# The Ring Star Problem: Polyhedral Analysis and Exact Algorithm

Martine Labbé*
Gilbert Laporte †
Inmaculada Rodríguez Martín‡
Juan José Salazar González‡

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**Abstract**

In the Ring Star Problem the aim is to locate a simple cycle through a subset of vertices of a graph with the objective of minimizing the sum of two costs: a routing cost proportional to the length of the cycle, and an assignment cost from the vertices not in the cycle to their closest vertex on the cycle. The problem has several applications in telecommunications network design and in rapid transit systems planning. It is an extension of the classical location-allocation problem introduced in the early sixties, and closely related versions have been recently studied by several authors. This article formulates the problem as a mixed integer linear program, and strengthens it with the introduction of several families of valid inequalities. These inequalities are shown to be facet defining and are used to develop a branch-and-cut algorithm. Computational results show that instances involving up to 300 vertices can be solved optimally using the proposed methodology.

*ISRO and SMG, Université Libre de Bruxelles, C.P. 210/01, B-1050,Bruxelles, Belgium.

†Canada Research Chair in Distribution Management, École des Hautes Études Commerciales, 3000 chemin de la Côte-Sainte-Catherine, Montréal, Canada H3T 2A7.

‡DEIOC, Facultad de Matemáticas, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain.

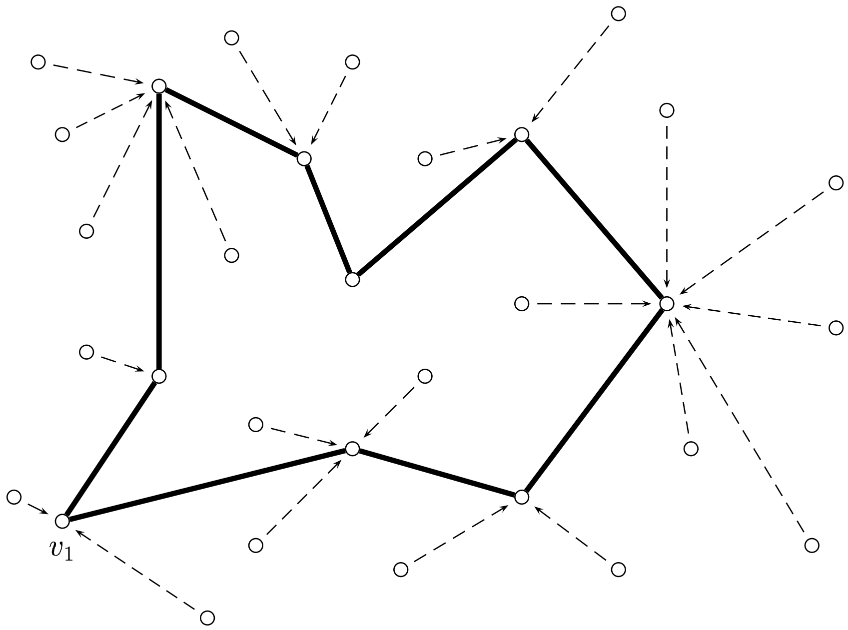


Figure 1: Ring-star solution

# 1 Introduction

In a typical telecommunications network, the traffic is gathered from many sources, progressively combined in order to fill links of increasing capacity and finally forwarded to its destination. Hence, most telecommunications networks are naturally structured in a multi-layer hierarchical structure.

Although real telecommunications (voice, video or data, nationwide or international networks) are usually structured into many such levels, most design problems considered by telecommunications companies concern only a part of the overall network. A generic telecommunications network consists of access networks which connect the terminals (user nodes) to concentrators (switches or multiplexers) and a backbone network which interconnects these concentrators or connects them to a central unit (root). Telecommunications networks differ in the designs of these access networks and the backbone network, see, e.g., Gourdin, Labbé and Yaman [7] or Klincewicz [10].

A solution to the design of some telecommunications networks is to connect terminals to concentrators by point-to-point links, which results in a star topology, and to interconnect the concentrators through a ring structure. This is the case in Digital Data Service (DDS) design, as mentioned in Xu, Chiu and Glover [17], for example. Roughly speaking, the problem then consists of selecting a subset of user nodes where concentrators will be installed, interconnect them by a ring network and assign the other user nodes to those concentrators. The total cost of all connections must be minimized. Figure 1 illustrates a potential solution, where the solid lines represent the *internet* (ring) and the dashed lines represent the *intranet* (assignments). This problem was introduced in Moreno Pérez, Moreno Vega and Rodríguez Martín [13] and solved heuristically.

Similar location-allocation problems arise in a number of planning contexts (see Cooper [3]). They consist of locating a number of facilities among the vertices of a graph so that the allocation costs (i.e., the sum of distances from the remaining vertices to their closest facility) is minimized. In some applications, as in rapid transit systems planning, the facilities must be routed by a vehicle. Thus, in addition to the allocation cost, the objective includes the cost of the shortest Hamiltonian cycle visiting all the facilities. See Labbé, Laporte and Rodríguez Martín [11] for a review of applications and a more general class of problems consisting of locating structures in a graph.

In this article we consider a problem common to both contexts, telecommunications and location-allocation, named *Ring Star Problem* (RSP). The problem can be formally stated as follows. Consider a mixed graph $G = (V, E \cup A)$ where $V = \{v_1, v_2, \ldots, v_n\}$ is the vertex set, $E = \{[v_i, v_j] : v_i, v_j \in V, i < j\}$ is the edge set, and $A = \{(v_i, v_j) : v_i, v_j \in V\}$ is the arc set (loops $(v_i, v_i)$ are included in $A$). We assume that $n \geq 5$ to eliminate degenerate and trivial instances. Vertex $v_1$ is referred to as the *root* (or *depot*). There is a non-negative *routing cost* $c_{ij}$ associated with each edge $[v_i, v_j]$, and a non-negative *assignment cost* $d_{ij}$ associated with each arc $(v_i, v_j)$. A solution to the RSP is a simple cycle through a subset $V'$ of $V$ including $v_1$ and at least two other vertices. The routing cost of a solution is the sum of the routing costs of all edges on the cycle. The assignment cost of a solution is defined as

$$\sum_{v_i \in V \setminus V'} \min_{v_j \in V'} d_{ij}. \tag{1}$$

The RSP consists of determining a solution for which the sum of routing and assignment costs is minimized. The problem is $\mathcal{NP}$-hard since the special case in which the assignment costs are very large compared to the routing cost is the classical *Traveling Salesman Problem* (TSP).

Lee, Chiu and Sanchez [12] defined a very closely related problem (referred to as *Steiner Ring Star Problem*) by considering an additional set of vertices $W$ (representing *terminals* in telecommunications and *customers* in location-allocation) and setting the assignment cost to

$$\sum_{v_i \in W} \min_{v_j \in V'} d_{ij}. \tag{2}$$

They developed a branch-and-cut that solved instances with $|V| \leq 50$, $|W| \leq 90$ and $|V| + |W| \leq 100$. Xu, Chiu and Glover [17] proposed a tabu search algorithm for this problem, and tested it on instances with $|V| \leq 300$ and $|W| \leq 300$. In addition, Current and Schilling [4] and Gendreau, Laporte and Semet [6] presented algorithms for variants of the RSP in which the routing cost is minimized subject to an upper bound on the cost of the largest assignment.

Our aim is to develop a polyhedral based exact algorithm for the RSP. The remainder of this article is organized as follows. In Section 2, the RSP is formulated as a mixed integer linear program to which several classes of valid inequalities are incorporated. In Section 3 we derive dimension and facet defining results for RSP. A branch-and-cut algorithm is described in Section 4. Section 5 presents extensive computational results on three classes of instances, one of them generated as to be able to compare our results with the ones in Lee, Chiu and Sanchez [12]. Conclusions follow in Section 6.

# 2 Mathematical Model

The RSP can be formulated as a mixed integer linear program as follows. For each edge $[v_i, v_j] \in E$, let $x_{ij}$ be a binary variable equal to 1 if and only if edge $[v_i, v_j]$ appears on the cycle. For each arc $(v_i, v_j) \in A$, let $y_{ij}$ be a binary variable equal to 1 if and only if vertex $v_i$ is assigned to vertex $v_j$ on the cycle. If a vertex $v_i$ is on the cycle, it is then assigned to itself, i.e., $y_{ii} = 1$. In addition, for $S \subset V$ define $E(S) := \{[v_i, v_j] \in E : v_i, v_j \in S\}$ and $\delta(S) := \{[v_i, v_j] \in E : v_i \in S,\ v_j \notin S\}$. If $S = \{v_i\}$, we simply write $\delta(i)$ instead of $\delta(\{v_i\})$. For $E' \subseteq E$, define $x(E') := \sum_{[v_i,v_j]\in E'} x_{ij}$.

The formulation is then:

$$\text{minimize} \sum_{[v_i,v_j]\in E} c_{ij}x_{ij} + \sum_{(v_i,v_j)\in A} d_{ij}y_{ij} \tag{3}$$

subject to:

$$\begin{aligned}
x(\delta(i)) &= 2y_{ii} && \text{for all } v_i \in V, && (4)\\
\sum_{v_j\in V} y_{ij} &= 1 && \text{for all } v_i \in V \setminus \{v_1\}, && (5)\\
x(\delta(S)) &\geq 2\sum_{v_j\in S} y_{ij} && \text{for all } S \subset V : v_1 \notin S, && (6)\\
x_{ij} &\in \{0,1\} && \text{for all } [v_i, v_j] \in E, && (7)\\
y_{ij} &\geq 0 && \text{for all } (v_i, v_j) \in A, && (8)\\
y_{11} &= 1 && && (9)\\
y_{1j} &= 0 && \text{for all } v_j \in V \setminus \{v_1\}, && (10)\\
y_{jj} &\text{ integer} && \text{for all } v_j \in V \setminus \{v_1\}. && (11)
\end{aligned}$$

In this formulation, constraints (4) are degree constraints. They ensure that the degree of a vertex $v_i$ is 2 if and only if it belongs to the cycle (i.e., $y_{ii} = 1$). Constraints (5) mean that either $v_i$ is a vertex on the cycle (i.e., $y_{ii} = 1$), or $v_i$ is assigned to a vertex

$v_j$ on the cycle (i.e., $y_{ii} = 0$ and $y_{ij} = 1$). Constraints (6) are connectivity constraints since they state that $S$ must be connected to its complement by at least two edges of the cycle whenever at least one vertex $v_i$ is assigned to $v_j \in S$. The combination of (4), (7), (9) and (10) guarantees that the solution will contain at least one cycle including the root. Constraints (6) limit the number of cycles to one, and the combination of (4), (5), (6), and (8) means that every vertex not belonging to the cycle is assigned to a vertex on the cycle. Integrality conditions on the $y_{ij}$ variables ($v_i \neq v_j$) are unnecessary since, for a given integer $x$, the objective is minimized when vertices not on the cycle are assigned to the closest vertices on the cycle. Integer solutions are trivially determined in case of ties. Implicitly the above model imposes that the cycle visits at least three vertices, including the root. Such a constraint guarantees that communication between vertices remains possible even if one vertex of the backbone network (e.g., the root) fails. Further, if reliability is unimportant, other potential solutions consisting of a degenerate cycle (i.e., only the root or only the root and another vertex) can easily be enumerated.

The linear relaxation of the model (3)-(11) can be strengthened through the introduction of additional valid inequalities. First observe that connectivity constraints (6) are very rich and reduce to the constraints

$$x_{ij} + y_{ij} \leq y_{jj} \tag{12}$$

whenever $S = \{v_i, v_j\} \subseteq V \setminus \{v_1\}$. Constraints (12) themselves dominate the classical inequalities:

$$x_{ij} \leq y_{ii} \text{ and } x_{ij} \leq y_{jj} \qquad \text{for all } v_i, v_j \in V \setminus \{v_1\}.$$

Furthermore, new useful constraints similar to (12) are

$$x_{1j} \leq y_{jj} \text{ and } x_{i1} + y_{i1} \leq 1. \tag{13}$$

Inequalities valid for the cycle polytope (Bauer [2]) are also clearly valid for the RSP. We restrict our attention to the following *2-matching inequalities*:

$$x(E(H)) + x(T) \leq \sum_{v_i \in H} y_{ii} + \frac{|T| - 1}{2} \tag{14}$$

for all $H \subset V$ and $T \subset \delta(H)$ satisfying:

(i) $\{v_i, v_j\} \cap \{v_k, v_\ell\} = \emptyset$ for $[v_i, v_j], [v_k, v_\ell] \in T$ and $[v_i, v_j] \neq [v_k, v_\ell]$,

(ii) $|T| \geq 3$ and odd.

An additional family of valid inequalities follows from the fact that some assignments are incompatible. More specifically, variables $y_{ij}$ and $y_{ik}$ are incompatible when $j \neq k$, thus the *Stable Set Problem* (SSP) associated to those incompatibility relations between the assignment variables is a relaxation of RSP. A similar observation has been made by

Avella and Sassano [1] for the p-Median Problem. This property leads (for example) to the following odd-hole (or clique) inequalities:

$$y_{ij} + y_{jk} + y_{ki} \leq 1 \tag{15}$$

for all three different vertices $v_i, v_j, v_k \in V \backslash \{v_1\}$. See Padberg [14] for a partial polyhedral description of the stable set polytope.

Finally, the combination of the two mentioned relaxations (i.e., the cycle and the stable set problems) produces new valid inequalities for RSP, such as:

$$x(\delta(S)) \geq 2(y_{ij} + y_{jk} + y_{ki}) \tag{16}$$

for all three different vertices $v_i, v_j, v_k \in V \setminus \{v_1\}$ and all $S \subseteq V \setminus \{v_1\}$ such that $v_i, v_j, v_k \in S$.

In next section we study conditions under which these inequalities are facet-defining for the RSP polytope.

# 3 Polyhedral Analysis

We now derive some polyhedral results for model (3)–(11). Denote by $P$ the polytope defined by the convex hull of feasible solutions to RSP and by $Q$ the associated TSP polytope, i.e.,

$$P := \text{conv}\left\{(x,y) \in \mathbb{R}^{|E|+|A|} : (x,y) \text{ satisfies } (4)\text{–}(11)\right\},$$

$$Q := \{(x,y) \in P : y_{ii} = 1 \text{ for all } v_i \in V \setminus \{v_1\}\}.$$

Both polytopes are linked by the intermediate polytopes:

$$P(F) := \{(x,y) \in P : y_{ii} = 1 \text{ for all } v_i \notin F \cup \{v_1\}\},$$

for all $F \subseteq V \setminus \{v_1\}$, in the sense that $P(\emptyset) = Q$ and $P(V \setminus \{v_1\}) = P$. This connection allows us to extend known results from $Q$ to $P$. For any $\alpha \in \mathbb{R}^{|E|}, \beta \in \mathbb{R}^{|A|}$ and $\gamma \in \mathbb{R}$, define the hyperplane

$$\mathcal{H}(\alpha, \beta, \gamma) := \{(x,y) \in \mathbb{R}^{|E|+|A|} : \alpha x + \beta y = \gamma\}.$$

**Lemma 3.1** *Let $v_{i_1}, v_{i_2}, ..., v_{i_{n-1}}$ be an ordering of the vertices of $V \setminus \{v_1\}$ and let $F_k = \{v_{i_1}, ..., v_{i_k}\}$ for all $k = 1, \ldots, n-1$. If for each $k = 1, ..., n-1$ and each $v_{i_t} \in V \setminus \{v_{i_k}\}$ there exists a feasible solution $(x,y)$ to the RSP such that*

*(i) $y_{i_l i_l} = 1$ for all $l > k$ (i.e., all vertices of $V \backslash F_k$ belong to the cycle),*

*(ii)* $y_{i_l i_l} + y_{i_l 1} = 1$ *for all* $l < k$ *(i.e., all vertices of* $F_k \backslash \{v_{i_k}\}$ *are either on the cycle or assigned to the root* $v_1$*),*

*(iii)* $y_{i_k i_t} = 1$ *(i.e., vertex* $v_{i_k}$ *does not belong to the cycle and is assigned to vertex* $v_{i_t}$*), and*

*(iv)* $\alpha x + \beta y = \gamma$*,*

*then* $\dim(\mathcal{H}(\alpha, \beta, \gamma) \cap P) \geq \dim(\mathcal{H}(\alpha, \beta, \gamma) \cap Q) + (|V| - 1)^2$.

**Proof.** We will prove by induction on $|F_k|$ that $\dim(\mathcal{H}(\alpha, \beta, \gamma) \cap P(F_k)) \geq \dim(\mathcal{H}(\alpha, \beta, \gamma) \cap Q) + |F_k|(|V| - 1)$. If $k = 0$, i.e., if $F_k = \emptyset$, then this inequality holds since $P(F_k) = Q$. Now, by the induction hypothesis, there exist $\dim(\mathcal{H}(\alpha, \beta, \gamma) \cap Q) + |F_{k-1}|(|V| - 1) + 1$ affinely independent points such that $y_{j_k j_k} = 1$ in $\mathcal{H}(\alpha, \beta, \gamma) \cap P(F_{k-1})$ and thus in $\mathcal{H}(\alpha, \beta, \gamma) \cap P(F_k)$. An additional set of $|V| - 1$ affinely independent points such that $y_{j_k j_k} = 0$ in $\mathcal{H}(\alpha, \beta, \gamma) \cap P(F_k)$ are obtained as in the lemma's hypothesis. $\square$

**Proposition 3.1** $\dim(P) = |E| + |V|^2 - 3|V| + 1$.

**Proof.** Recall that we are assuming $|V| \geq 5$. Then

$$\begin{aligned} \dim(P) &= \dim(\mathcal{H}(0,0,0) \cap P) \\ &\geq \dim(\mathcal{H}(0,0,0) \cap Q) + (|V| - 1)^2 && \text{(by Lemma 3.1 using any sequence)} \\ &= \dim(Q) + (|V| - 1)^2 \\ &= |E| - |V| + (|V| - 1)^2 && \text{(see Grötschel and Padberg [8]).} \end{aligned}$$

Now, $\dim(P) \leq |E| + |V|^2 - 3|V| + 1$ since the RSP can be described with $|E| + |V|^2$ variables and $|V|$ type (4), $|V| - 1$ type (5), and $|V|$ type (9)–(10) linearly independent equality constraints. $\square$

Another important consequence of Lemma 3.1 is that any facet-defining inequality $\alpha x + \beta y \leq \gamma$ for $Q$ satisfying (i)–(iv) is also facet-defining for $P$. We will use this result for the polyhedral analysis of $P$.

**Proposition 3.2** *The inequality* $x_{ij} \geq 0$ *defines a facet of* $P$ *for each* $[v_i, v_j] \in E$*.*

**Proof.** The hyperplane defined by $x_{ij} = 0$ satisfies the assumptions of Lemma 3.1 for any sequence. Indeed for any vertex $v_k \in V \setminus \{v_1\}$, a simple cycle spanning $V \setminus \{v_k\}$ and not containing edge $[v_i, v_j]$, together with the assignment of $v_k$ to any vertex of the cycle, yields an $(x, y)$ vector of $P$ such that $x_{ij} = 0$. Hence, $\dim(\{(x, y) \in P : x_{ij} = 0\}) \geq \dim(\{(x, y) \in Q : x_{ij} = 0\}) + (|V| - 1)^2 = (|E| - |V| - 1) + (|V| - 1)^2$, since $x_{ij} \geq 0$ induces a facet of $Q$ (see Grötschel and Padberg [8]). The thesis then follows from Proposition 3.2. $\square$

Note at this stage that the inequality $x_{ij} \leq 1$ is not facet inducing since it is dominated by the connectivity constraint (12).

**Proposition 3.3** *The inequality $y_{ij} \geq 0$ defines a facet of $P$ for each $(v_i, v_j) \in A$, $i \neq j$ and $i \neq 1$.*

**Proof.** The proof is obtained along the lines of the proof of Lemma 3.1 except that $F$ is now initialized as $F = \{v_i\}$. Since in this case $Q \subset P(F)$, there exist $\dim(Q) + 1 = |E| - |V| + 1$ affinely independent vectors (Hamiltonian cycles) such that $y_{ij} = 0$. An additional $|V| - 2$ affinely independent vectors are obtained by taking a simple cycle spanning $V \setminus \{v_i\}$ and assigning $v_i$ to each vertex of the cycle different from $v_j$ (so $y_{ij} = 0$), which proves that $y_{ij} \geq 0$ defines a facet of $P(F)$ for $F = \{v_i\}$. Moreover, for all $v_k \in V \setminus \{v_1, v_i\}$, any simple cycle spanning $V \setminus \{v_k\}$ and the successive assignment of $v_k$ to the vertices of the cycle yield $(x, y)$ vectors of $P$ satisfying conditions (i)–(iv) of Lemma 3.1. □

If $i = j$, the inequality $y_{ij} \geq 0$ is not facet inducing since it is implied by equalities (4) and $x_{ij} \geq 0$ for $(v_i, v_j) \in F$. If $i \neq j$ and $i = 1$, then $y_{ij} = 0$ since $y_{11} = 1$, and therefore $y_{ij} \geq 0$ is not facet inducing. The inequality $y_{ij} \leq 1$ is not facet inducing since it is implied by constraints (5) and (8).

**Proposition 3.4** *The valid inequality $x(\delta(S)) \geq 2\sum_{v_j \in S} y_{ij}$ defines a facet of $P$ for each $S \subseteq V \setminus \{v_1\}$, $2 \leq |S| \leq |V| - 3$, $v_i \in S$.*

**Proof.** The hyperplane defined by $x(\delta(S)) = 2\sum_{v_j \in S} y_{ij}$ satisfies the assumptions of Lemma 3.1 for any sequence containing $v_i$ in last position. Indeed, for any vertex $v_k \in V \setminus \{v_1, v_i\}$ there exists a simple cycle spanning $V \setminus \{v_k\}$ and with two edges in $\delta(S)$ such that, together with the assignment of $v_k$ to each vertex of the cycle, yields $(x, y)$ vectors of $P$ satisfying $x(\delta(S)) = 2\sum_{v_j \in S} y_{ij} (= 2)$. Furthermore, for $v_i$, the last vertex of the sequence, a simple cycle spanning $(S \cup \{v_1\} \setminus \{v_i\})$ with two edges in $\delta(S)$ can be constructed. The assignment of $v_i$ to each vertex of $S \setminus \{v_i\}$ yields $|S| - 1$ solutions $(x, y)$ satisfying the hypothesis (i) to (iv) of Lemma 3.1. An additional $|V \setminus S|$ such solutions $(x, y)$ are obtained by constructing a simple cycle spanning $V \setminus S$ (which contains at least three vertices), and successively assigning $v_i$ to each vertex of $V \setminus S$. The thesis follows by noting that in $Q$, the inequality $x(\delta(S)) \geq 2\sum_{v_j \in S} y_{ij}$ reduces to $x(\delta(S)) \geq 2$, which is facet defining (Grötschel and Padberg [8]). □

Proposition 3.4 does not hold if $|S| = 1$ or $|S| = |V| - 2$ since constraints (6) are then dominated by (4) or $x(\delta(S)) \geq 2$, respectively. The validity of the latter inequality comes from the fact that the cycle must contain at least one vertex from $S$ whenever $S \subset V \setminus \{v_1\}$ and $|S| \geq |V| - 2$. Proposition 3.4 is also invalid if $v_i \notin S$ since then

$$x(\delta(S \cup \{v_i\})) = x(\delta(S)) + x(\delta(i)) - 2\sum_{v_j \in S} x_{ij} \geq 2 \sum_{v_j \in S \cup \{v_i\}} y_{ij} = 2(y_{ii} + \sum_{v_j \in S} y_{ij}).$$

This implies that $x(\delta(S)) \geq 2\sum_{v_j \in S}(y_{ij} + x_{ij})$, which dominates the correspondent constraint (6).

**Proposition 3.5** *The inequality $x_{1j} \le y_{jj}$ defines a facet of $P$ for $j \neq 1$.*

**Proof.** The hyperplane defined by $x_{1j} = y_{jj}$ satisfies the assumptions of Lemma 3.1. This can be seen by taking a simple cycle spanning $V \setminus \{v_k\}$ and successively assigning $v_k$ to each vertex of the cycle. Furthermore, this cycle must contain edge $[v_1, v_j]$ whenever $v_k \neq v_j$. The thesis follows from the fact that $x_{1j} \le y_{jj} = 1$ induces a facet of $Q$ (Grötschel and Padberg [8]). $\square$

The constraints $x_{i1} + y_{i1} \le 1$ for all $i \neq 1$ do not induce facets of $P$ since these are dominated by the constraints $x_{1i} \le y_{ii}$. Indeed, $x_{1i} \le y_{ii} = 1 - \sum_{v_j \in \setminus\{v_i\}} y_{ij} \le 1 - y_{i1}$.

**Proposition 3.6** *The inequality $x(E(H)) + x(T) \le \sum_{v_i \in H} y_{ii} + (|T| - 1)/2$ for each $H \subset V$ and $T \subset \delta(H)$ satisfying:*

*(i) $\{v_i, v_j\} \cap \{v_k, v_\ell\} = \emptyset$ for $[v_i, v_j], [v_k, v_\ell] \in T$ and $[v_i, v_j] \neq [v_k, v_\ell]$,*

*(ii) $|T| \ge 3$ and odd,*

*defines a facet of $P$ when $|V| \ge 6$.*

**Proof.** The hyperplane defined by $x(E(H)) + x(T) \le \sum_{v_i \in H} y_{ii} + (|T| - 1)/2$ satisfies the assumptions of Lemma 3.1. Indeed, for each vertex $v_k \in V \setminus \{v_1\}$ it is trivial to design a simple cycle spanning $V \setminus \{v_k\}$ and satisfying hypothesis (i)–(iv) of Lemma 3.1, and $v_k$ can be assigned to each vertex of the cycle. The thesis follows from the fact that $x(E(H)) + x(T) \le \sum_{v_i \in H} y_{ii} + (|T| - 1)/2$ defines a facet of $Q$ (Grötschel and Padberg [8]). $\square$

We now turn to the two inequalities (15) and (16) related to the SSP relaxation.

**Proposition 3.7** *The inequality $y_{ij} + y_{jk} + y_{ki} \le 1$ defines a facet of $P$ for each three different vertices $v_i, v_j, v_k \in V \setminus \{v_1\}$.*

This proof is obtained using induction on the number $n := |V|$ of vertices. By using the software PORTA [15] we were able to enumerate all facet defining inequalities for $n = 5$ (see `http://webpages.ull.es/users/jjsalaza/ring`). This allows us to state that the result holds true for $n = 5$.

Assume now that (15) is facet defining for polytope $P$ associated to a mixed graph $G = (V, E \cup A)$ with $V = \{v_1, \dots, v_n\}$, $n \ge 5$. Let $v_{n+1} \notin V$ and define $P'$ as the polytope associated to $G' = (V', E' \cup A')$ where $V' = V \cup \{v_{n+1}\}$, $E' = E \cup \{[v_{n+1}, v_i] : i = 1, \dots, n\}$ and $A' = A \cup \{(v_{n+1}, v_i), (v_i, v_{n+1}) : i = 1, \dots, n\}$. First, consider $\dim(P)$ affinely independent vectors of $P$ satisfying (15) at equality. These vectors can be transformed into vectors of $P'$ by adding arc $(v_{n+1}, v_1)$ to the corresponding RSP solution graphs. Further, those $\dim(P)$ new vectors are still affinely independent and tight for (15). To complete the proof, we need to exhibit $\dim(P') - \dim(P) = 3n - 2$ affinely independent points of $P'$, tight for (15) and such that $y_{n+1,1} = 0$. Each mixed graph of Figure 2

| points | $y_{n+1,1}$ | $y_{n+1,n+1}$ | $y_{n+1,l} : l \neq 1, n+1$ | $y_{l,n+1} : l \neq n+1$ | $x_e : e \in \delta(n+1)$ |
|---|---|---|---|---|---|
| $P$ | $\begin{matrix}1\\ \vdots\\ 1\end{matrix}$ | 0 | 0 | 0 | 0 |
| $(a), (b)$ | $\begin{matrix}0\\ \vdots\\ 0\end{matrix}$ | $\begin{matrix}1\\ \vdots\\ 1\end{matrix}$ | 0 | 0 | $\begin{matrix}1 & 1 & 0 & 0 & \cdots & 0\\ \vdots & 0 & 1 & 0 & \cdots & 0\\ \vdots & 0 & 0 & 1 & & \vdots\\ \vdots & \vdots & & & \ddots & 0\\ 1 & 0 & 0 & \cdots & 0 & 1\end{matrix}$ |
| $(g)$ | 0 | 1 | 0 | 0 | $\begin{matrix}0 & 1 & 1 & 0 & \cdots & 0 & 0\end{matrix}$ |
| $(c), (d)$ | $\begin{matrix}0\\ \vdots\\ 0\end{matrix}$ | $\begin{matrix}1\\ \vdots\\ 1\end{matrix}$ | 0 | $\begin{matrix}1 & 0 & \cdots & 0\\ 0 & 1 & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & 1\end{matrix}$ | $\cdots$ |
| $(e), (f)$ | $\begin{matrix}0\\ \vdots\\ 0\end{matrix}$ | $\begin{matrix}0\\ \vdots\\ 0\end{matrix}$ | $\begin{matrix}1 & 0 & \cdots & 0\\ 0 & 1 & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & 1\end{matrix}$ | 0 | 0 |

Table 1: Affinely independence certificate for Proposition 3.7

(a), (c) and (e) provides three such points (obtained by permuting $v_i$, $v_j$ and $v_k$). This gives nine points. Next, each mixed graph of Figures 2 (b), (d) and (f) provides $n-4$ such points (one for each $l \neq 1, i, j, k$). Finally, Figure 2 (g) gives the last point. To see that all these $3n-2$ points are affinely independent, it suffices to check that the new components of their characteristic vectors, i.e. corresponding to the addition of $v_{n+1}$ to the graph, form a non-singular matrix (see Table 1). $\square$

If one vertex, say $v_k$, coincides with the root $v_1$, Proposition 3.7 does not hold since then (15) is dominated by (12). Indeed, in this case

$$y_{ij} + y_{j1} + y_{1i} = y_{ij} + y_{j1} \leq y_{jj} + y_{j1} \leq \sum_{v_l \in V} y_{jl} = 1.$$

**Proposition 3.8** *The inequality $x(\delta(S)) \geq 2(y_{ij} + y_{jk} + y_{ki})$ defines a facet of $P$ when $|V| \geq 7$ for each three different vertices $v_i, v_j, v_k \in V \setminus \{v_1\}$ and each $S \subseteq V \setminus \{v_1\}$ such that $v_i, v_j, v_k \in S$ and $4 \leq |S| \leq |V| - 3$.*

**Proof.** The proof is similar to that of Proposition 3.7.

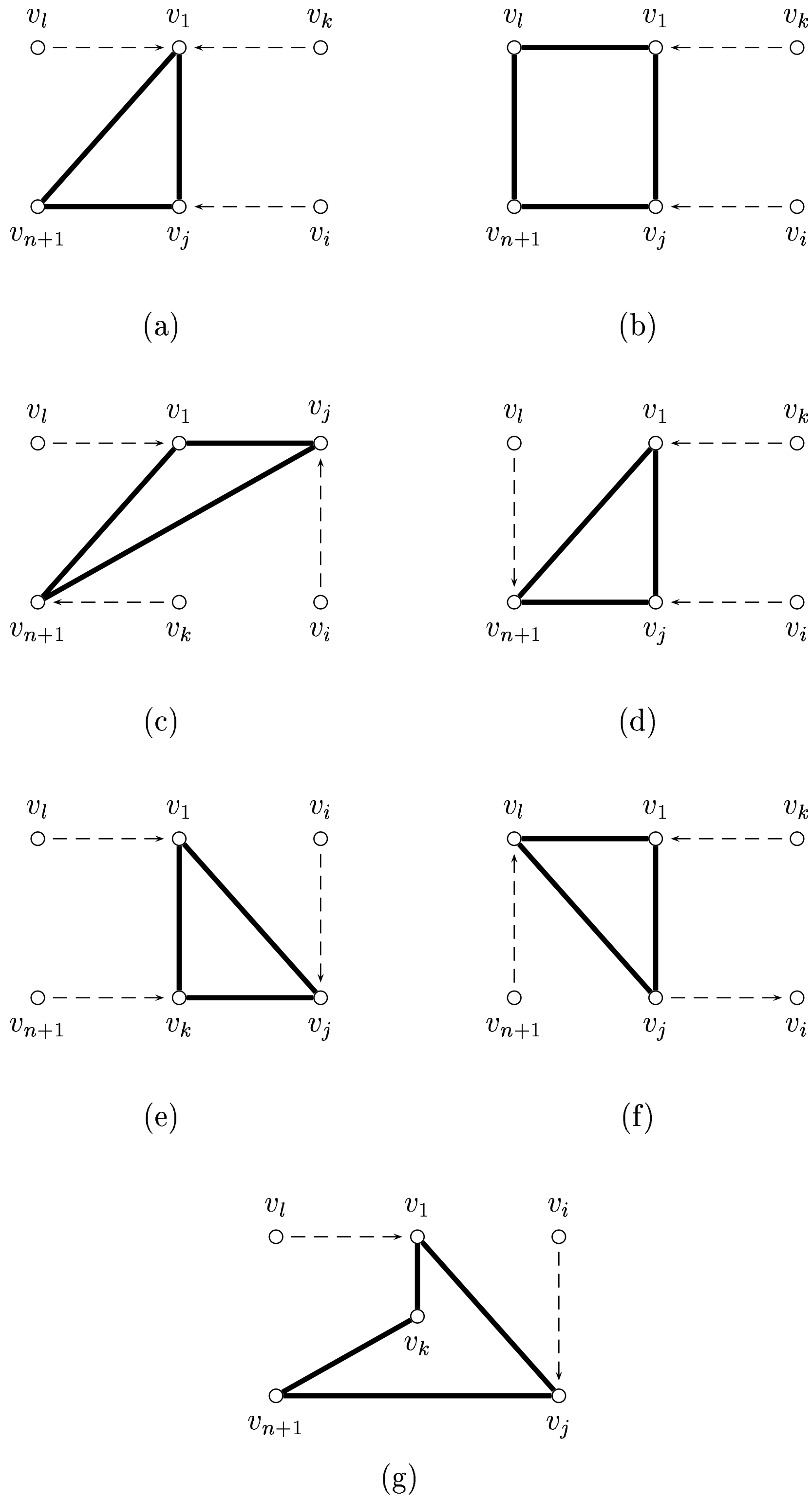


Figure 2: RSP solutions for the proof of Proposition 3.7

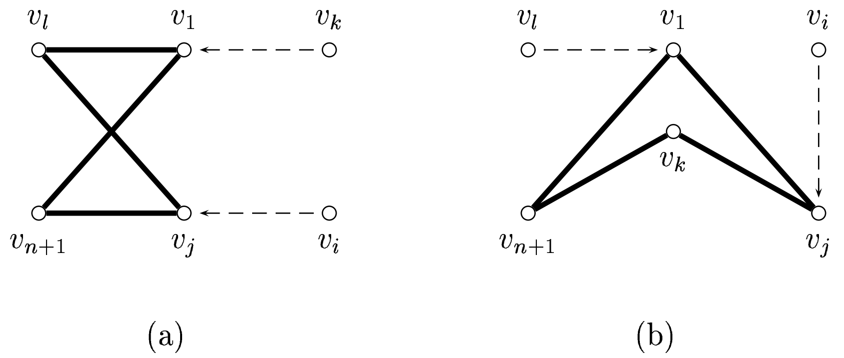


Figure 3: RSP solutions for the proof of Proposition 3.8

When $n = 7$, we can verify that (16) is facet defining by using PORTA [15] (see `http://webpages.ull.es/users/jjsalaza/ring`). Further the $\dim(P)$ affinely independent vectors of $P$ can be extended exactly in the same way.

When $v_{n+1} \in S$, the $3n - 2$ additional affinely independent vectors illustrated in Figure 2 are also tight for (16). When $v_{n+1} \notin S$, Figure 2 (b) yields $n - |S| - 1$ vectors tight for (16), i.e. only if $v_l \notin S$ and $v_l \neq v_1, v_{n+1}$. Figure 3 (a) provides $|S| - 3$ additional tight vectors, i.e. for $v_l \in S \backslash \{v_i, v_j, v_k\}$. Further, Figure 2 (g) must be replaced by Figure 3 (b). □

Observe that inequalities (16) are not facet defining when $|S| = 3$ since in that case they are the sum of inequalities (12) for arcs $(v_i, v_j)$, $(v_j, v_k)$ and $(v_k, v_i)$. Also, (16) cannot be facet defining if $|V \setminus S| \leq 2$ since then $x(\delta(S))$ must be equal to at least two.

# 4 Branch-and-cut algorithm

We now outline the main ingredients of our branch-and-cut algorithm for finding an optimal solution cycle $\overline{C}$ of the RSP. Let $V(\overline{C})$ be the set of vertices in the solution.

**Step 1** (Initialization). Set the iteration count $t := 1$. Compute an upper bound $\overline{z}$ on the optimal RSP as follows. Starting with $V(\overline{C}) := \{v_1\}$, successively insert a vertex $v_i \notin V(\overline{C})$ to minimize

$$L(i, \lambda) := \lambda \, inc(\overline{C}, i) + (1 - \lambda)(-dec(\overline{C}, i)),$$

where $inc(\overline{C}, i)$ is the minimum increment produced in the routing cost of cycle $\overline{C}$ when inserting $v_i$, and $dec(\overline{C}, i)$ is the decrement produced in the accessibility cost. This operation is repeated as long as $L(i, \lambda)$ decreases with a new insertion. This is done for each $\lambda \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and the best solution is selected. The initial linear subproblem is then defined as

$$\text{minimize} \sum_{[v_i,v_j]\in E} c_{ij}x_{ij} + \sum_{(v_i,v_j)\in A} d_{ij}y_{ij}$$

subject to:

$$\begin{aligned} x(\delta(i)) &= 2y_{ii} && \text{for all } v_i \in V \\ \sum_{v_j \in V} y_{ij} &= 1 && \text{for all } v_i \in V \setminus \{v_1\} \\ y_{11} &= 1 \\ x_{1i} &\leq y_{ii} && \text{for all } v_i \in V \setminus \{v_1\}, \end{aligned}$$

and the subproblem is inserted in a list $\mathcal{L}$.

**Step 2** (Termination check and subproblem selection). If the list $\mathcal{L}$ is empty, stop. Otherwise select a subproblem from the list according to a best-first policy.

**Step 3** (Subproblem solution). Set $t := t + 1$. Solve the subproblem and let $z$ be the solution objective value. If $z \geq \overline{z}$, go to Step 2. Otherwise, if the solution is feasible for the RSP, set $\overline{z} := z$, update $\overline{C}$, and go to Step 2.

**Step 4** (LP-based heuristic). If the solution is fractional and $t$ is a multiple of 5, apply the following heuristic. Given the fractional solution $(x^*, y^*)$, sort the $x^*_{ij}$ variables in decreasing order of their values and take in turn their associated edges until a cycle $C^*$ is obtained. If $v_1$ is not in $C^*$, introduce it in the best position. Finally, apply the classical 2-optimality procedure to further improve the routing cost of $C^*$. Let $z^*$ be the objective function value of solution $C^*$. If $z^* \leq \overline{z}$, set $\overline{z} := z^*$ and update $\overline{C}$ with $C^*$.

**Step 5** (Constraint separation and generation). Introduce up to 300 violated connectivity constraints (6) and 2-matching constraints (14). If no constraints can be generated, go to Step 6. Otherwise, go to Step 3.

**Step 6** (Branching). Create two subproblems by branching on a fractional $y_{ii}$ or $x_{ij}$ variable. The first branching strategy consists of finding a $y_{ii}$ variable. To this end, we applied the "strong branching" rule (see Jünger and Thienel [9] for details) within the five $y_{ii}$ variables with fractional value closest to 0.5. If all these variables are integer, select a $x_{ij}$ variable using the same criterion. Insert both subproblems in $\mathcal{L}$ and go to Step 2.

We have developed the following separation procedures for Step 5. Denote by $G^* = (V^*, E^*)$ the *support graph* associated with a given (fractional) solution $(x^*, y^*)$, i.e.,

$V^* := \{v_i \in V : 0 < y^*_{ii} < 1\}$ and $E^* := \{[v_i, v_j] \in E : 0 < x^*_{ij} < 1\}$. We also define $A^* := \{(v_i, v_j) \in A : i \neq j, 0 < y^*_{ij} < 1\}$.

### Separation of connectivity constraints (6).

The inequalities (6) involving sets $S$ such that $|S| = 2$ reduce to (12), and any of them can be violated only if the corresponding edge $[v_i, v_j]$ belongs to $E^*$ or the corresponding arc $(v_i, v_j)$ belongs to $A^*$. So, they are separated by examining only edges in $E^*$ and arcs in $A^*$. Next, if the support graph $G^*$ is not connected, then each subset $S \subset V$ corresponding to a connected component and such that $v_1 \notin S$ generates a violated connectivity constraint for each $v_i \notin S$. If $G^*$ is connected, finding a most violated connectivity constraint $x(\delta(S)) \geq 2\sum_{v_j \in S} y_{ij}$ is equivalent to finding the largest violation of $x(\delta(S)) + 2 \sum_{v_j \notin S} y_{ij} \geq 2$. For a given vertex $v_i \in V \setminus \{v_1\}$ this reduces to a maximum flow problem defined as follows. Let $G^{**} = (V^{**}, E^{**})$ be a graph such that $V^{**} = V^* \cup \{v_{n+1}\}$ and $E^{**} = E^* \cup \{[v_{n+1}, v_j] : v_j \in V^*\}$. The capacity of each edge $e \in E^*$ is equal to $x^*_e$ and that of the new edges $[v_{n+1}, v_j]$ is equal to $2y^*_{ij}$. Let $S' \subset V^{**}$ be such that $v_{n+1} \in S'$, $v_1 \notin S'$ and the capacity $\Delta$ of the cut $\delta(S')$ in $G^{**}$ is minimum. If $\Delta \geq 2$, there is no connectivity constraint involving $v_i$ violated by the current solution $(x^*, y^*)$. Otherwise, $S = S' \backslash \{v_{n+1}\}$ defines a most violated connectivity constraint (6) involving $v_i$.

### Separation of 2-matching constraints (14)

To separate the 2-matching inequalities, we use the heuristic procedure proposed in Fischetti, Salazar and Toth [5]. This algorithm can identify several violated 2-matching constraints for the current fractional solution.

### Separation of constraints (15) and (16)

Clearly, inequalities (15) can be separated through a complete enumeration of $v_i, v_j, v_k$ such that $y^*_{ij} > 0$, $y^*_{jk} > 0$, and $y^*_{ki} > 0$. Indeed, when (say) $y_{ij} = 0$ and (12) holds, then (15) also holds. In a similar way, for each $v_i$, $v_j$, $v_k$ such that $y^*_{ij} > 0$, $y^*_{jk} > 0$, and $y^*_{ki} > 0$, a min-cut separating $v_1$ from $\{v_i, v_j, v_k\}$ in $G^*$ gives the most-violated constraint (16), if any. Again, when (say) $y_{ij} = 0$ and (6) holds, then (16) also holds.

## 5 Computational results

The branch-and-cut algorithm described in the previous sections was implemented in the C++ programming language. ABACUS 2.3 linked with CPLEX 6.0 was used as a branch-and-cut framework. See Jünger and Thienel [9] for details on this software. The performance of the algorithm was tested on three different classes of test instances. The root was always chosen as the first vertex.

Class I is based on TSP instances from TSPLIB 2.1 (Reinelt [16]) involving between 50 and 200 vertices (problems *eil51* to *kroB200*). Denote by $l_{ij}$ the distance between vertices $v_i$ and $v_j$ given in the TSP files. To obtain optimal solutions visiting approximately 100%, 75%, 50% and 25 % of the total number of vertices in the instances, we set $c_{ij} = \lceil \alpha\ l_{ij} \rceil$, $d_{ij} = \lceil (10 - \alpha)\ l_{ij} \rceil$ for $\alpha \in \{3, 5, 7, 9\}$, and $d_{ii} = 0$ for all $v_i \in V$.

Class II was generated in the following way. We generated vertices with coordinates in $[0, 1000] \times [0, 1000]$ and computed $l_{ij}$ as the Euclidean distance between $v_i$ and $v_j$ rounded up to the nearest integer. Costs $c_{ij}$, $d_{ij}$ and $d_{ii}$ are defined as in Class I, using the same values of $\alpha$. For each pair $(|V|, \alpha)$ we generated ten instances.

Class III consists of instances generated as in Lee, Chiu and Sanchez [12]. We first generated vertices with coordinates in $[0, 1000] \times [0, 1000]$ and then computed $l_{ij}$ as the Euclidean distance between $v_i$ and $v_j$ rounded up to the nearest integer. Costs $c_{ij} := d_{ij} := l_{ij}$, and costs $d_{ii}$ were randomly generated in the interval [0,1000] for all $v_i \in V$. For each value $|V|$ we generated ten instances.

Tables 2 to 5 summarize the computational behavior of our branch-and-cut code on the three classes of instances. The column headings are defined as follows.

**Name:** problem name (for Class I);

$|V|$**:** number of vertices (for Classes II and III);

$\alpha$**:** scale factor described above (for Classes I and II);

**succ:** number of instances solved to optimality over 10 trials (for Classes II and III);

$p^*$ **:** percentage of vertices visited by the optimal cycle;

**opt:** optimal objective value (for Class I);

**%-UB0:** percentage ratio UB/opt, where UB is the objective function value of the initial heuristic solution.

**h-time:** initial heuristic time.

**%-LB:** percentage ratio LB/opt, where LB is the objective value of the LP-relaxation computed at the root node of the branch-decision tree;

**%-UB:** percentage ratio UB/opt, where UB is the objective value of the best solution at the root node of the branch-decision tree;

**pair:** number of Constraints (12) generated;

**sec:** number of Constraints (6) with $|S| > 2$ generated;

**2mat:** number of Constraints (14) generated;

**nodes:** total number of nodes examined (1 means that the problem required no branching);

**time:** total computing time.

The computing times reported are expressed in the format *h:mm:ss*, and refer to CPU time on Pentium III personal computer running at 866 MHz. We imposed a time limit of two hours for each run. For the instances exceeding the time limit, we report '>2:00:00' in the *time* column, and compute the corresponding results by considering the best available solution as optimal. Hence, for the time-limit instances the column %-LB gives an overestimation of the percentage approximation error. Tables 2 and 3 refer to Class I (TSPLIB based instances). Tables 4 and 5 refer to instances in Class II and Class III respectively, and contain average results over ten trials.

Preliminary experiments revealed that inequalities (15) and (16) are not useful in the sense that they were not violated after all others had been separated. Moreover, even when violations were detected at the root node, introducing these constraints did not help to reduce the gap between the lower and the upper bounds, and did not reduce the computing time. Hence we decided not to use these inequalities in our tests.

Results presented in Tables 2 to 4 indicate that the proposed branch-and-cut algorithm can solve instances involving up to 200 vertices within modest computing times. For a given problem size, the most difficult instances tend to be those where relatively few vertices belong to the optimal cycle. This can be explained by the fact that the location component of the problem is then important. At the other extreme, instances in which all vertices belong to the cycle reduce to a TSP which is a relatively easy problem for the values of $|V|$ considered. The %-LB column indicates that the lower bound developed at the root of the branch-and-cut tree is very tight, typically within 0.5% of the optimum. The heuristic is also very powerful and usually yields solution within 5% of the optimum. It seems to perform better when not all vertices belong to the cycle. All valid inequalities, apart from (15) and (16), are frequently generated within the search tree. The number of nodes in the tree is relatively low, and several instances (in particular the TSPLIB based instances) are solved without any branching. Overall we were able to solve to optimality 124 of the 132 TSPLIB based instances in Class I. For the random instances in Class II, the corresponding statistic is 267 out of 280, and the largest instances involved 200 vertices.

Table 5 shows that for random instances in Class III, i.e., those generated as in [17, 12], we were able to solve to optimality 122 out of 130 instances, involving up to 300 vertices. The largest instances we were able to solve contain three times as many vertices as those solved by Lee, Chiu and Sanchez [12], who consided a closely related problem.

## 6 Conclusions

We have presented what we believe to be the first exact algorithm for the *Ring Star Problem*, a problem arising in several network design contexts. An mixed integer linear programming formulation including several classes of facet defining inequalities was proposed, together with a branch-and-cut algorithm. The proposed approach was tested on three classes of instances. The largest solved involved 300 vertices. Recall that two of the classes are the benchmark instances in Moreno Pérez, Moreno Vega and Rodríguez Martín [13], while the third set of instances were generated as those used in Lee, Chiu and Sanchez [12] and in Xu, Chiu and Glover [17].

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| name | $\alpha$ | %-UB0 | h-time | $p^*$ | opt | %-LB | %-UB | pair | sec | 2mat | nodes | time |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| eil51 | 3 | 107.51 | 0:00:00 | 100.00 | 1278 | 100.00 | 107.51 | 4 | 155 | 91 | 3 | 0:00:02 |
| eil51 | 5 | 102.76 | 0:00:00 | 74.51 | 1995 | 100.00 | 100.00 | 61 | 42 | 0 | 1 | 0:00:01 |
| eil51 | 7 | 101.47 | 0:00:00 | 33.33 | 2113 | 100.00 | 100.00 | 238 | 1251 | 0 | 1 | 0:00:10 |
| eil51 | 9 | 101.29 | 0:00:00 | 11.76 | 1244 | 100.00 | 100.00 | 529 | 883 | 0 | 1 | 0:00:10 |
| berlin52 | 3 | 105.06 | 0:00:00 | 100.00 | 22626 | 100.00 | 100.00 | 12 | 46 | 0 | 1 | 0:00:01 |
| berlin52 | 5 | 102.78 | 0:00:00 | 78.85 | 36115 | 100.00 | 100.00 | 85 | 957 | 36 | 1 | 0:00:04 |
| berlin52 | 7 | 100.94 | 0:00:00 | 46.15 | 37376 | 100.00 | 100.00 | 221 | 655 | 1 | 1 | 0:00:06 |
| berlin52 | 9 | 100.78 | 0:00:00 | 9.62 | 20361 | 100.00 | 100.00 | 539 | 941 | 0 | 1 | 0:00:12 |
| brazil58 | 3 | 102.16 | 0:00:01 | 100.00 | 76185 | 100.00 | 100.00 | 18 | 105 | 3 | 1 | 0:00:02 |
| brazil58 | 5 | 101.46 | 0:00:01 | 68.97 | 115045 | 100.00 | 100.00 | 104 | 2994 | 285 | 1 | 0:00:14 |
| brazil58 | 7 | 100.24 | 0:00:00 | 48.28 | 126807 | 100.00 | 100.00 | 237 | 1543 | 2 | 1 | 0:00:12 |
| brazil58 | 9 | 100.00 | 0:00:00 | 15.52 | 83690 | 100.00 | 100.00 | 577 | 1502 | 0 | 1 | 0:00:28 |
| st70 | 3 | 103.41 | 0:00:01 | 100.00 | 2025 | 99.80 | 103.41 | 32 | 660 | 110 | 11 | 0:00:13 |
| st70 | 5 | 102.41 | 0:00:01 | 78.57 | 3110 | 99.84 | 102.41 | 110 | 2837 | 243 | 3 | 0:00:24 |
| st70 | 7 | 101.41 | 0:00:01 | 42.86 | 3402 | 100.00 | 100.00 | 283 | 1920 | 0 | 1 | 0:00:20 |
| st70 | 9 | 100.15 | 0:00:00 | 25.71 | 2610 | 100.00 | 100.00 | 806 | 1935 | 0 | 1 | 0:01:10 |
| eil76 | 3 | 107.25 | 0:00:02 | 100.00 | 1614 | 100.00 | 100.00 | 6 | 139 | 43 | 1 | 0:00:05 |
| eil76 | 5 | 100.61 | 0:00:02 | 73.68 | 2460 | 99.92 | 100.61 | 113 | 1858 | 56 | 5 | 0:00:21 |
| eil76 | 7 | 102.76 | 0:00:01 | 42.11 | 2504 | 100.00 | 100.00 | 343 | 2228 | 0 | 1 | 0:00:37 |
| eil76 | 9 | 101.35 | 0:00:00 | 15.79 | 1710 | 100.00 | 100.00 | 930 | 2440 | 0 | 1 | 0:02:28 |
| pr76 | 3 | 101.55 | 0:00:02 | 100.00 | 324477 | 98.59 | 101.55 | 128 | 15270 | 6043 | 727 | 0:17:39 |
| pr76 | 5 | 106.21 | 0:00:02 | 80.26 | 500395 | 99.66 | 106.21 | 171 | 8847 | 1489 | 71 | 0:02:27 |
| pr76 | 7 | 103.89 | 0:00:01 | 53.95 | 555858 | 99.80 | 103.89 | 350 | 6744 | 254 | 13 | 0:01:54 |
| pr76 | 9 | 100.72 | 0:00:00 | 17.11 | 424359 | 100.00 | 100.00 | 874 | 3011 | 0 | 1 | 0:03:04 |
| gr96 | 3 | 102.75 | 0:00:15 | 100.00 | 163935 | 99.30 | 102.75 | 38 | 1608 | 415 | 31 | 0:01:43 |
| gr96 | 5 | 104.31 | 0:00:13 | 79.17 | 252850 | 100.00 | 100.00 | 144 | 4802 | 292 | 1 | 0:01:06 |
| gr96 | 7 | 105.97 | 0:00:07 | 50.00 | 273599 | 100.00 | 100.00 | 409 | 3254 | 2 | 1 | 0:01:08 |
| gr96 | 9 | 105.99 | 0:00:02 | 27.08 | 232823 | 100.00 | 100.00 | 1236 | 3766 | 0 | 1 | 0:06:53 |
| rat99 | 3 | 107.35 | 0:00:06 | 100.00 | 3633 | 99.89 | 107.35 | 12 | 1178 | 234 | 13 | 0:00:38 |
| rat99 | 5 | 103.65 | 0:00:05 | 89.90 | 5885 | 100.00 | 100.00 | 150 | 2437 | 300 | 1 | 0:00:35 |
| rat99 | 7 | 104.26 | 0:00:02 | 42.42 | 6436 | 100.00 | 100.00 | 480 | 3197 | 0 | 1 | 0:01:20 |
| rat99 | 9 | 105.01 | 0:00:01 | 21.21 | 5150 | 100.00 | 100.00 | 1285 | 4541 | 0 | 1 | 0:07:19 |
| kroA100 | 3 | 100.86 | 0:00:06 | 100.00 | 63846 | 99.80 | 100.86 | 24 | 886 | 211 | 11 | 0:00:37 |
| kroA100 | 5 | 100.21 | 0:00:05 | 80.00 | 100785 | 99.78 | 100.21 | 149 | 6285 | 246 | 3 | 0:01:26 |
| kroA100 | 7 | 106.18 | 0:00:03 | 55.00 | 115388 | 100.00 | 100.00 | 428 | 3368 | 4 | 1 | 0:01:16 |
| kroA100 | 9 | 103.15 | 0:00:00 | 21.00 | 94265 | 100.00 | 100.00 | 1250 | 3150 | 0 | 1 | 0:05:57 |
| kroB100 | 3 | 105.42 | 0:00:06 | 100.00 | 66423 | 99.50 | 105.42 | 30 | 2051 | 310 | 39 | 0:01:38 |
| kroB100 | 5 | 104.35 | 0:00:05 | 77.00 | 104550 | 100.00 | 100.00 | 147 | 4319 | 271 | 1 | 0:00:57 |
| kroB100 | 7 | 103.95 | 0:00:02 | 46.00 | 118111 | 100.00 | 100.00 | 435 | 3322 | 0 | 1 | 0:01:17 |
| kroB100 | 9 | 100.99 | 0:00:00 | 16.00 | 93938 | 100.00 | 100.00 | 1303 | 4238 | 0 | 1 | 0:07:34 |
| kroC100 | 3 | 102.42 | 0:00:06 | 100.00 | 62247 | 99.81 | 102.42 | 48 | 1074 | 215 | 9 | 0:00:33 |
| kroC100 | 5 | 103.43 | 0:00:05 | 81.00 | 99065 | 100.00 | 100.00 | 141 | 3283 | 280 | 1 | 0:00:45 |
| kroC100 | 7 | 102.40 | 0:00:03 | 50.00 | 113533 | 100.00 | 100.00 | 438 | 3200 | 0 | 1 | 0:01:07 |
| kroC100 | 9 | 102.07 | 0:00:01 | 23.00 | 92894 | 100.00 | 100.00 | 1319 | 4056 | 0 | 1 | 0:07:05 |
| kroD100 | 3 | 105.58 | 0:00:06 | 100.00 | 63882 | 99.88 | 105.58 | 40 | 948 | 186 | 5 | 0:00:32 |
| kroD100 | 5 | 105.25 | 0:00:05 | 82.00 | 101645 | 100.00 | 100.00 | 154 | 2651 | 119 | 1 | 0:00:38 |
| kroD100 | 7 | 101.07 | 0:00:02 | 47.00 | 116849 | 99.94 | 101.07 | 440 | 4213 | 7 | 3 | 0:01:50 |
| kroD100 | 9 | 102.82 | 0:00:00 | 23.00 | 92102 | 100.00 | 100.00 | 1251 | 4123 | 0 | 1 | 0:06:58 |
| kroE100 | 3 | 104.55 | 0:00:06 | 100.00 | 66204 | 99.28 | 104.55 | 48 | 821 | 374 | 41 | 0:01:44 |
| kroE100 | 5 | 103.67 | 0:00:05 | 77.00 | 104915 | 99.94 | 103.67 | 170 | 6678 | 447 | 11 | 0:01:54 |
| kroE100 | 7 | 105.60 | 0:00:02 | 51.00 | 116471 | 100.00 | 100.00 | 441 | 3010 | 1 | 1 | 0:01:06 |
| kroE100 | 9 | 100.96 | 0:00:01 | 20.00 | 96116 | 100.00 | 100.00 | 1228 | 5068 | 0 | 1 | 0:08:25 |
| rd100 | 3 | 101.67 | 0:00:06 | 100.00 | 23730 | 100.00 | 100.00 | 28 | 524 | 122 | 1 | 0:00:16 |
| rd100 | 5 | 102.34 | 0:00:05 | 76.00 | 37975 | 99.87 | 102.34 | 165 | 6766 | 282 | 5 | 0:01:40 |
| rd100 | 7 | 102.61 | 0:00:02 | 44.00 | 40915 | 100.00 | 100.00 | 433 | 5848 | 26 | 1 | 0:01:53 |
| rd100 | 9 | 102.59 | 0:00:01 | 21.00 | 31776 | 100.00 | 100.00 | 1287 | 4368 | 0 | 1 | 0:07:01 |
| eil101 | 3 | 106.84 | 0:00:06 | 100.00 | 1887 | 99.84 | 106.84 | 16 | 824 | 227 | 19 | 0:00:46 |
| eil101 | 5 | 103.61 | 0:00:05 | 72.28 | 2905 | 100.00 | 100.00 | 301 | 1810 | 257 | 1 | 0:00:31 |
| eil101 | 7 | 102.29 | 0:00:02 | 38.61 | 2926 | 99.93 | 102.29 | 1680 | 7918 | 18 | 7 | 0:04:00 |
| eil101 | 9 | 101.48 | 0:00:00 | 17.82 | 1955 | 100.00 | 100.00 | 3291 | 4970 | 0 | 1 | 0:08:18 |
| lin105 | 3 | 103.43 | 0:00:07 | 100.00 | 43137 | 100.00 | 100.00 | 56 | 808 | 145 | 1 | 0:00:21 |
| lin105 | 5 | 100.81 | 0:00:06 | 80.95 | 69365 | 100.00 | 100.00 | 233 | 2588 | 341 | 1 | 0:00:43 |
| lin105 | 7 | 107.97 | 0:00:04 | 53.33 | 83597 | 100.00 | 100.00 | 1149 | 4047 | 6 | 1 | 0:01:38 |
| lin105 | 9 | 102.37 | 0:00:02 | 31.43 | 69920 | 100.00 | 100.00 | 3995 | 4998 | 0 | 1 | 0:07:30 |

Table 2: Computational results for instances in Class I

| name | $\alpha$ | %-UB0 | h-time | $p^*$ | opt | %-LB | %-UB | pair | sec | 2mat | nodes | time |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| pr107 | 3 | 100.36 | 0:00:08 | 100.00 | 132909 | 100.00 | 100.00 | 0 | 1171 | 15 | 1 | 0:00:23 |
| pr107 | 5 | 100.89 | 0:00:06 | 68.22 | 210465 | 100.00 | 100.00 | 327 | 3418 | 169 | 1 | 0:00:50 |
| pr107 | 7 | 100.63 | 0:00:03 | 42.99 | 259571 | 100.00 | 100.00 | 1198 | 4979 | 1 | 1 | 0:01:52 |
| pr107 | 9 | 101.08 | 0:00:01 | 26.17 | 264918 | 100.00 | 100.00 | 4218 | 5366 | 0 | 1 | 0:06:45 |
| gr120 | 3 | 103.26 | 0:00:14 | 100.00 | 20826 | 99.92 | 103.26 | 40 | 1337 | 186 | 9 | 0:01:01 |
| gr120 | 5 | 103.75 | 0:00:10 | 75.83 | 31480 | 100.00 | 100.00 | 420 | 8782 | 366 | 1 | 0:02:38 |
| gr120 | 7 | 102.19 | 0:00:06 | 41.67 | 32301 | 100.00 | 100.00 | 1211 | 6133 | 0 | 1 | 0:03:19 |
| gr120 | 9 | 103.19 | 0:00:01 | 22.50 | 24322 | 100.00 | 100.00 | 3799 | 6332 | 0 | 1 | 0:11:34 |
| pr124 | 3 | 104.96 | 0:00:14 | 100.00 | 177090 | 98.82 | 104.96 | 251 | 10055 | 1594 | 195 | 0:12:32 |
| pr124 | 5 | 103.45 | 0:00:13 | 90.32 | 286115 | 99.79 | 103.45 | 559 | 7777 | 782 | 17 | 0:03:58 |
| pr124 | 7 | 102.48 | 0:00:10 | 66.94 | 358853 | 100.00 | 100.00 | 1263 | 7590 | 21 | 1 | 0:03:27 |
| pr124 | 9 | 102.65 | 0:00:03 | 39.52 | 340153 | 100.00 | 100.00 | 5122 | 7999 | 1 | 1 | 0:16:05 |
| bier127 | 3 | 104.70 | 0:00:15 | 100.00 | 354846 | 99.84 | 104.70 | 64 | 1883 | 341 | 29 | 0:02:27 |
| bier127 | 5 | 108.23 | 0:00:13 | 76.38 | 539955 | 100.00 | 108.23 | 526 | 10150 | 638 | 3 | 0:03:32 |
| bier127 | 7 | 101.35 | 0:00:06 | 44.09 | 567110 | 99.99 | 101.35 | 2140 | 11081 | 27 | 5 | 0:07:11 |
| bier127 | 9 | 100.24 | 0:00:01 | 14.96 | 347845 | 100.00 | 100.00 | 5989 | 7333 | 0 | 1 | 0:36:52 |
| ch130 | 3 | 107.09 | 0:00:17 | 100.00 | 18330 | 99.85 | 107.09 | 100 | 2049 | 384 | 21 | 0:02:07 |
| ch130 | 5 | 106.60 | 0:00:15 | 83.08 | 28790 | 100.00 | 100.00 | 400 | 7044 | 608 | 1 | 0:02:38 |
| ch130 | 7 | 103.87 | 0:00:08 | 47.69 | 32707 | 99.64 | 100.77 | 3183 | 19575 | 143 | 15 | 0:11:24 |
| ch130 | 9 | 101.72 | 0:00:01 | 16.15 | 23639 | 100.00 | 100.00 | 5179 | 7112 | 0 | 1 | 0:21:56 |
| pr136 | 3 | 104.15 | 0:00:20 | 100.00 | 290316 | 99.45 | 104.15 | 42 | 11092 | 5600 | 147 | 0:15:04 |
| pr136 | 5 | 101.43 | 0:00:18 | 86.76 | 468520 | 100.00 | 100.00 | 401 | 6192 | 437 | 1 | 0:02:23 |
| pr136 | 7 | 102.59 | 0:00:06 | 43.38 | 491981 | 100.00 | 100.00 | 1786 | 8073 | 0 | 1 | 0:06:01 |
| pr136 | 9 | 104.14 | 0:00:02 | 25.74 | 387327 | 100.00 | 100.00 | 6152 | 10262 | 0 | 1 | 0:25:33 |
| gr137 | 3 | 104.13 | 0:01:02 | 100.00 | 208929 | 99.76 | 104.13 | 46 | 1764 | 348 | 13 | 0:02:36 |
| gr137 | 5 | 101.16 | 0:00:57 | 83.94 | 329465 | 99.74 | 101.16 | 603 | 20033 | 1097 | 15 | 0:08:59 |
| gr137 | 7 | 102.44 | 0:00:36 | 54.74 | 366022 | 100.00 | 100.00 | 1965 | 11516 | 16 | 1 | 0:07:06 |
| gr137 | 9 | 107.02 | 0:00:05 | 26.28 | 335009 | 100.00 | 100.00 | 6359 | 10498 | 0 | 1 | 0:31:09 |
| pr144 | 3 | 103.07 | 0:00:25 | 100.00 | 175611 | 99.83 | 103.07 | 126 | 1830 | 232 | 21 | 0:02:35 |
| pr144 | 5 | 104.87 | 0:00:24 | 94.44 | 290945 | 99.55 | 104.87 | 1108 | 45085 | 4218 | 323 | 0:51:01 |
| pr144 | 7 | 103.83 | 0:00:18 | 63.89 | 383041 | 100.00 | 100.00 | 1607 | 15046 | 236 | 1 | 0:07:22 |
| pr144 | 9 | 104.68 | 0:00:03 | 23.61 | 366833 | 100.00 | 100.00 | 7966 | 11221 | 0 | 1 | 0:25:44 |
| ch150 | 3 | 104.44 | 0:00:29 | 100.00 | 19584 | 99.79 | 104.44 | 77 | 4270 | 547 | 35 | 0:05:07 |
| ch150 | 5 | 104.64 | 0:00:25 | 77.33 | 31170 | 99.98 | 100.00 | 550 | 10366 | 456 | 3 | 0:05:15 |
| ch150 | 7 | 104.02 | 0:00:13 | 48.00 | 34930 | 99.99 | 104.02 | 1891 | 10208 | 3 | 3 | 0:09:21 |
| ch150 | 9 | 104.50 | 0:00:02 | 18.67 | 26371 | 100.00 | 100.00 | 7689 | 12746 | 0 | 1 | 0:54:12 |
| kroA150 | 3 | 103.53 | 0:00:29 | 100.00 | 79572 | 99.49 | 103.53 | 84 | 11656 | 2161 | 115 | 0:17:05 |
| kroA150 | 5 | 102.62 | 0:00:24 | 80.67 | 125435 | 100.00 | 100.00 | 399 | 7538 | 695 | 1 | 0:03:44 |
| kroA150 | 7 | 103.97 | 0:00:13 | 48.67 | 140961 | 99.76 | 103.97 | 2106 | 10498 | 90 | 5 | 0:09:05 |
| kroA150 | 9 | 102.79 | 0:00:03 | 19.33 | 113080 | 100.00 | 100.00 | 7254 | 10606 | 0 | 1 | 0:45:36 |
| kroB150 | 3 | 107.31 | 0:00:29 | 100.00 | 78390 | 99.51 | 107.31 | 78 | 4667 | 1491 | 65 | 0:10:33 |
| kroB150 | 5 | 104.52 | 0:00:25 | 79.33 | 122875 | 99.49 | 104.52 | 4171 | 181892 | 6524 | 221 | 1:52:12 |
| kroB150 | 7 | 104.48 | 0:00:12 | 50.67 | 135382 | 100.00 | 100.00 | 1729 | 8595 | 1 | 1 | 0:07:01 |
| kroB150 | 9 | 102.02 | 0:00:03 | 18.67 | 108885 | 100.00 | 100.00 | 7372 | 10483 | 0 | 1 | 0:43:22 |
| pr152 | 3 | 103.41 | 0:00:31 | 100.00 | 221046 | 99.51 | 103.41 | 162 | 5210 | 2296 | 45 | 0:09:42 |
| pr152 | 5 | 100.00 | 0:00:34 | 87.50 | 376155 | 96.05 | 100.00 | 1595 | 58290 | 6674 | 285 | >2:00:00 |
| pr152 | 7 | 100.00 | 0:00:18 | 51.32 | 475052 | 96.66 | 100.00 | 5576 | 84188 | 2042 | 59 | >2:00:00 |
| pr152 | 9 | 101.78 | 0:00:03 | 21.05 | 475440 | 100.00 | 100.00 | 10902 | 14875 | 0 | 1 | 0:52:15 |
| u159 | 3 | 107.83 | 0:00:37 | 100.00 | 126240 | 99.80 | 107.83 | 58 | 1346 | 610 | 7 | 0:02:27 |
| u159 | 5 | 104.35 | 0:00:33 | 84.28 | 204250 | 99.97 | 104.35 | 673 | 16748 | 897 | 7 | 0:09:08 |
| u159 | 7 | 101.92 | 0:00:20 | 55.97 | 235221 | 100.00 | 100.00 | 2040 | 10668 | 3 | 1 | 0:09:24 |
| u159 | 9 | 101.92 | 0:00:05 | 25.79 | 199552 | 100.00 | 100.00 | 8388 | 13736 | 0 | 1 | 0:59:50 |
| rat195 | 3 | 108.44 | 0:01:23 | 100.00 | 6969 | 99.68 | 108.44 | 78 | 17482 | 2048 | 61 | 0:29:41 |
| rat195 | 5 | 108.30 | 0:01:05 | 84.62 | 11320 | 99.92 | 108.30 | 562 | 10841 | 1033 | 11 | 0:12:21 |
| rat195 | 7 | 102.53 | 0:00:28 | 40.00 | 12319 | 100.00 | 100.00 | 2826 | 21795 | 14 | 1 | 0:34:25 |
| rat195 | 9 | 100.00 | 0:00:07 | 16.92 | 9395 | 94.21 | 100.00 | 8965 | 16115 | 0 | 1 | >2:00:00 |
| d198 | 3 | 102.50 | 0:01:28 | 100.00 | 47340 | 99.74 | 102.50 | 60 | 10573 | 1102 | 65 | 0:25:54 |
| d198 | 5 | 102.49 | 0:01:10 | 79.80 | 76945 | 99.94 | 102.49 | 2516 | 106574 | 7424 | 5 | 1:12:20 |
| d198 | 7 | 102.52 | 0:00:38 | 51.01 | 94300 | 99.87 | 101.34 | 8298 | 79000 | 716 | 59 | 1:48:59 |
| d198 | 9 | 100.00 | 0:00:09 | 19.19 | 97899 | 96.34 | 100.00 | 15536 | 26958 | 0 | 1 | >2:00:00 |
| kroA200 | 3 | 100.00 | 0:02:05 | 100.00 | 93699 | 93.59 | 100.00 | 234 | 23765 | 2930 | 129 | >2:00:00 |
| kroA200 | 5 | 107.78 | 0:01:12 | 74.00 | 138885 | 99.85 | 107.78 | 2410 | 149600 | 3181 | 39 | >2:00:00 |
| kroA200 | 7 | 103.77 | 0:00:43 | 49.00 | 158227 | 99.83 | 102.29 | 7169 | 72865 | 919 | 37 | 1:31:05 |
| kroA200 | 9 | 100.00 | 0:00:09 | 19.00 | 124678 | 97.15 | 100.00 | 10387 | 16377 | 0 | 1 | >2:00:00 |
| kroB200 | 3 | 106.26 | 0:01:32 | 100.00 | 88311 | 99.81 | 106.26 | 106 | 4080 | 868 | 33 | 0:13:44 |
| kroB200 | 5 | 104.97 | 0:01:18 | 77.00 | 138905 | 99.93 | 104.97 | 1025 | 29056 | 1023 | 7 | 0:25:44 |
| kroB200 | 7 | 101.96 | 0:00:48 | 49.00 | 156638 | 100.00 | 100.00 | 2854 | 16926 | 4 | 1 | 0:25:19 |
| kroB200 | 9 | 100.00 | 0:00:08 | 18.50 | 127800 | 95.13 | 100.00 | 10055 | 14849 | 0 | 1 | >2:00:00 |

Table 3: Computational results for instances in Class I (continued)

| $\|V\|$ | $\alpha$ | succ | $p^*$ | %-UB0 | h-time | %-LB | %-UB | pair | sec | 2mat | nodes | time |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 3 | 10 | 100.00 | 103.06 | 0:00:00 | 99.59 | 102.30 | 16.60 | 437.70 | 94.10 | 12.60 | 0:00:05 |
| | 5 | 10 | 78.60 | 100.94 | 0:00:00 | 99.91 | 100.11 | 70.70 | 941.30 | 69.10 | 2.00 | 0:00:04 |
| | 7 | 10 | 50.40 | 102.53 | 0:00:00 | 100.00 | 100.00 | 201.50 | 971.30 | 1.90 | 1.00 | 0:00:06 |
| | 9 | 10 | 12.40 | 100.95 | 0:00:00 | 100.00 | 100.00 | 543.50 | 977.90 | 0.00 | 1.00 | 0:00:15 |
| 75 | 3 | 10 | 100.00 | 104.30 | 0:00:02 | 99.83 | 102.70 | 22.20 | 548.70 | 148.40 | 9.20 | 0:00:15 |
| | 5 | 10 | 78.67 | 103.40 | 0:00:02 | 99.90 | 100.74 | 113.50 | 2777.90 | 207.60 | 7.00 | 0:00:32 |
| | 7 | 10 | 49.33 | 102.46 | 0:00:01 | 99.99 | 100.28 | 321.60 | 2364.80 | 5.00 | 1.40 | 0:00:33 |
| | 9 | 10 | 15.87 | 101.47 | 0:00:00 | 100.00 | 100.00 | 895.60 | 2279.60 | 0.00 | 1.00 | 0:02:02 |
| 100 | 3 | 10 | 100.00 | 104.40 | 0:00:06 | 99.74 | 104.08 | 35.30 | 1520.10 | 398.70 | 19.40 | 0:01:07 |
| | 5 | 10 | 77.70 | 103.96 | 0:00:05 | 99.87 | 102.08 | 159.90 | 8149.40 | 652.20 | 19.40 | 0:02:49 |
| | 7 | 10 | 46.00 | 102.35 | 0:00:02 | 99.78 | 100.89 | 514.60 | 31894.50 | 988.70 | 74.40 | 0:12:53 |
| | 9 | 10 | 19.40 | 102.31 | 0:00:01 | 100.00 | 100.00 | 1262.40 | 4051.60 | 0.00 | 1.00 | 0:07:36 |
| 125 | 3 | 10 | 100.00 | 106.17 | 0:00:14 | 99.77 | 105.31 | 51.00 | 2949.20 | 481.30 | 24.00 | 0:02:44 |
| | 5 | 10 | 80.08 | 104.92 | 0:00:12 | 99.92 | 103.48 | 520.40 | 13255.80 | 740.40 | 10.00 | 0:05:02 |
| | 7 | 10 | 45.44 | 103.09 | 0:00:06 | 99.87 | 100.51 | 1902.60 | 10757.70 | 140.50 | 6.40 | 0:06:23 |
| | 9 | 10 | 18.16 | 102.36 | 0:00:01 | 100.00 | 100.00 | 5409.70 | 7471.00 | 0.00 | 1.00 | 0:22:05 |
| 150 | 3 | 10 | 100.00 | 105.66 | 0:00:29 | 99.75 | 104.95 | 118.20 | 7356.40 | 1350.30 | 72.00 | 0:10:37 |
| | 5 | 10 | 80.20 | 105.07 | 0:00:25 | 99.91 | 102.41 | 775.90 | 32248.70 | 1494.70 | 21.00 | 0:16:30 |
| | 7 | 10 | 46.67 | 103.57 | 0:00:12 | 99.91 | 101.38 | 3718.50 | 23575.00 | 415.70 | 32.20 | 0:23:09 |
| | 9 | 10 | 18.47 | 102.09 | 0:00:02 | 100.00 | 100.00 | 7424.30 | 11163.00 | 0.00 | 1.00 | 0:54:17 |
| 175 | 3 | 10 | 100.00 | 106.69 | 0:00:54 | 99.84 | 105.96 | 94.90 | 4248.90 | 858.80 | 29.60 | 0:08:10 |
| | 5 | 8 | 79.71 | 105.12 | 0:00:46 | 99.88 | 103.20 | 1128.38 | 39025.38 | 1897.50 | 17.50 | 0:27:03 |
| | 7 | 9 | 46.16 | 103.53 | 0:00:23 | 99.89 | 101.90 | 4206.56 | 30322.00 | 478.78 | 19.00 | 0:34:58 |
| | 9 | 9 | 18.16 | 102.98 | 0:00:04 | 100.00 | 100.00 | 9630.22 | 15011.78 | 0.11 | 1.00 | 1:43:30 |
| 200 | 3 | 10 | 100.00 | 106.74 | 0:01:32 | 99.78 | 106.74 | 169.60 | 12025.50 | 1937.70 | 84.80 | 0:32:20 |
| | 5 | 9 | 79.56 | 104.57 | 0:01:19 | 99.93 | 103.53 | 1006.22 | 40507.22 | 1871.78 | 10.33 | 0:33:14 |
| | 7 | 8 | 47.75 | 104.17 | 0:00:38 | 99.92 | 102.45 | 4383.25 | 32413.25 | 369.13 | 14.00 | 0:48:01 |
| | 9 | 0 | | | | | | | | | | |

Table 4: Computational results for instances in Class II

| $\|V\|$ | succ | $p^*$ | %-UB0 | h-time | %-LB | %-UB | pair | sec | 2mat | nodes | time |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 10 | 36.00 | 101.12 | 0:00:00 | 100.00 | 100.00 | 12.20 | 12.50 | 0.00 | 1.00 | 0:00:00 |
| 20 | 10 | 25.00 | 101.74 | 0:00:00 | 100.00 | 100.00 | 61.90 | 92.70 | 0.00 | 1.00 | 0:00:00 |
| 30 | 10 | 25.33 | 103.29 | 0:00:00 | 100.00 | 100.00 | 104.20 | 133.50 | 0.50 | 1.00 | 0:00:00 |
| 40 | 10 | 23.00 | 101.13 | 0:00:00 | 100.00 | 100.00 | 154.00 | 258.10 | 1.70 | 1.00 | 0:00:01 |
| 50 | 10 | 20.80 | 103.74 | 0:00:00 | 100.00 | 100.00 | 225.90 | 338.10 | 1.50 | 1.00 | 0:00:02 |
| 75 | 10 | 20.53 | 101.87 | 0:00:00 | 99.99 | 100.03 | 401.20 | 1008.60 | 5.20 | 1.20 | 0:00:12 |
| 100 | 10 | 19.50 | 103.20 | 0:00:01 | 99.98 | 100.17 | 561.60 | 1777.40 | 14.30 | 1.60 | 0:00:35 |
| 125 | 10 | 18.08 | 102.55 | 0:00:01 | 100.00 | 100.18 | 1013.00 | 2882.60 | 22.10 | 1.40 | 0:01:16 |
| 150 | 10 | 17.20 | 102.37 | 0:00:03 | 99.98 | 100.44 | 1348.10 | 4785.40 | 25.10 | 2.00 | 0:02:53 |
| 175 | 10 | 17.01 | 103.29 | 0:00:05 | 99.96 | 100.65 | 2534.10 | 13865.80 | 98.00 | 6.20 | 0:12:33 |
| 200 | 10 | 16.50 | 102.32 | 0:00:08 | 99.96 | 100.79 | 3177.60 | 26594.00 | 165.40 | 11.00 | 0:28:22 |
| 250 | 8 | 15.80 | 103.39 | 0:00:17 | 99.98 | 101.32 | 3792.75 | 22781.75 | 133.63 | 4.25 | 0:32:54 |
| 300 | 4 | 14.75 | 102.62 | 0:00:30 | 99.99 | 100.25 | 9220.25 | 40934.25 | 191.25 | 1.50 | 1:08:41 |

Table 5: Computational results for instances in Class III