Logic Synthesis and Verification

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Boolean Function Representation

Reading:
Logic Synthesis in a Nutshell
Section 2

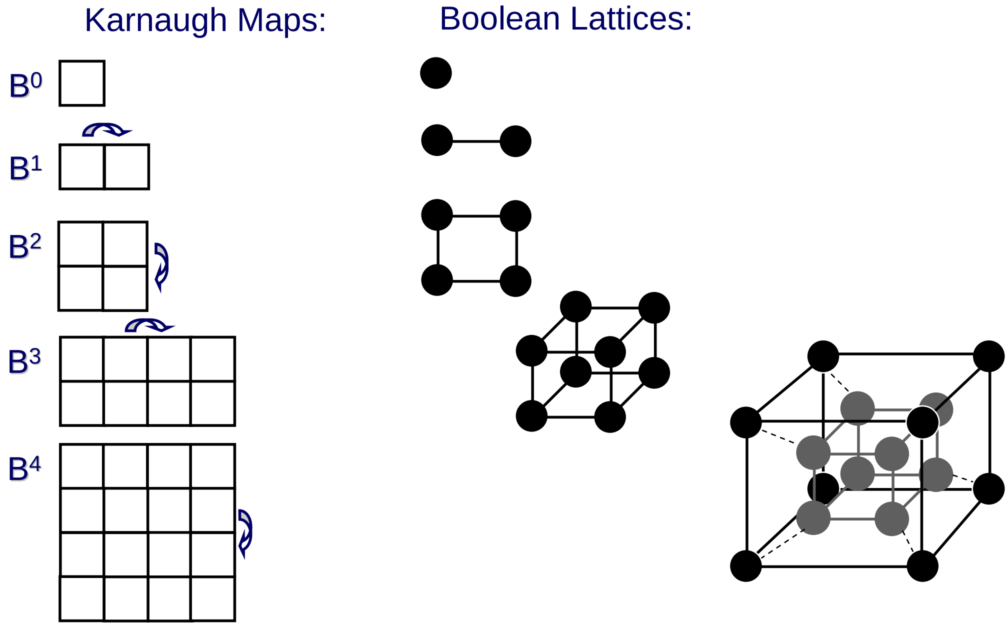
most of the following slides are by courtesy of Andreas Kuehlmann

Assumption

- Unless otherwise said, from now on we are concerned with two-element Boolean algebra (i.e. **B** = {0,1})

Boolean Space

- B = {0,1}
- $B^2 = \{0,1\}\times\{0,1\} = \{00, 01, 10, 11\}$

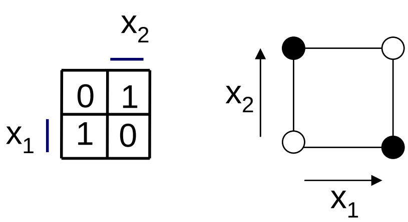


Boolean Function

- For **B** = {0,1}, a Boolean function f: $\mathbf{B}^n \rightarrow \mathbf{B}$ over variables $x_1,\ldots,x_n$ maps each Boolean valuation (truth assignment) in $\mathbf{B}^n$ to 0 or 1

Example

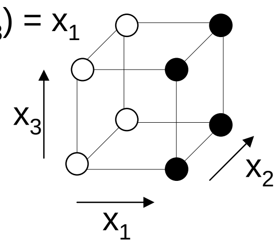
$f(x_1,x_2)$ with $f(0,0) = 0$, $f(0,1) = 1$, $f(1,0) = 1$, $f(1,1) = 0$



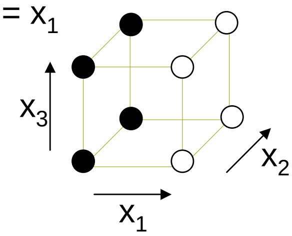
Boolean Function

- Onset of f, denoted as f^1, is $f^1 = \{v \in \mathbf{B}^n \mid f(v)=1\}$
  - If $f^1 = \mathbf{B}^n$, f is a tautology
- Offset of f, denoted as f^0, is $f^0 = \{v \in \mathbf{B}^n \mid f(v)=0\}$
  - If $f^0 = \mathbf{B}^n$, f is unsatisfiable. Otherwise, f is satisfiable.
- f^1 and f^0 are sets, not functions!
- Boolean functions f and g are equivalent if $\forall v \in \mathbf{B}^n$. $f(v) = g(v)$ where v is a truth assignment or Boolean valuation
- A literal is a Boolean variable *x* or its negation x' (or $\overline{x}$, $\neg x$) in a Boolean formula

$f(x_1, x_2, x_3) = x_1$

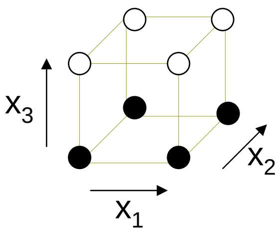


$f(x_1, x_2, x_3) = \overline{x_1}$



Boolean Function

- There are 2^n vertices in $\mathbf{B}^n$
- There are 2^{2^n} distinct Boolean functions
  - Each subset $f^1 \subseteq \mathbf{B}^n$ of vertices in $\mathbf{B}^n$ forms a distinct Boolean function f with onset f^1



$x_1x_2x_3$	f
0 0 0	1
0 0 1	0
0 1 0	1
0 1 1	0
1 0 0	$\Rightarrow$ 1
1 0 1	0
1 1 0	1
1 1 1	0

Boolean Operations

Given two Boolean functions:

$f : \mathbf{B}^n \to \mathbf{B}$

$g : \mathbf{B}^n \to \mathbf{B}$

- $h = f \wedge g$ from AND operation is defined as
  $h^1 = f^1 \cap g^1;\ h^0 = \mathbf{B}^n \setminus h^1$
- $h = f \vee g$ from OR operation is defined as
  $h^1 = f^1 \cup g^1;\ h^0 = \mathbf{B}^n \setminus h^1$
- $h = \neg f$ from COMPLEMENT operation is defined as
  $h^1 = f^0;\ h^0 = f^1$

Cofactor and Quantification

Given a Boolean function:

$f: \mathbf{B}^n \rightarrow \mathbf{B}$, with the input variable $(x_1,x_2,\ldots,x_i,\ldots,x_n)$

- Positive cofactor on variable x_i
  $h = f_{x_i}$ is defined as $h = f(x_1,x_2,\ldots,1,\ldots,x_n)$
- Negative cofactor on variable x_i
  $h = f_{\neg x_i}$ is defined as $h = f(x_1,x_2,\ldots,0,\ldots,x_n)$
- Existential quantification over variable x_i
  $h = \exists x_i.\ f$ is defined as $h = f(x_1,x_2,\ldots,0,\ldots,x_n) \vee f(x_1,x_2,\ldots,1,\ldots,x_n)$
- Universal quantification over variable x_i
  $h = \forall x_i.\ f$ is defined as $h = f(x_1,x_2,\ldots,0,\ldots,x_n) \wedge f(x_1,x_2,\ldots,1,\ldots,x_n)$
- Boolean difference over variable x_i
  $h = \partial f/\partial x_i$ is defined as $h = f(x_1,x_2,\ldots,0,\ldots,x_n) \oplus f(x_1,x_2,\ldots,1,\ldots,x_n)$

Representation of Boolean Function

- Represent Boolean functions for two reasons
  - to represent and manipulate the actual circuit we are implementing
  - to facilitate Boolean reasoning
- Data structures for representation
  - Truth table
  - Boolean formula
    - Sum of products (Disjunctive "normal" form, DNF)
    - Product of sums (Conjunctive "normal" form, CNF)
  - Boolean network
    - Circuit (network of Boolean primitives)
    - And-inverter graph (AIG)
  - Binary Decision Diagram (BDD)

Boolean Function Representation
Truth Table

- Truth table (function table for multi-valued functions):
  The truth table of a function f : $\mathbf{B}^n \rightarrow \mathbf{B}$ is a tabulation of its value at each of the 2^n vertices of $\mathbf{B}^n$.

  In other words the truth table lists all mintems
  Example: f = a′b′c′d + a′b′cd + a′bc′d + ab′c′d + ab′cd + abc′d + abcd′ + abcd

  The truth table representation is
  - impractical for large n
  - canonical

  If two functions are the same, then their canonical representations are isomorphic.

	abcd	f		abcd	f
0	0000	0	8	1000	0
1	0001	1	9	1001	1
2	0010	0	10	1010	0
3	0011	1	11	1011	1
4	0100	0	12	1100	0
5	0101	1	13	1101	1
6	0110	0	14	1110	1
7	0111	0	15	1111	1

Boolean Function Representation
Boolean Formula

- A Boolean formula is defined inductively as an expression with the following formation rules (syntax):

formula ::=	‘(‘ formula ‘)’	
|	Boolean constant	(**true** or **false**)
|	Boolean variable	
|	formula “+” formula	(OR operator)
|	formula “·” formula	(AND operator)
|	¬ formula	(complement)

Example

$f = (x_1 \cdot x_2) + (x_3) + \neg(\neg(x_4 \cdot (\neg x_1)))$

typically “·” is omitted and ‘(’, ‘)’ and ‘¬’ are simply reduced by priority,

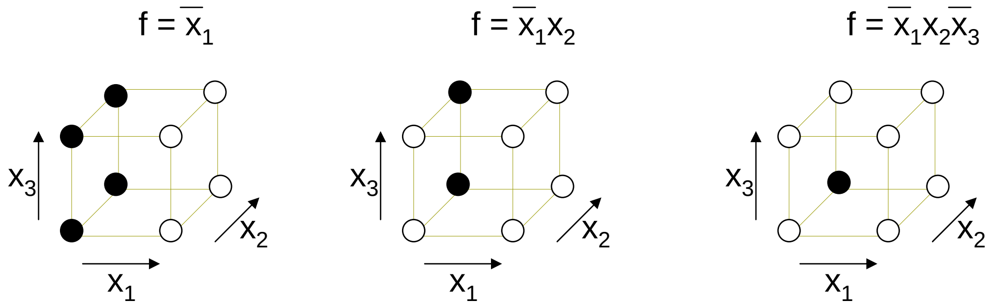
e.g. $f = x_1\ x_2 + x_3 + x_4 \neg x_1$

Boolean Function Representation
Boolean Formula in SOP

- A cube is defined as a conjunction of literals, i.e. a product term.

Example

C = $x_1x_2'x_3$ represents the function with onset: $f^1 = \{(x_1=1,x_2=0,x_3=1)\}$ in the Boolean space spanned by x_1,x_2,x_3, or $f^1 = \{(x_1=1,x_2=0,x_3=1, x_4=0), (x_1=1,x_2=0,x_3=1,x_4=1)\}$ in the Boolean space spanned by x_1,x_2,x_3,x_4, or …



Boolean Function Representation
Boolean Formula in SOP

- If $C \subseteq f^1$, C the onset of a cube c, then c is an implicant of f
- If $C \subseteq \mathbf{B}^n$, and c has k literals, then $|C| = 2^{n-k}$, i.e., C has 2^{n-k} elements

Example

$c = xy'$ ($c: \mathbf{B}^3 \to \mathbf{B}$), $C = \{100, 101\} \subseteq \mathbf{B}^3$

$k = 2$, $n = 3$ $\quad |C| = 2 = 2^{3-2}$

- An implicant with n literals is a minterm

Boolean Function Representation Boolean Formula in SOP

- A function can be represented by a sum-of-cubes (products):

  $f = ab + ac + bc$

  Since each cube is a product of literals, this is a sum-of-products (SOP) representation or disjunctive normal form (DNF)

- An SOP can be thought of as a set of cubes F

  $F = \{ab, ac, bc\}$

- A set of cubes that represents f is called a cover of f.

  $F_1 = \{ab, ac, bc\}$ and $F_2 = \{abc, abc', ab'c, a'bc\}$

  are covers of

  $f = ab + ac + bc$.

- Mainly used in circuit synthesis; seldom used in Boolean reasoning

Boolean Function Representation Boolean Formula in POS

- Product-of-sums (POS), or conjunctive normal form (CNF), representation of Boolean functions
  - Dual of the SOP representation

  Example

  $\varphi = (a+b'+c)\ (a'+b+c)\ (a+b'+c')\ (a+b+c)$

- A Boolean function in a POS representation can be derived from an SOP representation with De Morgan's law and the distributive law

- Mainly used in Boolean reasoning; rarely used in circuit synthesis (due to the asymmetric characteristics of NMOS and PMOS, and the easiness of adding design constraints)

Boolean Function Representation
Boolean Network

- Used for two main purposes
  - as target structure for logic implementation which gets restructured in a series of logic synthesis steps until result is acceptable
  - as representation for Boolean reasoning engine

- Efficient representation for most Boolean problems
  - memory complexity is similar to the size of circuits that we are actually building

- Close to the input and output representations of logic synthesis

Boolean Function Representation
Boolean Network

A Boolean network is a directed graph C(G,N) where G are the gates and $N \subseteq (G \times G)$ are the directed edges (nets) connecting the gates.

Some of the vertices are designated:

Inputs: $I \subseteq G$

Outputs: $O \subseteq G$

$I \cap O = \emptyset$

Each gate g is assigned a Boolean function f_g which computes the output of the gate in terms of its inputs.

Boolean Function Representation
Boolean Network

- The fanin FI(g) of a gate g are the predecessor gates of g:
  FI(g) = {g' | (g',g) ∈ N} (N: the set of nets)

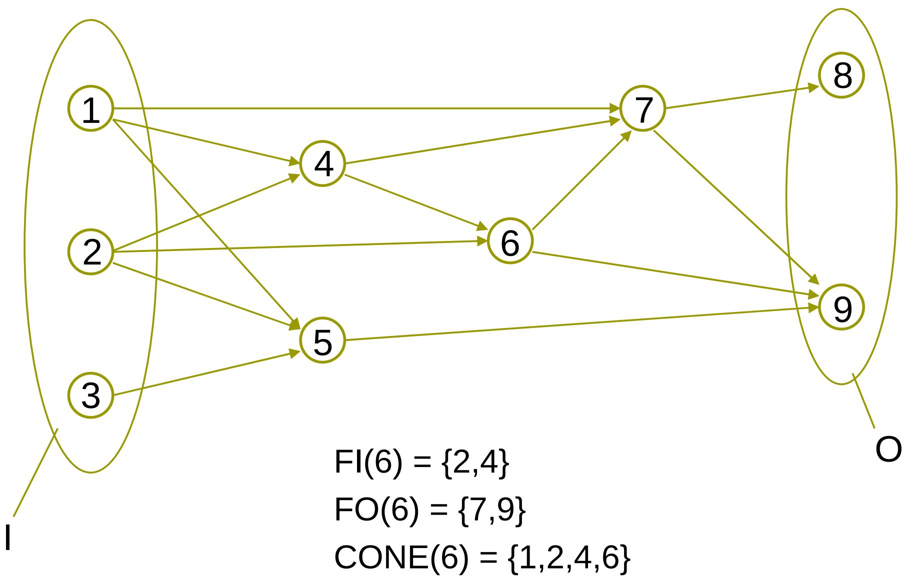
- The fanout FO(g) of a gate g are the successor gates of g:
  FO(g) = {g' | (g,g') ∈ N}

- The cone CONE(g) of a gate g is the transitive fanin (TFI) of g and g itself

- The support SUPPORT(g) of a gate g are all inputs in its cone:
  SUPPORT(g) = CONE(g) ∩ I

Boolean Function Representation
Boolean Network

Example



FI(6) = {2,4}
FO(6) = {7,9}
CONE(6) = {1,2,4,6}
SUPPORT(6) = {1,2}

Boolean Function Representation
Boolean Network

- Circuit functions are defined recursively:

$$h_{g_i} = \begin{cases} x_i & \text{if } g_i \in I \\ f_{g_i}(h_{g_j},\dots,h_{g_k}),\ \ g_j,\dots,g_k \in FI(g_i) & \text{otherwise} \end{cases}$$

If G is implemented using physical gates with positive (bounded) delays for their evaluation, the computation of h_g depends in general on those delays.

Definition

A circuit C is called combinational if for each input assignment of C for $t \rightarrow \infty$ the evaluation of h_g for all outputs is independent of the internal state of C.

Proposition

A circuit C is combinational if it is acyclic. (converse not true!)

Boolean Function Representation
Boolean Network

General Boolean network:

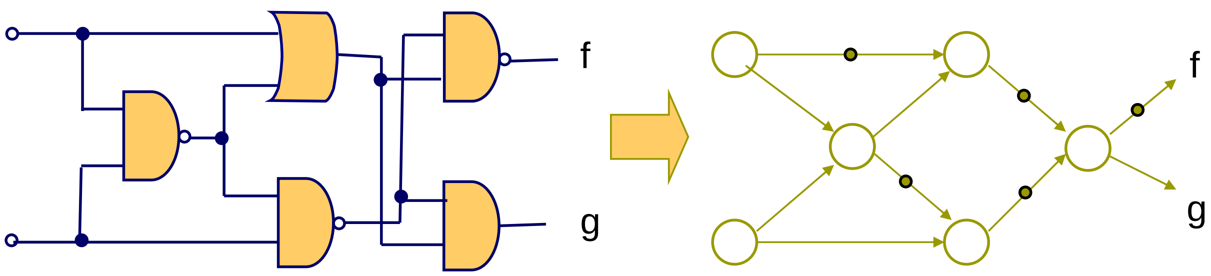
- Vertex can have an arbitrary finite number of inputs and outputs
- Vertex can represent any Boolean function stored in different ways, such as:
  - SOPs (e.g. in SIS, a logic synthesis package)
  - BDDs (to be introduced)
  - AIGs (to be introduced)
  - truth tables
  - Boolean expressions read from a library description
  - other sub-circuits (hierarchical representation)
- The data structure allows general manipulations for insertion and deletion of vertices, pins (connection ports of vertices), and nets
  - general but far too slow for Boolean reasoning

Boolean Function Representation
Boolean Network

Specialized Boolean network:
- Non-canonical representation in general
  - computational effort of Boolean reasoning is due to this non-canonicity (c.f. BDDs)
- Vertices have fixed number of inputs (e.g. two)
- Vertex function is stored as label (e.g. OR, AND, XOR)
- Allow on-the-fly compaction of circuit structure
  - Support incremental, subsequent reasoning on multiple problems

Boolean Function Representation
And-Inverter Graph

- AND-INVERTER graphs (AIGs)
  vertices: 2-input AND gates
  edges: interconnects with (optional) dots representing INVs
- Hash table to identify and reuse structurally isomorphic circuits

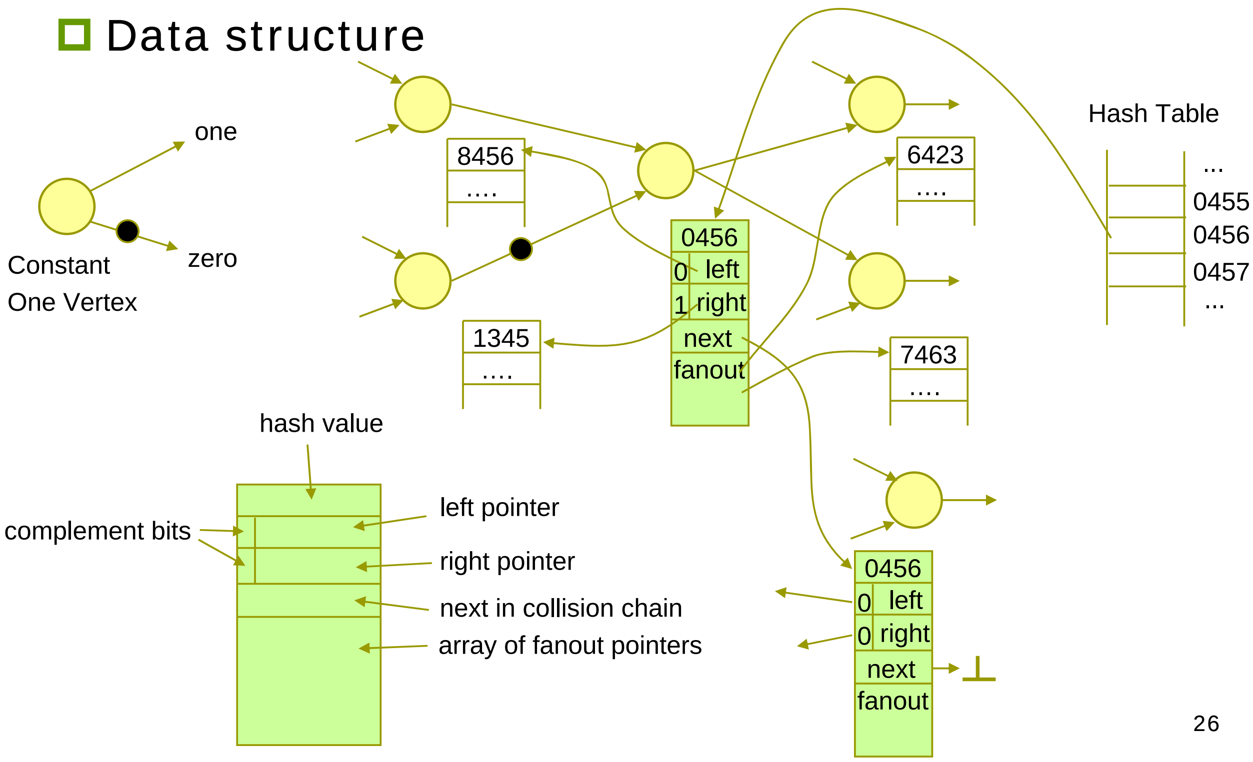


Boolean Function Representation
And-Inverter Graph

- Data structure for implementation
  - Vertex:
    - pointers (integer indices) to left- and right-child and fanout vertices
    - collision chain pointer
    - other data
  - Edge:
    - pointer or index into array
    - one bit to represent inversion
  - Global hash table holds each vertex to identify isomorphic structures
  - Garbage collection to regularly free un-referenced vertices

Boolean Function Representation
And-Inverter Graph

- Data structure



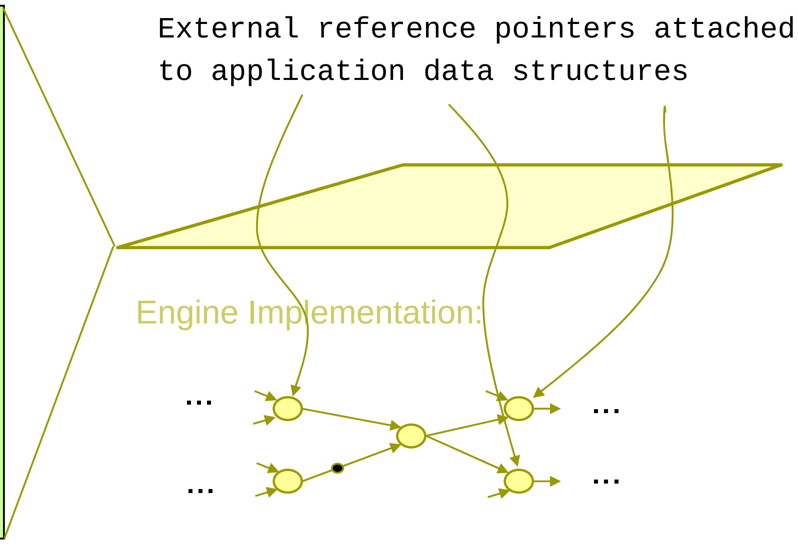
Boolean Function Representation
And-Inverter Graph

- AIG package for Boolean reasoning

  Engine application:

  - traverse problem data structure and build Boolean problem using the interface
  - call SAT to make decision

```
Engine Interface:
void INIT()
void QUIT()
Edge VAR()
Edge AND(Edge p1,
         Edge p2)
Edge NOT(Edge p1)
Edge OR(Edge p1
        Edge p2)
...
int  SAT(Edge p1)
```



Boolean Function Representation
And-Inverter Graph

- Hash table look-up

```
Algorithm HASH_LOOKUP(Edge p1, Edge p2) {
  index = HASH_FUNCTION(p1,p2)
  p     = &hash_table[index]
  while(p != NULL) {
   if(p->left == p1 && p->right == p2) return p;
   p = p->next;
  }
  return NULL;
}
```

- Tricks:
  - keep collision chain sorted by the address (or index) of p
  - use memory locations (or array indices) in topological order for better cache performance

Boolean Function Representation
And-Inverter Graph

- AND operation

```
Algorithm AND(Edge p1,Edge p2){
  if(p1 == const1) return p2
  if(p2 == const1) return p1
  if(p1 == p2)     return p1
  if(p1 == ¬p2)    return const0
  if(p1 == const0 || p2 == const0) return const0

  if(RANK(p1) > RANK(p2)) SWAP(p1,p2)

  if((p = HASH_LOOKUP(p1,p2)) return p
  return CREATE_AND_VERTEX(p1,p2)
}
```


Boolean Function Representation
And-Inverter Graph

- NOT operation

```
Algorithm NOT(Edge p) {
  return TOOGLE_COMPLEMENT_BIT(p)
}
```

- OR operation

```
Algorithm OR(Edge p1,Edge p2){
  return (NOT(AND(NOT(p1),NOT(p2))))
}
```

Boolean Function Representation
And-Inverter Graph

- Cofactor operation

```
Algorithm POSITIVE_COFACTOR(Edge p,Edge v){
  if(IS_VAR(p)) {
    if(p == v) {
      if(IS_INVERTED(v) == IS_INVERTED(p)) return const1
      else                                 return const0
    }
    else                                   return p
  }
  if((c = GET_COFACTOR(p,v)) == NULL) {
    left  = POSITIVE_COFACTOR(p->left, v)
    right = POSITIVE_COFACTOR(p->right, v)
    c = AND(left,right)
    SET_COFACTOR(p,v,c)
  }
  if(IS_INVERTED(p)) return NOT(c)
  else               return c
}
```


Boolean Function Representation
And-Inverter Graph

- Similar algorithm for **NEGATIVE_COFACTOR**

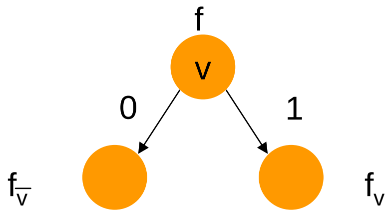
- Existential and universal quantifications can be built from AND, OR and COFACTORS

Exercise: Prove $(f \cdot g)_v = f_v \cdot g_v$ and $(\neg f)_v = \neg(f_v)$

Question: What is the worst-case complexity of performing quantifications over AIGs?

Boolean Function Representation
Binary Decision Diagram (BDD)

- A graphical representation of Boolean function
  - BDD is a Shannon cofactor tree:
    - $f = v\ f_v + v'\ f_{v'}$ (Shannon expansion)
    - vertices represent decision nodes (i.e. multiplexers) controlled by variables
    - leaves are constants "0" and "1"
    - two children of a vertex of f represent two subfunctions f_v and $f_{v'}$
  - Variable ordering restriction and reduction rules make (ROBDD) representation canonical



Boolean Function Representation
BDD – Canonicalization

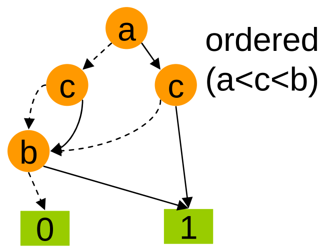
- General idea:
  - instead of exploring sub-cases by enumerating them in time, try to store sub-cases in memory
    - KEY: two hashing mechanisms:
      - unique table: find identical sub-cases and avoid replication
      - computed table: reduce redundant computation of sub-cases
- Represent logic functions as graphs (DAGs):
  - many logic functions can be represented compactly - usually better than SOPs
- Can be made canonical (ROBDD)
  - Shift the effort in a Boolean reasoning engine from SAT algorithm to data representation
- Many logic operations can be performed efficiently on BDD's:
  - usually linear in size of input BDDs
  - tautology checking and complement operation are constant time
- BDD size critically depends on variable ordering

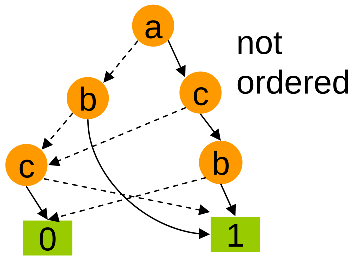
Boolean Function Representation
BDD – Canonicalization

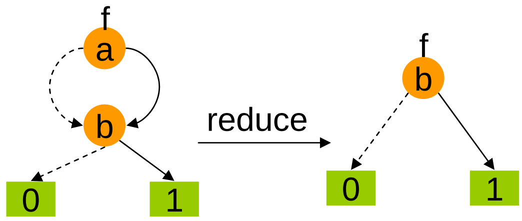
- Directed acyclic graph (DAG)
  - one root node, two terminal-nodes, 0 and 1
  - each node has two children and is controlled by a variable
- Shannon cofactor tree, except reduced and ordered (ROBDD)
  - Ordered:
    - cofactor variables (splitting variables) in the same order along all paths

      $x_{i_1} < x_{i_2} < x_{i_3} < \ldots < x_{i_n}$
  - Reduced:
    - any node with two identical children is removed
    - two nodes with isomorphic BDD's are merged

      These two rules make any node in an ROBDD represent a distinct logic function

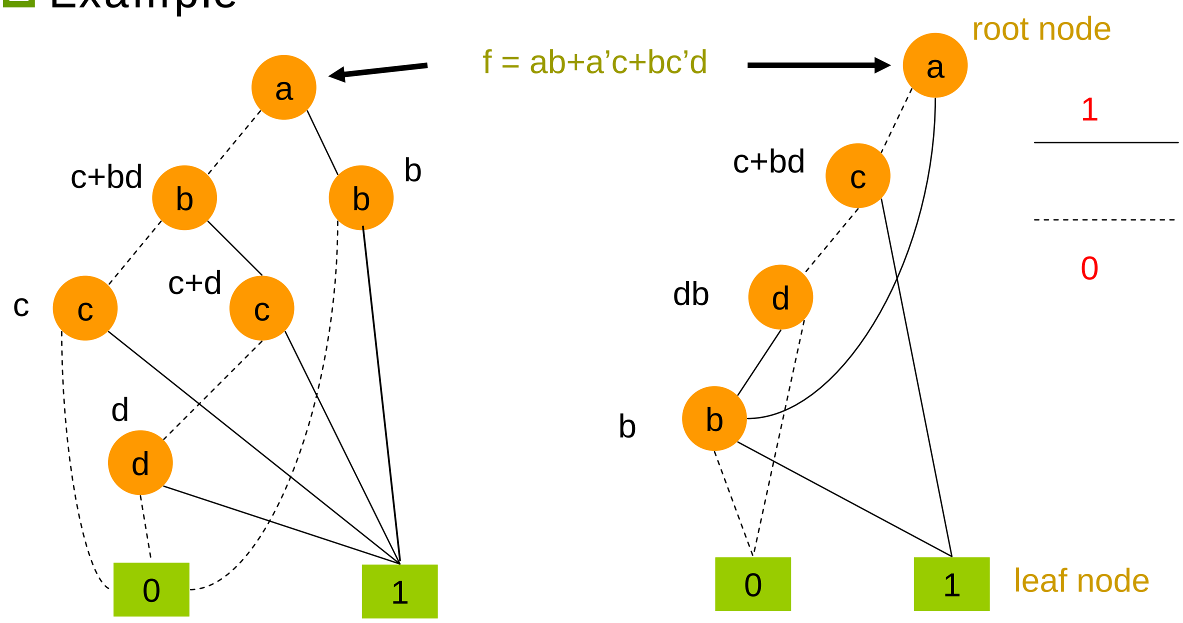






Boolean Function Representation
BDD

- Example



Same function with two different variable orders

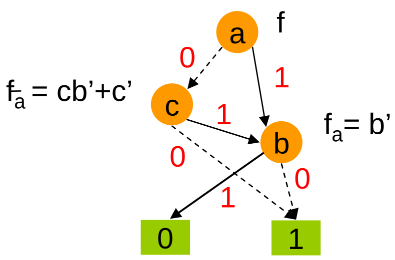
Boolean Function Representation BDD – Canonicity of ROBDD

- Three components make ROBDD canonical (Bryant 1986):
  - unique nodes for constant "0" and "1"
  - identical order of case-splitting variables along each path
  - a hash table that ensures
    - $(node(f_v) = node(g_v)) \wedge (node(f_{v'}) = node(g_{v'})) \Rightarrow node(f) = node(g)$

    and provides recursive argument that node(f) is unique when using the unique hash-table

Boolean Function Representation BDD – Onset Counting

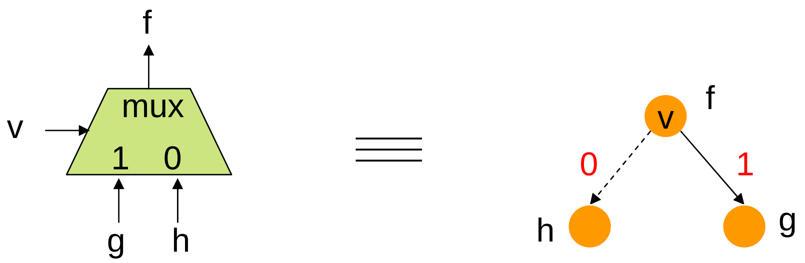
F = b'+a'c' = ab'+a'cb'+a'c' (all paths to the 1 node)



- By tracing all paths to the 1 node, we get a cover of pairwise disjoint cubes
- BDD does not explicitly enumerate all paths; rather it represents paths by a graph whose size is measures by its nodes
  - A DAG can represent an exponential number of paths with a linear number of nodes
- BDDs can be used to efficiently represent sets
  - interpret elements of the onset as elements of the set
  - f is called the characteristic function of that set

Boolean Function Representation
BDD – ITE Operator

- Each BDD node can be written as a triplet: $f = \text{ite}(v, g, h) = vg + v'h$, where $g = f_v$ and $h = f_{\bar{v}}$, meaning if v then g else h



(v is top variable of f)

Boolean Function Representation
BDD – ITE Operator

- $\text{ite}(f, g, h) = fg + f'h$
  - ITE operator can implement any two variable logic function. There are 16 such functions corresponding to all subsets of vertices of $\mathbf{B}^2$:

Table	Subset	Expression	Equivalent Form
0000	0	0	0
0001	AND(f, g)	f g	ite(f, g, 0)
0010	f > g	f g′	ite(f, g′, 0)
0011	f	f	f
0100	f < g	f′g	ite(f, 0, g)
0101	g	g	g
0110	XOR(f, g)	f ⊕ g	ite(f, g′, g)
0111	OR(f, g)	f + g	ite(f, 1, g)
1000	NOR(f, g)	(f + g)′	ite(f, 0, g′)
1001	XNOR(f, g)	f ⊕ g′	ite(f, g, g′)
1010	NOT(g)	g′	ite(g, 0, 1)
1011	f ≥ g	f + g′	ite(f, 1, g′)
1100	NOT(f)	f′	ite(f, 0, 1)
1101	f ≤ g	f′ + g	ite(f, g, 1)
1110	NAND(f, g)	(f g)′	ite(f, g′, 1)
1111	1	1	1

Boolean Function Representation
BDD – ITE Operator

- Recursive operation of ITE

$$
\begin{aligned}
&\mathrm{Ite}(f,g,h) \\
&= f\,g + f'\,h \\
&= v\,(f\,g + f'\,h)_v + v'\,(f\,g + f'\,h)_{v'} \\
&= v\,(f_v\,g_v + f'_v\,h_v) + v'\,(f_{v'}\,g_{v'} + f'_{v'}\,h_{v'}) \\
&= \mathrm{ite}(v,\ \mathrm{ite}(f_v,g_v,h_v),\ \mathrm{ite}(f_{v'},g_{v'},h_{v'}))
\end{aligned}
$$

- Let v be the top-most variable of BDDs f, g, h

Boolean Function Representation
BDD – ITE Operator

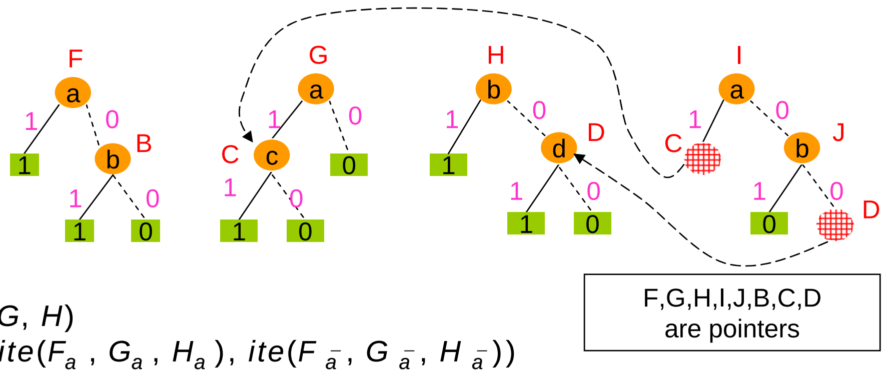
- Recursive computation of ITE

```
Algorithm ITE(f, g, h)
  if(f == 1) return g
  if(f == 0) return h
  if(g == h) return g

  if((p = HASH_LOOKUP_COMPUTED_TABLE(f,g,h)) return p
  v  = TOP_VARIABLE(f, g, h )  // top variable from f,g,h
  fn = ITE(f_v,g_v,h_v)            // recursive calls
  gn = ITE(f_v',g_v',h_v')
  if(fn == gn) return gn       // reduction
  if(!(p = HASH_LOOKUP_UNIQUE_TABLE(v,fn,gn)) {
     p = CREATE_NODE(v,fn,gn)  // and insert into UNIQUE_TABLE
  }
  INSERT_COMPUTED_TABLE(p,HASH_KEY{f,g,h})
  return p
}
```

Boolean Function Representation
BDD – ITE Operator

- Example



$$
\begin{aligned}
I &= ite(F, G, H) \\
&= ite(a, ite(F_a, G_a, H_a), ite(F_{\bar{a}}, G_{\bar{a}}, H_{\bar{a}})) \\
&= ite(a, ite(1, C, H), ite(B, 0, H)) \\
&= ite(a, C, ite(b, ite(B_b, 0_b, H_b), ite(B_{\bar{b}}, 0_{\bar{b}}, H_{\bar{b}}))) \\
&= ite(a, C, ite(b, ite(1, 0, 1), ite(0, 0, D))) \\
&= ite(a, C, ite(b, 0, D)) \\
&= ite(a, C, J)
\end{aligned}
$$

Check: F = a + b

G = ac

H = b + d

$ite(F, G, H) = (a + b)(ac) + a'b'(b + d) = ac + a'b'd$

Boolean Function Representation
BDD – ITE Operator

- Tautology checking using ITE

```
Algorithm ITE_CONSTANT(f,g,h) {   // returns 0,1, or NC
  if(TRIVIAL_CASE(f,g,h) return result (0,1, or NC)
  if((res = HASH_LOOKUP_COMPUTED_TABLE(f,g,h))) return res
  v = TOP_VARIABLE(f,g,h)
  i = ITE_CONSTANT(f_v,g_v,h_v)
  if(i == NC) {
    INSERT_COMPUTED_TABLE(NC, HASH_KEY{f,g,h}) // special table!!
    return NC
  }
  e = ITE_CONSTANT(f_v',g_v',h_v')
  if(e == NC) {
    INSERT_COMPUTED_TABLE(NC, HASH_KEY{f,g,h})
    return NC
  }
  if(e != i) {
    INSERT_COMPUTED_TABLE(NC, HASH_KEY{f,g,h})
    return NC
  }
  INSERT_COMPUTED_TABLE(e, HASH_KEY{f,g,h})
  return i;
}
```

Boolean Function Representation
BDD – ITE Operator

- Composition using ITE
  - Compose is an important operation, e.g. for building the BDD of a circuit backwards, Compose(*F*, *v*, *G*) : $F(v, x) \rightarrow F(G(x), x)$, means substitute $v = G(x)$

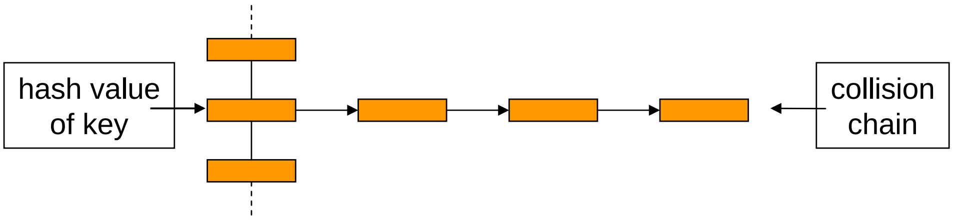
```
Algorithm COMPOSE(F,v,G) {
  if(TOP_VARIABLE(F) > v)  return F  // F does not depend on v
  if(TOP_VARIABLE(F) == v) return ITE(G,F1,F0)
  i = COMPOSE(F1,v,G)
  e = COMPOSE(F0,v,G)
  return ITE(TOP_VARIABLE(F),i,e)
}
```

Note:
1. F1 and F0 are the 1-child and 0-child of F, respectively
2. G, i, e are not functions of v
3. If TOP_VARIABLE of F is v, then ITE(G, F1, F0) does the replacement of v by G

Boolean Function Representation
BDD – Implementation Issues

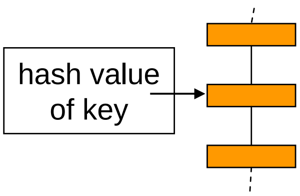
Unique table:

- avoids duplication of existing nodes
  - Hash-Table: hash-function(key) = value
  - identical to the use of a hash-table in AND/INVERTER circuits



Computed table:

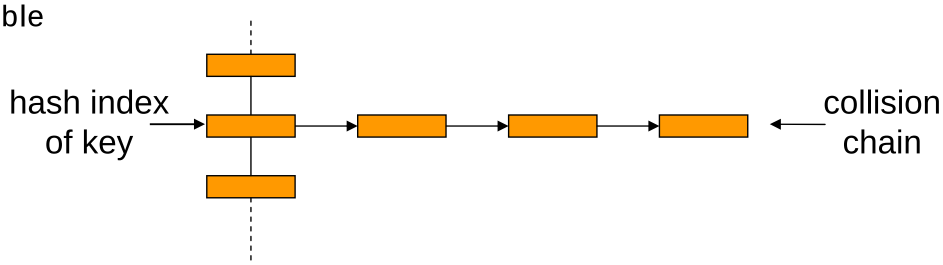
- avoids re-computation of existing results



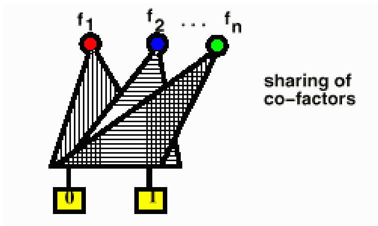
No collision chain

Boolean Function Representation
BDD – Implementation Issues

- Unique table

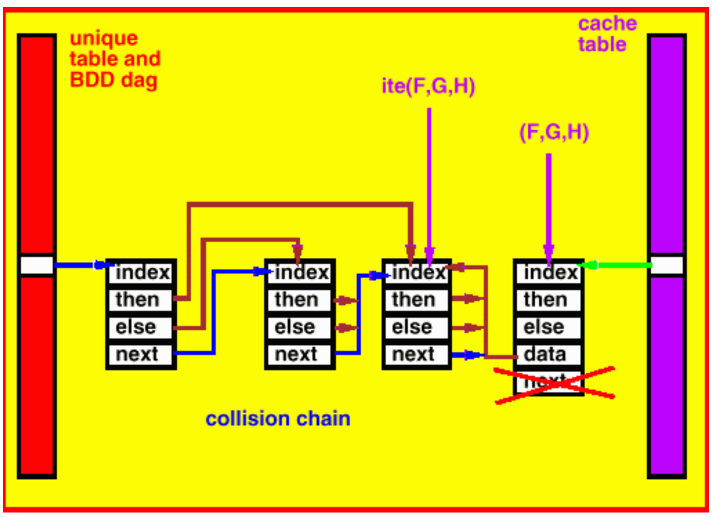


  - Before a node ite(*v, g, h*) is added to BDD database, it is looked up in the "unique-table". If it is there, then existing pointer to node is used to represent the logic function. Otherwise, a new node is added to the unique-table and the new pointer returned.
  - Thus a strong canonical form is maintained. The node for *f* = ite(*v, g, h*) exists iff ite(*v, g, h*) is in the unique-table. There is only one pointer for ite(*v, g, h*) and that is the address to the unique-table entry.
  - Unique-table allows single multi-rooted DAG to represent all users' functions



Boolean Function Representation
BDD – Implementation Issues

- Computed table
  - Keep a record of (*F*, *G*, *H*) triplets already computed by the ITE operator
    - software cache (*"cache" table*)
    - simply hash-table without collision chain (lossy cache)

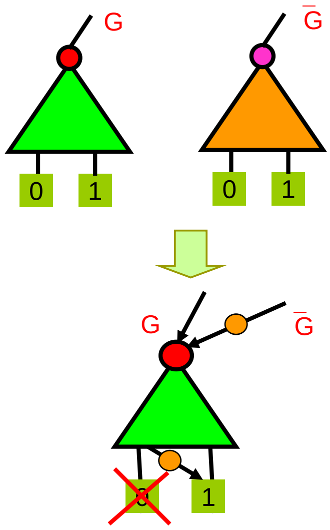


Boolean Function Representation
BDD – Implementation Issues

- Use of computed table
  - BDD packages often use optimized implementations for special operations
    - e.g. `ITE_Constant` (check whether the result would be a constant) `AND_Exist` (AND operation with existential quantification)
  - All operations need a cache for decent performance
    - local cache
      - for one operation only - cache will be thrown away after operation is finished (e.g. `AND_Exist`)
    - special cache for each operation
      - does not need to store operation type
    - shared cache for all operations
      - better memory handling
      - needs to store operation type

Boolean Function Representation
BDD – Implementation Issues

- Complemented edges
  - Combine inverted functions by using complemented edges
    - similar to AIG
    - reduces memory requirements
    - more importantly, makes operations NOT, ITE more efficient

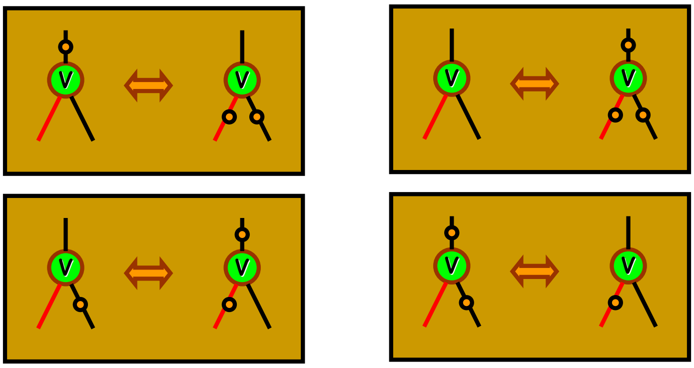


two different DAGs

only one DAG using complement pointer

Boolean Function Representation
BDD – Implementation Issues

- Complemented edges
  - To maintain strong canonical form, need to resolve 4 equivalences:



  - Solution: Always choose the ones on left, i.e. the “then” leg must have no complement edge.

Boolean Function Representation
BDD – Implementation Issues

- Complemented edges

Standard triples:
$ite(F, F, G) \Rightarrow ite(F, 1, G)$
$ite(F, G, F) \Rightarrow ite(F, G, 0)$
$ite(F, G, \neg F) \Rightarrow ite(F, G, 1)$
$ite(F, \neg F, G) \Rightarrow ite(F, 0, G)$

To resolve equivalences:
$ite(F, 1, G) \equiv ite(G, 1, F)$
$ite(F, 0, G) \equiv ite(\neg G, 1, \neg F)$
$ite(F, G, 0) \equiv ite(G, F, 0)$
$ite(F, G, 1) \equiv ite(\neg G, \neg F, 1)$
$ite(F, G, \neg G) \equiv ite(G, F, \neg F)$

To maximize matches on computed table:

1. First argument is chosen with smallest top variable.
2. Break ties with smallest address pointer. (breaks PORTABILITY!)

Triples:

$ite(F, G, H) \equiv ite(\neg F, H, G) \equiv \neg ite(F, \neg G, \neg H) \equiv \neg ite(\neg F, \neg H, \neg G)$

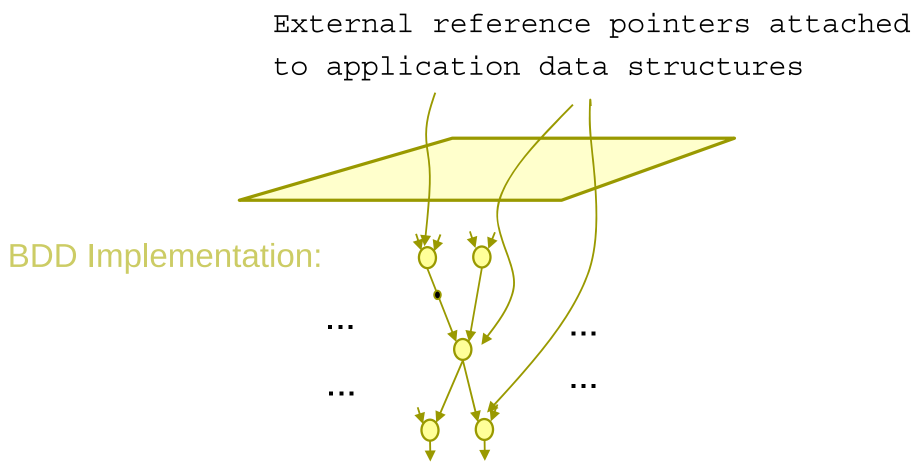
Choose the one such that the first and second argument of *ite* should not be complement edges (i.e. the first one above)

Boolean Function Representation
BDD – Implementation Issues

- Variable ordering – static
  - variable ordering is computed up-front based on the problem structure
  - works well for many practical combinational functions
    - general scheme: control variables first
    - DFS order is good for most cases
  - works bad for unstructured problems
    - e.g. using BDDs to represent arbitrary sets
  - lots of ordering algorithms
    - simulated annealing, genetic algorithms
    - give better results but extremely costly

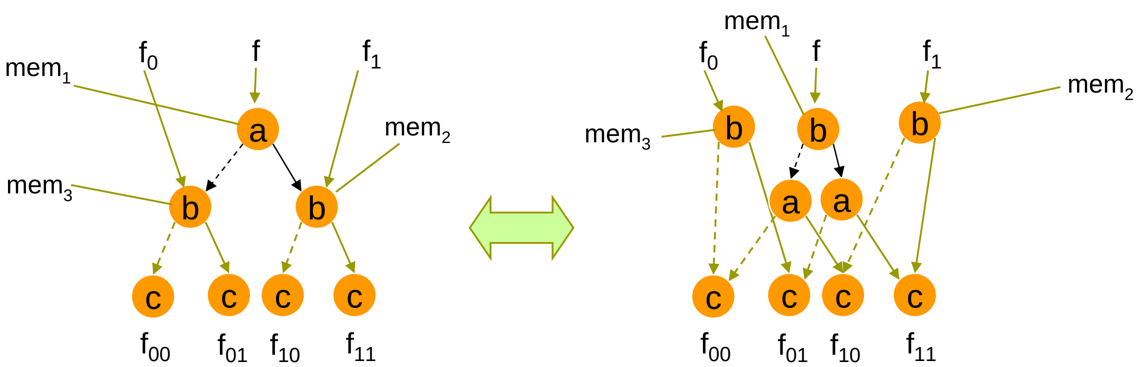
Boolean Function Representation
BDD – Implementation Issues

- Variable ordering – dynamic
  - Changes the order in the middle of BDD applications
    - must keep same global order
  - Problem: External pointers reference internal nodes!



Boolean Function Representation
BDD – Implementation Issues

- Variable ordering – dynamic
  - Theorem (Friedman):
    - Permuting any top part of the variable order has no effect on the nodes labeled by variables in the bottom part.
    - Permuting any bottom part of the variable order has no effect on the nodes labeled by variables in the top part.
  - Trick: Two adjacent variable layers can be exchanged by keeping the original memory locations for the nodes



Boolean Function Representation
BDD – Implementation Issues

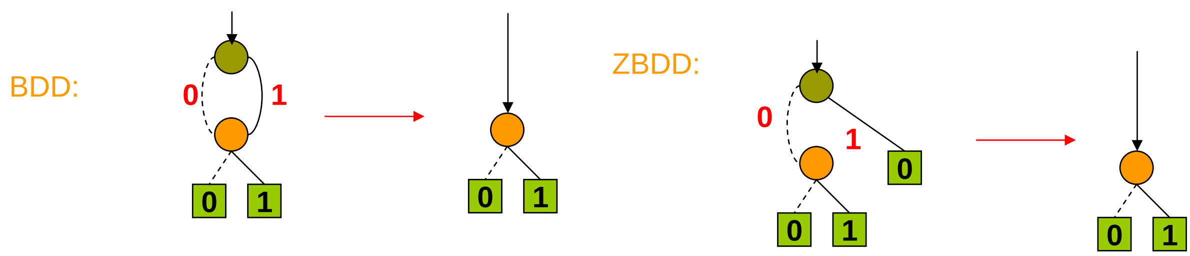
- Variable ordering – dynamic
  - BDD sifting:
    - shift each BDD variable to the top and then to the bottom and see which position had minimal number of BDD nodes
    - efficient if separate hash-table for each variable
    - can stop if lower bound on size is worse than the best found so far
    - shortcut: two layers can be swapped very cheaply if there is no interaction between them
    - expensive operation
  - grouping of BDD variables:
    - for many applications, grouping variables gives better ordering
      - e.g. current state and next state variables in state traversal
    - grouping variables for sifting

Boolean Function Representation
BDD – Variants

- MDD: Multi-valued DD
  - have more then two branches
  - can be implemented using a regular BDD package with binary encoding
    - the binary variables for one MV variable do not have to stay together and thus potentially better ordering
- ADD: (Algebraic BDDs) MTBDD
  - multi-terminal BDDs
  - decision tree is binary
  - multiple leaves, including real numbers, sets or arbitrary objects
  - efficient for matrix computations and other non-integer applications
- FDD: Free-order BDD
  - variable ordering differs
  - not canonical anymore
- ...

Boolean Function Representation
BDD – Variants

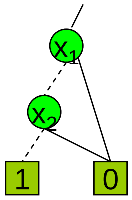
- Zero suppressed BDD (ZDD)
  - ZBDDs were invented by Minato to efficiently represent sparse sets. They have turned out to be useful in implicit methods for representing prime implicants (which usually are a sparse subset of all cubes).
  - Different reduction rules:
    - BDD: eliminate all nodes where then edge and else edge point to the same node.
    - ZBDD: eliminate all nodes where the then edge points to 0. Connect incoming edges to else node.
    - For both: share equivalent nodes.



Boolean Function Representation
BDD – Variants

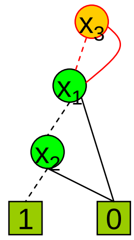
Theorem: ZBDDs are canonical given a variable ordering and the support set

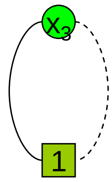
Example



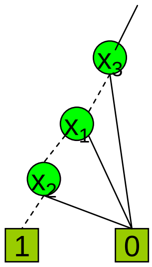
BDD

ZBDD if support is x_1, x_2





ZBDD if support is x_1, x_2, x_3



BDD

ZBDD if support is x_1, x_2 , x_3

Boolean Function Representation
BDD – Variants

- ZDD applications
  - Represent a set of subsets
  - Represent a cover of a Boolean function

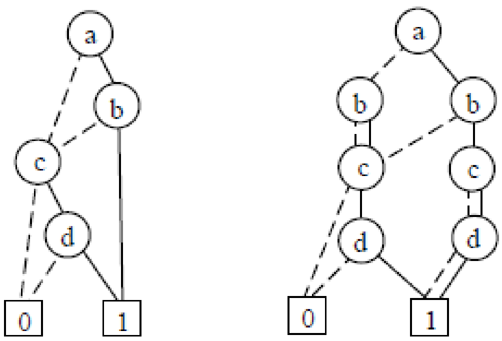


Figure 1. BDD and ZDD for F = ab + cd.

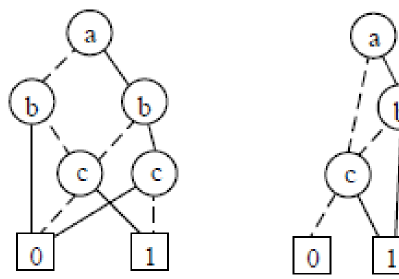


Figure 2. The BDD and the ZDD for the set of subsets {{a,b}, {a,c}, {c}}.

Courtesy: A. Mishchenko
https://en.wikipedia.org/wiki/Zero-suppressed_decision_diagram

Boolean Function Representation Summary

- Sum of products
  - Good for circuit synthesis
- Product of sums
  - Good for Boolean reasoning
- Boolean network
  - Generic network
    - Good for multi-level circuit synthesis
  - And-inverter graph
    - Good for Boolean reasoning
- Binary decision diagram
  - Good for Boolean reasoning

Boolean Reasoning

Reading:

Logic Synthesis in a Nutshell

Section 2

most of the following slides are by
courtesy of Andreas Kuehlmann

Boolean Reasoning
Satisfiability (SAT)

- Boolean reasoning engines need:
  - a data structure to represent problem instances
  - a decision procedure to decide about SAT or UNSAT

- Fundamental tradeoff
  - canonical data structure (e.g. truth table, ROBDD)
    - data structure uniquely represents function
    - decision procedure is trivial (e.g., just pointer comparison)
    - Problem: size of data structure is in general exponential
  - non-canonical data structure (e.g. AIG, CNF)
    - systematic search for satisfying assignment
    - size of data structure is linear
    - Problem: decision may take an exponential amount of time

Boolean Reasoning
SAT

- Basic SAT algorithms:
  - branch and bound algorithm
    - branching on the assignments of primary inputs only or those of all variables
      - E.g. PODEM vs. D-algorithms in ATPG

- Basic data structures:
  - circuits or CNF formulas
  - SAT on circuits is **identical** to the justification part in ATPG
    - 1st half of ATPG: justification
      - find an input assignment that forces an internal signal to a required value
    - 2nd half of ATPG: propagation
      - make a signal change at an internal signal observable at some outputs (can be easily formulated as SAT over CNF formulas)

Boolean Reasoning
SAT vs. Tautology

- SAT:
  - find a truth assignment to the inputs making a given Boolean formula true
  - NP-complete
- Tautology:
  - find a truth assignment to the inputs making a given Boolean formula false
  - coNP-complete
- SAT and Tautology are dual to each other
  - checking SAT on formula φ = checking Tautology on formula $\neg\varphi$, and vice versa

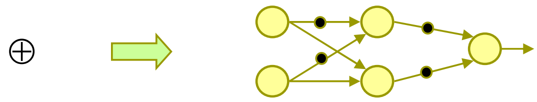
Boolean Reasoning
SAT – CNF-based Decision Procedure

- CNF
  - Product-of-Sums (POS) representation of Boolean function
  - Describes solution using a set of constraints
    - very handy in many applications because new constraints can be simply added to the list of existing constraints
    - very common in AI community
  - Example

    $\varphi = (a+b'+c)(a'+b+c)(a+b'+c')(a+b+c)$
  - SAT on CNF (POS) $\Leftrightarrow$ TAUTOLOGY on DNF (SOP)

Boolean Reasoning
SAT – CNF-based Decision Procedure

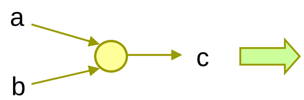
- Circuit to CNF conversion
  - Encountered often in practical applications
  - Naive conversion from circuit to CNF:
    - multiply out expressions of circuit until two level structure
    - Example: $y = x_1 \oplus x_2 \oplus x_3 \oplus \ldots \oplus x_n$ (parity function)
      - circuit size is linear in the number of variables



      - generated chess-board Karnaugh map
      - CNF (or DNF) formula has 2^{n-1} terms (exponential in the # vars)
  - Better approach:
    - introduce one variable per circuit vertex
    - formulate the circuit as a conjunction of constraints imposed on the vertex values by the gates
    - uses more variables but size of formula is linear in the size of the circuit

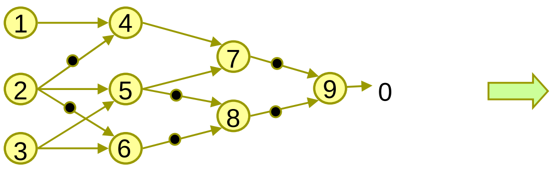
Boolean Reasoning
SAT – CNF-based Decision Procedure

- Circuit to CNF conversion
  - Example
    - Single gate



(¬a + ¬b + c) (a + ¬c) (b + ¬c)

    - Connected gates



Justify to "0"

(¬1 + 2 + 4) (1 + ¬4) (¬2 + ¬4)
(¬2 + ¬3 + 5) (2 + ¬5) (3 + ¬5)
(2 + ¬3 + 6) (¬2 + ¬6) (3 + ¬6)
(¬4 + ¬5 + 7) (4 + ¬7) (5 + ¬7)
(5 + 6 + 8) (¬5 + ¬8) (¬6 + ¬8)
(7 + 8 + 9) (¬7 + ¬9) (¬8 + ¬9)
(¬9)

Boolean Reasoning
SAT – CNF-based Decision Procedure

- DPLL procedure

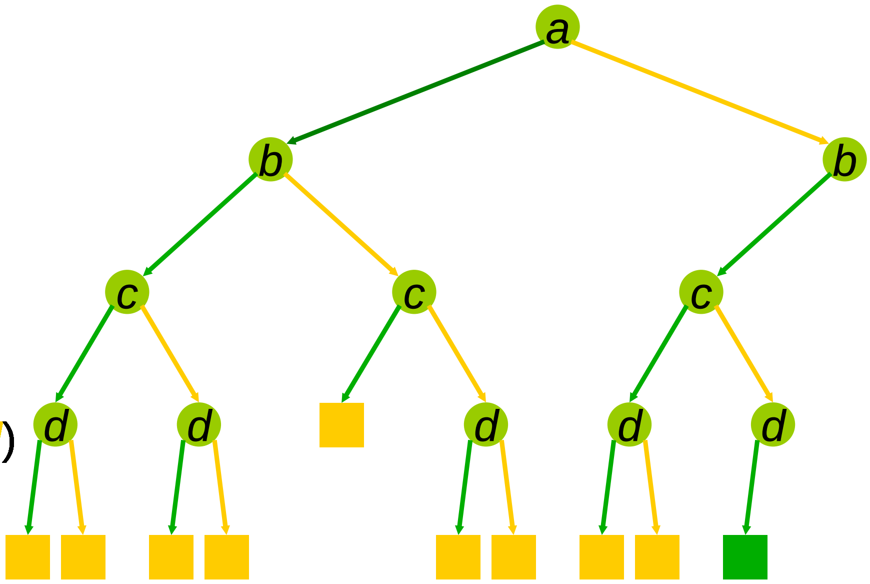
```
Algorithm DPLL() {
  while ChooseNextAssignment() {
     while Deduce() == CONFLICT  {
        blevel = AnalyzeConflict();
        if (blevel < 0) return UNSATISFIABLE;
        else Backtrack(blevel);
     }
   }
   return SATISFIABLE;
}
```

`ChooseNextAssignment` picks next decision variable and assignment
`Deduce` does Boolean Constraint Propagation (implications)
`AnalyzeConflict` backprocesses from conflict and produces learnt-clause
`Backtrack` undoes assignments

Boolean Reasoning
SAT – CNF-based Decision Procedure

- DPLL (basic case splitting)

1 $(a + b + c)$
2 $(a + b + \neg c)$
3 $(\neg a + b + \neg c)$
4 $(a + c + d)$
5 $(\neg a + c + d)$
6 $(\neg a + c + \neg d)$
7 $(\neg b + \neg c + \neg d)$
8 $(\neg b + \neg c + d)$



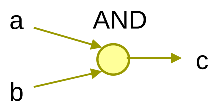
Source: Karem A. Sakallah, Univ. of Michigan

Boolean Reasoning
SAT – CNF-based Decision Procedure

- Implication
  - Implications in a CNF formula are caused by unit clauses
    - A unit clause is a CNF term for which all variables except one are assigned
      - the value of that clause can be implied immediately
    - Example
      clause (a+¬b+c)
      (a=0) (b=1) ⇒ (c=1)

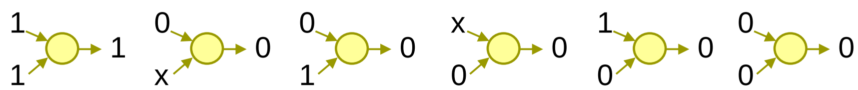
Boolean Reasoning
SAT – CNF-based Decision Procedure

- Implication
  - Example

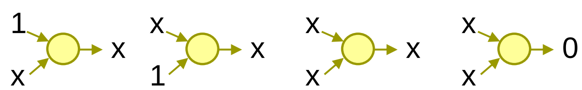


(¬a+¬b+c) (a+¬c) (b+¬c)

Non-implication cases:



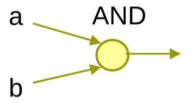
All clauses satisfied



Not all clauses satisfied (avoid exploring this part)

Boolean Reasoning
SAT – CNF-based Decision Procedure

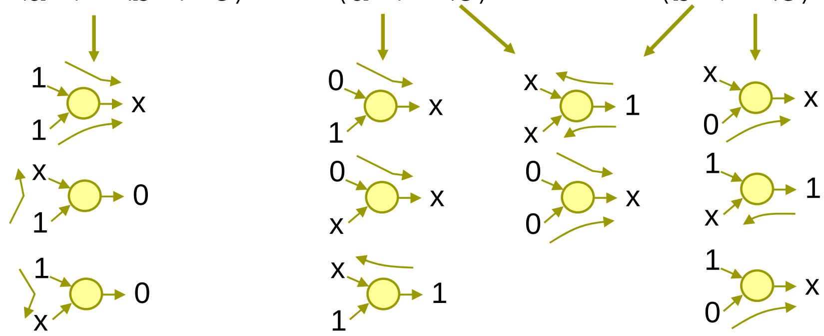
- Implication
  - Example (cont'd)



(¬a+¬b+c) (a+¬c) (b+¬c)

Implication cases:

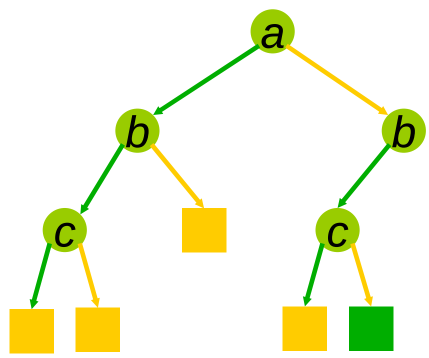
(¬a + ¬b + c)   (a + ¬c)   (b + ¬c)

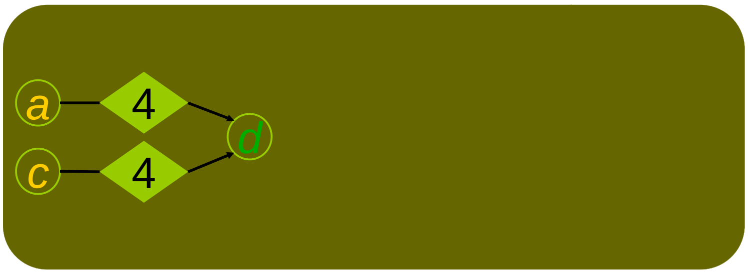


Boolean Reasoning
SAT – CNF-based Decision Procedure

- DPLL (w/ implication)

1. (a + b + c)
2. (a + b + ¬c)
3. (¬a + b + ¬c)
4. (a + c + d)
5. (¬a + c + d)
6. (¬a + c + ¬d)
7. (¬b + ¬c + ¬d)
8. (¬b + ¬c + d)





Source: Karem A. Sakallah, Univ. of Michigan

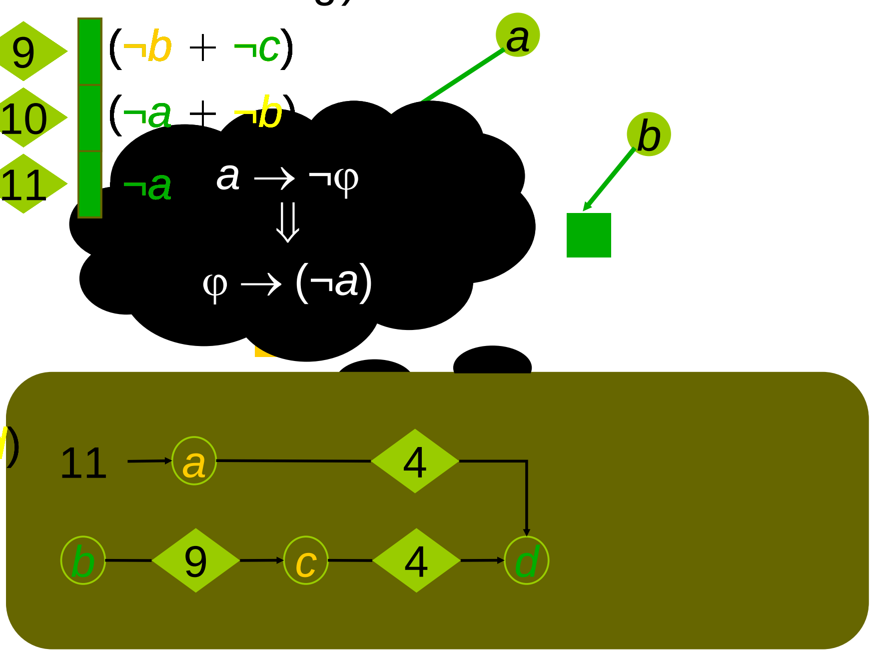
Boolean Reasoning
SAT – CNF-based Decision Procedure

- Conflict-based learning
  - Important detail for cut selection:
    - During implication processing, record decision level for each implication
    - At conflict, select earliest cut such that exactly one node of the implication graph lies on current decision level
      - Either decision variable itself
      - Or UIP (“unique implication point”) that represents a dominator node for current decision level in conflict graph
  - By selecting such cut, implication processing will automatically flip decision variable (or UIP variable) to its complementary value

Boolean Reasoning
SAT – CNF-based Decision Procedure

- DPLL (conflict-based learning)

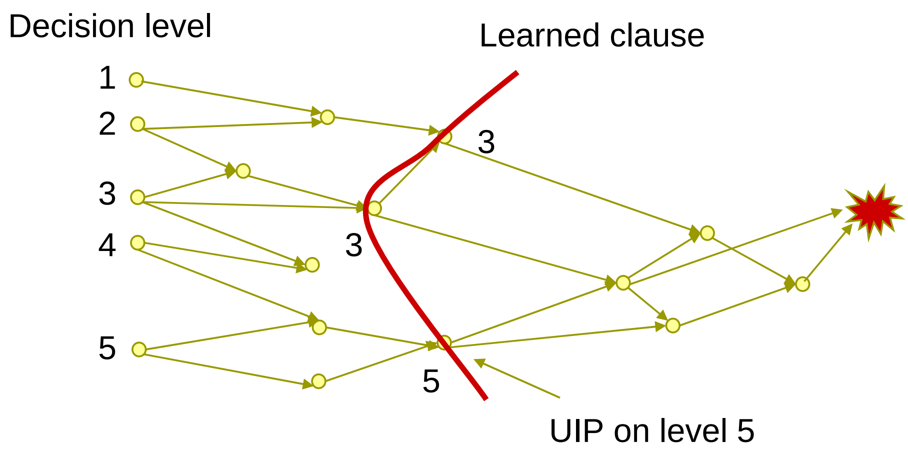
1. $(a + b + c)$
2. $(a + b + \neg c)$
3. $(\neg a + b + \neg c)$
4. $(a + c + d)$
5. $(\neg a + c + d)$
6. $(\neg a + c + \neg d)$
7. $(\neg b + \neg c + \neg d)$
8. $(\neg b + \neg c + d)$
9. $(\neg b + \neg c)$
10. $(\neg a + \neg b)$
11. $\neg a$



Source: Karem A. Sakallah, Univ. of Michigan

Boolean Reasoning
SAT – CNF-based Decision Procedure

- Conflict-based learning
  - UIP detection
    - Store with each implication the decision level, and a time stamp (integer that is incremented after each decision)
      - UIP on decision level l has the property that all following implications towards the conflict have a larger time stamp
      - When back processing from conflict, put all implications that are to be processed on heap, keeping the one with smallest time stamp on top
      - If during processing there is only one variable on current decision level on heap, then that variable must be a UIP



Boolean Reasoning
SAT – CNF-based Decision Procedure

- Implementation issues
  - Clauses are stores in arrays
  - Track change-sensitive clauses (two-literal watching)
    - all literals but one assigned -> implication
    - all literals but two assigned -> clause is sensitive to a change of either literal
    - all other clauses are insensitive and do not need to be observed
  - Learning:
    - learned implications are added to the CNF formula as additional clauses
      - limit the size of the clause
      - limit the "lifetime" of a clause, will be removed after some time
  - Non-chronological back-tracking
    - similar to circuit case

Boolean Reasoning
SAT – CNF-based Decision Procedure

- Implementation issues (cont'd)
  - Random restarts:
    - stop after a given number of backtracks
      - start search again with modified ordering heuristic
      - keep learned structures !
    - very effective for satisfiable formulas, often also effective for unsat formulas
  - Learning of equivalence relations:
    - if $(a \Rightarrow b) \wedge (b \Rightarrow a)$, then $(a = b)$
    - very powerful for formal equivalence checking
  - Incremental SAT solving
    - solving similar CNF formulas in a row
    - share learned clauses