Introduction to Mathematical Logic, Spring 2018

Homework 1

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(1) (a)

p	q	$q \to p$	$p \to (q \to p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

(b)

p	q	r	$p \to (q \to r)$	$(p \to q) \to (p \to r)$	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

(c)

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(d)

p	q	$\neg p \to q$	$p \to (\neg p \to q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

All formulas are tautology.

(2) The formulas are constructed as following:

$$\begin{aligned}\alpha_k = & \bigvee_{1\le i\le n-1} p_{k,i} \\ & \bigwedge_{1\le i<j\le n-1} \neg(p_{k,i}\wedge p_{k,j}) \\ & \bigwedge_{1\le l\le n}\bigwedge_{1\le i\le n-1} \neg(p_{k,i}\wedge p_{l,i}) \\ X = & \{\alpha_k | 1\le k\le n\}\end{aligned}$$

X corresponds to $(n-1)$-colorability of an n-clique G. The construction is a generalized version of the formula set in the proof of *Lemma 3.6*.

X itself is trivially unsatisfiable, since an n-clique is not $(n-1)$-colorable. However, every proper subset of X is satisfiable. Intuitively, a proper subset corresponds to a proper induced subgraph of G, which is an m-clique with $m < n$ and is thus $(n-1)$-colorable.

Here gives an truth assignment for a proper set Y of X. Let $|Y| = m$, $m < n$. W.L.O.G, assume that $Y = \{\alpha_k | 1 \le k \le m\}$. A satisfying assignment is

$$p_{k,i} = \begin{cases} \mathrm{T} & \text{, if } k \le m \text{ and } k = i. \\ \mathrm{F} & \text{, otherwise.} \end{cases}$$

That is, color the vertex i with color i.

(3)

$$\begin{aligned} X \models \alpha & \Rightarrow X \cup \{\neg\alpha\} \text{ is not satisfiable} \\ & \Rightarrow X \cup \{\neg\alpha\} \text{ is not finitely satisfiable} \\ & \Rightarrow \exists \mathrm{X}' \subseteq X \cup \{\neg\alpha\} \text{ s.t. } X' \text{ is finite and non-satisfiable} \\ & \Rightarrow X' \cup \{\neg\alpha\} \text{ is a finite subset of } X \text{ and non-satisfiable} \\ & \Rightarrow X' \setminus \{\neg\alpha\} \models \alpha \end{aligned}$$

i.e. the set $X' \setminus \{\neg\alpha\}$ is the desired X_0.