An Introduction to Fuzzy Logic
Part III

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Fuzzy systems

- *Fuzzy systems* (or *fuzzy inference systems*) are made up of several rules.
- Each rule is a conditional statement in which the antecedent and the consequent consist of fuzzy propositions containing linguistic variables and linguistic operators (such as *and*, *or*).

SISO (single-input single-output) fuzzy rules

$$
\begin{array}{ll}
R_1: & if\ X\ is\ A_1\ then\ Y\ is\ B_1 \\
R_2: & if\ X\ is\ A_2\ then\ Y\ is\ B_2 \\
 & \ldots\ldots\ldots \\
R_n: & if\ X\ is\ A_n\ then\ Y\ is\ B_n
\end{array}
$$

$$Fact: X\ is\ A'$$

$$\overline{\hspace{10em}}$$

$$Conclusion: \qquad Y\ is\ B'$$

- Each rule produces a consequence (which is a fuzzy set)*.* The *n* consequences must be *aggregated*.

Two approaches are possible:

- *FITA* (first infer then aggregate): each rule is evaluated individually and then the conclusions are aggregated (e.g., performing the union of the fuzzy sets produced by the various rules).
- *FATI* (first aggregate then infer): a single fuzzy relation is generated from the set of rules and combined with the fact.

FITA Strategy

Given a set of *n* rules $I(A_i(x), B_i(y)),\ i=1,\dots,n,$ and a fact $A'(x)$, the conclusion is

$$B^{(1)} = h(B_1^{'}, \dots, B_n^{'})$$

where $B_i^{'}(y) = A'(x) \circ I(A_i(x), B_i(y)) = \sup_{x \in U} T[A'(x), I(A_i(x), B_i(y))]$

and *h* is an aggregation operator (usually *min* or *max*).

Therefore, the FITA approach involves, in order:

- implication
- composition
- aggregation

FATI Strategy

Given a set of *n* rules $I(A_i(x), B_i(y)),\ i=1,\ldots,n,$ and a fact $A'(x)$, the conclusion is

$$B^{(2)} = A'(x) \circ R(x,y) = \sup_{x \in U} T(A'(x), R(x,y))$$

where $R(x,y) = h(I(A_1(x), B_1(y)), \ldots, I(A_n(x), B_n(y)))$

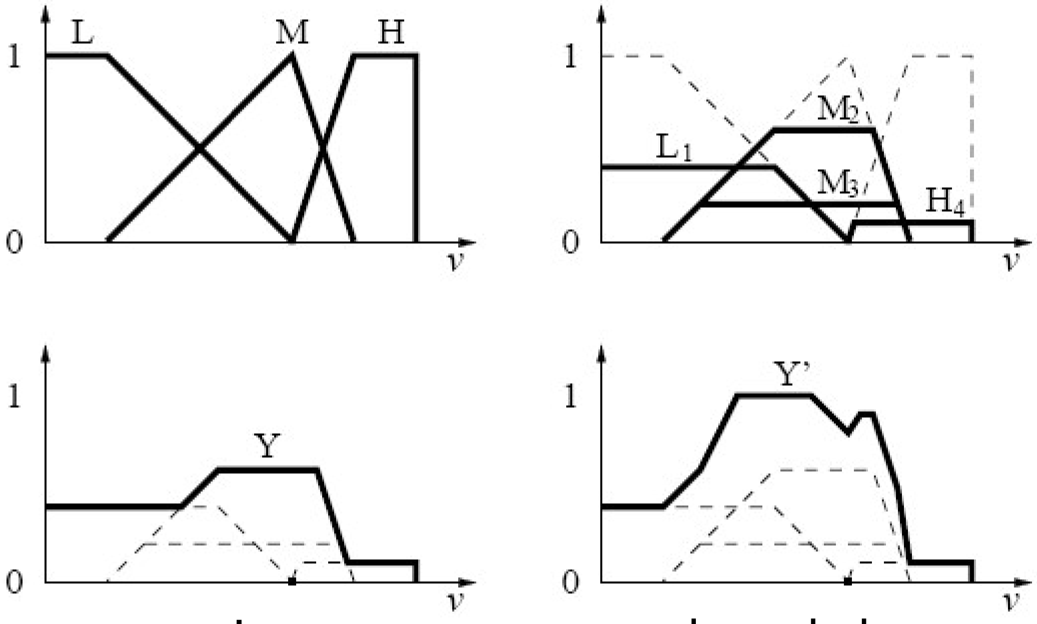
and *h* is an aggregation operator (usually *min* or *max*).

Therefore, the FATI approach involves, in order:

- implication
- aggregation
- composition

Example of aggregation

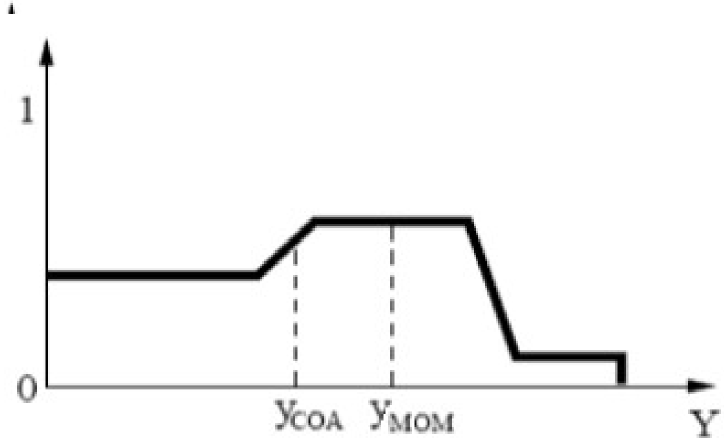
- The output variable of a fuzzy system has three fuzzy values: L, M and H.
- At a given instant, four rules are fired with activation levels 0.4, 0.6, 0.2 and 0.1, producing fuzzy sets L1, M2, M3 and H4, respectively.



maximum

bounded sum

- The final fuzzy set obtained by aggregating the conclusions of the rules represents a linguistic term that can be used as it is, or more often *defuzzified* to produce a numerical value.
- Typical defuzzifiers
  - *Center of area* (COA) (or *center of gravity* or *centroid method)*
  - *Mean of maxima* (MOM)



- The reason we need to defuzzify stems from the fact that we want, on the one hand, to be able to formalize and manage vague and imprecise concepts, and on the other hand to obtain precise results from vague and imprecise information.
- As a simple example, let us consider the following fuzzy rule:

If the temperature is high, then the conditioner speed is high

Depending on the specific temperature, I want to know exactly what the conditioner speed should be.

MISO (multi-input single-output) fuzzy rules

- In general a fuzzy rule takes the form

$$if\ X_1\ is\ A_1\ and\ X_2\ is\ A_2 \ldots and\ X_m\ is\ A_m\ then\ Y\ is\ B_1$$

- The antecedent is a fuzzy relation: we have to solve the connective *and* to produce a rule with a single fuzzy set in the if-part

$$R(X_1,\ldots,X_m)=T(A_1(x_1),\ldots,A_m(x_m))$$

where *T* is a t-norm (typically *min*).

- Therefore, given a fact

$$X_1 \; is \; A_1^{'} \; and \; \ldots \; and \; X_m \; is \; A_m^{'}$$

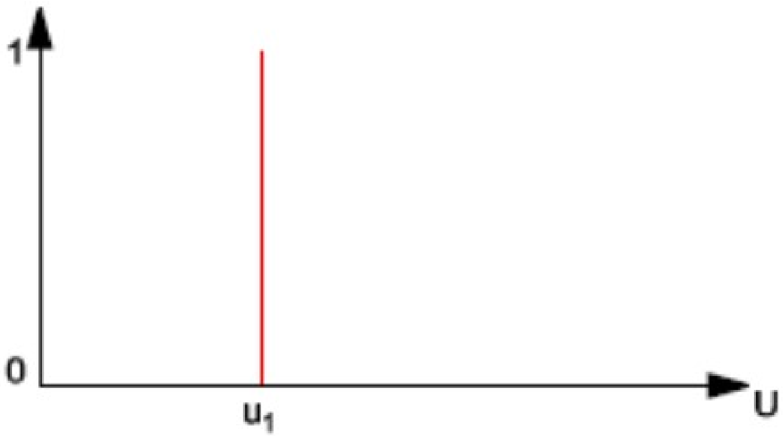
we have

$$B'(y) = \sup_{x_1,\ldots,x_m} t\left[T\left(A_1^{'}(x_1),\ldots,A_m^{'}(x_m) \right), I\left(T(A_1(x_1),\ldots,A_m(x_m)), B(y) \right) \right]$$

where *t* and *T* may be different t-norms.

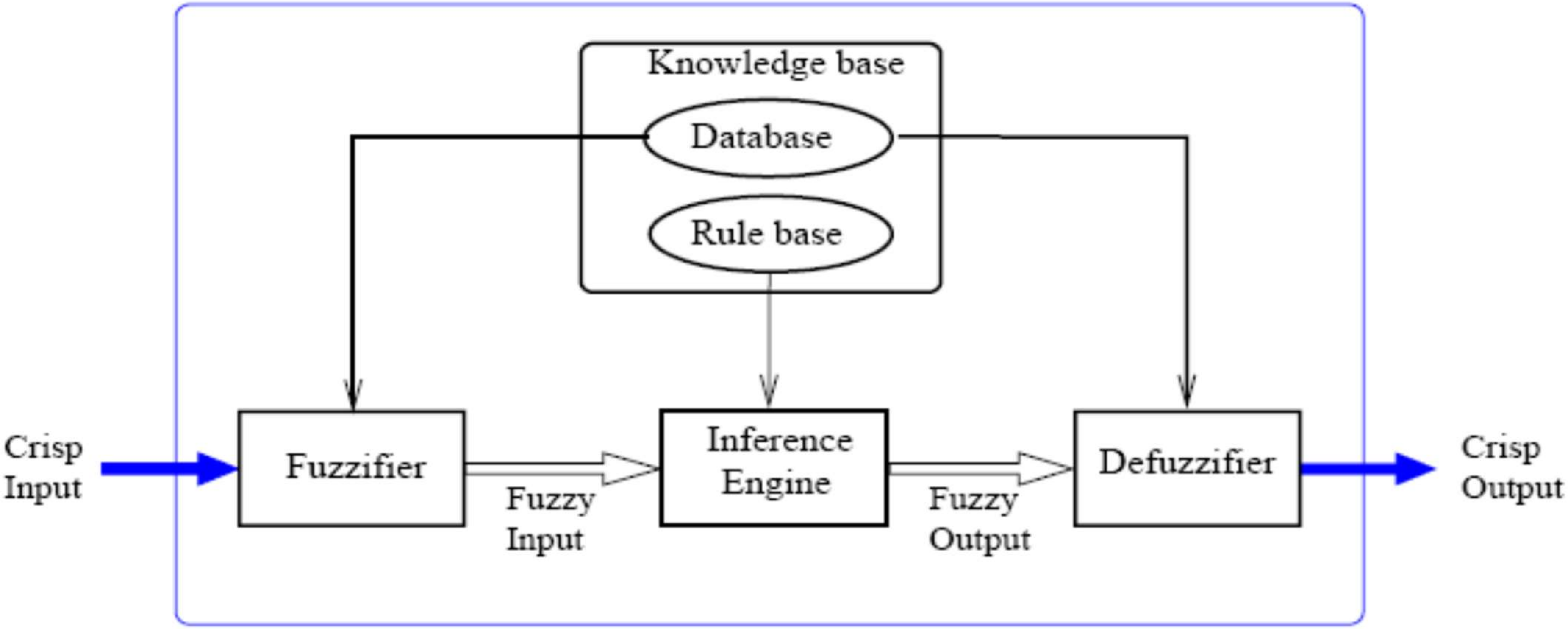
Fuzzification and defuzzification

- Usually the inputs to a fuzzy system are crisp values that need to be converted to fuzzy sets.
- The mapping from a crisp value to a fuzzy set is called *fuzzification*.
  - The most used fuzzifier is the *singleton* fuzzifier



- The inverse process is referred to as *defuzzification* (seen above).

Structure of a fuzzy rule-based system



- The *rule base* contains the fuzzy rules.
- The *database* contains the membership functions.

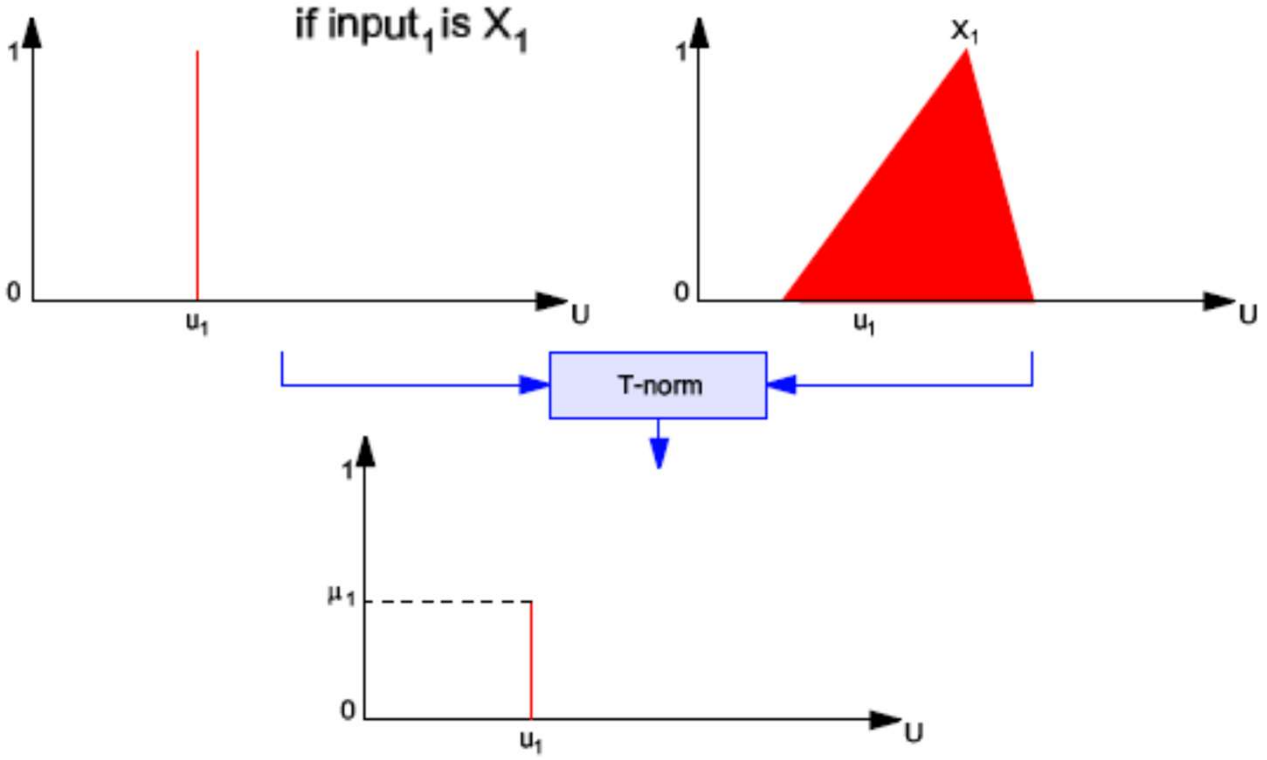
Basic inference steps (FITA strategy)

- 1. Fuzzification of the inputs.
- 2. Evaluation of the *firing strength* (or *activation degree* or *weight*) of each rule. This step includes two sub-steps 2.1 and 2.2:

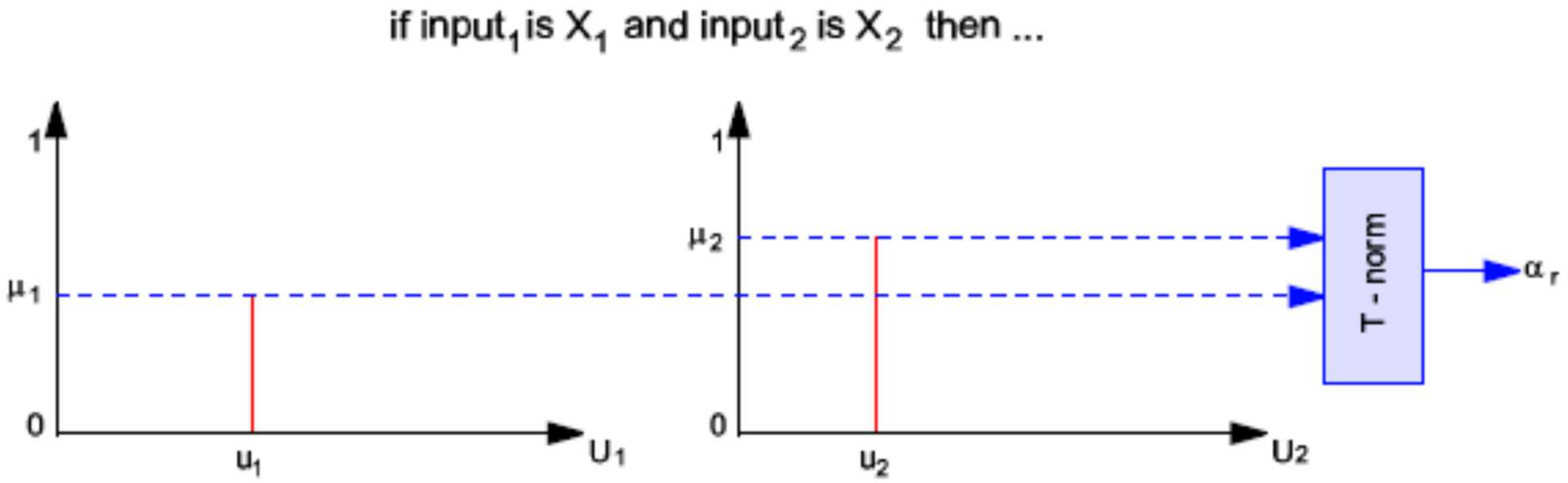
    2.1 compute the membership degrees of each input variable to the fuzzy sets in the antecedent of each rule (a *t*-norm is applied to the fuzzified input variable and to the antecedent fuzzy sets).

    - In other words, we must evaluate the extent to which the given inputs satisfy the linguistic concepts represented by the antecedent fuzzy sets.

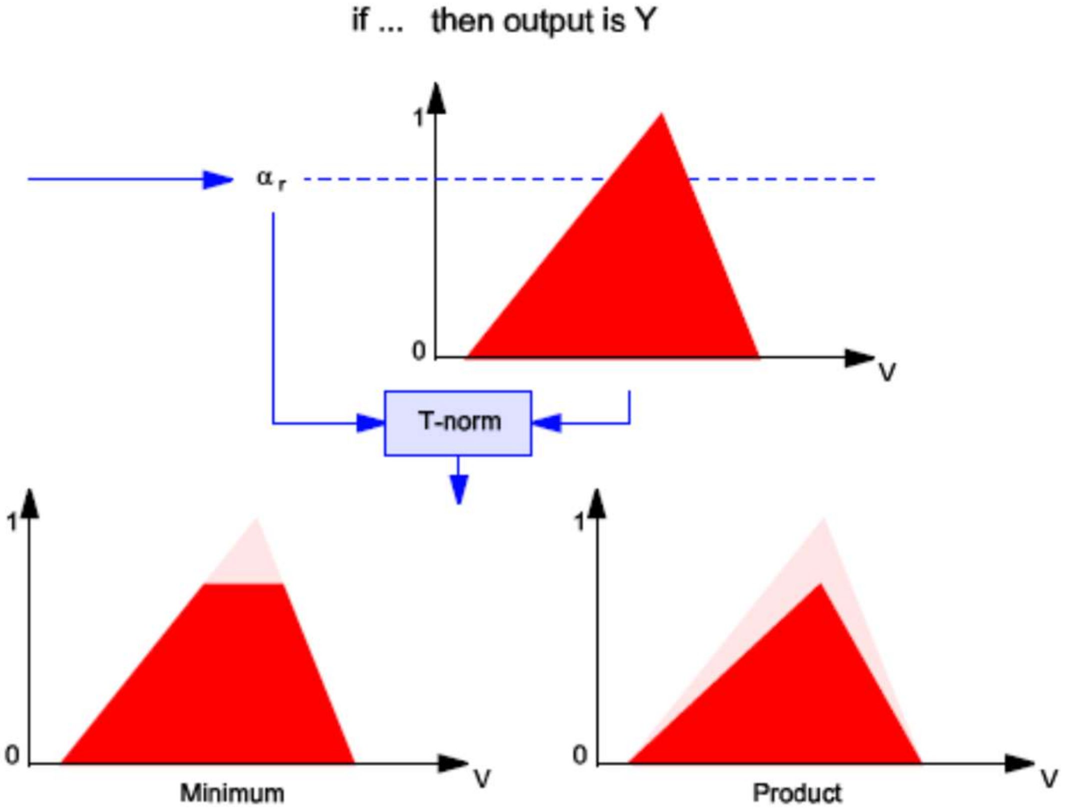
- Evaluation of the membership degree



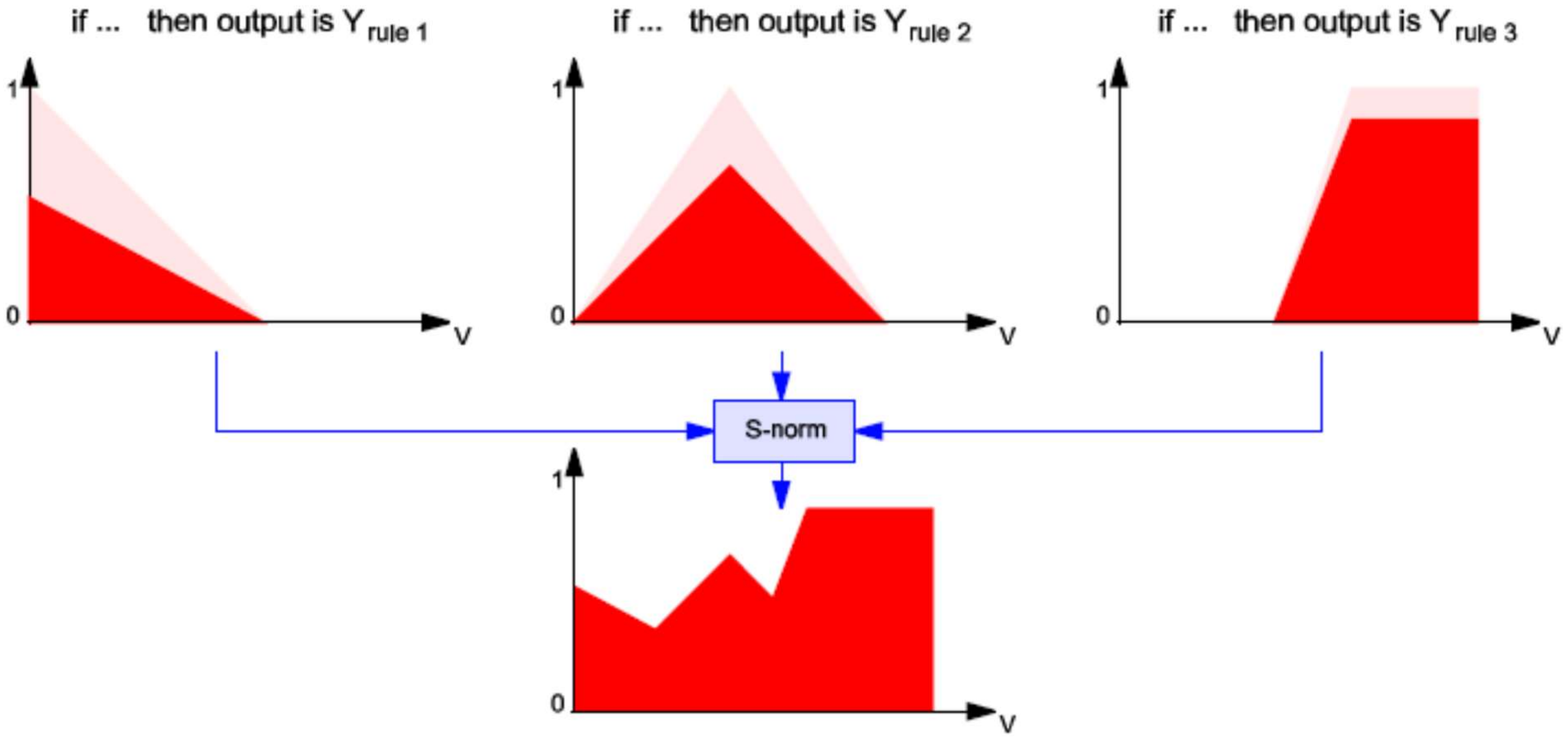
2.2 if *and* joins the input labels, a *t*-norm is applied to the membership degrees obtained in step 2.1), in order to obtain the *firing strength* of the rule



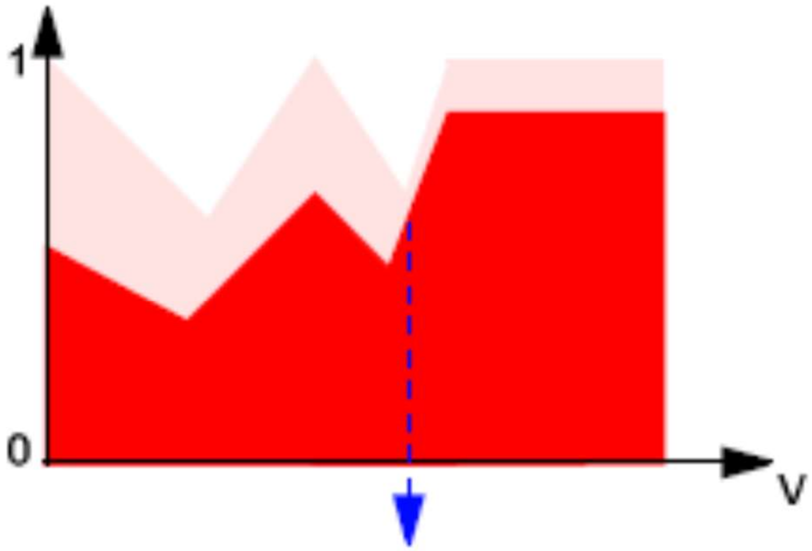
- 3. Implication and composition (we assume either minimum or product implication)



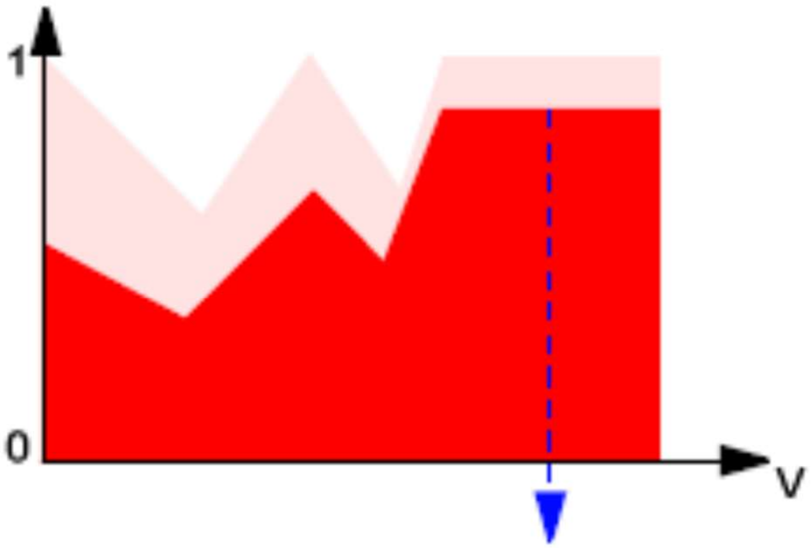
- 4. Rule aggregation (we assume three rules)


if ... then output is $Y_{rule\ 1}$
if ... then output is $Y_{rule\ 2}$
if ... then output is $Y_{rule\ 3}$
1
0
v
1
0
v
1
0
v
S-norm
1
0
v

- 5. Defuzzification



Center of gravity

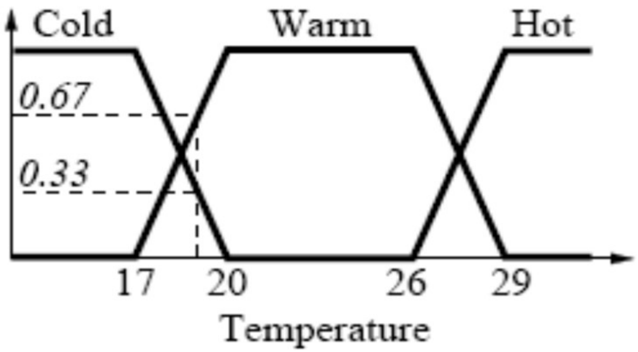


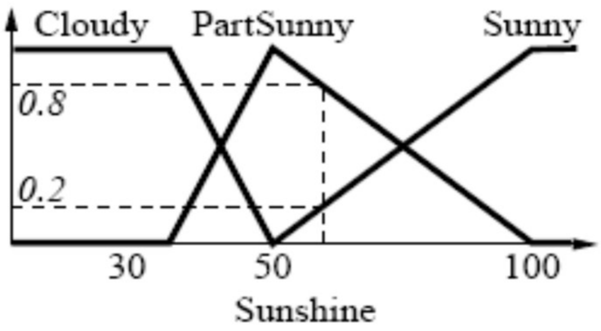
Mean of maxima

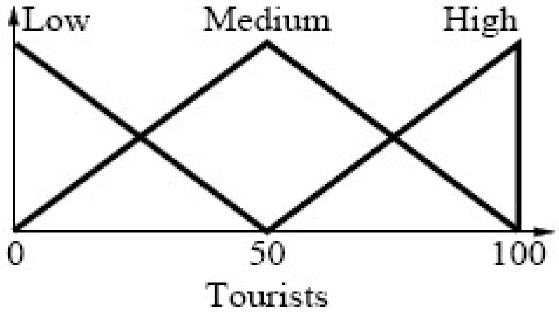
Example:
the tourist prediction system

- The fuzzy system predicts the number of tourists visiting a resort.
- 2 input variables: *Temperature* (in degrees) and *Sunshine* (a percentage of the maximum expected sunshine).
- 1 output variable: *Tourists*, which is the estimated amount of tourists (percentage of resort capacity).
- In the example, we refer to a slightly cold and partially sunny day: the temperature and sunshine are, respectively, 19 degrees and 60%.

Database


Cold
Warm
Hot
0.67
0.33
17
20
26
29
Temperature


Cloudy
PartSunny
Sunny
0.8
0.2
30
50
100
Sunshine


Low
Medium
High
0
50
100
Tourists

Rule base

- Rule 1: **if** (Temperature **is** Hot) **or** (Sunshine **is** Sunny) **then** (Tourists **is** High)
- Rule 2: **if** (Temperature **is** Warm) **and** (Sunshine **is** Partially Sunny) **then** (Tourists **is** Medium)
- Rule 3: **if** (Temperature **is** Cold) **or** (Sunshine **is** Cloudy) **then** (Tourists **is** Low)

Fuzzifier

- Temperature

$$\mu_{Cold}(19)=0.33$$

$$\mu_{Warm}(19)=0.67$$

$$\mu_{Hot}(19)=0$$

- Sunshine

$$\mu_{Cloudy}(60)=0$$

$$\mu_{PartSunny}(60)=0.8$$

$$\mu_{Sunny}(60)=0.2$$

Inference engine

- **Step 1. Antecedent activation**

- Rule 1: **if** (Temperature **is** Hot) **or** (Sunshine **is** Sunny)

$$\mu_{Rule1} = \mu_{Hot}(19) \vee \mu_{Sunny}(60)$$
$$= \max(0, 0.2) = 0.2$$

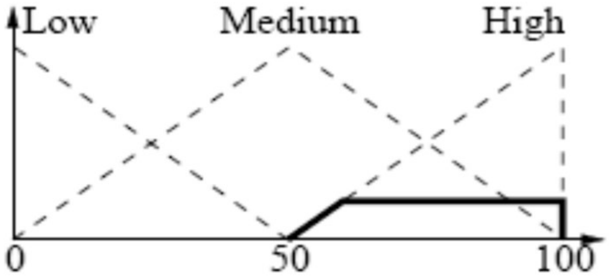
- Rule 2: **if** (Temperature **is** Warm) **and** (Sunshine **is** Partially Sunny)

$$\mu_{Rule2} = \mu_{Warm}(19) \wedge \mu_{PartSunny}(60)$$
$$= \min(0.67, 0.8) = 0.67$$

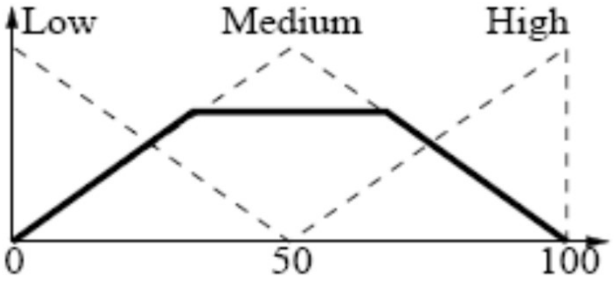
- Rule 3: **if** (Temperature **is** Cold) **or** (Sunshine **is** Cloudy)

$$\mu_{Rule3} = \mu_{Cold}(19) \vee \mu_{Cloudy}(60)$$
$$= \max(0.33, 0) = 0.33$$

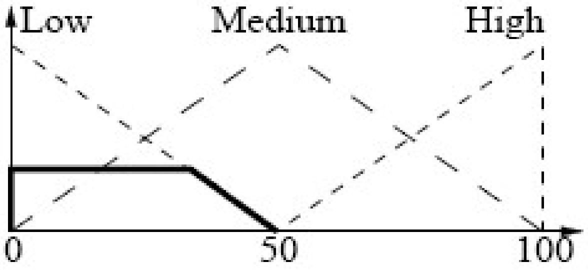
- **Step 2. Implication (*minimum* implication operator)**



Rule 1

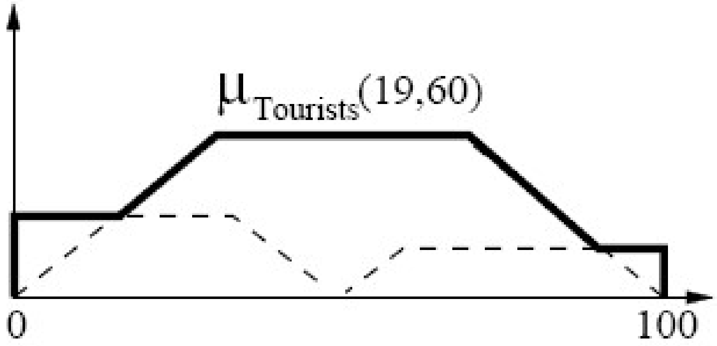


Rule 2



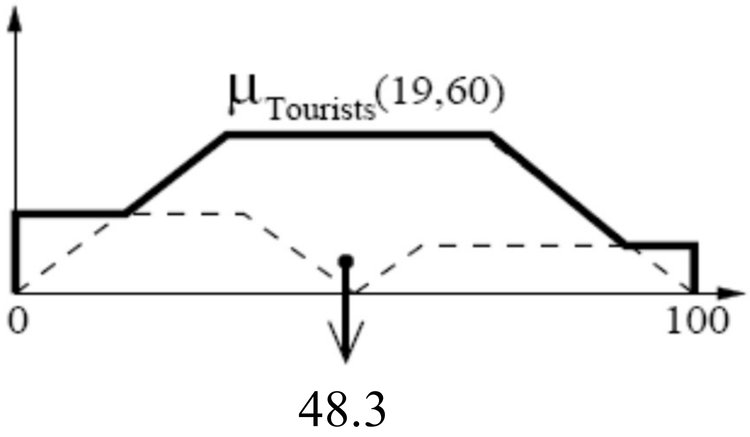
Rule 3

- **Step 3. Aggregation (*maximum* aggregation operator)**



Defuzzifier

- COA defuzzyfication method



- The final output value is approximately 48%.

Fuzzy versus Probability

$$Fuzzy \neq Probability$$

- Probability deals with uncertainty and likelihood.
- Fuzzy logic deals with ambiguity and vagueness.

- **Example 1**

- John owns three houses.
- The probability that John owns four houses is zero.
- The fuzzy membership of John in the set of people with four houses, however, is non-zero.

Example 2

- A bottle of liquid has a probability of ½ of being rat poison and ½ of being pure water.
- The contents of a second bottle have a membership degree ½ to the fuzzy set of liquids containing a lot of rat poison.
- The meaning of ½ for the two bottles clearly differs significantly and would affect your choice if you were dying of thirst.
- 50% probability means 50% chance the water is clean.
- 50% fuzzy membership means the water contains poison.