Foundations of Elliptic Curves Cryptosystems

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ECC in a nutshell

- Mid-1980s
- Same level of security of RSA and DL-system with considerably shorter operands
  - 160 – 256-bit vs 1024 – 3072 bit
- GDLP in ECC
  - DHKE and DL-systems can be redefined in ECCs
- Performance advantages over RSA and DL-systems
  - However, RSA with short public parameter is faster than ECC

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Elliptic Curves Cryptosystem

HOW TO COMPUTE WITH ECC

How to Compute with ECC

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- ECC is based on GDLP, so we have to accomplish two tasks
  - Task 1: Define an elliptic-curve-based cyclic group
    - Task 1.1: Define a set of elements
    - Task 1.2: Define the group operations
  - Task 2: Show that DLP is hard in that group

Polynomials and curves

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- We can form curves from polynomial equations
  - A curve is the set of points (x, y) which are the solutions of the equations
- Examples (in $\mathbb{R}$)
  - $x^2 + y^2 = r^2$ is a circle
  - $a \cdot x^2 + b \cdot y^2 = c$ is an ellipse

ECC – definition

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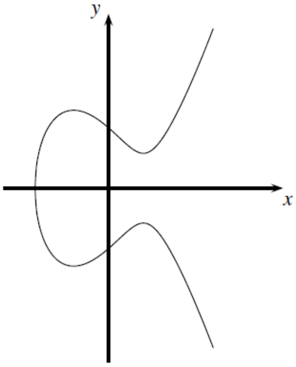
- We consider GF(p) = {0, 1, …. p – 1 }
  - Intuitively, GF is a finite set where you can add, subtract, multiply and invert
- Definition
  - The elliptic curve over $\mathbb{Z}_p$, p > 3, is the set of points $(x, y) \in \mathbb{Z}_p$ which fulfils

$$y^2 \equiv x^3 + a \cdot x + b \bmod p$$

  - together with an imaginary point of infinity $\mathcal{O}$, where $a, b \in \mathbb{Z}_p$, and $4 \cdot a^3 + 27 \cdot b^2 \neq 0 \bmod p$
    - The curve is non-singular (no vertices, no self-intersections)

Group elements (task 1.1)

- Plotting in $\mathbb{R}$ for the sake of illustration
- Observations
  - 1, 3 intersections with x axis
  - Symmetric with respect to x axis
- Task 1.1 solved
  - Group elements are the points of the curve



$y^2 = x^3 - 3x + 3$ over $\mathbb{R}$

Group operations (task 1.2)

- We call "addition" the group operation and denote it by "+" an operation that takes two points P = (x_1, y_1) and Q = (x_2, y_2) and produces a third point R = (x_3, y_3) as a result

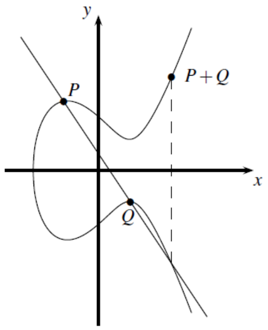
$$P + Q = R$$

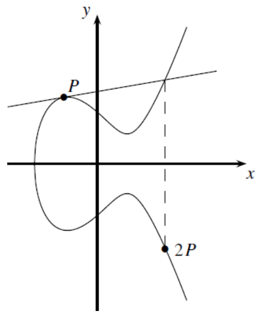
- Geometrical interpretation of + in $\mathbb{R}$
  - Point Addition P + Q, $Q \neq P$
  - Point Doubling P + P

Group operations (task 1.2)

- Geometrical interpretation of "+" operation: the *tangent-and-chord* method

Point addition **Point doubling**





Group operations (task 1.2)

- Geometrical interpretation of +
  - The tangent-and-chord method only uses the four standard operations
- FACT
  - If addition + is defined this way, the group points fulfil most of necessary conditions of a group: closure, associativity, existence of an identity element and existence of an inverse

Group operations (task 1.2)

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- Analytic expressions of Point Addition and Point Doubling
  - $x_3 \equiv s^2 - x_1 - x_2 \bmod p$
  - $y_3 \equiv s \cdot (x_1 - x_3) - y_1 \bmod p$

  where
  - $s \equiv \frac{y_2 - y_1}{x_2 - x_1} \bmod p$ if P ≠ Q (point addition)
  - $s \equiv \frac{3 \cdot x_1^2 + a}{2 \cdot y_1} \bmod p$ if P = Q (point doubling)
  - with s the slope of chord/tangent

Point at infinity (task 1.2)

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- An identity (neutral) element $\mathcal{O}$ is still missing
  - $\forall P \in E\colon\ P + \mathcal{O} = P$
- There exists not such a point on the curve
- Thus, we define $\mathcal{O}$ as the point at infinity
  - Located at “plus” infinity towards the y-axis or at “minus” infinity towards the y-axis
- Now, we also define –P (inverse)
  $P + (-P) = \mathcal{O}$

Group operations (task 1.2)

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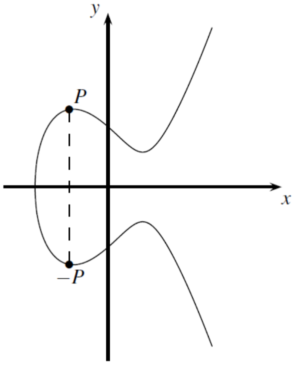
- Inverse of a point P on an elliptic curve
  - Apply the tangent-and-chord method
- In ECC over GF(p)
  - Given P = (x, y) then –P = (x, p – y)



Elliptic Curves Cryptosystem

BUILDING DLP ON EC

A useful theorem

- THM
  - The points on an elliptic curve together with $\mathcal{O}$ have cyclic subgroups. Under certain conditions all points on an elliptic curve form a cyclic group
    - A primitive element must exist such that its powers generate the entire group

Example (1/2)

- E: $y^2 \equiv x^3 + 2 \cdot x + 2 \bmod 17$
  - #E (order of E) = 19
  - P = (5, 1) primitive element
  - "Powers" of P
    - 2P = (6, 3) – point doubling
    - 3P = (10, 6) – point addition 2P + P
    - 4P = (3, 1)
    - 5P = (9, 16)
    - 6P = (16, 13)
    - 7P = (0, 6)
    - 8P = (13, 7)
    - 9P = (7, 6)
    - 10P = (7, 11)

11P = (13, 10)
12P = (0, 11)
13P = (16, 4)
14P = (9, 1)
15P = (3, 16)
16P = (10, 11)
17P = (6, 14)
18P = (5, 16)
19P = $\mathcal{O}$ = #E·P

Example (2/2)

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- The cyclic structure becomes visible
  - 20 P = 19P + P = O + P = P
  - 21P = 19P + 2P = 2P
  - …
- Furthermore
  - 19P = O, thus 18P + P = O, then 18P is the inverse of P and vice versa
    - Verification
      - P = (5, 1), 18P = (5, 16)
      - $x_p = x_{18P} = 5$
      - $y_p + y_{18p} \equiv 0 \bmod 17$

Hasse's Theorem

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- Hasse's theorem
  - Given an elliptic curve E modulo p, the number of points on the curve is denoted by #E and is bounded by:
    $$p + 1 - 2\sqrt{p} \leq \#E \leq p + 1 + \sqrt{p}$$
  - The number of points is roughly in the range of p (Hasse's bound)
  - Example – If you need an EC with 2^{160} points, you have to use a prime p of about 160 bit

ECDLP – point multiplication

- Elliptic Curve Discrete Logarithm Problem (ECDLP)
  - Given is an elliptic curve E. We consider a primitive element P and another element T. The DL problem is finding the integer d, where $1 \le d \le \#E$, such that:

$$\underbrace{P + P + \dots + P}_{\text{d times}} = d \cdot P = T$$

  - d is the private key, T is the public key
  - Point multiplication $\stackrel{\text{def}}{=}$ $T = d \cdot P$

Square-and-multiply

- Point multiplication is analogue to exponentiation in multiplicative groups $(\mathbb{Z}_p^*, \times)$
- We can adopt the square-and-multiply algorithm
- Example
  - $26P = (11010)_2 P = (d_4 d_3 d_2 d_1 d_0)2\ P$
  - Step
    - #0 P = **1**P — init setting, bit processed: $d_4 = 1$
    - #1a P+P = 2P = **10**P — DOUBLE, bit processed: d_3
    - #1b 2P+P = 3P = 10P+1P = **11**P — ADD, since $d_3 = 1$
    - #2a 3P+3P = 6P = 2(11P) = **110**P — DOUBLE, bit processed: d_2
    - #2b — no ADD, since $d_2 = 0$
    - #3a 6P+6P = 12P = 2(110P) = **1100**P — DOUBLE, bit processed: d_1
    - #3b 12P+P = 13P = 1100P+1P = **1101**P — ADD, since $d_1 = 1$
    - #4a 13P+13P = 26P = 2(1101P) = **11010**P — DOUBLE, bit processed: d_0
    - #4b — no ADD, since $d_0 = 0$

EC Cryptosystem

- Private key: d
- Public key: T
- Geometrical interpretation of ECDLP
  - Given P, we compute 2P, 3P,…, $d \cdot P = T$, we actually jump back and forth on the EC
  - Given the starting point P and the final point T (public key), the adversary has to figure out how often we "jumped" on the EC