Extension Principle

The extension principle is a basic concept of fuzzy set theory that provides a general procedure for transforming a fuzzy set from one universe of discourse to another universe of discourse provided we have point-to-point mapping of a function $f(.)$ known.

This procedure generalizes a common point-to-point mapping of a function $f(.)$ to a mapping between fuzzy sets. More specifically, suppose that f is a function from X to Y and A is a fuzzy set with the universe of discourse X defined as,

$$A(x) = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$$

Then, the extension principle states that the image of fuzzy set A under the mapping $f(.)$ can be expressed as a fuzzy set B as,

$$B(y) = \mu_A(x_1)/y_1 + \mu_A(x_2)/y_2 + \cdots + \mu_A(x_n)/y_n$$

where $y_i = f(x_i)$ or $x_i = f^{-1}(y_i)$; $\forall\, i = 1, \ldots, n$.

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Extension Principle

If $f(.)$ is a many-to-one mapping then there exist $x_1, x_2 \in X$, $x_1 \neq x_2$, such that

$$f(x_1) = f(x_2) = y^*, y^* \in Y$$

In this case, the membership value of fuzzy set B at $y = y^*$ will be:

$$\mu_B(y^*) = \max(\mu_A(x_1), \mu_A(x_2))$$

More generally, we have

$$\boxed{\mu_B(y) = \max_{x \in f^{-1}(y)} \mu_A(x)}$$

where $f^{-1}(y)$ denotes the set of all points in the universe of discourse $x \in X$ such that $f(x) = y$.

This is called the **extension principle**.

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Extension Principle

Example:

Let us consider a fuzzy set A with the universe of discourse $X = [-10,10]$ given as below.

$$A(x) = \sum_{x \in X} \mu_A(x)/x = 0.1/(-2) + 0.4/(-1) + 0.8/0 + 0.9/1 + 0.3/2$$

Find a fuzzy set B with the universe of discourse $Y = [-10,10]$ using the "Extension Principle" for mapping function defined as below?

$$y = f(x) = x^2 + x - 3$$

$$B(y) = \sum_{y \in Y} \mu_B(y)/y = ?$$

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Extension Principle

$$A(x) = \sum_{x \in X} \mu_A(x)/x = 0.1/(-2) + 0.4/(-1) + 0.8/0 + 0.9/1 + 0.3/2$$

$$y = f(x) = x^2 + x - 3$$

$$x = \{-2, -1, 0, 1, 2\}$$

$x = (-2)$	$x = (-1)$	$x = 0$	$x = 1$	$x = 2$
$y = (-2)^2 - 2 - 3 = (-1)$	$y = (-1)^2 - 1 - 3 = (-3)$	$y = (0)^2 + 0 - 3 = (-3)$	$y = (1)^2 + 1 - 3 = (-1)$	$y = (2)^2 + 2 - 3 = 3$
$\mu_B(-1) = 0.1$	$\mu_B(-3) = 0.4$	$\mu_B(-3) = 0.8$	$\mu_B(-1) = 0.9$	$\mu_B(3) = 0.3$

$$B(y) = 0.1/(-1) + 0.4/(-3) + 0.8/(-3) + 0.9/(-1) + 0.3/3$$

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Extension Principle

$$A(x) = \sum_{x \in X} \mu_A(x)/x = 0.1/(-2) + 0.4/(-1) + 0.8/0 + 0.9/1 + 0.3/2$$

$$y = f(x) = x^2 + x - 3$$

$$x = \{-2, -1, 0, 1, 2\}$$

$$B(y) = 0.1/(-1) + 0.4/(-3) + 0.8/(-3) + 0.9/(-1) + 0.3/3$$

After rearranging, we will have:

$$B(y) = (0.1 \vee 0.9)/(-1) + (0.4 \vee 0.8)/(-3) + 0.3/3$$

$$B(y) = 0.9/(-1) + 0.8/(-3) + 0.3/3$$

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$$P : \boldsymbol{If}\ x\ \boldsymbol{is}\ A\ \boldsymbol{then}\ y\ \boldsymbol{is}\ B$$

Let us consider two sets of variables x and y be

$X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$

Also, let us consider the following.

$$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$$
$$B = \{(y_1, 1), (y_2, 0.4)\}$$

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$$P: \textbf{\textit{If }} x \textbf{\textit{ is }} A \textbf{\textit{ then }} y \textbf{\textit{ is }} B$$

Suppose, given a fact expressed by the proposition x is A',

where $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

We are to derive a conclusion in the form y is B'

Here, we should use generalized modus ponens (GMP).

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If x is A Then y is B
x is A'

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y is B'

We are to find $B' = A' \circ R(x,y)$, where $R(x,y) = \max\{\mathrm{A} \times B, \bar{A} \times Y\}$

$$\mathrm{A} \times B = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \bar{A} \times Y = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Note. For $\mathrm{A} \times B$, $\mu_{A \times B}(x,y) = \min(\mu_A(x), \mu_B(y))$

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$$R(x,y) = (A \times B) \cup (\bar{A} \times Y) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

Now $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

Therefore $B' = A' \circ R(x,y) = [0.6 \quad 0.9 \quad 0.7] \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = [0.9 \quad 0.5]$

Thus we derive that y is B' where $B' = \{(y_1, 0.9), (y_2, 0.5)\}$

Center of area (COA)

$$x^* = \frac{\sum_{i=1}^{n} x_i \, .\mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)}$$

$$x^* = \frac{\int x \, \mu_A(\mathrm{x}) \, \mathrm{dx}}{\int \mu_A(\mathrm{x}) \, \mathrm{dx}}$$

Mean of Maxima

$$x^* = \frac{\sum_{x_i \in M} x_i}{|M|}$$