PECSN, 05/02/2019

Exercise 1

Consider function $f(x) = C \cdot (1 + |x|)^{-2}$, with *C* being a real constant.

1) Draw the function
2) Compute the values of *C* for which $f(x)$ is a PDF
3) Let X_1 and X_2 be two IID RVs with the above density. Compute $E[{X_1}^2 + X_1\, X_2 - {X_2}^2]$.
4) Compute the range of values and the PDF for RV $Y = \sqrt{|X|}$

Exercise 2

Consider a system that receives network packets and stores them into an infinite FIFO queue. Each packet consists of two *fragments*, that arrive independently one after the other and are stored in the queue. The server transmits *packets*, but <u>a new service is *only* initiated when the queue contains *fully received* packets</u>. In other words, after finishing a packet transmission, the server will *not initiate a new transmission* (of the head-of-line queued packet), regardless of the number of packets in the queue, if the last packet in the queue is missing its second fragment. Rather, it will sit idle until that packet has been received in full. *Fragments* arrive with an exponential rate, equal to λ. *Packet* transmission takes an exponential time, with a mean $1/\mu$.

1) Model the system as a queueing system and draw the transition-rate diagram.

2) Find the stability condition and compute the steady-state probabilities.

3) Compute the mean number of packets in the system. Do not count in incomplete packets.

4) Compute the system throughput (in packets per second). Justify your answer.

Exercise 1 – solution

1) The relevant thing about the function is that it is symmetric around 0, hence $E[X] = 0$

2) $\int_{-\infty}^{+\infty} f(x)\,dx = 2\int_0^{+\infty} f(x)\,dx = 2C\int_0^{+\infty}(1+x)^{-2}\,dx = 2C\int_1^{+\infty} t^{-2}\,dt = 2C\left[\frac{-1}{t}\right]_1^{+\infty} = 2C.$

Therefore, $C = \frac{1}{2}$

3) $E\left[{X_1}^2 + X_1 X_2 - {X_2}^2\right] = E[{X_1}^2] + E[X_1]\mathrm{E}[X_2] - E[{X_2}^2] = E[X_1]^2 = 0$, since the PDF is symmetric around 0.

4) the range of values for Y is $t \geq 0$. Moreover, we have:

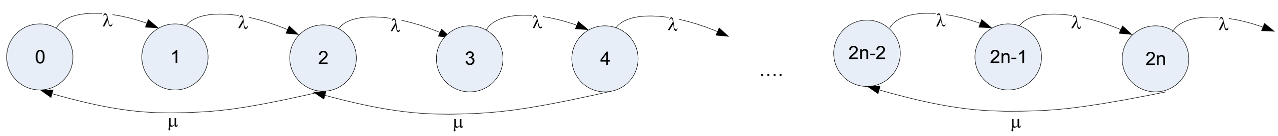
$$F_Y(t) = P\{Y \leq t\} = P\{|X| \leq t^2\} = P\{-t^2 \leq X \leq t^2\} = F_X(t^2) - F_X(-t^2) = 2F_X(t^2) - 1$$

(the last passage is due to symmetry of the PDF). Therefore we get:

$$f_Y(t) = \frac{d}{dt}(2F_X(t^2) - 1) = 2 \cdot 2t \cdot f_X(t^2) = \frac{2t}{(1+t^2)^2}$$

Exercise 2 – solution

We model the state of the system using the number of *fragments* as a state marker. The TR diagram is then as follows:



Using global equilibrium equations, one straightforwardly gets the following expressions:

$p_{2k} = p_0\left(\frac{\lambda}{\mu}\right)^k$, $k \geq 0$, $p_{2k+1} = p_{2k}$, $k \geq 0$

Normalization then reads:

$$\sum_{k=0}^{+\infty}(p_{2k} + p_{2k+1}) = 2p_0\sum_{k=0}^{+\infty}\left(\frac{\lambda}{\mu}\right)^k = 1$$

From which, defining $u = \frac{\lambda}{\mu}$ and assuming $u < 1$ as a stability condition:

$p_0 = \frac{1}{2}\cdot(1-u)$, $p_{2k} = p_{2k+1} = \frac{1}{2}\cdot(1-u)\cdot u^k$, $k \geq 0$

The mean number of packets in the system is:

$$E[Np] = \sum_{k=0}^{+\infty} k\cdot(p_{2k} + p_{2k+1}) = \sum_{k=0}^{+\infty} k\cdot(1-u)\cdot u^k = \frac{u}{1-u}$$

The throughput of the system is:

$$\gamma = \sum_{k=1}^{+\infty} \mu \cdot p_{2k} = \frac{\mu}{2} \cdot (1-u) \cdot \sum_{k=1}^{+\infty} u^k = \frac{\mu}{2} \cdot \frac{u \cdot (1-u)}{(1-u)} = \frac{\lambda}{2}$$

The result is expectable, since two fragments must arrive for a packet to be transmitted, hence the packet arrival rate is half the fragment arrival rate, and it is also the throughput.