PECSN 20/09/2019

Name and surname: ____ *Matricola:* __________________

I will sit for the oral examination in: ☐ *September* ☐ *November (*)*
This decision *cannot* *be changed later.*

(*) If you are ineligible, your written test will be discarded.

Exercise 1

A machine reads a string of random binary digits, which appear with probability p_0, p_1 respectively. The machine stops reading after it has read *at least one 0 and at least one 1*.

Assume that each binary digit is independent of the others, and call X_0, X_1 the RVs representing the number of 0s and 1s read *at the end* of an experiment.

1) Compute the PMFs of X_0, X_1. Check the normalization condition.
2) Compute $P\{X_0 > X_1\}$, and compute p_0 such that $P\{X_0 > X_1\} = 0.25$. Find a general relationship between $P\{X_0 > X_1\}$ and p_0.
3) Compute $E[X_0]$. Justify explicitly what happens in the limit cases $p_0 \to 0, p_0 \to 1$.
4) Are X_0, X_1 independent? Justify your answer

Exercise 2

An application server serves transactions with an exponential service time, at a rate μ. Its clients are aware of the number of transactions in the queue, hence tend to join with a smaller probability when the queueing is high. More specifically, the arrival rate for transactions when there are n jobs in the system is:

$$\frac{N-n}{N\cdot(n+1)}\cdot\lambda$$

For $0 \le n < N$, and 0 otherwise. N is a system constant.

1) Model the system as a queueing system and draw the transition rate diagram

2) Compute the stability condition and the steady-state probabilities

3) Compute the probabilities seen by an arriving transaction

4) Compute the mean number of transactions in the system and the mean response time

Note: It may be useful to observe that $k\binom{N}{k} = N\binom{N-1}{k-1}$

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Exercise 1 - Solution

1) This is a case of repeated trials, with $p_0 = 1 - p_1$. Let us start from X_0:

$$\begin{aligned}
P\{X_0 = 1\} &= \sum_{n=1}^{+\infty} P\{n \text{ consecutive 1s and one } 0\} + P\{0,1\} \\
&= \sum_{n=1}^{+\infty} p_1^{\,n} \cdot p_0 + p_0 \cdot p_1 \\
&= p_0 \cdot \left(\frac{1}{1-p_1} - 1 + p_1 \right) \\
&= 1 - p_0 + p_0 \cdot p_1 \\
&= p_1 \cdot (1 + p_0)
\end{aligned}$$

And, for $n > 1$, $P\{X_0 = n\} = P\{n \text{ consecutive 0s and one } 1\} = p_0^{\,n} \cdot p_1$

The normalization condition is:

$$\begin{aligned}
&P\{X_0 = 1\} + \sum_{n=2}^{+\infty} P\{X_0 = n\} = 1 \\
&\left[p_1 \cdot (1+p_0) \right] + \left[\sum_{n=2}^{+\infty} p_0^{\,n} \cdot p_1 \right] = \\
&\left[p_1 \cdot (1+p_0) \right] + \left[p_1 \cdot \left(\frac{1}{1-p_0} - 1 - p_0 \right) \right] = \\
&\left[p_1 \cdot (1+p_0) \right] + \left[1 - p_1 \cdot (1+p_0) \right] \\
&= p_1 + p_1 \cdot p_0 + 1 - p_1 - p_1 \cdot p_0 \\
&= 1
\end{aligned}$$

Symmetrically, for X_1 we have $P\{X_1 = 1\} = p_0 \cdot (1 + p_1)$, $P\{X_1 = n\} = p_1^{\,n} \cdot p_0$, and the normalization condition holds as well.

2) The event $\{X_0 > X_1\}$ occurs when 1-terminated sequences of three or more digits are observed. Hence:

$$\begin{aligned}
&P\{X_0 > X_1\} = \\
&\sum_{n=2}^{+\infty} P\{X_0 = n\} = \\
&\sum_{n=2}^{+\infty} p_0^{\,n} \cdot p_1 = \\
&1 - p_1 \cdot (1 + p_0)
\end{aligned}$$

Furthermore, the inequality that ensures that $P\{X_0 > X_1\} \geq 0.25$ is

$$\begin{aligned}
&1 - p_1 \cdot (1+p_0) = 0.25 \\
&0.75 = (1 - p_0) \cdot (1 + p_0) \\
&0.75 = 1 - p_0^{\,2} \\
&p_0^{\,2} = 0.25 \\
&p_0 = 0.5
\end{aligned}$$

Setting $P\{X_0 > X_1\} = \pi$, we easily obtain from the above $P\{X_0 > X_1\} = \pi \Leftrightarrow p_0 = \sqrt{\pi}$.

3) From the formula, we obtain:

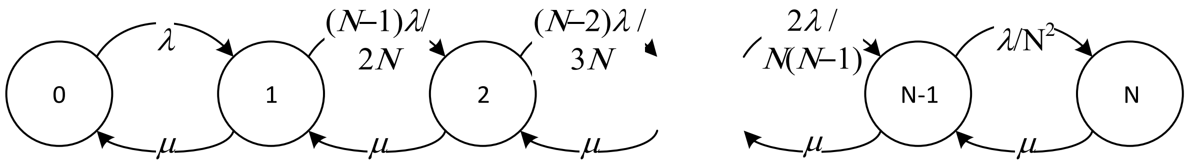
$$
\begin{aligned}
&E\left[X_0\right]= \\
&1\cdot P\left\{X_0=1\right\}+\left[\sum\nolimits_{n=2}^{+\infty} n\cdot P\left\{X_0=n\right\}\right]= \\
&p_1\cdot\left(1+p_0\right)+p_1\cdot\left[\left(\sum\nolimits_{n=2}^{+\infty} n\cdot p_0{}^n\right)\right]= \\
&p_1\cdot\left(1+p_0\right)+p_1\cdot\left[\left(\sum\nolimits_{n=1}^{+\infty} n\cdot p_0{}^n\right)-p_0\right]= \\
&p_1\cdot\left(1+p_0\right)+p_1\cdot\left[\left(\sum\nolimits_{n=1}^{+\infty} n\cdot p_0{}^n\right)-p_0\right]= \\
&p_1\cdot\left(1+p_0\right)+p_1\cdot\left[\left(p_0\cdot\frac{1}{\left(1-p_0\right)^2}-p_0\right)\right]= \\
&p_1\cdot\left(1+p_0\right)+\frac{p_0}{p_1}-p_0\cdot p_1= \\
&p_1+\frac{p_0}{p_1}
\end{aligned}
$$

We have $\lim_{p_0\to 1} E\left[X_0\right]=+\infty$. This can be explained by observing that, if no 1 ever appears, the sequence of 0s becomes infinitely long. Furthermore, we have $\lim_{p_0\to 0} E\left[X_0\right]=1$. In fact, if no 0 ever appears, the only sequences that we may ever obtain are infinitely long sequences of 1s, *terminated by a* 0 (this is mandatory for the experiment to terminate). Hence the mean number of 0s will be equal to one.

4) The two RVs are *not* independent. In fact, $P\left\{X_0=2, X_1=2\right\}=0$ if $a>1, b>1$. On the other hand, $P\left\{X_0=2\right\}\cdot P\left\{X_1=2\right\}\neq 0$.

Exercise 2 – solution

The TR diagram is the following:



The system has a finite number of states, hence it is always stable. Using local equations, one straightforwardly gets:

$$
\begin{aligned}
p_j &= \frac{\lambda^j}{\mu^j}\cdot\prod_{n=0}^{j-1}\frac{N-n}{N\cdot(n+1)}\cdot p_0 \\
&= \frac{\lambda^j}{\mu^j}\cdot\frac{1}{N^j}\cdot\frac{1}{j!}\cdot\frac{N!}{(N-j)!}\cdot p_0 \\
&= \frac{\lambda^j}{\mu^j}\cdot\frac{1}{N^j}\cdot\binom{N}{j}\cdot p_0 \\
&= \binom{N}{j}\cdot\left(\frac{\lambda}{N\cdot\mu}\right)^j\cdot p_0
\end{aligned}
$$

For any j, $0\le j\le N$.

Normalization then reads:

$$\sum_{j=0}^{N}\binom{N}{j}\cdot\left(\frac{\lambda}{N\cdot\mu}\right)^{j}\cdot p_0 = 1$$

$$p_0 = \frac{1}{\sum_{j=0}^{N}\binom{N}{j}\cdot\left(\frac{\lambda}{N\cdot\mu}\right)^{j}} = \frac{1}{\left(\frac{\lambda}{N\cdot\mu}+1\right)^{N}}$$

Therefore:

$$p_j = \frac{\binom{N}{j}\cdot\left(\frac{\lambda}{N\cdot\mu}\right)^{j}}{\left(\frac{\lambda}{N\cdot\mu}+1\right)^{N}}$$

The system is non-PASTA, hence the probabilities seen by an arriving job are different from the above ones. In particular, we have:

$$r_j = \frac{\lambda_j}{\bar{\lambda}}\cdot p_j$$

Now, it is

$$\bar{\lambda} = \gamma = \mu\cdot(1-p_0) = \mu\cdot\left(1-\frac{1}{\left(\frac{\lambda}{N\cdot\mu}+1\right)^{N}}\right)$$

Therefore, we straightforwardly get:

$$r_j = \frac{\lambda_j}{\bar{\lambda}}\cdot p_j = \frac{\frac{N-j}{N\cdot(j+1)}\cdot\lambda}{\mu\cdot\left(1-\frac{1}{\left(\frac{\lambda}{N\cdot\mu}+1\right)^{N}}\right)}\cdot\frac{\binom{N}{j}\cdot\left(\frac{\lambda}{N\cdot\mu}\right)^{j}}{\left(\frac{\lambda}{N\cdot\mu}+1\right)^{N}} =$$

$$= \frac{\frac{N-j}{N\cdot(j+1)}\cdot\lambda\cdot\binom{N}{j}\cdot\left(\frac{\lambda}{N\cdot\mu}\right)^{j}}{\mu\cdot\left(\left(\frac{\lambda}{N\cdot\mu}+1\right)^{N}-1\right)}$$

$$= \frac{\binom{N}{j+1}\cdot\left(\frac{\lambda}{N\cdot\mu}\right)^{j+1}}{\left(\frac{\lambda}{N\cdot\mu}+1\right)^{N}-1}$$

For $0 \le j < N$.

The mean number of transactions in the system is:

$$E[N] = \sum_{j=1}^{N} j\cdot\frac{\binom{N}{j}\cdot\left(\frac{\lambda}{N\cdot\mu}\right)^{j}}{\left(\frac{\lambda}{N\cdot\mu}+1\right)^{N}} = \frac{N\cdot\left(\frac{\lambda}{N\cdot\mu}\right)}{\left(\frac{\lambda}{N\cdot\mu}+1\right)^{N}}\cdot\sum_{i=0}^{N-1}\binom{N-1}{i}\cdot\left(\frac{\lambda}{N\cdot\mu}\right)^{i}$$

$$= \frac{N \cdot \left(\frac{\lambda}{N \cdot \mu}\right)}{\left(\frac{\lambda}{N \cdot \mu} + 1\right)^N} \cdot \left(\frac{\lambda}{N \cdot \mu} + 1\right)^{N-1} = \frac{\frac{\lambda}{\mu}}{\frac{\lambda}{N \cdot \mu} + 1} = \frac{\lambda}{\frac{\lambda}{N} + \mu}$$

By Little's theorem, the mean response time is:

$$E[R] = \frac{E[N]}{\gamma} = \frac{\lambda}{\mu} \cdot \frac{1}{\frac{\lambda}{N} + \mu} \cdot \frac{\left(\frac{\lambda}{N \cdot \mu} + 1\right)^N}{\left(\frac{\lambda}{N \cdot \mu} + 1\right)^N - 1}$$