new requests whenever the flow control allows it to. Transmission of requests to the receiver takes an exponential time, with a mean $1/\mu_1$. Transmission of responses to the sender takes an exponential time, with a mean $1/\mu_2$. It Is $\mu_1 > \mu_2$.

1) Model the system as a queueing network and find its steady-state probabilities.
2) Compute the throughput and the utilization of the sender and receiver, as a function of *K*
3) Compute the mean completion time of a request/response transaction, as a function of *K*
4) Describe what happens to the throughput, utilizations and transaction delay when $\mu_1 \gg \mu_2$

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Exercise 1 – Solution

1) In order for $F(x)$ to be a CDF, the following conditions should be verified:
   a) $F(x)$ must be monotonic
   b) $lim_{x\to-\infty} F(x) = 0$
   c) $lim_{x\to+\infty} F(x) = 1$

b) always holds. c) holds if and only if $\alpha = 1$.

As far as monotonicity is concerned, we observe that $F(x)$ is identically null for $x \leq -5$, and that its derivative is:

$$F'(x) = \begin{cases} \dfrac{\beta+5}{(\beta-x)^2} & -5 < x < 0 \\ \dfrac{\beta-5}{(\beta+x)^2} & x > 0 \end{cases}$$

when $x > -5$. Therefore, monotonicity is guaranteed if $\beta \geq 5$.
The conditions requested by 1) are $\alpha = 1, \beta \geq 5$.

2) As per the computations above, it is:

$$f(x) = \begin{cases} 0 & x < -5 \\ \dfrac{\beta+5}{(\beta-x)^2} & -5 < x < 0 \\ \dfrac{\beta-5}{(\beta+x)^2} & x > 0 \end{cases}$$

3) It is:

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f(x)dx = \int_{-5}^{0} x \cdot \frac{\beta+5}{(\beta-x)^2} dx + \int_{0}^{+\infty} x \cdot \frac{\beta-5}{(\beta+x)^2} dx$$
$$= (\beta+5) \cdot \int_{-5}^{0} \frac{x}{(\beta-x)^2} dx + (\beta-5) \cdot \int_{0}^{+\infty} \frac{x}{(\beta+x)^2} dx$$

Now, the first integral is always finite (since its limits are), whereas the second may not be. After few algebraic passages, we obtain:

$$\int_{0}^{+\infty} \frac{x}{(\beta+x)^2} dx = \int_{0}^{+\infty} \frac{(\beta+x)-\beta}{(\beta+x)^2} dx = \left[\frac{\beta}{\beta+x} + log|\beta+x|\right]_0^{+\infty} = +\infty$$

Therefore, the expectation exists only if $\beta = 5$.

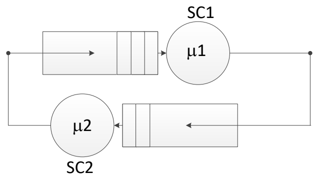
4) Assuming $\beta = 5$, the expectation is equal to:

$$E[X] = (\beta+5) \cdot \int_{-5}^{0} \frac{x}{(\beta-x)^2} dx + (\beta-5) \cdot \int_{0}^{+\infty} \frac{x}{(\beta+x)^2} dx$$

$$= 10 \cdot \int_{-5}^{0} \frac{x}{(5-x)^2} dx = 10 \cdot \left[\frac{-5}{x-5} + log|x-5|\right]_{-5}^{0}$$
$$= 5 - 10\, log(2)$$

Exercise 2 - Solution

The system can be modeled as a CJN with two SCs and K circulating jobs. Its routing matrix is $\underline{\mathbf{\Pi}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence the solution to the routing equation is $e_1 = e_2$.



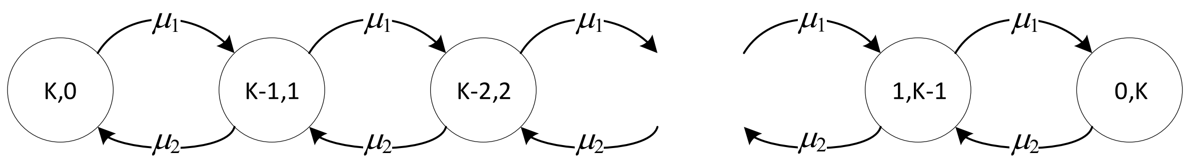
Setting $e_1 = \mu_1$, we get $\rho_1 = 1$, $\rho_2 = \mu_1/\mu_2$. Call $s = \mu_1/\mu_2$, $s > 1$, for simplicity from now on.

By GN's theorem, we get $p(n_1, n_2) = \frac{1}{G(2,K)} \cdot 1^{n_1} \cdot s^{n_2}$, which can be rewritten, having in mind that $n_1 + n_2 = K$, as $p(K-n, n) = \frac{1}{G(2,K)} \cdot s^n$, *n* being the number of jobs in SC2.

In this case, since $\mu_1 \neq \mu_2$, it is $G(2,K) = \frac{s^{K+1}-1}{s-1}$, thus:

$$p(K-n, n) = \frac{s-1}{s^{K+1}-1} \cdot s^n$$

Alternatively, the system can also be modeled via a continuous-time Markov Chain, whose states are $(K-n, n)$, $0 \leq n \leq K$, and whose transition rates are μ_1 (to the right) and μ_2 (to the left). Computing the SS probabilities in the latter yields the selfsame results.



The utilization of the sender (SC1) is:

$$U_1 = \rho_1 \cdot \frac{G(M, K-1)}{G(M, K)} = 1 \cdot \frac{1-s^K}{1-s} \cdot \frac{1-s}{1-s^{K+1}} = \frac{s^K - 1}{s^{K+1} - 1}$$

Alternatively, one may get to the same results by solving $U_1 = 1 - p(0, K)$.

The utilization of the receiver (SC2) is:

$$U_2 = \rho_2 \cdot \frac{G(2, K-1)}{G(2, K)} = s \cdot \frac{1-s^K}{1-s^{K+1}} = \frac{s - 1 + 1 - s^{K+1}}{1 - s^{K+1}} = 1 - \frac{s-1}{s^{K+1}-1}$$

Or, alternatively, $U_2 = 1 - p(K, 0)$, yielding the same result.

When $K \to +\infty$, U_1 approaches $1/s$, whereas U_2 approaches 1. When $\mu_1 \gg \mu_2$ (i.e., $s \to \infty$), $U_1 \to 0$ and $U_2 \to 1$, since the sender is always empty and the receiver is always full.

The throughput of both SCs is the same (there is only one path in the CJN) and it is equal to:

$$\gamma_1 = \gamma_2 = \gamma = e_1 \cdot \frac{G(2, K-1)}{G(2, K)} = \mu_1 \cdot \frac{s^K - 1}{s^{K+1} - 1} = \mu_2 \cdot \frac{s^{K+1} - s}{s^{K+1} - 1}$$

Or, alternatively, $\gamma_1 = \mu_1 \cdot (1 - p(0, K))$, $\gamma_2 = \mu_2 \cdot (1 - p(K, 0))$, yielding the same result.

When $K \to +\infty$ (i.e., when the flow control is ineffective) or $\mu_1 \gg \mu_2$ (i.e., $s \to \infty$), the throughput approaches μ_2, i.e. the slowest of the two systems, from below. This makes perfect sense.

To compute the mean delay of a transaction there are two ways: the short one and the lengthy one. The short one is to observe that:

$$E[R] = E[R_1] + E[R_2] = \frac{E[N_1]}{\gamma_1} + \frac{E[N_2]}{\gamma_2} = \frac{E[N_1] + E[N_2]}{\gamma} = \frac{K}{\gamma} = \frac{K \cdot (s^{K+1} - 1)}{\mu_1 \cdot (s^K - 1)} = \frac{K}{\mu_2} \cdot \frac{s^{K+1} - 1}{s^{K+1} - s}$$

If one fails to observe straightaway that $E[N_1] + E[N_2] = K$, they can still solve the problem computing $E[N_1]$ and $E[N_2]$ separately (but it will take longer). We have:

$$E[N_i] = \frac{1}{G(M,K)} \cdot \left[\sum_{h=1}^{K} {\rho_i}^h \cdot G(M, K-h)\right]$$

For the sender (SC1), the above formula reads:

$$E[N_1] = \frac{s-1}{s^{K+1}-1} \cdot \left[\sum_{h=1}^{K} 1^h \cdot \frac{s^{K+1-h} - 1}{s-1}\right]$$

$$= \frac{1}{s^{K+1}-1} \cdot \sum_{h=1}^{K} [s^h - 1]$$

$$= \frac{1}{s^{K+1}-1} \cdot \left[\frac{1 - s^{K+1}}{1-s} - 1 - K\right]$$

$$= \frac{1}{s-1} - \frac{K+1}{s^{K+1}-1}$$

For the receiver (SC2), the above formula reads:

$$E[N_2] = \frac{s-1}{s^{K+1}-1} \cdot \left[\sum_{h=1}^{K} s^h \cdot \frac{s^{K+1-h} - 1}{s-1}\right]$$

$$= \frac{1}{s^{K+1}-1} \cdot \sum_{h=1}^{K} [s^{K+1} - s^h]$$

$$= \frac{1}{s^{K+1}-1} \cdot \left[K \cdot s^{K+1} + 1 - \frac{s^{K+1} - 1}{s-1}\right]$$

$$= \frac{K \cdot s^{K+1} + 1}{s^{K+1} - 1} - \frac{1}{s-1}$$

And $E[N_1] + E[N_2] = K$, as expected. In any case,

$$E[R] = \frac{K}{\mu_2} \cdot \frac{s^{K+1} - 1}{s^{K+1} - s} = \frac{K}{\mu_2} \cdot \frac{1 - 1/s^{K+1}}{1 - 1/s^K}$$

The latter expression is lower bounded by $\frac{K}{\mu_2}$, which is approached when $\mu_1 \gg \mu_2$ (i.e., $s \to \infty$) or $K \to +\infty$. If the sender is considerably faster than the receiver, all the K jobs are always on the receiver, and the transaction delay is the time it takes to empty the receiver's queue.