Exercise 1

A voice codec alternates between two states (*idle* and *talkspurt*). When in *idle* state, it sends one packet after t_i seconds from the last transmission, and when in *talkspurt* it sends a packet after t_t seconds from the last transmission, $t_t < t_i$. The durations of idle and talkspurt states are exponentially distributed, with mean $1/\lambda_i$ and $1/\lambda_t$ respectively.

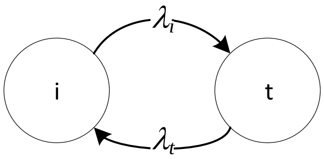
1) Compute the probability that the codec is in the *talkspurt* and *idle* states (assume an infinitely long observation window)
2) Compute the mean generation time of packets at the codec
3) You observed a sequence of *n* packets spaced t_t units of time. What is the probability that the codec will output the next packet after t_t more units of time? Draw a graph of the above probability as a function of *n*

Exercise 2

A server's main job is to handle transactions. These arrive exponentially at a rate λ, they are queued into an infinite FIFO buffer, and they are served at an exponential rate μ. However, transaction handling is sometimes interrupted by *garbage collection (GC) requests*. When one GC request arrives, the sever *stops accepting and serving transactions* and attends to the GC. When the GC request has been cleared, the server resumes accepting/serving transactions. GC request arrive at a rate γ and are served at a rate δ. However, a GC request may arrive only while the server is handling transactions, and there can only be one outstanding GC request (i.e., GC requests cannot queue up).

1) Model the system and draw the transition-rate diagram
2) Write the steady-state equations, find the stability condition and compute the SS probabilities. Explain the stability condition.
3) Express the fraction of time that the server spends handling GC requests.
4) Compute the mean number of transactions in the system.
5) Compute the transaction throughput.
6) Determine what happens of 3,4,5 in the limit cases $\gamma \ll \delta$ and $\gamma \gg \delta$.

Exercise 1



The codec can be seen as a 2-state Markov Chain, with transition rates λ_i and λ_t. Call P_i and P_t the steady-state probabilities of being in the idle and talkspurt states, we readily obtain:

$$P_i \cdot \lambda_i = P_t \cdot \lambda_t$$

$$P_i + P_t = 1$$

From which we obtain $P_i = \frac{\lambda_t}{\lambda_i + \lambda_t}$, $P_t = \frac{\lambda_i}{\lambda_i + \lambda_t}$.

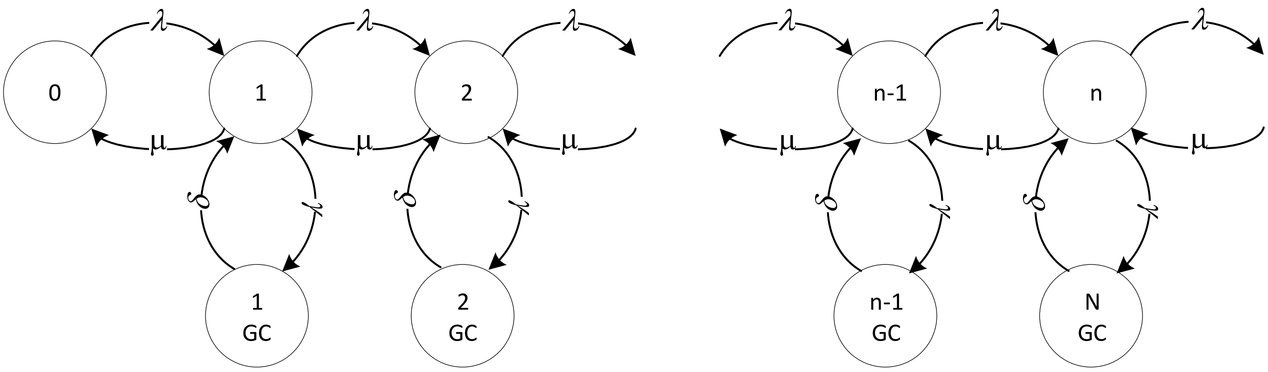
The mean packet generation time is $E[T] = t_i \cdot P_i + t_t \cdot P_t = t_i \cdot \frac{\lambda_t}{\lambda_i + \lambda_t} + t_t \cdot \frac{\lambda_i}{\lambda_i + \lambda_t}$

The CDF is the one in the figure below.

Call X the duration of a talkspurt period. The last question can be formulated as follows: $P\{X \geq (n+1) \cdot t_t | \, X \geq n \cdot t_t\}$. Since X is exponential, hence memoryless, this is equal to $P\{X \geq t_t\} = e^{-\lambda_t}$, and it is independent of *n*. The graph is thus a straight line.

Exercise 2 – solution

The TR diagram is the following:



Calling P_j the probabilities of the "upper" states and $P_j{}'$ those of "lower" states, the SS equations are the following:

$$P_0 \cdot \lambda = P_1 \cdot \mu$$

$$P_j \cdot (\lambda + \mu) = P_{j+1} \cdot \mu + P_{j-1} \cdot \lambda, \quad j > 0$$

$$P_j' \cdot \delta = P_j \cdot \gamma, \;\; j > 0$$

From these, one obtains that $P_j = P_0 \cdot \left(\frac{\lambda}{\mu}\right)^j$, and the following normalization equation:

$$P_0 \cdot \left[1 + \left(1 + \frac{\gamma}{\delta}\right) \cdot \sum_{j=1}^{+\infty} \left(\frac{\lambda}{\mu}\right)^j\right] = 1$$

From which the stability condition is $\lambda < \mu$, and it is independent of λ or δ. This is expectable, since during GC neither arrivals nor services occur, hence the transaction queue does not build up. Calling $\rho = \lambda/\mu$, and $\theta = \lambda/\delta$, we get:

$$P_0 = \frac{1}{1+(1+\theta)\cdot\frac{\rho}{1-\rho}} = \frac{1-\rho}{1+\theta\cdot\rho}, \quad P_j = \frac{1-\rho}{1+\theta\cdot\rho} \cdot \rho^j, \quad P_j' = \frac{1-\rho}{1+\theta\cdot\rho} \cdot \theta \cdot \rho^j$$

The fraction of time that the server spends attending GC requests is

$$P_{GC} = \sum_{j=1}^{+\infty} P_j' = \frac{1-\rho}{1+\theta\cdot\rho} \cdot \theta \cdot \sum_{j=1}^{+\infty} \rho^j = \frac{\theta \cdot \rho}{1+\theta\cdot\rho}$$

The mean number of transactions in the system is

$$E[N] = \sum_{j=1}^{+\infty} j \cdot (P_j + P_j') = \frac{1-\rho}{1+\theta\cdot\rho} \cdot (1+\theta) \cdot \sum_{j=1}^{+\infty} j \cdot \rho^j = \frac{\rho}{1-\rho} \cdot \frac{1+\theta}{1+\theta\cdot\rho}$$

The throughput is:

$$tpt = \sum_{j=1}^{+\infty} \mu_j \cdot P_j = \mu \cdot \frac{1-\rho}{1+\theta\cdot\rho} \cdot \sum_{j=1}^{+\infty} \rho^j = \mu \cdot \frac{1-\rho}{1+\theta\cdot\rho} \cdot \frac{\rho}{1-\rho} = \frac{\lambda}{1+\theta\cdot\rho}$$

The limit cases $\gamma \ll \delta$ and $\gamma \gg \delta$ correspond to $\theta \cong 0$ and $\theta \to \infty$.

a) $\theta \cong 0$: $P_{GC} \cong 0$, $E[N] = \frac{\rho}{1-\rho}$, $tpt = \lambda$. The system behaves like an M/M/1, which is obvious since GCs cause negligible overhead.

b) $\theta \to \infty$: $P_{GC} \to 1$, $E[N] \to \frac{1}{1-\rho}$, $tpt \to 0$. The system throughput goes down to zero. The transaction queue, however, does not grow indefinitely.