Homework 3

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Question 1. Construct a structure $\mathcal{A}$ and a formula φ such that $\mathcal{A} \models \forall x \exists y\, \varphi$, but $\mathcal{A} \not\models \exists x \forall y\, \varphi$.

Solution.

Consider $A = \{\mathtt{T}, \mathtt{F}\}$, $\varphi = x \approx y$.

For every x, there must exist y such that $x \approx y$. Just simply pick x for y.

However, there does not exist x such that for every y, $x \approx y$. If $y = \mathtt{T}$, then x must be $\mathtt{T}$; if $y = \mathtt{F}$, then x must be $\mathtt{F}$.

Question 2. There are many ways to "encode" propositional calculus in first-order logic. One way is to view a propositional variable p as a relation symbol of arity 0 as explained in the class.

Here is another way. Let $L = \{c\}$ be a vocabulary that consists of only one constant symbol c. Given a propositional formula α using variables $p_1, \ldots, p_k$, consider the FO formula Φ obtained from α by changing every p_i with an atomic $x_i \approx c$. For example, if α is

$$p_1 \wedge \neg p_2,$$

then Φ is

$$(x_1 \approx c) \;\wedge\; \neg(x_2 \approx c).$$

Prove that α is satisfiable (in the sense of propositional calculus) if and only if Φ is.

Solution.

Note that Φ and α are actually the same after replacing those $(x_i \approx c)$ with p_i in Φ.

Also note that satisfying assignments and models have the same recursive definition.

$\Rightarrow$:

α is satisfiable. Thus, there exists an assignment w that satisfies α.

Now consider the following interpretation $(\mathcal{A}, \mathsf{val})$.

$A = \{\mathtt{T}, \mathtt{F}\}$, $c = \mathtt{T}$, $\mathsf{val}(x_i) = w(p_i)$.

Then $(\mathcal{A}, \mathsf{val}) \models (x_i \approx c)$ if and only if $w(p_i) = \mathtt{T}$.

Thus, $w \models \alpha \Rightarrow (\mathcal{A}, \mathsf{val}) \models \Phi$, because satisfying assignments and models have the same recursive definition.

$\Leftarrow$:

Φ is satisfiable. Thus, there exists an interpretation $(\mathcal{A}, \mathsf{val})$ satisfying Φ.

Now consider the following assignment w for α.

$w(p_i) = \mathtt{T}$ if and only if $(\mathcal{A}, \mathsf{val}) \models (x_i \approx c)$.

Thus, $(\mathcal{A}, \mathsf{val}) \models \Phi \Rightarrow w \models \alpha$, because satisfying assignments and models have the same recursive definition.

Question 3. Consider a sentence of the following form:

$$\Phi \quad := \quad \exists x_1 \cdots \exists x_n \; \forall y_1 \cdots \forall y_m \; \varphi,$$

where φ is quantifier free and does *not* contain any function and constant symbol, and $n, m \geqslant 1$.

Prove that if Φ is satisfiable, then there is a structure $\mathcal{A}$ that satisfies Φ with $|A| \leqslant n$.

Solution.

Φ is a sentence, so each variable in Φ is one of the x_i or y_i.

Let $\Phi' = \forall y_1 \cdots \forall y_m \; \varphi$, then every model of Φ is a model of Φ', and vice versa.

Assume that Φ' is satisfiable, then there is a interpretation $(\mathcal{A}', \mathsf{val}) \models \Phi'$.

Let $X = \{\mathsf{val}(x_i)\}$. Consider the substructure $\mathcal{A}$ such that $A = A' \cap X$. It is obvious that $|A| \leqslant n$. Now prove that $(\mathcal{A}, \mathsf{val}) \models \Phi'$

$$\begin{aligned}
(\mathcal{A}', \mathsf{val}) \models \Phi' &\Rightarrow \forall\; Y \in A'^m, (\mathcal{A}', \mathsf{val}[\bar{y} \mapsto Y]) \models \phi \\
&\Rightarrow \forall\; Y \in A^m, (\mathcal{A}, \mathsf{val}[\bar{y} \mapsto Y]) \models \phi \text{ (because } A \subseteq A') \\
&\Rightarrow (\mathcal{A}, \mathsf{val}) \models \Phi' \\
&\Rightarrow (\mathcal{A}, \mathsf{val}) \models \Phi
\end{aligned}$$

Question 4. Let $h : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Define a relation $\sim_h$ on A as follows: $a \sim_h a'$ if and only if $h(a) = h(a')$. Prove that $\sim_h$ is an equivalence relation on A, and that $\sim_h$ is, in fact, a congruence in $\mathcal{A}$.

Solution.

- Reflexive

  $\forall a \in A,\; h(a) = h(a)$.

  Thus, $\forall a \in A,\; a \sim_h a$.

- Symmetric

$$\begin{aligned}
\forall a,\; b \in A,\; a \sim_h b &\Rightarrow h(a) = h(b) \\
&\Rightarrow h(b) = h(a) \\
&\Rightarrow b \sim_h a
\end{aligned}$$

- Transitive

$$\begin{aligned}
\forall a,\; b,\; c \in A,\; a \sim_h b,\; b \sim_h c &\Rightarrow h(a) = h(b),\; h(b) = h(c) \\
&\Rightarrow h(a) = h(c) \\
&\Rightarrow a \sim_h c
\end{aligned}$$

- Congruence

$$\begin{aligned}\forall a,\ b \in A,\ a \sim_h b &\Rightarrow h(a) = h(b)\\ &\Rightarrow f(h(a)) = f(h(b))\\ &\Rightarrow h(f(a)) = h(f(b))\\ &\Rightarrow f(a) \sim_h f(b)\end{aligned}$$

Question 5. Let $h : \mathcal{A} \to \mathcal{B}$ be a strong and surjective homomorphism, and let $\mathcal{A}/{\sim_h}$ be the factor of $\mathcal{A}$ modulo $\sim_h$. Define a function $\xi : A/{\sim_h} \to B$, where $\xi([a]_{\sim_h}) = h(a)$. Prove that ξ is an isomorphism.

Solution.
Let $\mathcal{C} = \mathcal{A}/\sim_h$.

- ξ is a homomorphism
  $\forall R,\ f,\ c \in L,$

$$\begin{aligned}forall\bar{a} \in A^{ar(f)}\ \xi(f^{\mathcal{C}}([\bar{a}]_{\sim_h}) &= \xi([f^{\mathcal{A}}(\bar{a})]_{\sim_h})\\ &=\end{aligned}$$