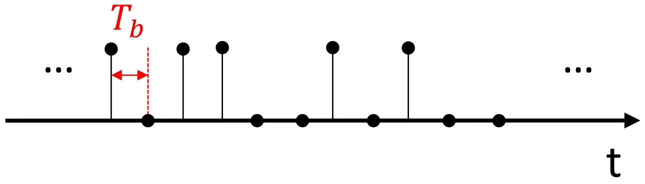
Digital communications

How can we transmit a sequence of bits?

- What happens if we want to transmit a *sequence of bits* in the place of an analog signal?

$d_k = \cdots 1,0,1,1,0,0,1,0,1,0,1,0,0, \ldots$



$$\sum_i d_k \delta(t - kT_b)$$

- A train of delta occupies infinite bandwidth, before transmission the bits need to be passed through a low pass filter.
- Moreover, any signal transmitted in the air needs to be translated in frequency.

How can we transmit a sequence of bits?

- Each bit of the sequence can be modelled as an equiprobable random variable

$$P\{d_k = 0\} = P\{d_k = 1\} = \frac{1}{2}$$

  so that $E\{d_k\} = \frac{1}{2}$.

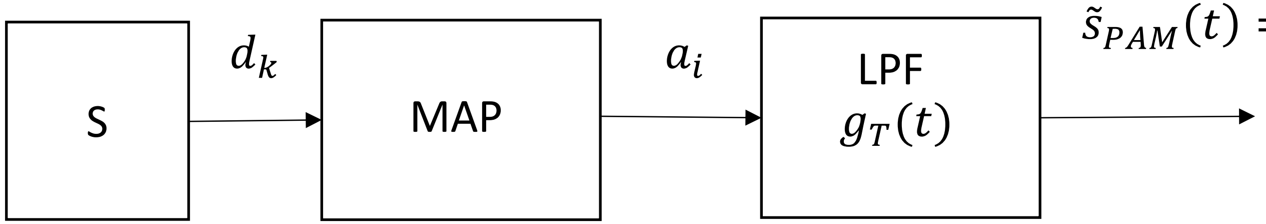
- In general, to save energy it is better to transmit 0-mean information. Bits d_k are mapped to 0-mean *information symbols*: $a_i = 2d_i - 1$

$$0 \rightarrow -1, \qquad 1 \rightarrow 1$$

- Using the same principles, one information symbol can be used to map more than just one bit.

Pulse amplitude modulation

- Pulse amplitude modulation (PAM) is the modulation obtained by
  1. Mapping the bits d_k to the information symbols a_i
  2. Filtering the symbols with a low pass filter with impulse response $g_T(t)$



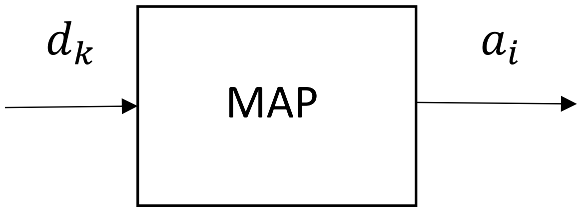
$$\tilde{s}_{PAM}(t) = \sum_i a_i g_T(t - iT)$$

- Since the mapper can map a sequence of m bits on just one information symbol, the bit duration T_b and the symbol duration T may be different.

Pulse amplitude modulation

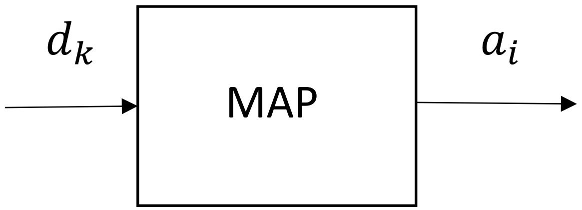
- The signal $\tilde{s}_{PAM}(t)$ is a *real* baseband signal that can be modulated at any frequency f_c
$$s_{PAM}(t) = \sum_i a_i g_T(t - iT) \cos(2\pi f_c t)$$
- The PAM signal is *equivalent* to an analog DSB where the modulating (and complex envelope) signal $m(t)$ is
$$m(t) = \sum_i a_i g_T(t - iT) = \tilde{s}_{PAM}(t)$$
- From now on we will only consider the signal *baseband equivalent* and we will omit the tilde for ease of notation, i.e. $s_{PAM}(t)$ indicates the complex envelope $s_{PAM}(t) = \sum_i a_i g_T(t - iT)$.

PAM: symbol mapping



- The mapper block associates a sequence of m bits d_k to a single symbol a_i.
- Since each information symbol maps m bits, the symbol constellation contains $M = 2^m$ symbols. M is always a power of 2.
- Usually, bit-to-symbol mapping is performed so that $E\{a_i\} = 0$.
- The larger the constellation size, the more bits are mapped to each symbol, the more efficient is the spectrum usage but also the larger is the average energy to transmit a bit.

PAM: symbol mapping

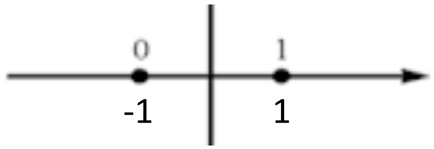


- The source generates bits with a rate $R_b = \frac{1}{T_b}$.
- Given the bit time T_b, since each symbol maps m bits, the symbol time T is

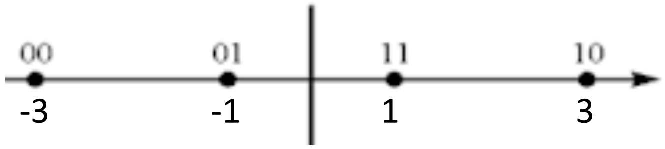
$$T = mT_b = \log_2 M\, T_b$$

- Accordingly, the symbol rate R is m times smaller than the bit rate R_b

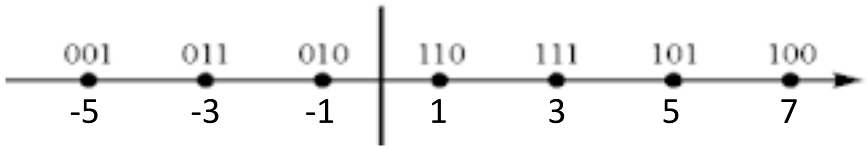
$$R = \frac{1}{T} = \frac{1}{mT_b} = \frac{R_b}{m} = \frac{R_b}{\log_2 M}$$



$M = 2$
$m = 1$



$M = 4$
$m = 2$



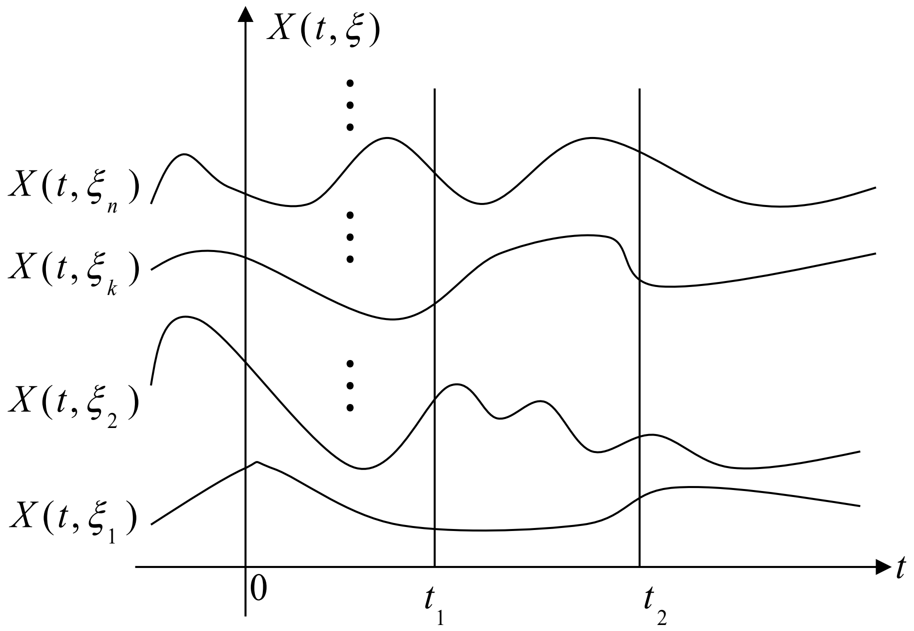
$M = 8$
$m = 3$

Stochastic processes

- *Deterministic process.* A deterministic process is represented by an explicit mathematical relation.
- *Stochastic process*. A stochastic process is the result of a large number of separate causes, described in probabilistic terms and by properties which are averages.

Stochastic processes

- A *stochastic process* is a set of random variables indexed in time
- Let ξ denote the random outcome of an experiment. To every such outcome suppose a waveform $X(t, \xi)$ is assigned. The collection of such waveforms form a *stochastic process*.
- For a fixed ξ (the set of all experimental outcomes), $X(t, \xi)$ is a specific time function.
- For fixed $t = t_0$, $X(t_0, \xi)$ is a random variable.
- The ensemble of all such realizations over time represents the stochastic process $X(t)$.



Categories of stochastic processes

- *Parameter space*: set T of indices $t \in T$.
- *State space*: set S of values $X(t) \in S$.
- Categories:
  - Based on the parameter space:
    - Discrete-time processes: parameter space discrete,
    - Continuous-time processes: parameter space continuous.
  - Based on the state space:
    - Discrete-state processes: state space discrete,
    - Continuous-state processes: state space continuous.

Distribution and probability density function

- If $X(t)$ is a stochastic process, then for fixed $t = t_0$, $X(t_0)$ represents a *random variable*.
- The *distribution function* is given by

$$F_{X(t_0)}(x) = \Pr\{X(t_0) \leq x\}$$

$F_{X(t_0)}(x)$ depends on the value of the time instant $t = t_0$. For different values of t, we obtain a different random variable.

- Further, the first-order *probability density function* of the process $X(t)$ is

$$f_{X(t_0)}(x) = \frac{d}{dx} F_{X(t_0)}(x)$$

Independence

- For an *independent* stochastic process, the random variables obtained by sampling the process at any n times $t_1, \ldots, t_n$ are independent random variables for any n.
- Accordingly, the distribution is
$$F_{X(t_1),\ldots,X(t_n)}(x_1, \ldots, x_n) = \Pr\{X(t_1) \le x_1\} \cdots \Pr\{X(t_n) \le x_n\}$$
$$= F_{X(t_1)}(x_1) \cdots F_{X(t_n)}(x_n)$$

  and the probability density function is
$$f_{X(t_1),\ldots,X(t_n)}(x_1, \ldots, x_n) = f_{X(t_1)}(x_1) \cdots f_{X(t_n)}(x_n)$$

Mean and autocorrelation

- **Mean** of a stochastic process**:**

$$\mu_X(t_0) = E\{X(t_0)\} = \int_{-\infty}^{+\infty} x f_{X(t_0)}(x)dx$$

is the mean value of the process $X(t)$ at time t_0. In general, the mean of a process depends on the time index t.

- **Autocorrelation function** of a process:

$$R_{XX}(t_1, t_2) = E\{X(t_1)X^*(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2^* f_{X(t_1),X(t_2)}(x_1, x_2)dx_1 dx_2$$

represents the interrelationship between the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ obtained by sampling the process $X(t)$ at times t_1 and t_2.

Stationarity

- A stationary process exhibits statistical properties that are invariant to shift in the time index.
- *First-order stationarity* implies that the statistical properties of $X(t_0)$ and $X(t_0 + c)$ are the same for any *c.*

$$f_{X(t_0)}(x) = f_X(x)$$

  - The *mean* is a constant and does not depend on t_0.
- *Second-order stationarity* implies that the statistical properties of the pairs $\{X(t_1), X(t_2)\}$ and $\{X(t_1 + c), X(t_2 + c)\}$ are the same for any c.

$$f_{X(t_1),X(t_2)}(x_1, x_2) = f_X(x_1, x_2, t_2 - t_1)$$

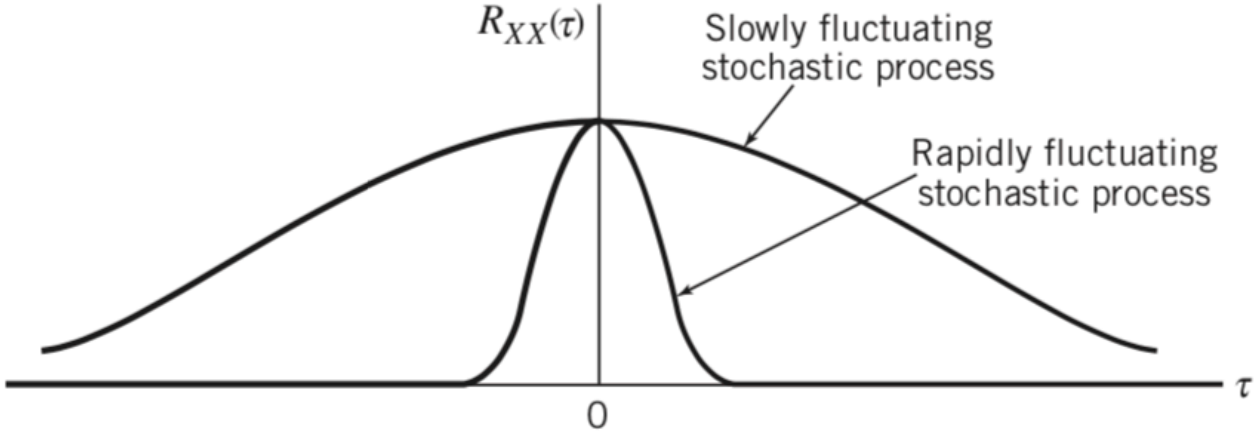
  - The autocorrelation depends only on the difference of the time indices.

Wide sense stationarity

- We can use a looser definition of stationarity. A process $X(t)$ is said to be *wide-sense stationary* (WSS) if only these two conditions hold:
    1) $E\{X(t)\} = \mu_X$
    2) $E\{X(t_1), X(t_2)\} = R_{XX}(t_2 - t_1)$
- For a wide-sense stationary process, the mean is a constant and the autocorrelation function depends only on the difference between the time indices.

Autocorrelation function

- The autocorrelation function $R_{X,X}(t_1, t_2)$ describes the interrelation between two random variables obtained by sampling the stochastic process $X(t)$ at times t_1 and t_2.
- The more rapidly the stochastic process $X(t)$ changes with time, the more rapidly will the autocorrelation function decrease from its maximum $R_{XX}(0)$.



Power spectral density

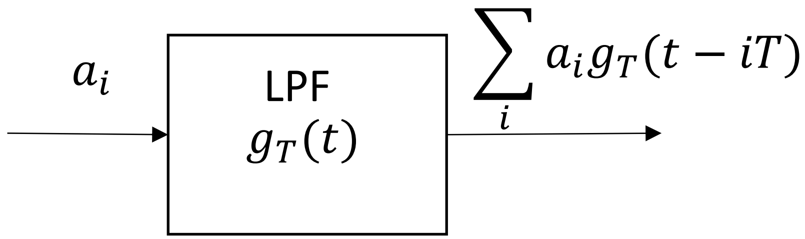
- The power spectral density $S_{XX}(f)$ of a WSS stochastic process $X(t)$ describes the distribution of power into frequency components composing that signal and is measured in watts per hertz (W/Hz).
- *Wiener-Kintchine theorem*. The power spectrum of $X(t)$ is obtained as the Fourier transform of the autocorrelation function $R_{XX}(\tau)$

$$S_{XX}(f) = \mathcal{F}\{R_{XX}(\tau)\} = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j2\pi f\tau} d\tau$$

- The signal power of $X(t)$ can be computed as

$$P_X = \int_{-\infty}^{+\infty} S_{XX}(f) df$$

PAM: power spectral density

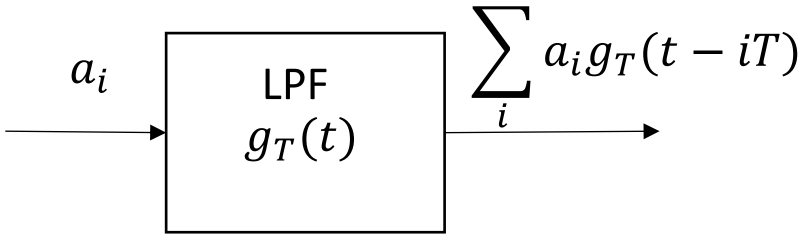


- A PAM signal is modelled as a *stochastic process* because the symbols a_i are samples of a discrete-time discrete-state stochastic process.
- The bandwidth occupied by a stochastic process is measured by its *power spectral density* (Fourier transform of its autocorrelation function).
- The PSD of the complex envelope $s(t)$ of a PAM signal is

$$S_s(f) = \frac{1}{T} S_a(f) |G_T(f)|^2$$

where $S_a(f)$ is the PSD of a_i and $G_T(f)$ is the frequency response of the transmit filter $g_T(t)$.

PAM: power spectral density



- $S_a(f)$ is computed as the Fourier transform of the autocorrelation function $R_a(m)$ of the stationary, discrete, independent process a_i.

$$R_a(m) = E\{a_i a_{i+m}\} = \begin{cases} E\{a_i^2\} = A & m = 0 \\ (E\{a_i\})^2 & m \neq 0 \end{cases}$$

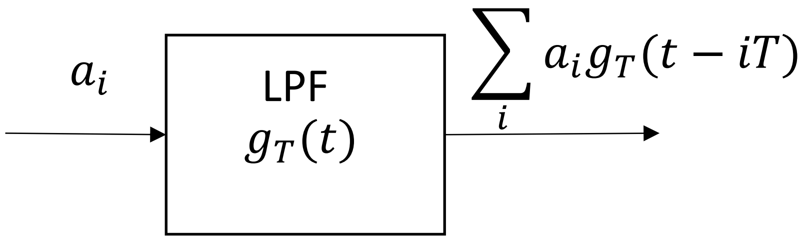
- When the symbols are zero-mean, i.e., $E\{a_i\} = 0$, it is

$$R_a(m) = A\delta(m) \rightleftharpoons S_a(f) = A,$$

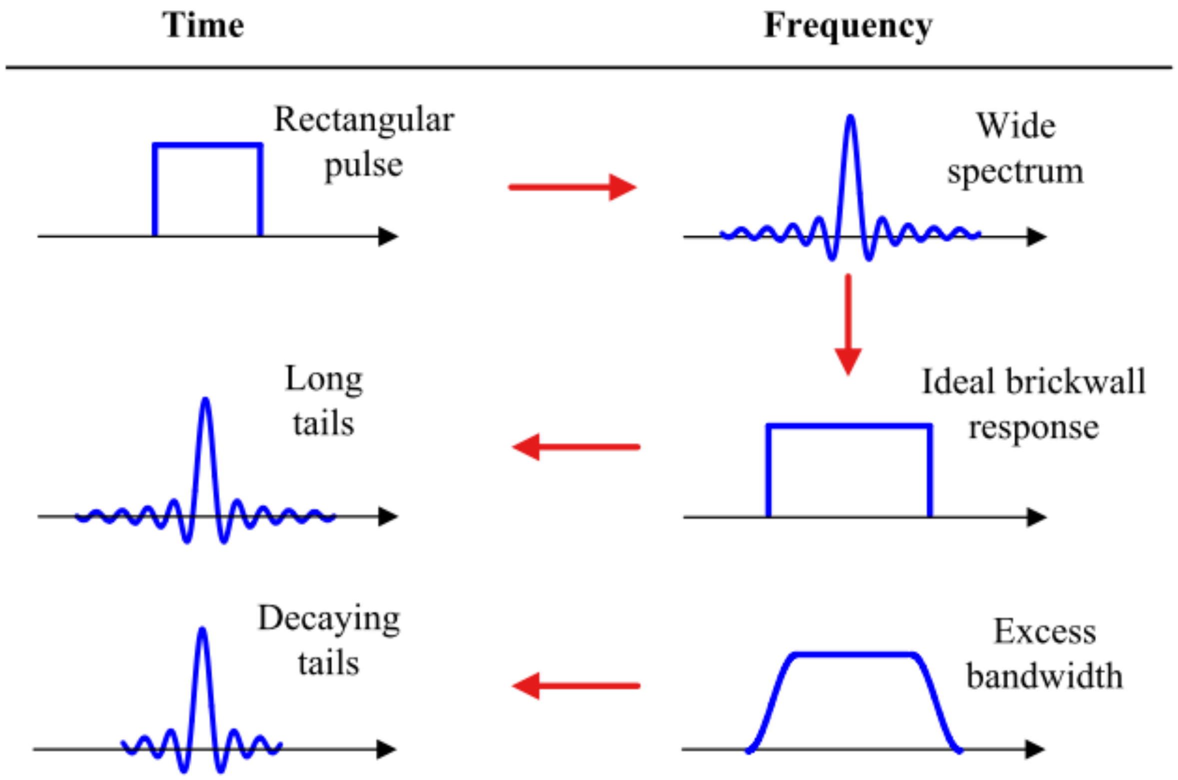
and the PSD of the PAM signal is

$$S_s(f) = \frac{A}{T}|G_T(f)|^2$$

PAM: pulse shaping

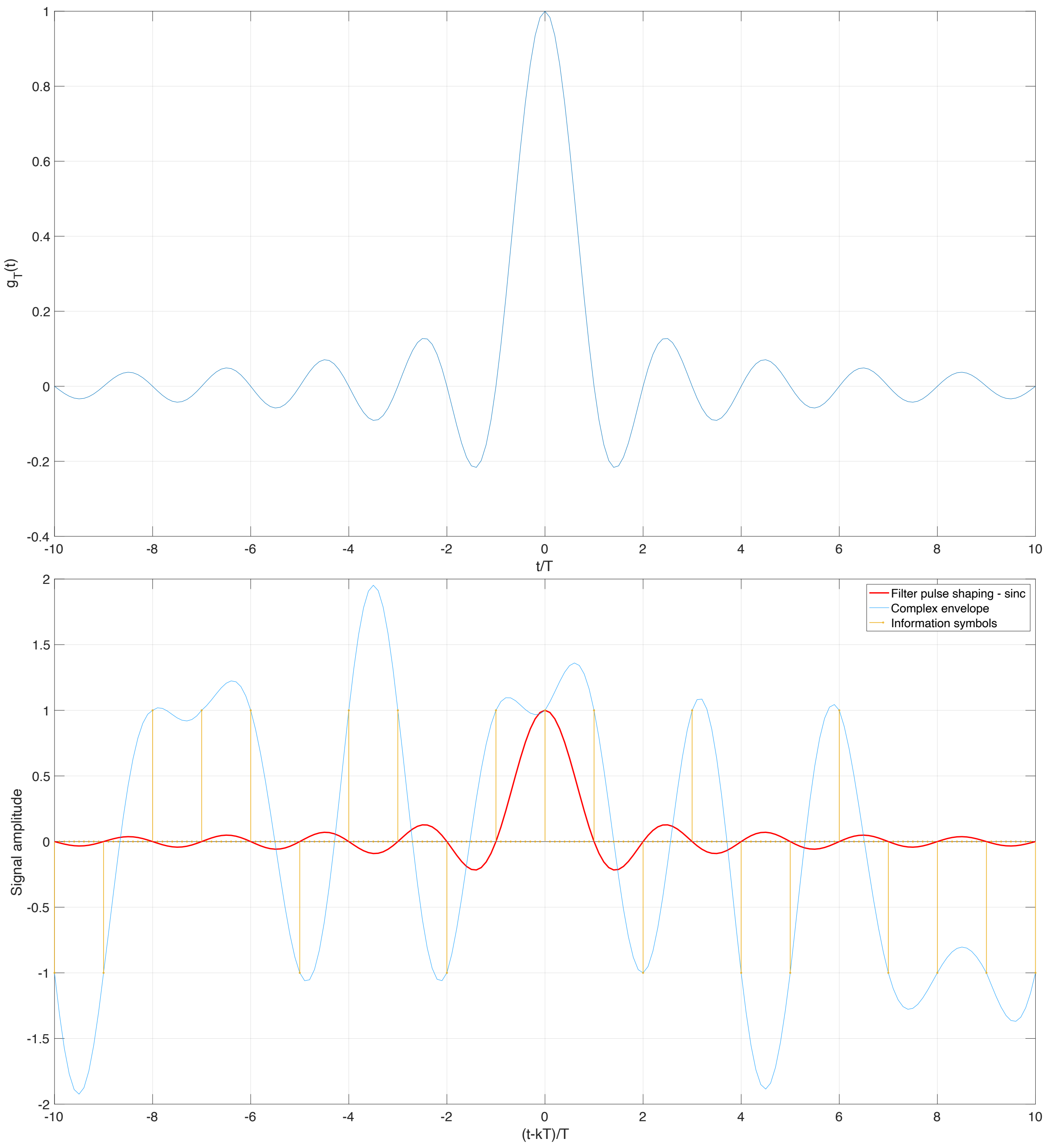


- The choice of the low-pass transmit filter determines the spectrum and the bandwidth of the PAM signal.
- If the pulse shape $g_T(t)$ has a duration longer than 1 symbol time T, the spectrum is more compact but the energy of one symbols is spread over several intervals.



PAM: pulse shaping

- The most compact spectrum is obtained when
$$G_T(f) = \mathrm{rect}(fT),$$
which in the time domain corresponds to
$$g_T(t) = \frac{1}{T}\,\mathrm{sinc}\left(\frac{t}{T}\right).$$
- The sinc spans an interval of several symbols.
- One single symbol ‘mixes’ its information with several adjacent symbols.
- This type of interference is denominated *inter-symbol interference* (ISI).

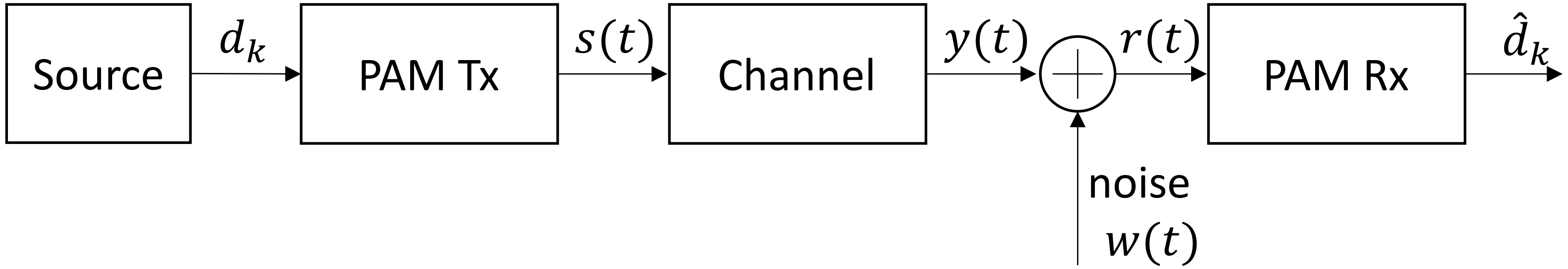


PAM: occupied bandwidth

- There is a trade-off to make:
  - compact spectrum → large amount of interference in the time domain (Extreme choice: a *rect* in the frequency domain and a *sinc* in time).
  - wide spectrum → most of the symbol energy is contained within one symbol interval (Extreme choice: a *rect* in the time domain and a *sinc* in frequency).

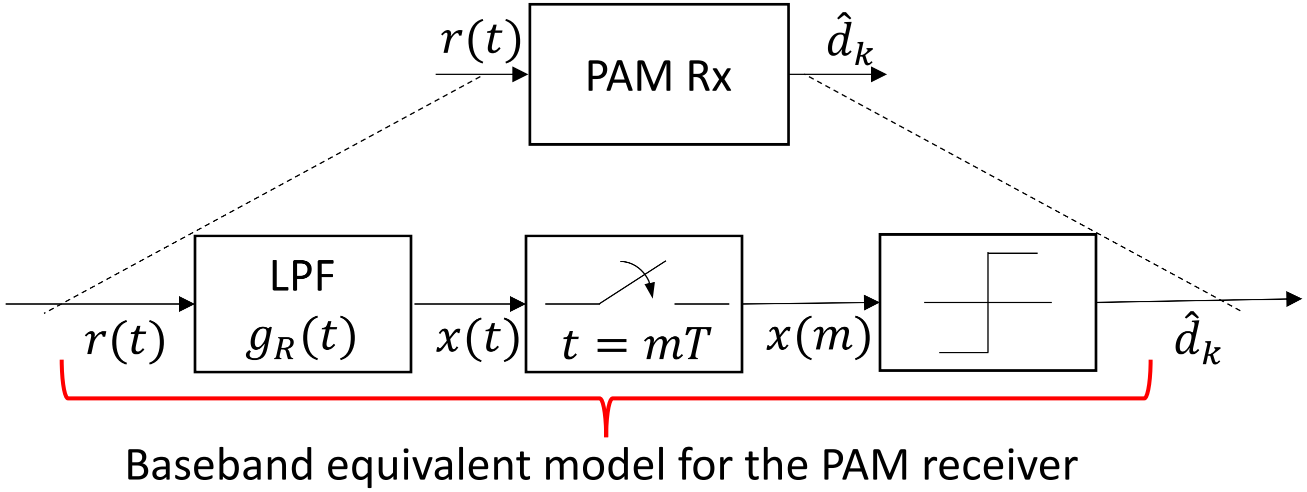
PAM: receiver architecture

- PAM system block diagram



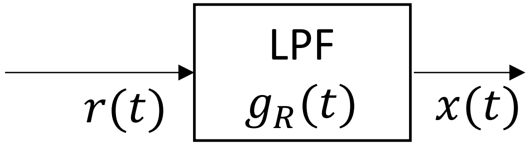
- The propagation channel is in general modelled as a LTI filter with impuls response $h(t)$. When the channel is ideal, it is $h(t) = \delta(t)$.
- The noise term is a white, zero-mean, Gaussian stationary process with PSD $S_w(f) = N_0/2$ ($S_w(f) = 2N_0$ for its complex envelope).
- The receiver's task is to reconstruct the sequence of transmitted bits from the received signal $r(t)$.

PAM: receiver architecture



- The PAM receiver performs the inverse operation of the transmitter: extract the transmitted bits from the analog received signal $r(t)$.
  1. Filters noise and interference from the received signal $r(t)$;
  2. Samples the filtered signal $x(t)$ once per symbol time T;
  3. Recovers the transmitted bits from the signal samples $x(m)$.

PAM: Receive filter



- The received baseband equivalent signal has the form

$$r(t) = s(t) \otimes h(t) + w(t)$$

- The filter output is

$$x(t) = r(t) \otimes g_R(t) = \sum_i a_i g(t - iT) + n(t)$$

where

- $g(t) = g_T(t) \otimes h(t) \otimes g_R(t)$ is the convolution of the impulse response of the channel, the transmit and the receiver filter,
- $n(t)$ is the filtered (and colored!) noise.

PAM receive filter

$$r(t) \longrightarrow \boxed{\begin{matrix}\text{LPF}\\ g_R(t)\end{matrix}} \longrightarrow x(t)$$

- One of the tasks of the receive filter $g_R(t)$ is to remove the intersymbol interference affecting the received samples.
- The samples of the received signal take this form:

$$x(m) = x(t)\Big|_{t=mT} = \sum_i a_i g(mT - iT) + n(mT)$$

$$= a_m\, g(0) + \sum_{\ell,\ell\neq 0} a_{m-\ell}\, g(\ell T) + n(mT)$$

ISI

- Neglecting the noise term, the condition on $g(\ell T) = g(t)|_{t=\ell T}$ to have zero ISI is

$$g(\ell T) = \begin{cases} 1 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$

- Under these conditions (Nyquist criterion), the received sample $x(m)$ is

$$x(m) = a_m + n(mT)$$

Nyquist criterion in the frequency domain

- The frequency response of the cascade of the channel, the transmit and the receive filter is $G(f)$, the Fourier transform of $g(t)$.
- Since sampling in time determines *periodicity* in the frequency domain, $\mathcal{F}\{g(\ell)\}$, the Fourier transform of $g(\ell)$, $g(t)$ sampled every T seconds, is

$$\mathcal{F}\{g(\ell)\} = \sum_{\ell} g(\ell) e^{-j2\pi f \ell T} = \frac{1}{T} \sum_{k} G\left(f - \frac{k}{T}\right)$$

Nyquist criterion in the frequency domain

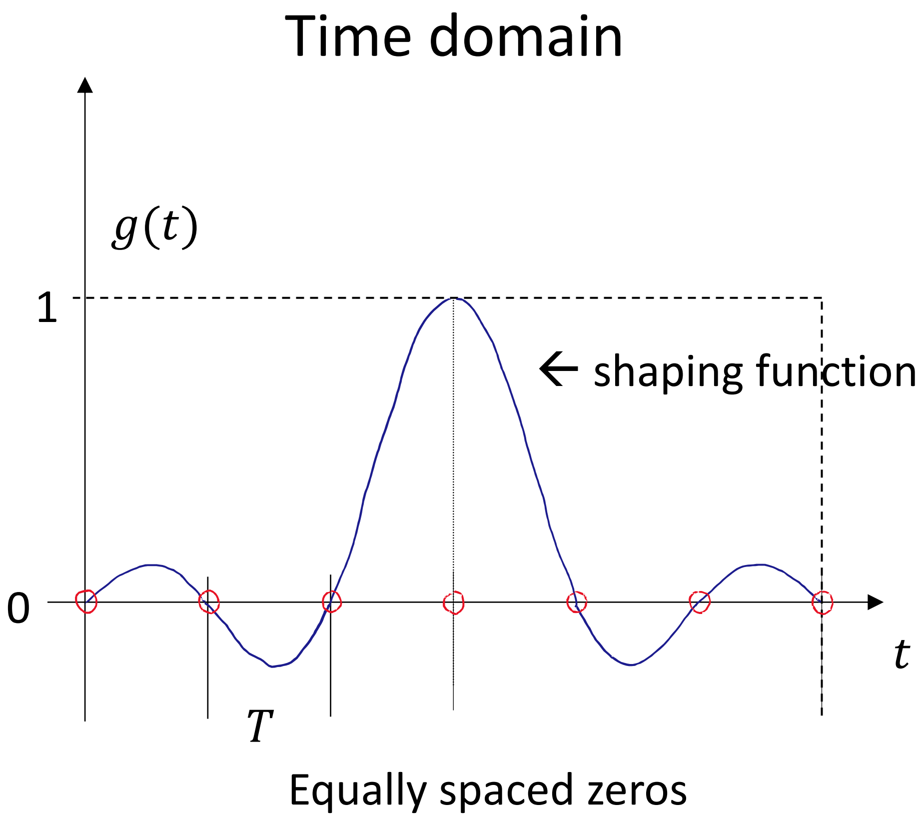
- On the other hand, if the sampled response $g(\ell)$ satisfies the Nyquist criterion, then it is a Kronecker delta, i.e. $g(\ell) = \delta(\ell)$.
- The Fourier transform of $\delta(\ell)$ is $\mathcal{F}\{\delta(\ell)\} = 1$.
- Accordingly, it is

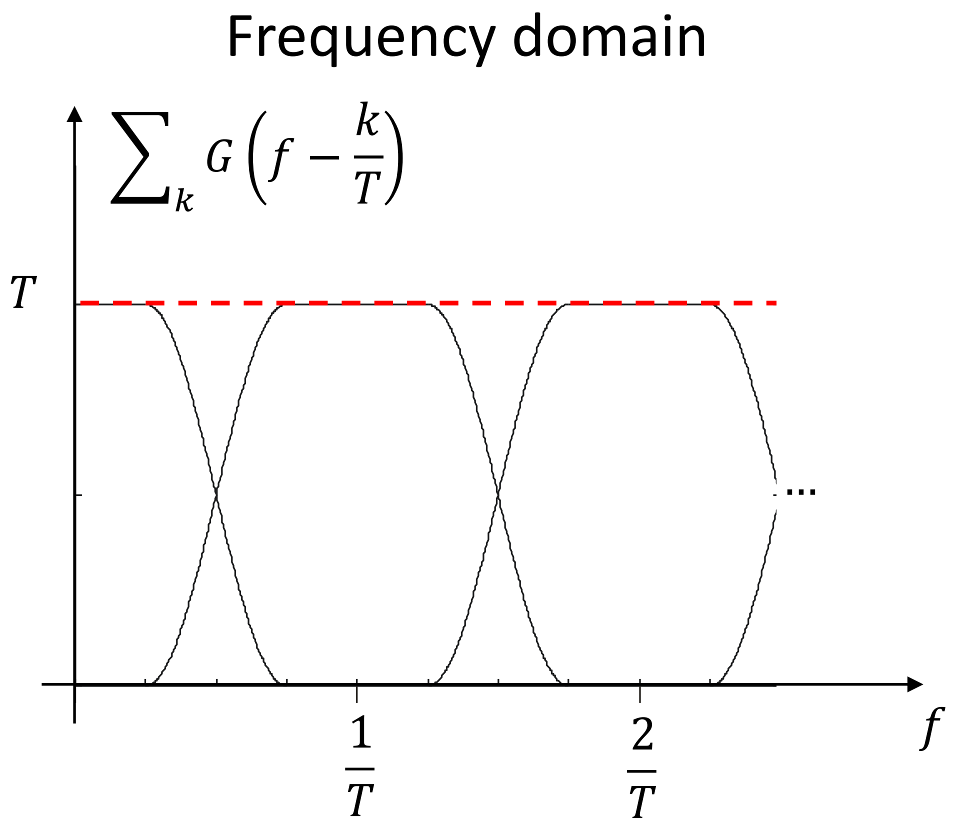
$$\mathcal{F}\{g(\ell)\} = \frac{1}{T}\sum\nolimits_k G\left(f - \frac{k}{T}\right) = \mathcal{F}\{\delta(\ell)\} = 1$$

- From which we can extrapolate the Nyquist criterion for zero ISI in the frequency domain

$$\sum\nolimits_k G\left(f - \frac{k}{T}\right) = T$$

Nyquist criterion



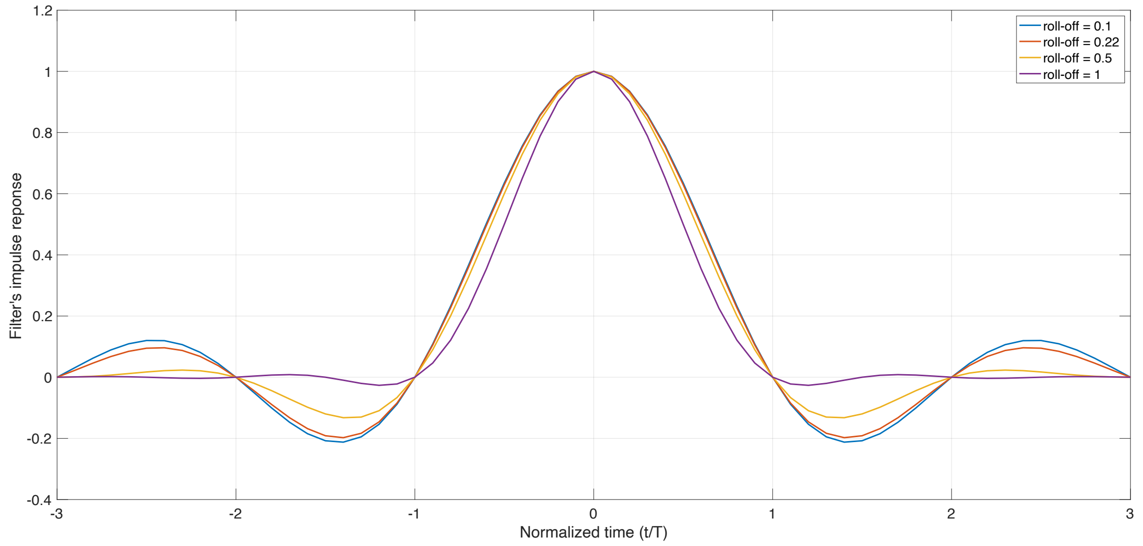


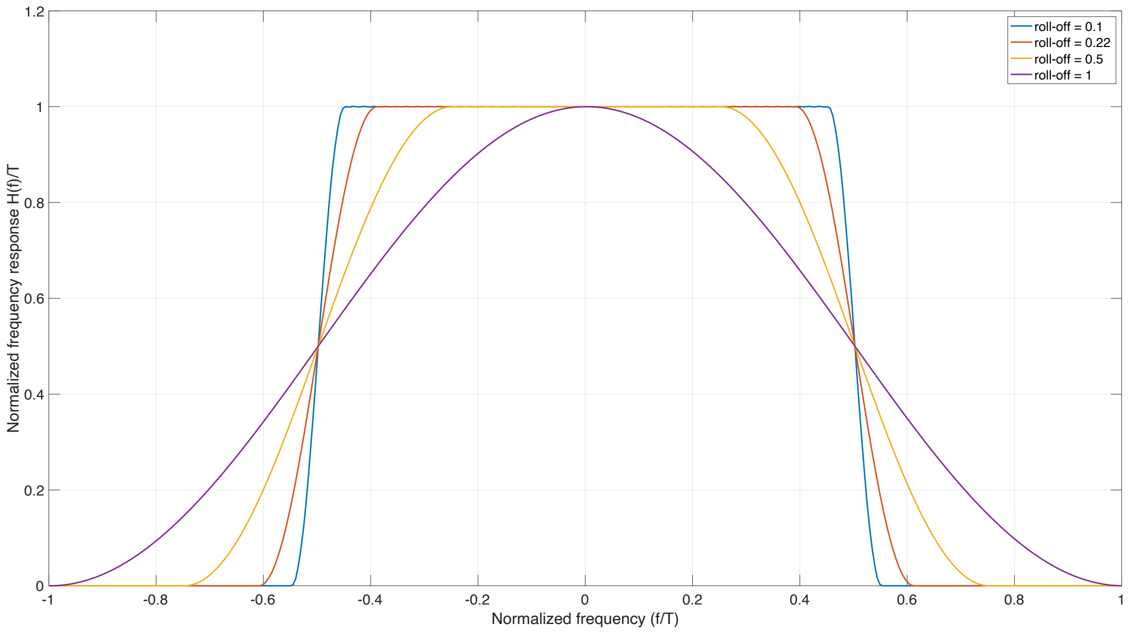
Raised cosine filters

Raised cosine filters satisfy the Nyquist criterion: the occupied bandwidth is

$$B_{RC} = \frac{1+\alpha}{2T}$$

The roll-off factor α is a design parameter, RC with $\alpha = 0$ is a rect and it is the *minimum bandwidth filter*.





Additive white Gaussian noise

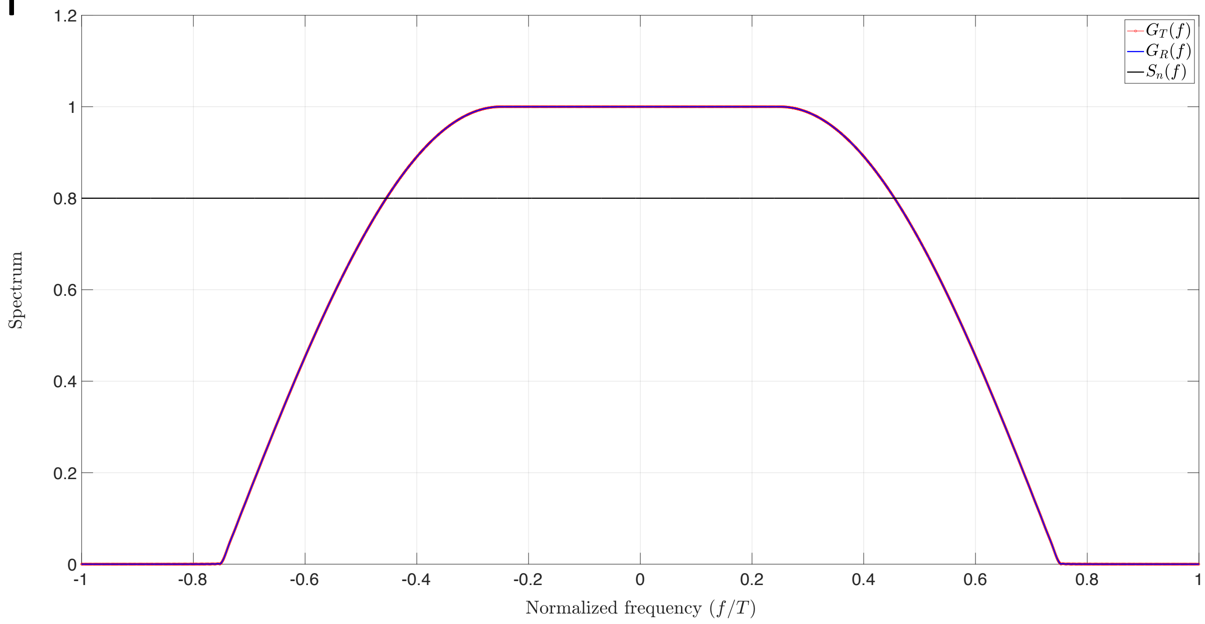
- The complex envelope of the noise is $w(t) = w_I(t) + jw_Q(t)$, a stationary zero-mean white Gaussian process with PSD $S_w(f) = 2N_0$.
- The in-phase and quadrature components are independent and it is $S_{w_I}(f) = S_{w_Q}(f) = N_0$.
- The noise $n(t) = n_I(t) + jn_Q(t) = g_R(t) \otimes w(t)$ is a zero-mean Gaussian complex stochastic process and its PSD is

$$S_n(f) = S_w(f)|G_R(f)|^2 = 2N_0|G_R(f)|^2$$

$$S_{n_I}(f) = S_{n_Q}(f) = N_0|G_R(f)|^2.$$

Receive filter design: matched filter

- Neglecting, for the moment, the effect of the channel, the other major impairment at the receiver is the presence of Gaussian noise.
- The receiver should be designed to minimize the negative effect of Gaussian noise.
- The choice
  $$g_R(t) = g_T(-t) \leftrightharpoons G_R(f) = G_T^*(f)$$
  maximizes the signal-to-noise ratio at the receiver.
- The receiver filter is *matched* to the transmit filter.
- If $g_T(t)$ is even, then it is $g_R(t) = g_T(t)$.



Root raised cosine filters

- Root raised cosine filters are filters whose frequency response is the square root of a raised cosine, i.e., $H_{RRC}(f,\alpha) = \sqrt{H_{RC}(f,\alpha)}$.
- If $G_R(f) = G_T(f) = H_{RRC}(f,\alpha)$, the transmit and receive filter pair satisfies the two independent *optimality* conditions:
  1. The cascade of $g_T(t)$ and $g_R(t)$ obeys the *Nyquist criterion*:
     $$G_R(f)G_T(f) = \left(H_{RRC}(f,\alpha)\right)^2 = H_{RC}(f,\alpha).$$
  2. The receive filter is matched to the transmit filter.
     Since it is $G_T(f) \in \Re \rightarrow G_R(f) = G_T(f) = G_T^*(f)$, the transmit and receive filter are *matched*.

Bandwidth of a PAM signal with RRC filtering

- If $G_T(f) = H_{RRC}(f, \alpha)$ and the symbols are zero-mean, it is
$$S_s(f) = \frac{1}{T} S_a(f) |G_T(f)|^2 = \frac{A}{T} H_{RC}(f, \alpha).$$
- The bandwidth occupied by the PAM *complex envelope* is
$$B_{PAM}^{(BB)} = \frac{1+\alpha}{2T} = \frac{1+\alpha}{2} \frac{1}{\log_2 M \, T_b} = \frac{1+\alpha}{2} \frac{R_b}{\log_2 M}$$
- The bandwidth occupied by the corresponding *passband signal* is
$$B_{PAM}^{(PB)} = 2B_{PAM}^{(BB)} = \frac{1+\alpha}{T} = (1+\alpha) \frac{R_b}{\log_2 M}$$

Power of a PAM signal with RRC filtering

- The mean power of the *complex envelope* of a PAM signal with zero-mean symbols and root raised cosine filtering is

$$P_s^{(BB)} = \int_{-\infty}^{+\infty} S_s(f)df = \frac{A}{T}\int_{-\infty}^{+\infty} |G_T(f)|^2 df = \frac{A}{T}\int_{-\infty}^{+\infty} H_{RC}(f,\alpha)df$$

since it is $\int_{-\infty}^{+\infty} H_{RC}(f,\alpha)df = h_{RC}(t,\alpha)|_{t=0} = 1$,

$$P_s^{(BB)} = \frac{A}{T}$$

- The power of the corresponding *passband signal* is

$$P_s = \frac{1}{2}P_s^{(BB)} = \frac{A}{2T}$$

Energy of a PAM symbol with RRC filtering

- The mean square value of the symbols for a PAM constellation is

$$A = E\{a_i^2\} = \frac{M^2 - 1}{3}$$

- The energy per symbol is computed as the power multiplied by the symbol duration

$$E_s = P_s T = \frac{A}{2T} T = \frac{M^2 - 1}{6}$$

Additive white Gaussian noise

- The sample $n(m) = n(t)|_{t=mT} = n_I(m) + jn_Q(m)$ is a zero-mean Gaussian complex *random variable* and its variance is

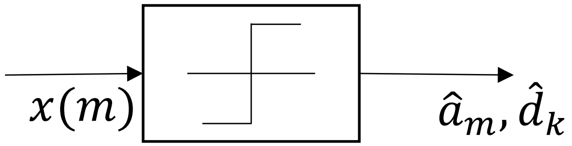
$$\sigma_n^2 = E\{|n(m)|^2\} = \int_{-\infty}^{+\infty} S_n(f)df = 2N_0 \int_{-\infty}^{+\infty} |G_R(f)|^2 df$$

- If the receive filter is a RRC, then it is $\int_{-\infty}^{+\infty} |G_R(f)|^2 df = 1$ and

$$\sigma_n^2 = 2N_0$$

- The in-phase and quadrature components $n_I(m)$ and $n_Q(m)$ are independent and the variance of each component is $\sigma^2 = \sigma_{n_I}^2 = \sigma_{n_Q}^2 = N_0$.

Decision strategy



- Under the hypothesis of RRC filtering at the transmit and at the receiver, the decision variable is

$$x(m) = a_m + n(m)$$

- The optimal decision strategy is the one that chooses the symbol that maximizes the probability conditioned on having received $x(m)$.

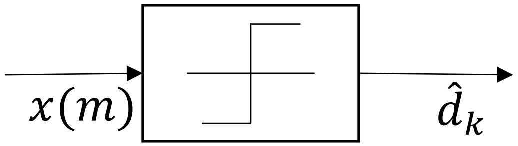
$$\hat{a}_m = \arg \max_{a^{(i)} \in \mathcal{A}} p\left(a^{(i)} \middle| x(m)\right)$$

- It can be shown that in case of equiprobable symbols it is

$$p\left(a^{(i)} \middle| x(m)\right) \approx p\left(x(m) | a^{(i)}\right)$$

so that the symbol $a^{(i)}$ that maximizes $p\left(x(m)|a^{(i)}\right)$ maximizes also $p\left(a^{(i)} \middle| x(m)\right)$, *maximum likelihood decision*.

PAM decision strategy



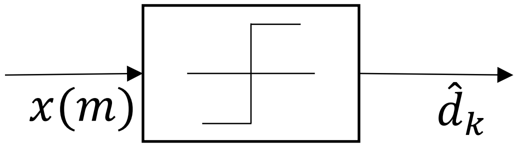
- Since PAM signal is *real*, we consider only the in-phase component of the received signal, i.e. $x(m) = x_I(m)$ and $n(m) = n_I(m)$.
- Because of the conditioning, the symbol value $a^{(i)}$ is fixed and $x(m) \in \mathcal{N}\left(a^{(i)}, N_0\right)$ is a Gaussian random variable with mean $a^{(i)}$ and variance $\sigma^2 = \sigma_{n_I}^2 = N_0$.
- The probability density function is

$$p\left(x(m)\middle|a^{(i)}\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(x(m)-a^{(i)}\right)^2}{2\sigma^2}}$$

so that the symbol $a^{(i)}$ that maximizes $p\left(x(m)\middle|a^{(i)}\right)$ is the one that minimizes the distance between the symbol and the received sample

$$\hat{a}_m = \arg \min_{a^{(i)} \in \mathcal{A}} \left|x(m) - a^{(i)}\right|$$

Decision strategy



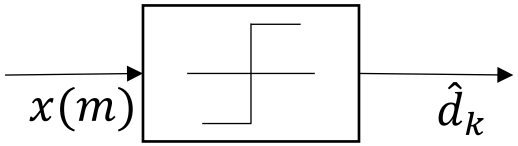
- Example. Consider a 2-PAM ($M = 2$) and assume that $x(m) = 0.5$. The two symbols are $a^{(0)} = -1$ and $a^{(1)} = 1$, i.e. $\mathcal{A} = \{-1,1\}$.
- The decision block will compute the two distances:

$$d\left(x(m), a^{(0)}\right) = |0.5 - (-1)| = 1.5$$
$$d\left(x(m), a^{(1)}\right) = |0.5 - 1| = 0.5$$

and will decide for the symbol that minimizes the distance in the signal space from the received sample $x(m)$, i.e.

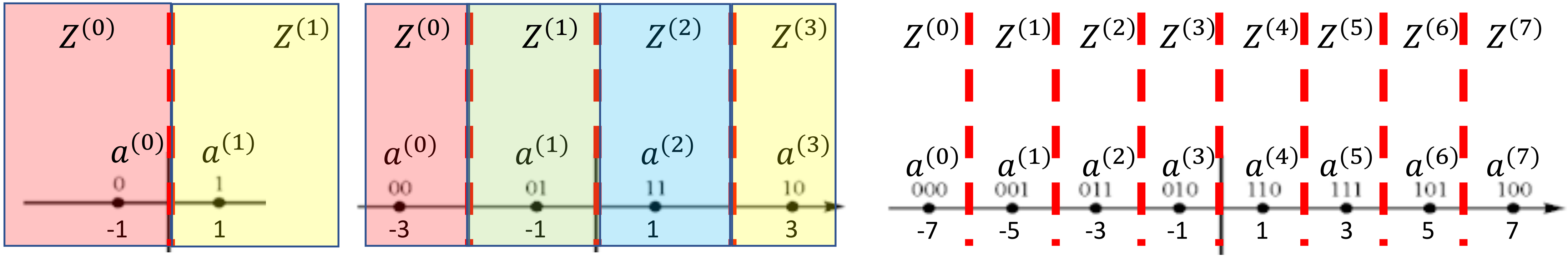
$$\hat{a}_m = \arg \min_{a^{(i)} \in \mathcal{A}} d(x(m), a^{(i)}) = 1\,.$$

Decision strategy



- Adopting the maximum likelihood criterion, we can partition the signal space in *zone of decisions*, where zone $Z^{(i)}$ is the set of points that are closer to the symbol $a^{(i)}$ than to any other symbol

$$Z^{(i)} = \{x | d(x, a^{(i)}) < d(x, a^{(j)}), j \neq i, j = 1, \dots, M\}$$



$M = 2$
$m = 1$

$M = 4$
$m = 2$

$M = 8$
$m = 3$

The decision threshold are in the midpoints of the segment connecting any two adjacent symbols. For example, for $M = 4$ the thresholds are in $-2{,}0$ and 2.

PAM error probability

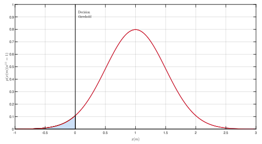
- Even if the maximum likelihood decision strategy is optimal, the receiver can still make some errors due to the presence of noise.
- The probability of error $P\left(e \middle| a^{(i)}\right)$ is the probability that, having transmitted $a^{(i)}$, the decision variable $x(m)$ due to the presence of noise does not fall in the decision region $Z^{(i)}$.
- The error probability is averaged over all the symbols of the constellation

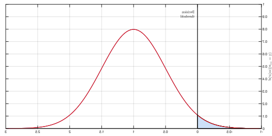
$$P_e = \frac{1}{M}\sum_{i=0}^{M-1} P\left(e \middle| a^{(i)}\right) = \lim_{N^{(s)}\to\infty} \frac{N_e^{(s)}}{N^{(s)}}$$

where $N^{(s)}$ is the number of transmitted symbols and $N_e^{(s)}$ is the number of symbol errors.

PAM error probability: Q-function

- The Q-function computes the integral of the *tail* of a Gaussian distribution.
- The probability that $x \in \mathcal{N}(m_x, \sigma_x^2)$ is smaller than $t_1 < m_x$ or larger than $t_2 > m_x$ are the integral of Gaussian tails and they are computed as





$$\left.\begin{aligned} \int_{-\infty}^{t_1} pdf(x)dx &= Q\left(\frac{m_x - t_1}{\sigma_x}\right) \\ \int_{t_2}^{+\infty} pdf(x)dx &= Q\left(\frac{t_2 - m_x}{\sigma_x}\right) \end{aligned}\right\} = Q\left(\frac{d(t_i, m_x)}{\sigma_x}\right), i = 1,2$$

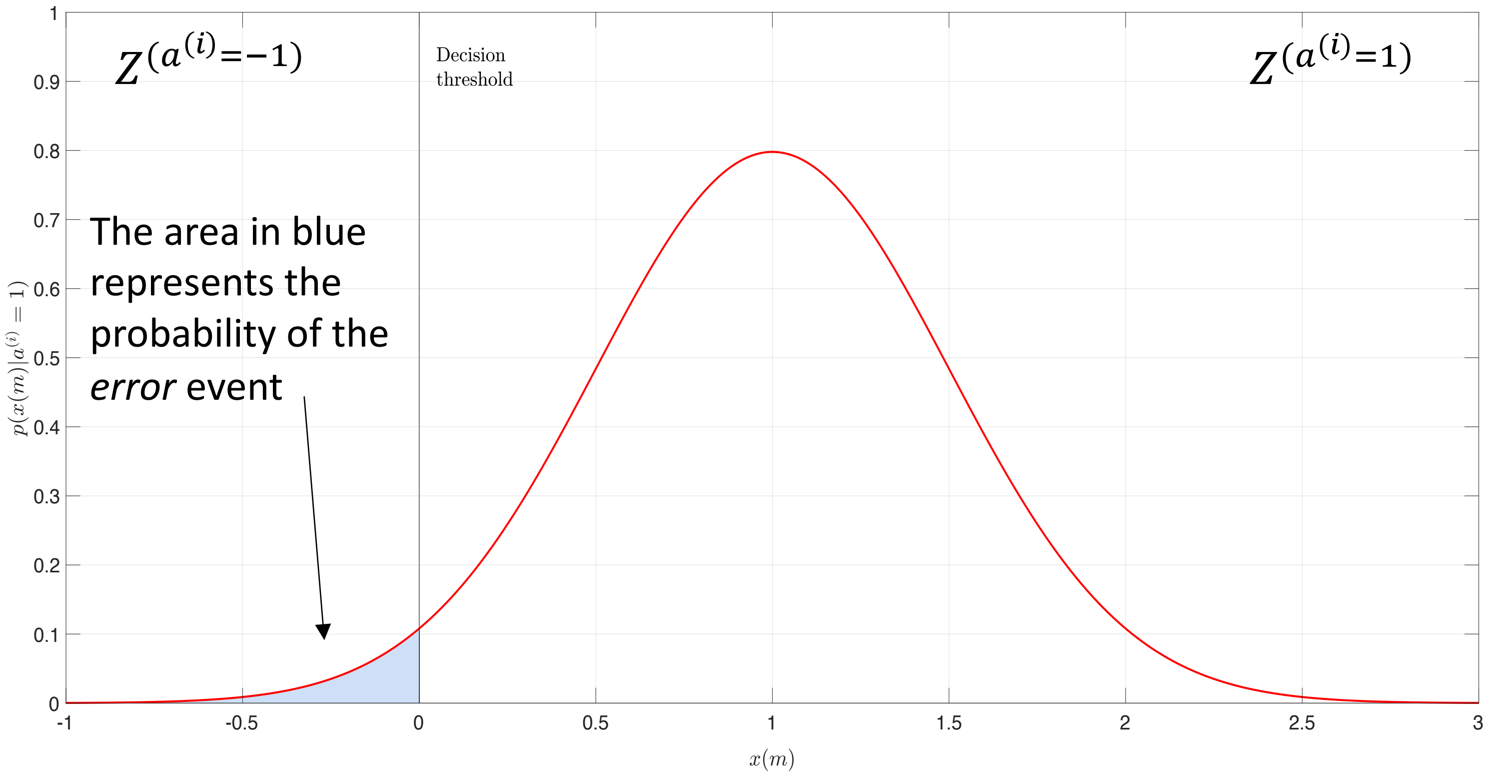
- In our case, m_x is the symbol $a^{(i)}$ and t_1 or t_2 are the detection thresholds.
- The main properties of the Q-function are

$$Q(-\infty) = 1, Q(\infty) = 0, Q(0) = 0.5, Q(-x) = 1 - Q(x).$$

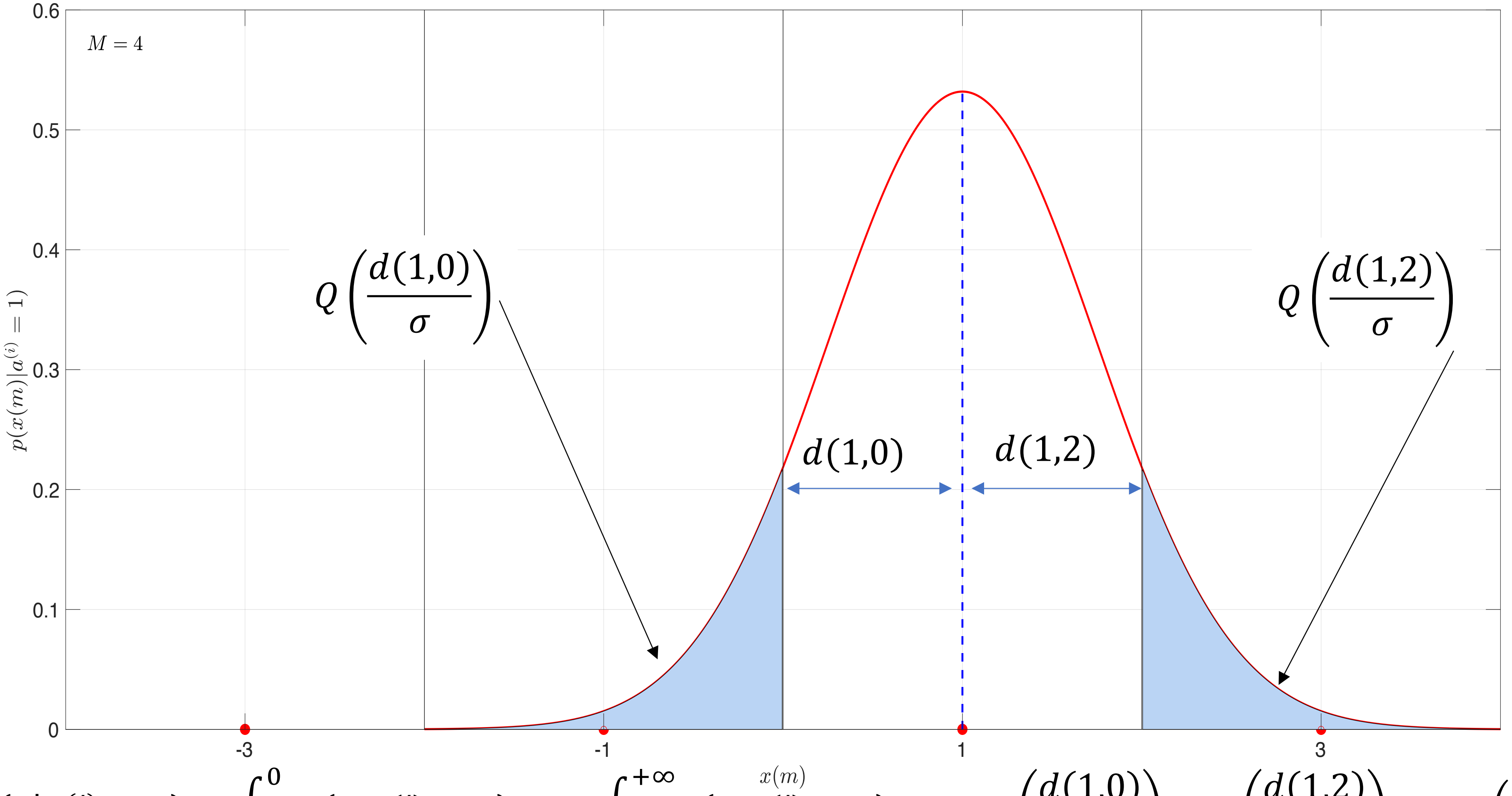
PAM error probability ($M = 2$)

- To compute $P(e|a^{(i)})$ we assume that $x(m) = a^{(i)} + n(m)$ and the probability of error is

$$P(e|a^{(i)}) = Pr\{x(m) \notin Z^{(i)}|a_m = a^{(i)}\} = Q\left(\frac{d(a^{(i)},0)}{\sigma}\right) = Q\left(\frac{1}{\sigma}\right)$$



PAM error probability ($M = 4$)



$$P(e|a^{(i)} = 1) = \int_{-\infty}^{0} p(x|a^{(i)} = 1)dx + \int_{2}^{+\infty} p(x|a^{(i)} = 1)dx = Q\left(\frac{d(1,0)}{\sigma}\right) + Q\left(\frac{d(1,2)}{\sigma}\right) = 2Q\left(\frac{1}{\sigma}\right)$$

PAM error probability

- 2-PAM

$$P_e^{(2-PAM)} = \frac{1}{2}\left(Q\left(\frac{d(-1,0)}{\sigma}\right) + Q\left(\frac{d(1,0)}{\sigma}\right)\right) = Q\left(\frac{1}{\sigma}\right)$$

- 4-PAM

$$P_e^{(4-PAM)} = \frac{1}{4}\left(Q\left(\frac{d(-3,-2)}{\sigma}\right) + Q\left(\frac{d(-1,-2)}{\sigma}\right) + Q\left(\frac{d(-1,0)}{\sigma}\right)\right.$$
$$\left. Q\left(\frac{d(1,0)}{\sigma}\right) + Q\left(\frac{d(1,2)}{\sigma}\right) + Q\left(\frac{d(3,2)}{\sigma}\right)\right) = \frac{3}{2}Q\left(\frac{1}{\sigma}\right)$$

PAM symbol error probability

- It is often useful to express the P_e in terms of E_s/N_0.
  - 2-PAM: $E_s = \frac{2^2-1}{6} = \frac{1}{2} \Longrightarrow 2E_s = 1$ and $\sigma^2 = N_0$, and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{2E_s}{N_0}}$.

$$P_e^{(2-PAM)} = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

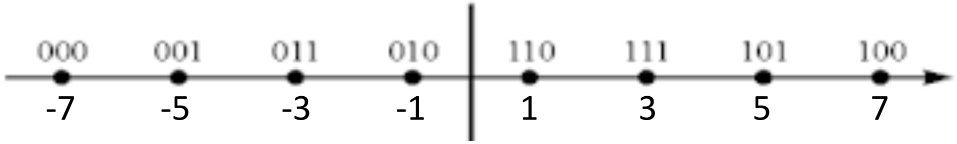
  - 4-PAM: $E_s = \frac{4^2-1}{6} = \frac{5}{2} \Longrightarrow \frac{2}{5}E_s = 1$, and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{2E_s}{5N_0}}$.

$$P_e^{(4-PAM)} = \frac{3}{2}Q\left(\sqrt{\frac{2E_s}{5N_0}}\right)$$

Gray Mapping

- The way the $m = \log_2 M$ bits are mapped on the M symbols of a PAM constellation is not unique.
- *Gray mapping* is a subset of all the possible mappings such that the string of m bits associated to adjacent symbols differ of only one bit.
- Example: $M = 8, m = 3$

$$
\begin{aligned}
a^{(0)} &= -7 \rightarrow 000 \\
a^{(1)} &= -5 \rightarrow 001 \\
a^{(2)} &= -3 \rightarrow 011 \\
a^{(3)} &= -1 \rightarrow 010 \\
a^{(4)} &= 1 \rightarrow 110 \\
a^{(5)} &= 3 \rightarrow 111 \\
a^{(6)} &= 5 \rightarrow 101 \\
a^{(7)} &= 7 \rightarrow 100
\end{aligned}
$$



$M = 8$
$m = 3$

PAM bit error probability

- To have a fair comparison, the modulation performance are expressed in terms of *bit error probability* $P_e^{(b)}$ as function of E_b/N_0.
- The energy E_b per bit is computed as the energy per symbol divided by the number of bits per symbol

$$E_b = \frac{E_s}{\log_2 M}$$

- Although one symbol carries $\log_2 M$ bits, it is reasonable to assume that in a well-designed system (*Gray mapping* and medium-high SNR) *a symbol error causes only one-bit errors*.
- If $N^{(b)}$ and $N_e^{(b)}$ are the number of transmitted bits and the number of bit errors, the bit error probability is computed as

$$P_e^{(b)} = \lim_{N^{(b)}\to\infty} \frac{N_e^{(b)}}{N^{(b)}} \approx \lim_{N^{(s)}\to\infty} \frac{N_e^{(s)}}{\log_2 M \, N^{(s)}} = \frac{1}{\log_2 M} \lim_{N^{(s)}\to\infty} \frac{N_e^{(s)}}{N^{(s)}} = \frac{1}{\log_2 M} P_e.$$

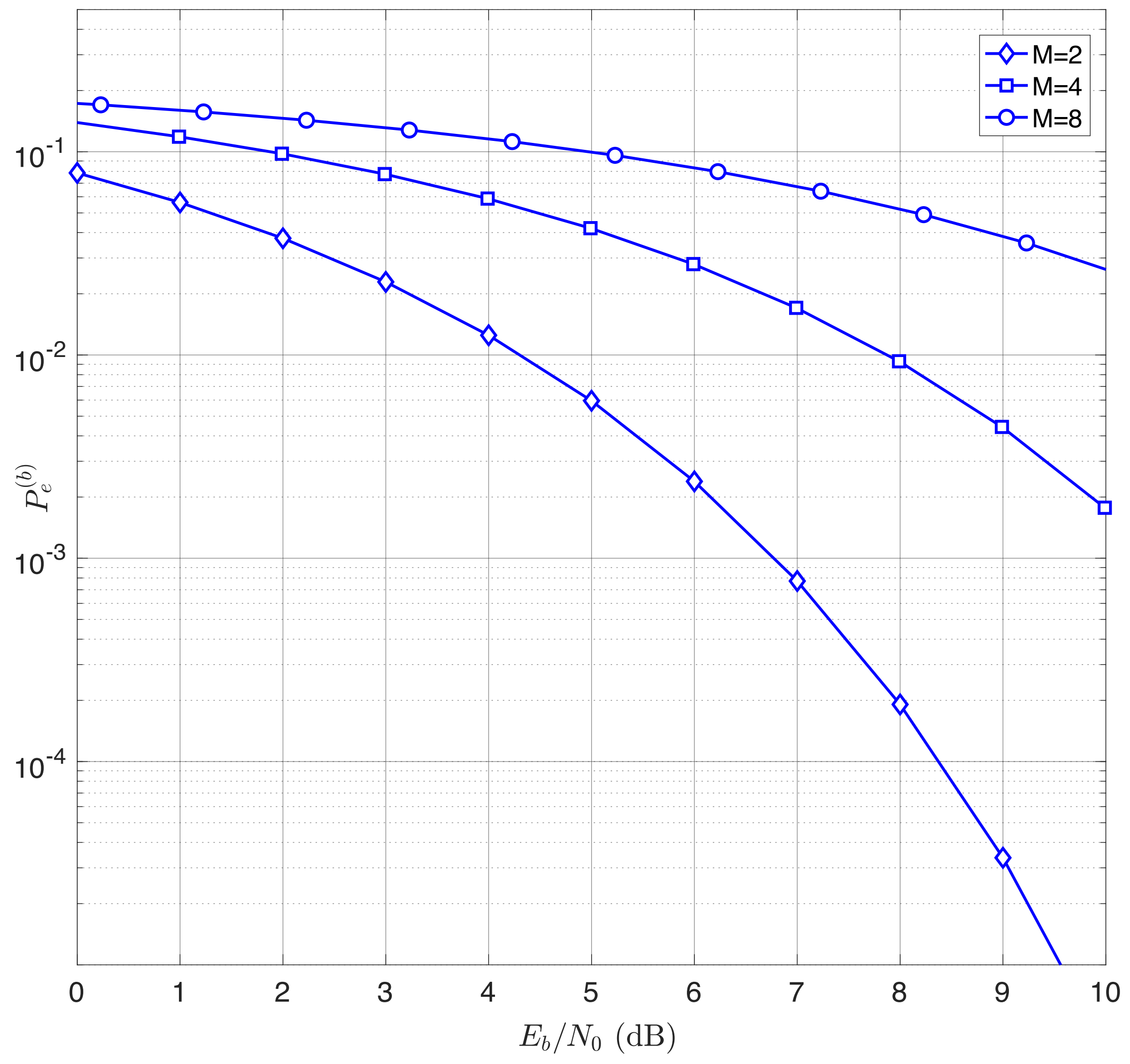
PAM bit error probability

- 2-PAM: $M = 2, m = 1$ bit per symbol $\Longrightarrow P_e^{(b)} = P_e, E_b = E_s$

$$P_e^{(2-PAM),b} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- 4-PAM: $M = 4, m = 2$ bit per symbol $\Longrightarrow P_e^{(b)} = \frac{1}{2}P_e, E_b = \frac{1}{2}E_s$

$$P_e^{(4-PAM),b} = \frac{3}{4}Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$



Quadrature Ampitude Modulation (QAM)

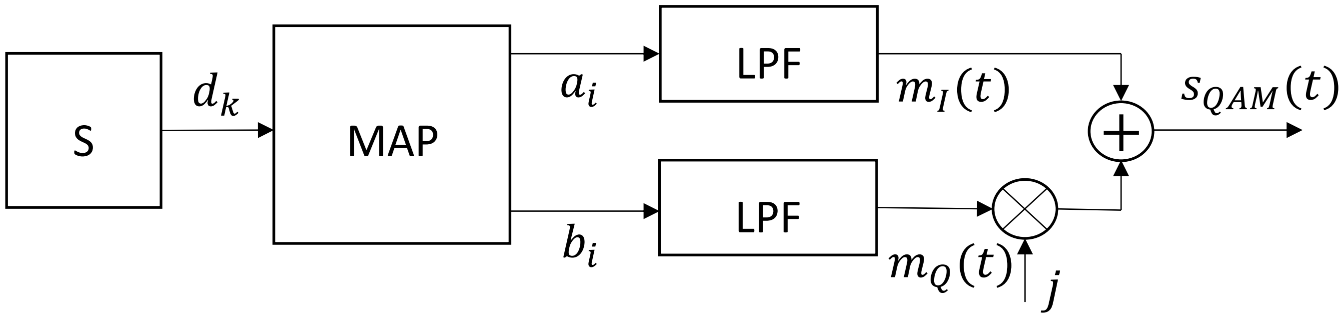
- In analog modulations, QAM is obtained by transmitting two orthogonal DSB signals $m_I(t), m_Q(t)$ and the complex envelope is

$$\tilde{s}_{QAM}(t) = m_I(t) + jm_Q(t)$$

- Quadrature PAM (QAM) is obtained exactly in the same manner by transmitting two PAM signals in quadrature $m_I(t) = \sum_i a_i g_T(t - iT)$ and $m_Q(t) = \sum_i b_i g_T(t - iT)$, with a_i, b_i PAM symbols.

$$s_{QAM}(t) = \sum_i a_i g_T(t - iT) + j \sum_i b_i g_T(t - iT)$$

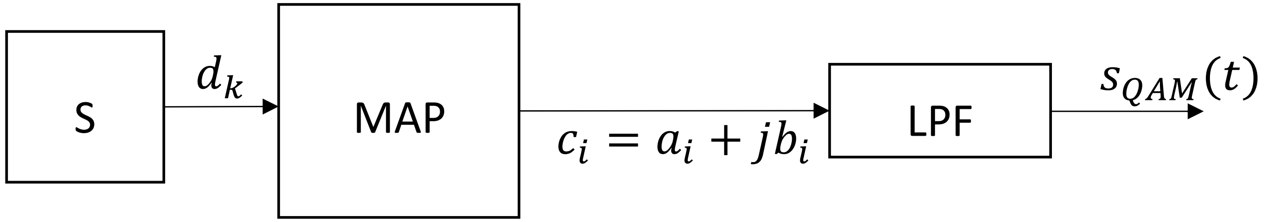
QAM complex envelope



- The complex envelope of a QAM signal is

$$s_{QAM}(t) = \sum_i (a_i + jb_i) g_T(t - iT) = \sum_i c_i g_T(t - iT)$$

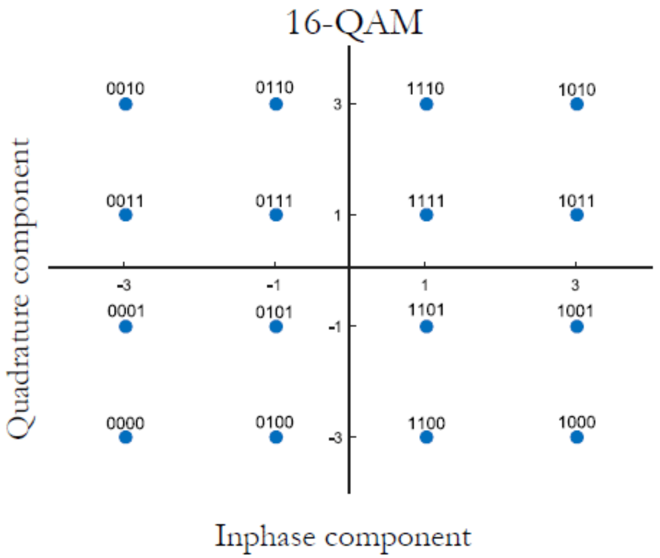
and the QAM *complex symbols* take the form $c_i = a_i + jb_i$.

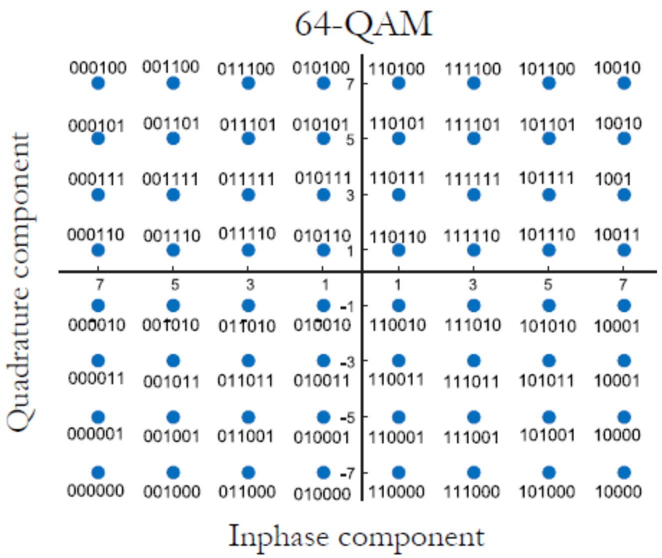


- The only difference between PAM and QAM is the use of complex symbols!

QAM symbols

- Since QAM is the combination of two orthogonal PAM with size M_{PAM}, the number of symbols is $M_{QAM} = M_{PAM}^2$ and the number of transmitted bits is $m_{QAM} = 2m_{PAM}$, always even.
  - If the QAM is obtained by combining two 4-PAMs, then it is $M_{QAM} = 16$ and $m_{QAM} = 4$, if $M_{PAM} = 8$ then $M_{QAM} = 64$ and $m_{QAM} = 6$.





Energy of a QAM symbol

- In the computation of power and energy the only difference between PAM and QAM is in the mean square value of the symbols c_m.
- Keeping in mind that the in-phase and quadrature symbols are indipendent and zero-mean, it is

$$A = E\{c_m c_m^*\} = E\{a_m^2\} + E\{b_m^2\} = 2\,\frac{M_{PAM}^2 - 1}{3} = 2\,\frac{M_{QAM} - 1}{3}$$

- QAM constellation is much more compact and requires less energy per symbol.

$$A^{(4-PAM)} = \frac{16-1}{3} = 5; \qquad A^{(4-QAM)} = 2\,\frac{4-1}{3} = 2.$$

$$A^{(16-PAM)} = \frac{256-1}{3} = 85; \qquad A^{(16-QAM)} = 2\,\frac{16-1}{3} = 10.$$

- The energy per symbol is

$$E_s = \frac{A}{2} = \frac{M_{QAM} - 1}{3}$$

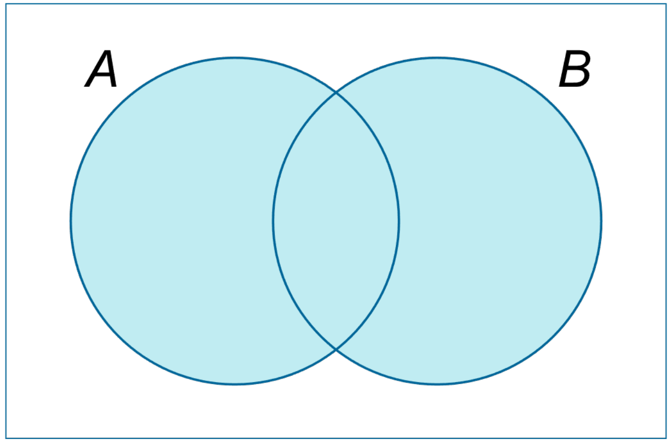
QAM error probability

- The QAM *complex* decision variable is

$$x(m) = c_m + n(m) = (a_m + jb_m) + \left(n_I(m) + jn_Q(m)\right)$$
$$= a_m + n_I(m) \; + j\left(b_m + n_Q(m)\right)$$

- The in-phase and quadrature noise components $n_I(m)$ and $n_Q(m)$ are *independent*.
- Error events depend on noise. If the *noise is independent* also *the error events on the two components are independent*.

Union bound



- The union of the sets A and B is computed as
$$A \cup B = A + B - A \cap B$$
- In terms of probabilities, it is
$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\}$$
- Since it is $\Pr\{A \cap B\} \geq 0$, a simple bound for $\Pr\{A \cup B\}$ is the *union bound*
$$\Pr\{A \cup B\} \leq \Pr\{A\} + \Pr\{B\}$$

QAM error probability – Union bound

- The event $\mathcal{E}^{(i)} = \left\{error \mid c^{(i)}\right\}$ of making an error for a given transmitted symbol $c^{(i)}$ can be computed as the union of the events
  $\mathcal{E}_I^{(i)} = \left\{error\ on\ channel\ I \middle| c^{(i)}\right\}$ and $\mathcal{E}_Q^{(i)} = \left\{error\ on\ channel\ Q \middle| c^{(i)}\right\}$
$$\mathcal{E}^{(i)} = \mathcal{E}_I^{(i)} \cup \mathcal{E}_Q^{(i)}$$
- Applying the union bound, the error probability can be *upper bounded* by
$$P\left(e \middle| c^{(i)}\right) = \Pr\left\{\mathcal{E}^{(i)}\right\} \le \Pr\left\{\mathcal{E}_I^{(i)}\right\} + \Pr\left\{\mathcal{E}_Q^{(i)}\right\}$$
- Since the noise components are independent, the error probability can be computed independently on the two quadrature channels.
- Since M-QAM is obtained as the composition of two PAM in quadrature, each with $\sqrt{M}$ symbols,
$$P_e^{(M-QAM)} < 2P_e^{(\sqrt{M}-PAM)}$$

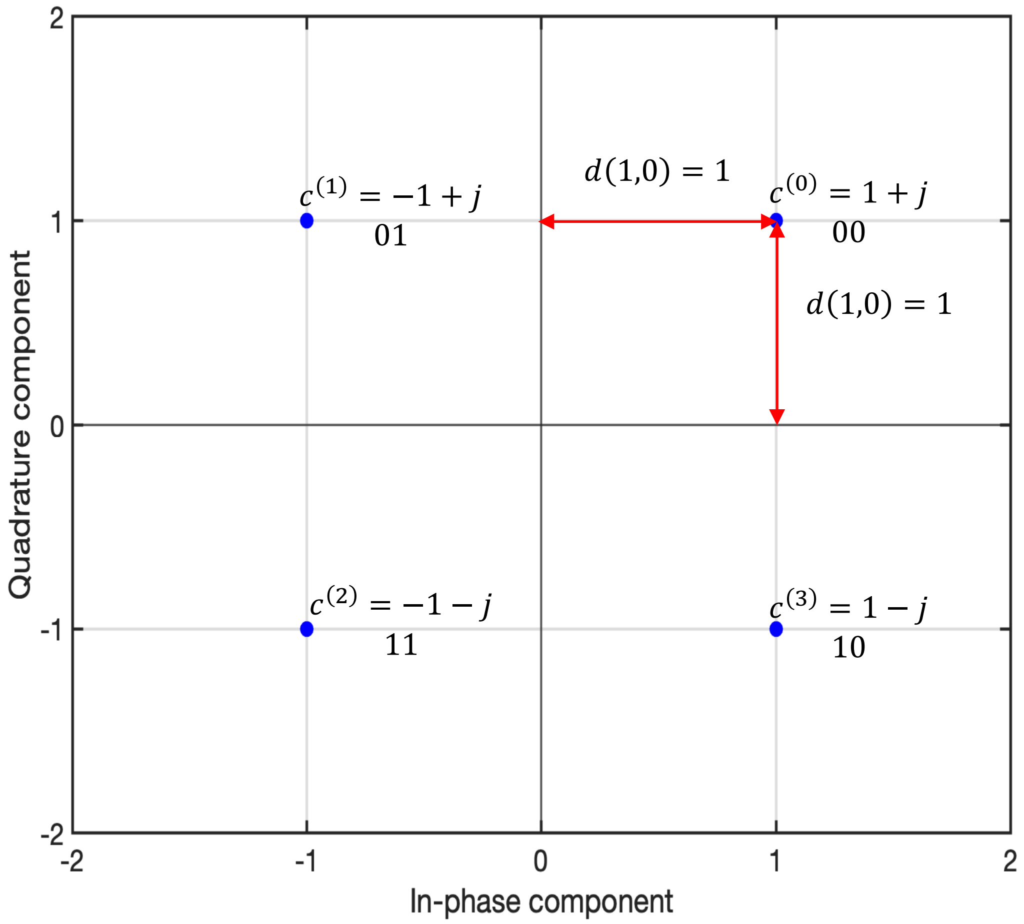
4-QAM error probability

- 4-QAM is obtained as the composition of two 2-PAM in quadrature.
- The symbol error probability is

$$P_e^{(4-QAM)} = \frac{1}{4}\sum_{i=0}^{3} P\left(e\middle|c^{(i)}\right) = P\left(e\middle|c^{(0)}\right)$$

$$P\left(e\middle|c^{(0)}\right) < Q\left(\frac{d(1\,,0)}{\sigma_{n_I}}\right) + Q\left(\frac{d(1{,}0)}{\sigma_{n_Q}}\right)$$

$$= 2Q\left(\frac{1}{\sigma}\right) = 2P_e^{(2-PAM)}$$



QAM symbol error probability

- 4-QAM: $M = 4, E_s = \frac{4-1}{3} = 1 \Longrightarrow E_s = 1$, and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{E_s}{N_0}}$
$$P_e^{(4-QAM)} < 2Q\left(\sqrt{\frac{E_s}{N_0}}\right);$$
- 16-QAM: $M = 16, E_s = \frac{16-1}{3} = 5 \Longrightarrow \frac{1}{5}E_s = 1$, and $\frac{1}{\sigma} = \sqrt{\frac{1}{\sigma^2}} = \sqrt{\frac{E_s}{5N_0}}$
$$P_e^{(16-QAM)} < 2\,\frac{3}{2}Q\left(\frac{1}{\sigma}\right) = 3Q\left(\sqrt{\frac{E_s}{5N_0}}\right).$$

M-QAM bit error probability

- The total number of M-QAM transmitted bits is the sum of the number of bits transmitted on the in-phase and quadrature channels, twice as many the bits transmitted by a PAM with $\sqrt{M}$ symbols
$$m_{QAM} = \log_2 M = 2\log_2\sqrt{M}$$
- Accordingly, $P_e^{(M-QAM,b)}$ can be computed as

$$P_e^{(M-QAM,b)} = \lim_{N^{(b)}\to\infty} \frac{N_e^{(b)}}{N^{(b)}} \approx \lim_{N^{(s)}\to\infty} \frac{N_e^{(s)}}{\log_2 M \, N^{(s)}} = \frac{1}{\log_2 M} \lim_{N^{(s)}\to\infty} \frac{N_e^{(s)}}{N^{(s)}} = \frac{1}{\log_2 M} P_e^{(M-QAM)}$$

$$= \frac{1}{2\log_2 \sqrt{M}} 2P_e^{(\sqrt{M}-PAM)} = P_e^{(\sqrt{M}-PAM,b)}$$

- Although M-QAM carries twice as many bits, the bit error probability $P_e^{(M-QAM,b)}$ is equal to the bit error probability of the corresponding $\sqrt{M} - PAM$ modulation.

QAM bit error probability

- 4-QAM:

$$\frac{1}{\sigma^2} = \frac{E_s}{N_0} = \frac{2E_b}{N_0}$$

$$P_e^{(4-QAM,b)} = P_e^{(2-PAM,b)} = Q\left(\frac{1}{\sigma}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- 16-QAM

$$\frac{1}{\sigma^2} = \frac{E_s}{5N_0} = \frac{4E_b}{5N_0}$$

$$P_e^{(16-QAM,b)} = P_e^{(4-PAM,b)} = \frac{3}{4}Q\left(\frac{1}{\sigma}\right) = \frac{3}{4}Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$