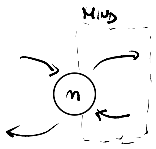
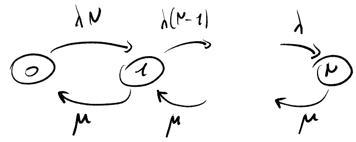
EXERCISE 2

TYPE 1 [CTMC]:

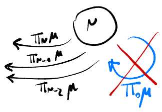
- NO FINITE # OF STATES:



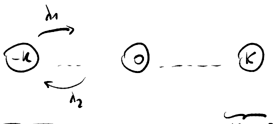
- N JOBS IN THE SYSTEM => FINITE POPULATION => CTMC:



- RANDOM SIMULTANEOUS SERVICE OF n JOBS DOESN'T COUNT 0 AS # OF JOBS SERVED:



- IF THERE ARE TWO KIND OF JOBS, WITH TWO DIFFERENT ARRIVAL RATE AND THEY LEAVE WHEN THERE ARE TWO OF THEM IN THE SYSTEM:



$\underbrace{\quad}_{\text{k JOBS OF TYPE 1}}$ $\underbrace{\quad}_{\text{k JOBS OF TYPE 2}}$

$2k+1$ STATES => FINITE NUMBER => $\boxed{\text{ALWAYS STABLE SYSTEM}}$

$$\lambda p_i = \mu p_{i+1} \quad -k \le i \le k-1 \quad => \quad p_j = p_{-k}\,\rho^{j+k} \quad -k \le j \le k$$

$$\rho = \frac{\lambda}{\mu}$$

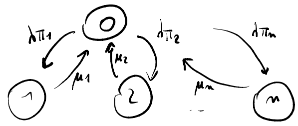
$$\sum_{-k}^{k} p_{-k}\,\rho^{j+k} = p_{-k}\sum_{0}^{2k}\rho^{j} = p_{-k}\begin{cases}\dfrac{1-\rho^{2k+1}}{1-\rho} & \rho \ne 1\\ (2k+1) & \rho = 1\end{cases}$$

$$\sum_{j=-k}^{k}\rho^{j+k}\,p_{-k} = 1 \;=>\; p_{-k} = \frac{1}{\sum_{j=-k}^{k}\rho^{j+k}} = \begin{cases}\dfrac{1-\rho}{1-\rho^{2k+1}} & \rho\ne 1\\ \dfrac{1}{2k+1} & \rho =\end{cases}$$

$$p_j = \begin{cases}\dfrac{1-\rho}{1-\rho^{2k+1}}\,\rho^{j+k} & \rho \ne 1\\ \dfrac{1}{2k+1} & \rho = 1\end{cases} \qquad -k \le j \le k$$

- IF WHEN A JOB IS SERVED: THE QUEUE IS FLUSHED

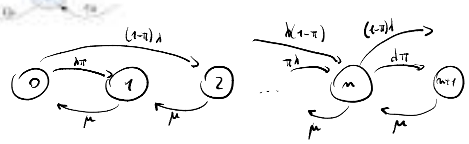
- ONE JOB IS SENT TO ONE OF n DIFFERENT SERVERS with π_i PROB.:



- M/M/C:

 - Arrival rates are constant;
 - The service rates will be $\mu_n = \begin{cases} n\cdot\mu & n \le C\\ C\cdot\mu & n \ge C\end{cases} = \min(C,n)\cdot\mu$

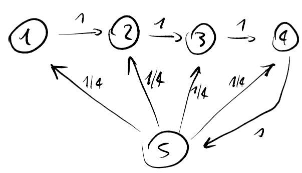
- M/M/1 WITH SINGLE (π PROB.) OR COUPLE ($1-\pi$ PROB.) ARRIVAL:



BULK ARRIVAL

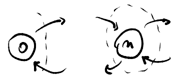
- 4 STAGES POSSIBLE, SAME PROBABILITY TO PROCESS ONE JOB AT STAGE J, K FIXED JOBS, WHEN ARRIVE AT STAGE 5 IT IS SENT BACK:

$\boxed{\text{CJN}}$



TYPE 2 [EQUILIBRIUM EQUATIONS]:

- GLOBAL EQUILIBRIUM EQUATIONS:



STATE 0 : OUTGOING = INCOMING
STATE J : " = " $J \geq 1$

- IF SERVICE TIME HAS PROBABILITY :

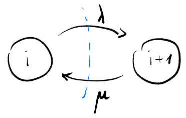
EQUATION :

$\mu \cdot \pi_J \cdot \underline{p_N} =$

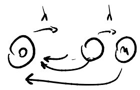
MIND

- IF $p_0 \cdot \ldots = p_N \cdot \ldots$ (NO OTHER p_J APPEARS) $\Rightarrow$ $\boxed{\text{PAY ATTENTION TO THE RELATIONSHIP BETWEEN } p_0 \text{ AND } p_N}$

- LOCAL EQUILIBRIUM EQUATIONS:



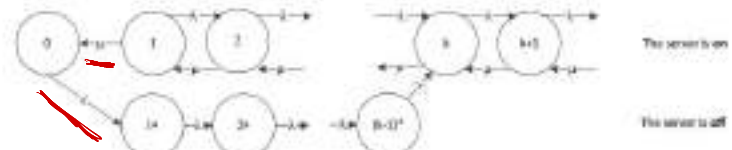
- IF WE HAVE ∞ STATES AND WHEN A JOB LEAVES THE QUEUE IS FLUSHED:



$\lambda p_0 = \sum_{i=1}^{\boxed{+\infty}} \mu_i p_i$ MIND

- IF IN SOME STATES $\lambda = \lambda_1$ AND IN OTHERS $\lambda = \lambda_2$ $\Rightarrow$ LOCAL EQUILIBRIUM EQUATIONS CAN BE WRITTEN FOR THESE TWO TYPE OF STATES

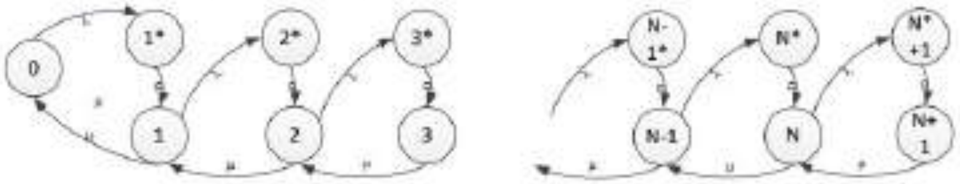
- 1) When j jobs are in the system, $0 < j < k$, the server may be either *on* or *off*. In either case, the system behaves differently. Thus, we need *two states* for each number of jobs in the system $0 < j < k$. The CTMC is therefore the following:



Starred states are traversed when the server is switched off. When $k = 1$, the interval $0 < j < k$ is empty, hence the system behaves like an M/M/1 and there are no starred states.

SERVER OFF UNTIL THE SYSTEM REACHES K JOBS

- 1) The CTMC is the following. Note that the state of the system cannot be described using only the number of jobs, since the fact that the system is *decrypting* or *serving* jobs is also relevant. Starred states are those when the system is decrypting. Furthermore, you can observe that when $\alpha \to \infty$ (i.e., decryption is much faster than the rate of arrival or service) the two states in the same column collapse to one, and the system becomes an M/M/1.



JOBS DURING DECRYPTION ARE LOST

ONGOING SERVICE ARE STOPPED IN ONE JOB HAS TO BE DECRYPTED

TYPE 3 [STEADY STATE PROBABILITIES & STABILITY CONDITION]

] FROM G.E.E. : $P_J = \dots \cdot P_{J-1} = \left(\boxed{\frac{\dots}{\dots}}\right)^J \cdot P_0 = \theta^J \cdot P_0$ CALL IT θ

PAY ATTENTION TO REDUNDANT EQUATIONS

BIRTH-DEATH SYSTEM GENERAL FORMULA: $\boxed{\prod_{i=0}^{J-1} \frac{\lambda_i}{\mu_{i+1}} \cdot P_0}$ VARIABLE

HOW TO FIND $P_0 \Rightarrow$ N.C. : $P_0 \left[1 + \sum_{J=1}^{+\infty} \theta^J\right] = 1 \Rightarrow P_0 = \frac{1}{[\dots]}$

SUBSTITUTING P_0 : $P_J = \dots \quad J \geq 0$

] CONDITIONS TO HAVE FINITE # OF JOBS : GIVEN θ

if $\theta < 1$: $\sum_{J=1}^{+\infty} \theta^J$ CONVERGES TO $\frac{\theta}{1-\theta} \Rightarrow P_0 = \frac{1}{1+\frac{\theta}{1-\theta}} = 1-\theta$

$\Downarrow$

STABILITY CONDITION IS WHEN $\theta < 1$

[FOR EXAMPLE: $\frac{\lambda}{\lambda+\mu} = \theta$, $\theta < 1$ WHEN $\mu \neq 0$]

] PMF OF NUMBER OF JOBS IN THE SYSTEM: $P_J = \dots \quad J \geq 0$

] IF # OF JOBS IS FINITE $\Rightarrow$ SYSTEM IS ALWAYS STABLE

] PMF OF SS-PROB. IS MONOTONIC SEQUENCE : $P_J < P_{J+1} \Rightarrow \frac{P_{J+1}}{P_J} > 1$ MOREOVER: $P_{N-1} < P_N$ IF $\dots < \dots$

] SS-PROB. SEEN BY A JOB : $r_J = \frac{\lambda_J P_J}{\sum_{i=0}^{N} \lambda_i P_i}$ $\boxed{\dots \leq J \leq \dots}$ MIND

IF $\boxed{r_J = P_J}$ IS PASTA

] IF FROM $J \geq k$ $\mu_J = 0$ WHAT DOES IT HAPPEN TO STABILITY?

N.C : $P_0 + \sum_{J=1}^{k-1} P_J + \sum_{J=k}^{+\infty} P_k = 1$ DIVERGES UNLESS $P_k = 0$

$\Downarrow$

UNSTABLE

$P_J = P_k \quad \forall J \geq k$

LOCALLY THE SYSTEM LOAD IS GROWING WITHOUT JOBS LEAVING

] IF APPLYING N.C. WE HAVE A SITUATION SIMILAR TO THE FOLLOWING: $P_0 \sum_{k=0}^{+\infty} \left[\left(\frac{\sqrt{2} \cdot \lambda}{\mu}\right)^2\right]^k \Rightarrow$ IT IS STABLE IF: $\sqrt{2} \cdot \lambda < \mu$

] IF A JOB IS SENT PROBABILISTICALLY TO 1 OF n SERVERS WITH PROBABILITY π_i, THE PROBABILITY TO OBSERVE A JOB RUNNED IN A SERVER IS EQUAL TO BE RUNNED IN ANOTHER ONE IF:

$\underset{\text{SS PROB.}}{P_i} = k \quad \forall i = 1..n$

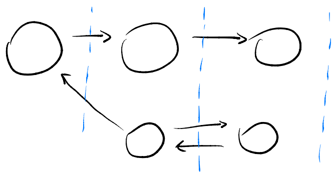
IF $P_i = \dots \pi_i \cdot \frac{1}{\mu_i}$ (MEDIUM SPEED OF SERVER i) THE ABOVE CONDITION IS VERIFIED IF $\pi_i \propto \mu_i$ (DEPENDS OF HOW FAST SERVERS ARE)

] 06/07/2022 FOR BULK ARRIVALS

] MOST LIKELY # OF JOBS AT STEADY STATE $\rightarrow P_J > P_i \quad \forall i \neq J$

CHECK ALSO HOW IT CHANGES DUE TO λ/μ

] IF WE HAVE A BIG QUEUE, BUT THE SERVER IS FASTER THAN THE ARRIVALS $\Rightarrow$ NO STABILITY PROBLEMS REGARDING THE # OF JOBS IN THE QUEUE

] 

LOCAL EQUILIBRIUM EQUATIONS HAVE TO BE PREFERRED IN THE CASE OF TWO PARALLEL CHAINS $\Rightarrow$ I HAVE TO DO "VERTICAL CUTS" $\Rightarrow$ IN THE N.C. : $P_0 + \sum P_J^* + \sum P_J = 1$

TYPE 4 [USE OF PGF]

] MEAN # OF JOBS IN THE SYSTEM OF A JOB $E[N]$: $P(z) = \sum_{J=0}^{+\infty} P_J z^J \longrightarrow P'(z) = \frac{\partial}{\partial z} P(z) \longrightarrow E[N] = P'(1)$

] CALCULATE $E[N]$, $Var(N)$: USE PGF: $P(z) = \sum_{J=0}^{+\infty} P_J z^J \Rightarrow P'(z) = \frac{\partial}{\partial z} P(z)$, $P''(z) = \frac{\partial}{\partial z} P'(z) \Rightarrow E[N] = P'(1)$

$Var(N) = P''(1) + P'(1) - P'(1)^2$

TYPE 5 [QUEUEING NETWORK]

POISSON PROCESSES IN GENERAL

POISSON PROCESS = EXPONENTIAL INTERARRIVAL TIME (MEMORY LESS) => We need just $N(t)$

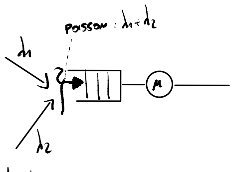
$\Downarrow$ NEXT ARRIVAL NOT INFLUENCED BY THE PAST

BURKE:
- ∞ QUEUE
- $\bar{\lambda}$ (AVERAGE ARRIVAL RATE)
- $M/M/c$ in STEADY STATE

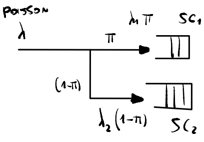
=> 1) DEPARTURE PROCESS IS POISSON, WITH RATE $\bar{\lambda}$
2) $\forall t$, $m(t)$ IS INDEPENDENT OF INTER-DEPARTURE TIMES IN $[0,t]$

BURKE'S THEOREM VALID => SS-PROB = PRODUCT OF SC_i SS-PROB.

[DEPARTURE POISSON <=> PRODUCT FORM]

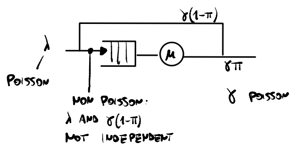


λ_1, λ_2 POISSON PROCESS



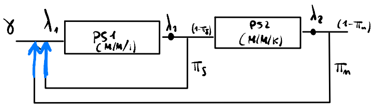
$\lambda = \lambda_1\pi + \lambda_2(1-\pi)$ STILL POISSON

IF I HAVE FEEDBACK LOOP:



IF λ IS POISSON, ALSO γ IS POISSON (EVEN WITH FEEDBACK LOOP)

QUEUEING NETWORK SCHEME



IF γ EXP. DISTRIBUTED => POISSON PROCESS => λ_1 AS DEPARTURE RATE (STILL POISSON)

$\lambda_2 = \lambda_1(1-\pi_1)$
$\lambda_1 = \sum \gamma + \uparrow\uparrow$

{ WHAT GOES IN MUST GO OUT (LITTLE LAW)

ROUTING MATRIX : $\Pi = \begin{bmatrix} \cancel{\pi_{11}} & \cdots \\ \cancel{\pi_{21}} & \cdots \end{bmatrix}$ NO NEED

$\lambda_1 = \dots$
$\lambda_2 = \dots$

{ ARE CORRECT IF JOBS COME OUT OF THE SYSTEM [IF QN IS <u>OPEN</u>], OTHERWISE THE LATTER DOESN'T REACH A STEADY STATE

IF NOT SPECIFIED: π_i ARE INDEPENDENT OF HOW JOBS ARE DISTRIBUTED IN THE SYSTEM

EXTERNAL ARRIVAL : $\gamma = [\;\;]$ } # OF SCs

BOTTLE NECK: the SC with <u>HIGHEST UTILIZATION</u> :

$\rho_i = \frac{\lambda_i}{m\mu_i}$ => $\rho_i > \rho_s \quad \forall s \neq i$ [SC_i BOTTLENECK]

m: NUMBER OF CPU IN A SC

MEAN NUMBER OF VISITS :

$E[V_i] = \frac{\lambda_i}{\gamma_{TOT}}$ $\quad \gamma_{TOT} = \sum \gamma_i$ TOTAL RATE OF EXTERNAL ARRIVALS

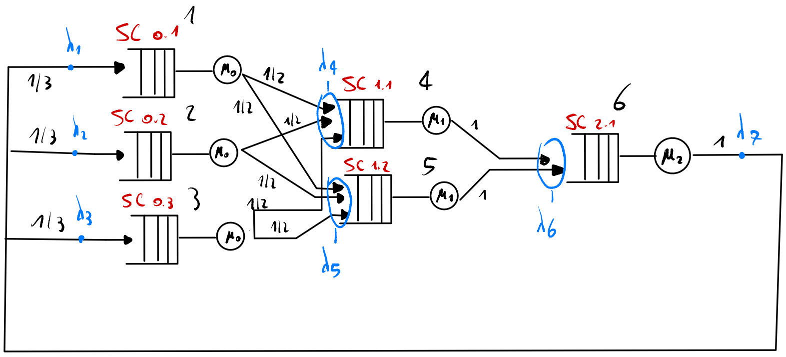
$E[V_i] < 1$, IF THE PATH THAT LEAVES THE SYSTEM IS UNIQUE, IS IMPOSSIBLE

STABILITY CONDITIONS: ALL SCs MUST BE STABLE : $\rho_i < 1 \quad \forall i$

$P\{N_2 = n \mid N_1 = k\}$ => WE ARE IN OJN: ADMITS PRODUCT FORM => N_1 INDEPENDENT OF N_2 => $P_2(n) = (1-\rho_2)\rho_2^n$ [M/M/1]

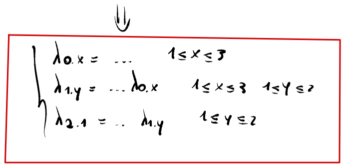
$[P_0 = P_1(n_1) \cdot P_2(n_2) \dots]$

QUEUEING NETWORK SCHEME WITH MULTIPLE STAGES:



$\lambda_7 = \frac{1}{3}\lambda_1 + \frac{1}{3}\lambda_2 + \frac{1}{3}\lambda_3$
$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}\lambda_7$
$\lambda_4 = \frac{3}{2}\lambda_1 = \frac{3}{2}\cdot\frac{1}{3}\lambda_7 = \frac{1}{2}\lambda_7 = \lambda_5$
$\lambda_6 = \lambda_4 + \lambda_5 = \frac{1}{2}\lambda_7 + \frac{1}{2}\lambda_7 = \lambda_7$

$\Downarrow$



IT IS A CLOSED JACKSON NETWORK:
- NO EXTERNAL ARRIVAL
- CONSTANT NUMBER OF JOBS
- NO DELAY
- MARKOVIAN ROUTING

-] SOLUTION FOR CJN: $\lambda = 0 + \Pi^T \lambda \Rightarrow e = [e_1 \dots e_M]^T$

HOW DO I CHOOSE?:
- WRITE Π^T
- FIND e THAT: $\Pi^T e = e$

$\rho_i = \frac{e_i}{\lambda_i}$ TO SIMPLIFY WE CAN CHOOSE: $e = \mu_0$

$\rho^T = [\dots]$

-] TO FIND SS PROB. OF CJN: $p(n_1 \dots n_M) = \frac{1}{G(M,K)} \cdot \prod_{i=1}^{M} f_i(n_i)$

$$f_i(n_i) = \begin{cases} \frac{(C_i \cdot \rho_i)^{n_i}}{n_i!} & n_i \le C_i \\ \frac{C_i^{C_i} \rho_i^{n_i}}{C_i!} & n_i \ge C_i \end{cases}$$

$M/M/1 \Rightarrow C_i = 1 \Rightarrow f_i(n_i) = \rho_i^{n_i}$

$[n_i < 1 \Rightarrow 0! = 1]$

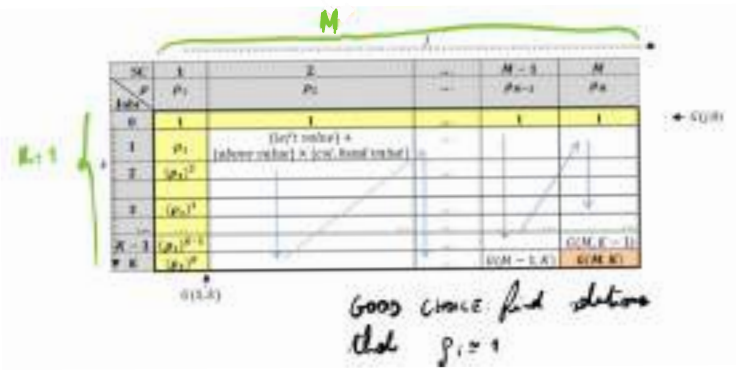
IF ONLY THE # OF JOBS AT EACH STAGE MATTERS:
(WITH 3 STAGES)

$$p(\underbrace{K-(N_1+N_2)}_{\text{JOBS IN THE CJN}}, N_1, N_2) = \underbrace{\frac{1}{G(M,K)} \rho^{N_1} \rho^{N_2}}_{\text{probabilities are equal for every distributions of jobs within a stage}}$$

-] IF $\rho = 1$ AT EACH STAGE $G(M,K) = |E| = \binom{K+M-1}{M-1}$

-] PROBABILITY OF EACH STATE: $p = \frac{1}{G(M,K)} \Rightarrow$ UPM, ALL JOBS AT STAGE j: $P_j = \frac{|E| \rightarrow \text{number of all possible combinations of } K \text{ jobs at } 3-j}{|E|} \Rightarrow |E| = \binom{K+(3-j)-1}{(3-j)-1}$

-] BUZEN ALGORITHM:



UTILIZATION OF SCs: $U_j = \rho_j \frac{G(M,K-1)}{G(M,K)}$

-] PROBABILITY THAT ALL JOBS ARE IN SC_j IN QUEUEING NETWORK:

$j = 1$ $\quad P_n = p(K, 0, \dots, 0) = \frac{1}{G(K,M)} \cdot \dots$

$\vdots$

$j = n$ $\quad P_n = p(0, 0, \dots, K) = \dots$

TYPE 6 [SIMULTANEOUS SERVICE]

-] $E[R] = \frac{1}{\mu}$ IF ALL REQUESTS ARE SERVED SIMULTANEOUSLY [TIME SPENT WAITING ISN'T REGARDED BECAUSE NO JOBS ARE BEING SERVED DURING THE WAIT]

-] THROUGHPUT: $X = \mu \cdot \underbrace{E[N]}_{\text{CALCULATED WITH PGF}}$ IF ALL JOBS ARE SERVED SIMULTANEOUSLY (# OF JOBS SERVED PER UNIT OF TIME)

TYPE 7 [SYSTEMS WITH FINITE MEMORY]:

] LOSS PROBABILITY: WHEN THE SYSTEM IS FULL

$$P_L = \underset{\text{TYPE 1}}{P_{-K}} + \underset{\text{TYPE 2}}{P_K} = \begin{cases} \dots & \rho = 1 \\ \dots & \rho \neq 1 \end{cases}$$

FULL OF : TYPE 1 TYPE 2

] E[R] to the PoV of TYPE 2: NON PASTA SYSTEM $(\lambda_n \neq \lambda)$

$\bar{\lambda}_2 = \lambda_2 (1 - P_K)$

$\bar{\lambda}_1 = \lambda_1 (1 - P_{-K})$

E[R] = 0 if all TYPE 1 with $p = \sum_{-K}^{-1} r_j$ FROM PoV of TYPE 2

$$r_j = \frac{\lambda_2}{\bar{\lambda}_2} \cdot P_j = \frac{P_j}{1 - P_K}$$

BECAUSE JOB 2 IS SERVED FIFO, JOB 1 and JOB 2 LEAVE IN COUPLE $\Leftarrow$ SUM OF A.T. OF $j+1$ JOBS (TYPE 1) IF SHE ARRIVES WHEN THE SYSTEM IS IN STATE j

$$E[R] = \sum_{j=0}^{K-1} \underbrace{\frac{j+1}{\lambda_1}}_{\text{SUM OF A.T. OF } j+1 \text{ JOBS}} \cdot \underbrace{\frac{1}{2K}}_{r_j} + \sum_{j=-K}^{-1} 0 \cdot \frac{1}{2K}$$

$\boxed{0 \leq j \leq K-1}$

TYPE 8 [INDEXES]

] $E[N] = \sum n P_n = 0 \cdot P_0 + 1 \cdot (1 - P_0) = 1 - P_0$

] $\bar{\lambda} = \lambda \cdot P_0$ IF THE SYSTEM DOESN'T ACCEPT JOBS WHEN A JOB IS RUNNING

] PROBABILITY THAT A JOB HAS TO WAIT IN a M/M/c:

IF IT'S PASTA => $j \geq c$ A CUSTOMER HAS TO WAIT:

$$P_{WAIT} = \sum_{j=c}^{+\infty} P_j = \frac{c^c}{c!} P_0 \sum_{j=c}^{\infty} \rho^j$$

] $E[N] = \sum_{n=1}^{3} n \cdot P_n = P_0 [1 \cdot \dots + 2 \dots + 3 \dots]$ WITH 4 STATES

] $E[N_q] = \underbrace{1 \cdot P_2 + 2 \cdot P_3}$ WITH 4 STATES

P_1 NO BECAUSE IS IN PROCESS WITH NO QUEUE

] UTILIZATION: $U = 1 - P_0 = \sum_{j=1}^{+\infty} P_j$

] THROUGHPUT: $\gamma = U \cdot \mu$

] E[R] CAN BE FOUND THROUGH LITTLE'S LAW IF HYPOTHESIS ARE MET

APPROXIMATION

- $\sum_{n=1}^{+\infty} n \cdot p^n = \frac{p}{(1-p)^2}$
- $\sum_{n=0}^{K} u^n = \begin{cases} \frac{1-u^{K+1}}{1-u} & u \neq 1 \\ K+1 & u = 1 \end{cases}$
- $e^{\log(b)} = b$
- $\sum_{K=0}^{N-1} \frac{K+1}{N-K} = \sum_{m=1}^{N} \frac{N-m+1}{m} = \sum_{m=1}^{N} \left(\frac{N+1}{m} - 1\right) = (N+1) H_N - N$

  H_N : N^{th} HARMONIC NUMBER
- $\sum j$ EXPONENTIALS = ERLANG DISTRIBUTION WITH MEAN $= \frac{j}{\lambda_{EXP}}$
- $P_0 \cdot N \cdot \lambda = P_N \cdot \pi_N \cdot \mu$

  $P_j \cdot (N-j) \cdot \lambda = P_{j-1} \cdot (N-(j-1)) \cdot \lambda + P_N \cdot \pi_{N-j} \cdot \mu, \ 1 \leq j \leq N-1$

  $P_N \cdot \mu = P_{N-1} \cdot \lambda$

  After a few algebraic manipulations, the following recurrence can be easily obtained:

  $$P_j = P_0 \cdot \frac{N}{N-j} \cdot \sum_{i=0}^{j} \frac{\pi_{N-i}}{\pi_N}$$

  With $0 \leq j \leq N-1$, and

  $$P_N = P_0 \cdot \frac{N \cdot \lambda}{\mu \cdot \pi_N}$$
- It may be useful to observe that $\lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{j!} = \lim_{n\to\infty} \left[\sum_{j=0}^{n} \frac{x^j}{j!}\right]_{x=1} = [e^x]_{x=1} = e$, and that $\sum_{j=0}^{n} \frac{1}{j!} \approx e$ when $n \geq 5$.