Alpha-Beta Pruning: Algorithm and Analysis

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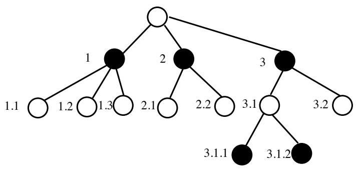
Introduction

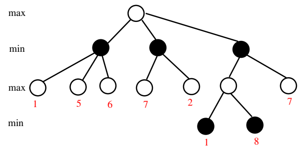
- Alpha-beta pruning is the standard searching procedure used for solving 2-person perfect-information zero sum games exactly.
- Definitions:
  - A *position* p.
  - The **value** of a position p, $f(p)$, is a numerical value computed from evaluating p.
    - ▷ *Value is computed from the root player's point of view.*
    - ▷ *Positive values mean in favor of the root player.*
    - ▷ *Negative values mean in favor of the opponent.*
    - ▷ *Since it is a zero sum game, thus from the opponent's point of view, the value can be assigned $-f(p)$.*
  - A **terminal** position: a position whose value can be decided.
    - ▷ *A position where win/loss/draw can be concluded.*
    - ▷ *A position where some constraints are met.*
  - A position p has b legal moves $p_1, p_2, \ldots, p_b$.

Tree node numbering



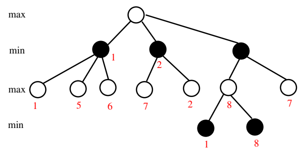
- From the root, number a node in a search tree by a sequence of integers $a_1.a_2.a_3.a_4\cdots$
  - Meaning from the root, you first take the a_1th branch, then the a_2th branch, and then the a_3th branch, and then the a_4th branch $\cdots$
  - The root is specified as an empty sequence.
  - The **depth** of a node is the length of the sequence of integers specifying it.
- This is called "Dewey decimal system."

Mini-max formulation



- Mini-max formulation:
  - $$F'(p) = \begin{cases} f(p) & \text{if } b = 0 \\ max\{G'(p_1), \ldots, G'(p_b)\} & \text{if } b > 0 \end{cases}$$
  - $$G'(p) = \begin{cases} f(p) & \text{if } b = 0 \\ min\{F'(p_1), \ldots, F'(p_b)\} & \text{if } b > 0 \end{cases}$$
  - An indirect recursive formula with a bottom-up evaluation!
  - Equivalent to AND-OR logic.

Mini-max formulation



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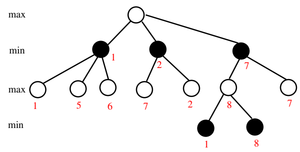

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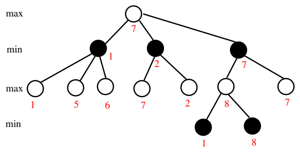

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  - An indirect recursive formula with a bottom-up evaluation!
  - Equivalent to AND-OR logic.

Algorithm: Mini-max

- Algorithm F'(position p) // max node
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$, then return $f(p)$ else begin
    - ▷ $m := -\infty$
    - ▷ *for* $i := 1$ *to* b *do*
    - ▷ $t := G'(p_i)$
    - ▷ *if* $t > m$ *then* $m := t$ // *find max value*
  - end;
  - return m
- Algorithm G'(position p) // min node
  - determine the successor positions $p_1, \ldots, p_b$
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    - ▷ $t := F'(p_i)$
    - ▷ *if* $t < m$ *then* $m := t$ // *find min value*
  - end;
  - return m

Mini-max: comments

- A brute-force method to try all possibilities!
  - May visit a position many times.
- Depth-first search
  - Move ordering is according to order the successor positions are generated.
  - Bottom-up evaluation.
  - Post-ordering traversal.
- Q:
  - Iterative deepening?
  - BFS?
  - Other types of searching?

Mini-max: revised (1/2)

- Algorithm F'(position p, integer $depth$) // max node
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
    then return $f(p)$// current board value
    else begin
    - ▷ $m := -\infty$ // *initial value*
    - ▷ *for* $i := 1$ *to* b *do* // *try each child*
    - ▷ *begin*
    - ▷ $t := G'(p_i, depth - 1)$
    - ▷ *if* $t > m$ *then* $m := t$ // *find max value*
    - ▷ *end*

    end
  - return m

Mini-max: revised (2/2)

- Algorithm G'(position p, integer $depth$) // min node
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or $depth = 0$ // remaining depth to search
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    then return $f(p)$// current board value
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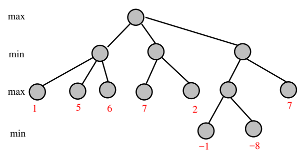
    - ▷ $m := \infty$ // *initial value*
    - ▷ *for* $i := 1$ *to* b *do* // *try each child*
    - ▷ *begin*
    - ▷ $t := F'(p_i, depth - 1)$
    - ▷ *if* $t < m$ *then* $m := t$ // *find min value*
    - ▷ *end*

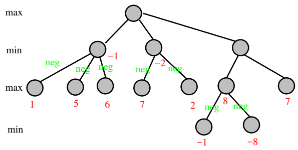
    end
  - return m

Nega-max formulation



- Nega-max formulation:
  Let $F(p)$ be the greatest possible value achievable from the position p against the optimal defensive strategy.
  - $$F(p) = \begin{cases} h(p) & \text{if } b = 0 \\ max\{-F(p_1), \ldots, -F(p_b)\} & \text{if } b > 0 \end{cases}$$
    - ▷ $$h(p) = \begin{cases} f(p) & \textit{if depth of } p \textit{ is 0 or even} \\ -f(p) & \textit{if depth of } p \textit{ is odd} \end{cases}$$

Nega-max formulation



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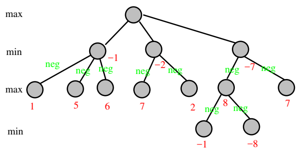

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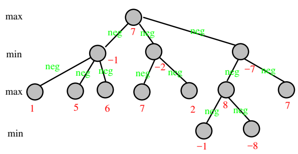

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Algorithm: Nega-max

- **Algorithm F(position p, integer $depth$)**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$ // a terminal node**
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - **then return $h(p)$ else**
  - **begin**
    - ▷ $m := -\infty$
    - ▷ *for $i := 1$ to b do*
    - ▷ *begin*
    - ▷ $t := -F(p_i, depth - 1)$ *// recursive call, the returned value is negated*
    - ▷ *if $t > m$ then $m := t$ // always find a max value*
    - ▷ *end*
  - **end**
  - **return m**

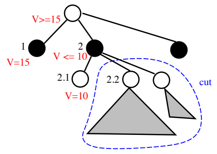
Nega-max: comments

- **Another brute-force method to try all possibilities.**
  - **Use $h(p)$ instead of $f(p)$.**
  - **Watch out the code in dealing with search termination conditions.**
    - ▷ *Reach a given searching depth.*
    - ▷ *Timing control.*
    - ▷ *Other constraints such as the score is good or bad enough.*
- **Notations:**
  - **F' means the Mini-max version.**
    - ▷ *Need a G' companion.*
    - ▷ *Easy to explain.*
  - **F means the Negamax version.**
    - ▷ *Simpler code.*
    - ▷ *Maybe difficult to explain.*

Intuition for improvements

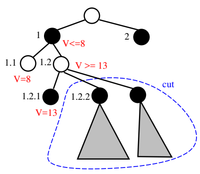
- **Branch-and-bound: using information you have so far to cut or prune branches.**
  - **A branch is cut means we do not need to search it anymore.**
  - **If you know for sure or almost sure the value of your result is more than x**
    and the current search result for this branch so far can give you no more than x,
    - ▷ *then there is no need to search this branch any further.*
- **Two types of approaches**
  - **Exact algorithms: through mathematical proof, it is guaranteed that the branches pruned won't contain the solution.**
    - ▷ *Alpha-beta pruning: reinvented by several researchers in the 1950's and 1960's.*
    - ▷ *Scout.*
    - ▷ $\cdots$
  - **Approximated heuristics: with a high probability that the solution won't be contained in the branches pruned.**
    - ▷ *Obtain a good estimation on the remaining cost.*
    - ▷ *Cut a branch when it is in a very bad position and there is little hope to gain back the advantage.*

Alpha cut-off



- **On the max node which is the root:**
  - *Assume you have finished exploring the branch at 1 and obtained the best value from it as* $bound$.
  - *You now search the branch at 2 by first searching the branch at 2.1.*
  - *Assume branch at 2.1 returns a value that is* $\le bound$.
  - *Then no need to evaluate the branch at 2.2 and all later branches of 2, if any, at all.*
  - *The best possible value for the branch at 2 must be* $\le bound$.
  - *Hence we should take value returned from the branch at 1 as the best possible solution.*

Beta cut-off

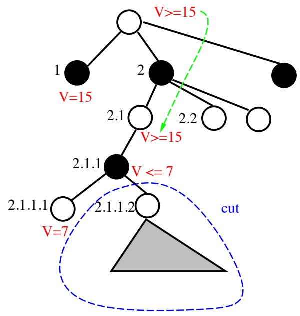


- **On the min node 1:**
  - *Assume you have finished exploring the branch at 1.1 and obtained the best value from it as* $bound$.
  - *You now search the branch at 1.2 by first exploring the branch at 1.2.1.*
  - *Assume the branch at 1.2.1 returns a value that is* $\ge bound$.
  - *Then no need to evaluate the branch at 1.2.2 and all later branches of 1.2, if any, at all.*
  - *The best possible value for the branch at 1.2 is* $\ge bound$.
  - *Hence we should take value returned from the branch at 1.1 as the best possible solution.*

Deep alpha cut-off

- **For alpha cut-off:**
  - *For a min node u, a branch of its ancestor (e.g., an elder brother of its parent) produces a lower bound V_l.*
  - *The first branch of u produces an upper bound V_u for v.*
  - *If $V_l \ge V_u$, then there is no need to evaluate the second branch and all later branches, of u.*
- **Deep alpha cut-off:**
  - *Def: For a node u in a tree and a positive integer g, Ancestor(g, u) is the direct ancestor of u by tracing the parent's link g times.*
  - *When the lower bound V_l is produced at and propagated from u's great grand parent, i.e., Ancestor(3,u), or any Ancestor($2i+1$,u), $i \ge 1$.*
  - *When an upper bound V_u is returned from the a branch of u and $V_l \ge V_u$, then there is no need to evaluate all later branches of u.*
- **We can find similar properties for deep beta cut-off.**

Illustration — Deep alpha cut-off



Ideas for refinements

- **During searching, maintain two values $alpha$ and $beta$ so that**
  - $alpha$ **is the current lower bound of the possible returned value;**
    - *This means to say you know a way to achieve the value $alpha$.*
  - $beta$ **is the current upper bound of the possible returned value.**
    - *This means to say your opponent knows a way to achieve a value of $beta$.*
- **If during searching, we know for sure $alpha > beta$, then there is no need to search any more in this branch.**
  - **The returned value cannot be in this branch.**
  - **Backtrack until it is the case $alpha \le beta$.**
- **The two values $alpha$ and $beta$ are called the ranges of the current search window.**
  - **These values are dynamic.**
  - **Initially, $alpha$ is $-\infty$ and $beta$ is ∞.**

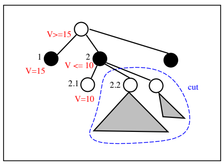
Alpha-beta pruning algorithm: Mini-Max

- **Algorithm $F1'$(position p, value $alpha$, value $beta$) // max node**
  - **determine the successor positions** $p_1, \ldots, p_b$
  - **if** $b = 0$, **then return** $f(p)$ **else begin**
    - $m := alpha$
    - **for** $i := 1$ **to** b **do**
    - $t := G1'(p_i, m, beta)$
    - **if** $t > m$ **then** $m := t$ // *improve the current best value*
    - **if** $m \ge beta$ **then return**($beta$) // *beta cut off*
  - **end; return** m
- **Algorithm $G1'$(position p, value $alpha$, value $beta$) // min node**
  - **determine the successor positions** $p_1, \ldots, p_b$
  - **if** $b = 0$, **then return** $f(p)$ **else begin**
    - $m := beta$
    - **for** $i := 1$ **to** b **do**
    - $t := F1'(p_i, alpha, m)$
    - **if** $t < m$ **then** $m := t$
    - **if** $m \le alpha$ **then return**($alpha$) // *alpha cut off*
  - **end; return** m

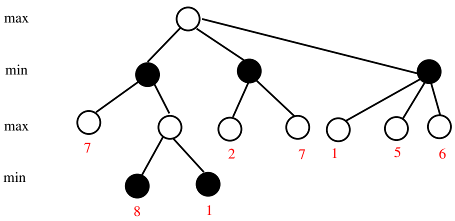
Example

Initial call: $F1'(\text{root},-\infty,\infty)$

- $m = -\infty$
- call $G1'$(node 1,$-\infty$,∞)
  - ▷ *it is a terminal node*
  - ▷ *return value* 15
- $t = 15$;
  - ▷ *since* $t > m$, m *is now* 15
- call $G1'$(node 2,15,∞)
  - ▷ *call* $F1'$*(node* **2.1***,15,*∞*)*
  - ▷ *it is a terminal node; return* 10
  - ▷ $t = 10$*; since* $t < \infty$, m *is now* 10
  - ▷ $alpha$ *is* 15, m *is* 10, *so we have an alpha cut off,*
  - ▷ *no need to call* $F1'$*(node* **2.2***,15,10)*
  - ▷ ***return 15***
  - ▷ $\cdots$



A complete example



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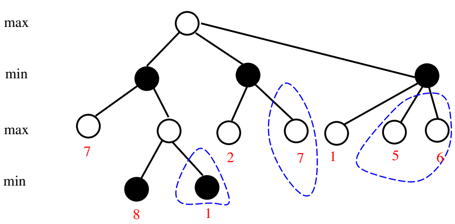
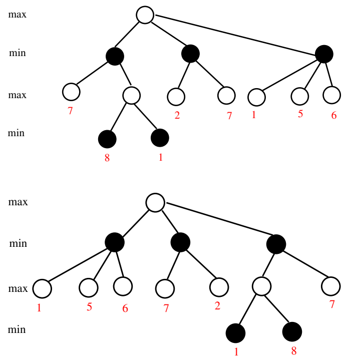


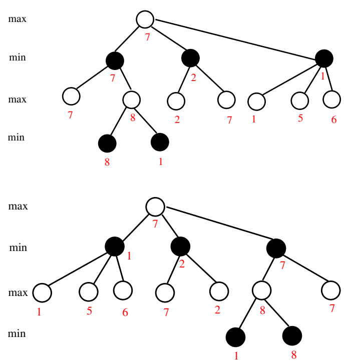
Alpha-beta pruning algorithm: Nega-max

- **Algorithm** $F1$**(position** p**, value** $alpha$**, value** $beta$**, integer** $depth$**)**
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - then return $h(p)$ else
  - begin
    - ▷ $m := alpha$
    - ▷ *for* $i := 1$ *to* b *do*
    - ▷ *begin*
    - ▷ $t := -F1(p_i, -beta, -m, depth-1)$
    - ▷ *if* $t > m$ *then* $m := t$
    - ▷ *if* $m \geq beta$ *then return(*$beta$*) // cut off*
    - ▷ *end*
  - end
  - return m

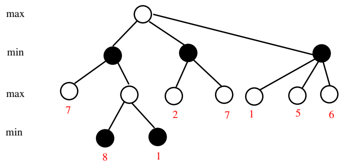
Examples (1/4)



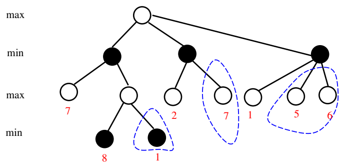
Examples (2/4)



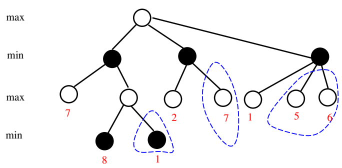
Examples (3/4)

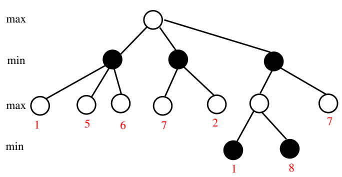


Examples (3/4)



Examples (4/4)





Lessons from the previous examples

- **It looks like for the same tree, different move orderings give very different cut branches.**
- **It looks like if a node can evaluate a child with the best possible outcome earlier, then it has a chance to cut earlier.**
  - **For a min node, this means to search the child branch that gives the lowest value first.**
  - **For a max node, this means to search the child branch that gives the highest value first.**
- **Comments:**
  - **It is impossible to always know which best branch is, or we do not have to do a brute-force search.**
  - **Q: In the best case scenario, how many nodes can be cut?**

Analysis of a possible best case

- **Definitions:**
  - **A path in a search tree is a sequence of numbers indicating the branches selected in each level using the Dewey decimal system.**
  - **A position is denoted as a path $a_1.a_2.\cdots.a_\ell$ from the root.**
  - **A position $a_1.a_2.\cdots.a_\ell$ is critical if**
    - ▷ *$a_i = 1$ for all even values of i or*
    - ▷ *$a_i = 1$ for all odd values of i.*
  - **Note: as a special case, the root is critical.**
  - **Examples:**
    - ▷ *2.1.4.1.2, 1.3.1.5.1.2, 1.1.1.2.1.1.1.3 and 1.1 are critical*
    - ▷ *1.2.1.1.2 is not critical*
- **Q: Why the root needs to be critical?**

Perfect-ordering tree

- **A perfect-ordering tree:**

$$F(a_1.\cdots.a_\ell) = \begin{cases} h(a_1.\cdots.a_\ell) & \textbf{if } a_1.\cdots.a_\ell \textbf{ is a terminal} \\ -F(a_1.\cdots.a_\ell.1) & \textbf{otherwise} \end{cases}$$

  - **The first successor of every non-terminal position gives the best possible value.**

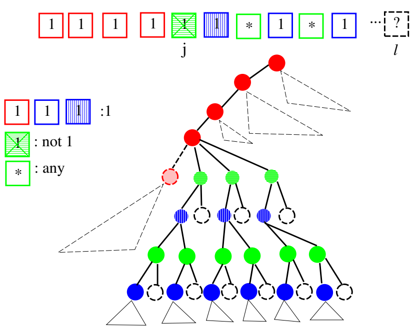
Theorem 1

- **Theorem 1: $F1$ examines precisely the critical positions of a perfect-ordering tree.**
- **Proof sketch:**
  - **Classify the critical positions, a.k.a. nodes, into different types.**
    - ▷ *You must evaluate the first branch from the root to the bottom.*
    - ▷ *Alpha cut off happens at odd-depth nodes as soon as the first branch of this node is evaluated.*
    - ▷ *Beta cut off happens at even-depth nodes as soon as the first branch of this node is evaluated.*
  - **For nodes of the same type, associate them with pruning of same characteristics occurred.**

Types of nodes

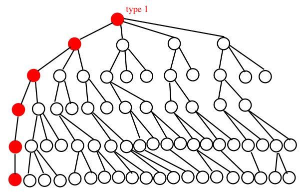
- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index, if exists, such that $a_j \neq 1$ and ℓ is the last index.**
  - **j will be the anchor in the analysis.**
  - **Def: let $IS1(a_i)$ be a boolean function so that it is 0 if it is not the value 1 and it is 1 if it is.**
    - ▷ *We call this $IS1$ parity of a number.*
  - **If j exists and $\ell > j$, then**
    - ▷ *$a_{j+1} = 1$ because this position is critical and thus the $IS1$ parities of a_j and a_{j+1} are different.*
  - **Since this position is critical, if $a_j \neq 1$, then $a_h = 1$ for any h such that $h - j$ is odd.**
- **We now classify critical nodes into 3 types.**
  - **Nodes of the same type share some common properties.**

Illustration — critical nodes



Type 1 nodes

- **type 1: the root, or a node with all the a_i are 1;**
  - **This means j does not exist.**
  - **Nodes on the leftmost branch.**
  - **The leftmost child of a type 1 node except the root.**



Type 2 nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 2: $\ell - j$ is zero or even;**
  - **type 2.1: $\ell - j = 0$ which means $\ell = j$.**
    - ▷ *It is in the form of $1.1.1.\cdots.1.1.1.a_\ell$ and $a_\ell \neq 1$.*
    - ▷ *The non-leftmost children of a type 1 node.*
  - **type 2.2: $\ell - j > 0$ and is even.**
    - ▷ *It is in the form of $1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1.a_\ell$.*
    - ▷ *Note, we have already defined $1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1$ to be a type 3 node.*
    - ▷ *All of the children of a type 3 node.*

Type 3 nodes

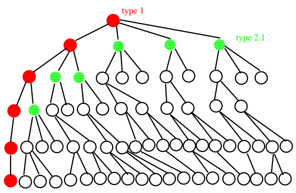
- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 3: $\ell - j$ is odd;**
  - **$a_j \neq 1$ and $\ell - j$ is odd**
    - ▷ *Since this position is critical, the $IS1$ parities of a_j and a_ℓ are different.*
      $\Longrightarrow a_\ell = 1$
      $\Longrightarrow a_{j+1} = 1$
  - **It is in the form of**
    - ▷ $1.1.\cdots.1.a_j.1.a_{j+2}.1.\cdots.1.a_{\ell-1}.1$.
  - **The leftmost child of a type 2 node.**
  - **type 3.1: $\ell - j = 1$.**
    - ▷ *It is of the form $1.1.\cdots.1.a_j.1$*
    - ▷ *The leftmost child of a type 2.1 node.*
  - **type 3.2: $\ell - j > 1$.**
    - ▷ *It is of the form $1.1.\cdots.1.a_j.1.a_{j+2}.1.\cdots.1.a_{\ell-1}.1$*
    - ▷ *The leftmost child of a type 2.2 node.*

Comments

- Nodes of the same type have common properties.
- These properties can be used in solving other problems.
  - Example: Efficient parallelization of alpha-beta based searching algorithms.
- Main techniques used:
  - You cannot have two consecutive non-1 numbers in the ID of a critical node.
  - For each non-1 number, any number appeared later and is odd distance away must be 1.

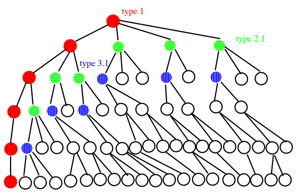
Type 2.1 nodes

- Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.
- type 2: $\ell - j$ is zero or even;
  - type 2.1: $\ell - j = 0$.
    - ▷ *Then $\ell = j$.*
    - ▷ *It is in the form of $\underline{1.1.1.\cdots.1.1.1}.a_\ell$ and $a_\ell \neq 1$.*
    - ▷ *The non-leftmost children of a type 1 node.*



Type 3.1 nodes

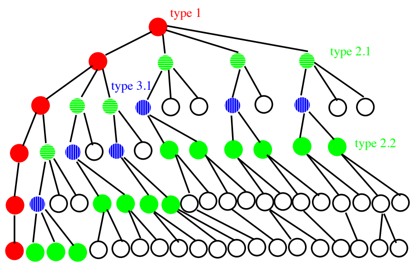
- Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.
- type 3: $\ell - j$ is odd;
  - type 3.1: $\ell - j = 1$.
    - ▷ *It is of the form $\underline{1.1.\cdots.1.a_j}.1$ and $a_\ell \neq 1$.*
    - ▷ *The leftmost child of a type 2.1 node.*



Type 2.2 nodes

- Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.
- type 2: $\ell - j$ is zero or even;
  - type 2.2: $\ell - j > 0$ and is even.
    - ▷ *The $IS1$ parties of a_j and a_{j+1} are different.*
      $\Longrightarrow$ *Since $a_j \neq 1$, $a_{j+1} = 1$.*
    - ▷ *$(\ell - 1) - j$ is odd:*
      $\Longrightarrow$ *The $IS1$ parties of $a_{\ell-1}$ and a_j are different.*
      $\Longrightarrow$ *Since $a_j \neq 1$, $a_{\ell-1} = 1$.*
    - ▷ *It is in the form of $\underline{1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1}.a_\ell$.*
    - ▷ *Note, we will show $1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1$ is a type 3 node later.*
    - ▷ *All of the children of a type 3 node.*

Illustration: Type 2.2 nodes



Type 3.2 nodes

- Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.
- type 3: $\ell - j$ is odd;
  - type 3.2: $\ell - j > 1$.
    - ▷ *It is of the form $\underline{1.1.\cdots.1.a_j.1.a_{j+2}.1.\cdots.1.a_{\ell-1}}.1$*
    - ▷ *The leftmost child of a type 2.2 node.*

Illustration: Type 3.2 nodes

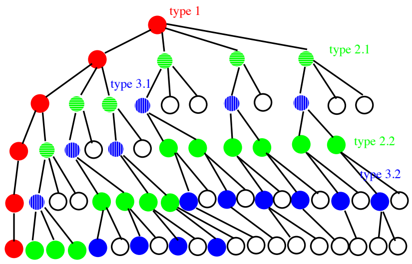


Illustration of all nodes

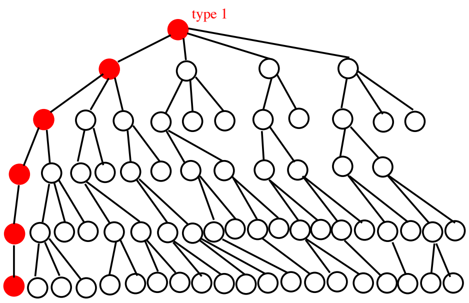


Illustration of all nodes

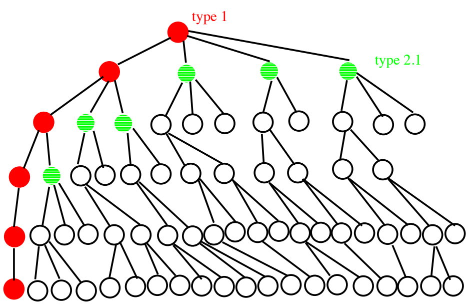


Illustration of all nodes

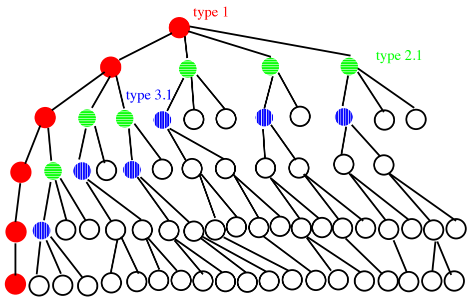


Illustration of all nodes

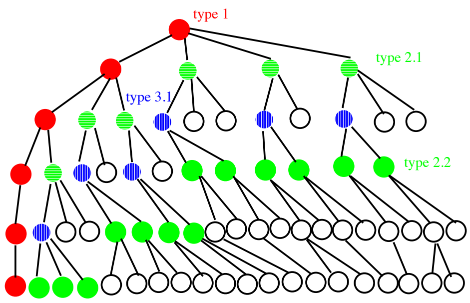


Illustration of all nodes

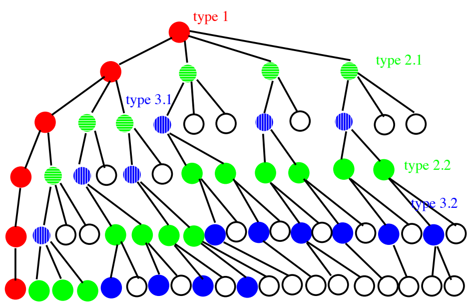
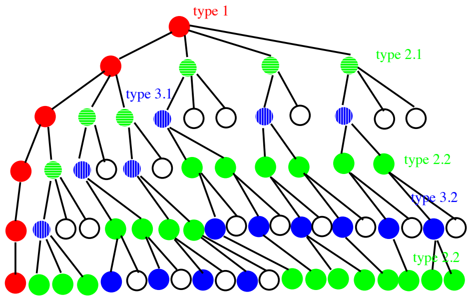


Illustration of all nodes



Theorem 1: Proof sketch

- Properties (invariants)
  - A type 1 position p is examined by calling $F1(p, -\infty, \infty, depth)$
    - ▷ *p's first successor p_1 is of type 1*
    - ▷ $F(p) = -F(p_1) \neq \pm\infty$
    - ▷ *p's other successors $p_2, \ldots, p_b$ are of type 2*
    - ▷ *p_i, $i > 1$, are examined by calling $F1(p_i, -\infty, F(p_1), depth)$*
  - A type 2 position p is examined by calling $F1(p, -\infty, beta, depth)$ where $-\infty < beta \leq F(p)$
    - ▷ *p's first successor p_1 is of type 3*
    - ▷ $F(p) = -F(p_1)$
    - ▷ *p's other successors $p_2, \ldots, p_b$ are not examined*
  - A type 3 position p is examined by calling $F1(p, alpha, \infty, depth)$ where $\infty > alpha \geq F(p)$
    - ▷ *p's successors $p_1, \ldots, p_b$ are of type 2*
    - ▷ *they are examined by calling $F1(p_1, -\infty, -alpha, depth)$, $F1(p_2, -\infty, -\max\{m_1, alpha\}, depth), \ldots,$ $F1(p_i, -\infty, -\max\{m_{i-1}, alpha\}, depth)$ where $m_i = F1(p_i, -\infty, -\max\{m_{i-1}, alpha\}, depth)$*
- Using an inductive argument to prove.

Analysis: best case

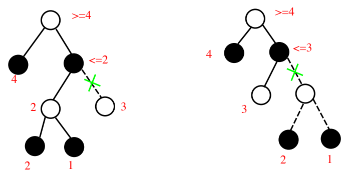
- Corollary 1: Assume each position has exactly b successors
  - The number of positions examined by the alpha-beta procedure on level i is exactly
$$b^{\lceil i/2 \rceil} + b^{\lfloor i/2 \rfloor} - 1.$$
- Proof:
  - There are $b^{\lfloor i/2 \rfloor}$ sequences of the form $a_1.\cdots.a_i$ with $1 \leq a_i \leq b$ for all i such that $a_i = 1$ for all odd values of i.
  - There are $b^{\lceil i/2 \rceil}$ sequences of the form $a_1.\cdots.a_i$ with $1 \leq a_i \leq b$ for all i such that $a_i = 1$ for all even values of i.
  - We subtract 1 for the sequence $1.1.\cdots.1.1$ which are counted twice.
- Total number of nodes visited is
$$\sum_{i=0}^{\ell} b^{\lceil i/2 \rceil} + b^{\lfloor i/2 \rfloor} - 1.$$

Analysis: average case

- Assumptions: Let a random game tree be generated in such a way that each position on level j
  - has a probability q_j of being nonterminal and
  - has an average of b_j successors.
- Properties of the above random game tree
  - Expected number of positions on level ℓ is $b_0 \cdot b_1 \cdots b_{\ell-1}$
  - Expected number of positions on level ℓ examined by an alpha-beta procedure assumed the random game tree is perfectly ordered is
$$b_0q_1b_2q_3\cdots b_{\ell-2}q_{\ell-1} + q_0b_1q_2b_3\cdots q_{\ell-2}b_{\ell-1} - q_0q_1\cdots q_{\ell-1} \text{if } \ell \text{ is even;}$$
$$b_0q_1b_2q_3\cdots q_{\ell-2}b_{\ell-1} + q_0b_1q_2b_3\cdots b_{\ell-2}q_{\ell-1} - q_0q_1\cdots q_{\ell-1} \text{if } \ell \text{ is odd}$$
- Proof sketch:
  - If x is the expected number of positions of a certain type on level j, then $x \times b_j$ is the expected number of successors of these positions, and $x \times q_j$ is the expected number of "numbered 1" successors.
  - The above numbers equal to those of Corollary 1 when $q_j = 1$ and $b_j = b$ for $0 \leq j < \ell$.

Perfect ordering is not always the best

- Intuitively, we may "think" alpha-beta pruning would be most effective when a game tree is perfectly ordered.
  - That is, when the first successor of every position is the best possible move.
  - This is not always the case!



- Truly optimum order of game trees traversal is not obvious.

When is a branch pruned?

- Assume a node r has two children u and v with u being visited before v using some move ordering.
  - Further assume u produced a new bound $bound$.
- Assume node v has a child w.
  - If the value new returned from w can cause a range conflict with $bound$, then branches of v later than w are cut.
- This means as long as the "relative" ordering of u and v are good enough, then we can have some cut-off.
  - There is no need for r to have the best move ordering.

Theorem 2

- **Theorem 2: Alpha-beta pruning is optimum in the following sense:**
  - **Given any game tree and any algorithm which computes the value of the root position, there is a way to permute the tree**
    - ▷ *by reordering successor positions if necessary;*
  - **so that every terminal position examined by the alpha-beta method under this permutation is examined by the given algorithm.**
  - **Furthermore if the value of the root is not ∞ or $-\infty$, the alpha-beta procedure examines precisely the positions which are critical under this permutation.**

Variations of alpha-beta search

- **Initially, to search a tree with the root r by calling $F1(r,-\infty,+\infty,depth)$.**
  - **What does it mean to search a tree with the root r by calling $F1(r,alpha,beta,depth)$?**
    - ▷ *To search the tree rooted at r requiring that the returned value to be within $alpha$ and $beta$.*
- **In an alpha-beta search with a pre-assigned window $[alpha, beta]$:**
  - **Failed-high means it returns a value that is larger than or equal to its upper bound $beta$.**
  - **Failed-low means it returns a value that is smaller than or equal to its lower bound $alpha$.**
- **Variations:**
  - **Brute force Nega-Max version: F**
    - ▷ *Always finds the correct answer according to the Nega-Max formula.*
  - **Original alpha-beta cut (Nega-Max) version: $F1$**
  - **Fail hard alpha-beta cut (Nega-Max) version: $F2$**
  - **Fail soft alpha-beta cut (Nega-Max) version: $F3$**

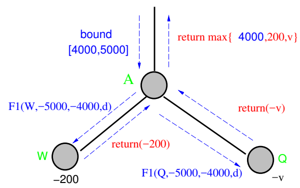
Original version

- **Requiring $alpha \le beta$**
- **Algorithm $F1$(position p, value $alpha$, value $beta$, integer $depth$)**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$ // a terminal node**
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - **then return $h(p)$ else**
  - **begin**
    - ▷ *$m := alpha$ // hard initial value*
    - ▷ *for $i := 1$ to b do*
    - ▷ *begin*
    - ▷ *$t := -F1(p_i, -beta, -m, depth - 1)$*
    - ▷ *if $t > m$ then $m := t$ // the returned value is "used"*
    - ▷ *if $m \ge beta$ then return($beta$) // cut off and return the hard bound*
    - ▷ *end*
  - **end**
  - **return m // if nothing over alpha, then alpha is returned**

Properties and comments

- **Properties:**
  - **Assumptions: (1) $alpha < beta$ and (2) p is not a leaf.**
  - $F1(p, alpha, beta, depth) = alpha$ **if** $F(p) \le alpha$
  - $F1(p, alpha, beta, depth) = F(p)$ **if** $alpha < F(p) < beta$
  - $F1(p, alpha, beta, depth) = beta$ **if** $F(p) \ge beta$
  - $F1(p, -\infty, +\infty, depth) = F(p)$
- **Comments:**
  - **$F1(p, alpha, beta, depth)$: find the best possible value according to a nega-max formula for the position p with the constraints that**
    - ▷ *If $F(p) \le alpha$, then $F1(p, alpha, beta, depth)$ returns with the value $alpha$ from a terminal position whose value is $\le alpha$.*
    - ▷ *If $F(p) \ge beta$, then $F1(p, alpha, beta, depth)$ returns the value $beta$ from a terminal position whose value is $\ge beta$.*
  - **The meanings of $alpha$ and $beta$ during searching:**
    - ▷ *For a max node: the current best value is at least $alpha$.*
    - ▷ *For a min node: the current best value is at most $beta$.*
  - **$F1$ always finds a value that is within $alpha$ and $beta$.**
    - ▷ *The bounds are hard, i.e., cannot be violated.*

Original version: Example



- **As long as the value of the leaf node W is less than the current $alpha$ value, the returned value of A will be $alpha$.**
- **If the value of the leaf node W is greater than the current $beta$ value, the returned value of A will be $beta$.**

Alpha-beta pruning algorithm: Fail hard

- **Algorithm $F2'$(position p, value $alpha$, value $beta$) // max node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ *$m := alpha$*
    - ▷ *for $i := 1$ to b do*
    - ▷ *$t := G2'(p_i, m, beta)$*
    - ▷ *if $t > m$ then $m := t$*
    - ▷ *if $m \ge beta$ then return(m) // beta cut off, return m*
  - **end; return m**
- **Algorithm $G2'$(position p, value $alpha$, value $beta$) // min node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ *$m := beta$*
    - ▷ *for $i := 1$ to b do*
    - ▷ *$t := F2'(p_i, alpha, m)$*
    - ▷ *if $t < m$ then $m := t$*
    - ▷ *if $m \le alpha$ then return(m) // alpha cut off, return m*
  - **end; return m**

Alpha-beta pruning algorithm: Fail hard

- **Algorithm $F2$(position p, value $alpha$, value $beta$, integer $depth$)**
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - then return $h(p)$ else
  - begin
    - ▷ *$m := alpha$*
    - ▷ *for $i := 1$ to b do*
    - ▷ *begin*
    - ▷ $t := -F2(p_i, -beta, -m, depth - 1)$
    - ▷ *if $t > m$ then $m := t$*
    - ▷ *if $m \geq beta$ then return(m) // cut off, return m that is $\geq beta$*
    - ▷ *end*
  - end
  - return m

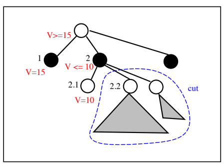
Properties and comments

- **Properties:**
  - Assumptions: (1) $alpha < beta$ and (2) p is not a leaf.
  - $F2(p, alpha, beta) = alpha$ if $F(p) \leq alpha$
  - $F2(p, alpha, beta) = F(p)$ if $alpha < F(p) < beta$
  - $F2(p, alpha, beta) \geq beta$ and $F(p) \geq F2(p, alpha, beta)$ if $F(p) \geq beta$
  - $F2(p, -\infty, +\infty) = F(p)$
- **Comments:**
  - $F2(p, alpha, beta)$: find the best possible value according to a nega-max formula for the position p with the constraints that
    - ▷ *If $F(p) \leq alpha$, then $F2(p, alpha, beta)$ returns with the value $alpha$ from a terminal position whose value is $\leq alpha$.*
    - ▷ *If $F(p) \geq beta$, then $F2(p, alpha, beta)$ returns a value $\geq beta$ from a terminal position whose value is $\geq beta$.*
  - An intermediate version.
    - ▷ *The lower bound is hard, cannot be violated.*
    - ▷ *Easier to find the branch where the returned value is coming from.*
    - ▷ *Always return something better than expected, but never something worse!!*
  - For historical reason [Fishburn 1983][Knuth & Moore 1975], this is called fail hard.

Example

Initial call: $F2'(\text{root}, -\infty, \infty)$

- $m = -\infty$
- call $G2'$(node 1,$-\infty$,∞)
  - ▷ *it is a terminal node*
  - ▷ *return value 15*
- $t = 15$;
  - ▷ *since $t > m$, m is now 15*
- call $G2'$(node 2,15,∞)
  - ▷ *call $F2'$(node 2.1,15,∞)*
  - ▷ *it is a terminal node; return 10*
  - ▷ *$t = 10$; since $t < \infty$, m is now 10*
  - ▷ *$alpha$ is 15, m is 10, so we have an alpha cut off,*
  - ▷ *no need to call $F2'$(node 2.2,15,10)*
  - ▷ *return 10*
  - ▷ $\cdots$



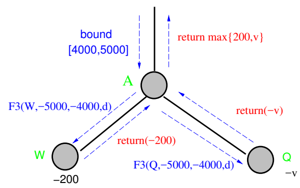
Fail soft version

- **Algorithm $F3$(position p, value $alpha$, value $beta$, integer $depth$)**
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - then return $h(p)$ else
  - begin
    - ▷ *$m := -\infty$ // soft initial value*
    - ▷ *for $i := 1$ to b do*
    - ▷ *begin*
    - ▷ $t := -F3(p_i, -beta, -\max\{m, alpha\}, depth - 1)$
    - ▷ *if $t > m$ then $m := t$ // the returned value is "used"*
    - ▷ *if $m \geq beta$ then return(m) // cut off*
    - ▷ *end*
  - end
  - return m

Properties and comments

- **Properties:**
  - Assumptions (1) $alpha < beta$ and (2) p is not a leaf
  - $F3(p, alpha, beta, depth) \leq alpha$ and $F(p) \leq F3(p, alpha, beta, depth)$ if $F(p) \leq alpha$
  - $F3(p, alpha, beta, depth) = F(p)$ if $alpha < F(p) < beta$
  - $F3(p, alpha, beta, depth) \geq beta$ and $F(p) \geq F3(p, alpha, beta, depth)$ if $F(p) \geq beta$
  - $F3(p, -\infty, +\infty, depth) = F(p)$
- **$F3$ finds a "better" value when the value is out of the search window.**
  - Better means a tighter bound.
    - ▷ *The bounds are soft, i.e., can be violated.*
  - When it is failed-high, $F3$ normally returns a value that is higher than that of $F1$ or $F2$.
    - ▷ *Never higher than that of F!*
  - When it is failed-low, $F3$ normally returns a value that is lower than that of $F1$ or $F2$.
    - ▷ *Never lower than that of F!*

Fail soft version: Example

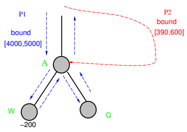


- **Let the value of the leaf node W be u.**
- **If $u < alpha$, then the returned value of A will be at least u.**

Comparisons between $F2$ and $F3$

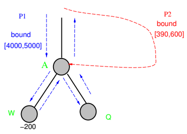
- Both versions find the corrected value v if v is within the window $[alpha, beta]$.
- Both versions scan the same set of nodes during searching.
  - ▷ *If the returned value of a subtree is decided by a cut, then $F2$ and $F3$ return the same value.*
- $F3$ provides more information when the true value is out of the pre-assigned search window.
  - Can provide a feeling on how bad or good the game tree is.
  - Use this "better" value to guide searching later on.
- $F3$ saves about 7% of time than that of $F2$ when a **transposition table** is used to save and re-use searched results [Fishburn 1983].
  - A transposition table is a data structure to record the results of previous searched results.
  - The entries of a transposition table can be efficiently accessed, i.e., read and write, during searching.
  - Need an efficient addressing scheme, e.g., hash, to translate between a position and its address.

$F2$ and $F3$: Example (1/2)



- Assume the node A can be reached from the starting position using path P_1 and path P_2.
  - If W is visited first along P_1 with a bound of $[4000, 5000]$, and returns a value of 200, then
    - ▷ *the returned value of W, 200, is stored into the transposition table.*
  - If A is visited again along P_2 with a bound of $[390, 600]$, then a better value of previously stored value of W helps to decide whether the subtree rooted at W needs to be searched again.

$F2$ and $F3$: Example (2/2)



- Fail soft version has a chance to record a better value to be used later when this position is revisited.
  - If A is visited again along P_2 with a bound of $[390, 600]$, then
    - ▷ *it does not need to be searched again, since the previous stored value of W is -200.*
  - However, if the value of W is 450, then it needs to be searched again.
- The fail hard version does not store the returned value of W after its first visit since this value is less than $alpha$.

Comments

- For historical reason, comparisons are made between $F2$ and $F3$, while we should compare $F1$ and $F3$.
  - To me, $F1$ fails really hard. $F2$ is only an intermediate version!
- What move ordering is good?
  - It may not be good to search the best possible move first.
  - It may be better to cut off a branch with more nodes first.
- How about the case when the tree is not uniform?
- What is the effect of using iterative-deepening alpha-beta cut off?
- How about the case for searching a game graph instead of a game tree?
  - Can some nodes be visited more than once?

References and further readings

* D. E. Knuth and R. W. Moore. An analysis of alpha-beta pruning. *Artificial Intelligence*, 6:293–326, 1975.
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- J. Pearl. The solution for the branching factor of the alpha-beta pruning algorithm and its optimality. *Communications of ACM*, 25(8):559–564, 1982.