Exercise 1

Mary is required to sell candy bars to raise money for a charity. She has *k* candy bars to sell door to door in her (infinite) neighborhood. At each house, there is a probability *p* of selling one candy bar and a probability 1-*p* of selling nothing. All visits are supposed to be independent.

1) Compute the probability $\pi_{j,k}$ that Mary sells her last candy bar at house number *j*.
2) Find the condition under which $\pi_{j,k}$ decreases with *j*
3) Compute the mean number of house calls that Mary has to do to sell all her candy bars
4) Compute the variance of the number of house calls and the CoV. Discuss what happens when *k* goes to infinity and justify your answer.

Exercise 2

A computer system has a pool of *M* functionally identical peripherals, that can be used to serve I/O requests. These *M* peripherals, however, exhibit different performance: each of them can be modeled as an exponential server, with a service rate μ_j, $1 \leq j \leq M$.

The system allocates peripherals according to the following strategy: all peripherals are kept switched off when there are no pending I/O requests. When the first I/O request occurs, the server switches on one peripheral at random (peripheral *j* is chosen with probability π_j), and that peripheral is kept on until there are no more pending I/O requests to be served. Assume that I/O requests occur with exponential interarrival times, at a rate λ. Assume that the switchon/switchoff operations are instantaneous.

1) Model the above system as a queueing system and draw the TR diagram (you can draw the diagram with *M*=2 for simplicity, and then discuss the generalization).
2) Compute the stability condition and the SS probabilities. Check the result in the limit cases i) *M*=1, and ii) *M* identical peripherals.
3) Assume that $\pi_j = 1 - \lambda/\mu_j, \;\; 1 \leq j \leq M$. Compute the mean number of pending I/O requests. Provide a physical interpretation for the result.
4) Compute the CDF of the response time for an I/O request.

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Exercise 1 – Solution

1) The probability that Mary sells her last (k-th) candy bar at house *j* is the probability that she sells *k*-1 in *j*-1 houses (however arranged) AND one in house *j*. This is:

$$\pi_{j,k} = \left[\binom{j-1}{k-1} \cdot (1-p)^{(j-1)-(k-1)} \cdot p^{k-1}\right] \cdot p = \binom{j-1}{k-1} \cdot (1-p)^{j-k} \cdot p^k$$

Where *p* is the success probability on the single trial, and $j \geq k$. With a little algebra, one gets:

$$\pi_{j,k} = \frac{k}{j} \cdot \binom{j}{k} \cdot (1-p)^{j-k} \cdot p^k$$

2) Using the second formulation for $\pi_{j,k}$, it is clear that:

$$\pi_{j+1,k} = \frac{k}{j+1} \cdot \binom{j+1}{k} \cdot (1-p)^{j+1-k} \cdot p^k$$

$$= \frac{k}{(j+1)} \cdot \frac{(j+1) \cdot j!}{k! \cdot (j+1-k) \cdot (j-k)!} \cdot (1-p) \cdot (1-p)^{j-k} \cdot p^k$$

$$= \frac{(1-p) \cdot k}{j+1-k} \cdot \binom{j}{k} \cdot (1-p)^{j-k} \cdot p^k$$

$$= \frac{(1-p) \cdot j}{j+1-k} \cdot \pi_{j,k}$$

The relationship holds if:

$$\frac{\pi_{j+1,k}}{\pi_{j,k}} = \frac{(1-p) \cdot j}{j+1-k} < 1$$

Which boils down to $j \cdot p > k - 1$

3) Call J the RV counting the number of house calls. The direct method to compute the mean value is:

$$E[J] = \sum_{j=k}^{+\infty} j \cdot \pi_{j,k} = \sum_{j=k}^{+\infty} \left[k \cdot \binom{j}{k} \cdot (1-p)^{j-k} \cdot p^k\right]$$

The above sum may be tricky to solve. A quicker solution can be obtained by observing that J is the sum of *k* IID geometric RVs, each one measuring the number of *trials* required to obtain the first success (hence having a support starting from 1). For such a variable, call it T, we have $p(i) = (1-p)^{i-1} \cdot p$. Therefore, we can compute its PGF as

$$G(z) = \sum_{i=1}^{+\infty} (1-p)^{i-1} \cdot p \cdot z^i = \frac{z \cdot p}{1 - z + z \cdot p}$$

hence we have:

$$E[T] = G'(1) = \left.\frac{p \cdot (1 - z + z \cdot p) - zp \cdot (-1 + p)}{(1 - z + z \cdot p)^2}\right|_{z=1} = \left.\frac{p}{(1 - z + z \cdot p)^2}\right|_{z=1} = \frac{1}{p}$$

For the above reason, the required mean value is $E[J] = k \cdot E[T] = \frac{k}{p}$.

4) Once the PGF has been computed, one may compute

$$Var(J) = k \cdot Var(T) = k \cdot [G''(1) + G'(1) - G'(1)^2]$$

because of independence. The only missing term in the above computation is $G''(1)$, which is:

$$G''(1) = \left. \frac{-p \cdot [2z(p-1)^2 + 2(p-1)]}{(1 - z + z \cdot p)^4} \right|_{z=1} = \frac{2(1-p)}{p^2} = \frac{2}{p^2} - \frac{2}{p}$$

Hence:

$$Var(T) = \frac{2}{p^2} - \frac{2}{p} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

And $Var(J) = k \cdot \frac{1-p}{p^2}$.

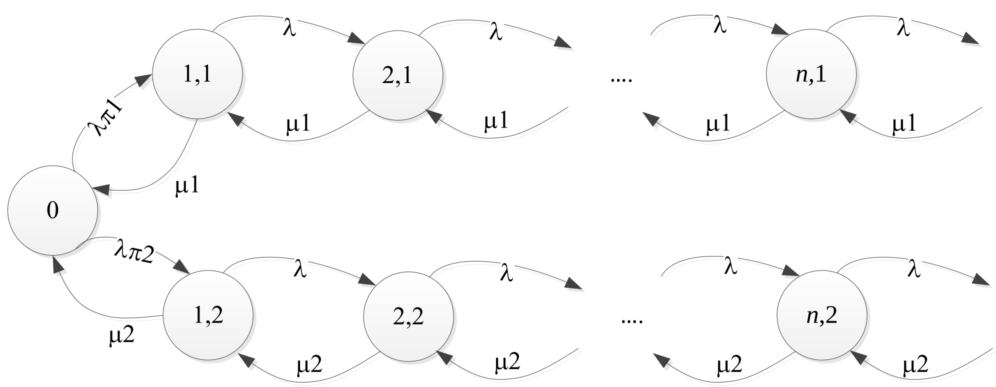
From the above, we obtain:

$$CoV(J) = \frac{std(J)}{E[J]} = \sqrt{k \cdot \frac{1-p}{p^2}} \bigg/ \frac{k}{p} = \sqrt{\frac{1-p}{k}}$$

The CoV goes to zero when *k* increases. Since J is a sum of IID RVs, this is a consequence of the CLT.

Exercise 2 – Solution

1) We represent the state of the system with a couple (*n*,*j*), where *n* is the number of pending I/O requests and *j* is the number of the active peripheral. The TR diagram (for the case *M*=2) is the following. The generalization is straightforward (*M* horizontal branches going to infinity, each one branching out of state 0 with a rate $\lambda \cdot \pi_j$, where service rates are μ_j and arrival rates are λ.



2) Using local equilibrium equations, one can write:

$$p_{n,j} = \left(\frac{\lambda}{\mu_j}\right)^n \cdot \pi_j \cdot p_0 = \rho_j{}^n \cdot \pi_j \cdot p_0, \quad n > 0, 1 \le j \le M$$

From which we get that the stability condition is clearly $\lambda < \min_j(\mu_j)$. Moreover, the following can be written:

$$p_0 \cdot \left[1 + \sum_{j=1}^{M} \pi_j \cdot \sum_{n=1}^{+\infty} {\rho_j}^n\right] = 1$$

Since $\sum_{j=1}^{M} \pi_j = 1$, under the stability condition the previous expression yields:

$$p_0 = \frac{1}{\sum_{j=1}^{M} \frac{\pi_j}{1-\rho_j}}$$

$$p_{n,j} = {\rho_j}^n \cdot \pi_j \cdot \frac{1}{\sum_{j=1}^{M} \frac{\pi_j}{1-\rho_j}}, \qquad n > 0, 1 \leq j \leq M$$

If *M*=1 we obtain the same SS probabilities as an M/M/1 system's. The same occurs if the peripherals are identical, in which case it is $\rho_j = \rho$, and we get $p_n = \sum_{j=1}^{M} p_{n,j} = (1-\rho) \cdot \rho^n$.

3) When $\pi_j = 1 - \lambda/\mu_j$, it is $\pi_j/(1-\rho_j) = 1$, the sum at the denominator becomes M, and we get $p_{n,j} = {\rho_j}^n \cdot \pi_j/M$. In this case we get:

$$E[N] = \sum_{j=1}^{M} \sum_{n=0}^{+\infty} n \cdot p_{n,j} = \frac{1}{M} \cdot \sum_{j=1}^{M} \pi_j \sum_{n=0}^{+\infty} n \cdot {\rho_j}^n$$

$$= \frac{1}{M} \cdot \sum_{j=1}^{M} \pi_j \frac{\rho_j}{(1-\rho_j)^2}$$

$$= \frac{1}{M} \cdot \sum_{j=1}^{M} \frac{\rho_j}{1-\rho_j}$$

The above result is the average of the Kleinrock functions of M/M/1 systems each having a single peripheral as a server.

4) The PDF of the response time R can be computed using total probability. When the system is in state (n,j), the response time is an $(n+1)$-stage Erlang with a rate μ_j.

$$f_R(t) = \sum_{j=1}^{M} \sum_{n=0}^{+\infty} f_{E_{n+1,j}}(t) \cdot r_{n,j}$$

Where $r_{n,j} = p_{n,j}$ and $f_{E_{n+1,j}}(t) = \mu_j e^{-\mu_j \cdot t} \cdot \frac{(\mu_j \cdot t)^n}{n!}$. After a few straightforward manipulations, one gets:

$$f_R(t) = \frac{1}{M} \cdot \sum_{j=1}^{M} \frac{1}{E[R_j]} e^{-\frac{t}{E[R_j]}}$$

Where $E[R_j] = 1/(\mu_j - \lambda)$. The above is the average of the PDFs of individual M/M/1 systems, each one having a service rate μ_j. The CDF is therefore

$$F_R(t) = \frac{1}{M} \cdot \sum_{j=1}^{M} \left(1 - e^{-\frac{t}{E[R_j]}}\right)$$