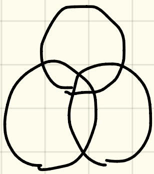
- Data Science

Application D

- MIA

Algorithm

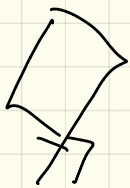
NLA, N. opt.,
stat, Prob, ...



Architecture

(GPU, system, network, software, ...)

- Linear Algebra
  - 4 Fundamental subspaces
    - The Dimensions of the 4 subspaces
    - The Orthogonality of the two pairs
    - The best bases for all 4 subspaces
  - 4 Central problems
    - $Ax = b$ Linear system
    - $Ax = b$ Least square
    - $Ax = \lambda x$ Eigenvalue problems
    - $Ax = \sigma x$, Singular value decomposition



$\boxed{\text{Space}}$

Fabric of the Cosmos

Vector Space

- "Vectors" : v, w

- If $v, w \in S$, every combinations $cv + wv \in S$

  vector space

- 一個"空間"究竟有多大? (e.g. size, dimension, rank)

  有什麼結構? (e.g. base, low-rank, sparsity, SVD, QR, LU)

  要如何找到這些結構?] L.A

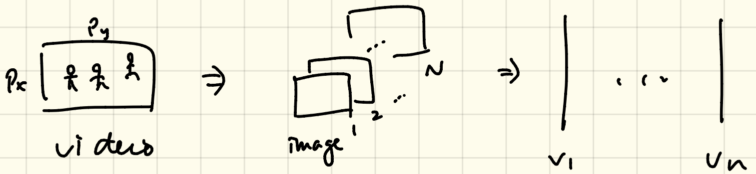
  要如何計算這些結構?] NLA

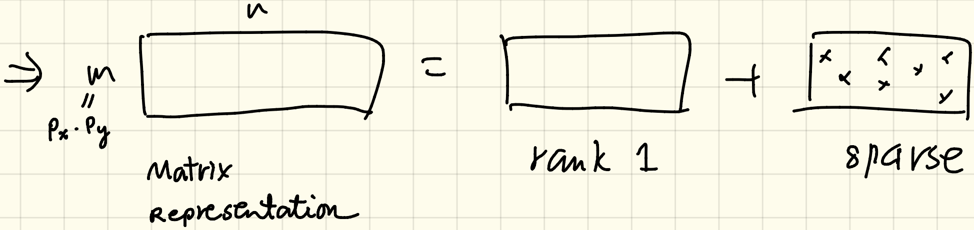
  解如何應用這些結構?] App

Common course

This course

- Example : Background Removal 空間 → 矩陣





$$\begin{bmatrix} a & a & b & b & c & c \\ a & a & b & b & c & c \\ a & a & b & b & c & c \\ a & a & b & b & c & c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [a \; b \; c \; d]$$

4×6

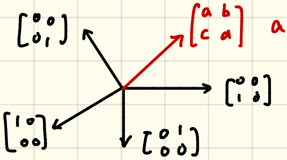
rank-1 matrix

SVD $\boxed{\text{rank-k matrix}}$

$\longrightarrow \mathbb{R}^{4 \times 6}$

$\mathbb{R}^{24}$

- Vectors in a vector subspace S can be matrices or functions



a "vector" in M (all real 2×2 matrix)

- $Ax \Rightarrow \begin{bmatrix} x & 0 & - \\ x & 0 & - \\ x & 0 & - \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} x \\ x \\ x \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} - \\ - \\ - \end{bmatrix}$

  $y^T A \Rightarrow (d\ e\ f)\begin{bmatrix} x & 0 & - \\ x & 0 & - \\ x & 0 & - \end{bmatrix} = d\,[x\ 0\ -] + e\,[x\ 0\ -] + f\,[x\ 0\ -]$

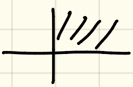
  Mtx-vector multiplication $\equiv$ linear combination

- $b \in C(A) \Rightarrow Ax = b$ is solvable

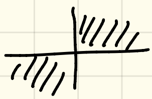
  Column space of A: combinations of the columns of A

  subspace: (i) $v + w \in S$ (ii) $cv \in S$

  for $\forall\ v, w \in S$



NOT



NOT subspace

- <u>Span</u>

S = set of vectors in V (may or <u>may not</u> be a subspace)

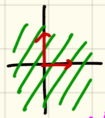
$= \{v_1, \cdots, v_N\}$

SS = All combinations in S

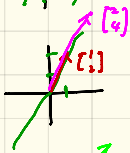
$= c_1 v_1 + \cdots + c_N v_N$

= the <u>subspace</u> of V spanned by S

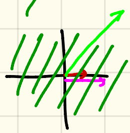
Example 1: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



Example 2: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$



Example 3: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$



看起来很多，其实只有一小部份

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{\text{multiple solutions}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

- Null Space $N(A)$ consists of all solutions to $Ax=0$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

free variable can be anything

- Example:

① $$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}}_{A \text{ is invertible}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}}_{\text{No free variables}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a-b \\ b/2 \end{pmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow N(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \dim(N(A)) = 0$$

② $$[A \ \ 2A] = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \quad *(-3) \; +$$

$$\Rightarrow U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad \underbrace{*\left(\tfrac{1}{2}\right)} \quad + \; *(-1)$$

Upper triangular — pivot columns (1, 2) — free columns (3, 4)

Special solution

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{matrix} \} \text{ pivot variables} \\ \} \text{ free variables} \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \} \text{ pivot variables} \\ \} \text{ free variables} \end{matrix}$$

$$\Rightarrow R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Reduced row echelon form

- Example

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} 1 & 0 & x & x & x & 0 & x \\ 0 & 1 & x & x & x & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$$

3 pivot variables : x_1, x_2, x_6

4 free variables : x_3, x_4, x_5, x_7

4 Special solutions in $N(R) \overset{\shortparallel}{} N(A)$

- The <u>Size</u> of A : 4×7

- The <u>rank</u> ("true size") of A : 3

$\Downarrow$

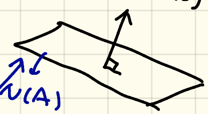
\# of pivots

- Every <u>free column</u> is a combination of (earlier) <u>pivot columns</u>

- Rank-1 Matrix

$$\underset{A}{\begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix}} = \underset{u}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \underset{v^T}{\begin{bmatrix} 1 & 3 & 10 \end{bmatrix}}$$

$$v = \begin{bmatrix} 1 \\ 3 \\ 10 \end{bmatrix}$$



$Ax = 0$

$\Rightarrow uv^T x = 0$

$\Rightarrow u(v^T x) = 0$

$\quad \overset{\shortparallel}{0} \quad \forall x \in N(A), \ x \perp v$

dim of null space : $n - r$

$= 3 - 1 = 2$

$\boxed{x + 3y + 10z = 0}$

$$\begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

row operation $\qquad$ column operation

\# of independent row

= # of independent column (= # of pivot column)

= 1

= dimension of the column space = dim of the row space