EXERCISE 1

TYPE 1 [PDF]:

IF I HAVE THE VALUE OF VAR:

$$Var(x) = E[X^2] - E[X]^2 = \int x^2 f_X(x)\,dx - \left[\int x f(x)\,dx\right]^2$$

MAIN

- APPLY NORMALIZATION $\rightarrow \int_{-\infty}^{+\infty} f(x)\,dx = 1$
- CHECK EVENTUAL SYMMETRY [EVEN FUNCTION]
- USE VAR(X) EXPRESSION IN WHICH APPEAR $f(x)$
- EXPRESS C IN FUNCTION OF λ, CHECK IF THE NEW EXPRESSION IS SIMILAR TO CANONICAL DISTRIBUTION (EX. EXPONENTIAL)
- REMEMBER THAT SOME VALUE (AS λ) HAS CONSTRAINTS ($\lambda > 0$ OR α INTEGER
- NON NEGATIVE

] GIVEN PDF EXPRESSION WITH PARAMETERS, HOW TO FIND THEM:

] GIVEN $f(x) = A_\lambda \begin{cases} e^{-t^2/2} & t<0 \text{ \{ STANDARD NORMAL SHAPE} \\ e^{-\lambda t} & t \geq 0 \text{ \{ EXPONENTIAL SHAPE DECAYING} \end{cases}$ { TRY TO FIND KNOWN DISTRIBUTIONS

IN 0? CONTINUOUS WITH VALUE A_λ

FIND $A_\lambda \Rightarrow$ N. C. $\rightarrow$ WE CAN SPLIT THE INTEGRAL:

$$\int_0^{+\infty} A\lambda \cdot e^{-\lambda t}\,dt = \frac{A\lambda}{\lambda}$$
$$+$$
$$\int_{-\infty}^{0} A\lambda \cdot e^{-t^2/2}\,dx = \frac{A\lambda}{2}\sqrt{2\pi}$$
$$= 1$$

] FIND CDF FROM PDF: $F_X(x) = \int_{-\infty}^{x} f(x')\,dx' \underset{\text{EXAMPLE}}{=} \begin{cases} 0 & x<0 \\ \left(\frac{x}{c}\right)^k & 0 \leq x \leq c \\ \boxed{1 \quad x > c} \end{cases}$

ATTENTION: $\lim_{a \to +\infty} F_X(a) = 1$ [NO MATTER IF $f_X(x)$ IS 0 ALWAYS EXCEPT FOR $0 \leq x \leq c$]

$\left(\frac{x}{c}\right)^k$ if $k \to +\infty$: CDF $\rightarrow$ STEP FUNCTION in $x = c$

] MEAN VALUE: $E[X] = \underbrace{\int_0^a}_{\text{WHERE } f_X(x) \text{ IS DEFINED AND } \neq 0} x \cdot f_X(x)\,dx$

if $f_X(x) = \text{cost}\,(a,b)$ $\quad a \leq x \leq b$ IS UNIFORM $\Rightarrow$ SYMMETRIC

$\Downarrow$

VALID ALSO FOR STEP FUNCTIONS { $E[X] = \underset{\text{MEDIAN}}{x_{0.5}}$

] MEAN SQUARE VALUE: $E[x^2] = \int_0^a x^2 f_X(x)\,dx$

] VARIANCE: $\sigma^2 = E[x^2] - E[X]$

] IF $S_5 = \sum_{i=1}^{5} X_i$, X_i IID ACCORDING TO $f(x)$, COMPUTE $C = \frac{\sigma}{E[S_5]}$ [COEFFICIENT OF VARIATION]:

$E[S_5] = 5\,E[X]$

$\sigma^2_{S_5} = 5 \cdot \sigma^2_X$

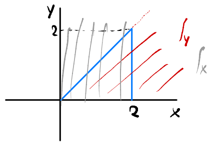
] FIND PDF FROM CDF: $f_X(x) = \frac{\partial}{\partial x} F_X(x)$ if $=$ cost $\Rightarrow X$ IS UNIFORM

[MIND THAT $F_X(x) = 1$ FOR $x > $ TOT DOESN'T HAVE TO BE REPRESENTED]

TYPE 2 [SPDF]:

] GIVEN SPDF $[f(x,y)]$, FIND "K" PARAMETER:

– REPRESENT INTEGRATION DOMAIN:



– IF y DEPENDS ON x: $0 \le y \le x$, $0 \le x \le 2$

USE SAME INTERVALS OF BEFORE

· INTEGRATING BEFORE $y \Rightarrow \int_0^2 \left[\int_0^x dy\right] dx \underbrace{= 1}_{\text{APPLY N.C.}} \Rightarrow$ <u>K FOUND</u>

· " " $x \Rightarrow \int_0^2 \left[\int_y^2 dx\right] dy = 1$

WHY? $\underbrace{0 \le y \le x \le 2}_{\text{NEW INTERVAL}}$

] FIND $f_X(x)$, $f_Y(y)$: $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$

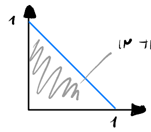
$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$

CHECKING INDEPENDENCE BETWEEN X, Y:

$f(x,y) = f_X(x)\, f_Y(y)$

] GIVEN SPDF $[f(x,y)]$, FIND "K" $[k \in \mathbb{R}]$ PARAMETER <u>WITHOUT INTEGRATION</u>: $f(x,y) = \begin{cases} k & 0 \le x \le 1,\ 0 < y < 1-x \\ 0 \end{cases}$

– REPRESENT INTEGRATION DOMAIN:



IN THIS DOMAIN $f(x,y) = k$

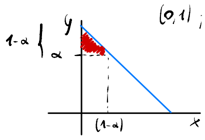
$A_{\text{TRIANGLE}} = \frac{1 \cdot 1}{2} = \frac{1}{2}$

$\int_{\text{TRIANGLE}} K = 1 = A_{\text{TRIANGLE}} \cdot K = 1$

[IT'S A VOLUME] $\frac{1}{2} \cdot k = 1 \Rightarrow k = 2$

] FIND $P\{Y > \alpha\}$ <u>WITHOUT INTEGRATION</u> $[0 < \alpha < 1]$:

$(0,1)$; $(0,\alpha)$; $(1-\alpha, \alpha)$: TRIANGLE

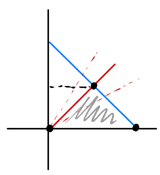


$P\{Y > \alpha\} = \frac{A_{\text{RED TRIANGLE}}}{A_{\text{TRIANGLE}}} = \frac{\frac{(1-\alpha)^2}{2}}{\frac{1}{2}} = (1-\alpha)^2$

SPDF UNIFORM, $\int_{\text{TRIANGLE}} \text{JPDF} = 1$

] COMPUTE PDF OF $W = \frac{Y}{X}$: WHERE IS W DEFINED? $0 \le X \le 1$, $0 < Y < 1-x$ $\left\{ W = \frac{Y}{X} \Rightarrow \begin{array}{l} W_{max} = \frac{Y}{0} = +\infty \\ W_{min} = 0 \end{array} \right. \Rightarrow W \in [0, +\infty)$

$P\{W \le a\} = P\{Y \le a \cdot X\} \Rightarrow$ REPRESENT DOMAIN:



$\left\{ \begin{array}{l} \boxed{y = ax} \\ y = -x+1 \end{array} \right. \Rightarrow$ INTERSECTION: $\left(\frac{1}{1+a}, \frac{a}{1+a}\right)$ + $(0,0)$; $(1,0)$ = TRIANGLE WHERE $Y \le a X$

$\Rightarrow P\{W \le a\} = \frac{A_{\text{GREY TRIANGLE}}}{A_{\text{TRIANGLE}}} = F_W(a)$

$f_W(a) = \frac{\partial}{\partial a} F_W(a)$

] X AND Y ARE IID? THEY MUST HAVE THE SAME PDF

] 95^{th} PERCENTILE OF X: $F_X(\pi_{95}) = \int_{-\infty}^{\pi_{95}} f_X(x)\,dx = 0.95 \Rightarrow \pi_{95} = ?$

] MEDIAN OF Y: $F_Y(\pi_{50}) = 0.5 = \int_0^{\pi_{50}} f_Y(y)\,dy$

] FIND THE REGION OF (x,y) PLANE WHERE $f(x,y)$ IS NON NULL $\Rightarrow$ IF X, Y ARE INDEPENDENT: $f(x,y) = f_X(x)\, f_Y(y)$

$R = \underbrace{[0 \le x \le 3] \times [0 \le y \le 2]}_{\text{INTERVALS WHERE } f_X(x), f_Y(y) \text{ ARE DEFINED}}$

] MARGINAL PROBABILITIES:

Given a pair of discrete RVs X, Y, whose JPMF is known, the two probabilities $P[X = x]$ and $P[Y = y]$ are often called **marginal probabilities**. The reason is that the JPMF is often given in a **table form** (with X, Y in row/column), hence the two above probabilities can be computed as row/column sums, and conveniently written on the **margin** of the table.

X \ Y	y1	y2	yk	
x1	p(x1,y1)	p(x1,y2)	p(x1,yk)	p(x1)
x2	p(x2,y1)	p(x2,y2)	p(x2,yk)	p(x2)
xh	p(xh,y1)	p(xh,y2)	p(xh,yk)	p(xh)
	p(y1)	p(y2)	p(yk)	

TO VERIFY INDEPENDENCE:

→ – TAKE $P(X_1)$, $P(Y_1)$

– COMPUTE $P(X=1) \cdot P(Y=1)$

– COMPARE IT TO THE TABLE WHERE $P(X=1, Y=1)$

] PMF OF $X+Y \Rightarrow S = X+Y \Rightarrow$

S	P(S)
0	P → Σp where $x+y=0$
1	
2	

TYPE 3 [COMPOSED RVs]

•] $\sum_{s=1}^{M} X_s$, $M \geq 30$, IID, KNOWN $\underbrace{E[X_s], Var(X_s)}_{\text{ALL THE SAME}}$ [BOTH FINITE], COMPUTE $P\{\sum_{s=1}^{M} X_s > \text{VALUE}\}$:

• $\boxed{\text{CENTRAL LIMIT THEOREM}}$ THEN: $P\left\{\frac{\sum_{i=1}^{M} X_i - M\cdot\mu}{\sqrt{M}\cdot\sigma} \leq \text{VALUE}\right\} \simeq \phi(\text{VALUE})$ OR

THE SUM IS APPROX NORMAL

• CDF OF NORMAL RV
$F(a) = P\{X \leq a\} = \phi\left(\frac{a-\mu}{\sigma}\right)$

$1 - F(a) = P\{X > a\} = 1 - \phi\left(\frac{a-\mu}{\sigma}\right)$

•] CDF/PDF OF $Y = |X|$ => $F_Y(k) = P\{|X| \leq k\} = P\{-k \leq X \leq k\} = \int_{-k}^{k} f(t)\,dt \longrightarrow$ IF $[-k,0]$ IS STANDARD NORMAL:

EXAMPLE: $\sqrt{2\pi}\int_{-k}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,d = \sqrt{2\pi}\,\underbrace{\left(\phi(k) - \frac{1}{2}\right)}$

$\phi(0) - \phi(-k) =$

$= \frac{1}{2} - (1 - \phi(k)) = \phi(k) - \frac{1}{2}$

•] IF $Y = \log(X)$ => $F_Y(y) = P\{Y \leq y\} = P\{\log(X) \leq y\} = P\{X \leq e^y\}$

IF I HAVE $f_X(x)$, CHECK THE INTERVAL WHERE X IS DEFINE, THEN FIND THE INTERVAL OF Y

•] $Z = \frac{1}{X}$, FIND $E[Z]$ AND $f_Z(z)$: $P\{X \leq x\} = F_X(x) = \underbrace{\int_0^2}_{\text{INTERVAL WHERE } x \text{ IS DEFINED}} f_X(x)\,dx$

$F_Z(z) = P\{\frac{1}{X} \leq z\} = P\{X \geq \frac{1}{z}\} = 1 - F_X(\frac{1}{z})$

$0 \leq X \leq 2$ => $\boxed{\frac{1}{X} = Z \geq \frac{1}{2}}$ CHECK ALWAY WHERE Z IS DEFINED

$\Downarrow$

$E[Z] = \int_{\frac{1}{2}}^{+\infty} f_Z(z)\,dz$

•] COMPUTE $P\{X_1 + X_2 = 0\}$

AND GENERALIZE THE FORMULA => $P\{X_1 = -a \underbrace{,}_{\text{AND = MULTIPLICATION (IID)}} X_2 = a\} + P\{X_1 = a, X_2 = -a\} = p_1(-a)\cdot p_2(a) + p_1(a)\cdot p_2(-a)$

$S_{m,k} = \frac{m!}{(m-k)!}$ ORDER MATTERS

$X_1, X_2 = \begin{cases} a \\ b \end{cases}$ JUST TWO VALUES

$P\{\sum_{s}^{2m} X_s = 0\}$ MEANS $\underbrace{-a \ldots -a}_{m}\ \underbrace{a \ldots a}_{m}$] → EACH OF THESE COMBINATIONS HAS $\frac{1}{2^{2m}}$ PROBABILITY, THERE ARE $\underbrace{\binom{2m}{m}}$ COMBINATIONS

m SUBSETS EQUAL TO ONE VALUE

$\boxed{P\{\Sigma = 0\} = \binom{2m}{m}\frac{1}{2^{2m}}}$

•] PMF of $S_m = \sum_{s}^{2m} X_s$ => POSSIBLE VALUES

$-2\cdot m\cdot a, \ldots, 2\cdot m\cdot a$ => $\{2a\cdot s, -m \leq s \leq m\}$ SYMMETRIC PMF

$2\cdot a\cdot s = \{m-s$ POSITIVE & $m+s$ NEGATIVE$\}$: THERE ARE $\binom{2m}{m-s} = \binom{2m}{m+s}$ DIFFERENT WAYS, EACH ONE HAS $\frac{1}{2^{2m}}$ PROBABILITY

$[m+s-(m-s) = 2s]$

$p_m(2a\cdot s) = \binom{2m}{m-s}\frac{1}{2^{2m}} \quad -m \leq s \leq m$

•] $Z = \min(T_1, T_2)$ with T_1, T_2 GEOMETRIC RV IID $\xrightarrow{\text{PMF}}$ $\underbrace{P\{Z > k\} = P\{T_1 > k, T_2 > k\}}_{\text{WHY?}} = P\{T_1 > k\}\cdot P\{T_2 > k\} =$

$\boxed{\text{BECAUSE IF THE } \min(T_1,T_2) \text{ IS } > k \text{ ALSO THE GREATER IS } > k}$

$= (1 - F_T(k))^2 = (1-p)^{2(k+1)}$

$F_Z(k) = 1 - (1-p)^{2(k+1)}$ => $\boxed{p_Z(k) = F_Z(k) - F_Z(k-1)}$

WE CAN USE $q = (1-p)^2$ OR SIMILAR TO FIND IF Z IS GEOMETRIC OR ANOTHER DISTRIBUTION

Like with exponential RVs (of which the geometric are the discrete counterparts), a *minimum* property can be formulated: the min of 2 IID geometric RVs is itself a geometric RV, whose success probability is $1 - (1-p)^{2}$, i.e. the complement of the probability that *both* trials will fail.

] $X_1, \dots, X_n$ IID WITH KNOWN DISTRIBUTION, COMPUTE $W = \max_i \left\{ \frac{1}{X_i} \right\} \Rightarrow F_W(w) = P\{\overbrace{Z_1 \le w, \dots, Z_n \le w}^{\text{IF THE BIGGER IS BELOW 'W' ALL OF THEM ARE BELOW}}\}$

$= [F_Z(w)]^n$

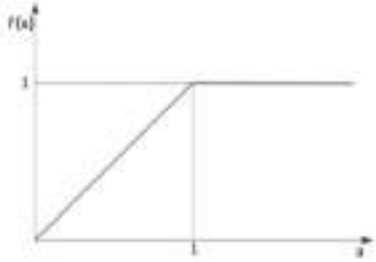
$Z = \frac{1}{X_i}$ $\quad F_Z(z) = P\{Z \le z\} = P\{X > \frac{1}{z}\} = 1 - F_X(\frac{1}{z})$

SINCE $X \ge 1 \Rightarrow \underbrace{0 \le Z \le 1}$

LIMITED SUPPORT OF $Z \Rightarrow$ CDF DECREAS WITH n, $n \to +\infty$ CDF = STEP FUNCTION

Assume that $F(a)$ is the following

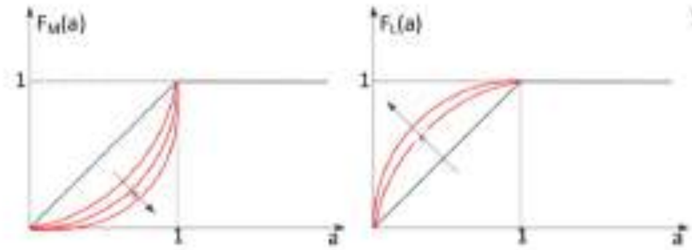
$$F(a) = \begin{cases} a & 0 \le a < 1 \\ 1 & a \ge 1 \\ 0 & a < 0 \end{cases}$$



Then it is easy to see the following:

$$F_M(a) = \begin{cases} a^n & 0 \le a < 1 \\ 1 & a \ge 1 \end{cases}, \quad F_L(a) = \begin{cases} 1 - [1-a]^n & 0 \le a < 1 \\ 1 & a \ge 1 \end{cases}$$

$y^n \to n x^{n-1}$



As n grows large, **the two distributions tend to a *step function*,** respectively in 1 (maximum) and 0 (minimum). There is a physical explanation for this. The n variables are *independent* and *uniformly distributed*. The fact that their PDF is uniform can be observed by deriving $F(a)$ (you get a constant function in [0,1]. If you take one sample at random, it can be anywhere in [0,1]. If you take n samples, there is an increasing probability that

- The **highest** sample will be near 1
- The **lowest** sample will be near 0.

] $Y \underset{(\alpha > 0)}{\text{EXP}}$, $X = x_m \cdot e^Y$, $x_m > 0$, COMPUTE CDF, PDF OF X: $F_X(x) = P\{X \le x\} = P\{x_m \cdot e^Y \le x\} = P\{Y \le \log(\frac{x}{x_m})\}$

$= 1 - e^{-\alpha(\log(\frac{x}{x_m}))} = 1 - \left(\frac{x_m}{x}\right)^\alpha$

$\left[e^{\log(\frac{x}{x_m})^{-\alpha}} = \left(\frac{x_m}{x}\right)^\alpha\right]$

$X \in [x_m, +\infty)$

$\min(Y) = 0 \Rightarrow \min(x) = x_m \cdot 1$

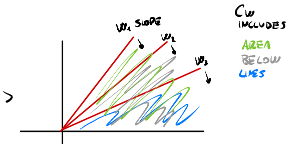
PDF: $f_X(x) = \frac{\partial}{\partial x} F_X(x)$

] RANGE OF VALUE OF $\frac{Y}{X}$: CHECK $\max \frac{Y}{X} = \frac{\max(Y)}{\min(X)} = a$

$\min \frac{Y}{X} = \frac{\min(Y)}{\max(X)} = b$

$\frac{Y}{X} \in [a, b]$ OR (a, b)

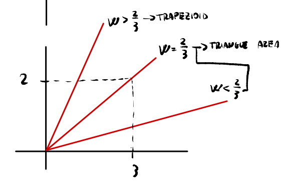
] C_w REGION WHERE $\frac{Y}{X} \le w$ ON (x,y) PLANE, $C_w \subseteq \mathbb{R}$: $C_w = \{(x,y) : (x,y) \in \mathbb{R}, \ y/x < w\} \Rightarrow$



] COMPUTE $P\{(X,Y) \in C_w\} = P\{\frac{Y}{X} \le w\} = F_{\frac{X}{Y}}(w)$:

$$F_{\frac{X}{Y}}(w) = \int_{C_w}\int f(x,y)\,dx\,dy \ \text{— TWO CASES} \begin{cases} \to 0 \le w \le \frac{2}{3} : \int_0^3 \left[\int_0^{wx} dy\right] dx \\ \to w > \frac{2}{3} : 1 - \underbrace{\int_0^2 \left[\int_0^{y/w} dx\right] dy}_{\text{EXCLUDE THE UPPER TRIANGULAR REGION AND INVERT THE ORDER OF INTEGRATION OF THE VARIABLES}} \end{cases}$$

$P\{X \ge \frac{Y}{w}\} = 1 - P\{X \le \frac{Y}{w}\}$



$[0 \le x \le 3] \times [0 \le y \le 2]$

TYPE 4 [PMF]:

- NUMBER OF TIMES BEFORE GETTING A RESULT → GEOMETRIC RV

$$P\left\{\underbrace{-\tfrac{1}{3}}_{=0} \le \underbrace{T}_{\text{COUNTER OF \# OF TIMES}} \le \underbrace{\sqrt{11}}_{=3}\right\} = \sum_{s=0}^{3} P\{T=s\} = F_T(3)$$

- PMF OF X, WHERE X_n IS THE NUMBER OF TRIALS TO GET:
  - 1 SUCCESS AT s^{th} TRIAL
  - $n-1$ SUCCESSES AT

  } n SUCCESSES IN TOTAL

X_n IS Σ OF GEOMETRIC RVs WITH $\frac{1}{p}$ MEAN EACH ONE

$$E[X_n] = \frac{k}{p}$$

$$P\{X_n = s\} = \underbrace{\binom{s-1}{n-1}} (1-p)^{s-n}\, p^n \qquad s \ge n$$

WITH $\binom{s}{n}$ IS NUMBER OF SUCCESS IN s REPEATED TRIALS, WE HAVE -1 BECAUSE WE WANT THE n^{th} SUCCESS AT THE LAST TRIAL [s^{th} IS FIXED] => ORDER DOESN'T MATTER

- THROW n FAIR DICE, IF $n=2$ => OUTCOME $=\{d_1, d_2\}$

$f_n(x)$ PMF, $X=x$ IS THE MAXIMUM VALUE OF ALL DICE

PMF: $f_2(1) = \frac{1}{36}$ $\{1,1\}$

$f_2(2) = \frac{3}{36}$ $\{1,2\}, \{2,1\}, \{2,2\}$

$\vdots$

$f_2(x) = \frac{2x-1}{36}$

BASIC PRINCIPLE OF COUNTING: $\frac{\text{FAVORABLE CASES}}{\text{TOTAL CASES}}$

PMF: $f_3(4)$ => COMPUTE CDF FIRST: $F_3(x) = P\{\max[d_1, d_2, d_3] \le x\}$, $1 \le x \le 6$ => $F_3(1) = \frac{1}{6^3}$ $\{1,1,1\}$

$F_3(2) = \frac{2^3}{6^3}$ $\{d_1, d_2, d_3\}$ where $d_i \le 2$

$\vdots$

$F_3(k) = \frac{k^3}{6^3}$

$\Downarrow$

$f_3(k) = F_3(k) - F_3(k-1)$ $\xrightarrow{\text{GENERALIZING}}$

$$F_n(k) = \frac{k^n}{6^n}$$

$$f_n(k) = F_n(k) - F_n(k-1)$$

- DRAW QUALITATIVE DIAGRAM OF $f_n(k)$:
  - $\lim_{n\to\infty} f_n(k) = \begin{cases} \dots & k < \\ & k = \dots \end{cases}$

- $f_n(6) = 0.9$ => THROWING n DICE, 6 IS THE MAXIMUM WITH >90% OF PROBABILITY

- $A \rightarrow p_A = 0.4$ duration $(A) = 4$ms

  $B \rightarrow p_B = 0.6$ duration $(B) = 5$ms

  S_n, n events of A or B

BERNOULLIAN EXPERIMENT: P of having k B events $\sim B(4, 0.6)$

PMF of S_4: $P\{\underbrace{S_4 = 16 + k}_{16, \dots, 20 \text{ CAN LAST}}\} = \overbrace{\binom{4}{k}}^{\text{4 tentatives to have k B}} 0.6^k \cdot 0.4^{4-k}$ $\quad 0 \le k \le 4$

$$E[S_4] = n \cdot p = 4 \cdot 0.6 + 16$$

$\downarrow$ GENERAL

$$P\{S_n = 4 \cdot n + k\} = \binom{n}{k} 0.6^k \cdot 0.4^{n-k}$$

$$E[S_n] = (4 + 0.6) \cdot n$$

$$4n \le S_n \le 5n$$

TYPE 5 [LIGHT/HEAVY TAILED]:

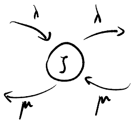
- IS Z HEAVY TAILED?

DEFINITION: $\forall \lambda: \lim_{z\to+\infty} e^{\lambda z}(1 - F_Z(z)) = \infty$

TYPE 6 [CDF]:

CDF OF EXPONENTIAL

•] FIND CDF OF T_S [TIME SPENT IN STATE S OF M/M/1]: $F_S(\tau) = P\{T_S \le \tau\} = P\{\underbrace{\min(T_A, T_B)}_{\substack{\text{min residual service time} \\ \text{is still EXP (MEMORYLESS)} \\ \text{equal to SUM OF RATES}}} \le \tau\} = 1 - e^{-(\lambda+\mu)\tau}$



$\boxed{\text{ONLY TIME OF ARRIVAL / DEPARTURE MATTERS}}$

•] FIND CDF OF EXP. RANDOM VARIABLE [S] THAT REPRESENTS SERVICE TIME OF A CLIENT (CLIENT GOES TO SERVER i WITH p_i PROB) $[1 \le i \le n]$

$$F_S(x) = P\{S \le x\} \overset{\text{LAW OF TOTAL PROBABILITY}}{=} \sum_{i=1}^{n} P\{S \le x \mid \text{serve} = i\}\, P\{\text{serve} = i\} = \sum_{i=1}^{n} \underbrace{\left(1 - e^{-\mu_i x}\right)}_{\substack{\text{CDF OF EXP.} \\ \text{WITH } \mu_i \text{ PARAMETER}}} p_i = \sum_{i=1}^{n} F_{S_i}(x)\, p_i$$

MEAN: $E[S] = \sum_{i=1}^{n} \underbrace{\frac{1}{\mu_i}}_{\substack{\text{MEAN OF} \\ \text{EXP. RV}}} p_i$ SINCE INTEGRALS AND SUMS COMMUTE

•] FIND CDF WHEN A JOB IS CONSIDERED SERVED IF AT LEAST ONE SERVER COMPUTES IT

$\tau = \sum_{i=1}^{n} \mu_i$ $\quad F_S(x) = 1 - e^{-\tau\mu}$ $\quad$ { EXPONENTIAL PROPERTY OF THE MIN BETWEEN EXP RVs

•] IDEM OF ABOVE BUT WHEN ALL SERVERS COMPUTE IT

$$F_S(x) = P\{\max\{S_i\} \le x\} = P\{S_1 \le x, S_2 \le x, \ldots, S_n \le x\} \underset{\text{INDEPENDENCE}}{=} \prod_{i=1}^{n} \left(1 - e^{-\mu x}\right) = \prod_{i=1}^{n} F_{S_i}(x)$$

•] OUTCOME TYPE K → p
" " a → 1−p

N_K # OF OUTCOME OF TYPE K
N_a " " " " a

$\underbrace{\qquad}_{\text{THE EXPERIMENT STOPS WHEN } N_K \ge 1 \text{ \& } N_a \ge 1}$

$$\Rightarrow P\{N_K = 1\} = \underbrace{\sum_{n=1}^{+\infty} P\{n \text{ CONSECUTIVE } a, 1\ K\}}_{\sum_{n=1}^{+\infty} (1-p)^n p} + \underbrace{P\{1\ a, 1\ K\}}_{(1-p)p}$$

$$P\{N_K = n\} = P\{n \text{ di } K, 1\ a\}$$

$$F_K(n) = P\{N_K = 1\} + \sum_{j=2}^{n} P\{N_K = j\} \qquad \underline{\forall n \ge 1}$$

•] $N_K > N_a$ if at least 3 OUTCOMES ARE OBSERVED: (WITH THE LAST ONE ACID)

$$P\{N_K > N_a\} = \sum_{n=2}^{+\infty} P\{N_K = n\}$$

$$E[N_K] = 1 \cdot P\{N_K = n\} + \sum_{n=2}^{+\infty} n \cdot P\{N_K = n\}$$

TYPE 7 [SET OF VALUES FOR RV]:

- X COUNTER UNTIL A CERTAIN CONDITION => WHAT IS THE MIN VALUE OF X WHEN THE CONDITION IS SATISFIED

TYPE 8 [MEMORYLESS PROPERTY]:

$$P\{X > a+b \mid X > a\} \underset{\text{BAYES THEOREM}}{=} \frac{P\{X > a+b\}}{P\{X > a\}}$$

if IT IS EQUAL TO: $P\{X > b\}$ THE DISTRIBUTION IS <u>MEMORYLESS</u>

TYPE 9 [UPM]

- A

1		...			N

IT HAS: $0 \le n_A$ elements $\le N$

I WANT THAT A CONTAINS ELEMENT 1:

$$P_A(E_1) = \frac{|E_1|}{N} = \frac{\binom{N-1}{n_A-1}}{\binom{N}{n_A}}$$

$\binom{N-1}{n_A-1}$: $n_A - 1$ different SUBSETS in $N-1$ because 1 IS FIXED → take $n_A - 1$ elements from a MAXIMUM OF $N-1$, where the missing one is the element 1/5 [NO MATTER] that you have to take mandatory

$\binom{N}{n_A}$: n_A different SUBSETS from N → take n_A elements, that is the Sample Space Cardinality

$0 \le n_A \le N$

- IF I TAKE B ARRAY EQUAL TO A WITH $0 \le n_B \le N$,

THE PROBABILITY OF HAVING THE SAME ELEMENT IN BOTH ARRAYS IS: $P_A(E_5) \cdot P_B(E_5)$

SINCE A,B ARE INDEPENDENT

- L, U BOUNDS OF # OF SAME ELEMENTS IN A,B GIVEN n_A, n_B =>

$$L = \max(0, n_A + n_B - N) = \begin{cases} 0 & n_A + n_B \le N \\ n_A + n_B - N & \text{OTHERWISE} \end{cases}$$

$$U = \min(n_A, n_B)$$

- PROBABILITY OF HAVING $L \le k \le U$ ELEMENTS IN COMMON:

$$P = \frac{\overbrace{\binom{N}{k}}^{\text{SUBSET OF K ELEMENTS IN COMMON}} \cdot \overbrace{\binom{N-k}{n_A-k}}^{\text{REMAINING WAYS TO ALLOCATE ELEMENTS OF A}} \cdot \overbrace{\binom{N-n_A}{n_B-k}}^{\text{REMAINING WAYS TO ALLOCATE ELEMENTS OF B}}}{\binom{N}{n_A} \cdot \binom{N}{n_B}}$$

REMAINING WAYS TO ALLOCATE ELEMENTS OF B WITHOUT CHOOSING THE ELEMENTS IN A TO AVOID OTHER INTERFERENCE

$\binom{N}{n_A} \cdot \binom{N}{n_B}$: SET OF ALL POSSIBLE ALLOCATIONS

- m FISH MARKED IN LAKE OF n FISH, I EXTRACT K FISH:

$$P\{X = x\} = \frac{\overbrace{\binom{m}{x}}^{\text{X FISH MARKED IN A SET OF m}} \overbrace{\binom{n-m}{k-x}}^{\text{EXTRACTING K-X FISH IN THE REMAINING SET OF n-m ELEMENTS}}}{\binom{n}{k}}$$

$\binom{n}{k}$: ALL POSSIBLE SUBSET OF K ELEMENTS in n TOTAL

- IF $n >> m >> k$, $\frac{m-\alpha}{n-\alpha} = \frac{m}{n}$, $0 \le \alpha \le k$, THE PROBABILITY THAT NEXT FISH IS RED DOESN'T CHANGE

ASSUME THE PICKING OF RED FISH AS REPEATED TRIAL EXPERIMENT IN INDEPENDENT CONDITION:

$$P\{X = x\} \approx \underbrace{\binom{k}{x} p^x (1-p)^{k-x}}_{\text{X SUCCESSES in K TRIALS}}$$

- 13 EMPLOYEES: $\underbrace{\text{8 HIRED, 5 INDEPENDENT}}_{\text{2 SICK}}$ → $\underbrace{P_{ILL} \text{ SAME FOR EVERYONE}}_{\text{UPM}}$, A # of HIRED SICK:

$$P\{A = s\} = \frac{\binom{8}{s}\binom{5}{2-s}}{\binom{13}{2}}$$

$s = 0, 1, 2$

PMF: $p(a) = p(0) + p(1) + p(2)$
$= P\{A=0\} + P\{A=1\} + P\{A=2\}$

B # OF HIRED ALLOCATE TO NEW TASK:

$$P\{B = b \mid A = k\} = \frac{\overbrace{\binom{8-k}{b}}^{\text{b FROM NON SICK}} \overbrace{\binom{(13-2)-(8-k)}{1-b}}^{\text{1-b FROM INDEPENDENT EMPLOYEES NON SICK}}}{\binom{13-2}{1}}$$

CAN BE EITHER INDEPENDENT NON SICK OR HIRED NON SICK WHICH ONE THAT GETS THE NEW TASK $\left[\frac{|E|}{N}\right]$

$\binom{13-2}{1}$: ONE FROM NON SICK

TOTAL PROBABILITY (?)

$b \in \{0, 1\}$, $k \in \{0, 1, 2\}$

$$\Downarrow$$

$$P\{B = b\} = \sum_{k=0}^{2} P\{B = b \mid Y = k\} P\{Y = k\}$$

TYPE 10 [POISSON]

- SEE 28/7/2021 [n LARGE, p SMALL]