# UBC CPSC 302 Num Comp for Alg Problems, 2019W

# Assignment 4

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*Tips: Use the tips from Assignment 1 to get the most credit possible on this and future assignments. Replace the [Preferred Name or CS-ID] and [Student Number] above.*

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## Acknowledgments

You are welcome to work with other students on this assignment, but everything in this file must clearly be your own original work. Complete the following section to recognize your collaborators on this assignment.

Begin acknowledgments


Thanks to Q5c1b for explaining me parts of question 2 and 4, and how to approach them.

Other than the contributions above, this work is my own.

— j0k0b

End acknowledgments

# Question 1 Vector norm

For any vector $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_{n-1}, u_n)^T \in \mathbb{R}^n$, let us define the difference vector $\mathbf{v} = (u_2 - u_1, u_3 - u_2, \ldots, u_n - u_{n-1})^T \in \mathbb{R}^{n-1}$. Next, define the function $g(\mathbf{u}) = \|\mathbf{v}\|_2$.

Is the function $g : \mathbb{R}^n \to \mathbb{R}$ a vector norm? Prove that it is or give an example showing that it isn't.

## Answer 1 Vector norm [2 marks]

Enter your solution in the space below.

Begin answer

Let us say that function $g : \mathbb{R}^n \to \mathbb{R}$ a vector norm, then it should satisfy all the properties of a vector norm.

The question also states that for any vector u, such that, $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_{n-1}, u_n)^T \in \mathbb{R}^n$ we can define a difference vector $\mathbf{v} = (u_2 - u_1, u_3 - u_2, \ldots, u_n - u_{n-1})^T \in \mathbb{R}^{n-1}$.

So, let us assume that $\forall u_x \in u$, s.t., $i \in [1, 2, \ldots, n]$, $u_x = \beta$, where $\beta$ is a real-valued constant.

$\therefore u = (\beta, \beta, \ldots, \beta)^T$ is a constant vector now. Now $\mathbf{v} = (u_2 - u_1, u_3 - u_2, \ldots, u_n - u_{n-1})^T$ = $(\beta - \beta, \beta - \beta, \ldots, \beta - \beta)^T$

= $(0, 0, \ldots, 0)$ = $\vec{0}$.

$\therefore$ $\mathrm{g(u)} = ||\mathrm{v}||_2$ = 0.

This fails the first property of vector norms, as $u \neq 0$, but $\mathrm{g(u)} = 0$ & $||\mathrm{v}||_2 = 0$. Hence, the function $g : \mathbb{R}^n \to \mathbb{R}$ is not a vector norm.

End answer

# Question 2

Let $c = \cos(\theta)$ and $s = \sin(\theta)$ for some angle $\theta$. In particular, $c^2 + s^2 = 1$.

# Question 2.1 Reflection matrix

Show that the so-called reflection matrix

$$G = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}$$

is orthogonal.

## Answer 2.1 Reflection matrix [2 marks]

Enter your solution in the space below.

Begin answer

The general formula for inverse of a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is $A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$. We can use the above to our specific problem.

We're given $G = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}$, so $G^{-1} = \frac{1}{c(-c) - ss} \begin{pmatrix} -c & -s \\ -s & c \end{pmatrix} = \frac{1}{-(c^2 + s^2)} \begin{pmatrix} -c & -s \\ -s & c \end{pmatrix} = \frac{1}{(c^2 + s^2)} \begin{pmatrix} c & s \\ s & -c \end{pmatrix}$. Now, we are given that $c^2 + s^2 = 1$, so $G^{-1}$ becomes $\frac{1}{1} \begin{pmatrix} c & s \\ s & -c \end{pmatrix} = G = G^T$. Since, we know the property of an orthogonal matrix, we can say that G, the reflection matrix is an orthogonal matrix as $G = G^{-1} = G^T$.

End answer

## Question 2.2 Orthogonality

Consider the linear system of equations

$$G\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.$$

Assume $\alpha > 0$.

Use orthogonality to express $\alpha$ in terms of only $\mathbf{a} = (a_1, a_2)^T$.

### Answer 2.2 Orthogonality [2 marks]

Enter your solution in the space below.

Begin answer

Since, we know that G is an orthogonal matrix, we can use the property that the vector norm for any orthogonal matrix U and vector x, $||Ux||_2 = ||x||_2$.

$\therefore$ For a given orthogonal matrix **G** and vector **a** = $(a_1, a_2)^T$, we can write $|\alpha| = ||\begin{pmatrix} \alpha \\ 0 \end{pmatrix}||_2 = ||Ga||_2$

Based on out deduction, we can use the property of an orthogonal matrix to say:

$||Ga||_2 = ||a||_2 = \sqrt{a_1^2 + a_2^2}$. $\therefore$ $|\alpha| = \sqrt{a_1^2 + a_2^2}$, and because $\alpha > 0,\ \alpha = |\alpha| = ||a||_2$.

End answer

## Question 2.3 Find *c* and *s*

Next, for a given vector $\mathbf{a}$, find *c* and *s* that satisfy these two linear equations.

### Answer 2.3 Find *c* and *s* [2 marks]

Enter your solution in the space below.

Begin answer

So, for this we need to use the sepcific equations we know from above.

$$||G\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}||_2 = ||\begin{pmatrix} \alpha \\ 0 \end{pmatrix}||_2 \text{ => } ||\begin{pmatrix} c & s \\ s & -c \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}||_2 = \begin{pmatrix} ||a||_2 \\ 0 \end{pmatrix}$$

=>

- $a_1c + a_2s \;=\; ||a||_2$
- $a_1s - a_2c \;=\; 0$

$$\text{=> } \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} ||a||_2 \\ 0 \end{pmatrix}$$

$$\text{=> } \begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}^{-1}\begin{pmatrix} ||a||_2 \\ 0 \end{pmatrix}$$

And, using the formula for inverse, we get

$$\begin{pmatrix} c \\ s \end{pmatrix} = \frac{1}{a_1a_1 - a_2(-a_2)}\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} ||a||_2 \\ 0 \end{pmatrix}$$

$$\text{=> } \begin{pmatrix} c \\ s \end{pmatrix} = \frac{1}{a_1a_1 - a_2(-a_2)}\begin{pmatrix} a_1.||a||_2 \\ a_2.||a||_2 \end{pmatrix}$$

$$\text{=> } \begin{pmatrix} c \\ s \end{pmatrix} = \frac{1}{\sqrt{a_1^2 + a_2^2}}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ [From 2.2]}$$

$$\therefore \begin{pmatrix} c \\ s \end{pmatrix} = \frac{1}{||a||_2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

End answer

## Question 3

For a real $m \times n$ matrix *A*, the *Frobenius norm* is defined by

$$||A||_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}.$$

## Question 3.1 Frobenius norm

Show that the Frobenius norm satisfies the first three properties of a norm (those that are common also to vector norms).

### Answer 3.1 Frobenius norm [2 marks]

Enter your solution in the space below.

Begin answer

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Let us define a $m \times n$ matrix A = $\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}$. We can apply the Frobenious norm on this general matrix to prove the above. Let's see the properties one-by-one, and in detail, below:

1. $||A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2} = \sqrt{a_{1,1}^2 + a_{2,1}^2 + \ldots + a_{m,1}^2 + a_{1,2}^2 + \ldots + a_{m,2}^2 + \ldots + a_{1,n}^2 + \ldots + a_{n,n}^2}$
   $\geq 0$ as it is the summation of positive (squared real numbers). This can only be 0 if $\forall a_{i,j} = 0;\ where\ \ i \in [1,n] \& j \in [1,m]$.
2. $||\beta A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a\beta_{ij}^2} = \sqrt{\beta^2 a_{1,1}^2 + \beta^2 a_{2,1}^2 + \ldots + \beta^2 a_{m,1}^2 + \beta^2 a_{1,2}^2 + \ldots + \beta^2 a_{m,2}^2 + \ldots + \beta^2 a_{1,n}^2 + \ldots + \beta^2 a_{n,n}^2}$
   = $\sqrt{\beta^2 (a_{1,1}^2 + a_{2,1}^2 + \ldots + a_{m,1}^2 + a_{1,2}^2 + \ldots + a_{m,2}^2 + \ldots + a_{1,n}^2 + \ldots + a_{n,n}^2)}$ = $|\beta|.||A||_F$. This shows that the second property of scalar multiplication for the Frobenius norm holds.
3. For this property, let us define a new $m \times n$ matrix B = $\begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m} \\ b_{2,1} & b_{2,2} & \dots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \dots & b_{n,m} \end{pmatrix}$. Now,
   $||A + B||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} + b_{ij})^2} = \sqrt{(a_{1,1} + b_{1,1})^2 + (a_{2,1} + b_{2,1})^2 + \ldots + (a_{n,n} + b_{n,n})^2}$. We already know the property that $\sqrt{(a_{ij} + b_{ij})^2} \leq \sqrt{a_{ij}^2} + \sqrt{b_{ij}^2}$. $\therefore\ ||A + B||_F \leq ||A||_F + ||B||_F$.

It is easy to deduce that we can write a general matrix A as a vector $\omega = (a_{1,1}^2 + a_{2,1}^2 + \ldots + a_{m,1}^2 + a_{1,2}^2 + \ldots + a_{m,2}^2 + \ldots + a_{1,n}^2 + \ldots + a_{n,n}^2)^T$ and then the $||A||_F = ||\omega||_2$. We can see that Frobenius norm satisfies the first three properties of a norm.

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End answer

## Question 3.2 Trace

Show that $||A||_F^2 = \mathrm{trace}(C)$, where $C = A^T A$ and $\mathrm{trace}(C)$ is defined as the sum of the diagonal elements of the matrix *C*.

### Answer 3.2 Trace [2 marks]

Enter your solution in the space below.

Begin answer

We are given $C = A^T A$

$$=> \; C = \begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{n,1} \\ a_{1,2} & a_{2,2} & \dots & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,m} & a_{2,m} & \dots & a_{n,m} \end{pmatrix} . \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}$$

The formula for $trace(C)$ is defined as the sum of the diagonal elements of the matrix ***C***. We can use this information as the following:

$$C[i][j] = \sum_{x=1}^{m} a_{x,i}.a_{x,j}; \; C[i][i] = \sum_{x=1}^{m} a_{x,i}.a_{x,i} = C[i][i] = \sum_{x=1}^{m} a_{x,i}^2$$

$\therefore trace(C) = \sum_{i=1}^{n} \sum_{x=1}^{m} a_{x,i}^2$ = **L.H.S.**

Now $\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{x=1}^{m} a_{xj}^2}$ ; $||A||_F^2 = \sum_{i=1}^{n} \sum_{x=1}^{m} a_{xj}^2$ = **R.H.S**

We can see that **R.H.S = L.H.S**

=> $||A||_F^2 = trace(C)$. hence, proved.

End answer

# Question 3.3 Singular values

Show that

$$\|A\|_F = \sqrt{\sum_{k=1}^{\min(m,n)} \sigma_k^2},$$

where $\sigma_k$ are the singular values of *A*.

## Answer 3.3 Singular values [2 marks]

Enter your solution in the space below.

Begin answer

We know that A is a $m \times n$ real-valued matrix, thus we can define A as $A = U\Sigma V^T$, which is the Singular Value Decomposition of A. Here, U and V are the orthogonal matrices, $\Sigma = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}, \; S = \text{diag}\{\sigma_1, \dots, \sigma_r\}$; $r = min(m, n)$. And, we also know the properties of orthogonal matrices, using which we can state that $U^T U = I$ and because $\Sigma$ is a diagonal matrix $\Sigma = \Sigma^T$. $\therefore$ The Matrix ***C*** as defined in the previous question as $C = A^T A$, can be now stated as $C = (U\Sigma V^T)^T.(U\Sigma V^T) = (V^T)^T\Sigma^T U^T.U.\Sigma.V^T = V\Sigma^T\Sigma V^T = V\Sigma^2 V^T$. And so, finally, we can define any diagonal element (i, i) of C as $c_{i,i} = \sum_{k=1}^{\min(m,n)} v_{i,k}\sigma_k^2 v_{i,k}$. From the previous problem, 3.2, we proved that $||A||_F^2 = trace(C)$, $\therefore ||A||_F^2 = \sum_{i=1}^{n} c_{i,i} = (\sum_{k=1}^{\min(m,n)} \sigma_k^2)(\sum_{i=1}^{n} v_{i,k}^2) = \sum_{k=1}^{\min(m,n)} \sigma_k^2$

=> $\|A\|_F = \sqrt{\sum_{k=1}^{\min(m,n)} \sigma_k^2}$. Hence, proved.

End answer

## Question 4

Let *A* be real and skew-symmetric, i.e., $A^T = -A$, and denote its singular values by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

## Question 4.1 Even/odd

Show that, if *n* is even then $\sigma_{2k} = \sigma_{2k-1} \geq 0$, $k = 1, 2, \ldots, n/2$. If *n* is odd then the same relationship holds up to $k = (n-1)/2$ and also $\sigma_n = 0$.

### Answer 4.1 Even/odd [2 marks]

Enter your solution in the space below.

Begin answer

So,*A*is a real and skew-symmetric matrix, i.e., $A^T = -A$, which also means that $A^T.A = -A.A$ =>$-A^T.A = A^2$. We also know that the singular values of A are $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \geq 0$, where the values are in ascending order, i.e., $\sigma_n$ is the maximal. Let us define the eigenvalues of A as $\lambda_x$,

$\therefore$ The eigenvalues of $A^2 = \lambda_1^2, \lambda_2^2, \lambda_3^2, \ldots, \lambda_n^2 = -\sigma_1^2, -\sigma_2^2, -\sigma_3^2 \ldots, -\sigma_n^2$, deduced from the equation mentioned above.

From this, we can state that $\lambda_x = \sqrt{-\sigma_x^2}$, and that $\lambda_x$ is either imaginary or 0. From the properties of imaginary eigenvalues if $\lambda_{2k-1}$ is an imaginary eigenvalue of A, then $-\lambda_{2k-1} = \lambda_{2k}$ is also an eigenvalue of A.

This is only possible in the case when n is even (i.e., $2k$ & $2k-1$both exists), and so in such cases $\sigma_{2k-1} = \sqrt{-\lambda_{2k-1}^2} = \sqrt{-\lambda_{2k}^2} = \sigma_{2k}$, i.e., $\sigma_{2k} = \sigma_{2k-1} \geq 0$.

When n is odd though, for any $2k-1$ we cannot have $2k$, and so the above relationship would only hold when $k = \frac{n-1}{2}$. And in this case, $\exists x \in n,\ s.t.\ \lambda_x = 0$, and so $\sigma_n = 0$.

End answer

## Question 4.2 Eigenvalues

Show that the eigenvalues $\lambda_j$ of *A* can be written as

$$\lambda_j = (-1)^j i\sigma_j, \;\; j = 1, \ldots, n,$$

where *i* is the imaginary unit, $i^2 = -1$.

### Answer 4.2 Eigenvalues [2 marks]

Enter your solution in the space below.

Begin answer

As seen above, any eigenvalue of A, $\lambda_j$ can be computed as $\lambda_j = \sqrt{-\sigma_j^2}$.

$\therefore \lambda_j^2 = -\sigma_j^2 \;\; => \dfrac{\lambda_j^2}{\sigma_j^2} = -1 \;\; => \dfrac{\lambda_j}{\sigma_j} = \pm\, i \;\; => \lambda_j = (-1)^j i\sigma_j$. Here $j = 1, \ldots, n$, and *i* is the imaginary unit, $i^2 = -1$.

End answer

## 5 My functions

MATLAB requires all functions to be at the end of the script. Enter functions you wrote for this assignment in the space below.

*Tips: You only need to complete this section if you developed and used your own MATLAB function(s) in your code above. If you do define your own functions, organize your code; use indenting and reasonable variables (e.g. no unused variables or ambiguous names). Document your code clearly and effectively with comments (`% This is a comment`). Note the specific purpose of each function; indicate input requirements and output results.*

Begin functions

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End functions

## 6 Provided functions

Additional utility functions are provided here. You should not need to modify these functions and may use them "as-is" in your code.

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