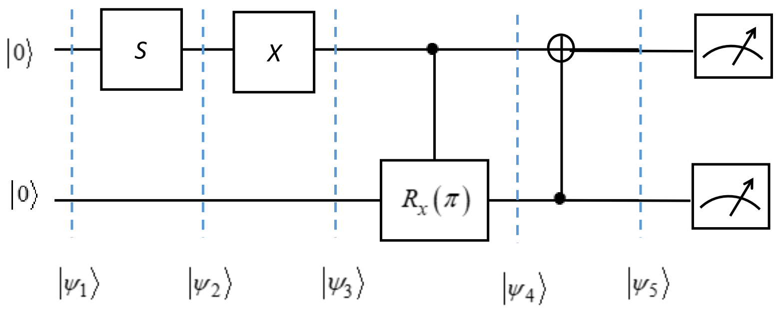
Computer Architecture course,
Quantum Computing part,
a.y. 2021/2022
18/07/2022

Exercise 1
Given the following quantum circuit,



the student:
1. calculate $R_x(\pi)|0\rangle$ and $S|0\rangle$;
2. compute all the intermediate 2-qubit states $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle$, and $|\psi_5\rangle$;
3. compute the final measurement probabilities (the measurement is along the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$);
4. write the $|\psi_5\rangle$ components along the standard basis.

Now suppose that the *S* and *X* gates are aggregated into a single gate that we call *U*. The student:

5. write the matrix for the *U*-gate.

The student can **optionally** implement the above circuit using the Qiskit platform and compare the results obtained theoretically with those estimated by the *qasm_simulator*.

Recall that:

$$R_x(\theta) = \begin{bmatrix} cos\frac{\vartheta}{2} & -isin\frac{\vartheta}{2} \\ -isin\frac{\vartheta}{2} & cos\frac{\vartheta}{2} \end{bmatrix}$$

Solution

1. $R_x(\pi)|0\rangle = \begin{bmatrix} cos\left(\frac{\pi}{2}\right) & -i\,sin\left(\frac{\pi}{2}\right) \\ -i\,sin\left(\frac{\pi}{2}\right) & cos\left(\frac{\pi}{2}\right) \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\-i\end{bmatrix} =$
   $-i\begin{bmatrix}0\\1\end{bmatrix} = -i|1\rangle$

   $S|0\rangle = \begin{bmatrix}1 & 0\\0 & i\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$

2. $|\psi_1\rangle = |0\rangle|0\rangle$
   $|\psi_2\rangle = (S \otimes I)|\psi_1\rangle = (S \otimes I)(|0\rangle \otimes |0\rangle) = (S|0\rangle) \otimes (I|0\rangle) = |0\rangle|0\rangle$
   $|\psi_3\rangle = (X \otimes I)|\psi_2\rangle = (X \otimes I)(|0\rangle \otimes |0\rangle) = (X|0\rangle) \otimes (I|0\rangle) = |1\rangle|0\rangle$
   $|\psi_4\rangle = C\big(R_x(\pi)\big)|\psi_3\rangle = C\big(R_x(\pi)\big)(|1\rangle \otimes |0\rangle) = |1\rangle \otimes R_x(\pi)|0\rangle$
   $= -i|1\rangle \otimes |1\rangle$
   $|\psi_5\rangle = (C \uparrow)CNOT|\psi_4\rangle = (C \uparrow)CNOT(-i|1\rangle \otimes |1\rangle) == -i|0\rangle \otimes |1\rangle$

3. $-i|0\rangle \otimes |1\rangle = -i\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = -i\begin{bmatrix}1\begin{bmatrix}0\\1\end{bmatrix}\\0\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = -i\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

4. Outcome probabilities
   $Pr(|00\rangle) = 0$
   $Pr(|01\rangle) = 1$
   $Pr(|10\rangle) = 0$
   $Pr(|11\rangle) = 0$

5. $U = XS = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & i\end{bmatrix} = \begin{bmatrix}0 & i\\1 & 0\end{bmatrix}$