Homework 2

Due: October 28, 2020 in class

Note: No late homework will be accepted. You may discuss with your classmates but you may not plagiarize. You need to turn in your analysis and also your code (printout) written in Octave or Matlab.

Part A. (20%)

We will consider three finite difference schemes for the first derivative, i.e.

forward difference

$$f'(x_j) = \frac{f(x_{j+1}) - f(x_j)}{h},$$

second-order central difference

$$f'(x_j) = \frac{f(x_{j+1}) - f(x_{j-1})}{2h},$$

fourth-order central difference

$$f'(x_j) = \frac{f(x_{j-2}) - 8f(x_{j-1}) + 8f(x_{j+1}) - f(x_{j+2})}{12h}.$$

A.1 Please use Taylor series to show that these schemes are first-order $O(h)$, second-order $O(h^2)$ and fourth-order $O(h^4)$, respectively. Here we use uniform grid points and h is the spacing between two consecutive grid points, i.e. $h = x_{j+1} - x_j$.

A.2 Now we take the function to be

$$f(x) = \frac{\sin x}{x^3},$$

so that the exact first derivative is known. Please use the above three finite difference schemes to numerically evaluate $f'(x_j)$ at $x_j = 4$ and compute the absolute values of error. You may choose $h = 1$, $h = 0.5$, $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$ and plot the absolute values of error versus grid spacing on a log-log plot.

Part B. (20%)

The fourth-order Padé scheme for the first derivative is

$$f'(x_{j-1}) + 4f'(x_j) + f'(x_{j+1}) = \frac{3}{h}(f(x_{j+1}) - f(x_{j-1})).$$

B.1 Please use Taylor series to show this Padé scheme is fourth-order $O(h^4)$.

B.2 Please derive the modified wavenumber k' for the second-order central difference (given in Part A), fourth-order central difference (given in Part A) and fourth-order Padé scheme for the first derivative. Plot $k'h$ versus kh for the three modified wavenumbers for $0 \le kh \le \pi$, where h is the grid spacing. In this plot, please also include $k'h = kh$, which is the exact wavenumber. (Note that $k = 2\pi n/L$, $n = 0, 1, 2, ..., N/2$ where L is the period and $h = L/N$ is the grid spacing. The grid points are $x_j = hj$, $j = 0, 1, 2, ..., N-1$.)

Part C. (20%)

We will use the fourth-order Padé scheme (given in Part B) to numerically evaluate the first derivative of a known function

$$f(x) = \sin(5x) \quad \text{for} \quad 0 \le x \le 3.$$

Fifteen uniformly spaced points are used here. Please note that $x_0 = 0$ and $x_N = 3$ while $N = 14$. For the left boundary ($x_0 = 0$) and right boundary ($x_N = 3$), use the following schemes

$$f'_0 + 2f'_1 = \frac{1}{h}(-\frac{5}{2}f_0 + 2f_1 + \frac{1}{2}f_2),$$

and

$$f'_N + 2f'_{N-1} = \frac{1}{h}(\frac{5}{2}f_N - 2f_{N-1} - \frac{1}{2}f_{N-2}).$$

C.1 Please use Taylor series to show the above two boundary schemes are third-order $O(h^3)$.

C.2 Using the fourth-order Padé scheme with the boundary schemes, you may derive fifteen equations for the first derivative at fifteen grid points. What are your solutions of $f'(x_j)$ for $j = 0, 1, 2, ..., 14$? Plot your solutions at the fifteen grid points with big dots and also plot the exact first derivative $f'(x) = 5\cos 5x$ as a continuous line for $0 \le x \le 3$.

Part D. (20%)

For a periodic function

$$f(x) = e^{ikx},$$

its second derivative is $-k^2 f$. A finite difference scheme for second derivative of $f(x)$ would lead to $-k'^2 f$, where k'^2 is the 'modified wavenumber' for the second-derivative.

D.1 Derive the 'modified wavenumber' for the central difference formula

$$f''(x_j) = \frac{f(x_{j+1}) - 2f(x_j) + f(x_{j-1})}{h^2}.$$

D.2 Use Taylor series to show that the following Padé scheme

$$\frac{1}{12}f''(x_{j-1}) + \frac{10}{12}f''(x_j) + \frac{1}{12}f''(x_{j+1}) = \frac{f(x_{j+1}) - 2f(x_j) + f(x_{j-1})}{h^2}$$

is fourth-order accurate.

D.3 Derive the 'modified wavenumber' for the Padé scheme given in D.2.

D.4 Plot the 'modified wavenumber' in D.1 and D.3 in terms of ${k'}^2h^2$ versus kh for $0 \leq kh \leq \pi$. In this plot, please also include ${k'}^2h^2 = k^2h^2$, which is the exact 'wavenumber'.

Part E. (20%)

Consider the central finite difference scheme

$$\frac{\delta u_n}{\delta x} = \frac{u_{n+1} - u_{n-1}}{2h}.$$

In calculus we have

$$\frac{duv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

E.1 Does the following finite difference expression hold?

$$\frac{\delta(u_n v_n)}{\delta x} = u_n\frac{\delta v_n}{\delta x} + v_n\frac{\delta u_n}{\delta x}$$

E.2 Please show that

$$\frac{\delta(u_n v_n)}{\delta x} = \bar{u}_n\frac{\delta v_n}{\delta x} + \bar{v}_n\frac{\delta u_n}{\delta x},$$

where

$$\bar{u}_n = \frac{1}{2}(u_{n+1} + u_{n-1}) \quad \text{and} \quad \bar{v}_n = \frac{1}{2}(v_{n+1} + v_{n-1}).$$