Homework 3

B04902012 Han Sheng, Liu

Question 1. Construct a structure $\mathcal{A}$ and a formula φ such that $\mathcal{A} \models \forall x \exists y\, \varphi$, but $\mathcal{A} \not\models \exists x \forall y\, \varphi$.

Solution.

Consider $A = \{\mathtt{T}, \mathtt{F}\}$, $\varphi = x \approx y$.
For every x, there must exist y such that $x \approx y$. Just simply pick x for y.
However, there does not exist x such that for every y, $x \approx y$. If $y = \mathtt{T}$, then x must be $\mathtt{T}$; if $y = \mathtt{F}$, then x must be $\mathtt{F}$.

Question 2. There are many ways to "encode" propositional calculus in first-order logic. One way is to view a propositional variable p as a relation symbol of arity 0 as explained in the class.

Here is another way. Let $L = \{c\}$ be a vocabulary that consists of only one constant symbol c. Given a propositional formula α using variables $p_1, \ldots, p_k$, consider the FO formula Φ obtained from α by changing every p_i with an atomic $x_i \approx c$. For example, if α is

$$p_1 \wedge \neg p_2,$$

then Φ is

$$(x_1 \approx c) \ \wedge \ \neg(x_2 \approx c).$$

Prove that α is satisfiable (in the sense of propositional calculus) if and only if Φ is.

Solution.

Note that Φ and α are actually the same after replacing those $(x_i \approx c)$ with p_i in Φ.

$\Rightarrow$:

α is satisfiable. Thus, there exists an assignment w that satisfies α.
Now consider the following interpretation $(\mathcal{A}, \mathsf{val})$.
$A = \{\mathtt{T}, \mathtt{F}\}$, $c = \mathtt{T}$, $\mathsf{val}(x_i) = w(p_i)$.
Then $(\mathcal{A}, \mathsf{val}) \models (x_i \approx c)$ if and only if $w(p_i) = \mathtt{T}$.

$$\begin{aligned} w \models \alpha &\Rightarrow w(\alpha) = \mathtt{T} \\ &\Rightarrow (\mathcal{A}, \mathsf{val}) \models \Phi \end{aligned}$$

$\Leftarrow$:

Φ is satisfiable. Thus, there exists an interpretation $(\mathcal{A}, \mathsf{val})$ satisfying Φ.
Now consider the following assignment w for α.
$w(p_i) = \mathtt{T}$ if and only if $(\mathcal{A}, \mathsf{val}) \models (x_i \approx c)$.

Question 3. Consider a sentence of the following form:

$$\Phi \quad := \quad \exists x_1 \cdots \exists x_n \; \forall y_1 \cdots \forall y_m \; \varphi,$$

where φ is quantifier free and does *not* contain any function and constant symbol, and $n, m \geqslant 1$.

Prove that if Φ is satisfiable, then there is a structure $\mathcal{A}$ that satisfies Φ with $|A| \leqslant n$.

Question 4. Let $h : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Define a relation $\sim_h$ on A as follows: $a \sim_h a'$ if and only if $h(a) = h(a')$. Prove that $\sim_h$ is an equivalence relation on A, and that $\sim_h$ is, in fact, a congruence in $\mathcal{A}$.

Question 5. Let $h : \mathcal{A} \to \mathcal{B}$ be a strong and surjective homomorphism, and let $\mathcal{A}/{\sim_h}$ be the factor of $\mathcal{A}$ modulo $\sim_h$. Define a function $\xi : A/{\sim_h} \to B$, where $\xi([a]_{\sim_h}) = h(a)$. Prove that ξ is an isomorphism.