REVIEW OF EIGENVALUES PROPERTIES

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Eigenvector Equations

- Also see PRML appendix for a summary of related properties.
- For a square matrix A of size $M \times M$, the eigenvector equation is defined by: $Au_i = \lambda_i u_i$.
- We solve the eigenvalue and eigenvector pairs by $A - \lambda_i I = 0$ having nontrivial solutions.
- The condition is $\det(A - \lambda_i) = 0$, which gives us characteristic polynomials.
- If A is real symmetric square matrix, then we can choose the eigenvectors to be orthonormal: $u_i u_j^T = I_{ij}$, I_{ij} are elements of the identity matrix I.

Eigenvector Equations (Cont'd.)

- Since we have $Au_i = \lambda_i u_i$ for each $i = 1, 2, \ldots, M$.
- $A[u_1\ u_2\ \ \ldots u_M] = [u_1\ u_2\ \ \ldots u_M] \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ & & \ddots & \vdots \\ 0 & & \cdots & \lambda_M \end{bmatrix}$
- Write it as $AU = U\Lambda$.
- Since $U^T U = I$, we have $U^T AU = U^T U\Lambda = \Lambda$.
- Moreover, $\det(U^T U) = \det(U^T)\det(U) = \det(I) = 1$
- ➔ $\det(U) = \frac{1}{\det(U^T)}$
- Thus $\det(U^T AU) = \det(U^T)\det(A)\det(U)$
- $= \det(A) = \det(\Lambda) = \prod_{i=1}^{M} \lambda_i$

Eigenvector Equations (Cont'd.)

- How does the eigenvalue of A and those of $A + gI$ related?

- Eigenvalues of A are roots to $\det(A - \lambda I) = 0$.
- Eigenvalues of $A + gI$ are roots to $\det(A + gI - \lambda I) = 0$
- Since $\det(A + gI - \lambda I) = \det(A - (\lambda - g)I) = 0$. Thus we can take the eigenvalues of A and add g, it will be the eigenvalue of $A + gI$.

Logarithm of Determinant

- Given a real symmetric squared matrix B, define
$$A = \alpha I + \mathrm{B}$$
- Want to know how to computing $\frac{\partial \ln|A|}{\partial \alpha}$.
- Let the eigenvalues of B is $\lambda_1, \lambda_2, \ldots, \lambda_M$.
- The eigenvalues of A is $\lambda_i + \alpha,\ i = 1, 2, \ldots, M$.
- Thus, $\ln|A| = \sum_{i=1}^{M} \ln(\lambda_i + \alpha)$
- $\frac{\partial \ln|A|}{\partial \alpha} = \sum_{i=1}^{M} \frac{1}{\lambda_i + \alpha}$