# Option Pricing Models

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## Terms

C: call value

P: put value

X: strike price

S: stock price

$\hat{r} > 0$: the continuously compounded riskless rate per year

$R \triangleq e^{\hat{r}}$: gross return

$\tau$: the time to expiration of the option measured in years

n: number of periods. (each period represents a time interval of $\tau/n$)

## Call on a Non-Dividend-Paying Stock: Single Period

If the current stock price is $S$, it can go to $Su$ with probability q and Sd with probability $1-q$.

$C_u$ is the call price at time 1 if the stock price moves to $Su$.

$C_d$ is the call price at time 1 if the stock price moves to $Sd$.

$$C_u = \max(0, Su - X)$$
$$C_d = \max(0, Sd - X)$$

Set up a portfolio of $h$ shares of stock and $B$ dollars in riskless bonds. This costs $hS + B$. Choose $h$ and $B$ such that the portfolio replicates the payoff of the call.

We call $h$ the hedge ratio or delta.

$$C_u = hSu + RB$$
$$C_d = hSd + RB$$
$$\Rightarrow h = \frac{C_u - C_d}{Su - Sd} \geq 0$$
$$B = \frac{uC_d - dC_u}{(u-d)R}$$

## Put Pricing in One Period

$$\begin{aligned} P_u &= \max(0, X - Su) \\ P_d &= \max(0, X - Sd) \\ \Rightarrow h &= \frac{P_u - P_d}{Su - Sd} \leq 0 \\ B &= \frac{uP_d - dP_u}{(u-d)R} \end{aligned}$$

The European put is worth $hS + B$

The American put is worth $\max(hS + B, X - S)$. (Early exercise is always possible.)

## Pseudo Probability

The option value is independent of $q$.

The set of possible stock prices is the same whatever $q$ is.

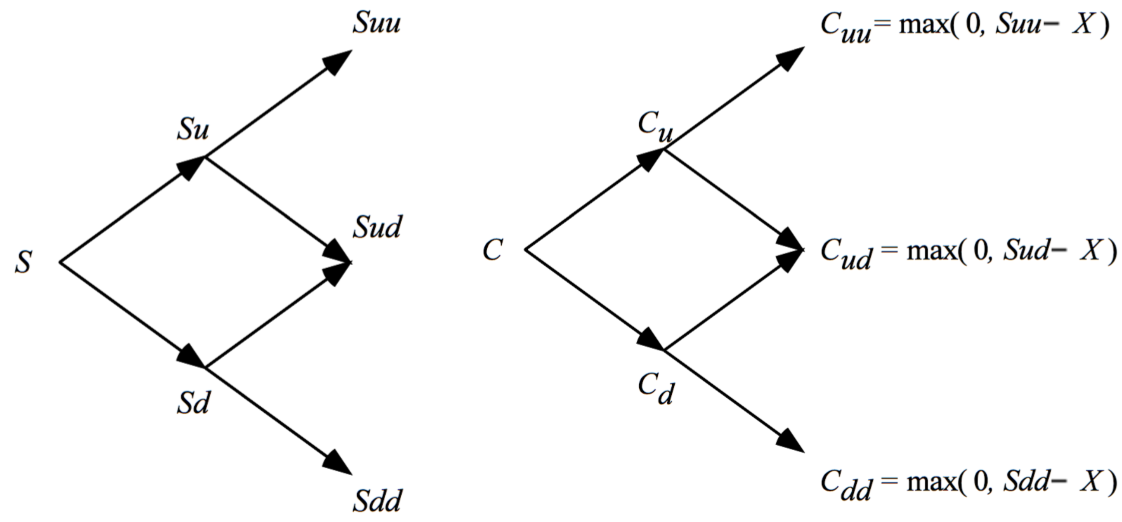
$$\begin{aligned} hS + B &= \frac{\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d}{R} \\ &= \frac{pC_u + (1-p)C_d}{R} \\ \Rightarrow p &\triangleq \frac{R-d}{u-d} \end{aligned}$$

As $0 < 1 < p$, it may be interpreted as a probability. The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under $p$ as

$$pS_u + (1-p)S_d = RS$$

The expected rates of return of all securities must be the riskless rate when investors are risk-neutral. For this reason, $p$ is called the risk-neutral probability.

## Option on a Non-Dividend-Paying Stock: Multi-Period




$$C_u = \frac{pC_{uu} + (1-p)C_{ud}}{R}$$
$$C_d = \frac{pC_{ud} + (1-p)C_{dd}}{R}$$

The current price is

$$C = \frac{pC_u + (1-p)C_d}{R} = \frac{p^2C_{uu} + 2p(1-p)C_{ud} + (1-p)^2C_{dd}}{R^2}$$

## Backward Induction

The value of a call or put on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy. In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max(0, Su^j d^{n-j} - X)}{R^n}$$
$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max(0, X - Su^j d^{n-j})}{R^n}$$

Let $p_j$ be the state price for the state $Su^jd^{n-j}, j = 0, 1, ..., n$.

$$p_j \triangleq \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j}}{R^n}$$

$$\text{option price } = \sum_j p_j \times \text{payoff at state } j$$

## The Binomial Option Pricing Formula

$$b(j;n,p) \triangleq \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$$

The stock prices at time $n$ are $Su^n, Su^{n-1}d, ..., Sd^n$.

Let $a$ be the minimum number of upward price moves for the call to finish in the money.

So $a$ is the smallest nonnegative integer $j$ such that

$$Su^j d^{n-j} \geq X$$

or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil$$

$$\begin{aligned} C &= \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times (Su^j d^{n-j} - X)}{R^n} \\ &= S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j [(1-p)d]^{n-j}}{R^n} - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\ &= S \sum_{j=a}^{n} b(j;n,\frac{pu}{R}) - Xe^{-\hat{r}n} \sum_{j=a}^{n} b(j;n,p) \end{aligned}$$

## Toward the Black-Scholes Formula

Risk-free continuously compounded rate per period: $\hat{r} = r\tau/n$

Gross return per period: $e^{\hat{r}}$

Expected rate of return per period: $\hat{\mu} = \frac{1}{n}\mathrm{E}\big[\ln \frac{S_\tau}{S}\big]$

Variance of the return per period: $\hat{\sigma} = \frac{1}{n}\mathrm{Var}\big[\ln \frac{S_\tau}{S}\big]$

Under the BOPM,

$$\hat{\mu} = q\ln(u/d) + \ln d$$

$$\hat{\sigma} = q(1-q)\ln^2(u/d)$$

Assume the stock's true continuously compounded rate of return over $\tau$ years has mean $\mu\tau$ and variance $\sigma^2\tau$, then

$$n\hat{\mu} = n\big[q\ln(u/d) + \ln d\big] \to \mu\tau$$

$$n\hat{\sigma} = n\big[q(1-q)\ln^2(u/d)\big] \to \sigma^2\tau$$

Impose $ud = 1$ (Other choices are possible). The above requirement can be statisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, d = e^{-\sigma\sqrt{\tau/n}}, \hat{r} = r\tau/n$$

## The Black-Scholes Formula

$$C = SN(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau})$$

$$P = Xe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - SN(-x)$$

where

$$x = \ln\left(\frac{S}{X}\right) + \frac{1}{2}\sigma^2\tau + r\tau$$

## BOPM and Black-Scholes Model

The Black-Scholes formula needs 5 parameters: $S$, $X$, $\sigma$, $\tau$, and $r$.

The Binomial tree algorithms take 6 inputs: $S$, $X$, $u$, $d$, $\hat{r}$, and $n$.

The connections are

$$u = e^{\sigma\sqrt{\tau/n}}$$

$$d = e^{-\sigma\sqrt{\tau/n}}$$

$$\hat{r} = r\tau/n$$

## Binomial Tree Algorithms for American Puts

1. Early exercise has to be considered.
2. The binomial tree algorithm starts with the terminal payoffs $\mathbf{max(0, X - Su^j d^{n-j})}$ and apply backward induction.
3. At each intermediate node, it compares the payoff if exercised and the continuation value. Exercise value of nodes at level i = $\max(0, X - Su^j d^{i-j})$

**Bermudan options** can be exercised only at discrete time points instead of continuously.