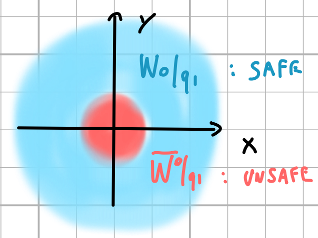
MAXIMAL SAFE SET

W^0



- $W^0|_{q_1}$: SAFE SET IN q_1 $\quad (x,y) \mid x^2 + y^2 \geq 25$
- $W^0|_{q_2}$: SAFE SET IN q_2 $\quad (x,y) \mid x^2 + y^2 \geq 25 \quad \forall z \in [0,\pi]$ "SEMI-CIRCLE TIME"
- $W^0|_{q_3} \equiv W^0|_{q_1}$

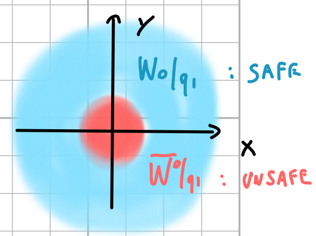
W^1

q_1

- $W^1|_{q_1} = W^0|_{q_1} \setminus CUP_{re}(\overline{W}^0|_{q_1}, DC\,Pre(W^0|_{q_1}))$

× $DC\,Pre(W^0|_{q_1})$: STATES IN $W^0|_{q_1}$ $\mid (\sigma, v_1)$ FORCES THE STATE IN $W^0|_{q_1}$

× THE ONLY WAY I CAN FORCE THE STATE IN THE SAFE SET $W^0|_{q_1}$ IS FROM $W^0|_{q_1}$ (THE SAFE SET ITSELF)



- THEREFORE $DC\,Pre(W^0|_{q_1}) \equiv W^0|_{q_1}$

× $CUP_{re}(\overline{W}^0|_{q_1}, W^0|_{q_1})$: $\forall v_1$ (CONTINUOUS CONTROL) $\exists v_2$ (CONT. DISTURBANCE) S.T. YOU CAN ENTER $\overline{W}^0|_{q_1}$ (UNSAFE) AVOIDING $W^0|_{q_1}$ (SAFE)

× SINCE $\overline{W}^0|_{q_1}$ IS "SURROUNDED" BY $W^0|_{q_1}$, YOU CAN ONLY ENTER IT FROM $W^0|_{q_1}$ AND HENCE $W^0|_{q_1}$ CANNOT BE AVOIDED

- THEREFORE $CUP_{re}(\overline{W}^0|_{q_1}, W^0|_{q_1}) \equiv \emptyset$

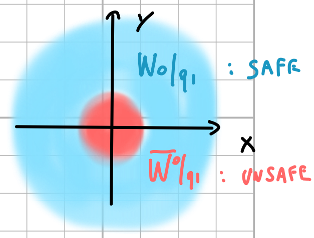
$W^1|_{q_1} = W^0|_{q_1} \setminus \emptyset \implies \boxed{W^1|_{q_1} \equiv W^0|_{q_1}}$

q_2

• $W^1|_{q_2} = W^0|_{q_2} \setminus CUPre(\bar{W}^0|_{q_2}, DCPre(W^0|_{q_2}))$

× $DCPre(W^0|_{q_2})$: STATES IN $W^0|_{q_2}$ | (σ, υ_1) FORCES THE STATE IN $W^0|_{q_2}$

× SINCE I'M IN q_2 AND σ CAN ONLY BE APPLIED IN q_1, THERE IS NO DISCRETE INPUT THAT FORCES ME TO STAY IN THE SAFE SET $W^0|_{q_2}$



- THEREFORE $DCPre(W^0|_{q_2}) \equiv \emptyset$

× $CUPre(\bar{W}^0|_{q_2}, \emptyset)$: $\forall \upsilon_1$ (CONTINUOUS CONTROL) $\exists \upsilon_2$ (CONT. DISTURBANCE) S.T. YOU CAN ENTER $\bar{W}^0|_{q_2}$ (UNSAFE) AVOIDING "NOTHING" $(DCPre(W^0|_{q_2}) \equiv \emptyset)$

- SINCE I HAVE TO AVOID NOTHING, THINK OF ALL THE STATES FOR WHICH I CAN ENTER $\bar{W}^0|_{q_2} = (x, y) \mid x^2 + y^2 \le 25 \ \ \forall \tau \in [0, \pi]$
"SEMI-CIRCLE TIME"

• TO DO THAT I NEED TO INTEGRATE BACKWARDS THE DYNAMICS IN q_2 STARTING FROM $t = 0$ AND GOING BACK UNTIL $t_0 = -\pi$

DYNAMICS IN q_2

$$\begin{cases} \dot{x}_r = -\upsilon' + \upsilon \cos\phi_r + y_r \\ \dot{y}_r = \upsilon' \sin\phi_r - x_r \\ \dot{\phi}_r = 0 \end{cases}$$

GENERAL EXPLICIT SOLUTION

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{A(t-t_0)} \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau$$

NOW CONSIDER, FOR THIS CASE:

$$t = 0$$
$$t_0 = \bar{t} \quad (\bar{t} \in [-\pi, 0))$$
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$B\upsilon = \begin{bmatrix} \upsilon'(\cos\phi_r - 1) \\ \upsilon' \sin\phi_r \end{bmatrix} = \text{CONSTANT}$$
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(0-\bar{t})} \begin{bmatrix} x(\bar{t}) \\ y(\bar{t}) \end{bmatrix} + \left[\int_{\bar{t}}^{0} e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(0-\tau)} d\tau \right] \begin{bmatrix} \upsilon'(\cos\phi_r - 1) \\ \upsilon' \sin\phi_r \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = e^{\begin{pmatrix} 0 & -\bar{t} \\ \bar{t} & 0 \end{pmatrix}} \begin{bmatrix} x(\bar{t}) \\ y(\bar{t}) \end{bmatrix} - \underbrace{\left[\int_{0}^{\bar{t}} e^{\begin{pmatrix} 0 & -\tau \\ \tau & 0 \end{pmatrix}} d\tau \right]}_{A^{-1}(e^{A(0-\bar{t})} - I)} \begin{bmatrix} \upsilon'(\cos\phi_r - 1) \\ \upsilon' \sin\phi_r \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} \cos(\bar{t})\, x(\bar{t}) - \sin(\bar{t})\, y(\bar{t}) \\ \sin(\bar{t})\, x(\bar{t}) + \cos(\bar{t})\, y(t) \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} \cos(\bar{t}) - 1 & -\sin(\bar{t}) \\ \sin(\bar{t}) & \cos(\bar{t}) - 1 \end{bmatrix} \begin{bmatrix} \upsilon'(\cos\phi_r - 1) \\ \upsilon' \sin\phi_r \end{bmatrix}$$

$\Downarrow$ $\upsilon' = WR = 1 \cdot R = R$ CURVATURE RADIUS

$$\begin{bmatrix} \cos(\bar{t})\, x(\bar{t}) - \sin(\bar{t})\, y(\bar{t}) \\ \sin(\bar{t})\, x(\bar{t}) + \cos(\bar{t})\, y(t) \end{bmatrix} = \begin{bmatrix} \sin(\bar{t}) & \cos(\bar{t}) - 1 \\ 1 - \cos(\bar{t}) & \sin(\bar{t}) \end{bmatrix} \begin{bmatrix} R(\cos\phi_r - 1) \\ R \sin\phi_r \end{bmatrix}$$

$$\Downarrow$$

$$\boxed{\begin{bmatrix} \cos(\bar{t})\, x(\bar{t}) - \sin(\bar{t})\, y(\bar{t}) \\ \sin(\bar{t})\, x(\bar{t}) + \cos(\bar{t})\, y(t) \end{bmatrix} = \begin{bmatrix} \sin(\bar{t})(\cos\phi_r - 1) + \sin\phi_r(\cos\bar{t} - 1) \\ (1 - \cos\bar{t})(\cos\phi_r - 1) + \sin\bar{t} \sin\phi_r \end{bmatrix} R}$$

AT THE BEGINNING OF THE CIRCLE MANOUVER, WE HAVE $\bar{t} = -\pi$

$$\bar{t} = -\pi \Rightarrow \begin{bmatrix} -x(\bar{t}) \\ -y(\bar{t}) \end{bmatrix} = \begin{bmatrix} -2\sin\phi_r \\ 2(\cos\phi_r - 1) \end{bmatrix} R \Rightarrow \begin{cases} x(\bar{t}) = 2R\sin\phi_r \\ y(\bar{t}) = 2R(1-\cos\phi_r) \end{cases}$$

$4R^2\sin^2\phi +$

$4R^2(1-\cos\phi)^2$

$4R^2(1-2\cos\phi+\cos^2\phi)$

$4R^2 + 4R^2 - 8R^2\cos\phi$

$R(\sqrt{2-2\cos\phi})$

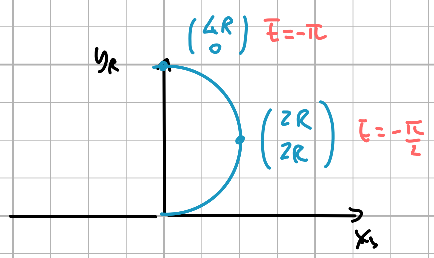
ϕ_r

$$\bar{t} = -\frac{\pi}{2} \Rightarrow \begin{bmatrix} y(\bar{t}) \\ -x(\bar{t}) \end{bmatrix} = \begin{bmatrix} 1-\cos\phi_r & -\sin\phi_r \\ \cos\phi_r - 1 - \sin\phi_r \end{bmatrix} R \Rightarrow \begin{cases} x(\bar{t}) = R(1+\sin\phi_r - \cos\phi_r) \\ y(\bar{t}) = R(1-\sin\phi_r - \cos\phi_r) \end{cases}$$

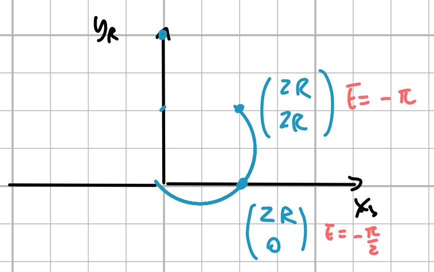
THEREFORE, WE CAN DRAW OUR RELATIVE TRAJECTORY TO FIND THE SAFE SET, ADDING A r-RADIUS TO THE SEMICIRCLE TRAJECTORY:

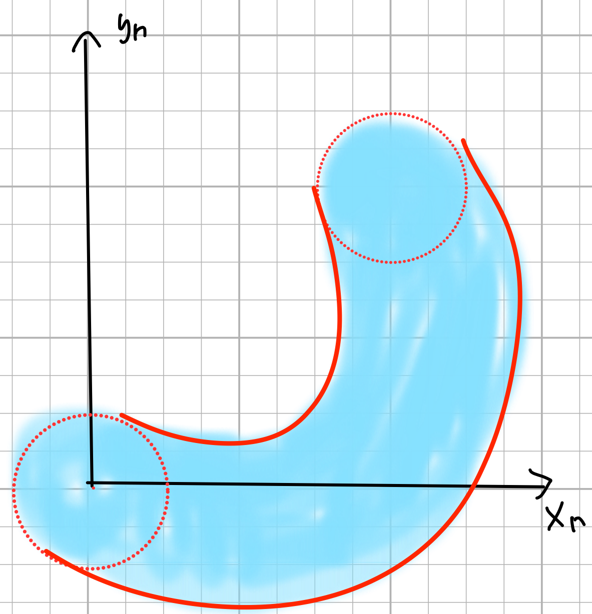
BACKWARDS INTEGRATED TRAJECTORY

$\phi_r = \pi$



$\phi_r = \frac{\pi}{2}$

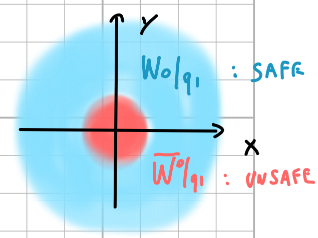




- $W^1|_{q_3} = W^0|_{q_3} \setminus CUP_{re}(\bar{W}^0|_{q_3}, DC\,P_{re}(W^0|_{q_3}))$

× $DC\,P_{re}(W^0|_{q_3})$: STATES IN $W^0|_{q_3}$ | (σ, v) FORCES THE STATE IN $W^0|_{q_3}$

× SINCE I'M IN q_3 AND σ CAN ONLY BE APPLIED IN q_1, THERE IS NO DISCRETE INPUT THAT FORCES ME TO STAY IN THE SAFE SET $W^0|_{q_3}$



- THEREFORE $DC\,P_{re}(W^0|_{q_3}) \equiv \emptyset$

× $CUP_{re}(\bar{W}^0|_{q_3}, \emptyset)$: $\forall v_1$ (CONTINUOUS CONTROL) $\exists v_2$ (CONT. DISTURBANCE) S.T. YOU CAN ENTER $\bar{W}^0|_{q_3}$ (UNSAFE) AVOIDING "NOTHING" $(DC\,P_{re}(W^0|_{q_3}) \equiv \emptyset)$

- SINCE I HAVE TO AVOID NOTHING, THINK OF ALL THE STATES FOR WHICH I CAN ENTER $\bar{W}^0|_{q_3} = (x,y) \mid x^2+y^2 \le 25$

- TO DO THAT I NEED TO INTEGRATE BACKWARDS THE DYNAMICS IN q_3 STARTING FROM $t = 0$ AND GOING BACK UNTIL $t_0 = \bar{t}$

DYNAMICS IN q_3

$$\begin{cases} \dot{x}_r = -v' + v\cos\phi_r \\ \dot{y}_r = v\sin\phi_r \\ \dot{\phi}_r = 0 \end{cases}$$

GENERAL EXPLICIT SOLUTION

$$\begin{bmatrix} X(t) \\ y(t) \end{bmatrix} = e^{A(t-t_0)} \begin{bmatrix} X(t_0) \\ y(t_0) \end{bmatrix} + \int_{t_0}^{t} e^{A(t-\tau)} B\,u(\tau)\,d\tau$$

NOW CONSIDER, FOR THIS CASE:

$t = 0$

$t_0 = \bar{t} < 0$

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$Bv = \begin{bmatrix} v(\cos\phi_r - 1) \\ v\sin\phi_r \end{bmatrix}$ = CONSTANT

$\begin{bmatrix} X(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} X(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} X(0) \\ y(0) \end{bmatrix} = e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}(0-\bar{t})} \begin{bmatrix} X(\bar{t}) \\ y(\bar{t}) \end{bmatrix} + \left[\int_{\bar{t}}^{0} e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}(0-\tau)} d\tau\right] \begin{bmatrix} v(\cos\phi_r - 1) \\ v\sin\phi_r \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = I_{2\times2} \begin{bmatrix} X(\bar{t}) \\ y(\bar{t}) \end{bmatrix} - \left[\int_{0}^{\bar{t}} I_{2\times2}\, d\tau\right] \begin{bmatrix} v(\cos\phi_r - 1) \\ v\sin\phi_r \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} X(\bar{t}) \\ y(\bar{t}) \end{bmatrix} = \begin{bmatrix} \bar{t} & 0 \\ 0 & \bar{t} \end{bmatrix} \begin{bmatrix} v(\cos\phi_r - 1) \\ v\sin\phi_r \end{bmatrix} \Rightarrow \begin{cases} X(\bar{t}) = v(\cos\phi_r - 1)\,\bar{t} \\ y(\bar{t}) = v\sin\phi_r\,\bar{t} \end{cases}$$

THEREFORE, WE CAN DRAW OUR RELATIVE TRAJECTORY TO FIND THE SAFE SET, ADDING A r-RADIUS TO THE LINEAR TRAJECTORY: