Compito PECSN 06/07/2022

Exercise 1

Consider a system where a client can be randomly routed to *one* (and only one) among *n* servers, with probability p_i, $1 \le i \le n$. Each server's service time is an exponentially distributed RV, with a mean $\frac{1}{\mu_i}$. All servers are independent.

1) Find the CDF and PDF of the service time *of a client*
2) Find the mean and variance of the service time
3) Consider an alternative design of the same system, where a client request is sent to *all the servers simultaneously*, and is only considered served when
    a. *at least one* server has processed it
    b. *all* the servers have processed it.

    Find the CDF of the service time of a client in these cases.
4) Assume that $\mu_i = \mu$. Is one design of the system (among the initial option and options *a* and *b* at point 3) faster? Why?

Exercise 2

A counseling practice offers individual advice to its clients. It admits both *singles* and *couples*, but counsels its clients *individually* (spouses are requested to wait outside the counseling room). The arrival rate at the counseling practice is λ. An arrival may be a *single*, with probability π, and a *couple*, with probability $1 - \pi$. Individual counseling takes an exponentially distributed time, with a rate μ.

1) Model the practice as a queueing system, draw the CTMC and write down the global equilibrium equations.
2) Find the stability condition, justify it, and compute the PGF of the SS probabilities.
3) Compute the mean number of jobs in the system. Verify in the limit cases.
4) Compute the probability p_0 that the practice is empty, and the probability that only one client is in, p_1

Solution of Exercise 1

1) Call S the service time RV. Due to the Law of Total Probability, we have:

$$F(s) = P\{S \le s\}$$
$$= \sum_{i=1}^{n} P\{S \le s \mid \text{server} = i\} \cdot P\{\text{server} = i\}$$
$$= \sum_{i=1}^{n} \left(1 - e^{-\mu_i \cdot s}\right) \cdot p_i$$
$$= \sum_{i=1}^{n} F_i(s) \cdot p_i$$

Hence:

$$f(s) = \frac{d}{ds} F(s) = \frac{d}{ds} \sum_{i=1}^{n} F_i(s) \cdot p_i = \sum_{i=1}^{n} p_i \cdot \mu_i \cdot e^{-\mu_1 \cdot s} = \sum_{i=1}^{n} f_i(s) \cdot p_i$$

2) The mean service time is $\sum_{i=1}^{n} \frac{1}{\mu_i} \cdot p_i$. The variance is $\sum_{i=1}^{n} \frac{1}{\mu_i{}^2} \cdot p_i$, due to independence.

3) Case a. is a textbook case of minimum of independent exponential RVs. The theory says that the answers are $F(s) = 1 - e^{-\tau \cdot s}$, where $\tau = \sum_{i=1}^{n} \mu_i$

For case b, we have:

$$F(s) = P\{S \le s\} = P\{\max\{S_i\} \le s\}$$
$$= P\{S_1 \le s, S_2 \le s, \ldots, S_n \le s\}$$
$$= \prod_{i=1}^{n} \left(1 - e^{-\mu_i \cdot s}\right)$$
$$= \prod_{i=1}^{n} F_i(s)$$

4) When all the servers are indistinguishable, the CDF for the three systems is, respectively:
- $F_o(s) = 1 - e^{-\mu \cdot s}$ (original)
- $F_a(s) = 1 - e^{-n \cdot \mu \cdot s}$ (design a)
- $F_b(s) = (1 - e^{-\mu \cdot s})^n$ (design b)

It is easy to see that $\forall s > 0$, $F_a(s) > F_o(s) > F_s(s)$. In fact:

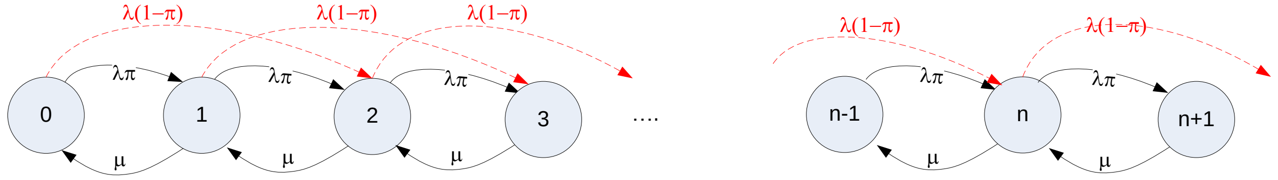
$$\begin{aligned} &F_a(s) > F_o(s) \\ &1 - e^{-n \cdot \mu \cdot s} > 1 - e^{-\mu \cdot s} \\ &e^{-n \cdot \mu \cdot s} < e^{-\mu \cdot s} \\ &-n\mu s < -\mu s \\ &n > 1 \end{aligned}$$

Moreover, $F_o(s) > F_s(s)$ iff $1 - e^{-\mu \cdot s} > (1 - e^{-\mu \cdot s})^n$, which is obvious since the l.h.s. is between 0 and 1.

The above implies that design a is (probabilistically) faster than the original design, which is faster than b.

Exercise 2 – Solution

1) The system is an M/M/1 with bulk arrivals, and the CTMC diagram is below.



The global equilibrium equations are:

- State 0: $\lambda \cdot p_0 = \mu \cdot p_1$
- State 1: $(\lambda + \mu) \cdot p_1 = \mu \cdot p_2 + \lambda \cdot \pi \cdot p_0$
- State 2: $(\lambda + \mu) \cdot p_2 = \mu \cdot p_3 + \lambda \cdot \pi \cdot p_1 + \lambda \cdot (1 - \pi) \cdot p_0$
- State n: $(\lambda + \mu) \cdot p_n = \mu \cdot p_{n+1} + \lambda \cdot \pi \cdot p_{n-1} + \lambda \cdot (1 - \pi) \cdot p_{n-2}$

2) The RV that describes the size of the arrival is a Bernoullian g, such that $g_1 = P\{g = 1\} = \pi$, $g_2 = P\{g = 2\} = 1 - \pi$, hence $E[g] = 1 \cdot \pi + 2 \cdot (1 - \pi) = 2 - \pi$, $\boldsymbol{G}(z) = \pi \cdot z + (1 - \pi) \cdot z^2$. The computations for a generic $\boldsymbol{G}(z)$ can be found on the QT notes, and read:

- $\rho = \frac{\lambda}{\mu} E[g] = \frac{\lambda}{\mu} \cdot (2 - \pi)$. Note that, if $\pi = 1$, then the system is an M/M/1, and the stability condition is the usual one. Instead, if $\pi = 0$, the system is one with constant-batch bulk arrivals.
- $\boldsymbol{P}(z) = \frac{\mu \cdot (1-\rho) \cdot (1-z)}{\mu \cdot (1-z) - \lambda \cdot z \cdot [1 - \boldsymbol{G}(z)]}$.

By substituting the above $\boldsymbol{G}(z)$ into the above expression, after a few straightforward computations, we get:

$$\mathbf{P}(z) = \frac{\mu \cdot (1-\rho) \cdot (1-z)}{\mu \cdot (1-z) - \lambda \cdot z \cdot \left[1 - \pi \cdot z - (1-\pi) \cdot z^2\right]}$$

$$= \frac{\mu \cdot (1-\rho) \cdot \cancel{(1-z)}}{\mu \cdot \cancel{(1-z)} - \lambda \cdot z \cdot \cancel{(1-z)} \left[1 + z - \pi \cdot z\right]}$$

$$= \frac{\mu - \lambda \cdot (2-\pi)}{\mu - \lambda \cdot z - \lambda \cdot z^2 \cdot (1-\pi)}$$

3) Both expressions require computing the first derivative of the above expression, which is:

$$\frac{\partial}{\partial z} \boldsymbol{P}(z) = \frac{\partial}{\partial z} \left(\frac{\mu - \lambda \cdot (2 - \pi)}{\mu - \lambda \cdot z - \lambda \cdot z^2 \cdot (1 - \pi)} \right) = \frac{\lambda \cdot [1 + 2z \cdot (1 - \pi)] \cdot [\mu - \lambda \cdot (2 - \pi)]}{[\mu - \lambda \cdot z - \lambda \cdot z^2 \cdot (1 - \pi)]^2}$$

From the above, we obtain:

$$E[N] = \frac{\partial}{\partial z} \boldsymbol{P}(z) \Big|_{z=1} = \frac{\lambda \cdot [3 - 2\pi] \cdot [\mu - (2 - \pi)\lambda]}{[\mu - (2 - \pi)\lambda]^2} = \frac{(3 - 2\pi)\lambda}{\mu - (2 - \pi)\lambda}$$

When $\pi = 1$ the system is an M/M/1, and the above expression reads $E[N] = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho}$.

When $\pi = 0$ the system is a constant-batch one, with $b = 2$, and the expression is $E[N] = \frac{3\lambda}{\mu - 2\lambda}$. The expression on the notes reads $E[N] = \frac{\rho \cdot (b+1)}{2 \cdot (1-\rho)}$, which is equal to the former after some straightforward substitutions.

4) It is $p_0 = \lim_{z \to 0} \boldsymbol{P}(z) = 1 - \frac{\lambda}{\mu} \cdot (2 - \pi) = 1 - \rho$. This was expected, since ρ is the system utilization. Moreover, it is $p_1 = \frac{\partial}{\partial z} \boldsymbol{P}(z) \Big|_{z=0} = \frac{\lambda \cdot [\mu - \lambda \cdot (2 - \pi)]}{\mu^2} = \frac{\lambda}{\mu} \cdot \left[1 - \left(\frac{\lambda}{\mu}\right) \cdot (2 - \pi)\right] = \frac{\rho \cdot (1 - \rho)}{2 - \pi}$. If $\pi = 1$, the expression is the one of an M/M/1 system.