PECSN 21/02/2014

Name and surname: ____ *Matricola:* __________________

I will sit for the oral examination in: ☐ *February* ☐ *April (*)*
This decision <u>cannot</u> be changed later.

(*) If you are ineligible, your written test will be discarded.

Exercise 1

Consider the following function of two variables: $f(x,y)=\begin{cases} C\cdot(x+y) & x\in[0,1],\ y\in[0,1], 0<x+y<1 \\ 0 & otherwise \end{cases}$

1) Compute C so that $f(x,y)$ is a JPDF for RVs *X* and *Y*
2) Compute the PDFs for RVs *X* and *Y* and compute $Cov(X,Y)$
3) Define $Z=X+Y$. Compute the PDF of *Z* and determine $E[Z]$

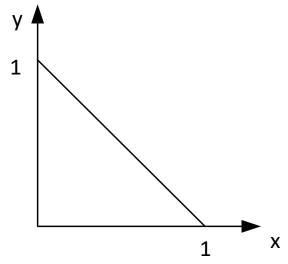
Exercise 2

Consider a distributed system providing service to a pool of *N* processes. The system has a pool of *M*≥*N* identical peripherals, which are kept idle until some process requests service to them. Every time a process issues a service request, a peripheral is activated and dedicated to that process. Service requests are blocking, i.e. the same process can only have at most an outstanding request at any time. When the service request is fulfilled, the peripheral is returned to the idle pool and the process carries on with its elaborations. Assume that a running process issues service requests with an interarrival time which is exponentially distributed, and let λ be its rate. Assume that all processes are independent. Assume that the service time of peripherals is exponentially distributed, and let μ be its rate. Assume μ ≠λ for simplicity.

1) Model the system and draw its Transition Rate diagram.
2) Compute the stability condition and find the steady-state probabilities.
3) Compute the mean number of blocked processes.
4) Compute the steady-state probabilities seen by a process issuing a service request.
5) Compute the CDF of the response time for a process.

Exercise 1 – Solution

The JPDF is non null in the triangle in the figure.



Therefore, normalization reads:

$$\int_0^1\left[\int_0^{1-x} C(x+y)\,dy\right]dx = C\cdot\int_0^1\left[\left[x\cdot y+\frac{y^2}{2}\right]_0^{1-x}\right]dx = C\cdot\int_0^1\left(\cancel{x}-x^2+\frac{1+x^2-\cancel{2x}}{2}\right)dx = \frac{C}{2}\cdot\left[x-\frac{x^3}{3}\right]_0^1 = \frac{C}{3}=1$$

Which yields *C*=3.

2) To obtain the PDF for a RV, we integrate in the other.

$f_X(x)=\int_0^{1-y}3(x+y)\,dy=\frac{3}{2}(1-x^2)$, with $x\in[0,1]$.

For obvious reasons of symmetry, $f_Y(y)=\frac{3}{2}(1-y^2)$, with $y\in[0,1]$.

It is $Cov(X,Y)=E[XY]-E[X]E[Y]$. Therefore:

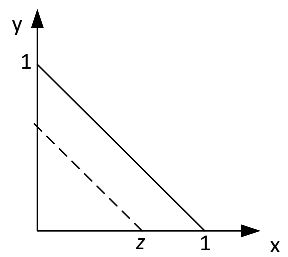
$$E[XY]=\int_0^1\left[\int_0^{1-x}3xy(x+y)\,dy\right]dx=3\int_0^1\left[\int_0^{1-x}(x^2\cdot y+x\cdot y^2)\,dy\right]dx$$
$$=3\int_0^1\left[x^2\cdot\frac{1+x^2-2x}{2}+x\cdot\frac{1-3x+3x^2-x^3}{3}\right]dx=\frac{1}{2}\left[x^2-x^3+\frac{x^5}{5}\right]_0^1=\frac{1}{10}$$

$$E[X]=E[Y]=\int_0^1 x\cdot\frac{3}{2}(1-x^2)\,dx=\frac{3}{8}$$

Therefore, $Cov(X,Y)=E[XY]-E[X]E[Y]=\frac{1}{10}-\frac{9}{64}=-\frac{13}{320}$

3) By the linearity of the mean value, one can immediately write $E[X]=E[Z]+E[Y]=\frac{6}{8}$.

The PDF of RV *Z* is non null in [0,1], and $F_Z(z)=P\{X+Y\le z\}$ is the probability that X and Y are in the triangle bounded by the dashed line in the figure.



Therefore, computing $F_Z(z)$ only entails substituting *z* for 1 in the computations already done for the normalization:

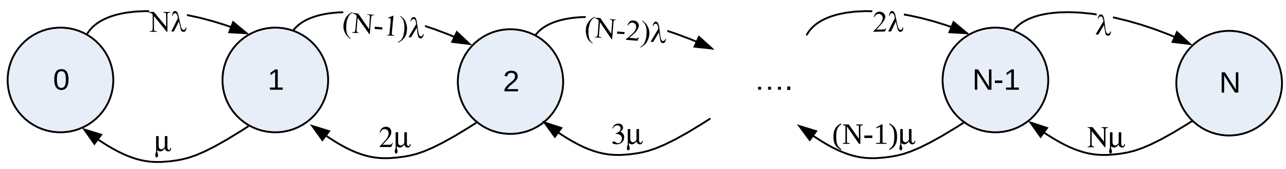
$$F_Z(z) = \int_0^z \left[\int_0^{z-x} 3(x+y)\,dy \right] dx = 3 \cdot \int_0^z \left[\left[x \cdot y + \frac{y^2}{2} \right]_0^{z-x} \right] dx$$

$$= 3 \cdot \int_0^z \left(\cancel{x \cdot z} - x^2 + \frac{z^2 + x^2 - \cancel{2z \cdot x}}{2} \right) dx = \frac{3}{2} \cdot \left[z^2 \cdot x - \frac{x^3}{3} \right]_0^z = \frac{3}{2} \cdot \frac{2z^3}{3} = z^3$$

From the above, we straightforwardly obtain $f_Z(z) = 3z^2$.

Exercise 2 – Solution

1) The system is a finite-population one. The service rate increases based on the number of jobs in the system, and there is never any queueing (an idle server is always available for an incoming job request).



This allows one to answer point 5 immediately: the distribution of the response time is the distribution of the service time, i.e. $F(t) = 1 - e^{-\mu \cdot t}$

2) The system is always stable, since there is never any queueing. The SS probabilities can be easily found by observing that the TR diagram has only nearest-neighbor transitions:

$$p_j = p_0 \cdot \left(\frac{N!}{(N-j)!} \cdot \lambda^j \right) \Big/ \left(\mu^j j! \right) = \binom{N}{j} \cdot u^j \cdot p_0 \text{, where } u = \lambda/\mu .$$

This said, normalization reads: $\sum_{j=0}^{N} p_j = p_0 \cdot \sum_{j=0}^{N} \binom{N}{j} u^j = p_0 \cdot (1+u)^N = 1$, hence $p_j = \binom{N}{j} \cdot \frac{u^j}{(1+u)^N}$, $0 \le j \le N$.

3) The mean number of blocked processes is:

$$\begin{aligned} E[n] &= \sum_{j=1}^{N} j \cdot \binom{N}{j} \cdot \frac{u^j}{(1+u)^N} \\ &= N \cdot \sum_{j=1}^{N} \binom{N-1}{j-1} \cdot \frac{u^j}{(1+u)^N} \\ &= N \frac{u}{1+u} \end{aligned}$$

4) The system is non-PASTA, since the arrival rates are not constant. It is:

$$\overline{\lambda} = \sum_{j=0}^{N} \lambda_j \cdot p_j = \sum_{j=0}^{N} (N-j) \cdot \lambda \cdot p_j = \left[N \cdot \lambda \cdot \sum_{j=0}^{N} p_j - \lambda \cdot \sum_{j=0}^{N} j \cdot p_j \right] = \lambda \cdot \left[N - E[n] \right].$$

Therefore, we have:

$$r_j = \frac{(N-j) \cdot \lambda}{\lambda \cdot \left[N - E[n] \right]} \cdot p_j = \frac{N-j}{N - E[n]} \cdot p_j = [\ldots] = \binom{N-1}{j} \cdot \frac{u^j}{(1+u)^{N-1}} \text{, } 0 \le j \le N-1$$

5) See point 1.