Performance evaluation of Computer Systems and Networks – 15/01/2020

Exercise 1

The JPDF of a couple of continuous RVs X and Y is uniform in a triangle of the cartesian plane whose vertexes are $(-1;0),(+1;0),(0;+1)$. Define RV $Z=Y-X$.

1) State what values Z may assume, and compute its PDF;
2) Compute the mean value and the median of Z;
3) Compute the PDF of $W=1/|Z|$. Is W a heavy-tailed RV? Justify your answer

Exercise 2

Consider a distributed system, where a central entity is connected to N independent computing agents. Each agent senses of a portion of space, and reports the sensed value to the central entity, which then elaborates an aggregate value by collating all the reports from the agents. The interaction between the components is as follows:

The **central entity**:

a) issues a "compute" request, which reaches all the agents simultaneously (in zero time);
b) waits until **all** the N sensed values have come in;
c) computes an aggregate value, taking an exponential time whose mean is $\frac{1}{\mu}$, and starts over.

Each of the $\boldsymbol{N}$ **agents**:

a) is initially idle, waiting for a "compute" request;
b) when it receives the "compute" request, it performs the sensing, which takes an exponential time whose mean is $\frac{1}{\lambda}$;
c) reports the sensed value to the central entity (in zero time), and then starts over.

The candidate should:

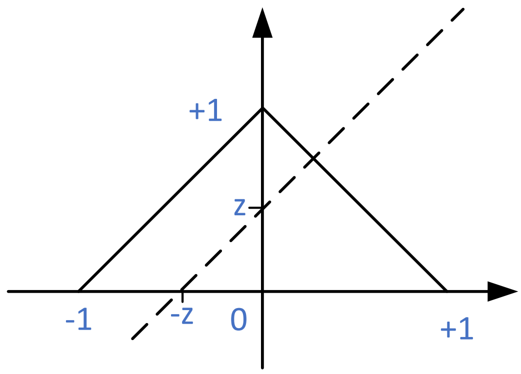
1) Model the above system as a queueing system and draw its CTMC.
2) Compute the steady-state probabilities and the stability condition.
3) Compute the utilization of the central entity, and explain how it depends on N,λ,μ
4) Compute the probability that it takes less than α for all the N sensed values to come in, starting from a compute request.

Assume now that, instead of sending the "compute" request to all agents simultaneously, the central entity polls the agents *sequentially*, i.e., it schedules agent $j+1$ *right after* agent j has completed its sensing. Assume that polling takes zero time.

5) Draw the resulting CTMC and answer point 4) again, assuming as a reference instant the time when agent 1 is polled.

Performance evaluation of Computer Systems and Networks – 15/01/2020

Exercise 1 – Solution



The JPDF is constant by definition, hence normalization on the triangle area implies that $f(x,y) = 1$.

RV Z is equal to value z when $Y - X = z$, i.e., for all the points on the dashed line in the figure. The values that Z may assume are therefore all the ordinates such that the dashed line intersects the triangle, i.e., $[-1; +1]$.

It is $P\{Z \leq z\} = P\{Y - X \leq z\}$. The latter is the probability that a point belongs to the triangle defined by the intersection of the original triangle with the dashed line, i.e., the one whose vertexes are $(-z; 0)$, $(+1; 0)$, $((1-z)/2; (1+z)/2)$. Since the JPDF is uniform and equal to one, this is just the area of the smaller triangle (normalized to the area of the larger triangle, which is however equal to one). That area is:

$$P\{Z \leq z\} = P\{Y - X \leq z\} = \frac{1}{2} \cdot \left[(1+z) \cdot \frac{(1+z)}{2}\right] = \frac{(1+z)^2}{4}$$

The PDF of Z is obtained by differentiating the above:

$$f_Z(z) = \frac{1+z}{2}$$

The mean value of Z is:

$$E[Z] = \int_{-1}^{+1} z \cdot \frac{1+z}{2} dz = \left[\frac{z^2}{4} + \frac{z^3}{6}\right]_{-1}^{+1} = \frac{1}{3}$$

The median of Z is the solution to:

$$P\{Z \leq z_{0.5}\} = \frac{1}{2}$$

Which is, after a few straightforward computations, $z_{0.5} = \sqrt{2} - 1$.

RV $|Z|$ may take on values in $[0; +1]$. Therefore, RV $W = 1/|Z|$ may take on values in $[1; +\infty]$, where we have $P\{W \leq w\} = P\{1/|Z| \leq w\} = P\{|Z| \geq 1/w\} = P\{Z \geq 1/w\} + P\{Z \leq -1/w\}$.

Therefore, it is:

$$P\{W \leq w\} = 1 - \frac{\left(1 + \frac{1}{w}\right)^2}{4} + \frac{\left(1 - \frac{1}{w}\right)^2}{4} = 1 - \frac{1}{w}$$

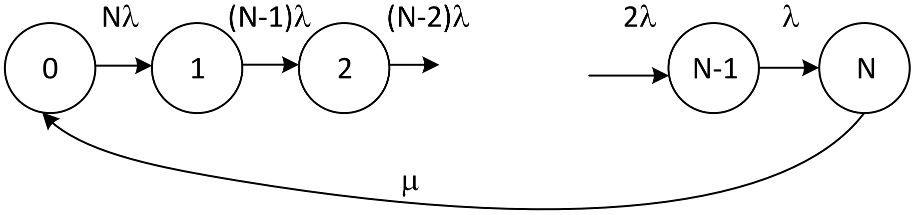
Therefore, it is $f_W(w) = \frac{1}{w^2}$. This is a heavy-tailed RV. In fact,

$$\forall \lambda > 0, \quad \lim_{x \to \infty} e^{\lambda x} \cdot \left(1 - F_W(x)\right) = \lim_{x \to \infty} \frac{e^{\lambda x}}{x} = \infty$$

Performance evaluation of Computer Systems and Networks – 15/01/2020

Exercise 2 – Solution

1) The system is a finite-population one with batch services. The CTMC is the following.



The system is always stable, since it has a finite number of states.

2) From the CTMC it is easy to write down equilibrium equations:

$$p_o \cdot N\lambda = p_N \cdot \mu$$
$$p_1 \cdot (N-1)\lambda = p_o \cdot N\lambda$$
$$\dots$$
$$p_j \cdot (N-j)\lambda = p_{j-1} \cdot (N-j+1)\lambda$$
$$\dots$$
$$p_N \cdot \mu = p_{N-1} \cdot \lambda$$

From which it is immediate to obtain:

$$p_j = p_0 \cdot \frac{N}{N-j} \quad (0 \le j < N)$$
$$p_N = p_o \cdot N\frac{\lambda}{\mu}$$

We impose the normalization condition, and obtain:

$$p_0 \cdot N \cdot \left\{\frac{\lambda}{\mu} + \sum_{j=0}^{N-1} \frac{1}{N-j}\right\} = 1$$
$$p_0 \cdot N \cdot \left\{\frac{\lambda}{\mu} + \sum_{i=1}^{N} \frac{1}{i}\right\} = 1$$
$$p_0 = \frac{1}{N\left(H_N + \frac{\lambda}{\mu}\right)}$$

Where $H_N = \sum_{i=1}^{N} \frac{1}{i}$. From the above, we obtain:

$$p_j = \frac{1}{(N-j)\left(H_N + \frac{\lambda}{\mu}\right)} \quad (0 \le j < N)$$
$$p_N = \frac{\frac{\lambda}{\mu}}{H_N + \frac{\lambda}{\mu}}$$

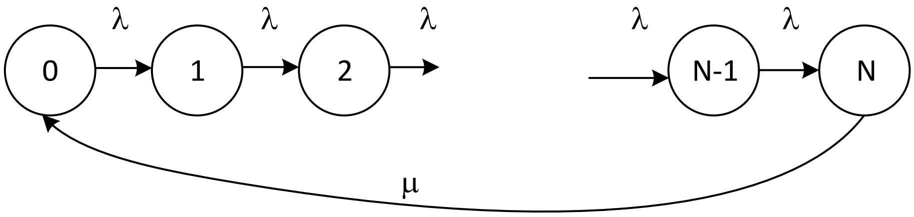
3) The central entity is idle except in state N (when it is computing the aggregate value). The answer is therefore:

$$p_{idle} = p_N = \frac{\frac{\lambda}{\mu}}{H_N + \frac{\lambda}{\mu}}$$

The utilization increases with λ, since when sensing is faster the central entity will spend comparatively more time doing its own computations. For the same reason, it decreases with μ. The utilization also decreases with N, since N influences the time before all the sensed values are in.

4) If X_i is the RV that models the sensing time at agent i, the time it takes for the central entity to obtain all the responses is $Z = \max_i\{X_i\}$. Since all agents are IID, It is $P\{Z \leq \alpha\} = (P\{X_i \leq \alpha\})^N$, i.e. $F_Z(\alpha) = \left(1 - e^{-\lambda\alpha}\right)^N$

5) In this case the CTMC is the following:



The time it takes to get all the sensed values in is the *sum* of N IID exponentials, which is an N-stage Erlang. Therefore,

$$F_Z(\alpha) = 1 - \sum_{k=0}^{N-1} e^{-\lambda\alpha} \frac{(\lambda\alpha)^k}{k!}$$

PECSN, 4/2/2020

Exercise 1

You have two independent unfair coins. The probability of *heads* are p and q, respectively, for coin 1 and 2. You play the following game: flip coin 1 until *heads* come up, and then flip coin 2 once. If coin 2 is *heads,* the game ends, otherwise you resume flipping coin 1, and so on.
Let T be the RV that counts the number of flips of coin 1 before the game ends.

1) Compute the PMF and the CDF of RV T.
2) Compute the mean value of T and its variance

Exercise 2

A network server handles *job lists*, which are sent from the outside world at a constant rate λ, with exponential interarrival times. A list includes an arbitrary number of *jobs*. The probability that a list includes n jobs is equal to π_n. The server processes individual jobs in FIFO order. A job's service time is an exponential RV with a mean equal to $\frac{1}{\mu}$. The server only accepts a new job list when it is idle.

1) model the system and draw the CTMC;
2) compute the steady-state probabilities and the stability condition; provide an interpretation for your findings.
3) compute the probability that a job list is rejected;
4) compute the *distribution* of the job's response time;
5) compute the server utilization.

Exercise 1 – Solution

The model is no different from the following: you *always* flip both coins simultaneously, and you win when you get two heads simultaneously. Since the coins are independent, the probability of getting two heads simultaneously is pq, hence T is a geometric RV with a probability of success pq (to be more specific, the version of a geometric RV that counts the number of *trials* before the first success). For the latter, we have:

$$p_k = P\{T = k\} = (1 - pq)^{k-1} \cdot pq$$
$$F(k) = P\{T \leq k\} = 1 - (1 - pq)^k$$
$$E[T] = \frac{1}{pq}$$
$$Var(T) = \frac{1 - pq}{(pq)^2}$$

To compute the mean and variance, one may resort to the PGF:

$$G(z) = \sum_{k=1}^{+\infty} z^k \cdot (1 - pq)^{k-1} \cdot pq = \frac{zpq}{1 - z + zpq}$$

And use the well-known relationships $E[T] = G'(1)$, $Var(T) = G"(1) + G'(1) - G'(1)^2$

If one misses the above trick, the PMF and CDF can still be found by taking an alternative (though considerably longer) route. Start from small values of k and compute the PMF and CDF manually:

- $k = 1$: $p_1 = pq$ and $F(1) = pq$
- $k = 2$:

$$p_2 = (1 - p)pq + p(1 - q)pq = pq(1 - pq)$$
$$F(2) = F(1) + p_2 = 1 - (1 - pq)^2$$

One can then observe the following general relationships:

$$p_k = P\{T = k | T > k - 1\} \cdot P\{T > k - 1\} = pq \cdot [1 - F(k - 1)]$$
$$F(k) = F(k - 1) + p_k = pq + (1 - pq) \cdot F(k - 1)$$

From which one gets:

$$p_3 = pq \cdot [1 - F(k - 1)] = pq \cdot (1 - pq)^2$$
$$F(3) = pq + (1 - pq) \cdot F(2) = 1 - (1 - pq)^3$$

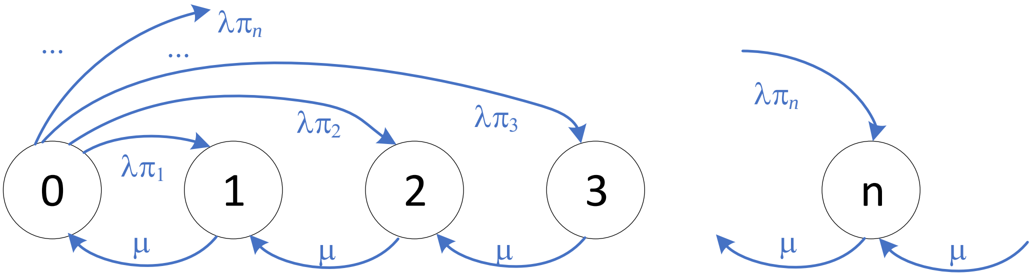
At this point, it is easy to venture that the general forms for the PMF and CDF should be:

$$p_k = pq \cdot (1 - pq)^{k-1}$$
$$F(k) = 1 - (1 - pq)^k$$

The above thesis can be proved by induction, exploiting the two general relationships above.

Exercise 2 - Solution

1) The CTMC is as follows. Normalization must hold in the π_n probabilities, i.e., $\sum_{n=1}^{+\infty} \pi_n = 1$.



2) The steady state probabilities are computed by writing down the global equilibrium equations:

$$p_0 \cdot \lambda = p_1 \cdot \mu$$
$$p_j \cdot \mu = p_{j+1} \cdot \mu + p_0 \cdot \lambda \cdot \pi_j, \quad j \geq 1$$

From the above one readily obtains:

$$p_j = p_0 \cdot \frac{\lambda}{\mu} \cdot \sum_{i=j}^{+\infty} \pi_i\,, \quad j \geq 0$$

By imposing the normalization condition $\sum_{j=0}^{+\infty} p_j = 1$, one obtains the following:

$$p_0 + \sum_{j=1}^{+\infty} \left(p_0 \cdot \frac{\lambda}{\mu} \cdot \sum_{i=j}^{+\infty} \pi_i \right) = 1$$
$$p_0 \left[1 + \frac{\lambda}{\mu} \cdot \sum_{j=1}^{+\infty} \left(\sum_{i=j}^{+\infty} \pi_i \right) \right] = 1$$
$$p_0 \left[1 + \frac{\lambda}{\mu} \cdot \sum_{j=1}^{+\infty} j \cdot \pi_j \right] = 1$$
$$p_0 \left[1 + \frac{\lambda}{\mu} \cdot E[\Pi] \right] = 1$$
$$p_0 = \frac{1}{1 + \frac{\lambda}{\mu} \cdot E[\Pi]}$$

The stability condition is that $E[\Pi]$ should be finite. Stability does not depend on λ or μ, since the server does not accept job lists unless idle, hence the rate of services and arrivals are immaterial. Moreover, we get:

$$p_j = \frac{\frac{\lambda}{\mu} \cdot \sum_{i=j}^{+\infty} \pi_i}{1 + \frac{\lambda}{\mu} \cdot E[\Pi]} = \frac{1 - F_\Pi(j-1)}{\frac{\mu}{\lambda} + E[\Pi]}, \quad j \geq 1$$

3) The probability that a job list is rejected is:

$$p_L = 1 - p_0 = \frac{E[\Pi]}{\frac{\mu}{\lambda} + E[\Pi]}$$

4) Whenever a list arrives, the system is empty by hypothesis. Therefore, the response time of a job arriving in a list that has $n-1$ jobs ahead of it will be an Erlang with n stages. The probability that an arriving job is the n^{th}in a list is:

$$\sum_{j=n}^{+\infty} \pi_j \cdot \frac{1}{j}$$

Since i) that list must include at least n jobs, and ii) that job must be the n^{th}in that list (while it could be any other with the same probability). Therefore, the required answer is:

$$P\{R \leq t\} = \sum\nolimits_{n=1}^{+\infty} P\{R \leq t | n^{th}\, jobs\} \cdot P\{n^{th}\, job\} =$$

$$\sum\nolimits_{n=1}^{+\infty} \left\{ \left[1 - \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \right] \cdot \left[\sum_{j=n}^{+\infty} \pi_j \cdot \frac{1}{j} \right] \right\}$$

5) The server utilization is equal to the loss probability:

$$U = p_L = 1 - p_0 = \frac{E[\Pi]}{\frac{\mu}{\lambda} + E[\Pi]}$$

PECSN 25/02/2020

Name and surname: ____ *Matricola:* ________________

I will sit for the oral examination in: ☐ *February* ☐ *April (*)*

The above decision <u>*cannot*</u> *be changed later.*

(*) If you are ineligible, your written test will be discarded.

Exercise 1

Consider function:

$$f(x) = \begin{cases} -\alpha \cdot x^2 + 1 & x \in \left[-\frac{1}{\sqrt{\alpha}}; +\frac{1}{\sqrt{\alpha}}\right] \\ 0 & otherwise \end{cases}$$

1) Find α such that the above one is a PDF.
2) Let X be a RV having $f(x)$ as a PDF. Compute its mean, median and variance.
3) Let $Y = \log(X + \frac{1}{\sqrt{\alpha}})$. Compute $f_Y(y)$.

Exercise 2

A smart-service industry operates a fleet of N identical *robots*, each one composed of m electro-mechanic subsystems. When a robot is switched on, the mean-time-to-failure of subsystem j is exponential, with a rate λ_j, $1 \leq j \leq m$. As soon as one subsystem fails, the robot must be switched off, removed from operation and repaired. Repairs occur in a maintenance bay, which has a FIFO queue, and they take an exponential time whose mean is $1/\mu$, regardless of which subsystem is being repaired. After the repairment, the robot is switched on and put back into operation.

1) Model the system and draw its CTMC (or *transition-rate diagram*)
2) Compute the probability that there are n operating robots at the steady state, and the stability condition under which a steady state is reached.
3) Compute the mean number of operating robots.
4) Compute the mean downtime for a robot following a failure of one of its subsystems.

Exercise 1 – Solution

1) One should compute α such that:

$$\int_{-\frac{1}{\sqrt{\alpha}}}^{+\frac{1}{\sqrt{\alpha}}}(-\alpha\cdot x^2+1)dx=[\frac{-\alpha\cdot x^3}{3}+x]_{-\frac{1}{\sqrt{\alpha}}}^{+\frac{1}{\sqrt{\alpha}}}=1$$

After a few straightforward manipulations, one gets $\alpha = 16/9$.

2)Function $f(x)$ is even, hence its mean and median are null. From the above, one obtains

$$Var(X)=E[X^2]=\int_{-\frac{3}{4}}^{+\frac{3}{4}}x^2\left(-\frac{16}{9}\cdot x^2+1\right)dx=\frac{3}{80}$$

3) $Y=\log(X+\frac{3}{4})$. Therefore, Y is defined in $[-\infty;\log(3/2)]$, and

$$F_Y(k)=P\{Y\le k\}=P\left\{\log\left(X+\frac{3}{4}\right)\le k\right\}=P\left\{X+\frac{3}{4}\le e^k\right\}=P\left\{X\le e^k-\frac{3}{4}\right\}=F_X(e^k-\frac{3}{4})$$

It is:

$$F_X(x)=\int_{-\frac{3}{4}}^{x}\left(-\frac{16}{9}\cdot y^2+1\right)dy=[\frac{-16}{27}y^3+y]_{-\frac{3}{4}}^{x}=\left[-\frac{16}{27}x^3+x\right]-\left[\frac{16}{27}\cdot\frac{27}{64}-\frac{3}{4}\right]=-\frac{16}{27}x^3+x+\frac{1}{2}$$

Therefore:

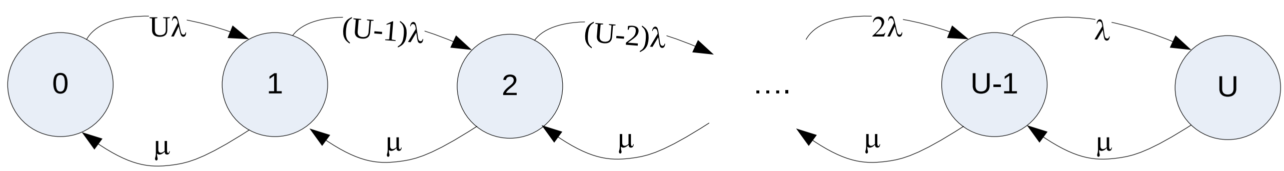
$$F_Y(k)=-\frac{16}{27}\left(e^k-\frac{3}{4}\right)^3+\left(e^k-\frac{3}{4}\right)+\frac{1}{2}=-\frac{16}{27}\left(e^{3k}-\frac{27}{64}-\frac{9}{4}e^{2k}+\frac{27}{16}e^k\right)+e^k-\frac{1}{4}$$
$$=-\frac{16}{27}e^{3k}+\frac{4}{3}e^{2k}$$

From the above, we get:

$$f_Y(k)=\frac{dF_Y(k)}{dk}=-\frac{16}{9}e^{3k}+\frac{8}{3}e^{2k}=\frac{8}{3}e^{2k}\left(1-\frac{2}{3}e^k\right)$$

Exercise 2 - Solution

1) The system is a finite population one, the population being N robots. Call $\lambda=\sum_{j=1}^{m}\lambda_j$. When all robots are operating, the arrival (i.e., subsystem failure) rate is thus $N\cdot\lambda$ due to independence. However, whenever one subsystem fails, the robot it belongs to is halted, hence there will be $N-1$ operating robots, hence an arrival rate of $(N-1)\cdot\lambda$, etc.
All in all, the CTMC can be deduced from the one of a finite-population system having N individuals (i.e., products) and an arrival rate λ. This is also consistent with the view that a robot can individually fail with a rate λ, since it contains independent subsystems whose failure rate are λ_j (recall the well-known theorem about the minimum of independent exponential distributions). We use the number of *repaired* robots as a state characterization (substitute N for U in the graph below):



2, 3,4) Once the above interpretation is understood, the computation for the above results can be found on my handouts (see “finite-population systems”). Here are the results:

- The system is always stable (since it has a finite population). The number of *operating* robots is the complement to N of the number of *repaired* robots.
  $p_{N-n} = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{N!}{(N-n)!} \cdot p_0, 0 \le n \le N$ with $p_0 = \frac{1}{\sum_{j=0}^{N}\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{N!}{(N-j)!}}$, is the steady-state probability to have $N-n$ repaired robots, hence: $\pi_x = \frac{\left(\frac{\lambda}{\mu}\right)^{N-x} \cdot \frac{N!}{x!}}{\sum_{j=0}^{N}\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{N!}{(N-j)!}}$, $0 \le x \le N$, is the SS probability to have x operating robots.
- The mean number of operating robots is $N - E[n] = \frac{\mu}{\lambda} \cdot (1 - p_0)$
- The mean downtime of a repaired robot is $E[R] = \frac{N}{\mu \cdot (1-p_0)} - \frac{1}{\lambda}$

PECSN, 16/6/20

Exercise 1

A hospital in a large city admits both CoVid19 and standard patients. The mean daily intake of patients of either type is λ and μ respectively. The numbers of patients of the two types are independent of each other.

1) Compute the probability that the hospital admits patients in a day.
2) Compute the probability that the total number of patients is m, and h of them are CoVid19 patients.
3) The hospital registry shows that the total number of patients admitted today is m. What is the probability that h of them are CoVid19 patients? Justify your result.
4) Assume that $\lambda = \mu = 20$. Compute the probability of having at least 60 patients of either type, given that you have 100 patients. Discuss the correctness of the approximation.

Exercise 2

Following CoVid19 restrictions, a restaurant operates by hosting at most K customers simultaneously. Customers can order a meal including one to three courses (*starter*, *main*, *dessert*). After eating, they pay at the counter and leave. As soon as a customer leaves, a new one is admitted in (there is always a queue outside the restaurant).

Half of the customers choose to begin their meal with a starter; one fourth of the customers forgo the starter and order directly the main, and the remaining fourth order only the dessert. After eating a starter or a main, half of the customers choose to follow on with the next course, whereas the other half check out.

Assume that the restaurant has three cooks, each specialized in one course, which process their orders sequentially. Each cook takes an exponentially distributed time to serve one order, and let its mean be $1/\mu$. Assume that the check-out operation is twice as fast.

1) Model the system as a queueing network;
2) Solve the routing equations and compute the SS probabilities in their general form;
3) Compute the normalizing constant as a function of the number of customers K;
4) Compute the utilization of the cooks and the cashier; check the formula in limit cases.
5) Compute the number of customers served per unit of time;

Exercise 1 - Solution

1) It stands to reason that the number of patients of either type is distributed as a Poisson RV - large number of "trials" (citizens), low "success probability" (i.e., getting ill). The sum of independent Poisson variables is a Poisson variable, whose mean is $E[Z] = \lambda + \mu$. Therefore, $P\{Z > 0\} = 1 - P\{Z = 0\} = 1 - e^{-(\lambda+\mu)}$

2) The probability is the following:

$$
\begin{aligned}
P\{X = h, Z = m\} &= P\{X = h, Y = m - h\} \\
&= P\{X = h\} \cdot P\{Y = m - h\} \\
&= \left(e^{-\lambda} \cdot \frac{\lambda^h}{h!}\right) \cdot \left(e^{-\mu} \cdot \frac{\mu^{m-h}}{(m-h)!}\right) \\
&= e^{-(\lambda+\mu)} \cdot \frac{\lambda^h \mu^{m-h}}{h!\,(m-h)!}
\end{aligned}
$$

3) Using the former result, we get:

$$
P\{X = h | Z = m\} = \frac{P\{X = h, Y = m - h\}}{P\{Z = m\}} = \frac{e^{-(\lambda+\mu)} \cdot \frac{\lambda^h \mu^{m-h}}{h!\,(m-h)!}}{e^{-(\lambda+\mu)} \cdot \frac{(\lambda+\mu)^m}{m!}} = \binom{m}{h} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^h \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{m-h}
$$

The above distribution is clearly a binomial, with $p = \frac{\lambda}{\lambda+\mu} < 1$. In fact, once you fix the total number of patients, the number of CoVid19 ones is the count of a repeated Bernoullian experiment, whose success rate is the probability that an admitted patient has CoVid19. This probability is in fact $\frac{\lambda}{\lambda+\mu}$.

4) The requested probability is:

$$
\begin{aligned}
&P\{\{X \geq 60 | Z = 100\} \cup \{X \leq 40 | Z = 100\}\} \\
&= 1 - P\{40 \leq X \leq 60 | Z = 100\}
\end{aligned}
$$

The above probability can be approximated via a Gaussian, as long as $m \cdot p \cdot (1-p) > 10$. In fact, with $m = 100$ and $p = \frac{\lambda}{\lambda+\mu} = 0.5$ we get $m \cdot p \cdot (1-p) = 25 > 10$, hence the approximation is sound. Thus, we have:

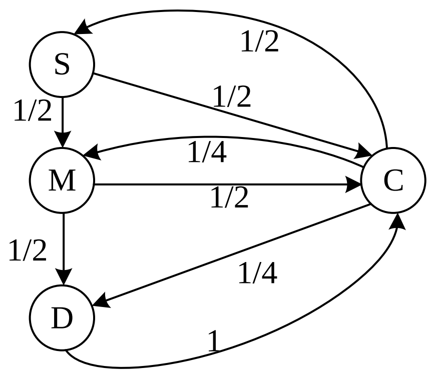
$$
\begin{aligned}
P\{40 \leq X \leq 60 | Z = 100\} &= P\left\{\frac{39.5 - m \cdot p}{\sqrt{m \cdot p \cdot (1-p)}} \leq \frac{X - m \cdot p}{\sqrt{m \cdot p \cdot (1-p)}} \leq \frac{60.5 - m \cdot p}{\sqrt{m \cdot p \cdot (1-p)}} | Z = 100\right\} \\
&= \Phi(2.1) - \Phi(-2.1) = -1 + 2 \cdot \Phi(2.1)
\end{aligned}
$$

The requested probability is thus $1 - [-1 + 2\Phi(2.1)] = 2(1 - \Phi(2.1)) \simeq 0.0357$.

Exercise 2 - Solution

The system can be modeled by a closed QN, with four SCs (Starters' cook, Mains' cook, Desserts' cook, Cashier –S, M, D, C for short from now on).

The QN diagram and routing matrix are the following:



$$
\begin{array}{c|cccc}
 & S & M & D & C \\
\hline
S & 0 & 1/2 & 0 & 1/2 \\
M & 0 & 0 & 1/2 & 1/2 \\
D & 0 & 0 & 0 & 1 \\
C & 1/2 & 1/4 & 1/4 & 0
\end{array}
$$

The input/output balance relationships yield the following:

$$\lambda_S = \lambda_C/2$$
$$\lambda_M = \frac{\lambda_C}{4} + \frac{\lambda_S}{2} = \frac{\lambda_C}{2}$$
$$\lambda_D = \frac{\lambda_C}{4} + \frac{\lambda_M}{2} = \frac{\lambda_C}{2}$$

Therefore, a solution to the routing equations is $e^T = [e, e, e, 2e]^T$. Setting $e = \mu$, one finds $\rho^T = [1,1,1,1]^T$.

Therefore, we have $p(n_S, n_M, n_D, n_C) = \frac{1}{G(M,K)} \cdot 1^{n_S} \cdot 1^{n_M} \cdot 1^{n_D} \cdot 1^{n_C} = \frac{1}{G(M,K)}$

The normalizing constant is the cardinality of set $|\mathcal{E}|$, since all states are equally likely. Therefore, it is $G(M,K) = \binom{K+3}{3}$.

The utilization of the cooks and the cashiers is the same, and it is equal to $U = \frac{G(4,K-1)}{G(4,K)} = \frac{\binom{K+2}{3}}{\binom{K+3}{3}} = \frac{K}{K+3}$. The expression makes perfect sense, since the utilization grows to 1 with K.

The number of customers served per units of time is the throughput at the cashier. The latter is: $\gamma = 2\mu \cdot \frac{K}{K+3}$.

Exercise 1

1. Flip a fair coin twice. What is the probability that you get two heads (HH)? What is the probability that you get heads followed by tails (HT)? Are these probabilities the same?
2. Flip a fair coin repeatedly until you get heads and tails in a row (HT). What is the probability that it takes n flips to win?
3. Flip a fair coin repeatedly until you get two heads in a row (HH). What is the probability that it takes n flips to win? (*suggestion*: go all the way up to $n = 8$ before making conclusions).
4. Based on the answers to points 2 and 3, is the probability of a *large* value of n equal in the two cases? If it is not, which probability is the highest?
5. Player A and B play the following game: they flip a coin repeatedly until either HH occurs (A wins) or HT occurs (B wins). Is the game fair (i.e., are the two players equally likely to win)?

Exercise 2

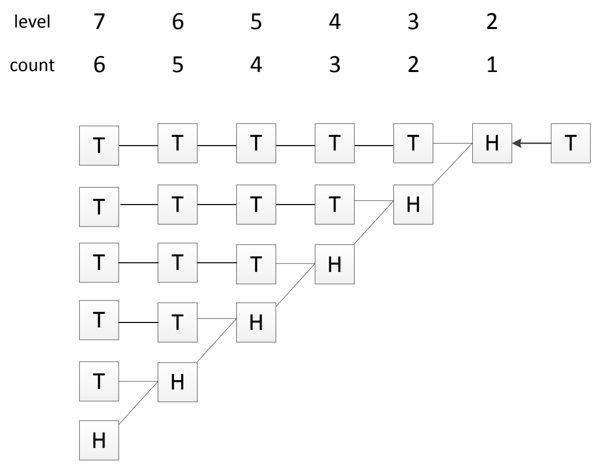
Consider a system where *messages* arrive (exponentially, at a rate λ) and *packets* are buffered and served (exponentially, at a rate μ). Each message carries two packets. The system has enough memory to store two packets, and will reject a message unless it can store both packets.

1. Model the system as a queuing system and draw the CTTC (or transition-rate diagram)
2. Compute the stability condition, the SS probabilities and the mean number of packets in the system
3. Compute the probability that a message is lost, and, from the latter, the mean *rate* of accepted packets
4. Compute the system throughput (in packets per second)
5. Compute the z-transform of the number of packets in the system and, using the latter, the mean and the variance of the number of packets in the system.

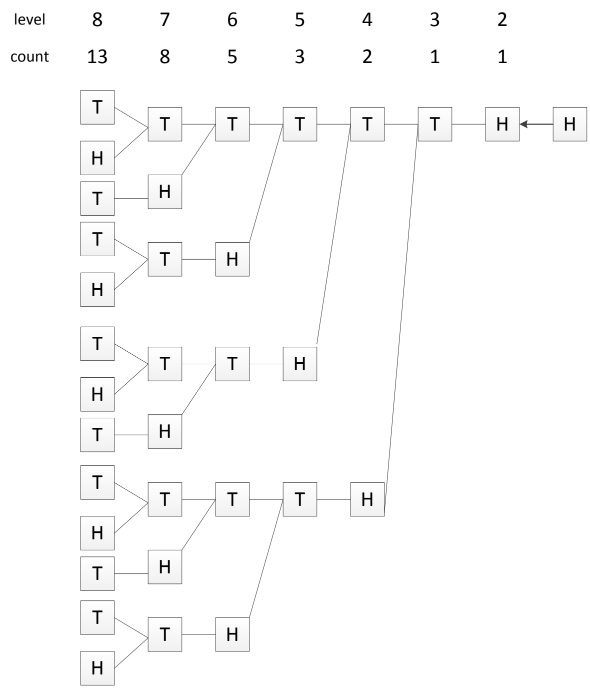
Exercise 1 - Solution

1. The two probabilities are obviously the same, and each one is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

2. This is clearly a uniform probability model. The number of n-sequences is 2^n, whereas the number of *good* n-sequences can be computed by constructing a tree. The topmost two levels of the tree are HT, and every time a level j is added, only feasible sequences are generated (i.e., you only add a child T to a parent T, whereas you add both children H and T to a parent H). The tree is in the figure besides. The number of *good* n-sequences increases by one at each step, i.e., it grows linearly and is equal to $n-1$. Hence:
$P\{N_{HT} = n\} = \frac{n-1}{2^n}, n \geq 2$



3. We repeat the same argument as before and find that the number of *good* n-sequences is different. The highest two levels of the tree are HH, and every time a level j is added, you only add a child T to a parent H, whereas you add both children H and T to a parent T. The tree is in the figure below. By counting the number of nodes at each level, one immediately gets that the number of *good* n-sequences is the $n-1$ number in the Fibonacci sequence S_{n-1}. Therefore:
$P\{N_{HH} = n\} = \frac{S_{n-1}}{2^n}, n \geq 2$



4. Given the above, we can observe that $S_{n-1} > n - 1$ starting from $n = 6$ (The fact that the Fibonacci sequence is superlinear is well known). Therefore, the probability of observing long sequences is *larger* if you bet on HH.

5. Counterintuitively, the game *is* fair, despite the fact that you observe longer sequences more often with HH than with HT. This can be proved by observing that the *only* n-sequence that can go on indefinitely without either player winning is {TT...TT} (every other sequence has at least one H in the middle, hence leads to one of the two winning). As soon as *one* H appears, the winner is decided by the next flip. So, there is only *one* sequence that leads to the decisive flip, and it is {TTT…TT}H. After that, both players have the same 50% chance to win.

Exercise 2 – solution

It is expedient to use the number of *packets* as a state characterization. This way, the system has a finite memory, equal to 2, and only admits arrivals in state 0. The CTTC is the following:

The SS equations are:

$$p_0 \cdot \lambda = p_1 \cdot \mu$$
$$p_1 \cdot \mu = p_2 \cdot \mu$$
$$p_2 \cdot \mu = p_0 \cdot \lambda$$

And the normalization condition is:

$$p_0 + p_1 + p_2 = 1$$

Using two out of three SS equations (one is linearly dependent on the other two) and the normalization, one straightforwardly obtains:

$$p_1 = p_2 = \frac{\lambda}{2\lambda + \mu}$$
$$p_0 = \frac{\mu}{2\lambda + \mu}$$

The system is always stable, as are all a finite-memory ones. The mean number of packets in the system is:

$$E[N] = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 = \frac{3\lambda}{2\lambda + \mu}$$

The loss probability is the probability that the system is in states 1 or 2. Therefore, it is:

$$p_L = p_1 + p_2 = \frac{2\lambda}{2\lambda + \mu}$$

The rate at which the system accepts packets is instead:

$$\lambda_{pkt} = 2 \cdot \lambda \cdot (1 - p_L) = \frac{2\lambda \cdot \mu}{2\lambda + \mu}$$

The throughput (in packets per second) is obviously $\gamma = \lambda_{pkt}$. The definition yields:

$$\gamma = \mu \cdot (p_1 + p_2) = \mu \cdot \frac{2\lambda}{2\lambda + \mu}$$

Which is obviously the same expression.

By definition, we have $P(z) = \sum_{k=0}^{+\infty} p_k \cdot z^k$, i.e.:

$$P(z) = \frac{\mu}{2\lambda + \mu} + (z + z^2) \cdot \frac{\lambda}{2\lambda + \mu} = \frac{\lambda \cdot z^2 + \lambda \cdot z + \mu}{2\lambda + \mu}$$

We compute:

$$P'(z) = \frac{2\lambda \cdot z + \lambda}{2\lambda + \mu}, \qquad P''(z) = \frac{2\lambda}{2\lambda + \mu}$$

And we know from the theory that:

$$E[N] = P'(1) = \frac{3\lambda}{2\lambda + \mu}$$

$$Var(N) = P''(1) + P'(1) - P'(1)^2 = \frac{2\lambda}{2\lambda + \mu} + \frac{3\lambda}{2\lambda + \mu} - \left(\frac{3\lambda}{2\lambda + \mu}\right)^2 = \frac{\lambda^2 + 5\lambda \cdot \mu}{(2\lambda + \mu)^2}$$

Exercise 1

Consider a MAC-layer protocol where a transmitter sends frames whose length is L bits over a line whose transmission speed is C bits per second. The receiver acknowledges the frames by sending back an ACK, if the frame has been received correctly, or a NACK, if the frame is corrupted. A frame is corrupted if at least one bit is corrupted, and the link's *bit error rate* (i.e., the probability that a *single bit* is corrupted) is constant and equal to p. The transmitter retransmits the same frame until it receives an ACK.
Assume that:

- all transmitted bits are independent of each other;
- an ACK/NACK never gets corrupted;
- the propagation time along both directions of the link is constant and equal to t;
- the time it takes to transmit an ACK/NACK is equal to t'.

1) Compute the probability p_{err} that a frame is corrupted;
2) Compute the values that the RV T, "correct transmission time of a frame" can assume;
3) Compute the PMF of the above RV;
4) Compute the mean time of correct transmission of a frame;
5) Assuming that a frame has been corrupted $k-1$ consecutive times, find the conditional probability that it will be corrupted at the subsequent transmission attempt. Justify the result.

Exercise 2

Consider a switch, whose output link has a capacity of C bits per second, connected to 50 input terminals. Each terminal sends packets whose length is exponentially distributed, with a mean of 1000 bits. The interarrival times of packets *from a terminal* are exponentially distributed. For half of the terminals, the mean interarrival time is 10 seconds. For the other half, it is 5 seconds.

1) Compute the value of C so that the mean number of packets in the switch is equal to 5
2) Compute the value of C so that the 99^{th} percentile of the number of packets in the switch is equal to 5, call it C'.
3) *This point differs depending on the year when you attended the course*
    a. <u>2019 (last edition)</u>: assume that the length of packets is *constant*. Solve again point 1). Comment on your findings.
    b. <u>Previous editions</u>: Find the condition on C by which the standard deviation of the number of jobs in the system is larger than the mean.
4) Compute the 99^{th} percentile of the response time of a packet as a function of C.

Exercise 1 - Solution

1) $p_{err} = 1 - \binom{L}{0} p^0 \cdot (1-p)^{L-0} = 1 - (1-p)^L$

2) T can be equal to $t_k = k \cdot \left(2t + t' + \frac{L}{C}\right)$, $k \geq 1$ being the number of required transmissions before an ACK is received back.

3) Call $P_j = P\{T = t_j\}$. It is:

$$P_j = \left(\prod_{i=1}^{j-1} p_{err,i}\right) \cdot \left(1 - p_{err,j}\right) = {p_{err}}^{j-1} \cdot (1 - p_{err})$$

4) It can be easily seen that T is a (scaled) geometric variable, hence its mean value is

$$E[T] = \frac{(2t + t' + L/C)}{(1 - p_{err})}$$

5) The conditional probability is

$$P\{\text{corrupted k-th time} \mid \text{corrupted k-1 times}\} = \frac{P\{\text{corrupted k consecutive times}\}}{P\{\text{corrupted k-1 times}\}}$$

$$= \frac{{p_{err}}^k}{{p_{err}}^{k-1}} = p_{err} = P\{\text{corrupted}\}$$

This is obvious, since retransmissions are independent of each other.

Exercise 2 - Solution

The system can be modeled as an M/M/1 queueing system where:

- $\mu = \frac{C}{E[L]} = \frac{C}{1000}$
- $\lambda = 25 \cdot \frac{1}{10} + 25 \cdot \frac{1}{5} = 7.5$

1) The mean number of jobs in an M/M/1 is given by Kleinrock's formula, i.e. $E[N] = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$. Imposing $E[N] = 5$ and solving the latter for C yields $C = 9000$.

2) We also know that, in an M/M/1, SS probabilities are $p_k = \rho^k \cdot (1-\rho)$, hence we need to impose that:

$$\sum_{k=0}^{5} p_k = (1-\rho) \cdot \frac{1-\rho^6}{1-\rho} = 0.99$$

This yields $\rho^6 = 0.01$, i.e. $\rho = \frac{1}{\sqrt[3]{10}} \cong 0.464$. Solving this equality for C yields

$$C' \cong 16163$$

3) The variance of the number of jobs in the system is the variance of a geometric RV, i.e. $Var(N) = \frac{\rho}{(1-\rho)^2}$. Hence the inequality that checks if the std is larger than the mean is the following:

$$\sqrt{\frac{\rho}{(1-\rho)^2}} > \frac{\rho}{1-\rho}$$

Which quickly boils down to $\rho < 1$. Therefore, the required condition is the same as the stability condition, i.e. $C > 7500$.

On the other hand, if the length of the packets is constant, the system is an M/D/1, for which PK's formula yields the mean number of jobs in the system:

$$E[N] = \rho + \frac{\rho^2}{2(1-\rho)}$$

Imposing $E[N] = 5$ and solving the latter for ρ yields the following: $\rho = 6 \pm \sqrt{26}$. Since stability requires $\rho < 1$, the only good solution is $\rho = 6 - \sqrt{26} \cong 0.901$. Solving the latter for C yields $C \cong 8324$.
The value required is *smaller* than at point 1), since in this case there is no randomness in the transmission. Randomness increases queueing, as PK's formula shows.

4) In an M/M/1 system, the response time is an exponential RV, whose CDF is the following:

$$R(t) = 1 - e^{-\mu(1-\rho)t}$$

Hence the required equation is $1 - e^{-(\mu-\lambda)\pi_{99}} = 0.99$, to be solved for π_{99}. After a few straightforward computations, one gets:

$$\pi_{99} = \frac{2log10 \cdot E[L]}{C - 7.5 \cdot E[L]}$$

PECSN, classwork 22/09/2020

Exercise 1

Consider the following function:

$$F(x) = \begin{cases} 0 & x \leq -5 \\ \dfrac{\alpha \cdot x + 5}{\beta + |x|} & x > -5 \end{cases}$$

Where α, β are *positive* constants.

1) Determine under what conditions $F(x)$ is a CDF.

Assume from now on that we are in the above conditions.

2) Compute the PDF of RV X, whose CDF is $F(x)$.
3) Determine under what further conditions $E[X]$ is finite.
4) Assuming $\beta = 5$, compute $E[X]$.

Exercise 2

A service is hosted on a system having K identical servers. A job dispatcher routes each arriving job to an idle server, if there exists one, and *rejects it* otherwise. Assume that the service time and interarrival time are exponentially distributed RVs, whose rates are μ, λ respectively.

1) Draw the CTMC (or transition rate diagram).
2) Compute the stability condition and an expression for the steady-state probabilities.
3) Find under what conditions the PMF of the steady-state probability is:
    a. strictly increasing , i.e., $p_{j+1} > p_j$, $0 \leq j < K$.
    b. strictly decreasing.
    c. Neither of the above.

    Explain your findings.

4) Compute the loss probability
5) Compute the mean number of busy servers and, from that, the mean response time. Discuss both expressions when $K \to \infty$.
6) Compute the steady-state probabilities and the performance indexes for $K = 1$. Discuss the result.

Exercise 1 – Solution

1) In order to be a CDF, the following should happen:
   a) $F(x)$ must be monotonic
   b) $lim_{x\to-\infty} F(x) = 0$
   c) $lim_{x\to+\infty} F(x) = 1$

b) always holds. c) holds if and only if $\alpha = 1$.
As far as monotonicity is concerned, we observe that $F(x)$ is identically null for $x \le -5$, and that its derivative is:

$$F'(x) = \begin{cases} \dfrac{\beta+5}{(\beta-x)^2} & -5 < x < 0 \\ \dfrac{\beta-5}{(\beta+x)^2} & x > 0 \end{cases}$$

when $x > -5$. Therefore, monotonicity is guaranteed if $\beta \ge 5$.
The conditions requested by 1) are $\alpha = 1$, $\beta \ge 5$.

2) As per the computations above, it is:

$$f(x) = \begin{cases} 0 & x < -5 \\ \dfrac{\beta+5}{(\beta-x)^2} & -5 < x < 0 \\ \dfrac{\beta-5}{(\beta+x)^2} & x > 0 \end{cases}$$

3) It is:

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f(x)dx = \int_{-5}^{0} x \cdot \frac{\beta+5}{(\beta-x)^2}dx + \int_{0}^{+\infty} x \cdot \frac{\beta-5}{(\beta+x)^2}dx =$$
$$(\beta+5) \cdot \int_{-5}^{0} \frac{x}{(\beta-x)^2}dx + (\beta-5) \cdot \int_{0}^{+\infty} \frac{x}{(\beta+x)^2}dx$$

Now, the first integral is always finite (since its limits are), whereas the second may not be. After few algebraic passages, we obtain:

$$\int_{0}^{+\infty} \frac{x}{(\beta+x)^2}dx = \int_{0}^{+\infty} \frac{(\beta+x)-\beta}{(\beta+x)^2}dx = \left[\frac{\beta}{\beta+x} + log|\beta+x|\right]_{0}^{+\infty} = +\infty$$

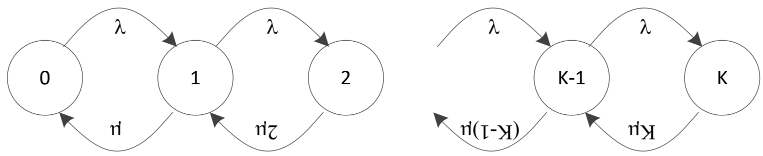
Therefore, the expectation exists only if $\beta = 5$.

4) Assuming $\beta = 5$, the expectation is equal to:

$$E[X] = (\beta+5) \cdot \int_{-5}^{0} \frac{x}{(\beta-x)^2}dx + (\beta-5) \cdot \int_{0}^{+\infty} \frac{x}{(\beta+x)^2}dx$$
$$= 10 \cdot \int_{-5}^{0} \frac{x}{(5-x)^2}dx = 10 \cdot \left[\frac{-5}{x-5} + log|x-5|\right]_{-5}^{0}$$
$$= 5 - 10\, log(2)$$

Exercise 2 – Solution

1) The diagram is shown below



2) The system is finite, hence always stable. The steady-state probabilities can be computed by writing the local balance equations: $\lambda \cdot p_j = (j+1)\cdot \mu \cdot p_{(j+1)}$, $0 \le j \le K-1$. From the above, we quickly obtain:

$$p_j = \frac{1}{j!}\cdot\left(\frac{\lambda}{\mu}\right)^j \cdot p_0,\ 0 \le j \le K.$$

Call $u = \lambda/\mu$. By imposing the normalization condition, we get:

$$p_0 = \frac{1}{\sum_{i=0}^{K} \frac{u^i}{i!}},\ p_j = \frac{\frac{u^j}{j!}}{\sum_{i=0}^{K} \frac{u^i}{i!}}.$$

3) $p_{j+1} > p_j \Leftrightarrow \frac{\frac{u^{j+1}}{(j+1)!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} > \frac{\frac{u^j}{(j)!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} \Leftrightarrow u > j+1$. Since the above must hold for every j up to $K-1$ included, the required condition is $u > K$. This can be explained by observing that, under this condition, the arrival rate is such that the serving capacity of the system (which is $K\mu$ under a heavy load) is smaller than the arrival rate.

Using the same reasoning, we find that the PMF is decreasing when $u < 1$. This means that a single server has enough capacity to serve the arrivals, hence it is increasingly more unlikely that more servers will be busy. When $1 \le u \le K$, the PMF is neither (strictly) increasing nor (strictly) decreasing.

4) The loss probability is $p_K = \frac{\frac{u^K}{K!}}{\sum_{i=0}^{K} \frac{u^i}{i!}}$.

5) The mean number of busy servers is:

$$E[N] = \sum_{j=1}^{K} j \cdot p_j = \frac{\sum_{j=1}^{K} j\frac{u^j}{j!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} = \frac{u\cdot\sum_{j=1}^{K} \frac{u^{j-1}}{(j-1)!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} = u\cdot\frac{\left[\sum_{j=0}^{K} \frac{u^j}{j!} - \frac{u^K}{K!}\right]}{\sum_{i=0}^{K} \frac{u^i}{i!}} = u\cdot(1-p_K)$$

The mean response time can be obtained through Little's theorem, using the *mean* arrival rate $\bar{\lambda} = \lambda \cdot (1 - p_K)$. Thus, the average response time is $E[R] = E[N]/\bar{\lambda} = \frac{u \cdot (1 - p_K)}{\lambda \cdot (1 - p_K)} = \frac{1}{\mu}$. In fact, there is never any queueing in this system.

When $K \to \infty$ the system becomes an $M/M/\infty$. For this system, the mean number of busy servers is in fact $u = \lambda/\mu$. This can only happen if $\lim_{K \to \infty} p_k = 0$. Note that this condition is necessary of *any* PMF with an infinite support, hence must hold for this PMF as well. We can check it explicitly:

$$\lim_{K \to \infty} p_K = \lim_{K \to \infty} \frac{\frac{u^K}{K!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} = e^{-u} \cdot \lim_{K \to \infty} \frac{u^K}{K!} = 0$$

6) When $K = 1$ we have $p_0 = \frac{1}{\sum_{i=0}^{K} \frac{u^i}{i!}} = \frac{1}{1+u}$, $p_1 = \frac{\frac{u^1}{1!}}{1+u} = \frac{u}{1+u}$. It is $p_K = p_1 = \frac{u}{1+u}$, $E[N] = p_1 = \frac{u}{1+u}$, and also $E[N] = u \cdot (1 - p_K) = u \cdot \left(1 - \frac{u}{1+u}\right) = u \cdot \left(\frac{1}{1+u}\right) = \frac{u}{1+u}$, $E[R] = \frac{1}{\mu}$. In this case, the system is an $M/M/1/1$ (or a 1-buffer), and the reader can easily check that the above expressions match those of such a system.

Exercise 1

A chemical reaction may yield either an alkaline or an acid result. The probability that the result is alkaline is p. A lab technician repeats the above reaction in independent conditions, and stops when both an alkaline and an acid result have occurred at least once. Let N_k and N_a be the number of alkaline and acid outcomes recorded by the technician at the end of her experiment.

1) Compute the CDF of N_k and N_a
2) Find the probability that N_k exceeds N_a, as a function of p. Draw a graph of the latter.
3) Compute the mean values of N_k and N_a. Check limit case and justify the answer.
4) State whether or not N_k and N_a are independent and justify your answer.

Exercise 2

A system serves job at a rate μ. When it is idle, it goes into power saving (immediately). On arrival of a new job, the system wakes up again. The wake-up operation takes an exponentially distributed time, with a mean $1/\beta$. During wake-up, the system does not accept jobs. Jobs arrive at a rate λ.

1) Model the system as a queueing system and draw its CTMC;
2) Compute the steady-state probabilities and the stability condition. State explicitly what happens of the stability condition when β increases;
3) Compute the mean number of jobs in the system;
4) Compute the mean response time of a job. Draw a graph of the latter with β on the abscissa.

Exercise 1 – Solution

1) Let us start from N_k.

$$P\{N_k = 1\} = \sum_{n=1}^{+\infty} P\{n \text{ consecutive acid and one alkaline}\} + P\{\text{one alkaline and one acid}\}$$
$$= \sum_{n=1}^{+\infty} (1-p)^n \cdot p + p \cdot (1-p) = (1-p) \cdot p \cdot \left(\frac{1}{p} + 1\right) = (1-p) \cdot (1+p) = 1 - p^2$$

And, for $n > 1$, $P\{N_k = n\} = P\{n \text{ consecutive acid and one alkaline}\} = p^n \cdot (1-p)$

The CDF of N_k is:

$$F_k(1) = 1 - p^2$$

$$F_k(n) = 1 - p^2 + \sum_{i=2}^{n} p^i \cdot (1-p) = (1-p)\left[\frac{1-p^{n+1}}{1-p} - (1+p) + (1+p)\right] = 1 - p^{n+1}$$

Therefore, it is $F_k(n) = 1 - p^{n+1} \;\; \forall n \geq 1$

Define $q = 1 - p$: for N_a it is $P\{N_a = 1\} = (1-q) \cdot (1+q) = p \cdot (2-p)$, $P\{N_a = n\} = q^n \cdot p = (1-p)^n \cdot p$.

Symmetrically, it is $F_a(n) = 1 - (1-p)^{n+1} \;\; \forall n \geq 1$

2) Event $\{N_k > N_a\}$ occurs when three or more reactions are observed, and the last one is acid. Hence:

$$P\{N_k > N_a\} = \sum_{n=2}^{+\infty} P\{N_k = n\} = \sum_{n=2}^{+\infty} p^n \cdot q = 1 - q \cdot (1+p) = p^2$$

3) From the formula, we obtain:

$$E[N_k] = 1 \cdot P\{N_k = 1\} + \left[\sum_{n=2}^{+\infty} n \cdot P\{N_k = n\}\right] =$$
$$(1-p) \cdot (1+p) + (1-p) \cdot \left[\left(\sum_{n=2}^{+\infty} n \cdot p^n\right)\right] =$$
$$(1-p) \cdot (1+p) + (1-p) \cdot \left[\left(\sum_{n=1}^{+\infty} n \cdot p^n\right) - p\right] =$$
$$(1-p) \cdot \left[\left(\frac{p}{(1-p)^2} - p + 1 + p\right)\right] =$$
$$\frac{p}{1-p} + (1-p)$$

We have $lim_{p\to 1} E\,[N_k] = +\infty$. This can be explained by observing that, if no acid outcome ever appears, the sequence of alkaline ones becomes infinitely long. Furthermore, we have $lim_{p\to 0} E\,[N_k] = 1$. In fact, if no alkaline outcome ever appears, the only sequences that we may ever obtain are infinitely long sequences of acid outcomes, *terminated by an alkaline one anyway* (this is mandatory for the experiment to terminate). Hence the mean number of alkaline outcomes will be equal to one.

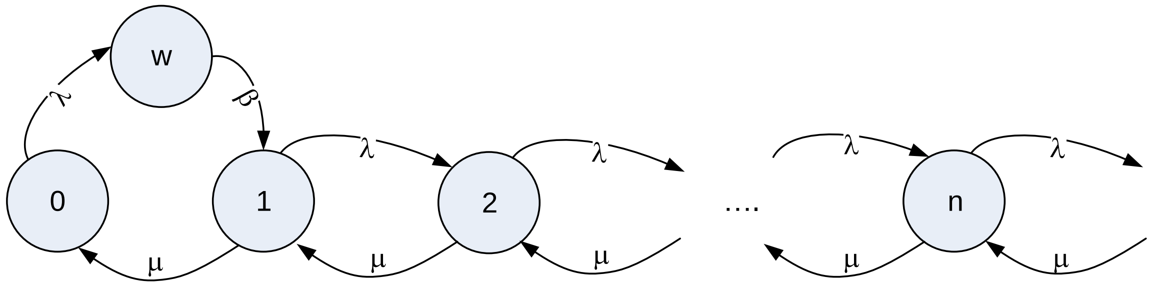
Symmetrically, we get:

$$E[N_a] = \frac{1-p}{p} + p$$

And symmetric explanations hold in limit cases, mutatis mutandis.

4) The two RVs are *not* independent. In fact, $P\{N_k = a, N_a = b\} = 0$ if $a > 1, b > 1$. On the other hand, it is $P\{N_k = a\} \cdot P\{N_a = b\} \neq 0$.

Exercise 2 - Solution



State w is the one during which the system wakes up. Global equilibrium equations in states 0 and w yield the following:

$$p_0 \cdot \lambda = p_1 \cdot \mu, \; p_w \cdot \beta = p_0 \cdot \lambda$$

And then we have

$$p_j \cdot \lambda = p_{j+1} \cdot \mu, \;\; j \geq 1$$

After a few straightforward manipulations, normalization reads:

$$p_0 \cdot \left[1 + \frac{\lambda}{\beta} + \sum_{j=1}^{+\infty} \left(\frac{\lambda}{\mu}\right)^j \right] = 1$$

Call $u = \lambda/\mu$ for simplicity. The stability condition is $u < 1$, and does not depend on β. SS probabilities are:

$$p_0 = \frac{(1-u) \cdot \beta}{\beta + \lambda \cdot (1-u)}$$

$$p_w = \frac{(1-u) \cdot \lambda}{\beta + \lambda \cdot (1-u)}$$

$$p_j = \frac{(1-u) \cdot \beta \cdot u^j}{\beta + \lambda \cdot (1-u)}, \;\; j \geq 1$$

To compute the mean number of jobs in the system, one has to remember that the system contains one job in state w as well. Therefore:

$$E[N] = \sum_{j=1}^{+\infty} j \cdot p_j + 1 \cdot p_w = \cdots = 1 + \frac{2u-1}{1-u} \cdot \frac{1}{1 + \frac{\lambda}{\beta}(1-u)}$$

One can check that $\lim_{\beta \to +\infty} E[N] = \frac{u}{1-u}$, and that $\lim_{\beta \to 0} E[N] = 1$. Both limits make sense intuitively.

In order to compute the mean response time, one needs to remember that this is not a PASTA system, since the system does not accept jobs when in state w. Therefore, it is $\bar{\lambda} = \lambda \cdot (1 - p_w) = \frac{\lambda}{1+\frac{\lambda}{\beta}(1-u)}$. This said, one can apply Little's law and find:

$$E[R] = \frac{E[N]}{\bar{\lambda}} = \cdots = \frac{1}{\lambda} \cdot \frac{u}{1-u} + \frac{1-u}{\beta}$$

The first term in the above expression is the mean response time of a standard M/M/1. The second term is due to the added wake-up time, and goes to zero as β increases.

Exercise 1

ACME manufacturing has 13 staff. Eight are hired by ACME directly, whereas five are independent contractors working for ACME. Two staff are on sick leave. We do not know whether they are ACME's own employees or contractors, but we do know that all staff have the same probability to be ill. ACME allocates to a new task the first staff that arrives at their premises in the morning (the order of arrival is random).

1) Call A the number of ACME's <u>own</u> employees that are on sick leave. Compute the PMF of A.
2) Call B the number of ACME's <u>own</u> employees that are allocated to the new task. Compute the PMF of B.

Exercise 2

Consider a computing system that accepts a workload consisting of exponential requests, arriving at a rate λ. The system serves requests using one server, whose rate is μ. However, when the number of jobs in the system exceeds a threshold $K \geq 1$, a *second* server is activated.

1) Model the system and draw its CTMC
2) Identify the stability condition and state explicitly its dependence on K.
3) compute the steady-state probabilities as a function of K.
4) Compute the probability that an arriving job finds exactly K jobs in the system, and compute under what conditions K is the most likely number of jobs that an arriving job will see. Justify the result.
5) Compute the average number of jobs in the system when $\lambda = \mu$, as a function of K. Justify the result.

Exercise 1 - Solution

1) RV A can take values 0, 1, 2. We are in a UPM, hence the PMF is given by the following expression:

$$P\{A = j\} = \frac{\binom{8}{j}\binom{5}{2-j}}{\binom{13}{2}}.$$

More specifically, it is $P\{A = 0\} = 5/39 \cong 0.128$, $P\{A = 1\} = 20/39 \cong 0.513$, $P\{A = 2\} = 14/39 \cong 0.359$

2) The PDF of RV B can be computed using total probability, by conditioning to A. In fact, it is:

$$P\{B = b|A = k\} = \frac{\binom{8-k}{b}\binom{(13-2)-(8-k)}{1-b}}{\binom{13-2}{1}}.$$

Once the above has been computed with $b \in \{0,1\}$ and $k \in \{0,1,2\}$, the following expression computes the PDF of B:

$$P\{B = b\} = \sum_{k=0}^{2} P\{B = b|Y = k\} \cdot P\{Y = k\}$$

The numerators of the above conditional probabilities are in the table below (the denominator is 11):

		b	
		0	**1**
k	**0**	3	8
	1	4	7
	2	5	6

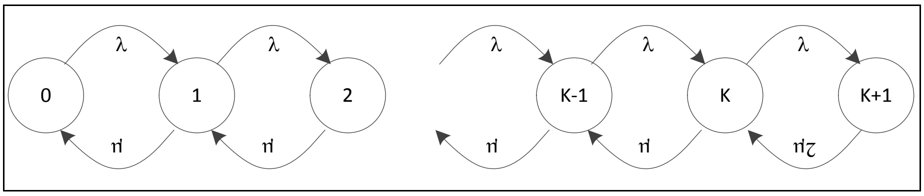
We obtain:

$$P\{B = 0\} = \frac{3}{11} \cdot \frac{5}{39} + \frac{4}{11} \cdot \frac{20}{39} + \frac{5}{11} \cdot \frac{14}{39} = \frac{165}{11 \cdot 39} = \frac{15}{39} \cong 0.384$$

$$P\{B = 1\} = \frac{24}{39} \cong 0.615$$

Exercise 2 - Solution

1) The CTMC is as follows:



2) The stability condition can be easily inferred to be *independent of K*. In fact, the fact that the service rate is μ up to state K is irrelevant, as long as the remaining states (up to infinity) have a different service rate 2μ. This said, the stability condition can only be $\boxed{\lambda < 2\mu}$. The same result should emerge from the computations as well. In fact, the local balance equations yield:

$$p_i = \begin{cases} \frac{\lambda}{\mu} p_{i-1} & i \le K \\ \frac{\lambda}{2\mu} p_{i-1} & i > K \end{cases}, \text{ hence } p_i = \begin{cases} \left(\frac{\lambda}{\mu}\right)^i \cdot p_0 & i \le K \\ \left(\frac{\lambda}{2\mu}\right)^{i-K} \cdot \left(\frac{\lambda}{\mu}\right)^K \cdot p_0 & i > K \end{cases}$$

Therefore, the normalization condition is as follows:

$$p_0 \cdot \left[\sum_{i=0}^{K} \left(\frac{\lambda}{\mu}\right)^i + \sum_{i=K+1}^{+\infty} \left(\frac{\lambda}{\mu}\right)^K \cdot \left(\frac{\lambda}{2\mu}\right)^{i-K} \right] = 1$$

Which can be rewritten as:

$$p_0 \cdot \left[\sum_{i=0}^{K} \left(\frac{\lambda}{\mu}\right)^i + \left(\frac{\lambda}{\mu}\right)^K \cdot \sum_{j=1}^{+\infty} \left(\frac{\lambda}{2\mu}\right)^j \right] = 1.$$

The first sum is always finite, and the second one converges if and only if $\boxed{\lambda < 2\mu}$. Call $u = \lambda/2\mu$.

3) We need to treat separately the two cases $\lambda = \mu, \lambda \ne \mu$ (because of the first sum).

Case 1: $\lambda = \mu$

In this case, we have $p_0 \cdot \left[(K+1) + \frac{u}{1-u} \right] = 1$, hence:

$$p_0 = \frac{1-u}{K \cdot (1-u) + 1}, \boxed{p_i = \begin{cases} p_0 & i \le K \\ u^{i-K} \cdot p_0 & i > K \end{cases}}$$

In this case, all s.s. probabilities are equal up to state K. This should not surprise us, since the states at the leftmost part of the diagram have equal transition rates to the left and to the right.

Case 2: $\lambda \neq \mu$

We get: $p_0 \cdot \left[\frac{1-(2u)^{K+1}}{1-2u} + (2u)^K \cdot \frac{u}{1-u} \right] = 1$, i.e. $p_0 = \frac{1}{\frac{1-(2u)^{K+1}}{1-2u} + \frac{(2u)^K \cdot u}{1-u}}$

The above expression can be simplified to $p_0 = \frac{(1-2u)\cdot(1-u)}{(1-u) - u \cdot (2u)^K}$.

$$\boxed{p_i = \begin{cases} (2u)^i \cdot p_0 & i \leq K \\ u^{i-K} \cdot (2u)^K \cdot p_0 & i > K \end{cases}}.$$

4) This system has constant arrival rates, hence it possesses the PASTA property, i.e., $\boxed{r_K = p_K}$. The answer is, thus, to find the condition under which $p_K > p_i, \quad i \neq K$. Note that, if $\lambda = \mu$, the above condition is false (all states up to *K* have the same probability), therefore we can safely assume $\lambda \neq \mu$

.

In this case, we observe that $u^{i-K} \cdot (2u)^K \cdot p_0$ is by definition a decreasing sequence (otherwise the system would not be stable). On the other hand, $(2u)^i \cdot p_0$ is an increasing sequence if $\mu < \lambda < 2\mu$. Therefore, state *K* will be the most likely number of jobs if and only if $\boxed{\mu < \lambda < 2\mu}$.

5) When $\lambda = \mu$ we have:

$$p_0 = \frac{1-u}{K \cdot (1-u) + 1}, \quad p_i = \begin{cases} p_0 & i \leq K \\ u^{i-K} \cdot p_0 & i > K \end{cases}$$

Hence,

$$\begin{aligned} E[N] &= \sum_{i=0}^{K} i \cdot p_0 + \sum_{i=K+1}^{+\infty} i \cdot u^{i-K} \cdot p_0 \\ &= p_0 \cdot \left[\sum_{i=0}^{K} i + \sum_{j=1}^{+\infty} (K+j) \cdot u^j \right] \\ &= p_0 \cdot \left[\frac{K \cdot (K+1)}{2} + K \cdot \frac{u}{1-u} + \frac{u}{(1-u)^2} \right] \end{aligned}$$

By substituting $u = 1/2$ we get

$$E[N] = \frac{1}{K+2} \cdot \left[\frac{K \cdot (K+1)}{2} + (K+2) \right] = \frac{K}{2} \cdot \frac{K+1}{K+2} + 1$$

The result makes sense. In fact, if $u = 1/2$, all the states up to *K* are equally likely (hence the first term), but states *above K* have non null probability. Therefore, the result must be larger than *K*/2, which would be the result if those state did not exist.

PECSN, 23/2/21

Exercise 1

Two bank clerks are assigned *standard* and *urgent* customers respectively. Let X denote the number of customers being attended to by the first clerk, and Y denote the number of customers of the second one at the same time. Let the JPMF of X and Y be the following:

X / Y	0	1	2	3
0	0.08	0.07	0.04	0.00
1	0.06	0.15	0.05	0.04
2	0.05	0.04	0.10	0.06
3	0.00	0.03	0.04	0.07
4	0.00	0.01	0.05	0.06

1) What is the probability that there is exactly one customer in each line?

2) What is the probability that the number of customers in the two lines are identical?

3) Let A denote the event that there are at least two more customers in one line than in the other line. Calculate the probability of A.

4) Determine the marginal PMF of X and then calculate the expected number of standard customers in line.

5) Determine the marginal PMF of Y.

6) Are X and Y independent random variables? Explain your answer.

7) Determine the PMF of the *overall* number of customers in line at the bank.

Exercise 2

Consider a single-server queueing system where the arrival and departure rates are ($k \geq 0$):

$$\lambda_j = \begin{cases} \frac{\lambda}{(j+1)} & 0 \leq j < k \\ \lambda & j \geq k \end{cases}, \qquad \mu_j = \begin{cases} \mu & 1 \leq j \leq k \\ j \cdot \mu & j > k \end{cases}.$$

1) Draw the CTMC.
2) Compute the stability condition and the steady-state probabilities. State explicitly whether and how both depend on k.
3) Express the condition by which s.s. probabilities are a decreasing sequence.
4) Compute the mean number of jobs in the system and in the queue. State explicitly how both depend on k.
5) Compute the mean arrival rate as a function of k. Discuss what happens when $k = 0, k \to +\infty$ and justify your results.
6) Compute the mean response time when $k = 0, k \to +\infty$.

PECSN, 23/2/21

Exercise 1 - Solution

1) What is the probability that there is exactly one customer in each line?

$$P(X = 1, Y = 1) = p(1,1) = 0.15$$

2) What is $P(X = Y)$, that is, the probability that the number of customers in the two lines are identical?

$$P(X = Y) = p(0,0) + p(1,1) + p(2,2) + p(3,3) = 0.08 + 0.15 + 0.1 + 0.07 = 0.4$$

3) Let A denote the event that there are at least two more customers in one line than in the other line. Calculate the probability of A.

$$\begin{aligned} A &= \{(x,y): x \geq y+2\} \cup \{(x,y): y \geq x+2\} \\ &= \{(2,0),(3,0),(4,0),(3,1),(4,1),(4,2),(0,2)(0,3)(1,3)\} \end{aligned}$$

$$\begin{aligned} P(A) &= p(2,0) + p(3,0) + p(4,0) + p(3,1) + p(4,1) + p(4,2) + p(0,2) + p(0,3) + p(1,3) \\ &= 0.22 \end{aligned}$$

4) Determine the marginal PMF of X and then calculate the expected number of customers in the standard queue.

$$p_X(n) = \sum_{i=-\infty}^{+\infty} p(n,i) = p(n,0) + p(n,1) + p(n,2) + p(n,3)$$

x	0	1	2	3	4
$p_X(x)$	0.19	0.30	0.25	0.14	0.12

Hence, $E(X) = \sum_{x=1}^{4} x \cdot p_X(x) = 1 \cdot 0.19 + 2 \cdot 0.25 + 3 \cdot 0.14 + 4 \cdot 0.12 = 1.7$

5) Determine the marginal PMF of Y .

$$p_Y(n) = \sum_{i=-\infty}^{+\infty} p(i,n) = p(0,n) + p(1,n) + p(2,n) + p(3,n) + p(4,n)$$

y	0	1	2	3
$p_Y(y)$	0.19	0.30	0.28	0.23

6) Are X and Y independent random variables? Explain.

They are not. By counterexample: $P(X = 4) = 0.12$, $P(Y = 0) = 0.19$, $P(X = 4, Y = 0) = 0$.

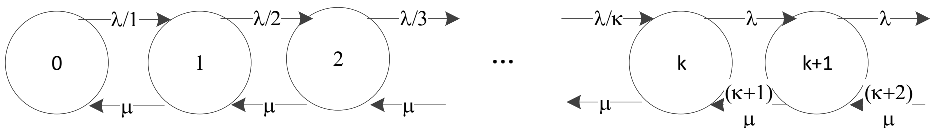
7) Determine the PMF of the *overall* number of customers in line at the bank.

The PMF is obtained by summing up all the values (x, y) having the same sum $s = x + y$, i.e.

$\boldsymbol{s}$	$\boldsymbol{p(s)}$
0	0.08
1	0.13
2	0.24
3	0.09
4	0.17
5	0.11
6	0.12
7	0.06

Exercise 2 - Solution

1) The CTMC is the following:



2) Call $u = \frac{\lambda}{\mu}$. The local equilibrium equations are always $P_{j+1} = \frac{u}{j+1} \cdot P_j$, regardless of whether $j < k$ or $j \geq k$, hence $P_j = \frac{u^j}{j!} \cdot P_0, j \geq 0$. Therefore, the system is always stable, and the s.s. probabilities are the same as those of a discouraged arrival system or an M/M/∞ one, i.e. $P_j = e^{-u} \frac{u^j}{j!}, \forall j, \geq 0$. The s.s. probabilities and the stability condition do not depend on k.

3) The s.s. probabilities are a Poisson distribution with a parameter u. We know from the theory that the Poisson PDF is monotonically decreasing if $u < 1$. The same result can be obtained by imposing that $\forall j, P_j > P_{j+1}$, which yields $\forall j, j+1 > u$, hence $u < 1$.

4) From the theory on Poisson distribution, we readily obtain $E[N] = u$. Moreover, in every single-server system it is: $E[N_q] = E[N] - (1 - P_0)$, hence $E[N_q] = u - (1 - e^{-u})$. Neither of the above depend on k.

5) The system is non-PASTA, since the arrival rate depends on the state. It is, in fact:

$\bar{\lambda} = \lambda \cdot \left[\sum_{j=0}^{k-1} \frac{1}{j+1} e^{-u} \frac{u^j}{j!} + \sum_{j=k}^{+\infty} e^{-u} \frac{u^j}{j!}\right] = \lambda \cdot e^{-u} \cdot \left[\frac{1}{u} \cdot \left(\sum_{j=0}^{k} \frac{u^j}{j!} - 1\right) + e^u - \sum_{j=0}^{k} \frac{u^j}{j!} + \frac{u^k}{k!}\right] = \lambda \cdot e^{-u} \cdot \left[\left(\frac{1}{u} - 1\right) S_k + e^u - \frac{1}{u} + \frac{u^k}{k!}\right]$,

where we use $S_k = \sum_{j=0}^{k} \frac{u^j}{j!}$ for the sake of conciseness. Note that $S_0 = 1$, and $\lim_{k \to +\infty} S_k = e^u$.

The mean arrival rate does depend on k. We deal with the two cases $k = 0, k \to +\infty$ separately:

a) $k = 0$. In this case, the arrival rate is constant and equal to λ. In fact, from the above formula we get:

$$\bar{\lambda} = \lambda \cdot e^{-u} \cdot \left[\left(\frac{1}{u} - 1\right) S_k + e^u - \frac{1}{u} + \frac{u^k}{k!}\right] = \lambda \cdot e^{-u} \cdot \left[\frac{1}{u} - 1 + e^u - \frac{1}{u} + 1\right] = \lambda$$

b) $k \to +\infty$. In this case, the mean arrival rate is the one of a discouraged arrival system. In fact we get:

$$\bar{\lambda} = \lim_{k \to +\infty} \lambda \cdot e^{-u} \cdot \left[\left(\frac{1}{u} - 1 \right) S_k + e^u - \frac{1}{u} + \frac{u^k}{k!} \right] = \lambda \cdot e^{-u} \cdot \left[\left(\frac{1}{u} - 1 \right) e^u + e^u - \frac{1}{u} \right]$$
$$= \mu \cdot [1 - e^{-u}]$$

6) We deal with the two cases $k = 0, k \to +\infty$ separately:

a) $k = 0$. In this case we get $E[R] = \frac{E[N]}{\bar{\lambda}} = \frac{u}{\lambda} = \frac{1}{\mu}$. This makes sense, since when $k = 0$ the system is akin to a load-dependent server with infinite capacity.

b) $k \to +\infty$. In this case we get $E[R] = \frac{E[N]}{\bar{\lambda}} = \frac{u}{\mu \cdot [1 - e^{-u}]} = \frac{\lambda}{\mu^2 \cdot [1 - e^{-u}]}$. This is in fact the mean response time of a discouraged-arrival system.

Performance Evaluation of Computer Systems and Networks, 15/06/2021

Exercise 1

A *schedule* is a sequence of *events*, which can be of two types, *A* and *B*:

- The probability of an event of type *A* is 0.4, and the one of an event *B* is 0.6;
- Event *A* lasts for 4ms, and event *B* lasts for 5 ms.
- Events are independent of each other.

Let S_n denote the RV that measures the time duration of a schedule of n events.

1) Find the probability mass function of S_4 and its mean value $E[S_4]$.
2) Compute the PMF of S_n and $E[S_n]$, for a generic value of n.

3) Assume now that the distribution of the *number* of events in a schedule is as follows: there are never less than five events, and the probability of having more than five events is $p_{5+n} = \left(\frac{1}{2}\right)^{n+1}$, with $n \geq 0$. Under the above hypothesis, you measure 40ms as the duration of the schedule. Compute the PMF of the number of events in that schedule.

Exercise 2

A multiprogrammed computer system has N running processes. These processes request I/O operations independently, each at a rate λ, with exponentially distributed inter-request times. I/O operations are served by an array of N identical I/O peripherals. Each I/O operation has a duration which is exponential with a mean $\frac{1}{\mu}$.

1) Model the system as a queueing system and draw the transition rate diagram
2) Compute the steady-state probabilities and the stability condition
3) Compute the mean response time for a process
4) Compute the mean number of processes blocked on an I/O operation
5) Compute the throughput of the I/O subsystem. Interpret the result.

Exercise 1 - Solution

1) A sequence of four events can last for 16, 17, 18, 19, 20 ms, depending on the number of Bs in it. If you consider each digit as a Bernoullian experiment, the probability of having k Bs is $\sim B(4{,}0.6)$, hence we have:

$P\{S_4 = 16 + k\} = \binom{4}{k} 0.6^k \cdot 0.4^{4-k}$, with $0 \le k \le 4$.

The mean value for S_4 is $16 + 4 \cdot 0.6 = 18.4$.

2) in the general case, S_n is an integer ranging from $4n$ to $5n$, and we have:

$P\{S_n = 4 \cdot n + k\} = \binom{n}{k} 0.6^k \cdot 0.4^{n-k}$, with $0 \le k \le n$.

The mean value for S_n is $(4 + 0.6) \cdot n = 4.6 \cdot n$.

3) We first need to compute all the values of n for which $4n \le 40 \le 5n$. They are the intersection of $n \le 10$ (left-hand inequality) with $n \ge 8$ (right-hand inequality). Hence, the transmission can be of 8, 9 or 10 events. The "*a priori*" probabilities of a transmission of 8, 9, 10 digits are $\frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, respectively. After you observe that the transmission is 40ms long, you can apply Bayes's theorem and compute the "*a posteriori*" probabilities as follows:

$$P(n|40) = \frac{P(40|n) \cdot P_n}{\sum_{k=8}^{10} P(40|k) \cdot P_k},$$

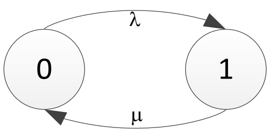
with $n = 8{,}9{,}10$.

The *a posteriori* probabilities can be found in the rightmost column of the following table:

n	P(40|n)	P(40|n)	Pn	Pn	product	P(n|40)
8	$\binom{8}{8}0.6^8$	0.016796	$\frac{1}{16}$	0.0625	0.00105	16.72%
9	$\binom{9}{4}0.6^4 \cdot 0.4^5$	0.167215	$\frac{1}{32}$	0.03125	0.005225	83.25%
10	$\binom{10}{0}0.4^{10}$	0.000105	$\frac{1}{64}$	0.015625	1.64E-06	0.03%
				total	0.006277	

Exercise 2 - Solution

There are two ways to model this system. The simplest is to observe that, since there are as many I/O peripherals as running processes, no queueing ever occurs, hence the system can be seen as the juxtaposition of N independent M/M/1/1 systems as the one in the figure:



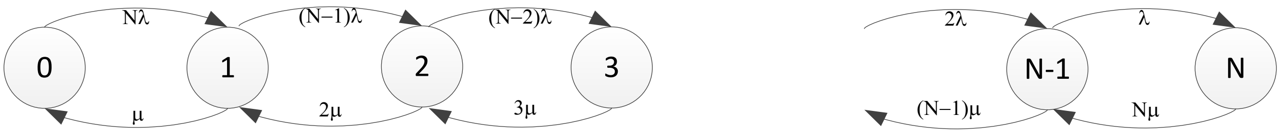
State "1" is the one where the process is doing some I/O operation, state "0" is when it is not occupying the I/O peripheral. The above system admits two SS probabilities, $\pi_0 = \frac{1}{(1+u)}$, $\pi_1 = \frac{u}{(1+u)}$, with $u = \frac{\lambda}{\mu}$, and is always stable. Therefore, we can define the state of the multiprogrammed system as the number of occupied I/O peripherals, ranging from 0 to N.

The probability that k peripherals are occupied is thus a binomial RV, with a number of trials N and a probability of success equal to π_1, hence:

$P_k = \binom{N}{k} {\pi_1}^k \cdot (1-\pi_1)^{N-k} = \binom{N}{k} {\pi_1}^k \cdot {\pi_0}^{N-k} = \binom{N}{k} \cdot \frac{u^k}{(1+u)^N}$.

The system has a finite number of states, hence it is always stable.

The same solution could be found using the canonical procedure, i.e. seeing the system as an M/M/N/./N one as follows:



Developing the computations yields the selfsame results.

3) The mean response time for a process is $\frac{1}{\mu}$, since there is no queueing.

4) The mean number of blocked processes is the mean of the binomial, i.e., $N \cdot \pi_1 = \frac{N \cdot u}{(1+u)}$.

5) The throughput of the I/O subsystem is

$$\gamma = \sum_{n=1}^{N} \mu_n \cdot P_n = \sum_{n=1}^{N} n \cdot \mu \cdot \binom{N}{n} \cdot \frac{u^n}{(1+u)^N} =$$

$$= \sum_{n=1}^{N} \lambda \cdot \frac{N!}{(n-1)! \cdot (N-n)!} \cdot \frac{u^{n-1}}{(1+u)^N} =$$

$$= N \cdot \frac{\lambda}{1+u} \cdot \sum_{n=1}^{N} \frac{(N-1)!}{(n-1)! \cdot \big(N-1-(n-1)\big)!} \cdot \frac{u^{n-1}}{(1+u)^{N-1}} =$$

$$= N \cdot \frac{\lambda}{1+u} \cdot \sum_{n=0}^{N-1} \binom{N-1}{n} \cdot \left(\frac{u}{1+u}\right)^n \cdot \left(\frac{1}{1+u}\right)^{(N-1)-n} =$$

$$= N \cdot \frac{\lambda}{1+u} = N \cdot \mu \cdot \frac{u}{1+u}$$

The last result is easily explained, since it states that the throughput is equal to the serving rate of each I/O peripheral, times the probability that the latter is occupied, times their overall number N.

PECSN 6/7/2021

Exercise 1

Consider the following function:

$$f(x) = \begin{cases} \frac{k}{c} \cdot \left(\frac{x}{c}\right)^{k-1} & 0 \le x \le c \\ 0 & otherwise \end{cases} .$$

1) Express the conditions under which $f(x)$ is a PDF
2) Compute the CDF $F(x)$, draw it for $k=1$ and $k=2$, and discuss what happens when $k \to \infty$.
3) Compute the mean value and the median. Justify the result when $k=1$ and $k \to \infty$.
4) Compute the variance. Justify the result.
5) Consider RV $S_j = \sum_{i=1}^{j} X_i$, where the X_i s are IID distributed according to $f(x)$. Compute the coefficient of variation *C* of S_j (CoV: $C = \sigma / E[S_j]$). Justify the result.

Exercise 2

A car repair service company has *n* repair bays, and expects customers' cars to come in for repair with exponentially distributed interarrivals, at a rate λ. The repair of a car takes an exponentially distributed time with a mean of $1/\mu$. The company wants to man the smallest possible number of repair bays (so as to save money), but knows that its customers find it unacceptable to have to wait.

1) Model the above system as a birth-death process and draw the transition diagram (CTMC)
2) Compute the steady-state probabilities. Express the stability condition
3) Compute the probability P_{wait} that a car that breaks has to *wait* before entering a repair bay
4) Assume $\lambda = \mu$. Compute P_{wait} as a function of *n* and study its behavior with *n*.
5) Under the above hypothesis, state whether 6 manned repair bays are enough to have P_{wait} smaller than $5 \cdot 10^{-4}$.

It may be useful to observe that $\lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{j!} = \lim_{n\to\infty} \left[\sum_{j=0}^{n} \frac{x^j}{j!} \right]_{x=1} = \left[e^x \right]_{x=1} = e$, and that $\sum_{j=0}^{n} \frac{1}{j!} \approx e$ when $n \ge 5$.

Exercise 1 - Solution

1) The conditions that must hold are that:

a) the function must be positive, hence $k > 0$

b) its integral must be equal to 1, i.e.: $\int_{-\infty}^{+\infty} f(x)dx = \frac{k}{c^k} \cdot \int_0^c x^{k-1}dx = \frac{k}{c^k} \cdot \left[\frac{x^k}{k}\right]_0^c = 1$.

Therefore, the only required condition is that $\boxed{k > 0}$.

2) The CDF can be easily obtained as: $\boxed{F(x) = \begin{cases} 0 & x < 0 \\ \left(\frac{x}{c}\right)^k & 0 \le x \le c \\ 1 & x > c \end{cases}}$.

As *k* grows large, the CDF tends to become more and more similar to a step function in $x = c$. Note that this can be interpreted as the distribution of the *maximum* of *k* IID RVs ~U(0,*c*), as seen during classes.

3) The mean value is $\boxed{E[X] = \int_0^c x \cdot f(x)dx = \int_0^c k \cdot \left(\frac{x}{c}\right)^k dx = \frac{k}{c^k} \cdot \left[\frac{x^{k+1}}{k+1}\right]_0^c = \frac{k \cdot c}{k+1}}$.

The median is instead obtained by equating $F(x) = 0.5$, i.e., $\left(\frac{x}{c}\right)^k = \frac{1}{2}$, i.e. $\boxed{x_{0.5} = \frac{c}{\sqrt[k]{2}}}$. When $k = 1$ the distribution is uniform, hence symmetric, thus $E[X] = x_{0.5}$. As *k* grows large, it is again $E[X] = x_{0.5} = c$, since the distribution becomes a step function.

4) The mean square value is: $E[X^2] = \int_0^c x^2 \cdot f(x)dx = \frac{k}{c^k} \cdot \left[\frac{x^{k+2}}{k+2}\right]_0^c = \frac{k \cdot c^2}{k+2}$.

Hence the variance is: $\boxed{\sigma^2 = E[X^2] - E[X]^2 = \frac{k \cdot c^2}{k+2} - \frac{k^2 \cdot c^2}{(k+1)^2} = \frac{k \cdot c^2}{(k+2) \cdot (k+1)^2}}$.

The variance depends on the square of *c*, since the latter is the range of the variable. Moreover, it decreases with *k*. This makes perfect sense, since, as *k* grows large, the variability of the distribution decreases as well, as already observed. When $k = 1$, we obtain again the well-known variance of a uniform variable.

5) The mean and variance are $j \cdot \frac{k \cdot c}{k+1}$ and $j \cdot \frac{k \cdot c^2}{(k+2) \cdot (k+1)^2}$ respectively. Thus, it is:

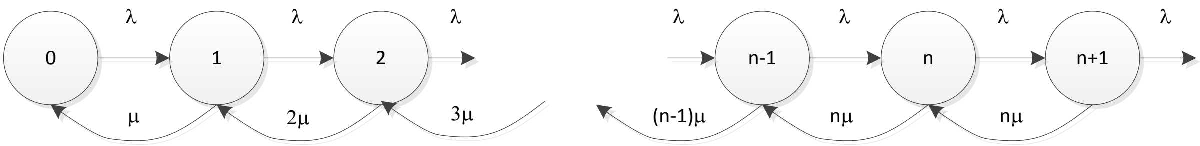
$$C = \frac{\sqrt{j \cdot \frac{k \cdot c^2}{(k+2)\cdot(k+1)^2}}}{j \cdot \frac{k \cdot c}{k+1}} = \frac{\sqrt{j \cdot \frac{k}{(k+2)}}}{j \cdot k} = \frac{1}{\sqrt{j \cdot k \cdot (k+2)}}$$

The result is in accord with the Central Limit Theorem, which is applicable since variables are IID with finite mean and variance. The CoV decreases with the number of variables, and it decreases *faster* if *k* is large. In fact, as *k* grows large, the mean converges, but the variance goes to zero. When *k* is large, the following approximation holds:

$$\boxed{C = \frac{1}{\sqrt{j \cdot k \cdot (k+2)}} \approx \frac{1}{k} \cdot \frac{1}{\sqrt{j}}}.$$

Exercise 2 - Solution

1) The system is an *M*/*M*/*n* one, hence the diagram is the following:



2) We know from the theory that the system is stable if $\rho = \lambda/(n \cdot \mu) < 1$. This should also emerge from the computation of the steady-state probabilities. The global equilibrium equations are the following:

$$
\begin{aligned}
&P_0 \cdot \lambda = P_1 \cdot \mu \\
&P_1 \cdot \lambda = P_2 \cdot 2\mu \\
&\ldots \\
&P_{n-1} \cdot \lambda = P_n \cdot n \cdot \mu \\
&P_n \cdot \lambda = P_{n+1} \cdot n \cdot \mu \\
&\ldots \\
&P_{n+j} \cdot \lambda = P_{n+j+1} \cdot n \cdot \mu \quad j \geq 0
\end{aligned}
$$

From which we get:

$$P_j = \begin{cases} \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \cdot P_0 & j < n \\ \rho^j \cdot \frac{n^n}{n!} \cdot P_0 & j \geq n \end{cases}$$

Hence, the normalization condition is:

$$P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{n^n}{n!} \cdot \sum_{j=n}^{\infty} \rho^n \right\} = 1$$

The infinite sum converges if and only if $\rho < 1$, as expected. This said,

$$P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu} \right)^j \cdot \frac{1}{j!} \right] + \frac{n^n}{n!} \cdot \left[\sum_{j=0}^{\infty} \rho^n - \sum_{j=0}^{n-1} \rho^n \right] \right\} = 1$$

$$P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu} \right)^j \cdot \frac{1}{j!} \right] + \frac{1}{n!} \cdot \frac{(n \cdot \rho)^n}{1-\rho} \right\} = 1 \qquad .$$

$$P_0 = \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu} \right)^j \cdot \frac{1}{j!} \right] + \frac{1}{n!} \cdot \frac{(n \cdot \rho)^n}{1-\rho} \right\}^{-1}$$

If *n* is large, the following approximation is reasonable: $P_0 = \dfrac{1}{e^{\lambda/\mu} + \dfrac{1}{n!} \cdot \dfrac{(n \cdot \rho)^n}{1-\rho}}$

3) Since the system enjoys the PASTA property, the probability that a car that breaks has to wait before entering service is the probability that $j \geq n$ customers are in the system, i.e. $\sum_{j=n}^{\infty} r_j = \sum_{j=n}^{\infty} P_j$. This can be written as:

$$P_{wait} = \sum_{j=n}^{\infty} P_j = \frac{n^n}{n!} \cdot P_0 \cdot \sum_{j=n}^{\infty} \rho^j = \frac{(n \cdot \rho)^n}{n!(1-\rho)} \cdot P_0 = \frac{\dfrac{(n \cdot \rho)^n}{n!(1-\rho)}}{\sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu} \right)^j \cdot \frac{1}{j!} \right] + \dfrac{(n \cdot \rho)^n}{n!(1-\rho)}} .$$

Again, if *n* is large, the following approximation is reasonable:

$$P_{wait} \approx \frac{1}{\dfrac{n!(1-\rho) \cdot e^{\lambda/\mu}}{(n \cdot \rho)^n} + 1}$$

4) Note that $\lambda = \mu$ implies $n > 1$, otherwise the system is unstable. When $\lambda = \mu$, we get:

$$P_{wait} = \frac{\dfrac{\left(n \cdot \frac{1}{n} \right)^n}{n! \left(1 - \frac{1}{n} \right)}}{\sum_{j=0}^{n-1} \left[\frac{1}{j!} \right] + \dfrac{\left(n \cdot \frac{1}{n} \right)^n}{n! \left(1 - \frac{1}{n} \right)}} = \frac{1}{(n-1)!(n-1) \cdot \sum_{j=0}^{n-1} \left[\frac{1}{j!} \right] + 1} .$$

The above statement is confirmed by the fact that $(n-1)$ appears in the denominator.

P_{wait} is obviously decreasing with *n* (since the denominator increases with *n*). Moreover, $P_{wait}(n) \geq \dfrac{1}{(n-1)!(n-1) \cdot e + 1}$, with $P_{wait}(n) \approx \dfrac{1}{(n-1)!(n-1) \cdot e + 1}$ when $n \geq 5$. The first few values are reported in the table:

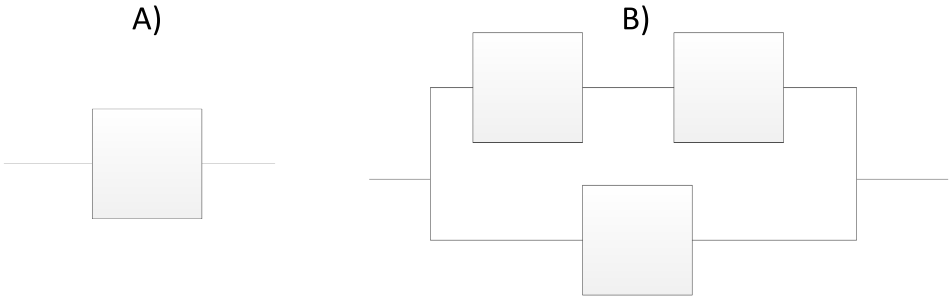
n	Pwait expr.	Numerical value
2	$1/3$	0.333333
3	$1/11$	0.090909
4	$1/49$	0.020408
5	$1/261$	0.003831

5) When $n=6$, it is $P_{wait}(n) \approx \frac{1}{600 \cdot e + 1}$. Since $e < 3$, it is $P_{wait}(n) > \frac{1}{1800} > \frac{1}{2000} = 5 \cdot 10^{-4}$, so the answer is no.

Exercise 1

ACME components owns two switch production plants. In plant 1, each unit is defective with probability $p_1 = 10^{-5}$, independently from the others. In plant 2, the mean weekly number of defective units is equal to 5. The production of each plant is $n = 4 \cdot 10^5$ units per week.

1) Compute mean and variance of the number of defective units produced by ACME in a week.
2) Draw a *qualitative* plot (with as many details are possible) of the PMF of the number of defective units in a week.
3) Compute the probability that the weekly number of defective units produced by ACME is equal to 5.
4) Compute the probability that the weekly number of defective units produced by ACME is less than 3.
5) Compute the probability that a randomly chosen unit is defective.



Suppose now that ACME units can be connected in series or in parallel as above, and that the resulting system works if there exists a way that connects both extremities traversing only non-defective systems.

6) Explain which of the two systems has a higher chance to be functioning. Justify your findings.

Exercise 2

A network buffer has enough space for three packets. It employs a *gated* policy, meaning that it only accepts ingresses when the system is *empty*. Ingresses come in the form of *messages*, each one containing *one*, *two or three* packets, with probability q_1, q_2, q_3 respectively. The interarrival time of messages is an exponentially distributed variable with a mean equal to $\frac{1}{\lambda}$. The buffer processes *packets* (not messages), and the service time of a packet is an exponentially distributed variable with a mean equal to $\frac{1}{\mu}$.

1) Model the system and draw the CTMC
2) Compute the steady-state probabilities and the stability condition
3) Determine which number of packets in the system is the most likely at the steady state
4) Compute the mean number of packets in the system and in the queue. State the conditions under which the mean number of packets in the system is larger than one.
5) Compute the system utilization.

Exercise 1 - Solution

1) Given that n is large and p is small, we can approximate the failure probability of each plant using a Poisson variable, whose average is $\lambda_i = n_i \cdot p_i$. Hence, it is $\lambda_1 = 4, \quad \lambda_2 = 5$. Thus, there are on average 9 defective units in a weekly production of $2n = 8 \cdot 10^5$ pieces. As for the variance, it is all the more reasonable to approximate the whole production using a Poisson variable, whose average and variance is equal to 9.
2) The Poisson variable has a bell shape, with an infinite right tail. It peaks around its mean value, which is equal to 9.

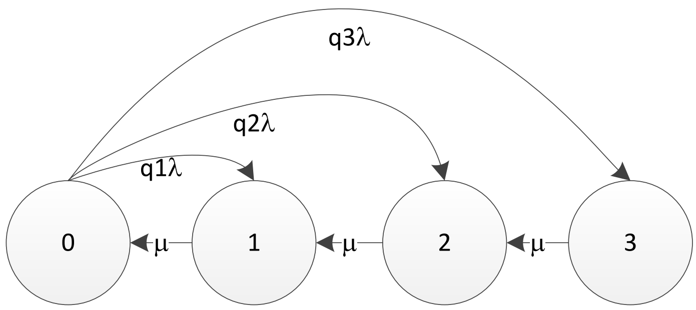
3) The probability that 5 pieces are defective is equal to $p_5 = e^{-9} \cdot \frac{9^5}{5!} = 0.060727$
4) The probability that less than 3 pieces are defective is equal to $p_0 + p_1 + p_2 = 1.23 \cdot 10^{-4} + 1.11 \cdot 10^{-3} + 4.998 \cdot 10^{-3} = 6.232 \cdot 10^{-3}$
5) The probability is the following:
$$\begin{aligned} p_d &= P\{\text{defective}\} \\ &= P\{\text{defective}|\text{plant 1}\} \cdot P\{\text{plant 1}\} + P\{\text{defective}|\text{plant 2}\} \cdot P\{\text{plant 2}\} \\ &= 10^{-5} \cdot 0.5 + \frac{5}{4 \cdot 10^5} \cdot 0.5 \\ &= 1.125 \cdot 10^{-5} \end{aligned}$$
6) System a) works with probability $p_a = 1 - p_d$. System b) works with probability
$$\begin{aligned} p_b &= 1 - P\{\text{upper branch fails}\} \cdot P\{\text{lower branch fails}\} \\ &= 1 - (1 - (1 - p_d)^2) \cdot p_d \end{aligned}$$

   Thus, $p_b > p_a$ if and only if
$$\begin{aligned} &1 - (1 - (1 - p_d)^2) \cdot p_d > 1 - p_d \\ &p_d > (1 - (1 - p_d)^2) \cdot p_d \\ &1 > 1 - (1 - p_d)^2 \\ &p_d < 1 \end{aligned}$$

   which is always true. System b) is always more reliable than system a), no matter what the failure probability of a single component is.

Exercise 2 – Solution

1) The CTMC is as follows. Note that $q_1 + q_2 + q_3 = 1$, obviously.



2) The steady state probabilities are computed by writing down the global equilibrium equations:
$$P_0 \cdot (q_1 + q_2 + q_3) \cdot \lambda = P_1 \cdot \mu$$
$$P_1 \cdot \mu = P_2 \cdot \mu + P_0 \cdot q_1 \cdot \lambda$$
$$P_2 \cdot \mu = P_3 \cdot \mu + P_0 \cdot q_2 \cdot \lambda$$
$$P_3 \cdot \mu = P_0 \cdot q_3 \cdot \mu$$

One of the above equations is redundant. The system is always stable, since it has a finite queue. By solving the above system, we obtain:

$$P_0 = \frac{1}{1+\frac{\lambda}{\mu}\cdot(q_1+2q_2+3q_3)} = \frac{1}{1+\frac{\lambda}{\mu}\cdot(1+q_2+2q_3)}$$

$$P_1 = \frac{\lambda}{\mu}\cdot P_0 = \frac{\lambda}{\mu}\cdot(q_1+q_2+q_3)P_0$$

$$P_2 = \frac{\lambda}{\mu}(1-q_1)\cdot P_0 = \frac{\lambda}{\mu}(q_2+q_3)\cdot P_0$$

$$P_3 = \frac{\lambda}{\mu}q_3\cdot P_0$$

3) From the above, it is clear that $P_1 \geq P_2 \geq P_3$, whatever the values q_i. Whether $P_0 > P_1$ or vice versa instead depends on whether $\lambda > \mu$ or vice versa. Thus, the answer is P_0 if $\lambda > \mu$, and P_1 otherwise.
4) It is:

$$E[N] = \sum_{n=1}^{3} n\cdot P_n = \frac{\lambda}{\mu}\cdot P_0\cdot[1\cdot(q_1+q_2+q_3)+2(q_2+q_3)+3q_3]$$

$$= \frac{q_1+3q_2+6q_3}{\frac{\mu}{\lambda}+q_1+2q_2+3q_3} = \frac{1+2q_2+5q_3}{\frac{\mu}{\lambda}+1+q_2+2q_3}$$

$$E[N_q] = 1\cdot P_2 + 2\cdot P_3 = \frac{\lambda}{\mu}\cdot P_0\cdot(q_2+3q_3) = \frac{q_2+3q_3}{\frac{\mu}{\lambda}+q_1+2q_2+3q_3} = \frac{q_2+3q_3}{\frac{\mu}{\lambda}+1+q_2+2q_3}$$

In order to have $E[N] > 1$ we need to have $q_1+3q_2+6q_3 > \frac{\mu}{\lambda}+q_1+2q_2+3q_3$, which translates to
$q_2+3q_3 > \frac{\mu}{\lambda}$.

The system utilization is $1-P_0$, i.e. $U = \frac{q_1+2q_2+3q_3}{\frac{\mu}{\lambda}+q_1+2q_2+3q_3} = \frac{1+q_2+2q_3}{\frac{\mu}{\lambda}+1+q_2+2q_3}$.

PECSN, 21/09/2021

Exercise 1

Consider two *independent* continuous RVs *X* and *Y*, whose CDFs are the following:

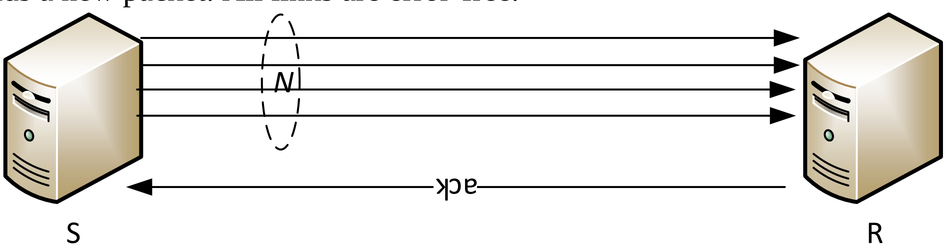
$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/3 & 0 \le x \le 3 \\ 1 & x > 3 \end{cases}, \quad F_Y(y) = \begin{cases} 0 & y < 0 \\ y^2/4 & 0 \le y \le 2 \\ 1 & y > 2 \end{cases}$$

1) Compute and draw the PDF of both RVs.
2) Draw the region *R* of the Cartesian plane x, y where the JPDF $f(x,y)$ is non null.
3) Identify the range of values for RV Y/X .
4) Let *w* be a value in the interval computed at bullet 3). Plot the region $C_w \subseteq R$, where inequality $Y/X \le w$ holds, on the Cartesian plane x, y. [Hint: try at least $w = 1/2$, $w = 1$].
5) Compute $P\{(X,Y) \in C_w\} = P\{Y/X \le w\} = F_{Y/X}(w)$.
6) Compute and draw the PDF of Y/X .

Exercise 2

Consider a network where a sender S sends packets to a receiver R, using *N* parallel forward links. On sending a packet, S chooses one forward link *at random* and sends the packet to it. Each forward link has a FIFO queue to buffer packets. On receipt of a packet (from any forward link), R sends an *ack* message through the return link. The transmission of both a packet and an ack takes an exponential time with a mean $1/\mu$.

The above communication is subject to a *flow-control protocol*. Flow is controlled by allowing at most *K* packets to be un-acked at any time. Therefore, at the beginning S generates *K* packets, sends each to a forward link at random, and then stops and waits for an ack to come back. Whenever an ack returns, S sends a new packet. All links are error-free.



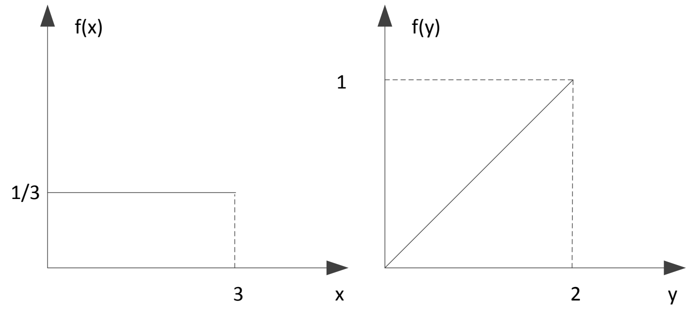
1) Model the above system as a Closed Jackson's Network.
2) Solve the routing equation system and compute the values of ρ_i.
3) Compute the steady-state probabilities as a function of *N* and *K* and interpret the result.

Assume *N*=3 and *K*=6

4) Compute G(M,K) using Buzen's algorithm
5) Compute the probability that S cannot send a packet due to flow control, even though all forward links are idle.
6) Compute the probability that all the forward links are simultaneously busy

Exercise 1 - Solution

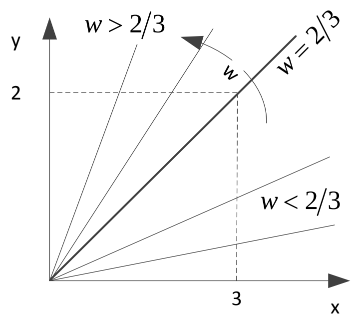
1) X is clearly a uniform RV, hence $f(x)=1/3$, for $0 \le x \le 3$, whereas Y's PDF is $f(y)=y/2$, for $0 \le y \le 2$. The graphs of the PDFs are the following:



2) The region of the Cartesian plane where the JPDF $f(x,y)=f(x)\cdot f(y)$ is non null is the one where the couple of RV (X,Y) may take on values, i.e. box $R \equiv [0 \le x \le 3]\times[0 \le y \le 2]$.

3) Since X and Y can only assume nonnegative values, Y/X can only assume nonnegative values. Moreover, since X's infimum is zero, then Y/X takes on values in $[0,+\infty[$.

4) Set C_w is $C_w = \{(x,y):(x,y)\in R,\ y/x \le w\}$, and it includes the area *below* each of the lines drawn in the figure, whose slope is w. Set C_w covers a triangle if $0 \le w \le 2/3$, and a trapezoid if $w > 2/3$.



5) It is $P\{(X,Y)\in C_w\} = \iint_{C_w} f(x,y)\,dxdy$, and $f(x,y)=f(x)\cdot f(y)=y/6$ for $(x,y)\in R$. We thus need to integrate $f(x,y)$ on set C_w. We need to distinguish the cases $0 \le w \le 2/3$ and $w > 2/3$.

a) $0 \le w \le 2/3$. We integrate y from 0 to $w\cdot x$, and x from 0 to 3.

$$P\{(X,Y)\in C_w\} = \iint_{C_w} \frac{y}{6}\,dx\,dy = \int_0^3 \left[\int_0^{w\cdot x} \frac{y}{6}\,dy\right] dx = \int_0^3 \left[\frac{w^2 x^2}{12}\right] dx = \frac{3}{4} w^2$$

b) $w > 2/3$. In this case, it is quicker to use the complement, i.e. to exclude the upper triangular region.

We need to invert the order of integration of the variables.

$$P\{(X,Y)\in C_w\} = \iint_{C_w} \frac{y}{6}\,dx\,dy = 1-\int_0^2\int_0^{y/w}\frac{y}{6}\,dxdy == 1-\int_0^2 \frac{y^2}{6w}\,dy = 1-\left[\frac{y^3}{18w}\right]_0^2 = 1-\frac{4}{9w}$$

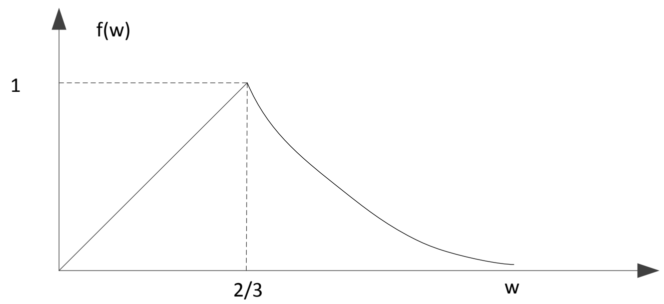
6) Since $P\{(X,Y)\in C_w\} = P\{Y/X \le w\} = F_{Y/X}(w)$, we have:

$$F_{Y/X}(w) = \begin{cases} \frac{3}{4}w^2 & 0 \le w \le 2/3 \\ 1-\frac{4}{9w} & w > 2/3 \end{cases}.$$

Note that $F_{Y/X}(w)$ is continuous, and $\lim_{w\to+\infty} F_{Y/X}(w) = 1$. From that, we get:

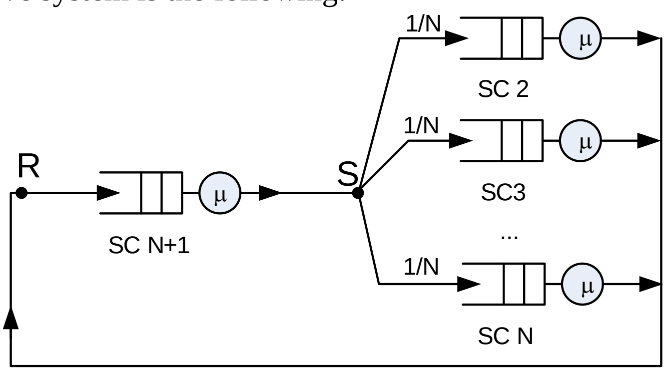
$$f_{Y/X}(w) = \begin{cases} \frac{3}{2}w & 0 \le w \le 2/3 \\ \frac{4}{9w^2} & w > 2/3 \end{cases}$$

The graph of the PDF is the following:



Exercise 2 - Solution

1) Let SCs 1 to N be each output interface at S, and let SC $N+1=M$ be the output interface at R. Then a CJN model of the above system is the following:



2) The routing matrix is the following: $\underline{\mathbf{\Pi}} = \begin{bmatrix} 0 & 0 & ... & 1 \\ 0 & 0 & ... & 1 \\ ... & ... & ... & ... \\ 1/N & 1/N & ... & 0 \end{bmatrix}$

And the equations are $\lambda_i = \lambda_{N+1}/N$, $1 \le i \le N$. It is fairly obvious that solutions are of the form $\underline{\mathbf{e}}^T = [e, e, ..., e, N \cdot e]^T$. Hence, we select $e = \mu$ and obtain $\underline{\boldsymbol{\rho}}^T = [1, 1, ..., 1, N]^T$.

3) The SS probabilities are $P(n_1, n_2, ..., n_{N+1})|_{\sum_{i=1}^{N+1} n_i = K} = \frac{1}{G(N+1, K)} \cdot \prod_{i=1}^{N+1} \rho_i^{n_i} = \frac{1}{G(N+1, K)} \cdot N^{n_{N+1}}$.

Since the above expression only depends on n_{N+1}, this means that any distribution of packets on the forward link queues having the same number of outstanding acks n_{N+1} is equally likely.

4) Buzen's algorithm yields the following table:

rho	**1**	**1**	**1**	**3**
s.c.	**1**	**2**	**3**	**4**
jobs				
0	1	1	1	1
1	1	2	3	6
2	1	3	6	24
3	1	4	10	82
4	1	5	15	261
5	1	6	21	804
6	1	7	28	2440

Hence $G(4,6) = 2440$.

Note that, due to the findings at point 3), G could also be computed using the normalization condition instead of Buzen's algorithm:

$$\sum_{\bar{n} \in \mathbf{e}} P(n_1, n_2, ..., n_{N+1}) = \sum_{\bar{n} \in \mathbf{e}} \left(\frac{1}{G(N+1, K)} \cdot N^{n_{N+1}} \right) = 1, \text{ from which:}$$

$$G(N+1, K) = \sum_{\bar{n} \in \mathbf{e}} N^{n_{N+1}} = \sum_{n_{N+1}=0}^{K} \binom{K - n_{N+1} + N - 1}{N - 1} N^{n_{N+1}}.$$

The binomial coefficient is the number of ways to split $K - n_{N+1}$ jobs (i.e., those that are not on SC N+1) among N SCs, and can be found on the course notes. The two methods obviously yield the same result.

5) The probability that the sender cannot send a packet due to flow control is the probability that *K acks* are queued at the return link, hence $p_{block} = \frac{1}{G(N+1, K)} \cdot N^K = \frac{3^6}{2440} = \frac{729}{2440} \approx 0.3$

6) The probability that all the forward links are simultaneously busy is:

$$p_{busy} = \frac{G(M, K-N)}{G(M, K)} \prod_{i=1}^{N} \rho_i^1 = \frac{G(4,3)}{G(4,6)} = \frac{82}{2440} \approx 0.033.$$

PECSN, 18/1/2022

Exercise 1

A safety-critical distributed application exchanges messages via a noisy channel. The sender keeps sending copies of the same message until it receives k ACKs from the receiver. The protocol is run on a slotted network: on every slot the sender sends a new copy of the message, and it receives an ACK with probability p. Transmissions in different slots are independent.
Call X_k the random variable that counts the number of slots until the communication is considered completed by the sender.

1) Find the set of values for X_k;
2) Compute the PMF of X_1, and explain your result;
3) Compute the PMF of X_2;
4) Compute the PMF of X_3 and generalize the above result to any value of k;
5) Compute the mean value of X_k.

Exercise 2

Consider a processing system where jobs arrive with exponential interarrival times with a mean $1/\lambda$ and request a service time which is also exponentially distributed with a mean $1/\mu$. To save power, the system server is switched *off* whenever the system is empty, and it is switched back *on* again when k jobs are in the system.

1) Model the system and draw the CTMC. Verify your model in the limit case $k = 1$;
2) Write the equilibrium equations, find the stability condition, and state explicitly its dependence on k. Justify your answer;
3) Compute the probability of having n jobs in the system at the steady state;
4) Compute the server utilization and the throughput.

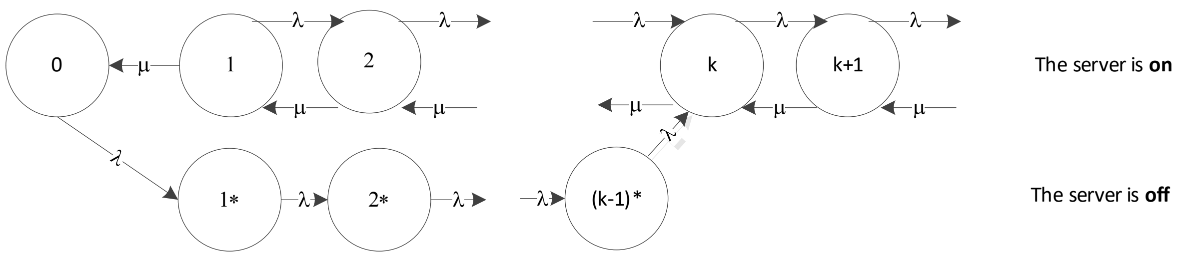
Exercise 1 – Solution

This is a repeated-trial experiment, with slots being trials.

1) The set of values for X_k is $k, +\infty)$.
2) X_1 is the number of trials to the first success, hence is a geometric RV: $P\{X_1 = j\} = (1-p)^{j-1} \cdot p$. Note that there is only one sequence of Bernoullian outcomes that terminates the repeated-trial experiment on trial j.
3) X_2 is the number of trials to the *second* success. $P\{X_2 = j\}$ can be computed by reasoning in terms of a composite experiment: one experiment consists in obtaining a success at the j^{th} trial, and the other consists in obtaining *one* success *in* $j-1$ trials. The first probability is equal to p, whereas the second one is equal to $\binom{j-1}{1}(1-p)^{j-2} \cdot p = (j-1) \cdot (1-p)^{j-2} \cdot p$. Therefore, we have: $P\{X_2 = j\} = (j-1) \cdot (1-p)^{j-2} \cdot p^2$.
4) X_3 is the number of trials to the *third* success. Again, this can be split into the probability of having a success on the j^{th} trial, times the probability of having $3-1=2$ successes in the first $j-1$ trials. The result is therefore: $P\{X_3 = j\} = p \cdot \left[\binom{j-1}{2} \cdot (1-p)^{j-3} \cdot p^2\right] = \binom{j-1}{2} \cdot (1-p)^{j-3} \cdot p^3$. From the above, it is clear that the correct expression is $P\{X_k = j\} = p \cdot \left[\binom{j-1}{k-1} \cdot (1-p)^{j-k} \cdot p^{k-1}\right] = \binom{j-1}{k-1} \cdot (1-p)^{j-k} \cdot p^k, j \geq k$.
5) One can observe that X_k is the sum of k geometric RVs, each one having a mean $1/p$. Therefore, it is $E[X_k] = k/p$. The same conclusion can be reached even without the above observation, via a modicum of algebra.

Exercise 2 - Solution

1) When j jobs are in the system, $0 < j < k$, the server may be either *on* or *off*. In either case, the system behaves differently. Thus, we need *two states* for each number of jobs in the system $0 < j < k$. The CTMC is therefore the following:



Starred states are traversed when the server is switched off. When $k = 1$, the interval $0 < j < k$ is empty, hence the system behaves like an M/M/1 and there are no starred states.

2) The stability condition should not depend on k. In fact, if the server is faster than the arrivals, the fact that k jobs are allowed to queue up does not thwart its ability to empty the queue. This must be confirmed by the computations.

The global equilibrium equations for the starred states are: $\lambda \cdot P_0 = \lambda \cdot P_{1^*} = \ldots = \lambda \cdot P_{(k-1)^*}$. Therefore, it is $P_{j^*} = P_0$, $1 < j \leq k-1$.

Moreover, it is easy to write *local* equilibrium equations along "vertical" cuts, as follows:

$$\begin{aligned} &\lambda \cdot P_o = \mu \cdot P_1 \\ &\lambda \cdot (P_j + P_{j^*}) = \mu \cdot P_{j+1} \quad 1 \leq j \leq k-1 \\ &\lambda \cdot P_j = \mu \cdot P_{j+1} \qquad\qquad j \geq k \end{aligned}$$

Setting $\rho = \frac{\lambda}{\mu}$ for ease of writing, we get:

$$\begin{aligned} &P_1 = \rho \cdot P_o \\ &P_{j+1} = \rho \cdot \left(P_j + P_0\right) \quad 1 \le j \le k-1 \\ &P_{j+1} = \rho \cdot P_j \qquad\qquad j \ge k \end{aligned}$$

Therefore, we get:

$$\begin{aligned} &P_1 = \rho \cdot P_o \\ &P_2 = \rho \cdot (P_1 + P_0) = (\rho^2 + \rho) \cdot P_0 \\ &P_3 = \rho \cdot (P_2 + P_0) = (\rho^3 + \rho^2 + \rho) \cdot P_0 \\ &\dots \\ &P_j = \left(\sum_{i=1}^{j} \rho^i\right) \cdot P_0 = \frac{\rho - \rho^{j+1}}{1-\rho} \cdot P_0 \qquad 1 \le j \le k \\ &P_j = \rho^{j-k} \cdot P_k = \rho^{j-k} \cdot \frac{\rho - \rho^{k+1}}{1-\rho} \cdot P_0 \qquad j > k \end{aligned}$$

[The last two expressions are correct only if $\rho \neq 1$. This condition is necessary for stability in any case.]

The normalization condition is therefore the following:

$$P_0 + \sum_{j=1}^{k-1} P_{j^*} + \sum_{j=1}^{k} P_j + \sum_{j=k+1}^{\infty} P_j = 1$$

Which can be rewritten as:

$$P_0 \cdot \left[k + \sum_{j=1}^{k} \frac{\rho - \rho^{j+1}}{1-\rho} + \frac{\rho - \rho^{k+1}}{1-\rho} \cdot \sum_{j=1}^{\infty} \rho^j\right] = 1$$

From the latter it clearly appears that the stability condition is $\rho < 1$. In fact, the first two terms are finite. The condition is thus independent of k, as was expected. Through a few algebraic manipulations, we get:

$$\begin{aligned} &P_0 \cdot \left[k + \sum_{j=1}^{k} \frac{\rho - \rho^{j+1}}{1-\rho} + \frac{\rho - \rho^{k+1}}{1-\rho} \cdot \sum_{j=1}^{\infty} \rho^j\right] = \\ &P_0 \cdot \left[k + \sum_{j=1}^{k} \frac{\rho}{1-\rho} - \sum_{j=1}^{k} \frac{\rho^{j+1}}{1-\rho} + \frac{\rho - \rho^{k+1}}{1-\rho} \cdot \frac{\rho}{1-\rho}\right] = \\ &P_0 \cdot \left[k + k \cdot \frac{\rho}{1-\rho} - \frac{\rho}{1-\rho} \cdot \frac{\rho - \rho^{k+1}}{1-\rho} + \frac{\rho - \rho^{k+1}}{1-\rho} \cdot \frac{\rho}{1-\rho}\right] = \\ &P_0 \cdot \frac{k}{1-\rho} = 1 \end{aligned}$$

3) From the above expressions we obtain:

$$\begin{aligned} &P_0 = \frac{1-\rho}{k} \\ &P_{j^*} = \frac{1-\rho}{k} \qquad 1 \le j \le k-1 \\ &P_j = \rho \cdot \frac{1-\rho^j}{k} \qquad 1 \le j \le k \end{aligned}$$

$$P_j = \rho^{j-k+1} \cdot \frac{1-\rho^k}{k} \qquad j > k$$

(Note that, when $k = 1$, we get the usual M/M/1 steady-state probabilities). Call π_j the SS probability to have j jobs in the system. It is:

$$\pi_j = \begin{cases} P_j & j = 0 \\ P_j + P_{j^*} & 1 \le j \le k-1 \\ P_j & j \ge k \end{cases}$$

Hence:

$$\pi_j = \begin{cases} \dfrac{1-\rho}{k} & j = 0 \\ \dfrac{1-\rho^{j+1}}{k} & 1 \le j \le k-1 \\ \rho^{j-k+1} \cdot \dfrac{1-\rho^k}{k} & j \ge k \end{cases}$$

4) When the system is stable, it is $\gamma = \lambda$. Since $\gamma = U \cdot \mu$, we get $U = \rho$. The same result can be obtained by observing that:

$$U = \sum_{j=1}^{+\infty} P_j = 1 - P_0 - \sum_{j=1}^{k-1} P_j^* = 1 - k \cdot \frac{1-\rho}{k} = \rho$$

PECSN 2/2/2022

Exercise 1

Consider a set of n six-faced fair dice, which are thrown independently. Call X_n the random variable denoting the *maximum* value obtained in the throw of all the dice.

1) Compute (as a formula) the PMF $f_2(x)$ of X_2;
2) Compute (as a formula) the PMF $f_3(x)$ of X_3. Hint: compute the CDF *first*;
3) Compute $f_n(x)$ for any number of dice n. Verify the normalization condition;
4) Draw a qualitative diagram of $f_n(x)$. Explain your findings;
5) Find the smallest number of dice *n* such that you exceed 90% probability that you will get a 6 as the maximum.

Exercise 2

A network buffer has enough space for three packets. It employs a *gated* policy, meaning that it only accepts ingresses when the system is *empty*. Ingresses come in the form of *messages*, each one containing *one, two or three* packets, with probability q_1, q_2, q_3 respectively. The interarrival time of messages is an exponentially distributed variable with a mean equal to $\frac{1}{\lambda}$. The buffer processes *packets* (not messages), and the service time of a packet is an exponentially distributed variable with a mean equal to $\frac{1}{\mu}$.

1) Model the system and draw the CTMC (or transition rate diagram);
2) Compute the steady-state probabilities and the stability condition;
3) Determine which number of packets in the system is the most likely at the steady-state;
4) Compute the mean number of packets in the system and in the queue. State the conditions under which the mean number of packets in the system is larger than one;
5) Compute the system utilization.

Exercise 1 – Solution

1) Since the dice are fair, each outcome $\{d_1, d_2\}$ is equally likely. Therefore, you can apply the basic principle of counting and obtain:
   - $f_2(1) = \frac{1}{36}$, (the "good" outcome is {1,1})
   - $f_2(2) = \frac{3}{36}$, (the "good" outcomes are {1,2}, {2,1}, {2,2})
   - $f_2(3) = \frac{5}{36}$, etc.,

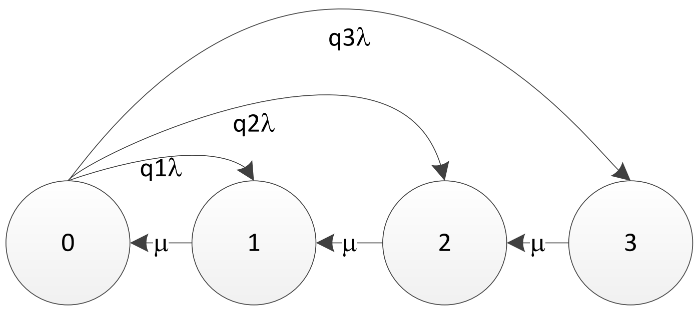
   Which leads to $f_2(x) = \frac{2x-1}{36}$, $1 \le x \le 6$.
2) The same reasoning applies. However, in order to compute (say) $f_3(4)$ we would need to count really many outcomes (i.e., all the permutations of {1,1,4}, {1,2,4}, etc.), and this is cumbersome, hence error-prone. Following the hint, it is considerably easier to compute $F_3(x) = P\{max[d_1, d_2, d_3] \le x\}$, $1 \le x \le 6$. In fact:
   - $F_3(1) = \frac{1}{6^3}\left(= \frac{1^3}{6^3}\right)$, (the only "good" outcome being {1,1,1})
   - $F_3(2) = \frac{2^3}{6^3}$, since the "good" outcomes are all the sequences $\{d_1, d_2, d_3\}$ where $d_i \le 2$.
   - Similarly, $F_3(k) = \frac{k^3}{6^3}$, $1 \le x \le 6$.

   Therefore, $f_3(k) = F_3(k) - F_3(k-1) = \frac{k^3-(k-1)^3}{6^3}$.

3) The above expressions immediately generalize to $F_n(k) = \frac{k^n}{6^n}$, and $f_n(k) = F_n(k) - F_n(k-1) = \frac{k^n-(k-1)^n}{6^n}$, $1 \le x \le 6$. The normalization condition is verified, since $F_n(6) = \frac{6^n}{6^n} = 1 \quad \forall n$. Note that the above formulas are compatible with the one found at point 1), once the required (obvious) simplifications are performed.
4) $f_n(k)$ is increasing with *k*, for any *n*. It is easy to see that:
$$\lim_{n\to\infty} f_n(k) = \lim_{n\to\infty} \frac{k^n - (k-1)^n}{6^n} = \lim_{n\to\infty}\left[\left(\frac{k}{6}\right)^n - \left(\frac{k-1}{6}\right)^n\right] = \begin{cases} 0 & k < 6 \\ 1 & k = 6 \end{cases}$$
   Therefore, as *n* grows large, the diagram tends to be *flat* towards zero for $k < 6$ and have a spike in *k*=6. This is because, as the number of dice grows large, it becomes increasingly unlikely that the maximum value obtained when throwing *n* dice is smaller than 6.
5) It is $f_n(6) = 0.9$, i.e. $1 - \left(\frac{5}{6}\right)^n = 0.9$. This yields $\left(\frac{5}{6}\right)^n = 0.1$, i.e. $n[log_{10} 5 - log_{10} 6] = -1$, i.e. $n = \left\lceil \frac{1}{log_{10} 6 - log_{10} 5} \right\rceil = 13$.

Exercise 2 - Solution

1) The CTMC is as follows. Note that $q_1 + q_2 + q_3 = 1$, obviously.



2) The steady state probabilities are computed by writing down the global equilibrium equations:

$$P_0 \cdot (q_1 + q_2 + q_3) \cdot \lambda = P_1 \cdot \mu$$
$$P_1 \cdot \mu = P_2 \cdot \mu + P_0 \cdot q_1 \cdot \lambda$$
$$P_2 \cdot \mu = P_3 \cdot \mu + P_0 \cdot q_2 \cdot \lambda$$
$$P_3 \cdot \mu = P_0 \cdot q_3 \cdot \mu$$

One of the above equations is redundant. The system is always stable, since it has a finite queue. By solving the above system, we obtain:

$$P_0 = \frac{1}{1 + \frac{\lambda}{\mu} \cdot (q_1 + 2q_2 + 3q_3)} = \frac{1}{1 + \frac{\lambda}{\mu} \cdot (1 + q_2 + 2q_3)}$$
$$P_1 = \frac{\lambda}{\mu} \cdot P_0 = \frac{\lambda}{\mu} \cdot (q_1 + q_2 + q_3) P_0$$
$$P_2 = \frac{\lambda}{\mu}(1 - q_1) \cdot P_0 = \frac{\lambda}{\mu}(q_2 + q_3) \cdot P_0$$
$$P_3 = \frac{\lambda}{\mu} q_3 \cdot P_0$$

3) From the above, it is clear that $P_1 \geq P_2 \geq P_3$, whatever the values q_i. Whether $P_0 > P_1$ or vice versa instead depends on whether $\lambda > \mu$. Thus, the answer is P_0 if $\lambda > \mu$, and P_1 otherwise.

4) It is:

$$E[N] = \sum_{n=1}^{3} n \cdot P_n = \frac{\lambda}{\mu} \cdot P_0 \cdot [1 \cdot (q_1 + q_2 + q_3) + 2(q_2 + q_3) + 3q_3]$$
$$= \frac{q_1 + 3q_2 + 6q_3}{\frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3} = \frac{1 + 2q_2 + 5q_3}{\frac{\mu}{\lambda} + 1 + q_2 + 2q_3}$$
$$E[N_q] = 1 \cdot P_2 + 2 \cdot P_3 = \frac{\lambda}{\mu} \cdot P_0 \cdot (q_2 + 3q_3) = \frac{q_2 + 3q_3}{\frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3} = \frac{q_2 + 3q_3}{\frac{\mu}{\lambda} + 1 + q_2 + 2q_3}$$

In order to have $E[N] > 1$ we need to have $q_1 + 3q_2 + 6q_3 > \frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3$, which translates to
$q_2 + 3q_3 > \frac{\mu}{\lambda}$.

5) The system utilization is $1 - P_0$, i.e.

$$U = \frac{q_1 + 2q_2 + 3q_3}{\frac{\mu}{\lambda} + q_1 + 2q_2 + 3q_3} = \frac{1 + q_2 + 2q_3}{\frac{\mu}{\lambda} + 1 + q_2 + 2q_3}$$

PECSN, 23/2/2022

Exercise 1

Consider function $f(x,y) = \begin{cases} e^{3x-\frac{y}{k}} & x \le 0, y \ge 0 \\ 0 & otherwise \end{cases}$, with $k > 0$.

1) Determine k so that the above is a JPDF for RVs X and Y
2) Compute the single PDFs of RVs X and Y
3) Are X and Y independent RVs? Are they identically distributed?
4) Compute the 95th percentile of X and the median of Y.

Exercise 2

Consider a manufacturing plant where K products are being processed at any time. As soon as a product is completed, another one is started. New products may require *one to four* stages of processing, with the same probability. When a new product arrives, it is examined at SC 5, and then sent to SCs 1,2,3,4, depending on the number of stages of processing it requires. Each SC, on terminating its processing of a product, sends it to the *next* SC for processing (i.e., SC 1 sends it to SC 2, etc.). A product that reaches SC 5 again is considered completed, and it leaves the plant (to be replaced by a new product, so as to keep K constant). Assume that the service rates at SCs 1-5 are all the same and equal to μ, and all SCs can be modeled as M/M/1 systems.

1) Model the system as a queueing network
2) Solve the routing equation system and compute the SS probabilities in its general form

Assume $K = 4$

3) Compute the normalizing constant using Buzen's algorithm
4) Compute the utilization of SC j, $j = 1, \ldots, 5$
5) Compute the probability that all jobs are on a single SC

PECSN, 23/2/2022

Exercise 1 – Solution

1) In order for f to be a JPDF, the normalization condition must hold, hence:

$$\int_0^{+\infty}\int_{-\infty}^{0} e^{3x-\frac{y}{k}}\,dxdy = 1$$
$$\int_0^{+\infty} e^{-\frac{y}{k}}\left[\int_{-\infty}^{0} e^{3x}dx\right]dy = 1$$
$$\frac{1}{3}\int_0^{+\infty} e^{-\frac{y}{k}}\,dy = 1$$
$$\frac{k}{3} = 1$$
$$k = 3$$

2) The PDFs for the single variables are:

$$f_X(x) = \int_0^{+\infty} e^{3x-\frac{y}{3}}\,dy = e^{3x}\int_0^{+\infty} e^{-\frac{y}{3}}dy = 3\cdot e^{3x}$$

$$f_Y(y) = \int_{-\infty}^{0} e^{3x-\frac{y}{3}}\,dx = e^{-\frac{y}{3}}\int_{-\infty}^{0} e^{3x}\,dx = \frac{1}{3}\cdot e^{-\frac{y}{3}}$$

3) The two PDFs are clearly independent, since $f(x,y) = f_X(x)\cdot f_Y(y)$. However, they are not equal, hence the two RVs are not IID.

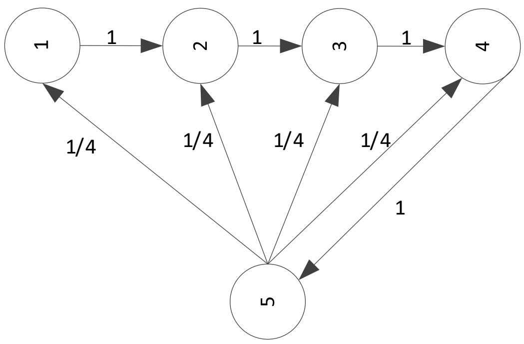
4) The 95th percentile of X can be found by imposing that $F_X(\pi_{95}) = \int_{-\infty}^{\pi_{95}} f_X(x)dx = 0.95$, i.e.

$$\int_{-\infty}^{\pi_{95}} 3e^{3x}dx = 0.95$$
$$e^{3\pi_{95}} = 0.95$$
$$\pi_{95} = \frac{1}{3}log(0.95)$$

Similarly, the median of Y is found by imposing $F_Y(\pi_{50}) = 0.5$, which yields $\pi_{50} = -3\,log(0.5)$.

Exercise 2

The network is a closed Jackson's network. A routing diagram is the following:



The routing matrix is :

$$\boldsymbol{\Pi} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

Hence the linear system to be solved is the following:

$$\begin{cases} \lambda_1 = \lambda_5/4 \\ \lambda_2 = \lambda_1 + \lambda_5/4 \\ \lambda_3 = \lambda_2 + \lambda_5/4 \\ \lambda_4 = \lambda_3 + \lambda_5/4 \\ \lambda_5 = \lambda_4 \end{cases}$$

A solution of the above is $\boldsymbol{e}^T = \begin{bmatrix} \frac{\mu}{4} & \frac{2\mu}{4} & \frac{3\mu}{4} & \mu & \mu \end{bmatrix}^T$, from which we obtain $\boldsymbol{\rho}^T = \begin{bmatrix} \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & 1 \end{bmatrix}^T$.

From this, we obtain the following SS probabilities:

$$p_{\underline{n}} = p(n_1, n_2, n_3, n_4, n_5) = \frac{1}{G(5, K)} \cdot \left[\frac{1}{4^{n_1}} \cdot \frac{2^{n_2}}{4^{n_2}} \cdot \frac{3^{n_3}}{4^{n_3}} \right] = \frac{1}{G(5, K)} \cdot \frac{3^{n_3}}{2^{2n_1+n_2+2n_3}}$$

Constant $G(5,4)$ can be computed through Buzen's algorithm.

rho	1/4	2/4	3/4	1	1
s.c.	1	2	3	4	5
jobs					
0	1	1	1	1	1
1	1/4	3/4	6/4	10/4	14/4
2	1/16	7/16	25/16	65/16	121/16
3	1/64	15/64	90/64	350/64	834/64
4	1/256	31/256	301/256	1701/256	5037/256

$$G(5,4) = \frac{5037}{256} \cong 19.68$$

The utilization of the SCs is $U_j = \rho_j \cdot \frac{G(5,3)}{G(5,4)} = \rho_j \cdot \frac{834}{64} \cdot \frac{256}{5037} = \rho_j \cdot \frac{3336}{5037} \cong 0.66 \cdot \rho_j$. Thus, the utilizations are: $\boldsymbol{U}^T = \begin{bmatrix} \frac{834}{5037} & \frac{1668}{5037} & \frac{2502}{5037} & \frac{3336}{5037} & \frac{3336}{5037} \end{bmatrix}^T \cong [0.17 \quad 0.33 \quad 0.5 \quad 0.66 \quad 0.66]^T$.

The probability that all jobs are on SC j is:

- $j = 1: p_{\underline{n}} = p(4,0,0,0,0) = \frac{1}{G(5,4)} \cdot \frac{3^0}{2^{8+0+0}} = \frac{256}{5037} \cdot \frac{1}{256} = \frac{1}{5037}$
- $j = 2: p_{\underline{n}} = p(0,4,0,0,0) = \frac{1}{G(5,4)} \cdot \frac{3^0}{2^{0+4+0}} = \frac{256}{5037} \cdot \frac{1}{16} = \frac{16}{5037}$
- $j = 3: p_{\underline{n}} = p(0,0,4,0,0) = \frac{1}{G(5,4)} \cdot \frac{3^4}{2^{0+0+8}} = \frac{256}{5037} \cdot \frac{81}{256} = \frac{81}{5037}$
- $j = 4: p_{\underline{n}} = p(0,0,0,4,0) = \frac{1}{G(5,4)} \cdot \frac{3^0}{2^{0+0+0}} = \frac{256}{5037}$
- $j = 5: p_{\underline{n}} = p(0,0,0,0,4) = \frac{1}{G(5,4)} \cdot \frac{3^0}{2^{0+0+0}} = \frac{256}{5037}$

Therefore, the probabilities that all jobs are at a single SC is the sum of the above, i.e.

$$p = \frac{1 + 16 + 81 + 256 + 256}{5037} = \frac{610}{5037} \cong 0.12$$

PECSN 15/06/2022

Exercise 1

Given an exponential RV Y, whose rate is α, consider RV $X = x_m \cdot e^Y$, $x_m > 0$.

1) Compute the interval of values for X, its CDF and its PDF
2) Compute the mean value of X and its variance
3) Is X heavy-tailed? Justify your answer
4) Compute conditional probability $P\{X > a + b | X > a\}$, with a,b positive values. Is the distribution memoryless? What happens to the above probability if $a \to +\infty$?

Exercise 2

Consider a computer system to which jobs are sent through an encrypted channel, with exponential interarrival times at a rate λ. When a *new* job arrives, the system first *decrypts* it. Decryption takes an exponential time, with a mean $\frac{1}{\alpha}$. Decrypted jobs are served in FIFO order with an exponential rate equal to μ. However, the system *cannot* perform any other task when decrypting a new job: either it is accepting/serving jobs, or it is decrypting one. More specifically, any arrival that occurs when the system is decrypting a job is lost, and any ongoing service is suspended and then resumed afterwards.

1) Model the system as a queueing system and draw the transition-rate diagram
2) Find the stability condition and compute the steady-state probabilities. State explicitly their dependence on α, and justify your answer.
3) Compute the mean arrival rate at the system and the tagged-arrival probabilities.
4) Compute the mean number of jobs in the system and the mean response time.

Exercise 1 – Solution

1) The interval of values for X is $x_m, +\infty)$.

$$F_X(x) = P\{X \le x\} = P\{x_m \cdot e^Y \le x\} = P\left\{Y \le log\left(\frac{x}{x_m}\right)\right\} = 1 - e^{-\alpha \cdot log\left(\frac{x}{x_m}\right)} = 1 - \left(\frac{x_m}{x}\right)^\alpha$$

$$f_X(x) = \frac{\partial}{\partial x} F_X(x) = \frac{\alpha \cdot {x_m}^\alpha}{x^{\alpha+1}}$$

2) The first two moments are:

$$E[X] = \int_{x_m}^{+\infty} x \cdot \frac{\alpha \cdot {x_m}^\alpha}{x^{\alpha+1}} dx = \frac{\alpha \cdot {x_m}^\alpha}{1-\alpha} \cdot \left[x^{(1-\alpha)}\right]_{x_m}^{+\infty} = \begin{cases} \frac{\alpha}{\alpha-1} \cdot x_m & \alpha > 1 \\ +\infty & otherwise \end{cases}$$

$$E[X^2] = \int_{x_m}^{+\infty} x^2 \cdot \frac{\alpha \cdot {x_m}^\alpha}{x^{\alpha+1}} dx == \alpha \cdot {x_m}^\alpha \int_{x_m}^{+\infty} x^{-\alpha+1} dx = \frac{\alpha \cdot {x_m}^\alpha}{2-\alpha} \cdot \left[x^{(2-\alpha)}\right]_{x_m}^{+\infty}$$

$$= \begin{cases} \frac{\alpha}{\alpha-2} \cdot {x_m}^2 & \alpha > 2 \\ +\infty & otherwise \end{cases}$$

From the above, we obtain:

$$\sigma^2 = E[X^2] - E[X]^2 = \begin{cases} \frac{\alpha}{\alpha-2} \cdot {x_m}^2 - \frac{\alpha^2}{(\alpha-1)^2} \cdot {x_m}^2 & \alpha > 2 \\ +\infty & otherwise \end{cases}$$

$$= \begin{cases} \frac{\alpha}{(\alpha-2)} \cdot \left(\frac{x_m}{\alpha-1}\right)^2 & \alpha > 2 \\ +\infty & otherwise \end{cases}$$

3) The definition of heavy-tailed distribution is: $\forall \lambda > 0, \quad \lim_{x \to \infty} e^{\lambda x} \cdot \left(1 - F(x)\right) = \infty$

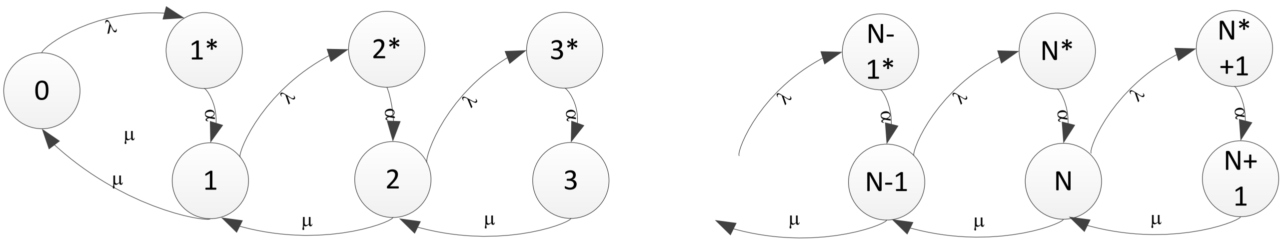
In this case, we have $\forall \lambda > 0, \quad \lim_{x \to \infty} e^{\lambda x} \cdot \left(\frac{x_m}{x}\right)^\alpha = +\infty$, hence the above distribution is heavy-tailed.

4) The above conditional probability is computed as follows:

$P\{X > a + b | X > a\} = \frac{P\{X > a+b\}}{P\{X > a\}} = \left(\frac{x_m}{a+b}\right)^\alpha \cdot \left(\frac{a}{x_m}\right)^\alpha = \left(\frac{a}{a+b}\right)^\alpha$. The distribution is not memoryless, since the above is not equal to $P\{X > b\} = \left(\frac{x_m}{b}\right)^\alpha$. When $a \to +\infty$, the conditional probability approaches 1. This means that if you have exceeded a very large value, then the probability that you will exceed an even larger one approaches 1.

Exercise 2 - Solution

1) The CTMC is the following. Note that the state of the system cannot be described using only the number of jobs, since the fact that the system is *decrypting* or *serving* jobs is also relevant. Starred states are those when the system is decrypting. Furthermore, you can observe that when $\alpha \to \infty$ (i.e., decryption is much faster than the rate of arrival or service) the two states in the same column collapse to one, and the system becomes an M/M/1.



2) Using local equilibrium equations, one can easily check that $p_j \cdot \lambda = p_{j+1} \cdot \mu$, hence $p_j = \left(\frac{\lambda}{\mu}\right)^j \cdot p_0$, $j \geq 0$. Moreover, using global equilibrium equation around the decrypting states, one obtains $p_j{}^* \cdot \alpha = p_{j-1} \cdot \lambda, j \geq 1$, hence $p_j{}^* = \frac{\mu}{\alpha} \cdot \left(\frac{\lambda}{\mu}\right)^j \cdot p_0,\ j \geq 1$.

The normalization condition then reads $p_0 \left[\sum_{j=0}^{+\infty} \left(\frac{\lambda}{\mu}\right)^j + \frac{\mu}{\alpha} \cdot \sum_{j=1}^{+\infty} \left(\frac{\lambda}{\mu}\right)^j\right] = 1$. Call $\rho = \frac{\lambda}{\mu}$, the stability condition is then $\rho < 1$, and it is independent of α. This is as expected, since during starred states the system is neither accepting nor serving jobs, so the decryption rate cannot have any influence on stability. After a few straightforward computations, one obtains:

$p_j = (1-\rho) \cdot \rho^j \cdot \frac{\alpha}{\lambda+\alpha},\ j \geq 0.$

$p_j{}^* = (1-\rho) \cdot \rho^j \cdot \frac{\mu}{\lambda+\alpha}, j \geq 1.$

3) The average arrival rate in the system is $\overline{\lambda} = \lambda \cdot \sum_{j=0}^{+\infty} p_j + 0 \cdot \sum_{j=0}^{+\infty} p_j{}^* = \lambda \cdot \frac{\alpha}{\lambda+\alpha}$. The tagged arrival probabilities are $r_j = p_j \cdot \frac{\lambda}{\overline{\lambda}} = (1-\rho) \cdot \rho^j, r_j{}^* = 0$.

4) The mean number of jobs in the system (which includes a job being decrypted, if there is one) is:

$$E[N] = \sum_{j=1}^{+\infty} j \cdot \left(p_j + p_j{}^*\right) = \frac{\mu+\alpha}{\lambda+\alpha} \cdot \sum_{j=1}^{+\infty} j \cdot (1-\rho) \cdot \rho^j = \frac{\mu+\alpha}{\lambda+\alpha} \cdot \frac{\rho}{1-\rho}$$

The mean traversal time for a job can be found by applying Little's Law, and it is:

$$E[R] = \frac{E[N]}{\overline{\lambda}} = \frac{\mu+\alpha}{\lambda+\alpha} \cdot \frac{\rho}{1-\rho} \cdot \frac{\lambda+\alpha}{\lambda \cdot \alpha} = \frac{\mu+\alpha}{\alpha} \cdot \frac{1}{\mu-\lambda}$$

Compito PECSN 06/07/2022

Exercise 1

Consider a system where a client can be randomly routed to *one* (and only one) among *n* servers, with probability p_i, $1 \leq i \leq n$. Each server's service time is an exponentially distributed RV, with a mean $\frac{1}{\mu_i}$. All servers are independent.

1) Find the CDF and PDF of the service time *of a client*
2) Find the mean of the service time
3) Consider an alternative design of the same system, where a client request is sent to *all the servers simultaneously*, and is only considered served when
   a. *at least one* server has processed it
   b. *all* the servers have processed it.

   Find the CDF of the service time of a client in these cases.
4) Assume that $\mu_i = \mu$. Is one design of the system (among the initial option and options *a* and *b* at point 3) faster? Why?

Exercise 2

A counseling practice offers individual advice to its clients. It admits both *singles* and *couples*, but counsels its clients *individually* (spouses are requested to wait outside the counseling room). The arrival rate at the counseling practice is λ. An arrival may be a *single*, with probability π, and a *couple*, with probability $1 - \pi$. Individual counseling takes an exponentially distributed time, with a rate μ.

1) Model the practice as a queueing system, draw the CTMC and write down the global equilibrium equations.
2) Find the stability condition, justify it, and compute the PGF of the SS probabilities.
3) Compute the mean number of jobs in the system. Verify in the limit cases.
4) Compute the probability p_0 that the practice is empty, and the probability that only one client is in, p_1

Solution of Exercise 1

1) Call S the service time RV. Due to the Law of Total Probability, we have:

$$F(s) = P\{S \le s\}$$
$$= \sum_{i=1}^{n} P\{S \le s \mid \text{server} = i\} \cdot P\{\text{server} = i\}$$
$$= \sum_{i=1}^{n} \left(1 - e^{-\mu_i \cdot s}\right) \cdot p_i$$
$$= \sum_{i=1}^{n} F_i(s) \cdot p_i$$

Hence:

$$f(s) = \frac{d}{ds} F(s) = \frac{d}{ds} \sum_{i=1}^{n} F_i(s) \cdot p_i = \sum_{i=1}^{n} p_i \cdot \mu_i \cdot e^{-\mu_1 \cdot s} = \sum_{i=1}^{n} f_i(s) \cdot p_i$$

2) The mean service time is $\sum_{i=1}^{n} \frac{1}{\mu_i} \cdot p_i$, since integrals and sums commute.

3) Case a. is a textbook case of minimum of independent exponential RVs. The theory says that the answers are $F(s) = 1 - e^{-\tau \cdot s}$, where $\tau = \sum_{i=1}^{n} \mu_i$

For case b, we have:

$$F(s) = P\{S \le s\} = P\{\max\{S_i\} \le s\}$$
$$= P\{S_1 \le s, S_2 \le s, \ldots, S_n \le s\}$$
$$= \prod_{i=1}^{n} \left(1 - e^{-\mu_i \cdot s}\right)$$
$$= \prod_{i=1}^{n} F_i(s)$$

4) When all the servers are indistinguishable, the CDF for the three systems is, respectively:
- $F_o(s) = 1 - e^{-\mu \cdot s}$ (original)
- $F_a(s) = 1 - e^{-n \cdot \mu \cdot s}$ (design a)
- $F_b(s) = (1 - e^{-\mu \cdot s})^n$ (design b)

It is easy to see that $\forall s > 0$, $F_a(s) > F_o(s) > F_s(s)$. In fact:

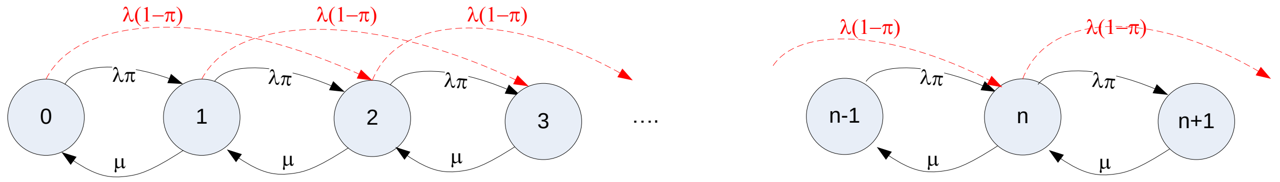
$$\begin{aligned} F_a(s) &> F_o(s) \\ 1 - e^{-n \cdot \mu \cdot s} &> 1 - e^{-\mu \cdot s} \\ e^{-n \cdot \mu \cdot s} &< e^{-\mu \cdot s} \\ -n\mu s &< -\mu s \\ n &> 1 \end{aligned}$$

Moreover, $F_o(s) > F_s(s)$ iff $1 - e^{-\mu \cdot s} > (1 - e^{-\mu \cdot s})^n$, which is obvious since the l.h.s. is between 0 and 1.

The above implies that design a is (probabilistically) faster than the original design, which is faster than b.

Exercise 2 – Solution

1) The system is an M/M/1 with bulk arrivals, and the CTMC diagram is below.



The global equilibrium equations are:

- State 0: $\lambda \cdot p_0 = \mu \cdot p_1$
- State 1: $(\lambda + \mu) \cdot p_1 = \mu \cdot p_2 + \lambda \cdot \pi \cdot p_0$
- State 2: $(\lambda + \mu) \cdot p_2 = \mu \cdot p_3 + \lambda \cdot \pi \cdot p_1 + \lambda \cdot (1 - \pi) \cdot p_0$
- State n: $(\lambda + \mu) \cdot p_n = \mu \cdot p_{n+1} + \lambda \cdot \pi \cdot p_{n-1} + \lambda \cdot (1 - \pi) \cdot p_{n-2}$

2) The RV that describes the size of the arrival is a Bernoullian g, such that $g_1 = P\{g = 1\} = \pi$, $g_2 = P\{g = 2\} = 1 - \pi$, hence $E[g] = 1 \cdot \pi + 2 \cdot (1 - \pi) = 2 - \pi$, $\boldsymbol{G}(z) = \pi \cdot z + (1 - \pi) \cdot z^2$. The computations for a generic $\boldsymbol{G}(z)$ can be found on the QT notes, and read:
- $\rho = \frac{\lambda}{\mu} E[g] = \frac{\lambda}{\mu} \cdot (2 - \pi)$. Note that, if $\pi = 1$, then the system is an M/M/1, and the stability condition is the usual one. Instead, if $\pi = 0$, the system is one with constant-batch bulk arrivals.
- $\boldsymbol{P}(z) = \frac{\mu \cdot (1-\rho) \cdot (1-z)}{\mu \cdot (1-z) - \lambda \cdot z \cdot [1 - \boldsymbol{G}(z)]}$.

By substituting the above $\boldsymbol{G}(z)$ into the above expression, after a few straightforward computations, we get:

$$\mathbf{P}(z) = \frac{\mu \cdot (1-\rho) \cdot (1-z)}{\mu \cdot (1-z) - \lambda \cdot z \cdot \left[1 - \pi \cdot z - (1-\pi) \cdot z^2\right]}$$
$$= \frac{\mu \cdot (1-\rho) \cdot \cancel{(1-z)}}{\mu \cdot \cancel{(1-z)} - \lambda \cdot z \cdot \cancel{(1-z)}\,[1 + z - \pi \cdot z]}$$
$$= \frac{\mu - \lambda \cdot (2-\pi)}{\mu - \lambda \cdot z - \lambda \cdot z^2 \cdot (1-\pi)}$$

3) Both expressions require computing the first derivative of the above expression, which is:

$$\frac{\partial}{\partial z}\boldsymbol{P}(z) = \frac{\partial}{\partial z}\left(\frac{\mu - \lambda \cdot (2-\pi)}{\mu - \lambda \cdot z - \lambda \cdot z^2 \cdot (1-\pi)}\right) = \frac{\lambda \cdot [1 + 2z \cdot (1-\pi)] \cdot [\mu - \lambda \cdot (2-\pi)]}{[\mu - \lambda \cdot z - \lambda \cdot z^2 \cdot (1-\pi)]^2}$$

From the above, we obtain:

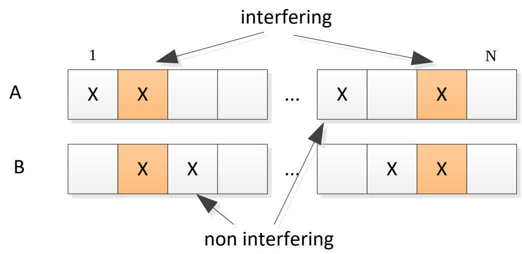
$$E[N] = \frac{\partial}{\partial z}\boldsymbol{P}(z)\bigg|_{z=1} = \frac{\lambda \cdot [3 - 2\pi] \cdot [\mu - (2-\pi)\lambda]}{[\mu - (2-\pi)\lambda]^2} = \frac{(3-2\pi)\lambda}{\mu - (2-\pi)\lambda}$$

When $\pi = 1$ the system is an M/M/1, and the above expression reads $E[N] = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1-\rho}$.
When $\pi = 0$ the system is a constant-batch one, with $b = 2$, and the expression is $E[N] = \frac{3\lambda}{\mu - 2\lambda}$. The expression on the notes reads $E[N] = \frac{\rho \cdot (b+1)}{2 \cdot (1-\rho)}$, which is equal to the former after some straightforward substitutions.

4) It is $p_0 = \lim_{z \to 0} \boldsymbol{P}(z) = 1 - \frac{\lambda}{\mu} \cdot (2 - \pi) = 1 - \rho$. This was expected, since ρ is the system utilization. Moreover, it is $p_1 = \frac{\partial}{\partial z}\boldsymbol{P}(z)\Big|_{z=0} = \frac{\lambda \cdot [\mu - \lambda \cdot (2-\pi)]}{\mu^2} = \frac{\lambda}{\mu} \cdot \left[1 - \left(\frac{\lambda}{\mu}\right) \cdot (2-\pi)\right] = \frac{\rho \cdot (1-\rho)}{2-\pi}$. If $\pi = 1$, the expression is the one of an M/M/1 system.

Exercise 1

Consider a system where two entities A and B allocate elements in their own vectors of N elements, numbered from 1 to N. We say that there is *interference* on element j if it is allocated in both A and B's vectors. Assume that each entity allocates its elements *at random*, and *independently*, and call n_A, n_B the *number of elements* allocated by each entity, $0 \le n_x \le N$.



1) Compute the probability that the allocation of A includes element 1.
2) Compute the probability that the allocation of A includes element j.
3) Compute the probability that there is interference on element j. Verify your answer in limit cases.
4) Compute L, U, i.e., the minimum and maximum number of interfering elements among all possible allocations.
5) Compute the probability that the allocations of A and B have exactly k interfering elements, for a generic k, $L \le k \le U$.
6) Assume that $n_A = n_B = n$. Compute the probability that the two allocations are completely overlapping. Find a combinatorial rationale for the result.

Exercise 2

A car repair service company has n repair bays, and expects customers' cars to come in for repair with exponentially distributed interarrivals, at a rate λ. The repair of a car takes an exponentially distributed time with a mean of $\frac{1}{\mu}$. The company wants to man the smallest possible number of repair bays (so as to save money), but knows that its customers find it unacceptable to have to wait.

1) Model the above system as a birth-death process and draw its CTMC.
2) Compute the steady-state probabilities. Express the stability condition.
3) Compute the probability P_{wait} that a car that breaks has to *wait* before entering a repair bay.
4) Assume $\lambda = \mu$. Compute P_{wait} as a function of n and study its behavior with n.
5) Under the above hypothesis, state whether 6 manned repair bays are enough to have P_{wait} smaller than $5 \cdot 10^{-4}$.

It may be useful to observe that $\lim\limits_{n\to\infty} \sum_{j=0}^{n} \frac{1}{j!} = \lim\limits_{n\to\infty} \left[\sum_{j=0}^{n} \frac{x^j}{j!}\right]_{x=1} = [e^x]_{x=1} = e$, and that $\sum_{j=0}^{n} \frac{1}{j!} \approx e$ when $n \ge 5$.

Exercise 1 - Solution

1) This is a uniform probability model. Therefore, the answer is $P = \frac{\binom{N-1}{n_A-1}}{\binom{N}{n_A}} = \frac{n_A}{N}$.

2) The answer is the same as before, there being nothing special about any particular element in the vectors.

3) Since the two allocations are independent, the answer is $\frac{n_A}{N} \cdot \frac{n_B}{N} = \frac{n_A \cdot n_B}{N^2}$. The result is *null* if either of the two allocations is null, and it is equal to one only if both entities allocate the whole vector.

4) It is fairly obvious that the number of interfering elements is upper bounded by $U = min(n_A, n_B)$. The lower bound is zero, if $n_A + n_B \leq N$, and $n_A + n_B - N$ otherwise. Thus, $L = max(0, n_A + n_B - N)$.

5) We are in a UPM. The sample space is the set of all possible allocations, whose cardinality is: $|\boldsymbol{S}| = \binom{N}{n_A} \cdot \binom{N}{n_B}$. There are $\binom{N}{k}$ subset of k interfering elements. These must be common to both allocations. This leaves A with $\binom{N-k}{n_A-k}$ ways to allocate the remaining $n_A - k$ non-interfering elements, and B with $\binom{N-n_A}{n_B-k}$ possible ways to allocate the remaining $n_B - k$ (note that the two expressions are different, since we cannot allow interference between the remaining $n_A - k$ of vector A and the remaining $n_B - k$ elements of vector B). By applying the basic principle of counting, we obtain:

$$P = \frac{\binom{N}{k} \cdot \binom{N-k}{n_A-k} \cdot \binom{N-n_A}{n_B-k}}{\binom{N}{n_A} \cdot \binom{N}{n_B}}$$

The alert reader can easily check that – despite the appearances – the above formula is symmetric (i.e., swapping n_A, n_B yields the same result).

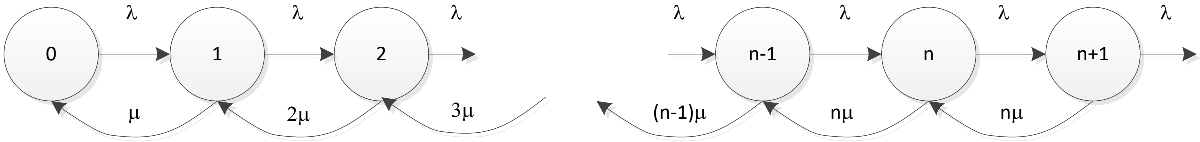
6) From the above, we obtain:

$$P = \frac{\binom{N}{n} \cdot \binom{N-n}{n-n} \cdot \binom{N-n}{n-n}}{\binom{N}{n} \cdot \binom{N}{n}} = \frac{1}{\binom{N}{n}}$$

The result can be easily explained as follows: assume A allocates n elements. Of all the possible allocations at B, which are $\binom{N}{n}$, there is only *one* that has exactly the same elements as the other.

Exercise 2 - Solution

1) The system is an *M*/*M*/*n* one, hence the CTMC is the following:



2) We know from the theory that the system is stable if $\rho = \frac{\lambda}{(n \cdot \mu)} < 1$. This should also emerge from the computation of the steady-state probabilities. The global equilibrium equations are the following:

$$
\begin{aligned}
&P_0 \cdot \lambda = P_1 \cdot \mu \\
&P_1 \cdot \lambda = P_2 \cdot 2\mu \\
&\dots \\
&P_{n-1} \cdot \lambda = P_n \cdot n \cdot \mu \\
&P_n \cdot \lambda = P_{n+1} \cdot n \cdot \mu \\
&\dots \\
&P_{n+j} \cdot \lambda = P_{n+j+1} \cdot n \cdot \mu \quad j \geq 0
\end{aligned}
$$

From which we get:

$$
P_j = \begin{cases} \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \cdot P_0 & j < n \\ \rho^j \cdot \frac{n^n}{n!} \cdot P_0 & j \geq n \end{cases}
$$

Hence, the normalization condition is:

$$
P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{n^n}{n!} \cdot \sum_{j=n}^{\infty} \rho^n \right\} = 1
$$

The infinite sum converges if and only if $\rho < 1$, as expected. This said,

$$
P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{n^n}{n!} \cdot \left[\sum_{j=0}^{\infty} \rho^n - \sum_{j=0}^{n-1} \rho^n \right] \right\} = 1
$$

$$
P_0 \cdot \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{1}{n!} \cdot \frac{(n \cdot \rho)^n}{1-\rho} \right\} = 1
$$

$P_0 = \left\{ \sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{1}{n!} \cdot \frac{(n \cdot \rho)^n}{1-\rho} \right\}^{-1}$.

If *n* is large, the following approximation is reasonable: $P_0 = \frac{1}{e^{\frac{\lambda}{\mu}} + \frac{1}{n!} \cdot \frac{(n \cdot \rho)^n}{1-\rho}}$

3) Since the system enjoys the PASTA property, the probability that a car that breaks has to wait before entering service is the probability that $j \geq n$ customers are in the system, i.e. $\sum_{j=n}^{\infty} r_j = \sum_{j=n}^{\infty} P_j$. This can be written as:

$$
P_{wait} = \sum_{j=n}^{\infty} P_j = \frac{n^n}{n!} \cdot P_0 \cdot \sum_{j=n}^{\infty} \rho^j = \frac{(n \cdot \rho)^n}{n!\,(1-\rho)} \cdot P_0 = \frac{\frac{(n \cdot \rho)^n}{n!\,(1-\rho)}}{\sum_{j=0}^{n-1} \left[\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1}{j!} \right] + \frac{(n \cdot \rho)^n}{n!\,(1-\rho)}}
$$

Again, if n is large, the following approximation is reasonable:

$$P_{wait} \approx \frac{1}{\frac{n!\,(1-\rho)\cdot e^{\frac{\lambda}{\mu}}}{(n\cdot\rho)^n}+1}$$

4) Note that $\lambda = \mu$ implies $n > 1$, otherwise the system is unstable. When $\lambda = \mu$, we get:

$$P_{wait} = \frac{\frac{\left(n\cdot\frac{1}{n}\right)^n}{n!\left(1-\frac{1}{n}\right)}}{\sum_{j=0}^{n-1}\left[\frac{1}{j!}\right]+\frac{\left(n\cdot\frac{1}{n}\right)^n}{n!\left(1-\frac{1}{n}\right)}} = \frac{1}{(n-1)!\,(n-1)\cdot\sum_{j=0}^{n-1}\left[\frac{1}{j!}\right]+1}$$

The above statement is confirmed by the fact that $(n-1)$ appears in the denominator. P_{wait} is obviously decreasing with n (since the denominator increases with n). Moreover, $P_{wait}(n) \geq \frac{1}{(n-1)!(n-1)\cdot e+1}$, with $P_{wait}(n) \approx \frac{1}{(n-1)!(n-1)\cdot e+1}$ when $n \geq 5$. The first few values are reported in the table:

n	Pwait expr.	Numerical value
2	$\frac{1}{3}$	0.333333
3	$\frac{1}{11}$	0.090909
4	$\frac{1}{49}$	0.020408
5	$\frac{1}{261}$	0.003831

5) When $n = 6$, it is $P_{wait}(n) \approx \frac{1}{600\cdot e+1}$. Since $e < 3$, it is $P_{wait}(n) > \frac{1}{1800} > \frac{1}{2000} = 5\cdot 10^{-4}$, so the answer is no.

PECSN, 7° appello 2022

Exercise 1

A lake contains an unknown number of fishes, call it n. The environment protection officers take a (random) sample of m fish from the lake, *mark* each of them with red ink (so that it will be recognizable later on), and return it to the lake.
Later, they take a second (random) sample of k fish.
Call X the RV that counts the number of red fishes in the second sample

1) Assume that $k = 1$. Compute the probability that $X = 1$.
2) Assume that $k = 2$. Compute the probability that $X = 0,1,2$.
3) For a generic k, and compute the probability that $X = x$, $0 \leq x \leq k$ (hint: reason about the probability model *first*).

Assume that $n \gg m \gg k$ from now on.

4) Find a suitable approximation for the previous expression;
5) Assume you observe x red fish in a sample of k. Using the result of point 4), compute an estimate of n and justify your result.

Exercise 2

Consider a system that can solve the same problem by running one among n different algorithms for that problem on a single processor. The system solves one problem at a time, and only accepts a new problem when it is idle. Problems arrive at rate λ. When a problem is admitted, the system selects the algorithm to be run in a probabilistic way: the probability that algorithm j is selected is equal to π_j. Each algorithm j has an exponential running time, whose mean is $\frac{1}{\mu_j}$.

1) Provide a suitable model for the system

2) find the steady-state probabilities and the stability condition

3) Compute the condition on the probabilities π_j such that it is *equally likely* to observe the system running any of the n algorithms at the steady state

4) Compute the mean number of jobs in the system and the mean response time. Justify the result for the latter.

Exercise 1 – solution

1) The probability that a fish is red is clearly $\frac{m}{n}$, hence this is the answer.

2) The probability that both fishes are red is $p_2 = \frac{m}{n} \cdot \frac{m-1}{n-1}$. The probability that none are is $p_0 = \frac{n-m}{n} \cdot \frac{n-m-1}{n-1}$. The probability that one fish is marked is

$$
\begin{aligned}
1 - p_2 - p_0 &= 1 - \frac{m}{n} \cdot \frac{m-1}{n-1} - \frac{n-m}{n} \cdot \frac{n-m-1}{n-1} \\
&= \frac{n^2 - n - \left(m^2 - m\right) - \left[\left(n-m\right)^2 - \left(n-m\right)\right]}{n \cdot \left(n-1\right)} \\
&= \frac{n^2 - n - m^2 + m - \left[n^2 + m^2 - 2nm - n + m\right]}{n \cdot \left(n-1\right)} \\
&= \frac{-m^2 - m^2 + 2nm}{n \cdot \left(n-1\right)} \\
&= 2 \cdot \frac{\left(n-m\right) \cdot m}{n \cdot \left(n-1\right)}
\end{aligned}
$$

3) The sample space is the set of all possible subset of k fish taken from a set of n. Each subset is equally likely (sampling is done at random), hence we are in a UPM. The number of possible outcomes is therefore $\binom{n}{k}$. In order to count the favorable outcomes, we use the basic principle of counting: the favorable outcomes will be $A \cdot B$, where A is the number of subsets of x red fish taken from a set of m, and B is the number of $k - x$ non-red fish taken from a set of $n - m$. Therefore, the answer is:

$P\{X = x\} = \frac{\binom{m}{x} \cdot \binom{n-m}{k-x}}{\binom{n}{k}}$.

4) There are at least two ways to answer this question. The first one is to observe that, if $n \gg m \gg k$, then $\frac{m-\alpha}{n-\alpha} \approx \frac{m}{n}$, $0 \leq \alpha \leq k$, hence the probability that the next fish will be red does not change significantly after you removed some fish from the lake. Therefore, you can regard picking red fish as a repeated trial experiment in (almost) independent conditions. Accordingly, the probability that you catch x red fish in a set of k will be (approximately) binomial, i.e.:

$P\{X = x\} \approx \binom{k}{x} p^x (1 - p)^{k-x}$, with $p = \frac{m}{n}$.

You can get to the same result by simplifying the previous formula according to the approximations:

$$P\{X = x\} = \frac{\binom{m}{x} \cdot \binom{n-m}{k-x}}{\binom{n}{k}}$$

$$= \frac{\left(\frac{m!}{x! \cdot (m-x)!}\right) \cdot \left(\frac{(n-m)!}{(k-x)! \cdot [(n-m)-(k-x)]!}\right)}{\frac{n!}{k! \cdot (n-k)!}}$$

$$\approx \frac{\left(\frac{m^x}{x!}\right) \cdot \left(\frac{(n-m)^{k-x}}{(k-x)!}\right)}{\frac{n^k}{k!}}$$

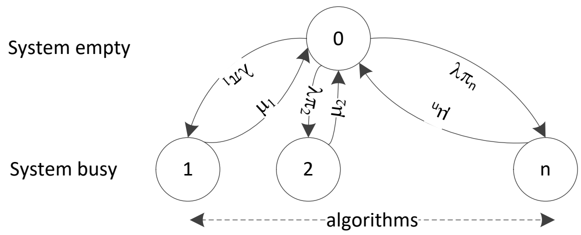
$$= \binom{k}{x} \cdot \left(\frac{m}{n}\right)^x \cdot \left(\frac{n-m}{m}\right)^{k-x}$$

5) Again, there are two ways to answer this question. The first one is acknowledging that taking the second sample is a bernoullian experiment. An estimate for the success probability of a bernoullian, given a sample of k observations, is $p = \frac{x}{k}$. Since we know that $p = \frac{m}{n}$, then it follows that $n = \frac{m \cdot k}{x}$.

Alternatively, you can reason that, since you observed x red fish in a sample of k, this can expected be the most likely outcome. The mode of a binomial distribution is around its mean value, which is $k \cdot p = k \cdot \frac{m}{n}$. Hence $x = k \cdot \frac{m}{n}$, which yields the same result.

Exercise 2 – solution

1) The system can be modeled by splitting probabilistically the arrival (Poisson) process using probabilities π_j. Algorithms are modeled as servers. The CTMC is the one below.



2) The system is always stable, since it has a finite queue (of one job). The SS probabilities can be computed using global equilibrium equations as follows:

$$\begin{cases} P_0 \cdot \lambda = \sum_{i=1}^{n} P_i \cdot \mu_i \\ P_i \cdot \mu_i = P_0 \cdot \lambda \cdot \pi_i \qquad 1 \le i \le n \end{cases}$$

From which – by imposing normalization - one easily finds:

$$\begin{cases} P_0 = \dfrac{1}{1 + \sum_{j=1}^{n} \dfrac{\lambda \cdot \pi_j}{\mu_j}} \\ P_i = \dfrac{\lambda \cdot \pi_i}{\mu_i} \cdot \dfrac{1}{1 + \sum_{j=1}^{n} \dfrac{\lambda \cdot \pi_j}{\mu_j}} \qquad 1 \le i \le n \end{cases}$$

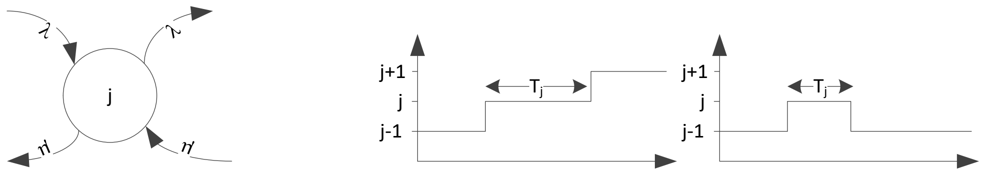
3) The condition by which all algorithms have the same probability to be observed running at the steady state is the condition by which $P_i = K$, $1 \le i \le n$. This is achieved if $\pi_i \propto \mu_i$: algorithms are as likely to be run as they are fast.

4) The system is empty in state 0 and holds one job in every other state. Hence $E[N] = 0 \cdot P_0 + (1 - P_0) \cdot 1 = 1 - P_0$. Moreover, it is $\bar{\lambda} = \lambda \cdot P_0$, since the system does not accept jobs while an algorithm is running, hence $E[R] = \frac{E[N]}{\bar{\lambda}} = \frac{1}{\lambda}\left(\frac{1}{P_0} - 1\right) = \sum_{i=1}^{n} \frac{\pi_i}{\mu_i}$. This last result has a straightforward interpretation: the only component of the response time is the service time, which is $\frac{1}{\mu_j}$ for algorithm j. However, algorithm j is run with probability π_j, hence the sum.

PECSN, 18 Jan 2023

Exercise 1

Consider a state $j > 0$ of a birth-death system like the one in the figure to the left, with exponential interarrival and service times, assumed to be independent. Let λ be the arrival rate and μ be the service rate, both constant.



Call T_j the RV that describes the <u>*time*</u> spent by the system in a <u>*single visit*</u> to state j, two samples of which are shown in the figure to the right. Do not confuse T_j with the proportion of time spent in state j at the steady state.

1) Compute the CDF of T_j;
2) Compute the mean value and variance of T_j;
3) Assume now that the system has *bulk* arrivals/services: the arrival rate is still λ and the service rate is still μ, but each arrival (service) can jump k states to the left (right) with some probability p_k (r_k). Answer again questions 1 and 2.

Exercise 2

Consider a system where service times are exponentially distributed with a mean $\frac{1}{\mu}$. The system is fed by *two independent and identical input processes*. Both inputs are Poisson processes with a rate λ. The first one is always active, and the second one is active only when an *odd* number of customers is in the system.

1) Model the system as a queueing system and draw the CTMC;
2) Write down the *local* equilibrium equations;
3) Derive the stability conditions and the steady-state probabilities;
4) Find the system throughput. Compare it to λ and justify the result;
5) Compute the steady-state probabilities r_j seen by an arriving customer.

PECSN, 18 Jan 2023

Exercise 1 - solution

1) Assume the system enters state j (say, at time t). It then leaves that state because either an arrival occurs, or a departure does. The time of either event is an exponential RV, with a rate λ and μ respectively. The fact that exponential RVs are memoryless implies that the *residual* interarrival/service times *seen at time* t are still exponential, and with the same rates.

Call T_A and T_B the RVs of an interarrival and service time. It is:

$$F_j(\tau) = P\{T_j \leq \tau\} = P\{min(T_A, T_B) \leq \tau\} = 1 - e^{-(\lambda+\mu)\cdot\tau}$$

Since the minimum among n independent exponential RVs is an exponential RV whose rate is the sum of the rates.

2) The answer is straightforward: $E[T_j] = \frac{1}{\lambda+\mu}$, $Var(T_j) = \frac{1}{(\lambda+\mu)^2}$.

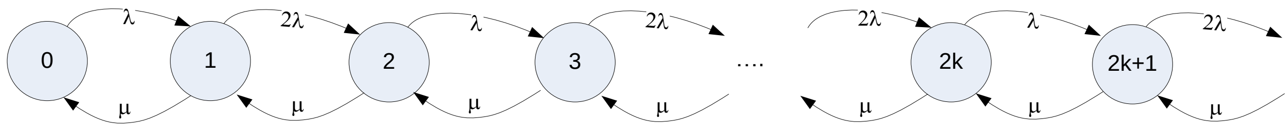
3) The answer is still the same, since:

a) the computations are unaffected by the probability of *entering* state j.

b) only the time at which arrivals and departures occur matters, whereas the length of the (right/left) arcs does not.

Exercise 2 - Solution

1) The CTMC is the following (note that the superimposition of two independent Poisson processes is a Poisson process):



2) In order to write down the equations, it pays to distinguish *odd* and *even* states. The local equilibrium equations are:

$$\begin{cases} \lambda \cdot p_{2k} = \mu \cdot p_{2k+1} & k \geq 0 \\ 2\lambda \cdot p_{2k-1} = \mu \cdot p_{2k} & k > 0 \end{cases}$$

from which we obtain – after some straightforward algebraic manipulations:

$$\begin{cases} p_{2k} = 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k} \cdot p_0 & k \geq 0 \\ p_{2k+1} = 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k+1} \cdot p_0 & k \geq 0 \end{cases}$$

3) The stability conditions is the following:

$$p_0 \cdot \left[\sum_{k=0}^{+\infty} 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k} + \sum_{k=0}^{+\infty} 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k+1}\right] = 1$$

$$p_0 \cdot \left(1 + \frac{\lambda}{\mu}\right) \cdot \sum_{k=0}^{+\infty} \left(\frac{\sqrt{2}\cdot\lambda}{\mu}\right)^{2k} = 1$$

$$p_0 \cdot \left(1 + \frac{\lambda}{\mu}\right) \cdot \sum_{k=0}^{+\infty} \left[\left(\frac{\sqrt{2}\cdot\lambda}{\mu}\right)^{2}\right]^{k} = 1$$

The series converges if and only if $\sqrt{2} \cdot \lambda < \mu$. Under the above condition, we obtain:

$$p_0 \cdot \left(1 + \frac{\lambda}{\mu}\right) \cdot \frac{1}{1 - 2\frac{\lambda^2}{\mu^2}} = 1$$

$$p_0 \cdot \frac{\lambda + \mu}{\cancel{\mu}} \cdot \frac{\mu^{\cancel{2}}}{\mu^2 - 2\lambda^2} = 1$$

$$p_0 = \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda + \mu)}$$

Thus, by applying the above formulas:

$$\begin{cases} p_{2k} = 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda + \mu)} & k \geq 0 \\ p_{2k+1} = 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k+1} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda + \mu)} & k \geq 0 \end{cases}$$

4) The system throughput can be found by applying the formula: $\gamma = \sum_{j=1}^{+\infty} \mu_j \cdot p_j = \mu \cdot (1 - p_0) = \mu \cdot \left[1 - \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda + \mu)}\right] = \lambda \cdot \frac{2\lambda + \mu}{\lambda + \mu}$.

   It is $\gamma > \lambda$, which makes sense since there are states when the arrival rate is 2λ.

5) The system is non-PASTA, hence $r_j = \frac{\lambda_j}{\bar{\lambda}} \cdot p_j$. We already have $\bar{\lambda}$, which can only be equal to γ ($\bar{\lambda}$ can also be obtained from the definition $\bar{\lambda} = \sum_{j=0}^{+\infty} \lambda_j \cdot p_j$, albeit with some more computations). Thus, we have:

$$r_{2k} = \frac{\lambda}{\gamma} \cdot p_{2k} = \frac{\lambda + \mu}{2\lambda + \mu} \cdot 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda + \mu)} = 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (2\lambda + \mu)}, \quad k \geq 0$$

$$r_{2k+1} = \frac{2\lambda}{\gamma} \cdot p_{2k+1} = 2 \cdot \frac{\lambda + \mu}{2\lambda + \mu} \cdot 2^k \cdot \left(\frac{\lambda}{\mu}\right)^{2k+1} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (\lambda + \mu)}$$

$$= 2^{k+1} \cdot \left(\frac{\lambda}{\mu}\right)^{2k+1} \cdot \frac{\mu^2 - 2\lambda^2}{\mu \cdot (2\lambda + \mu)}, \quad k \geq 0$$

Exercise 1

Consider the following JPDF of two RVs X and Y:

$$f(x,y) = \begin{cases} K & 0 \le x \le 1, \quad 0 < y < 1 - x \\ 0 & otherwise \end{cases}$$

where K is a real-valued constant.

1) Compute K *without using integration*;
2) Compute $P\{Y > \alpha\}$,$0 < \alpha < 1$, *without using integration*;
3) Compute $f_X(x), f_Y(y)$, and state if the two RVs are independent;
4) Compute the PDF of RV $W = \frac{Y}{X}$.

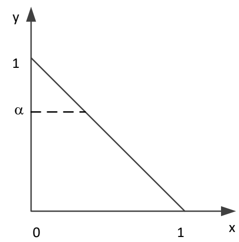
Exercise 2

Consider a system where a constant (and configurable) number of jobs K is processed cyclically through *three stages*. Stage j, $0 \le j \le 2$, is populated by $3 - j$ *identical* Service Centers, each one having a single server with a serving rate μ_j. Jobs leaving a SC in stage *j* go to stage $|j+1|_3$, and they are routed to each of the SCs of the new stage with the same probability.

1) Model the system as a closed queueing network
2) Find the routing matrix and compute the SS probabilities in their general form.
3) Explain the SS probabilities and find an alternative formulation that only includes the number of jobs at each *stage*.
4) Find $G(M,K)$ for a generic K assuming $\mu_1 = \frac{3}{2}\mu_0$, $\mu_2 = 3\mu_0$. Instantiate the result for $K = 9$.
5) Find the probability that all the jobs are at stage *j*

Exercise 1 – Solution

1) As per the figure, the JPDF is non null and equal to K within the triangle. The integral of that constant in a triangular area must be equal to 1. Therefore, it must be that $K = 2$. The same result can be obtained (more tediously) through integration.



2) With reference to the above drawing, $\{Y > \alpha\}$ is the triangle having the $(0,1)$,$(0,\alpha)$,$(1-\ \alpha,\alpha)$ as vertices. Since the JPDF is uniform, and its integral on the whole triangle is equal to one, the requested probability is just the proportion of the areas, i.e.

$$P\{Y > \alpha\} = \frac{\frac{(1-\alpha)^2}{2}}{\frac{1}{2}} = (1-\alpha)^2$$

3) By integration:

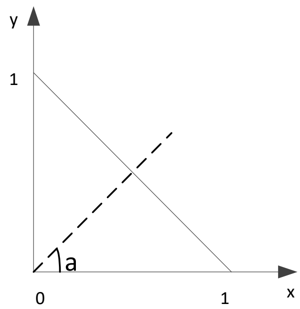
$$f_X(x) = \int_0^{1-x} f(x,y)dy = \int_0^{1-x} 2dy = 2(1-x),\ f_Y(y) = \int_0^{1-y} f(x,y)dx = \int_0^{1-y} 2dx = 2(1-y)$$

Thus, $f(x,y) \neq f_X(x) \cdot f_Y(y)$, hence the two RVs are not independent.

4) RV W is defined in $0,+\infty)$. It is $P\{W \leq a\} = P\{Y \leq a \cdot X\}$. The above inequality defines a triangle whose vertexes are $(0,0)$, $(1,0)$, $\left(\frac{1}{(1+a)},\frac{a}{(1+a)}\right)$, hence its base is equal to 1 and its height is equal to $\frac{a}{(1+a)}$. Following the same reasoning as for points 1 and 2, the probability is the area of that triangle normalized to the area of the initial triangle, i.e.

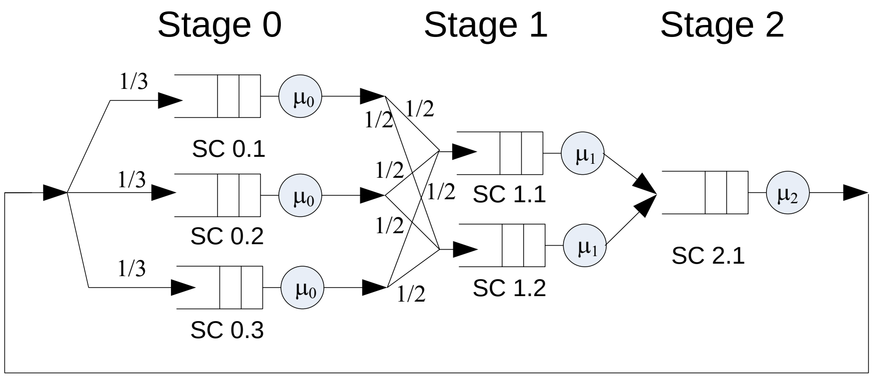
$$P\{W \leq a\} = P\{Y \leq a \cdot X\} = \frac{\frac{\left[\frac{a}{(1+a)}\right]}{2}}{\frac{1}{2}} = \frac{a}{(1+a)}$$

The PDF for W is the following: $f_W(a) = \frac{\partial}{\partial a}\left(\frac{a}{1+a}\right) = \frac{1}{(1+a)^2}$



Exercise 2 - solution

1) A model for the above system is the following:



The routing matrix is the following (row indexes are also reported for ease of reading):

$$\boldsymbol{\Pi} = \begin{array}{c} 0.1 \\ 0.2 \\ 0.3 \\ 1.1 \\ 1.2 \\ 2.1 \end{array} \begin{bmatrix} 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \end{bmatrix}$$

Hence the routing equations are:

$$\begin{cases} \lambda_{0.x} = \frac{1}{3}\lambda_{2.1} \quad 1 \le x \le 3 \\ \lambda_{1.y} = \frac{3}{2}\lambda_{0.x} \quad 1 \le x \le 3, \quad 1 \le y \le 2 \\ \lambda_{2.1} = 2 \cdot \lambda_{1.y} \quad 1 \le y \le 2 \end{cases}$$

2) From the above system (also due to reasons of symmetry), it is clear that $\lambda_{0.x} = \lambda_0, \lambda_{1.y} = \lambda_1$. A solution of the above system is $\boldsymbol{e}^T = \left[e, \quad e, \quad e, \quad \frac{3}{2}e, \quad \frac{3}{2}e, \quad 3e\right]$, with e being an arbitrary constant. Therefore, we can select $e = \mu_0$ and obtain $\boldsymbol{\rho}^T = \left[1, \quad 1, \quad 1, \quad \frac{3\mu_0}{2\mu_1}, \quad \frac{3\mu_0}{2\mu_1}, \quad 3\frac{\mu_0}{\mu_2}\right]$. By Gordon and Newell's Theorem, the SS probabilities are:

$$p(n_{0.1}, n_{0.2}, \dots, n_{2.1}) = \frac{1}{G(M,K)} \cdot \left(\frac{3\mu_0}{2\mu_1}\right)^{(n_{1.1}+n_{1.2})} \cdot \left(\frac{3\mu_0}{\mu_2}\right)^{n_{2.1}}$$

3) The above expression can be rewritten by observing that only the number of jobs *at each stage* matters, since probabilities are equal for every distribution of jobs within a stage:

$$p(K-(N_1+N_2), N_1, N_2) = \frac{1}{G(M,K)} \cdot \left(\frac{3\mu_0}{2\mu_1}\right)^{N_1} \cdot \left(\frac{3\mu_0}{\mu_2}\right)^{N_2}$$

4) Under the hypotheses, it is $\rho = 1$ at each stage. This means that $G(M,K) = |\mathcal{E}| = \binom{K+M-1}{M-1}$. With $K = 9$ we have $G(M,K) = \binom{14}{5} = 2002$.

5) The probability of each state is the same, and it is equal to

$$p = \frac{1}{G(M,K)} = \frac{1}{2002}$$

Therefore, we are in a UPM. The probability that all jobs are at stage j is $P_j = \frac{|E|}{|\mathcal{E}|}$, where $|E|$ is the number of possible combinations of K jobs at $3-j$ SCs, i.e., $|E| = \binom{K+(3-j)-1}{(3-j)-1}$. This yields:

$$P_0 = \frac{\binom{11}{2}}{2002} = \frac{55}{2002}$$

$$P_1 = \frac{\binom{10}{1}}{2002} = \frac{10}{2002}$$

$$P_2 = \frac{\binom{9}{0}}{2002} = \frac{1}{2002}$$

PECSN, 22/02/23

Exercise 1
Consider the following JPDF:

$$f(x,y) = \begin{cases} k \cdot x \cdot y & 0 \le x \le 2, \quad 0 \le y \le x \\ 0 & otherwise \end{cases}$$

1) Find k
2) Compute the PDFs of RVs X and Y. Are X and Y independent RVs? Justify your answer.
3) Compute the PDF of RV $Z = \frac{1}{X}$ and its mean value. Is Z heavy-tailed? Justify your answer.

Exercise 2
Consider a queueing system where jobs arrive with exponential interarrival times. The arrival rate is independent of the number of jobs currently in the system. The service time is also exponential, and the service rates may depend on the number of jobs in the system (and also be null in some states). Whenever a job ends its service, *all the waiting jobs are flushed*, i.e., the queue is emptied.

1) Model the system as a queueing system and draw the CTMC.
2) Write down the global equilibrium equations and the normalization condition in their general form.

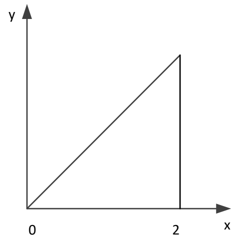
Assume that service rates are *constant*:

3) Compute the stability condition.
4) Compute the steady-state probabilities and the mean and variance of the number of jobs in the system.
5) What happens to stability if $\mu_j = 0 \ \forall j \ge k$? Justify your answer.

PECSN, 22/02/23

Exercise 1 – solution

The JPDF is defined in the triangle in the figure.



1) The normalization condition reads $\int_0^2 \left[\int_0^x k \cdot x \cdot y \ \ dy\right] \ dx = 1$, hence:

$$\int_0^2 \left[\int_0^x k \cdot x \cdot y \ \ dy\right] \ dx = k \cdot \int_0^2 x \cdot \left[\int_0^x y \ \ dy\right] \ dx = \ k \cdot \int_0^2 x \cdot \frac{x^2}{2} \ dx = k \cdot \left[\frac{x^4}{8}\right]_0^2 = 2k$$

The result is $k = \frac{1}{2}$.

The same result can be obtained inverting the order of the variables in the double integral, taking care to define the extremes correctly:

$$\int_0^2 \left[\int_y^2 k \cdot x \cdot y \ \ dx\right] \ dy = \cdots = 2k$$

2) The PDFs are:

$$f_X(x) = \frac{1}{2} x \cdot \int_0^x y \ \ dy = \frac{1}{4} x^3$$

$$f_Y(y) = \frac{1}{2} y \cdot \int_y^2 x \ \ dx = \frac{1}{2} y \cdot \left[2 - \frac{y^2}{2}\right] = y - \frac{y^3}{4}$$

It can easily be checked that $f(x, y) \neq f_X(x) \cdot f_Y(y)$, hence the two RVs are not independent.

3) $P\{X \leq x\} = F_X(x) = \frac{x^4}{16} = P\left\{\frac{1}{X} \geq \frac{1}{x}\right\} = 1 - F_Z\left(\frac{1}{x}\right)$. Therefore, $F_Z(z) = 1 - \frac{1}{16 \cdot z^4}$, $z \geq \frac{1}{2}$.

The PDF is $f_Z(z) = \frac{1}{4z^5}$. The mean value of Z is

$$E[Z] = \int_{\frac{1}{2}}^{+\infty} \frac{1}{4z^5} \cdot z \ \ dz = \frac{1}{4} \int_{\frac{1}{2}}^{+\infty} \frac{1}{z^4} \ \ dz = \frac{1}{4}\left[-\frac{1}{3z^3}\right]_{\frac{1}{2}}^{+\infty} = \frac{2}{3}$$

The definition of heavy-tail is the following: $\forall \lambda > 0, \quad \lim_{z \to \infty} e^{\lambda z} \cdot \left(1 - F_Z(z)\right) = \infty$

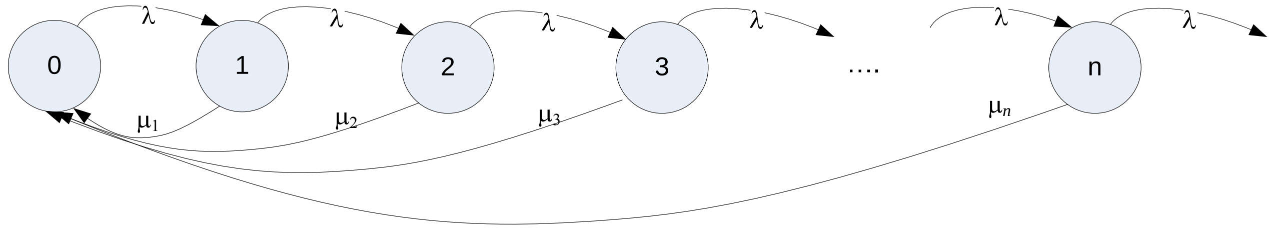
By substitution, we get:

$$\lim_{z \to \infty} e^{\lambda z} \cdot \frac{1}{16z^4} = +\infty, \forall \lambda > 0.$$

Therefore, Z is heavy-tailed.

Exercise 2 – solution

1) The CTMC is the following:



2) The global equilibrium equations in their general form are:

$$\begin{cases} p_0 \cdot \lambda = \sum_{i=1}^{+\infty} p_i \cdot \mu_i \\ p_j \cdot (\lambda + \mu_j) = p_{j-1} \cdot \lambda \quad j \geq 1 \end{cases}$$

From the second one we easily obtain $p_j = p_0 \cdot \frac{\lambda^j}{\prod_{i=1}^{j}(\lambda+\mu_i)}$.

From the above, normalization reads: $p_0 \cdot \left[1 + \sum_{j=1}^{+\infty} \frac{\lambda^j}{\prod_{i=1}^{j}(\lambda+\mu_i)}\right] = 1$.

3) If service rates are state-independent, it is $\mu_j = \mu$, hence the infinite sum in the normalization becomes: $\sum_{j=1}^{+\infty} \frac{\lambda^j}{(\lambda+\mu)^j}$

Call $\theta = \frac{\lambda}{\lambda+\mu}$, If $\mu \neq 0$ it is $\theta < 1$, hence the above sum converges to $\sum_{j=1}^{+\infty} \theta^j = \frac{\theta}{1-\theta} = \frac{\frac{\lambda}{\lambda+\mu}}{1-\frac{\lambda}{\lambda+\mu}} = \frac{\lambda}{\mu}$.

Therefore, the stability condition is $\mu \neq 0$.

4) With non-null, state-independent service rates, it is $p_0 = \frac{\mu}{\lambda+\mu}$, $p_j = p_0 \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^j = \frac{\mu}{\lambda} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{j+1}$.

Hence, $p_j = \frac{\mu}{\lambda} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{j+1}$, $j \geq 0$.

Therefore, we have $P(z) = \sum_{j=0}^{+\infty} p_j \cdot z^j = \frac{\mu}{\lambda+\mu} \cdot \sum_{j=0}^{+\infty} \left(\frac{\lambda \cdot z}{\lambda+\mu}\right)^j = \frac{\mu}{\lambda(1-z)+\mu}$. From the latter, we get:

$$P'(z) = \frac{\partial}{\partial z} \frac{\mu}{\lambda(1-z)+\mu} = \frac{\lambda \cdot \mu}{[\lambda(1-z)+\mu]^2}$$

$$P''(z) = \frac{\partial}{\partial z} P'(z) = \frac{\lambda \cdot \mu \cdot [2\lambda^2 z - 2\lambda(\lambda+\mu)]}{[\lambda(1-z)+\mu]^4}$$

And it is:

$$E[N] = P'(1) = \frac{\lambda}{\mu},$$

$$Var(N) = P''(1) + P'(1) - P'(1)^2 = \frac{2\lambda^2}{\mu^2} + \frac{\lambda}{\mu} - \frac{\lambda^2}{\mu^2} = \frac{\lambda}{\mu} \cdot \left(1 + \frac{\lambda}{\mu}\right)$$

5) If the service rates are identically null starting from some state k onward, then it is $p_j = p_k \; \forall j \geq k$. Then, normalization reads:

$$p_0 + \sum_{j=1}^{k-1} p_j + \sum_{j=k}^{+\infty} p_k = 1$$

The infinite sum diverges, unless $p_k = 0$. This means that the system is unstable. This could have been observed right from the CTMC: if the system hits state k, which happens with nonnull probability, then it will never empty again, because there is no longer a path leading to state zero.

PECSN – 14/06/23

Exercise 1

Consider the following function:

$$f(x) = \begin{cases} 0 & x < 1 \\ \alpha \cdot x^{-(\alpha+1)} & x \geq 1 \end{cases}$$

With α being an integer.

1) State the conditions under which $f(x)$ is a PDF;
2) Compute the CDF and the mean.

Let X be a RV distributed according to the above PDF.

3) Compute the PDF for RV $Y = log(X)$;
4) Assume that RVs $X_1, X_2, \dots, X_n$ are IID according to $F_X(x)$ computed at point 2. Compute the CDF for $W = \max_i\{1/X_i\}$. Discuss the behavior of the CDF as a function of n.

Exercise 2

SpeedyMatch is a blind speed-dating agency that arranges dates between strangers of opposite sexes. A male that wants to use their services shows up at their premises, and:

- If there are already females at the premises, he picks up the longest-waiting one and they leave (instantly);
- Otherwise, he queues up (FIFO) behind other males.

Dually, a female that wants to use SpeedyMatch shows up at their premises, and:

- If there are already males at the premises, she picks up the longest-waiting one and they leave (instantly);
- Otherwise, she queues up (FIFO) behind other females.

Speedymatch has K seats available for queueing at its premises, and it does not let customers queue up if they cannot seat. Males and females arrive with exponential interarrival times, at a rate λ_M and λ_F respectively.

1) Model the system as a queueing system and draw a CTMC;
2) Write down the local equilibrium equations and find the stability condition and the SS probabilities;
3) Find the probability that a customer is rejected;
4) Find the mean response time for a female customer (assuming λ_M = λ_F). Verify your answer in the limit case $K = 1$ and explain the result.

Exercise 1 - Solution

f(x) is a PDF if it is always non-negative and normalization holds. The first condition is verified if $\alpha \geq 0$. The second is verified if:

$$\int_{1}^{+\infty} \alpha \cdot x^{-(\alpha+1)} dx = 1$$

The above integral converges if and only if $\alpha \geq 1$, which is therefore the required condition. Under the latter hypothesis, we have:

$$F(x) = \int_{1}^{x} \alpha \cdot y^{-(\alpha+1)} dx \, [-y^{-\alpha}]_{1}^{x} = 1 - \frac{1}{x^{\alpha}}$$

The mean value is:

$$E[x] = \int_{1}^{+\infty} x \cdot \alpha \cdot x^{-(\alpha+1)} dx = \int_{1}^{+\infty} \alpha \cdot x^{-\alpha} dx$$

The above integral is infinite unless $\alpha \geq 2$. Under the latter hypothesis, we have:

$$E[x] = \frac{\alpha}{\alpha - 1} \cdot \left[-x^{-(\alpha-1)}\right]_{1}^{+\infty} = \frac{\alpha}{\alpha - 1}$$

$$F_Y(y) = P\{Y \leq y\} = P\{\log(X) \leq y\} = P\{\mathrm{X} \leq e^{y}\} = 1 - \frac{1}{(e^{y})^{\alpha}} = 1 - e^{-\alpha y}$$

And, since $X \geq 1$, then $Y \geq 0$. Therefore, it is $f_Y(y) = \alpha \cdot e^{-\alpha y}$.

Call $Z = 1/X_i$. We have:

$F_Z(z) = P\{Z \leq z\} = P\{\mathrm{X} > 1/z\} = 1 - F_X(1/z) = z^{\alpha}$.

And, since $X \geq 1$, then $0 \leq Z \leq 1$.

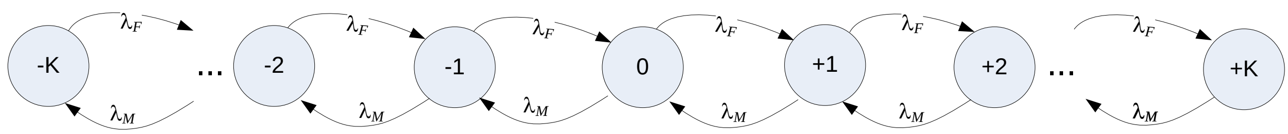
Now, it is:

$$F_W(w) = P\{Z_1 \leq w, Z_2 \leq w, \dots, Z_n \leq w\} = [F_Z(w)]^n = w^{n\alpha}$$

Since the support of RV Z is limited, then the CDF will decrease with n, and will tend to a step function as n goes to infinity.

Exercise 2 - Solution

1) Given the requirements, it is impossible that males and females queue up simultaneously. Therefore, if there is a queue, it will *only* contain either males or females, and it will be of at most K customers. When there is a queue, the arrival of a customer of the same sex increases the queue, and the arrival of a customer of the opposite sex decreases the queue. Thus, a state description for this system is given by the length of the queue and a binary information (male, female), which can be represented by a *sign* (arbitrarily, + is female and – is male). The system has $2K + 1$ states, and the transition diagram is the following:



The system is always stable since it has a finite number of states.

2) The local equilibrium equations are $p_j \cdot \lambda_F = p_{j+1} \cdot \lambda_M,\ -K \le j < K.$

From the above, one obtains straightforwardly that $p_j = p_{-K} \cdot \rho^{j+K}$, $-K \le j \le K$, where $\rho = \frac{\lambda_F}{\lambda_M}$.

The normalization condition reads $\sum_{-K}^{+K} p_j = 1$, hence:

$$\sum_{-K}^{K} p_{-K} \cdot \rho^{j+K} = p_{-K} \cdot \sum_{0}^{2K} \rho^j = \begin{cases} p_{-K} \cdot \dfrac{1-\rho^{2K+1}}{1-\rho} & if\ \rho \ne 1 \\ p_{-K} \cdot (2K+1) & if\ \rho = 1 \end{cases}$$

Therefore,

$$p_j = \begin{cases} \dfrac{1-\rho}{1-\rho^{2K+1}} \cdot \rho^{j+K} & if \rho \ne 1 \\ \dfrac{1}{2K+1} & if \rho = 1 \end{cases}, \qquad -K \le j \le K,$$

3) The probability that a customer is rejected is $p_R = p_{-K} + p_{+K}$. Therefore:

$$p_R = \begin{cases} \dfrac{1-\rho}{1-\rho^{2K+1}} (1+\rho^{2K}) & \rho \ne 1 \\ \dfrac{2}{2K+1} & \rho = 1 \end{cases}$$

4) This is a non-PASTA system. The mean arrival rates for females and males are, respectively:

$$\overline{\lambda_F} = \lambda_F \cdot (1 - p_K)$$

$$\overline{\lambda_M} = \lambda_M \cdot (1 - p_{-K})$$

The response time of a female client is:

- null, if males are queued, i.e., with probability $p = \sum_{-K}^{-1} r_j$, where

$$r_j = \frac{\lambda_F}{\overline{\lambda_F}} \cdot p_j = \frac{p_j}{1 - p_K} = \frac{1}{2K}$$

- equal to the sum of the arrival times of $j+1$ males if she arrives when the system is in state j (i.e., with probability r_j, $0 \le j \le K-1$). The sum of $j+1$ exponentials is an Erlang distribution with as many stages, and its mean value is $(j+1)/\lambda_M$. Therefore:

$$E[R] = \sum_{j=0}^{K-1} \frac{j+1}{\lambda_M} \cdot \frac{1}{2K} + \sum_{j=-K}^{-1} 0 \cdot \frac{1}{2K} = \frac{1}{\lambda_M} \cdot \frac{1}{2K} \cdot \frac{K \cdot (K+1)}{2} = \frac{1}{\lambda_M} \cdot \frac{K+1}{4}$$

In the limit case $K = 1$ the above expression boils down to $\frac{1}{\lambda_M} \cdot \frac{1}{2}$. In fact, in this case an arrival of a female may occur only in state -1 or 0, with equal probability. In the first case, the response time will be zero, and in the second case it will be the mean arrival time of a male customer, which explains the result.

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Exercise 1

Consider the following function, where $\lambda > 0$:

$$f(t) = A_\lambda \cdot \begin{cases} e^{-t^2/2}, & t < 0 \\ e^{-\lambda t}, & t \geq 0 \end{cases}$$

1) Draw $f(t)$ with the maximum possible accuracy;
2) Compute constant A_λ (as a function of λ) so that the above function is a PDF for RV X;
3) Compute the expectation of X and find the values of λ for which $E[X]$ is, respectively, negative, null and positive;
4) Compute the CDF and the PDF of $|X|$

Exercise 2

Consider a manufacturing plant where there are two processing stations (PS). Jobs arrive at PS 1, they are put on a conveyor belt, and they are processed FIFO by two identical machines, which may work in parallel. After leaving PS1, jobs get to PS2. Before arriving at PS2, they are inspected, and sent back to PS1 if they are found not compliant. This happens with probability π_1. Jobs entering PS2 are processed by a single machine. After leaving PS2, jobs are either completed or they can be sent back to PS1 again, this time with a probability π_2.

Interarrivals at the plant are exponentially distributed, and so are the service times of single machines at PS1 and PS2. Call γ the arrival rate of jobs, μ_1, μ_2 the serving rates of the machines at PS1, PS2.

1) Model the above plant as a queueing network. Compute the routing matrix and the arrival rates. State explicitly the conditions under which the computations are correct.

2) Compute the conditions under which PS1 (PS2) is the bottleneck. Verify your answer in limit cases and write down an intuitive justification.

3) Compute an upper bound on the arrival rate for the system to achieve stability, and the steady-state probability to have *n* jobs in PS2 *conditioned to the fact that there are k jobs in PS1*.

4) Compute the average number of visits to PS1 and PS2. State the conditions under which

a) PS *j* has less than one visit on average

b) PS1 is visited more often than PS2

Exercise 1 – solution

1) Function $f(t)$ has a standard normal shape on the left semi-axis, and an exponentially decaying one in the positive semi-axis. It is continuous in 0, where its value is equal to A.
2) In order for $f(t)$ to be a PDF, normalization must hold. The integral can be split into two, and the following observations are in order:
   - $\int_0^{+\infty} A_\lambda \cdot e^{-\lambda t} dt = A_\lambda/\lambda$
   - $\int_{-\infty}^0 A_\lambda \cdot e^{-t^2/2} dx = \frac{A_\lambda}{2}\sqrt{2\pi}$

The last observation is due to the fact that a standard normal is symmetric. Therefore, normalization reads $A_\lambda \cdot (\frac{1}{\lambda} + \frac{1}{2}\sqrt{2\pi}) = 1$, i.e.,

$$A_\lambda = \frac{1}{\frac{1}{\lambda} + \frac{1}{2}\sqrt{2\pi}}$$

3)

$$E[X] = \int_{-\infty}^{+\infty} t \cdot f(t) dt$$

$$= \frac{1}{\frac{1}{\lambda} + \frac{1}{2}\sqrt{2\pi}} \left(\int_{-\infty}^0 t \cdot e^{-t^2/2} dt + \int_0^{+\infty} t \cdot e^{-\lambda t} dt \right)$$

$$= \frac{1}{\frac{1}{\lambda} + \frac{1}{2}\sqrt{2\pi}} \left(-\int_{-\infty}^0 -t \cdot e^{-t^2/2} dt + \frac{1}{\lambda} \int_0^{+\infty} t \cdot \lambda \cdot e^{-\lambda t} dt \right)$$

The first integral can be written in the form $f'(x)e^{f(x)}$, hence a primitive is $e^{f(x)}$, whereas the second integral can be obtained from the mean value of an exponential.

$$= \frac{1}{\frac{1}{\lambda} + \frac{1}{2}\sqrt{2\pi}} \left(-1 + \frac{1}{\lambda^2} \right) = \frac{1 - \lambda^2}{\lambda + \frac{\lambda^2}{2}\sqrt{2\pi}}$$

We easily obtain that E[X] is positive when $0 < \lambda < 1$, null when $\lambda = 1$ and negative when $\lambda > 1$.

4) Call $Y = |X|$.We easily obtain

$$F_Y(k) = P\{|X| \le k\} = P\{-k \le X \le k\} = \int_{-k}^{k} f(t) dt = \int_{-k}^{0} f(t) dt + \int_0^k f(t) dt$$

$$= A_\lambda \cdot \sqrt{2\pi} \int_{-k}^0 \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt + \frac{A_\lambda}{\lambda} \cdot \int_0^k \lambda \cdot e^{-\lambda t} dt$$

$$= A_\lambda \cdot \sqrt{2\pi} \left(\Phi(k) - \frac{1}{2} \right) + \frac{A_\lambda}{\lambda} \cdot \left(1 - e^{-\lambda k} \right)$$

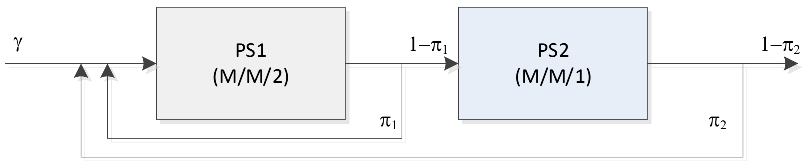
By derivation,

$$f_Y(k) = A_\lambda \cdot \left(e^{-k^2/2} + e^{-\lambda k} \right)$$

And the latter is just the sum of the two branches of X's PDF, which makes sense since the support is halved and the underlying area must still be one.

Exercise 2 - Solution

1) The system can be modeled as a queueing network as follows:



With the following routing matrix and arrival vector

$$\mathbf{\Pi} = \begin{bmatrix} \pi_1 & 1-\pi_1 \\ \pi_2 & 0 \end{bmatrix}, \boldsymbol{\gamma} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}$$

From the figure it is clear that:

$$\begin{cases} \lambda_2 = \lambda_1 \cdot (1-\pi_1) \\ \lambda_1 = \gamma + \pi_1 \cdot \lambda_1 + \pi_2 \cdot \lambda_2 \end{cases}$$

From which one obtains

$$\begin{cases} \lambda_1 = \dfrac{\gamma}{(1-\pi_1)\cdot(1-\pi_2)} \\ \lambda_2 = \dfrac{\gamma}{1-\pi_2} \end{cases}$$

The above computations are correct if $\pi_j < 1$. In fact, condition $\pi_j = 1$ implies that no job come out of the plant, which means that the system cannot reach a steady state.

2) The bottleneck is the system with the highest utilization. The latter is:

$$\begin{cases} \rho_1 = \dfrac{\lambda_1}{2\mu_1} = \dfrac{\gamma}{2\mu_1 \cdot (1-\pi_1)\cdot(1-\pi_2)} \\ \rho_2 = \dfrac{\lambda_2}{\mu_2} = \dfrac{\gamma}{\mu_2 \cdot (1-\pi_2)} \end{cases}$$

Therefore, $\rho_1 > \rho_2$ implies

$$\frac{\gamma}{2\mu_1 \cdot (1-\pi_1)\cdot(1-\pi_2)} > \frac{\gamma}{\mu_2 \cdot (1-\pi_2)}$$

$$2\mu_1 \cdot (1-\pi_1) < \mu_2$$

$$\frac{\mu_1}{\mu_2} < \frac{1}{2\cdot(1-\pi_1)}$$

If $\pi_1 = 0$, then PS1 and PS2 have exactly the same arrivals (they are in a tandem). Therefore, in this case, the server speed at PS1 needs be at least one half of PS2's, since PS1 has two servers. If, instead, $\pi_1 > 0$, PS1 needs to be faster than that, since it will also see arrivals that PS2 does not see, due to the first feedback loop.

3) The stability condition is that both SCs are stable, i.e., $\rho_1 < 1, \rho_2 < 1.$, i.e.

$$\gamma < \min\{2\mu_1 \cdot (1-\pi_1)\cdot(1-\pi_2), \mu_2 \cdot (1-\pi_2)\}$$

Under the above, the SS probabilities related to PS2 are independent of those of PS1, since OJN admit product forms. Hence conditioning makes no matter. The SS probabilities are those of an M/M/1 system, hence

$$p_n = \left(\frac{\gamma}{\mu_2 \cdot (1-\pi_2)}\right)^n \cdot \left(1 - \frac{\gamma}{\mu_2 \cdot (1-\pi_2)}\right)$$

4) The mean number of visits to each PS is the arrival rate scaled by the total rate of external arrivals, i.e., γ. Thus we have:

$$\begin{cases} v_1 = \dfrac{1}{(1-\pi_1)\cdot(1-\pi_2)} \\ v_2 = \dfrac{1}{1-\pi_2} \end{cases}$$

Condition a) is plainly impossible, since a job *must* traverse both PSs at least once in order to leave. Trying to impose it on the above equalities leads to the routing probabilities taking negative values. Condition b) is instead true if $\pi_1 > 0$.

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Exercise 1

Consider an *unfair* die, where the probability of obtaining 6 is $p \neq \frac{1}{6}$. The die is thrown several times. Call T the RV that counts the number of throws before a 6 appears for the first time, and assume that $E[T] = 3$.

1) Find the distribution of T and compute $P\left\{-\frac{1}{3} \leq T \leq \sqrt{11}\right\}$.
2) Let T_1, T_2 be two IID RVs having the same distribution as T. Find the PMF of $Z = min(T_1, T_2)$. Explain your findings.
3) Compute the 95th percentile of Z and $Var\left(\frac{25}{9}Z - \sqrt{6}\right)$.

Exercise 2

Consider an operating system where N tasks may issue a blocking request to a server. When *all* the tasks are blocked, the server unblocks *a random number of them simultaneously,* which then resume their operation. The other tasks remain blocked until the next service epoch. Each unblocked task issues blocking requests at a rate λ, and the service rate of the server is equal to μ. Call π_n the probability of unblocking n tasks.

1) Draw the CTMC
2) Compute the stability condition and the steady-state probabilities
   Assuming from now on that $\pi_n = const$,
3) Specialize the SS probabilities
4) Find the condition under which the PMF of the SS probabilities is a monotonic sequence.
5) Compute the SS probabilities seen by a blocking task.

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Exercise 1 – solution

1) Quite obviously *T* is a geometric RV. Since $E(T)=\frac{1-p}{p}=3$. Therefore, it is $p=\frac{1}{4}$. For a geometric RV, we have $F_T(k)=1-(1-p)^{k+1}$. This means that $P\left\{-\frac{1}{3}\le T\le\sqrt{11}\right\}=\sum_{j=0}^{3}P\{T=j\}=F_T(3)=1-(1-p)^4=\frac{175}{256}$.

2) We have:

$$
\begin{aligned}
P\{Z>k\}&=P\{T_1>k,T_2>k\}\\
&=P\{T_1>k\}\cdot P\{T_2>k\}\\
&=\left(1-F_T(k)\right)^2\\
&=(1-p)^{2\cdot(k+1)}
\end{aligned}
$$

Therefore, it is $F_Z(k)=1-(1-p)^{2\cdot(k+1)}=F_T(2k)$. From the latter, we obtain:

$$
\begin{aligned}
p_Z(k)&=F_Z(k)-F_Z(k-1)\\
&=\left[1-(1-p)^{2\cdot(k+1)}\right]-\left[1-(1-p)^{2k}\right]\\
&=(1-p)^{2k}\left[1-(1-p)^{2\cdot}\right]
\end{aligned}
$$

Call $q=(1-p)^2$, and the latter becomes $p_Z(k)=q^k(1-q)$, which is again a geometric RV, with a success probability equal to $1-q=1-(1-p)^{2\cdot}$.

Like with exponential RVs (of which the geometric are the discrete counterparts), a *minimum* property can be formulated: the min of 2 IID geometric RVs is itself a geometric RV, whose success probability is $1-(1-p)^{2\cdot}$, i.e. the complement of the probability that *both* trials will fail.

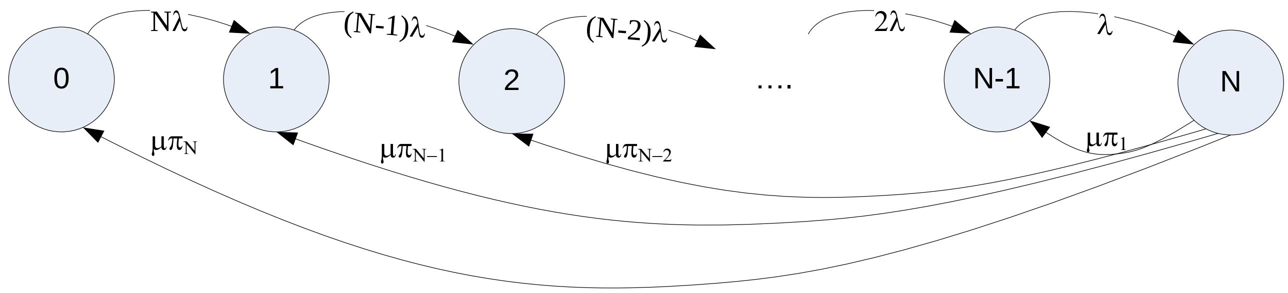
3) The 95th percentile of *Z* is obtained by solving the following equation $F_Z(k)=0.95$ for *k*:

$$
\begin{aligned}
F_Z(k)&=0.95\\
1-(1-p)^{2\cdot(k+1)}&=0.95\\
\left(\frac{3}{4}\right)^{2\cdot(k+1)}&=\frac{1}{20}\\
2(k+1)\cdot\left[\log(3)-\log(4)\right]&=-\log 20\\
k&=\left\lceil\frac{1+\log(2)}{2\left[\log(4)-\log(3)\right]}\right\rceil-1=5
\end{aligned}
$$

Moreover, it is:

$$
\begin{aligned}
Var\left(\frac{25}{9}Z-\sqrt{6}\right)&=\left(\frac{25}{9}\right)^2 Var(Z)\\
&=\left(\frac{25}{9}\right)^2\cdot\frac{1-\left[1-q^{2\cdot}\right]}{\left[1-q^{2\cdot}\right]^2},\\
&=\frac{25^2}{81}\cdot\frac{81}{256}\cdot\frac{256^2}{25^2\cdot 7^2}\\
&=\frac{256}{49}
\end{aligned}
$$

Exercise 2 - Solution



Note that the system has a finite capacity, hence it is always stable.

2) The steady-state global equations are the following:

$P_0 \cdot N \cdot \lambda = P_N \cdot \pi_N \cdot \mu,$

$P_j \cdot (N-j) \cdot \lambda = P_{j-1} \cdot (N-(j-1)) \cdot \lambda + P_N \cdot \pi_{N-j} \cdot \mu, \;\; 1 \le j \le \mathrm{N}-1$

$P_N \cdot \mu = P_{N-1} \cdot \lambda.$

After a few algebraic manipulations, the following recurrence can be easily obtained:

$$P_j = P_0 \cdot \frac{N}{N-j} \cdot \sum_{i=0}^{j} \frac{\pi_{N-i}}{\pi_N}$$

With $0 \le j \le \mathrm{N}-1$, and

$$P_N = P_0 \cdot \frac{N \cdot \lambda}{\mu \cdot \pi_N}$$

From the above we obtain the following normalization condition:

$$P_0 \cdot \left(\frac{N \cdot \lambda}{\mu \cdot \pi_N} + \sum_{j=0}^{N-1} \left[\frac{N}{N-j} \cdot \sum_{i=0}^{j} \frac{\pi_{N-i}}{\pi_N} \right] \right) = 1$$

From which we obtain the SS probabilities:

$$P_j = \frac{\frac{1}{N-j} \cdot \sum_{i=0}^{j} \frac{\pi_{N-i}}{\pi_N}}{\frac{\lambda}{\mu \cdot \pi_N} + \sum_{k=0}^{N-1} \left[\frac{1}{N-k} \cdot \sum_{i=0}^{k} \frac{\pi_{N-i}}{\pi_N} \right]}$$

$$P_N = \frac{\frac{\lambda}{\mu \cdot \pi_N}}{\frac{\lambda}{\mu \cdot \pi_N} + \sum_{k=0}^{N-1} \left[\frac{1}{N-k} \cdot \sum_{i=0}^{k} \frac{\pi_{N-i}}{\pi_N} \right]}$$

3) If $\pi_n = const$ it is $\pi_n = 1/N$, hence the above SS probabilities become:

$$P_j = \frac{\frac{j+1}{N-j}}{\frac{N\lambda}{\mu}+\sum_{k=0}^{N-1}\frac{k+1}{N-k}}, P_N = \frac{\frac{\lambda}{\mu\cdot\pi_N}}{\frac{N\lambda}{\mu}+\sum_{k=0}^{N-1}\frac{k+1}{N-k}}$$

Considering that $\sum_{k=0}^{N-1}\frac{k+1}{N-k} = \sum_{n=1}^{N}\frac{N-n+1}{n} = \sum_{n=1}^{N}\left(\frac{N+1}{n}-1\right) = (N+1)H_N - N$, we obtain:

$$P_j = \frac{\frac{j+1}{N-j}}{(N+1)H_N+N\left(\frac{\lambda}{\mu}-1\right)}, P_N = \frac{\frac{N\lambda}{\mu}}{(N+1)H_N+N\left(\frac{\lambda}{\mu}-1\right)}$$

Where H_N is the *N-th* harmonic number.

4) Let us check that $P_j < P_{j+1}$. In fact, $\frac{P_{j+1}}{P_j} = \frac{j+2}{N-j-1-}\frac{N-j}{j+1} > 1$. Moreover, it is $P_{N-1} < P_N$ if

$$\frac{(N-1)+1}{N-(N-1)} < \frac{N\lambda}{\mu}$$

i.e., if $\lambda > \mu$. Therefore, under the above condition the PMF of the SS probabilities is an increasing sequence.

5) The system is non PASTA, hence:

$$r_j = \frac{\lambda_j P_j}{\sum_{i=0}^{N}\lambda_i P_i} = \frac{(N-j)P_j}{\sum_{i=0}^{N-1}(N-i)P_i} = \frac{j+1}{\sum_{i=1}^{N} i} = 2\frac{j+1}{N\cdot(N+1)}$$

With $0 \leq j \leq N-1$.

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Exercise 1

Let $X_j, j = 1,2, \ldots$ be IID RVs, with $E[X_j] = 0$ and $Var(X_j) = 2$;

1) Assume that X_j can only take two values, call them a, b. Find $a \cdot b$. Assume then that the two values are equally likely, and find *both* a and b;
2) Assume *instead* that X_j has the following PDF: $f(x) = C \cdot e^{-\lambda|x|}$. Compute C and λ;
3) Estimate $P\left\{\sum_{j=1}^{100} X_j > 10\right\}$ in both cases 1) and 2);
4) Assuming that X_j can only take two equally likely values, compute $P\{X_1 + X_2 = 0\}$. Generalize the above formula to $P\left\{\sum_{j=1}^{2n} X_j = 0\right\}$, for any positive n;
5) Still in the hypotheses of point 4), find the PMF of $S_n = \sum_{j=1}^{2n} X_j$.

Exercise 2

In a publish-subscribe system, subscribers issue *subscribe requests* to a publisher, and then wait for the corresponding notification. Subscribe requests arrive at exponential interarrival times with a rate λ. The publisher publishes *notifications* with a rate μ. All waiting subscribers receive a notification simultaneously, they all leave the system and they have to subscribe again to receive another notification.

1) Draw the CTMC;
2) Find the global equilibrium equations;
3) Compute the conditions according to which the number of waiting requests stays finite, and the PMF of the number of subscribe requests waiting;
4) Find the mean response time for a subscriber;
5) Compute the throughput, i.e., the number of subscribers notified per unit of time.

Exercise 1 – solution

1) The values must satisfy two constraints, one for the mean and one for the variance. Let p be the probability of value a. These constraints are:

$$\begin{cases} a \cdot p + b \cdot (1-p) = 0 \\ a^2 \cdot p + b^2 \cdot (1-p) = 2 \end{cases}$$

From the above equations, we easily obtain $a \cdot b = -2$. If the two values are equally likely, it must be $b = -a$, hence $a = \sqrt{2}$.

2) We need to impose that $\int_{-\infty}^{+\infty} f(x) = 1$. Since the PDF is symmetric around 0, we have $2C \cdot \int_0^{+\infty} e^{-\lambda \cdot x} = 1$, hence $\frac{2C}{\lambda} = 1$, hence $C = \frac{\lambda}{2}$. Moreover, we have $Var(X_j) = E[X_j^2] - E[X_j]^2 = E[X_j^2] = \int_{-\infty}^{+\infty} x^2 f(x) = 2$. From this we derive $\int_{-\infty}^{+\infty} \frac{\lambda}{2} x^2 e^{-\lambda \cdot |x|} dx = 2 \int_0^{+\infty} \frac{\lambda}{2} x^2 e^{-\lambda \cdot x} dx = 2$. However, the latter is the expression of the mean square value of an exponential RV with a rate λ. For the latter (call it *Y*), we know that $E[Y^2] = Var(Y) + E[Y]^2 = \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2}$. Hence, we have $\frac{2}{\lambda^2} = 2$, or $\lambda = 1$ (λ must in fact be positive), and $C = \frac{1}{2}$.

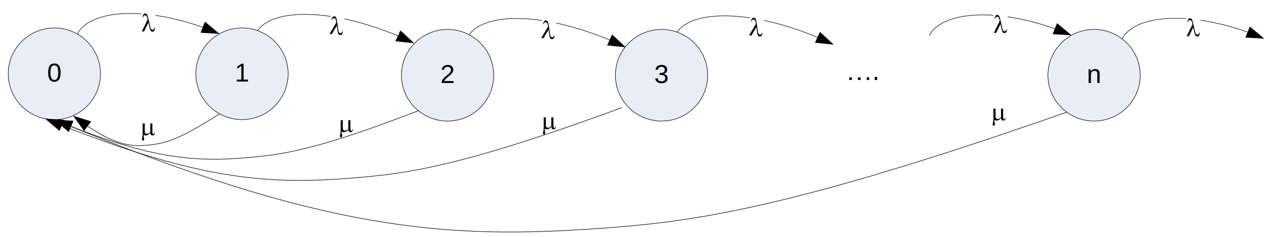
3) The result is given by the CLT: RV $Y = \sum_{j=1}^{100} X_j$ is approximately Normal, with a null mean and a variance 200 (for both cases 1) and 2), since the CLT requires only that the variance is finite). Therefore, it is: $P\{Y > 10\} = P\left\{\frac{Y-0}{\sqrt{200}} > \frac{10-0}{10\sqrt{2}}\right\} = P\left\{Z > \frac{1}{\sqrt{2}}\right\} = 1 - \Phi\left(\frac{1}{\sqrt{2}}\right) \simeq 0.239$.

4) It is $P\{X_1 + X_2 = 0\} = P\{X_1 = a, X_2 = -a\} + P\{X_1 = -a, X_2 = +a\} = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$. Moreover, the sum of $2n$ RVs is null if half of them are equal to $-a$, and the other half to $+a$. Each combination of "half and half" has a probability $\frac{1}{2^{2n}}$, and there are $\binom{2n}{n}$ such combinations, so the result is $P\{\sum_{j=1}^{2n} X_j = 0\} = \binom{2n}{n} \frac{1}{2^{2n}}$.

5) RV $S_n = \sum_{j=1}^{2n} X_j$ can take on values $-2n \cdot a$, $-2(n-1) \cdot a$, …, 0, …, $+2(n-1) \cdot a$, $+2n \cdot a$, i.e. $\{2a \cdot j,\ -n \le j \le +n\}$, and its PMF is obviously symmetric. Each of the above values $2a \cdot j$ can be obtained by selecting $n-j$ "positives" and $n+j$ "negatives", which can happen in $\binom{2n}{n-j} = \binom{2n}{n+j}$ different ways, each with probability $\frac{1}{2^{2n}}$. Therefore, the PMF is $p_n(2 \cdot a \cdot j) = \binom{2n}{n-j} \cdot \frac{1}{2^{2n}}$, $-n \le j \le +n$.

Exercise 2 – solution

The CTMC is the following:



Where each state represents the number of requests waiting.

2) The global equilibrium equations are:

$$\begin{cases} p_0 \cdot \lambda = \sum_{i=1}^{+\infty} p_i \cdot \mu \\ p_j \cdot (\lambda + \mu) = p_{j-1} \cdot \lambda \quad j \geq 1 \end{cases}$$

From the second one we easily obtain $p_j = p_0 \cdot \frac{\lambda^j}{(\lambda+\mu)^j}$.

From the above, normalization reads: $p_0 \cdot \left[1 + \sum_{j=1}^{+\infty} \frac{\lambda^j}{(\lambda+\mu)^j}\right] = 1$.

3) Call $\theta = \frac{\lambda}{\lambda+\mu}$, If $\mu \neq 0$ it is $\theta < 1$, hence the above sum converges to $\sum_{j=1}^{+\infty} \theta^j = \frac{\theta}{1-\theta} = \frac{\frac{\lambda}{\lambda+\mu}}{1-\frac{\lambda}{\lambda+\mu}} = \frac{\lambda}{\mu}$.
Therefore, the stability condition is $\mu \neq 0$.

We have $p_0 = \frac{\mu}{\lambda+\mu}$, $p_j = p_0 \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^j = \frac{\mu}{\lambda} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{j+1}$. Hence, $p_j = \frac{\mu}{\lambda} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^{j+1}, j \geq 0$.

Therefore, we have $P(z) = \sum_{j=0}^{+\infty} p_j \cdot z^j = \frac{\mu}{\lambda+\mu} \cdot \sum_{j=0}^{+\infty} \left(\frac{\lambda \cdot z}{\lambda+\mu}\right)^j = \frac{\mu}{\lambda(1-z)+\mu}$. From the latter, we get:

$P'(z) = \frac{\partial}{\partial z} \frac{\mu}{\lambda(1-z)+\mu} = \frac{\lambda \cdot \mu}{[\lambda(1-z)+\mu]^2}$, and it is: $E[N] = P'(1) = \frac{\lambda}{\mu}$.

4) The mean response time is the mean service time (all requests are served simultaneously), i.e. $E[R] = \frac{1}{\mu}$.

5) The throughput is $\mu \cdot E[N] = \mu \cdot \frac{\lambda}{\mu} = \lambda$ (which is obvious, given that the system is stable).