An Efficient Transformation of the Generalized Traveling Salesman Problem into the Traveling Salesman Problem on Digraphs.

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A New Efficient Transformation of Generalized Traveling Salesman Problem into Traveling Salesman Problem

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Abstract—We develop a method to obtain an optimal solution for generalized traveling salesman problem (GTSP). The approach is to transform GTSP into a traveling salesman problem (TSP), whereby the number of nodes in the transformed TSP does not exceed the number of nodes in the original GTSP. This can be compared with some previous methods using the same approach, which the number of nodes in the transformed TSP is at least twice and in some methods three times the number of nodes in GTSP. Since GTSP as well as TSP are NP-hard, decreasing the size of the problem into half makes it much more efficient in comparison with the other methods.

I. Introduction

Generalized traveling salesman problem (GTSP) was introduced by Henry-Labordere [3]. Although a wide variety of real world problems can be modeled as GTSP ([1], [7], [8]), many practitioners are reluctant to use it for practical modeling purposes because of the complexity of finding an optimal or near-optimal solution. Thus, a great deal of efforts have been devoted by many researchers to develop efficient methods for this interesting model and so far a variety of approaches have been proposed ([2], [5], [9], [10]).

Laporte and Nobert [9] developed an exact algorithm for GTSP by formulating an integer programming and finding the shortest Hamiltonian circuit through some clusters of nodes. Noon and Bean [5] proposed a Lagrangean relaxation algorithm and Fischetti *et al.* developed a branch and cut algorithm for the asymmetric version of the problem. In turn, transformation of a GTSP into traveling salesman problem (TSP) was first introduced by Lien and Ma [10], in which the number of nodes of the transformed TSP was relatively very large, in fact more than three times the number of nodes in the associated GTSP. Later, Dimitrijevic and Saric [2] developed another transformation to decrease the size of the corresponding TSP. In their method, the number of nodes of the TSP was twice the number of nodes of the original GTSP.

In this paper, we propose a new method to obtain an optimal solution of GTSP. This is done by transforming GTSP into a single TSP. There are two reasons to believe that this is an appropriate approach. First, GTSP is an extension of TSP and we can naturally take advantage of the similarities between them. Second, there exist many efficient methods for TSP. As a matter of fact, most of the existing methods for GTSP, exact or approximate, are some kind of extensions of TSP methods. Since TSP is NP-hard by nature, increasing the number of nodes increases the process time of the algorithm exponentially and, hence, it is vital to have a transformation with less number of nodes. Specifically, in this paper, we develop and investigate a new transformation, in which the number of nodes of TSP is equal to that of the original GTSP.

Although our method is a modification of the one proposed by Dimitrijevic and Saric [2], it is clearly much more efficient due to the reduction in number of nodes.

The remainder of the paper is structured as the following: In Section II, we review the definition as well as some properties of both GTSP and TSP. In Section III, our proposed transformation of GTSP into TSP is introduced. Finally, we provide an example in Section IV to illustrate the method.

II. GTSP and TSP Preliminaries

Traveling salesman problem (TSP) is represented by a graph $G = (V, A)$, where V is the set of nodes and A is the set of arcs (or edges). Associated with each arc $(i, j) \in A$, a cost (or distance) of $c(i, j)$ is defined. A TSP is defined to be *symmetric* if and only if $c(i, j) = c(j, i), \forall i, j \in V$. Furthermore, a TSP is *Euclidean* if and only if $c(i, j) + c(j, k) \geq c(i, k), \forall i, j, k \in V$. The objective in TSP is to find a Hamiltonian cycle with the minimal cost, where a *Hamiltonian cycle* is a closed path that visits each node once and only once.

In turn, in generalized traveling salesman problem (GTSP) the set of nodes (V) is the union of p clusters, which may or may not be intersected. Each feasible solution of GTSP, called a *g-tour*, is a closed path that includes at least one node from each cluster and the objective is to find a g-tour with the minimum cost. In a special case of GTSP, called E-GTSP, each cluster is visited exactly once.

III. Transformation of GTSP into a TSP

In this section, we develop a novel method to solve GTSP. This is done by transforming the problem into a TSP. As a matter of fact, the method we propose is a modification of the transformation developed by Dimitrijevic and Saric [2]. It should be mentioned that although in our transformation the underlying TSP is represented by a directed graph, the graph associated with the GTSP is not necessarily a directed one.

We do not restrict the problem to be symmetric or Euclidean. However, for the sake of presentation simplicity, we assume that the graph of GTSP is directed and that there are no intra-cluster arcs in the graph. We further assume that the clusters are non-intersected and the objective is to find a minimal g-tour that visits each cluster exactly once (that is E-GTSP). As Lien and Ma [10] have shown, the aforementioned assumptions are not restrictive and every GTSP can be easily transformed into our problem.

Definition 1: Consider a GTSP with p clusters, represented by a graph $G = (V, E)$. Let $v^r{}_i$ represent the *ith* node of cluster r. We define a TSP on a directed graph of G' associated with G such that,

a) The set of nodes of G and G' are identical.

b) All nodes of each cluster of G are connected by arcs into a cycle in G'. We refer to the node that succeeds $v^r{}_i$ in the cycle as $v^r{}_{i(s)}$.

c) The elements of the cost matrix of G' are defined as

$$c'(v^r{}_i, v^r{}_{i(s)}) = 0 \quad (1)$$

and

$$c'(v^r{}_i, v^t{}_j) = c(v^r{}_{i(s)}, v^t{}_j) + M, \quad r \neq t, \quad (2)$$

where

$$M > \sum_{(i,j)\in E} c(i,j). \qquad (3)$$

Clearly, if based on relation (2), the cost $c'(v^r{}_i, v^t{}_j)$ becomes 'infinity,' it implies that $v^r{}_i$ is not connected to $v^t{}_j$.

Definition 2: We define a path in *G'* as a *Cluster Path* if it consists of all nodes of only one cluster.

Definition 3: We refer to a Hamiltonian cycle in *G'* whose cost is less than *(p + 1)M* as a *Clustered Hamiltonian Cycle* (or simply, *C-Hamiltonian Cycle*).

Lemma 1: Every C-Hamiltonian Cycle only traverses along cluster paths (That is, it enters and leaves each cluster exactly once).

Proof. The definition of cost matrix *G'* implies that the cost of moving from one cluster to another is at least *M*. Since every Hamiltonian cycle visits all the clusters, its cost is at least *pM.* However, if it does not traverse along the cluster paths, it enters at least one of the clusters more than once. The latter implies that the total cost of the Hamiltonian cycle cannot be less than *(p+1)M*, which completes the proof.

Definition 4: We define a one-to-one correspondence between C-Hamiltonian cycles in *G'* and g-tours in *G* in the following fashion:

a) Consider a C-Hamiltonian of *G'* and connect the first nodes of its cluster paths together in the order of their corresponding clusters. The result is a g-tour of GTSP (or more precisely, E-GTSP).

b) Consider a g-tour in *G* that includes $\cdots \rightarrow v^r{}_i \rightarrow v^t{}_j \rightarrow \cdots, r \neq t$. Replace $v^r{}_i$ with the *rth* cluster path of *G'* starting with $v^r{}_i$, and then connect the last node of this path to the next cluster path starting with $v^t{}_j$. Clearly, the result of the above procedure forms a Hamiltonian cycle.

Theorem 2: The cost of every g-tour in *G* is equal to the cost of the corresponding C-Hamiltonian cycle in *G'* less *pM.*

Proof. The proof is straight forward based on the definition of the cost matrix, definition 4, and Lemma 1.

Clearly, since *pM* is constant, it has no impact on the optimal solution of GTSP.

IV. Numerical Example

Consider a GTSP defined on graph *G,* as shown in Fig. 1. A TSP is defined on the basis of definition 1 and is represented by *G'* in Fig. 2. The nodes of each cluster are cycled in *G'* by solid arcs as depicted in Fig. 2. The matrices *C* and *C*' are the cost matrices of GTSP and TSP, respectively. The optimal solution of TSP as well as its corresponding solution in GTSP are illustrated by dash-dot arcs in Fig. 1. The optimal cost of TSP and GTSP are equal to 7 + 3*M* and 7, respectively.

V. Conclusions

In this paper, we introduced a new transformation of GTSP into TSP. What makes this approach distinguished from the other transformations is that the number of nodes of the transformed TSP does not exceed the number of nodes of the original GTSP.

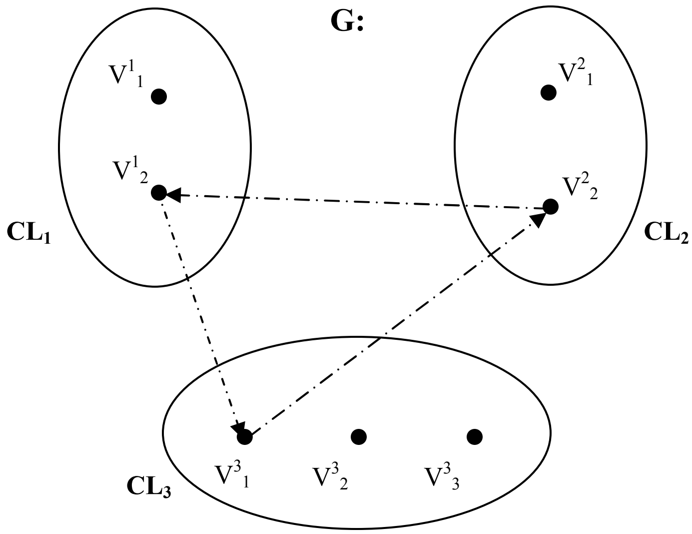

Figure 1. Illustration of the GTSP and its optimal solution.

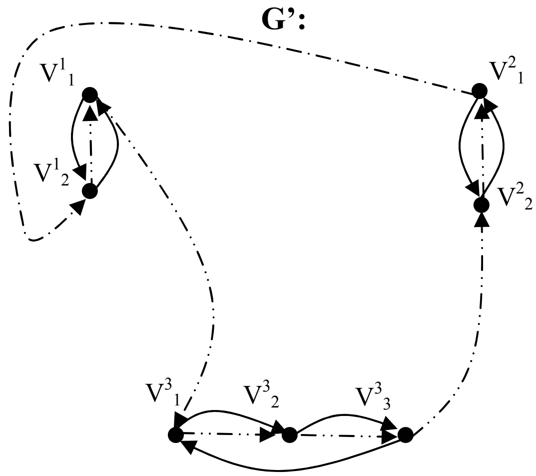

Figure 2. Illustration of the associated TSP and its optimal solution.

$$C=\begin{bmatrix}
c(v^1{}_1,v^1{}_1)=\infty & c(v^1{}_1,v^1{}_2)=\infty & c(v^1{}_1,v^2{}_1)=5 & c(v^1{}_1,v^2{}_2)=7 & c(v^1{}_1,v^3{}_1)=4 & c(v^1{}_1,v^3{}_2)=6 & c(v^1{}_1,v^3{}_3)=2 \\
c(v^1{}_2,v^1{}_1)=\infty & c(v^1{}_2,v^1{}_2)=\infty & c(v^1{}_2,v^2{}_1)=8 & c(v^1{}_2,v^2{}_2)=9 & c(v^1{}_2,v^3{}_1)=3 & c(v^1{}_2,v^3{}_2)=1 & c(v^1{}_2,v^3{}_3)=5 \\
c(v^2{}_1,v^1{}_1)=6 & c(v^2{}_1,v^1{}_2)=3 & c(v^2{}_1,v^2{}_1)=\infty & c(v^2{}_1,v^2{}_2)=\infty & c(v^2{}_1,v^3{}_1)=4 & c(v^2{}_1,v^3{}_2)=2 & c(v^2{}_1,v^3{}_3)=4 \\
c(v^2{}_2,v^1{}_1)=2 & c(v^2{}_2,v^1{}_2)=1 & c(v^2{}_2,v^2{}_1)=\infty & c(v^2{}_2,v^2{}_2)=\infty & c(v^2{}_2,v^3{}_1)=5 & c(v^2{}_2,v^3{}_2)=6 & c(v^2{}_2,v^3{}_3)=1 \\
c(v^3{}_1,v^1{}_1)=9 & c(v^3{}_1,v^1{}_2)=7 & c(v^3{}_1,v^2{}_1)=9 & c(v^3{}_1,v^2{}_2)=3 & c(v^3{}_1,v^3{}_1)=\infty & c(v^3{}_1,v^3{}_2)=\infty & c(v^3{}_1,v^3{}_3)=\infty \\
c(v^3{}_2,v^1{}_1)=8 & c(v^3{}_2,v^1{}_2)=4 & c(v^3{}_2,v^2{}_1)=6 & c(v^3{}_2,v^2{}_2)=5 & c(v^3{}_2,v^3{}_1)=\infty & c(v^3{}_2,v^3{}_2)=\infty & c(v^3{}_2,v^3{}_3)=\infty \\
c(v^3{}_3,v^1{}_1)=4 & c(v^3{}_3,v^1{}_2)=7 & c(v^3{}_3,v^2{}_1)=9 & c(v^3{}_3,v^2{}_2)=7 & c(v^3{}_3,v^3{}_1)=\infty & c(v^3{}_3,v^3{}_2)=\infty & c(v^3{}_3,v^3{}_3)=\infty
\end{bmatrix}$$

$$C' = \begin{bmatrix}
c(v^1{}_1,v^1{}_1)=\infty & c(v^1{}_1,v^1{}_2)=0 & c(v^1{}_1,v^2{}_1)=8+M & c(v^1{}_1,v^2{}_2)=9+M & c(v^1{}_1,v^3{}_1)=3+M & c(v^1{}_1,v^3{}_2)=1+M & c(v^1{}_1,v^3{}_3)=5+M \\
c(v^1{}_2,v^1{}_1)=0 & c(v^1{}_2,v^1{}_2)=\infty & c(v^1{}_2,v^2{}_1)=5+M & c(v^1{}_2,v^2{}_2)=7+M & c(v^1{}_2,v^3{}_1)=4+M & c(v^1{}_2,v^3{}_2)=6+M & c(v^1{}_2,v^3{}_3)=2+M \\
c(v^2{}_1,v^1{}_1)=2+M & c(v^2{}_1,v^1{}_2)=1+M & c(v^2{}_1,v^2{}_1)=\infty & c(v^2{}_1,v^2{}_2)=0 & c(v^2{}_1,v^3{}_1)=5+M & c(v^2{}_1,v^3{}_2)=6+M & c(v^2{}_1,v^3{}_3)=1+M \\
c(v^2{}_2,v^1{}_1)=6+M & c(v^2{}_2,v^1{}_2)=3+M & c(v^2{}_2,v^2{}_1)=0 & c(v^2{}_2,v^2{}_2)=\infty & c(v^2{}_2,v^3{}_1)=4+M & c(v^2{}_2,v^3{}_2)=2+M & c(v^2{}_2,v^3{}_3)=4+M \\
c(v^3{}_1,v^1{}_1)=8+M & c(v^3{}_1,v^1{}_2)=4+M & c(v^3{}_1,v^2{}_1)=6+M & c(v^3{}_1,v^2{}_2)=5+M & c(v^3{}_1,v^3{}_1)=\infty & c(v^3{}_1,v^3{}_2)=0 & c(v^3{}_1,v^3{}_3)=\infty \\
c(v^3{}_2,v^1{}_1)=4+M & c(v^3{}_2,v^1{}_2)=7+M & c(v^3{}_2,v^2{}_1)=9+M & c(v^3{}_2,v^2{}_2)=7+M & c(v^3{}_2,v^3{}_1)=\infty & c(v^3{}_2,v^3{}_2)=\infty & c(v^3{}_2,v^3{}_3)=0 \\
c(v^3{}_3,v^1{}_1)=9+M & c(v^3{}_3,v^1{}_2)=7+M & c(v^3{}_3,v^2{}_1)=9+M & c(v^3{}_3,v^2{}_2)=3+M & c(v^3{}_3,v^3{}_1)=0 & c(v^3{}_3,v^3{}_2)=\infty & c(v^3{}_3,v^3{}_3)=\infty
\end{bmatrix}$$

References

[1] J. Bovet, "The Selective Traveling Salesman Problem ," in *Proc. Of EURO VI Conference*, Vienna, 1993.
[2] V. Dimitrijevic and Z. Saric, "Efficient Transformation of the Generalized Traveling Salesman Problem into the Traveling Salesman Problem on Digraphs," *Information Sci*., 102, 65-110, 1997.
[3] A. L. Henry-Labordere, "The Record Balancing Problem: A Dynamic Programming Solution of a Generalized Traveling Salesman Problem," *RIBO B-2*, 736-743, 1969.
[4] J. J. Fischetti, S. Gonzalez and P. Tota, "A Branch-and-Cut Algorithm for the Symmetric Generalized Traveling Salesman Problem," *Opns. Res*. 45, 378-394, 1997.
[5] C. E. Noon and J. C. Bean, "A Lagrangean based Approach for Asymmetric Generalized Traveling Salesman Problem," *Opns. Res*. 39, 623-632, 1991.
[6] G. Laporte, Location-Routing Problems. In B. L. Golden and A. A. Assad (eds.), *Vehicle Routing: Methods and Studies*, North-Holland, Amsterdam, 1988.
[7] G. Laporte, "The Traveling Salesman Problem: An Overview of Exact and Approximate Algorithms," *European J. Opnl. Res*, 59, 231-247, 1992.
[8] G. Laporte, A. Asef-Vaziri and C. Srikandarajah, "Some Applications of the Generalized Traveling Salesman Problem," *J. Opnl. Res. Soc*. 47, 1461-1467, 1996.
[9] G. Laporte, Y. Nobert and S. Taillefer, "Solving a Family of Multi-Depot Vehicle Routing and Location-Routing Problems," *Transp. Sci*. 22, 161-172, 1988.
[10] Y. Lien, E. Ma and B. W. Wah, Transformation of the Generalized Traveling Salesman Problem into the Standard Traveling Salesman Problem," *Information Sci.* 74, 177-189, 1993.