

Logic Synthesis and Verification

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Multi-Level Logic Minimization

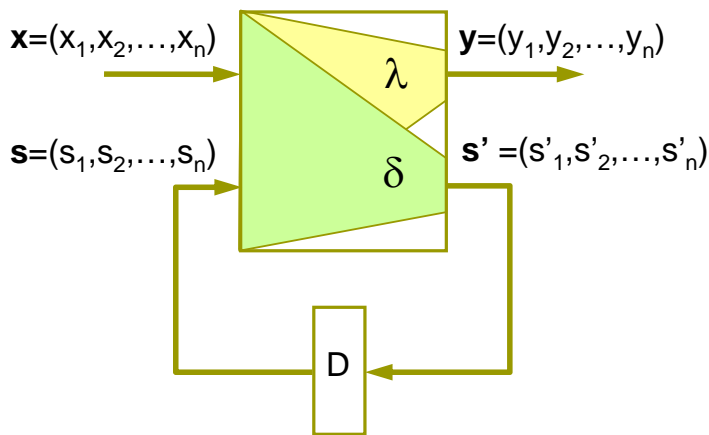
Reading:

Logic Synthesis in a Nutshell
Section 3 (§3.3)

most of the following slides are by
courtesy of Andreas Kuehlmann

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Finite State Machine



Finite-State Machine $F(Q, Q_0, X, Y, \delta, \lambda)$
where:

Q : Set of internal states

Q_0 : Set of initial states

X : Input alphabet

Y : Output alphabet

$\delta: X \times Q \rightarrow Q$ (next state function)

$\lambda: X \times Q \rightarrow Y$ (output function)

Delay element:

- Clocked: synchronous circuit
 - single-phase clock, multiple-phase clocks
- Clockless: asynchronous circuit

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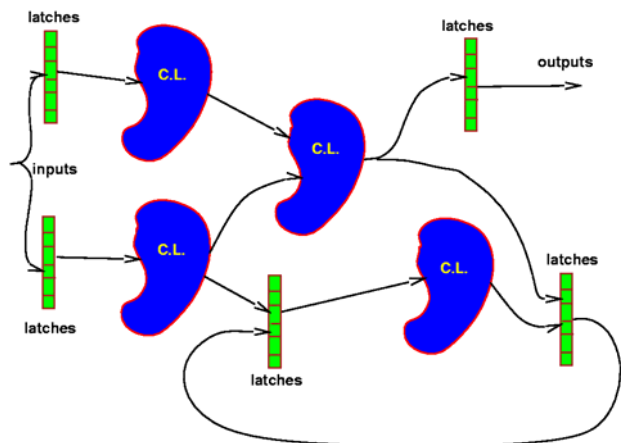
General Logic Structure

Combinational optimization

- keep latches/registers at current positions, keep their function
- optimize combinational logic in between

Sequential optimization

- change latch position/function



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Optimization Criteria for Synthesis

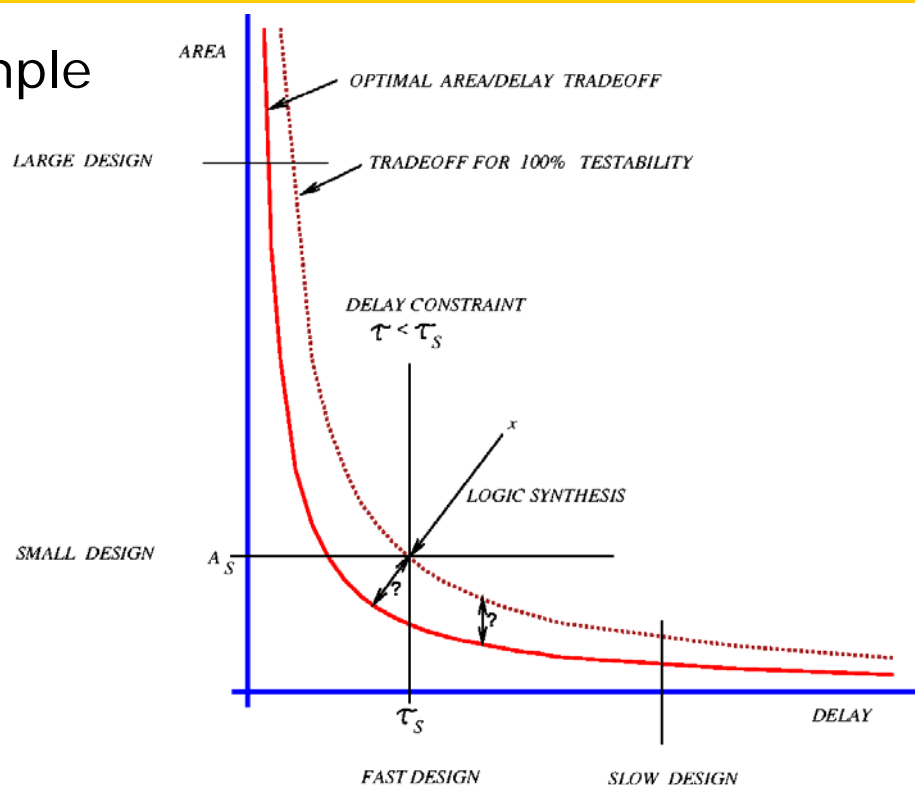
- The optimization criteria for multi-level logic is to *minimize* some function of:
 1. Area occupied by the logic gates and interconnect (approximated by literals = transistors in technology independent optimization)
 2. Critical path delay of the longest path through the logic
 3. Degree of testability of the circuit, measured in terms of the percentage of faults covered by a specified set of test vectors for an approximate fault model (e.g. single or multiple stuck-at faults)
 4. Power consumed by the logic gates
 5. Noise immunity
 6. Placeability, routability

while simultaneously satisfying upper or lower bound constraints placed on these physical quantities

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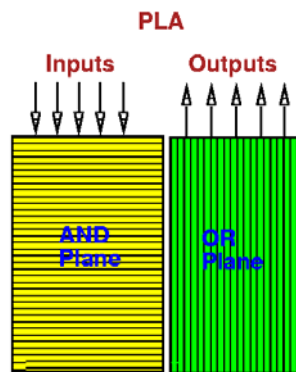
Area-Delay Trade-off

□ Example

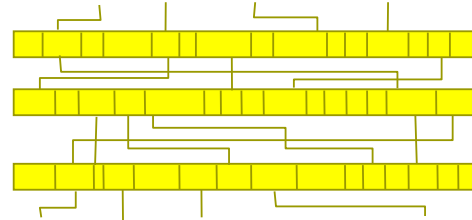


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Two-Level (PLA) vs. Multi-Level



E.g. Standard Cell Layout



□ PLA

- Control logic
- Constrained layout
- Highly automatic
- Technology independent
- Multi-valued logic
- Input, output, state encoding
- Predictable

□ Multi-level logic

- Control logic, data path
- General layout
- Automatic
- Partially technology independent
- Some ideas of multi-valued logic
- Occasionally involving encoding
- Hard to predict

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General Approaches to Synthesis

□ PLA synthesis:

- theory well understood
- predictable results in a top-down flow

□ Multi-level synthesis:

- optimization criteria very complex
 - except special cases, no general theory available
- greedy optimization approach
 - incrementally improve along various dimensions of the criteria
- works on common design representation (circuit or network representation)
 - attempt a change, accept if criteria improve, reject otherwise

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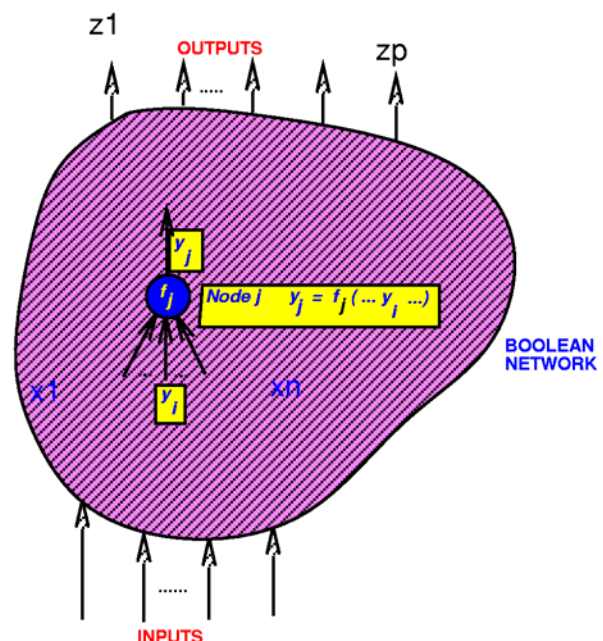
Transformation-based Synthesis

- All modern synthesis systems are transformation based
 - set of transformations that change network representation
 - work on uniform network representation
 - “script” of “scenario” that can orchestrate various transformations
- Transformations differ in:
 - the scope they are applied
 - Local vs. global restructuring
 - the domain they optimize
 - combinational vs. sequential
 - timing vs. area
 - technology independent vs. technology dependent
 - the underlying algorithms they use
 - BDD based, SAT based, structure based

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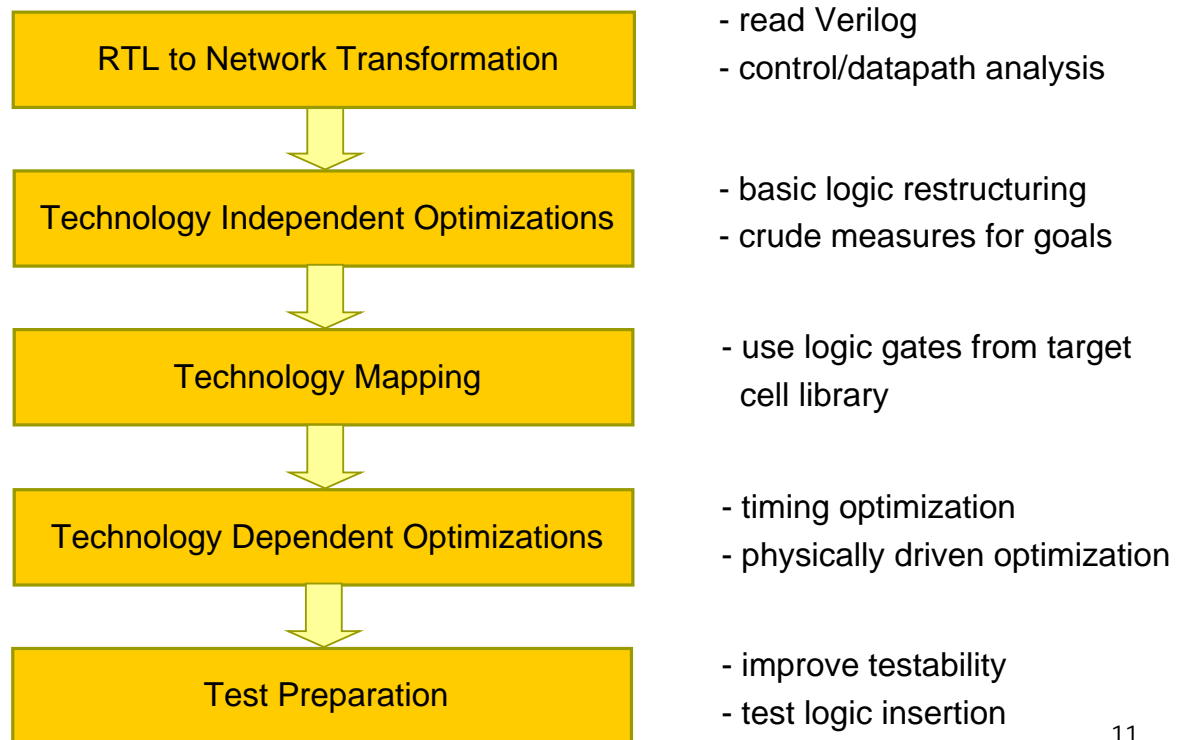
Network Representation

- Boolean network
 - Directed acyclic graph (DAG)
 - Node logic function representation $f_j(x,y)$
 - Node variable y_j : $y_j = f_j(x,y)$
 - Edge (i,j) if f_j depends explicitly on y_i
- Inputs: $x = (x_1, \dots, x_n)$
- Outputs: $z = (z_1, \dots, z_p)$
- External don't cares: $d_1(x), \dots, d_p(x)$ for outputs



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Typical Synthesis Scenario



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Local vs. Global Transformation

- Local transformations optimize one node's function in the network
 - smaller area considered
 - faster performance
 - map to a particular set of cells
- Global transformations restructure the entire network
 - merging nodes
 - splitting nodes
 - removing/changing connections between nodes
- Node representation:
 - keep size bounded to avoid blow-up of local transformations
 - SOP, POS
 - BDD
 - Factored forms
 - AIG + cut computation (modern logic synthesis method)

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Sum-of-Products (SOP)

□ Example

$$abc' + a'bd + b'd' + b'e'f$$

□ Advantages:

- Easy to manipulate and minimize
- many algorithms available (e.g. AND, OR, TAUTOLOGY)
- two-level theory applies

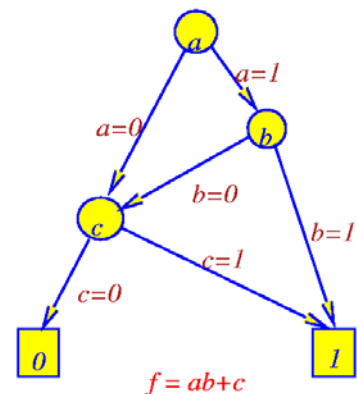
□ Disadvantages:

- Not representative of logic complexity
 - E.g., $f = ad + ae + bd + be + cd + ce$ and $f' = a'b'c' + d'e'$ differ in their implementation by an **inverter**
- Not easy to **estimate** logic; difficult to estimate **progress** during logic manipulation

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Reduced Ordered BDD

- Represents both function and its **complement**, like **factored forms** to be discussed
- Like network of muxes, but restricted since controlled by **primary input** variables
 - **not really a good estimator** for implementation complexity
- Given an ordering, reduced BDD is **canonical**, hence a good replacement for truth tables
- For a good **ordering**, BDDs remain reasonably small for complicated functions (but not multipliers, for instance)
- **Manipulations** are well defined and efficient
- Only **true** support variables (dependency on primary input variables) are displayed



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Factor Form

□ Example

$$(ad+b'c)(c+d'(e+ac')) + (d+e)fg$$

□ Advantages

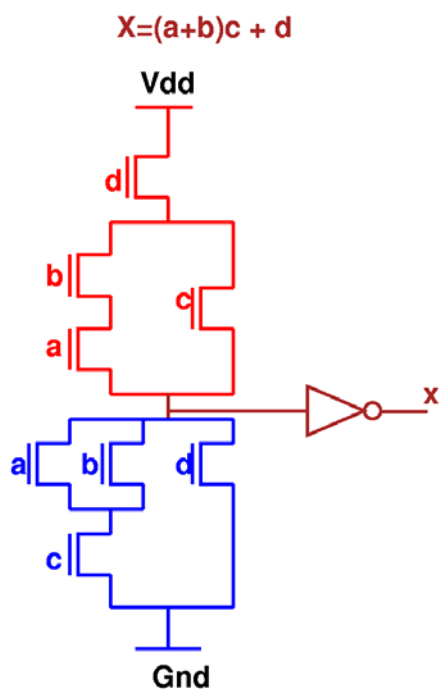
- good representative of logic **complexity**
 $f=ad+ae+bd+be+cd+ce$
 $f=a'b'c'+d'e' \Rightarrow f=(a+b+c)(d+e)$
- in many designs (e.g. complex gate CMOS) the **implementation** of a function corresponds directly to its factored form
- good **estimator** of logic implementation complexity
- doesn't **blow up** easily

□ Disadvantages

- not as many algorithms available for **manipulation**
- usually **converted** into SOP before manipulation

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Factor Form



Note:

literal count \approx transistor count \approx area

- however, area also depends on wiring, gate size, etc.
- therefore very crude measure

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Factored Form

- **Definition:** f is an **algebraic expression** if f is a set of cubes (SOP), such that no single cube contains another (minimal with respect to single cube containment)

- **Example**

$a+ab$ is not an algebraic expression (factoring gives $a(1+b)$)

- **Definition:** The **product** of two expressions f and g is a set defined by $fg = \{cd \mid c \in f \text{ and } d \in g \text{ and } cd \neq 0\}$

- **Example**

$(a+b)(c+d+a')=ac+ad+bc+bd+a'b$

- **Definition:** fg is an **algebraic product** if f and g are algebraic expressions and have **disjoint** support (that is, they have no input variables in common)

- **Example**

$(a+b)(c+d)=ac+ad+bc+bd$ is an algebraic product

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Factored Form

- **Definition:** A **factored form** can be defined recursively by the following rules. A factored form is either a product or sum where:

- a product is either a single **literal** or a **product** of factored forms

- a sum is either a single **literal** or a **sum** of factored forms

- A **factored form is a parenthesized algebraic expression**

- In effect a factored form is a **product of sums of products** or a **sum of products of sums**

- **Any** logic **function** can be represented by a factored form, and **any** factored form is a representation of some logic function

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Factored Form

□ Example

- $x, y', abc', a+b'c, ((a'+b)cd+e)(a+b')+e'$ are factored forms
- $(a+b)'c$ is not a factored form since complement is not allowed, except on literals

□ Factored forms are not unique

- Three equivalent factored forms
 $ab+c(a+b), \quad bc+a(b+c), \quad ac+b(a+c)$

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Factored Form

- Definition: The factorization value of an algebraic factorization $F=G_1G_2+R$ is defined to be

$$\begin{aligned} fact_val(F, G_2) &= lits(F) - (lits(G_1) + lits(G_2) + lits(R)) \\ &= (|G_1|-1) lits(G_2) + (|G_2|-1) lits(G_1) \end{aligned}$$

- Assuming G_1, G_2 and R are algebraic expressions, where $|H|$ is the number of cubes in the SOP form of H

■ Example

$$F = ae+af+ag+bce+bcf+bcg+bde+bdf+bdg$$

can be expressed in the form $F = (a+b(c+d))(e+f+g)$, which requires 7 literals, rather than 24

- If $G_1=(a+bc+bd)$ and $G_2=(e+f+g)$, then $R=\emptyset$ and

$$fact_val(F, G_2) = 2 \times 3 + 2 \times 5 = 16$$

- The above factored form saves 17 literals, not 16. The extra literal saving comes from recursively applying the formula to the factored form of G_1 .

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Factored Form

- Factored forms are more **compact** representations of logic functions than the traditional SOP forms

- Example:

$$(a+b)(c+d(e+f(g+h+i+j)))$$

when represented as an SOP form is

$$ac+ade+adfg+adfh+adfi+adfj+bc+bde+bdfg+bdfh+bdfi+bdfj$$

- SOP is a factored form, but it may not be a good factorization

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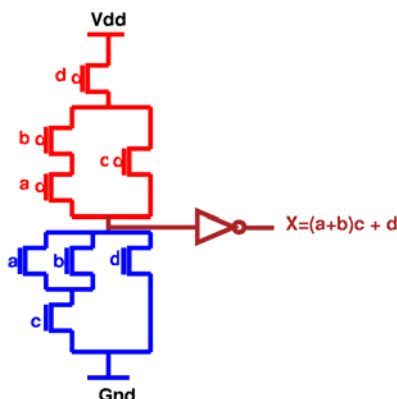
Factored Form

- There are functions whose size is **exponential** in SOP representation, but **polynomial** in factored form

- Example:

Achilles' heel function $\prod_{i=1}^{i=n/2} (x_{2i-1} + x_{2i})$

n literals in factored form and $(n/2) \times 2^{n/2}$ literals in SOP form



Factored forms are useful in **estimating** area and delay in a multi-level synthesis and optimization system. In many design styles (e.g. complex gate CMOS design) the implementation of a function corresponds directly to its factored form.

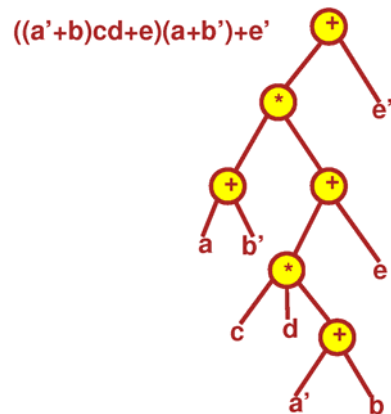
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Factored Form

- Factored forms can be graphically represented as labeled **trees**, called factoring trees, in which each internal node including the root is labeled with either $+$ or \times , and each leaf has a label of either a variable or its complement

- Example**

factoring tree of $((a'+b)cd+e)(a+b')+e'$



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Factored Form

- Definition:** The **size** of a factored form F (denoted $\rho(F)$) is the number of literals in the factored form
 - E.g., $\rho((a+b)ca') = 4$, $\rho((a+b+cd)(a'+b')) = 6$
- A factored form of a function is **optimal** if no other factored form has less literals
- A factored form is **positive unate** in x , if x appears in F , but x' does not. A factored form is **negative unate** in x , if x' appears in F , but x does not.
- F is **unate** in x if it is either positive or negative unate in x , otherwise F is **binate** in x
 - E.g., $F = (a+b')c+a'$
positive unate in c ; negative unate in b ; binate in a

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Factored Form Cofactor

- The cofactor of a factored form F , with respect to a literal x_1 (or x_1'), is the factored form $F_{x_1} = F_{x_1=1}(x)$ (or $F_{x_1'} = F_{x_1=0}(x)$) obtained by
 - replacing all occurrences of x_1 by 1, and x_1' by 0
 - simplifying the factored form using the Boolean algebra identities
$$1y=y \quad 1+y=1 \quad 0y=0 \quad 0+y=y$$
 - after constant propagation (all constants are removed), part of the factored form may appear as $G+G$. In general, G is in a factored form.

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Factored Form Cofactor

- The cofactor of a factored form F , with respect to a cube c , is a factored form F_c obtained by successively cofactoring F with each literal in c

- Example

$$F = (x+y'+z)(x'u+z'y'(v+u')) \text{ and } c = vz'.$$

Then

$$F_{z'} = (x+y')(x'u+y'(v+u'))$$

$$F_{z'v} = (x+y')(x'u+y')$$

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Factored Form Optimality

□ Definition

Let f be a completely specified Boolean function, and $\rho(f)$ is the minimum number of literals in any factored form of f

■ Recall $\rho(F)$ is the number of literals of a factored form F

□ Definition

Let $\text{sup}(f)$ be the true support variable of f , i.e. the set of variables that f depends on. Two functions f and g are **orthogonal**, denoted $f \perp g$, if $\text{sup}(f) \cap \text{sup}(g) = \emptyset$

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Factored Form Optimality

□ Lemma: Let $f = g + h$ such that $g \perp h$, then $\rho(f) = \rho(g) + \rho(h)$

■ Proof:

Let F , G and H be the optimum factored forms of f , g and h . Since $G+H$ is a factored form, $\rho(f) = \rho(F) \leq \rho(G+H) = \rho(g) + \rho(h)$.

Let c be a minterm, on $\text{sup}(g)$, of g' . Since g and h have disjoint support, we have $f_c = (g+h)_c = g_c + h_c = 0 + h_c = h_c = h$. Similarly, if d is a minterm of h' , $f_d = g$. Because $\rho(h) = \rho(f_c) \leq \rho(F_c)$ and $\rho(g) = \rho(f_d) \leq \rho(F_d)$, $\rho(h) + \rho(g) \leq \rho(F_c) + \rho(F_d)$.

Let m (n) be the number of literals in F that are from $\text{SUPPORT}(g)$ ($\text{SUPPORT}(h)$). When computing F_c (F_d), we replace all the literals from $\text{SUPPORT}(g)$ ($\text{SUPPORT}(h)$) by the appropriate values and simplify the factored form by eliminating all the constants and possibly some literals from $\text{sup}(g)$ ($\text{sup}(h)$) by using the Boolean identities. Hence $\rho(F_c) \leq n$ and $\rho(F_d) \leq m$. Since $\rho(F) = m+n$, $\rho(F_c) + \rho(F_d) \leq m+n = \rho(F)$.

We have $\rho(f) \leq \rho(g) + \rho(h) \leq \rho(F_c) + \rho(F_d) \leq \rho(F) \Rightarrow \rho(f) = \rho(g) + \rho(h)$ since $\rho(f) = \rho(F)$.

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Factored Form

Optimality

- Note, the previous result does not imply that **all** minimum literal factored forms of f are sums of the minimum literal factored forms of g and h

- Corollary: Let $f = gh$ such that $g \perp h$, then $\rho(f) = \rho(g) + \rho(h)$

- Proof:

Let F' denote the factored form obtained using DeMorgan's law. Then $\rho(F) = \rho(F')$, and therefore $\rho(f) = \rho(F')$. From the above lemma, we have $\rho(f) = \rho(F') = \rho(g' + h') = \rho(g') + \rho(h') = \rho(g) + \rho(h)$.

- Theorem: Let $f = \sum_{i=1}^n \prod_{j=1}^m f_{ij}$ such that $f_{ij} \perp f_{kl}, \forall i \neq k \text{ or } j \neq l$, then

$$\rho(f) = \sum_{i=1}^n \sum_{j=1}^m \rho(f_{ij})$$

- Proof:

Use induction on m and then n , and the above lemma and corollary.

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Factored Form

- SOP forms are used as the internal representation of logic functions in most multi-level logic optimization systems
- Advantages
 - good algorithms for manipulating them are available
- Disadvantages
 - performance is unpredictable - may accidentally generate a function whose SOP form is too large
 - factoring algorithms have to be used constantly to provide an estimate for the size of the Boolean network, and the time spent on factoring may become significant
- Possible solution
 - **avoid** SOP representation by using factored forms as the internal representation
 - still not practical unless we know how to perform logic operations **directly** on factored forms without converting to SOP forms
 - the most common logic operations over factored form have been partially provided

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Boolean Network Manipulation

□ Basic techniques

- Structural operations (change topology)
 - Algebraic
 - Boolean
- Node simplification (change node functions)
 - Node minimization using don't cares

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Structural Operation

□ Restructuring: Given initial network, find best network

■ Example

$$f_1 = abcd + ab'cd' + acd'e + ab'c'd' + a'c + cdf + abc'd'e' + ab'c'df'$$

$$f_2 = bdg + b'dfg + b'd'g + bd'eg$$

minimizing

$$f_1 = bcd + b'cd' + cd'e + a'c + cdf + abc'd'e' + ab'c'df'$$

$$f_2 = bdg + dfg + b'd'g + d'eg$$

factoring

$$f_1 = c(d(b+f) + d'(b'+e) + a') + ac'(bd'e' + b'df')$$

$$f_2 = g(d(b+f) + d'(b'+e))$$

decompose

$$f_1 = c(x+a') + ac'x'$$

$$f_2 = gx$$

$$x = d(b+f) + d'(b'+e)$$

□ Two problems:

- find good common subfunctions
- effect the division

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Structural Operation

Basic Operations:

- Decomposition (single function)

$$f = abc + abd + a'c'd' + b'c'd' \Rightarrow$$

$$f = xy + x'y' \quad x = ab \quad y = c + d$$
- Extraction (multiple functions)

$$f = (az + bz')cd + e \quad g = (az + bz')e' \quad h = cde \Rightarrow$$

$$f = xy + e \quad g = xe' \quad h = ye \quad x = az + bz' \quad y = cd$$
- Factoring (series-parallel decomposition)

$$f = ac + ad + bc + bd + e \Rightarrow$$

$$f = (a + b)(c + d) + e$$
- Substitution

$$g = a + b \quad f = a + bc \Rightarrow$$

$$f = g(a + c)$$
- Collapsing (also called elimination)

$$f = ga + g'b \quad g = c + d \Rightarrow$$

$$f = ac + ad + bc'd' \quad g = c + d$$

"Division" plays a key role in all of these operations

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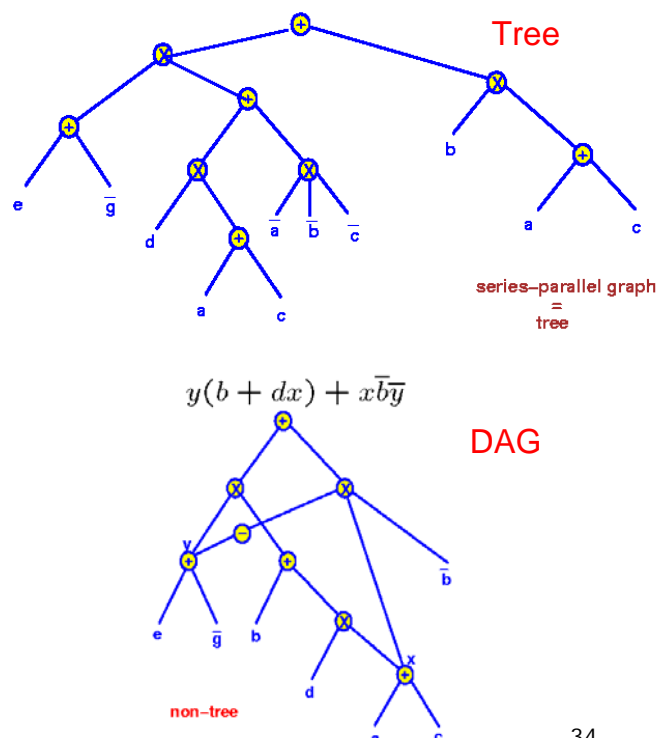
Factoring vs. Decomposition

Factoring:

- $f = (e + g')(d(a + c) + a'b'c') + b(a + c)$

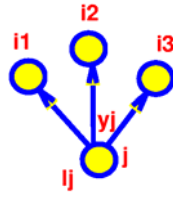
Decomposition:

- $y(b + dx) + xb'y'$
 - Similar to merging common nodes and using negative pointers in BDD. However, **not** canonical, so have no perfect identification of common nodes.



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Structural Operation Node Elimination



$$value(j) = \left(\sum_{i \in FO(j)} n_i \right) (l_j - 1) - l_j$$

where

n_i = number of times literals y_j and y_j' occur in factored form f_i

■ can treat y_j and y_j' the same since $\rho(F_j) = \rho(F_j')$

l_j = number of literals in factored f_j

with factoring

$$l_j + \sum_{i \in FO(j)} n_i + c$$

without factoring

$$l_j \sum_{i \in FO(j)} n_i + c$$

value = (without factoring) - (with factoring)

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Structural Operation Node Elimination

Example

■ Literals before

$$5 + 7 + 5 = 17$$

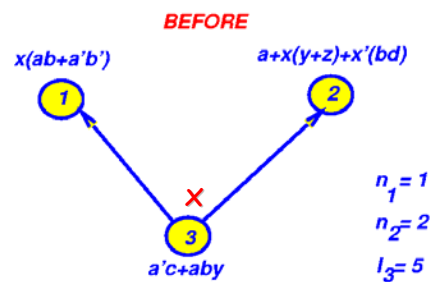
■ Literals after

$$9 + 15 = 24$$

■ Difference:

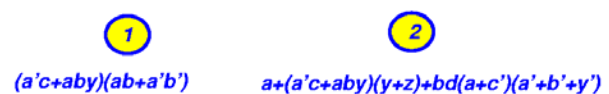
after - before =

$$value = 7$$



$$value(3) = (1+2-1)(5-1) - 1 = 7$$

AFTER



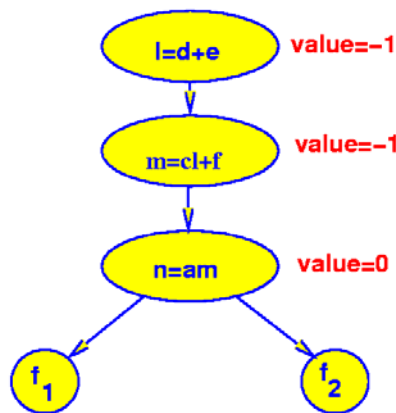
$$value(j) = \left(\sum_{i \in FO(j)} n_i \right) (l_j - 1) - l_j$$

$$= (n_1 + n_2)(l_3 - 1) - l_3$$

$$= (1 + 2)(5 - 1) - 5 = 7$$

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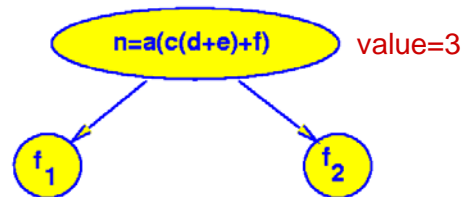
Structural Operation Node Elimination



$$n = a(c(d + e) + f)$$

$$f_1 = b(n + ag) + h$$

$$f_2 = i(n + aj) + k$$



Note: Value of a node can change during elimination

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Factorization

□ Given a SOP, how do we generate a “good” factored form

□ Division operation:

- is central in many operations
- find a good divisor
- apply division
 - results in quotient and remainder

□ Applications:

- factoring
- decomposition
- substitution
- extraction

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Division

□ **Definition:** An operation **op** is called **division** if, given two SOP expressions F and G, it generates expressions H and R ($\langle H, R \rangle = \text{op}(F, G)$) such that **$F = GH + R$**

- G is called the divisor
- H is called the quotient
- R is called the remainder

□ **Definition:** If GH is an algebraic product, then **op** is called an algebraic division (denoted **$F // G$**), otherwise GH is a Boolean product and **op** is called a Boolean division (denoted **$F \div G$**)

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Division

□ **Example:**

$$f = ad + ae + bcd + j$$

$$g_1 = a + bc$$

$$g_2 = a + b$$

■ Algebraic division:

□ $f // a = d + e, r = bcd + j$

Also, $f // a = d$ or $f // a = e$, i.e. algebraic division is not unique

□ $f // (bc) = d, r = ad + ae + j$

□ $h_1 = f // g_1 = d, r_1 = ae + j$

■ Boolean division:

□ $h_2 = f \div g_2 = (a + c)d, r_2 = ae + j.$
i.e. $f = (a+b)(a+c)d + ae + j$

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Division

□ Definition:

G is an **algebraic factor** of F if there exists an algebraic expression H such that $F = GH$ (using algebraic multiplication)

□ Definition:

G is a **Boolean factor** of F if there exists an expression H such that $F = GH$ (using Boolean multiplication)

□ Example

■ $f = ac + ad + bc + bd$

□ $(a+b)$ is an algebraic factor of f since $f = (a+b)(c+d)$

■ $f = -ab + ac + bc$

□ $(a+b)$ is a Boolean factor of f since $f = (a+b)(-a+c)$

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Why Algebraic Methods?

□ Algebraic methods provide fast algorithms for various operations

■ Treat logic functions as polynomials

■ Fast algorithms for polynomials exist

■ Lost of optimality but results are still good

■ Can iterate and interleave with Boolean operations

□ In specific instances, slight extensions are available to include Boolean methods

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Weak Division

□ **Weak division** is a specific example of algebraic division

□ **Definition:**

Given two algebraic expressions F and G, a division is called a **weak division** if

1. it is **algebraic** and
 2. remainder R has **as few cubes as possible**
- The **quotient** H resulting from weak division is denoted by **F/G**

□ **Theorem:**

Given expressions F and G, H and R generated by weak division are unique

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Weak Division

```
ALGORITHM WEAK_DIV(F,G) {
  // G = {g1,g2,...}, F = {f1,f2,...} are sets of cubes
  foreach gi {
    Vgi = ∅
    foreach fj {
      if(fj contains all literals of gi) {
        vij = fj - literals of gi
        Vgi = Vgi ∪ vij
      }
    }
  }
  H = ∩i Vgi
  R = F - GH
  return (H,R);
}
```

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Weak Division

□ Example

$$F = ace + ade + bc + bd + be + a'b + ab$$

$$G = ae + b$$

$$V^{ae} = c + d$$

$$V^b = c + d + e + a' + a$$

$$H = c + d = F/G$$

$$H = \cap V^{g_i}$$

$$R = be + a'b + ab$$

$$R = F \setminus GH$$

$$F = (ae + b)(c + d) + be + a'b + ab$$

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Weak Division

□ We use **filters** to prevent trying a division

■ G is not an algebraic divisor of F if

□ G contains a literal not in F,

□ G has more terms than F,

□ For any literal, its count in G exceeds that in F, or

□ F is in the transitive fanin of G.

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Weak Division

- Weak_Div provides a method to divide an expression for a given divisor
- How do we find a “good” divisor?
 - Restrict to algebraic divisors
 - Generalize to Boolean divisors
- Problem:
Given a set of functions $\{ F_i \}$, find common weak (algebraic) divisors

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Divisor Identification

Primary Divisor

- Definition:
An expression is **cube-free** if no cube divides the expression evenly (i.e., there is no literal that is common to all the cubes)
 - “ab+c” is cube-free
 - “ab+ac” and “abc” are not cube-free
 - Note: A cube-free expression **must** have more than one cube
- Definition:
The **primary divisors** of an expression F are the set of expressions
$$D(F) = \{ F/c \mid c \text{ is a cube} \}$$
Note that F/c is the **quotient of a weak division**

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Divisor Identification

Kernel and Co-Kernel

□ Definition:

The **kernels** of an expression F are the set of expressions

$$K(F) = \{G \mid G \in D(F) \text{ and } G \text{ is cube-free}\}$$

- In other words, the kernels of an expression F are the **cube-free primary divisors** of F

□ Definition:

A cube c used to obtain the kernel $K = F/c$ is called a **co-kernel** of K

- $C(F)$ is used to denote the **set of co-kernels** of F

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Divisor Identification

Kernel and Co-Kernel

□ Example

$$\begin{aligned} x &= adf + aef + bdf + bef + cdf + cef + g \\ &= (a + b + c)(d + e)f + g \end{aligned}$$

kernels

$a+b+c$

$d+e$

$(a+b+c)(d+e)$

$(a+b+c)(d+e)f+g$

co-kernels

df, ef

af, bf, cf

f

1

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Divisor Identification

Kernel and Kernel Intersection

□ Fundamental Theorem

If two expressions F and G have the property that

$\forall k_F \in K(F), \forall k_G \in K(G) \rightarrow |k_G \cap k_F| \leq 1$
(k_G and k_F have at most one term in common),

then F and G have **no common** algebraic divisors with **more than one cube**

■ Important:

If we “kernel” all functions and there are no nontrivial intersections, then the only common algebraic divisors left are **single cube divisors**

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Divisor Identification

Kernel Level

□ Definition:

A kernel is of **level 0** (K^0) if it contains no kernels except itself

A kernel is of **level n** or less (K^n) if it contains at least one kernel of level (n-1) or less, but no kernels (except itself) of level n or greater

- $K^n(F)$ is the set of kernels of level n or less
- $K^0(F) \subset K^1(F) \subset K^2(F) \subset \dots \subset K^n(F) \subset K(F)$
- level-n kernels = $K^n(F) \setminus K^{n-1}(F)$

□ Example:

$$F = (a + b(c + d))(e + g)$$

$$k_1 = a + b(c + d) \in K^1 \\ \notin K^0 \implies \text{level-1}$$

$$k_2 = c + d \in K^0$$

$$k_3 = e + g \in K^0$$

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Divisor Identification Kerneling Algorithm

```
Algorithm KERNEL(j, G) {  
  R =  $\emptyset$   
  if(CUBE_FREE(G)) R = {G}  
  for(i=j+1,...,n) {  
    if( $l_i$  appears only in one term) continue  
    if( $\exists k \leq i, l_k \in$  all cubes of  $G/l_i$ ) continue  
    R = R  $\cup$  KERNEL(i, MAKE_CUBE_FREE( $G/l_i$ ))  
  }  
  return R  
}
```

MAKE_CUBE_FREE(F) removes algebraic cube factor from F

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Divisor Identification Kerneling Algorithm

□ **KERNEL**(0, F) returns all the kernels of F

□ Note:

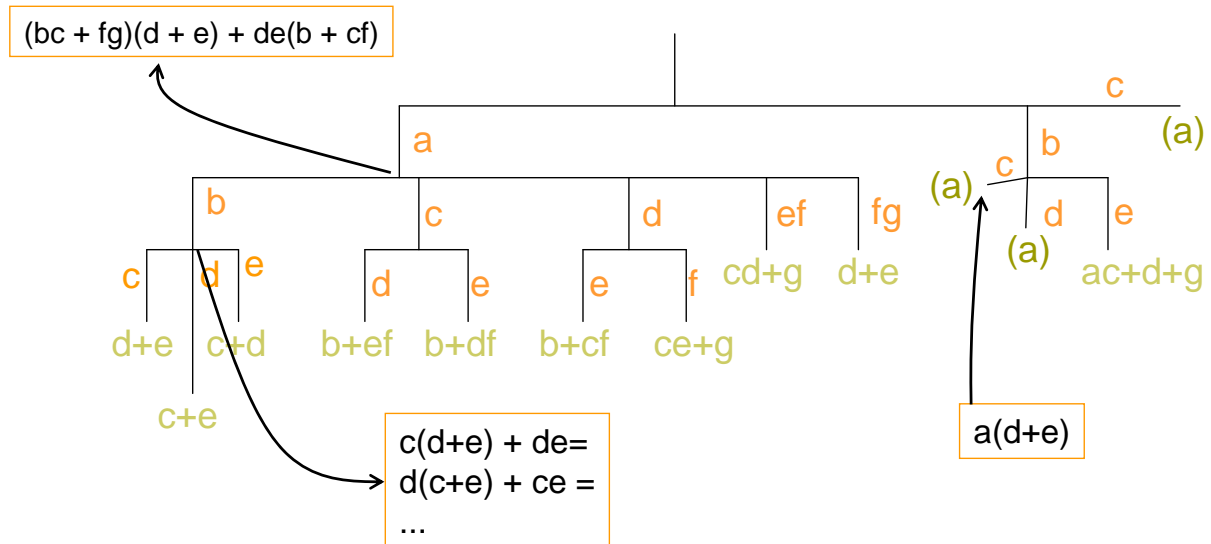
- The test “($\exists k \leq i, l_k \in$ all cubes of G/l_i)” in the kerneling algorithm is a **major** efficiency factor. It also guarantees that no co-kernel is tried more than once.
- Can be used to generate all co-kernels

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Divisor Identification Kerneling Algorithm

□ Example

$F = abcd + abce + adfg + aefg + adbe + acdef + beg$
(Let a, b, c, d, e, f, g be $l_1, l_2, l_3, l_4, l_5, l_6, l_7$, respectively.)



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Divisor Identification Kerneling Algorithm

□ Example

co-kernels

1
a
ab
abc
abd
abe
ac
acd

kernels

$a((bc + fg)(d + e) + de(b + cf)) + beg$
 $(bc + fg)(d + e) + de(b + cf)$
 $c(d+e) + de$
 $d + e$
 $c + e$
 $c + d$
 $b(d + e) + def$
 $b + ef$

Note: $F/bc = ad + ae = a(d + e)$

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Factor

```
Algorithm FACTOR(F) {  
  if(F has no factor) return F  
  // e.g. if |F|=1, or F is an OR of single literals  
  // or of no literal appears more than once  
  D      = CHOOSE_DIVISOR(F)  
  (Q,R) = DIVIDE(F,D)  
  return FACTOR(Q)×FACTOR(D) + FACTOR(R) //recur  
}
```

- different heuristics can be applied for CHOOSE_DIVISOR
- different DIVIDE routines may be applied (algebraic division, Boolean division)

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Factor

□ Example:

$F = abc + abd + ae + af + g$

$D = c + d$

$Q = ab$

$P = ab(c + d) + ae + af + g$

$O = ab(c + d) + a(e + f) + g$

Notation:

F = original function

D = divisor

Q = quotient

P = partial factored form

O = final factored form by
FACTOR restricting to
algebraic operations only

■ Problem 1:

O is not optimal since not maximally factored and can be further factored to “ $a(b(c + d) + e + f) + g$ ”

- It occurs when quotient Q is a single cube, and some of the literals of Q also appear in the remainder R

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Factor

□ To solve Problem 1

■ Check if the quotient Q is not a single cube, then done

■ Else, pick a literal l_1 in Q which occurs most frequently in cubes of F . Divide F by l_1 to obtain a new divisor D_1 .

Now, F has a new partial factored form

$$(l_1)(D_1) + (R_1)$$

and literal l_1 does not appear in R_1 .

■ **Note:** The new divisor D_1 contains the original D as a divisor because l_1 is a literal of Q . When recursively factoring D_1 , D can be discovered again.

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Factor

□ Example:

$$F = ace + ade + bce + bde + cf + df$$

$$D = a + b$$

$$Q = ce + de$$

$$P = (ce + de)(a + b) + (c + d)f$$

$$O = e(c + d)(a + b) + (c + d)f$$

Notation:

F = original function

D = divisor

Q = quotient

P = partial factored form

O = final factored form by

FACTOR restricting to

algebraic operations only

■ Problem 2:

O is not maximally factored because “ $(c + d)$ ” is common to both products “ $e(c + d)(a + b)$ ” and “ $(c + d)f$ ”

■ The final factored form should have been “ $(c+d)(e(a + b) + f)$ ”

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Factor

□ To solve Problem 2

■ Essentially, we reverse D and Q!!

- Make Q **cube-free** to get Q_1
- Obtain a new divisor D_1 by dividing F by Q_1
- If D_1 is cube-free, the partial factored form is $F = (Q_1)(D_1) + R_1$, and can recursively factor Q_1 , D_1 , and R_1
- If D_1 is not cube-free, let $D_1 = cD_2$ and $D_3 = Q_1D_2$. We have the partial factoring $F = cD_3 + R_1$. Now recursively factor D_3 and R_1 .

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Factor

```
Algorithm GFACOR(F, DIVISOR, DIVIDE) { // good factor
  D = DIVISOR(F)
  if(D = 0) return F
  Q = DIVIDE(F,D)
  if (|Q| = 1) return LF(F, Q, DIVISOR, DIVIDE)
  Q = MAKE_CUBE_FREE(Q)
  (D, R) = DIVIDE(F,Q)
  if (CUBE_FREE(D)) {
    Q = GFACOR(Q, DIVISOR, DIVIDE)
    D = GFACOR(D, DIVISOR, DIVIDE)
    R = GFACOR(R, DIVISOR, DIVIDE)
    return Q × D + R
  }
  else {
    C = COMMON_CUBE(D) // common cube factor
    return LF(F, C, DIVISOR, DIVIDE)
  }
}
```

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Factor

```
Algorithm LF(F, C, DIVISOR, DIVIDE) { // literal
  factor
  L = BEST_LITERAL(F, C) //L ∈ C most frequent in F
  (Q, R) = DIVIDE(F, L)
  C = COMMON_CUBE(Q) // largest one
  Q = CUBE_FREE(Q)
  Q = GFACTOR(Q, DIVISOR, DIVIDE)
  R = GFACTOR(R, DIVISOR, DIVIDE)
  return L × C × Q + R
}
```

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Factor

- ❑ Various kinds of factoring can be obtained by choosing different forms of **DIVISOR** and **DIVIDE**
- ❑ **CHOOSE_DIVISOR**:
 - LITERAL** - chooses most frequent literal
 - QUICK_DIVISOR** - chooses the first level-0 kernel
 - BEST_DIVISOR** - chooses the best kernel
- ❑ **DIVIDE**:
 - Algebraic Division
 - Boolean Division

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Factor

□ Example

$$x = ac + ad + ae + ag + bc + bd + be + bf + ce + cf + df + dg$$

LITERAL_FACTOR:

$$x = a(c + d + e + g) + b(c + d + e + f) + c(e + f) + d(f + g)$$

QUICK_FACTOR:

$$x = g(a + d) + (a + b)(c + d + e) + c(e + f) + f(b + d)$$

GOOD_FACTOR:

$$(c + d + e)(a + b) + f(b + c + d) + g(a + d) + ce$$

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Factor

- QUICK_FACTOR uses GFACTOR, first level-0 kernel DIVISOR, and WEAK_DIV

□ Example

$$x = ae + afg + afh + bce + bcfg + bcfh + bde + bdfg + bcfh$$

$$D = c + d \quad \text{---- level-0 kernel (first found)}$$

$$Q = x/D = b(e + f(g + h)) \quad \text{---- weak division}$$

$$Q = e + f(g + h) \quad \text{---- make cube-free}$$

$$(D, R) = \text{WEAK_DIV}(x, Q) \quad \text{---- second division}$$

$$D = a + b(c + d)$$

$$x = QD + R, \quad R = 0$$

$$x = (e + f(g + h)) (a + b(c + d))$$

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Decomposition

- Decomposition is the same as factoring **except**:
 - divisors are added as **new** nodes in the network
 - the new nodes may **fan out** elsewhere in the network in both **positive** and **negative** phases

```
Algorithm DECOMP( $f_i$ ) {  
   $k = \text{CHOOSE\_KERNEL}(f_i)$   
  if ( $k == 0$ ) return  
   $f_{m+j} = k$  // create new node  $m + j$   
   $f_i = (f_i/k)y_{m+j} + (f_i/k')y'_{m+j} + r$  // change node  $i$  using  
  // new node for kernel  
  DECOMP( $f_i$ )  
  DECOMP( $f_{m+j}$ )  
}
```

Similar to factoring, we can define

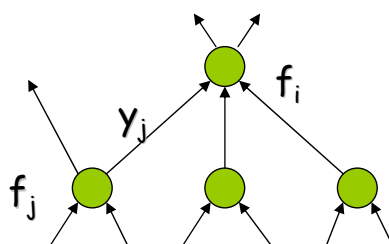
QUICK_DECOMP: pick a level 0 kernel and improve it

GOOD_DECOMP: pick the best kernel

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Substitution

- **Idea**: An existing node in a network may be a useful divisor in another node. If so, no loss in using it (unless delay is a factor).
- Algebraic substitution consists of the process of algebraically dividing the function f_i at node i in the network by the function f_j (or by f'_j) at node j . During substitution, if f_j is an algebraic divisor of f_i , then f_i is transformed into
$$f_i = qy_j + r \quad (\text{or } f_i = q_1y_j + q_0y'_j + r)$$
- In practice, this is tried for each node pair of the network. n nodes in the network $\Rightarrow O(n^2)$ divisions.



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Extraction

- **Recall:** Extraction operation identifies **common** sub-expressions and restructures a Boolean network
 - Combine **decomposition** and **substitution** to provide an effective extraction algorithm

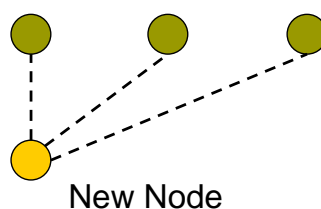
Algorithm **EXTRACT**

```
foreach node n {  
    DECOMP(n) // decompose all network nodes  
}  
foreach node n {  
    RESUB(n) // resubstitute using existing nodes  
}  
ELIMINATE_NODES_WITH_SMALL_VALUE  
}
```

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Extraction

- **Kernel Extraction:**
 1. Find **all** kernels of all functions
 2. **Choose** kernel intersection with best “value”
 3. **Create** new node with this as function
 4. Algebraically **substitute** new node everywhere
 5. Repeat 1,2,3,4 until best value \leq threshold



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Extraction

□ Example

$$f_1 = ab(c(d + e) + f + g) + h$$

$$f_2 = ai(c(d + e) + f + j) + k$$

(only level-0 kernels used in this example)

1. Extraction:

$$K^0(f_1) = K^0(f_2) = \{d + e\}$$

$$K^0(f_1) \cap K^0(f_2) = \{d + e\}$$

$$l = d + e$$

$$f_1 = ab(cl + f + g) + h$$

$$f_2 = ai(cl + f + j) + k$$

2. Extraction:

$$K^0(f_1) = \{cl + f + g\}; K^0(f_2) = \{cl + f + j\}$$

$$K^0(f_1) \cap K^0(f_2) = cl + f$$

$$m = cl + f$$

$$f_1 = ab(m + g) + h$$

$$f_2 = ai(m + j) + k$$

No kernel intersections anymore!!

3. Cube extraction:

$$n = am$$

$$f_1 = b(n + ag) + h$$

$$f_2 = i(n + aj) + k$$

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Extraction Rectangle Covering

□ Alternative method for extraction

□ Build co-kernel cube matrix $M = R^T C$

- rows correspond to co-kernels of individual functions
- columns correspond to individual cubes of kernel
- m_{ij} = cubes of functions
- $m_{ij} = 0$ if cube not there

□ Rectangle covering:

- identify sub-matrix $M^* = R^{*T} C^*$, where $R^* \subseteq R$, $C^* \subseteq C$, and $m^*_{ij} \neq 0$
- construct divisor d corresponding to M^* as new node
- extract d from all functions

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Extraction Rectangle Covering

Example

$$F = af + bf + ag + cg + ade + bde + cde$$

$$G = af + bf + ace + bce$$

$$H = ade + cde$$

Kernels/Co-kernels:

$$F: (de+f+g)/a$$

$$(de + f)/b$$

$$(a+b+c)/de$$

$$(a + b)/f$$

$$(de+g)/c$$

$$(a+c)/g$$

$$G: (ce+f)/\{a,b\}$$

$$(a+b)/\{f,ce\}$$

$$H: (a+c)/de$$

		<i>a</i>	<i>b</i>	<i>c</i>	<i>ce</i>	<i>de</i>	<i>f</i>	<i>g</i>
<i>F</i>	<i>a</i>					<i>ade</i>	<i>af</i>	<i>ag</i>
<i>F</i>	<i>b</i>					<i>bde</i>	<i>bf</i>	
<i>F</i>	<i>de</i>	<i>ade</i>	<i>bde</i>	<i>cde</i>				
<i>F</i>	<i>f</i>	<i>af</i>	<i>bf</i>					
<i>M = F</i>	<i>c</i>					<i>cde</i>		<i>cg</i>
<i>F</i>	<i>g</i>	<i>ag</i>		<i>cg</i>				
<i>G</i>	<i>a</i>				<i>ace</i>		<i>af</i>	
<i>G</i>	<i>b</i>				<i>bce</i>		<i>bf</i>	
<i>G</i>	<i>ce</i>	<i>ace</i>	<i>bce</i>					
<i>G</i>	<i>f</i>	<i>af</i>	<i>bf</i>					
<i>H</i>	<i>de</i>	<i>ade</i>		<i>cde</i>				

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Extraction Rectangle Covering

Example (cont'd)

$$F = af + bf + ag + cg + ade + bde + cde$$

$$G = af + bf + ace + bce$$

$$H = ade + cde$$

■ Pick sub-matrix *M'*

■ Extract new expression *X*

$$F = fx + ag + cg + dex + cde$$

$$G = fx + cex$$

$$H = ade + cde$$

$$X = a + b$$

■ Update *M*

		<i>a</i>	<i>b</i>	<i>c</i>	<i>ce</i>	<i>de</i>	<i>f</i>	<i>g</i>
<i>F</i>	<i>a</i>					<i>ade</i>	<i>af</i>	<i>ag</i>
<i>F</i>	<i>b</i>					<i>bde</i>	<i>bf</i>	
<i>F</i>	<i>de</i>	<i>ade</i>	<i>bde</i>	<i>cde</i>				
<i>F</i>	<i>f</i>	<i>af</i>	<i>bf</i>					
<i>M = F</i>	<i>c</i>					<i>cde</i>		<i>cg</i>
<i>F</i>	<i>g</i>	<i>ag</i>		<i>cg</i>				
<i>G</i>	<i>a</i>				<i>ace</i>		<i>af</i>	
<i>G</i>	<i>b</i>				<i>bce</i>		<i>bf</i>	
<i>G</i>	<i>ce</i>	<i>ace</i>	<i>bce</i>					
<i>G</i>	<i>f</i>	<i>af</i>	<i>bf</i>					
<i>H</i>	<i>de</i>	<i>ade</i>		<i>cde</i>				

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Extraction Rectangle Covering

- Number literals before - Number of literals after

$$V(R', C') = \sum_{i \in R, j \in C} v_{ij} - \sum_{i \in R'} w_i^r - \sum_{j \in C'} w_j^c$$

v_{ij} : Number of literals of cube m_{ij}

w_i^r : (Number of literals of the cube associated with row i) + 1

w_j^c : Number of literals of the cube associated with column j

- For prior example

- $V = 20 - 10 - 2 = 8$

		<i>a</i>	<i>b</i>	<i>c</i>	<i>ce</i>	<i>de</i>	<i>f</i>	<i>g</i>
<i>F</i>	<i>a</i>					<i>ade</i>	<i>af</i>	<i>ag</i>
<i>F</i>	<i>b</i>					<i>bde</i>	<i>bf</i>	
<i>F</i>	<i>de</i>	<i>ade</i>	<i>bde</i>	<i>cde</i>				
<i>F</i>	<i>f</i>	<i>af</i>	<i>bf</i>					
<i>M = F</i>	<i>c</i>					<i>cde</i>		<i>cg</i>
<i>F</i>	<i>g</i>	<i>ag</i>		<i>cg</i>				
<i>G</i>	<i>a</i>				<i>ace</i>		<i>af</i>	
<i>G</i>	<i>b</i>				<i>bce</i>		<i>bf</i>	
<i>G</i>	<i>ce</i>	<i>ace</i>	<i>bce</i>					
<i>G</i>	<i>f</i>	<i>af</i>	<i>bf</i>					
<i>H</i>	<i>de</i>	<i>ade</i>		<i>cde</i>				

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Extraction Rectangle Covering

- Pseudo Boolean Division

- Idea: consider entries in covering matrix that are don't cares

- overlap of rectangles ($a + a = a$)

- product that cancel each other out ($a \cdot a' = 0$)

- Example:

$$F = ab' + ac' + a'b + a'c + bc' + b'$$

Result:

$$X = a' + b' + c'$$

$$F = ax + bx + cx$$

		<i>a</i>	<i>b</i>	<i>c</i>	<i>a'</i>	<i>b'</i>	<i>c'</i>
<i>F</i>	<i>a</i>				*	<i>ab'</i>	<i>ac'</i>
<i>F</i>	<i>b</i>				<i>a'b</i>	*	<i>bc'</i>
<i>M = F</i>	<i>c</i>				<i>a'c</i>	<i>b'c</i>	*
<i>F</i>	<i>a'</i>	*	<i>a'b</i>	<i>a'c</i>			
<i>F</i>	<i>b'</i>	<i>ab'</i>	*	<i>b'c</i>			
<i>F</i>	<i>c'</i>	<i>ac'</i>	<i>bc'</i>	*			

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Fast Kernel Computation

- Non-robustness of kernel extraction
 - Recomputation of kernels after every substitution: expensive
 - Some functions may have many kernels (e.g. symmetric functions)
- Cannot measure if kernel can be used as complemented node
- Solution: compute only subset of kernels:
 - Two-cube “kernel” extraction [Rajski et al '90]
 - Objects:
 - 2-cube divisors
 - 2-literal cube divisors
 - **Example:** $f = abd + a'b'd + a'cd$
 - $ab + a'b'$, $b' + c$ and $ab + a'c$ are 2-cube divisors.
 - $a'd$ is a 2-literal cube divisor.

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Fast Kernel Computation

- Properties of fast divisor (kernel) extraction:
 - $O(n^2)$ number of 2-cube divisors in an n -cube Boolean expression
 - Concurrent extraction of 2-cube divisors and 2-literal cube divisors
 - Handle divisor and complemented divisor simultaneously
- **Example:**
 - $f = abd + a'b'd + a'cd$
 - $k = ab + a'b'$, $k' = ab' + a'b$ (both 2-cube divisors)
 - $j = ab + a'c$, $j' = ab' + a'c'$ (both 2-cube divisors)
 - $c = ab$ (2-literal cube), $c' = a' + b'$ (2-cube divisor)

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Fast Kernel Computation

□ Generating all two cube divisors

$$F = \{c_i\}$$

$$D(F) = \{d \mid d = \text{make_cube_free}(c_i + c_j)\}$$

- c_i, c_j are any pair of cubes in F
 - I.e., take all pairs of cubes in F and makes them cube-free
- Divisor generation is $O(n^2)$, where n = number of cubes in F

□ Example:

$$F = ax^3 + ag^3 + bcx^3 + bcg^3$$

$$\text{make_cube_free}(c_i + c_j) = \{x^3 + g^3, a^3 + bc^3, ax^3 + bcg^3, ag^3 + bcx^3\}$$

- **Note:** Function F is made into an algebraic expression before generating double-cube divisors
- Not all 2-cube divisors are kernels (why?)

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Fast Kernel Computation

□ Key results of 2-cube divisors

Theorem: Expressions F and G have a common multiple-cube divisors **if and only if** $D(F) \cap D(G) \neq \emptyset$

Proof:

If:

If $D(F) \cap D(G) \neq \emptyset$ then $\exists d \in D(F) \cap D(G)$ which is a double-cube divisor of F and G . d is a multiple-cube divisor of F and of G .

Only if:

Suppose $C = \{c_1, c_2, \dots, c_m\}$ is a multiple-cube divisor of F and of G . Take any $e = (c_i + c_j)$. If e is cube-free, then $e \in D(F) \cap D(G)$. If e is not cube-free, then let $d = \text{make_cube_free}(c_i + c_j)$. d has 2 cubes since F and G are algebraic expressions. Hence $d \in D(F) \cap D(G)$.

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Fast Kernel Computation

□ Example:

Suppose that $C = ab + ac + f$ is a multiple divisor of F and G

If $e = ac + f$, e is cube-free and $e \in D(F) \cap D(G)$

If $e = ab + ac$, $d = \{b + c\} \in D(F) \cap D(G)$

As a result of the Theorem, all multiple-cube divisors can be “discovered” by using just double-cube divisors

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Fast Kernel Computation

□ Algorithm:

- Generate and store all 2-cube kernels (2-literal cubes) and recognize complement divisors
- Find the best 2-cube kernel or 2-literal cube divisor at each stage and extract it
- Update 2-cube divisor (2-literal cubes) set after extraction
- Iterate extraction of divisors until no more improvement

□ Results:

- Much faster
- Quality as good as that of kernel extraction

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Boolean Division

□ What's wrong with algebraic division?

- Divisor and quotient are orthogonal!

- Better factored form might be:

$$(g_1 + g_2 + \dots + g_n) (d_1 + d_2 + \dots + d_m)$$

- g_i and d_j may share same literals

- redundant product literals

- Example

$$abe + ace + abd + cd / (ae + d) = \emptyset$$

But: $aabe + ace + abd + cd / (ae + d) = (ab + c)$

- g_i and d_j may share opposite literals

- product terms are non-existing

- Example

$$a'b + ac + bc / (a' + c) = \emptyset$$

But: $a'a + a'b + ac + bc / (a' + c) = (a + b)$

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Boolean Division

□ Definition:

g is a **Boolean divisor** of f if h and r exist such that $f = gh + r$, $gh \neq 0$

g is said to be a **factor** of f if, in addition, $r = 0$, i.e., $f = gh$

- h is called the **quotient**

- r is called the **remainder**

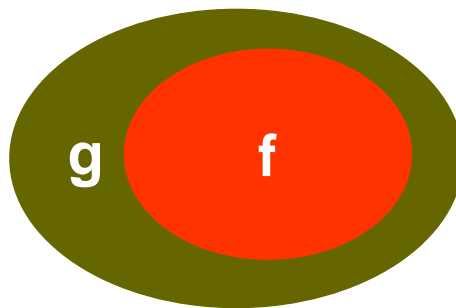
- h and r may **not** be unique

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Boolean Division

□ Theorem:

A logic function g is a **Boolean factor** of a logic function f if and only if $f \subseteq g$ (i.e. $fg' = 0$, i.e. $g' \subseteq f'$)



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Boolean Division

Proof:

(\Rightarrow) g is a Boolean factor of f . Then $\exists h$ such that $f = gh$;
Hence, $f \subseteq g$ (as well as h).

(\Leftarrow) $f \subseteq g \Rightarrow f = gf = g(f + r) = gh$. (Here r is any function $r \subseteq g'$.)

□ Note:

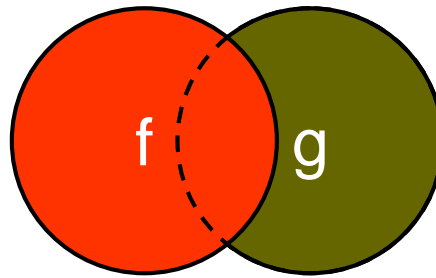
- $h = f$ works fine for the proof
- Given f and g , h is not unique
- To get a small h is the same as to get a small $f + r$. Since $rg = 0$, this is the same as minimizing (simplifying) f with $DC = g'$.

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Boolean Division

□ Theorem:

g is a Boolean divisor of f if and only if $fg \neq 0$



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Boolean Division

Proof:

(\Rightarrow) $f = gh + r$, $gh \neq 0 \Rightarrow fg = gh + gr$. Since $gh \neq 0$, $fg \neq 0$.

(\Leftarrow) Assume that $fg \neq 0$. $f = fg + fg' = g(f + k) + fg'$. (Here $k \subseteq g'$.)

Then $f = gh + r$, with $h = f + k$, $r = fg'$. Since $gh = fg \neq 0$, then $gh \neq 0$.

□ Note:

- f has many divisors. We are looking for some g such that $f = gh + r$, where g, h, r are simple functions. (simplify f with DC = g')

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Boolean Division Incomplete Specified Function

□ $F = (f, d, r)$

□ Definition:

A completely specified logic function g is a **Boolean divisor of F** if there exist h, e (completely specified) such that

$$f \subseteq gh + e \subseteq f + d$$

and $gh \not\subseteq d$.

□ Definition:

g is a **Boolean factor of F** if there exists h such that

$$f \subseteq gh \subseteq f + d$$

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Boolean Division Incomplete Specified Function

□ Lemma:

$f \subseteq g$ if and only if g is a Boolean factor of F .

Proof:

(\Rightarrow) Assume that $f \subseteq g$. Let $h = f + k$ where $kg \subseteq d$.

Then $hg = (f + k)g \subseteq (f + d)$.

Since $f \subseteq g$, $fg = f$ and thus $f \subseteq (f + k)g = gh$.

Thus

$$f \subseteq (f + k)g \subseteq f + d$$

(\Leftarrow) Assume that $f = gh$.

Suppose \exists minterm m such that $f(m) = 1$ but $g(m) = 0$.

Then $f(m) = 1$ but $g(m)h(m) = 0$ implying that $f \not\subseteq gh$.

Thus $f(m) = 1$ implies $g(m) = 1$, i.e. $f \subseteq g$

□ Note:

- Since $kg \subseteq d$, $k \subseteq (d + g')$. Hence obtain $h = f + k$ by simplifying f with $DC = (d + g')$.

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Boolean Division

Incomplete Specified Function

□ Lemma:

$fg \neq 0$ if and only if g is a Boolean divisor of F .

Proof:

(\Rightarrow) Assume $fg \neq 0$.

Let $fg \subseteq h \subseteq (f + d + g')$ and $fg' \subseteq e \subseteq (f + d)$.

Then $f = fg + fg' \subseteq gh + e \subseteq g(f + d + g') + f + d = f + d$

Also, $0 \neq fg \subseteq gh \rightarrow ghf \neq 0$.

Now $gh \not\subseteq d$, since otherwise $ghf = 0$ (since $fd = 0$),
verifying the conditions of Boolean division.

(\Leftarrow) Assume that g is a Boolean divisor.

Then $\exists h$ such that $gh \not\subseteq d$ and

$f \subseteq gh + e \subseteq f + d$

Since $gh = (ghf + gh'd) \not\subseteq d$, then $fgh \neq 0$ implying that $fg \neq 0$.

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Boolean Division

Incomplete Specified Function

□ Recipe for Boolean division:

$(f \subseteq gh + e \subseteq f + d)$

■ Choose g such that $fg \neq 0$

■ Simplify fg with DC = $(d + g')$ to get h

■ Simplify fg' with DC = $(d + fg)$ to get e (could use DC = $d + gh$)

□ $fg \subseteq h \subseteq f + d + g'$

$fg' \subseteq e \subseteq fg' + d + fg = f + d$

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SAT & Logic Synthesis

Functional Dependency as Boolean Division

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Functional Dependency

- **$f(x)$ functionally depends** on $g_1(x), g_2(x), \dots, g_m(x)$ if $f(x) = h(g_1(x), g_2(x), \dots, g_m(x))$, denoted $h(G(x))$
 - Under what condition can function f be expressed as some function h over a set $G = \{g_1, \dots, g_m\}$ of functions ?
 - h exists $\Leftrightarrow \nexists a, b$ such that $f(a) \neq f(b)$ and $G(a) = G(b)$

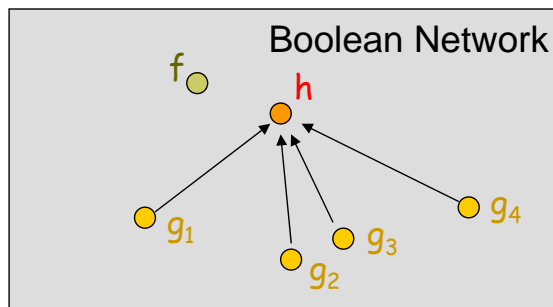
i.e., G is more distinguishing than f

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Motivation

Applications of functional dependency

- Resynthesis/rewiring
- Redundant register removal
- BDD minimization
- Verification reduction
- ...



- target function
- base functions

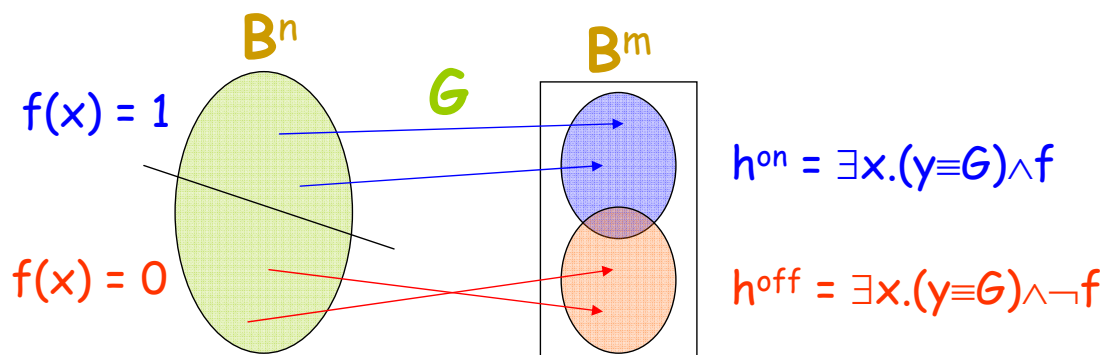
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BDD-Based Computation

BDD-based computation of h

$$h^{on} = \{y \in \mathbf{B}^m : y = G(x) \text{ and } f(x) = 1, x \in \mathbf{B}^n\}$$

$$h^{off} = \{y \in \mathbf{B}^m : y = G(x) \text{ and } f(x) = 0, x \in \mathbf{B}^n\}$$



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BDD-Based Computation

□ Pros

- Exact computation of h^{on} and h^{off}
- Better support for don't care minimization

□ Cons

- 2 image computations for every choice of G
- Inefficient when $|G|$ is large or when there are many choices of G

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SAT-Based Computation

□ h exists \Leftrightarrow

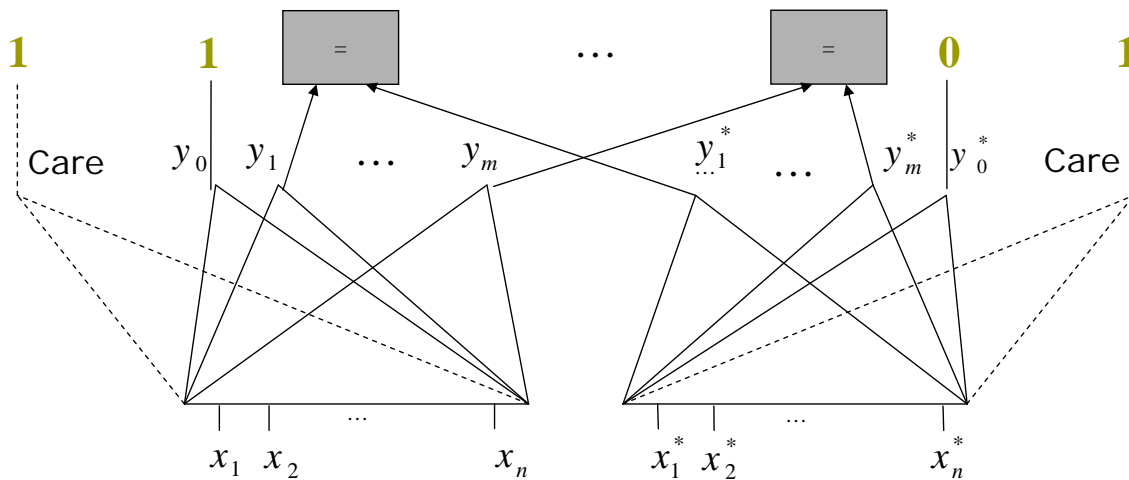
$\nexists a, b$ such that $f(a) \neq f(b)$ and $G(a) = G(b)$,
i.e., $(f(x) \neq f(x^*)) \wedge (G(x) = G(x^*))$ is **UNSAT**

□ How to derive h ? How to select G ?

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SAT-Based Computation

□ $(f(x) \neq f(x^*)) \wedge (G(x) \equiv G(x^*))$ is UNSAT



y_0 is the output variable of f , y_i is the output variable of g_i , $i > 0$

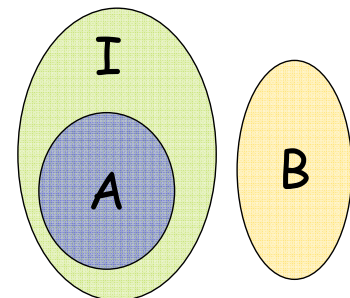
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Craig Interpolation

□ [Craig Interpolation Thm, 1957]

If $A \wedge B$ is UNSAT for formulae A and B , there exists an interpolant I of A such that

1. $A \Rightarrow I$
2. $I \wedge B$ is UNSAT
3. I refers only to the common variables of A and B

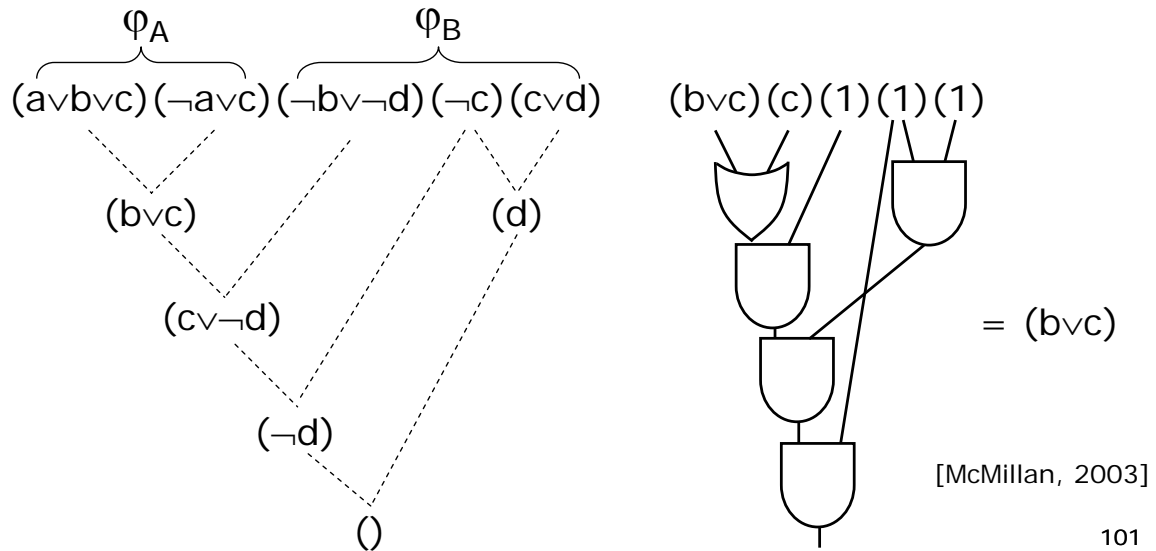


I is an abstraction of A

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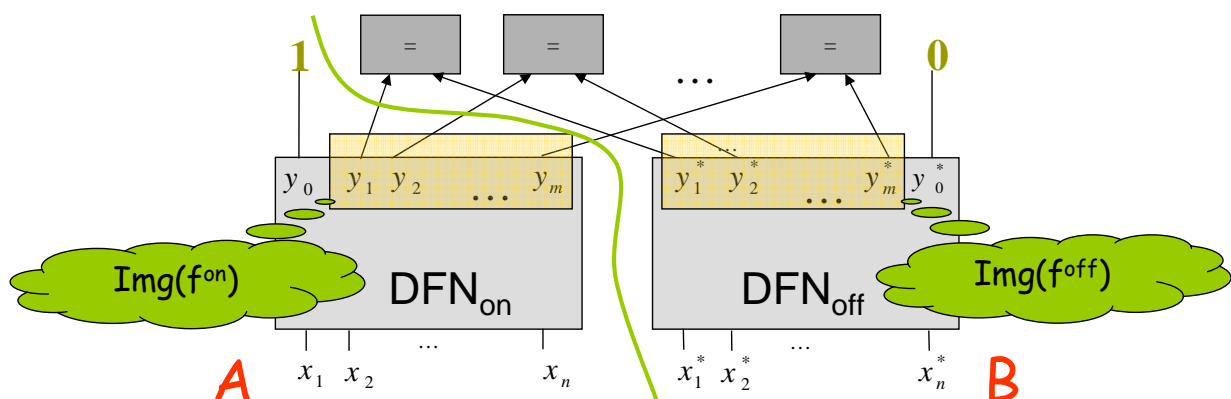
Interpolant and Resolution Proof

- SAT solver may produce the resolution proof of an UNSAT CNF φ
- For $\varphi = \varphi_A \wedge \varphi_B$ specified, the corresponding interpolant can be obtained in time linear in the resolution proof



SAT-Based Computation

- Clause set A : $C_{\text{DFN}_{\text{on}}}, y_0$
- Clause set B : $C_{\text{DFN}_{\text{off}}}, \neg y_0^*, (y_i \equiv y_i^*)$ for $i=1, \dots, m$
- I is an overapproximation of $\text{Img}(f^{\text{on}})$ and is disjoint from $\text{Img}(f^{\text{off}})$
- I only refers to y_1, \dots, y_m
- Therefore, I corresponds to a feasible implementation of h



Incremental SAT Solving

□ Controlled equality constraints

$$(y_i \equiv y_i^*) \rightarrow (\neg y_i \vee y_i^* \vee \alpha_i)(y_i \vee \neg y_i^* \vee \alpha_i)$$

with auxiliary variables α_i

$\alpha_i = \text{true} \Rightarrow i^{\text{th}}$ equality constraint is disabled

- Fast switch between target and base functions by unit assumptions over control variables
- Fast enumeration of different base functions
- Share learned clauses

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SAT vs. BDD

□ SAT

■ Pros

- Detect multiple choices of \mathcal{G} automatically
- Scalable to large $|\mathcal{G}|$
- Fast enumeration of different target functions f
- Fast enumeration of different base functions \mathcal{G}

■ Cons

- Single feasible implementation of h

□ BDD

■ Cons

- Detect one choice of \mathcal{G} at a time
- Limited to small $|\mathcal{G}|$
- Slow enumeration of different target functions f
- Slow enumeration of different base functions \mathcal{G}

■ Pros

- All possible implementations of h

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Practical Evaluation

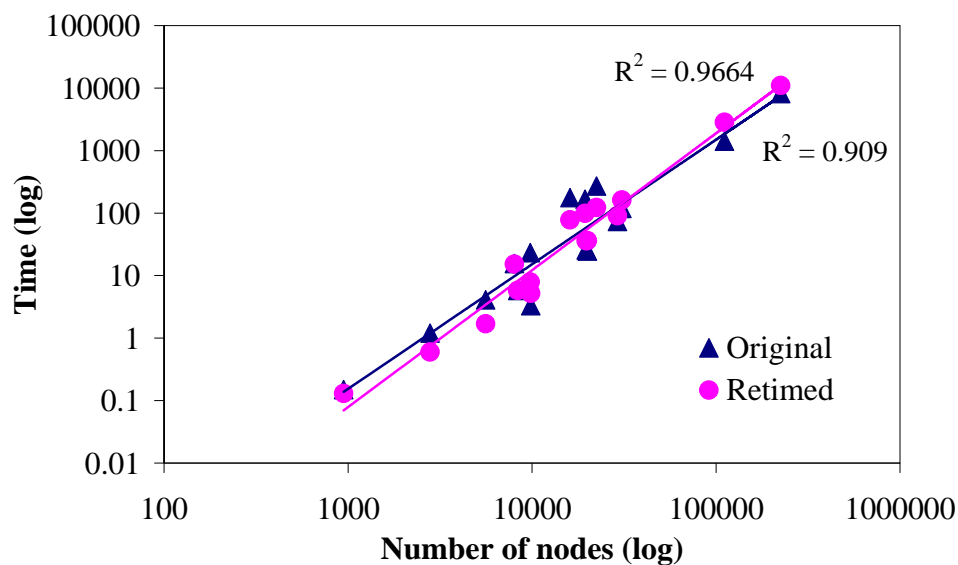
SAT vs. BDD

Circuit	#Nodes	Original			Retimed			SAT (original)		BDD (original)		SAT (retimed)		BDD (retimed)	
		#FF.	#Dep-S	#Dep-B	#FF.	#Dep-S	#Dep-B	Time	Mem	Time	Mem	Time	Mem	Time	Mem
s5378	2794	179	52	25	398	283	173	1.2	18	1.6	20	0.6	18	7	51
s9234.1	5597	211	46	x	459	301	201	4.1	19	x	x	1.7	19	194.6	149
s13207.1	8022	638	190	136	1930	802	x	15.6	22	31.4	78	15.3	22	x	x
s15850.1	9785	534	18	9	907	402	x	23.3	22	82.6	94	7.9	22	x	x
s35932	16065	1728	0	--	2026	1170	--	176.7	27	1117	164	78.1	27	--	--
s38417	22397	1636	95	--	5016	243	--	270.3	30	--	--	123.1	32	--	--
s38584	19407	1452	24	--	4350	2569	--	166.5	21	--	--	99.4	30	1117	164
b12	946	121	4	2	170	66	33	0.15	17	12.8	38	0.13	17	2.5	42
b14	9847	245	2	--	245	2	--	3.3	22	--	--	5.2	22	--	--
b15	8367	449	0	--	1134	793	--	5.8	22	--	--	5.8	22	--	--
b17	30777	1415	0	--	3967	2350	--	119.1	28	--	--	161.7	42	--	--
b18	111241	3320	5	--	9254	5723	--	1414	100	--	--	2842.6	100	--	--
b19	224624	6642	0	--	7164	337	--	8184.8	217	--	--	11040.6	234	--	--
b20	19682	490	4	--	1604	1167	--	25.7	28	--	--	36	30	--	--
b21	20027	490	4	--	1950	1434	--	24.6	29	--	--	36.3	31	--	--
b22	29162	735	6	--	3013	2217	--	73.4	36	--	--	90.6	37	--	--

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Practical Evaluation

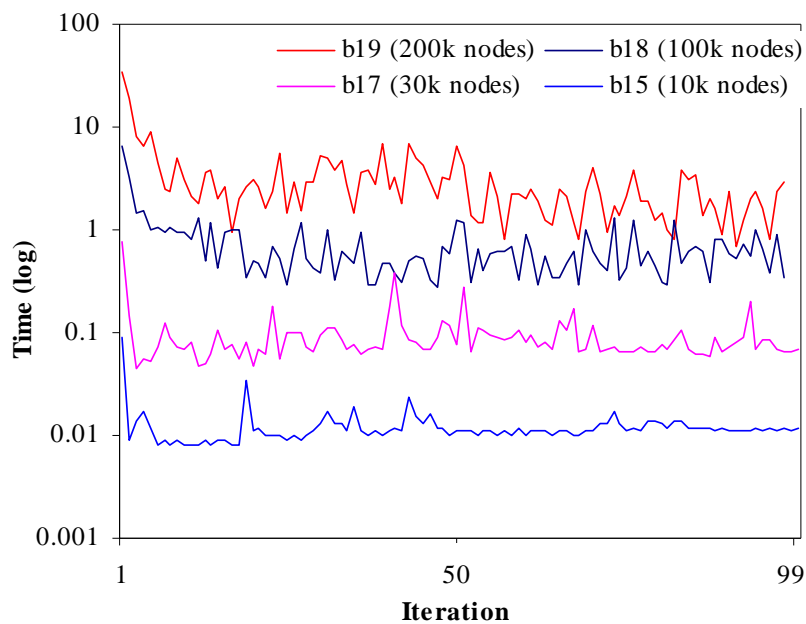
circuit size vs. runtime



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Practical Evaluation

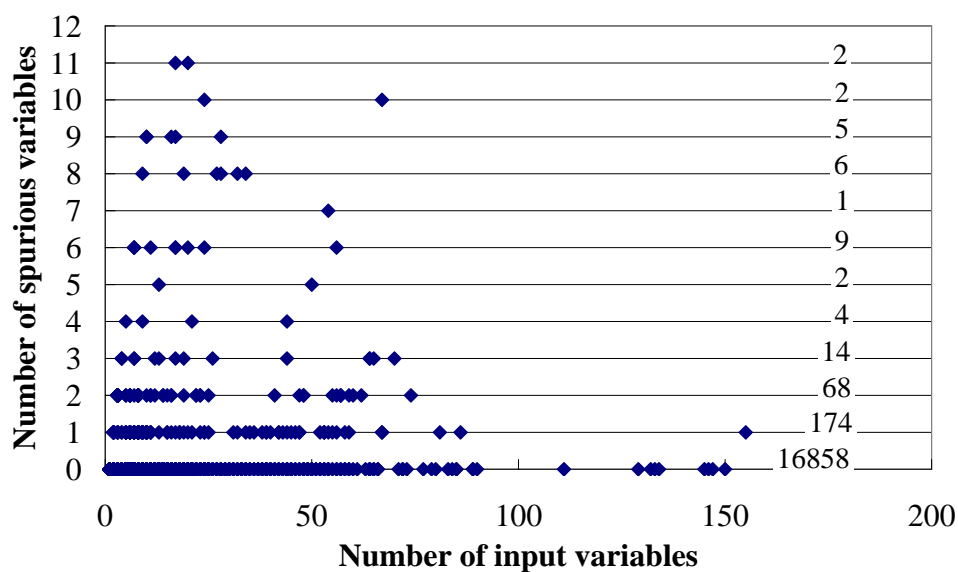
Incremental SAT



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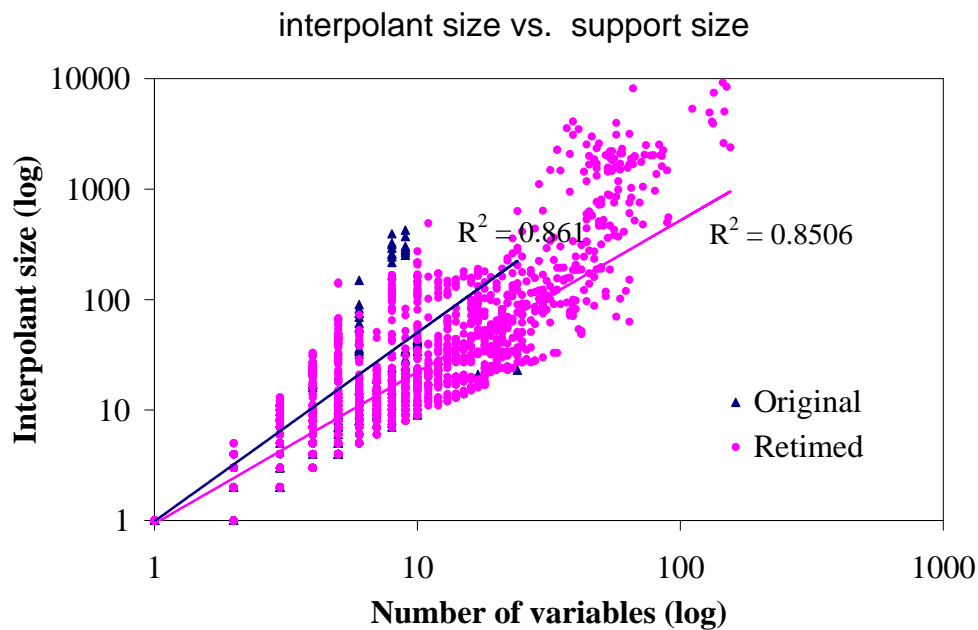
Practical Evaluation

#total input vs. #redundant inputs



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Practical Evaluation



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Summary

- Functional dependency is computable with pure SAT solving (with the help of Craig interpolation)
- Compared to BDD-based computation, it is much scalable to large designs

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