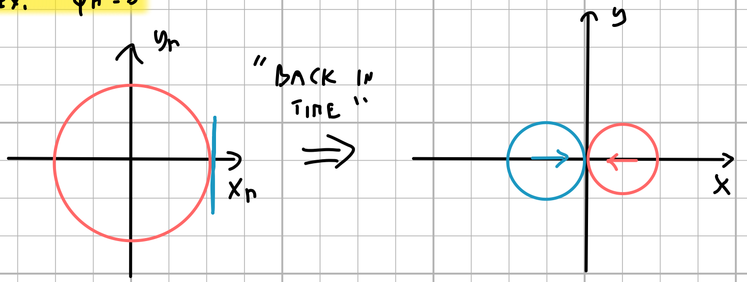
FIND MINIMUM TIME / RADIUS

i) MINIMUM TIME

CONSIDER A COLLISION AT TIME $t = 0$ OF q_1. "GO BACK IN TIME" UNTIL YOU EXIT THE UNSAFE AREA

EX. $\phi_r = 0$



DYNAMICS

$$\begin{cases} \dot{x}_r = -v + v\cos\phi_r \\ \dot{y}_r = v\sin\phi_r \\ \dot{\phi}_r = \omega \end{cases}$$

EXPLICIT SOLUTION

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{A(t-t_0)} \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} + \int_{t_0}^{t} e^{A(t-\tau)} B\,u(\tau)\,d\tau$$

IN THIS CASE

$t = 0$

$t_0 = t_{min} < 0$

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$Bv = \begin{pmatrix} \cos\phi_r - 1 \\ \sin\phi_r \end{pmatrix} v$

$x(t) = x(0) = 0$

$y(t) = y(0) = 0$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}(0-t_{min})} \begin{bmatrix} x(t_{min}) \\ y(t_{min}) \end{bmatrix} + \left[\int_{t_{min}}^{0} e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}(0-\tau)} d\tau\right] \begin{bmatrix} \cos\phi_r - 1 \\ \sin\phi_r \end{bmatrix} v$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = I_{2\times2} \begin{bmatrix} x(t_{min}) \\ y(t_{min}) \end{bmatrix} - \left[\int_{0}^{t_{min}} \begin{pmatrix} d\tau & 0 \\ 0 & d\tau \end{pmatrix}\right] Bv$$

$$\boxed{\begin{bmatrix} x(t_{min}) \\ y(t_{min}) \end{bmatrix} = \begin{bmatrix} (\cos\phi_r - 1) \\ \sin\phi_r \end{bmatrix} v\,t_{min}}$$

- NOW CONSIDER THE SAFETY CONSTRAINT $x(t)^2 + y(t)^2 \geq r^2$

$$x(t_{min})^2 + y(t_{min})^2 = r^2$$

$$v^2 t_{min}^2 \left[(\cos\phi_r - 1)^2 + \sin^2\phi_r\right] = r^2$$

$$t_{min}^2 \left(\cancel{\cos^2\phi_r} - 2\cos\phi_r + 1 + \cancel{\sin^2\phi_r}\right) = \frac{r^2}{v^2}$$

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$$\boxed{t_{min} = -\frac{r}{v}\sqrt{\frac{1}{2(1-\cos\phi_r)}}}$$

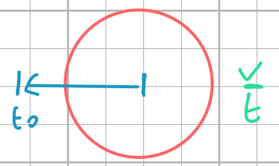
THE AIRCRAFTS MUST START THE MANOUVER AT LEAST $|t_{min}|$ SECONDS BEFORE THE ESTIMATED COLLISION TIME

- ANY MANOUVER TIME CHOSEN SUCH THAT $t_{man} \leq t_{coll} - |t_{min}|$ IS SAFE WITH A CURVATURE RADIUS LARGE ENOUGH

ii) TIME–RADIUS RELATIONSHIP

- CONSIDER A COLLISION AT TIME $t=0$ OF q_1. "GO BACK IN TIME" UNTIL YOU EXIT THE UNSAFE AREA AT TIME $t_0<0$.
- THEN GO FORWARD FROM t_0 TO $\bar{t}>0$, IN q_2, TO AVOID THE COLLISION

$\bar{t}-t_0=\pi$



FOR THE BACKWARDS PHASE IN q_1 WE HAVE:

$$\boxed{\begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} (\cos\phi_r - 1) \\ \sin\phi_r \end{bmatrix} v t_0}$$

- THEN WE HAVE A "BACKWARD RESET" OF $R(-\pi/2)$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = R\left(-\frac{\pi}{2}\right) \begin{bmatrix} \cos\phi_r - 1 \\ \sin\phi_r \end{bmatrix} v t_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos\phi_r - 1 \\ \sin\phi_r \end{bmatrix} v t_0 = \begin{bmatrix} \sin\phi_r \\ 1-\cos\phi_r \end{bmatrix} v t_0$$

- INTEGRATE FORWARD THE q_2 DYNAMICS BETWEEN t_0 AND t, WITH $t-t_0=\pi$

$$\begin{cases} \dot{x}_r = -v' + v'\cos\phi_r + \omega \\ \dot{y}_r = v'\sin\phi_r - x \\ \dot{\phi}_r = \phi \end{cases}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\pi} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \underbrace{\left[\int_{t_0}^{t} e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(t-\tau)} d\tau\right]}_{A^{-1}(e^{A(\pi)}-I)} B v'$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\begin{bmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{bmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & - \end{pmatrix} \right)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = -\begin{bmatrix} \sin\phi_r \\ 1-\cos\phi_r \end{bmatrix} v t_0 + \begin{bmatrix} 2v'\sin\phi \\ 2v'(1-\cos\phi) \end{bmatrix}$$

$v' = \omega R = R$

$$\boxed{\begin{aligned} x(t) &= \sin\phi_r\,(4R - v t_0) \\ y(t) &= (1-\cos\phi_r)(4R - v t_0) \end{aligned}}$$

- ROTATE FORWARD $R(\pi/2)$.

$$\begin{bmatrix} x(t^+) \\ y(t^+) \end{bmatrix} = R(\pi/2) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sin\phi_r\,(4R - v t_0) \\ (1-\cos\phi_r)(4R - v t_0) \end{bmatrix} = \begin{bmatrix} \cos\phi_r - 1 \\ \sin\phi_r \end{bmatrix} (4R - v t_0)$$

- NOW CONSIDER THE SAFETY CONSTRAINT $x(t)^2 + y(t)^2 \geq r^2$

$$x(t^*)^2 + y(t^*)^2 \geq r^2$$

$$(4R - v t_0)^2 \left[\cos\phi_r^2 - 2\cos\phi_r + 1 + \sin^2\phi_r\right] \geq r^2$$

$$\boxed{(4R - v t_0)^2 \geq \frac{r^2}{2 - 2\cos\phi_r}}$$

- FIXING ANY TIME $t_0 < t_{min}$ WE FIND A R_{min} THAT SATISFIES THE EQUALITY.
  THEREFORE, ANY $R \in [R_{min}, +\infty)$ GUARANTEES SAFETY
- FIXING ANY RADIUS $R > R_{min}$, $\exists\ \underline{t}, \bar{t}$ S.T. $\forall t \in [\underline{t}, \bar{t}]$ SAFETY IS GUARANTEED