Multivariable Feedback Control
Homework 02 - due Sunday April 28th, 2024.

The goal of this homework is to explore ***limitations*** due to the presence of non-minimum phase zeros and study ***robust stability***.

We want to design a control system for the single flexible arm of Fig. 1 which is characterized by having uniform linear density ρ, length ℓ, cross-section inertia I and Young modulus E. The motor at the base has an inertia J_0 while the payload at the tip has mass m_p and inertia J_p. We represent the beam deflection in a rotating frame (x, y) having the x axis passing through the instantaneous arm center of mass (CoM) and making an angle $\vartheta(t)$ with the (X, Y) frame.

Under proper hypotheses and for small deformations, if we consider a finite number of flexible modes n_e, the deflection $w(x, t)$ can be written as $w(x, t) = \sum_{i=1}^{n_e} \phi_i(x)\delta_i(t)$ where $\phi_i(x)$ is the spatial shape of the deformed arm associated to the ith mode and $\delta_i(t)$ is the time dependent weight of the shape $\phi_i(x)$. Assuming also the presence of modal damping, the resulting finite dimensional dynamic model is given by

$$J\ddot{\vartheta} = \tau$$
$$\ddot{\delta}_i + 2\zeta_i\omega_i\dot{\delta}_i + \omega_i^2\delta_i = \phi_i'(0)\tau, \qquad i = 1, \ldots, n_e$$

where $J = J_0 + (\rho\ell^3)/3 + J_p + m_p\ell^2$, while $\zeta_i \in [0, 1)$ and ω_i are respectively the damping coefficient and the eigenfrequency of the ith mode while τ is the motor torque. For example, considering a single flexible mode $n_e = 1$, one has the following state space representation

$$z = \begin{pmatrix} \vartheta \\ \delta_1 \\ \dot{\vartheta} \\ \dot{\delta}_1 \end{pmatrix}, \quad \dot{z} = Az + B\tau, \quad \text{with} \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -\omega_1^2 & 0 & -2\zeta_1\omega_1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{J} \\ \phi_1'(0) \end{pmatrix}$$

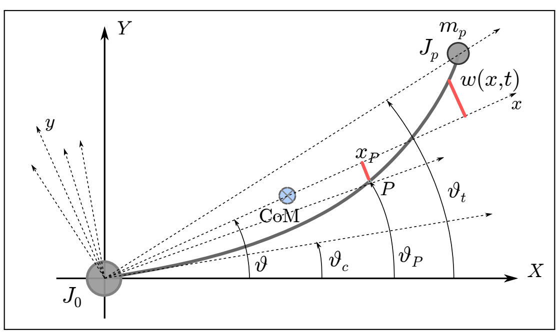

Figure 1: The single flexible link robotic arm

There are different possible choices of the output and therefore different transfer functions.

a) Joint angle

$$y_c = \vartheta_c = \vartheta + \sum_{i=1}^{n_e} \phi_i'(0)\delta_i$$

The resulting system is minimum-phase. Let's define $P_c(s)$ the resulting transfer function.

b) Tip angle

$$y_t = \vartheta_t = \vartheta + \sum_{i=1}^{n_e} \frac{\phi_i(\ell)}{\ell}\delta_i$$

which leads in general to a non-minimum phase system (depends on the parameter values and in particular on the payload inertia). Let's define $P_{tip}(s)$ the resulting transfer function.

c) Point P angle

$$y_P = \vartheta_P = \vartheta + \sum_{i=1}^{n_e} \frac{\phi_i(x_P)}{x_P}\delta_i, \qquad x_P \in (0, \ell)$$

which, for a given set of parameters, can be either minimum-phase or not depending on where P is. Let's define $P_P(s)$ the resulting transfer function.

Main objectives:

1. Start with $n_e = 1$ (only one elastic mode). Choose a set of parameters (get inspiration from the Matlab file) so that you have a reasonable values for the eigenfrequency (the higher the more numerical problems you may encounter) and the system is non-minimum phase when choosing the tip output ($P_{tip}(s)$). Explore and apply as many results as possible regarding the limitations arising from the presence of a non-minimum phase zero. Some ideas you could explore:

   (a) evaluate numerically the sensitivity integrals and other quantities of interest for both the minimum and non-minimum phase cases. In both systems we assume that the measured variable coincides with the controlled one (for example, when considering $P_c(s)$, the output is the joint angle $y = \vartheta_c$ in a standard output feedback scheme).

   (b) Put in evidence through simulations the limitation arising from the presence of the non-minimum phase zero(s).

   (c) (Extra) You may want to explore if there is any difference/improvement using the augmented plant framework, with ϑ_c being the measured variable and the tip tracking error as performance variable.

   (d) (Extra) What happens if you start including more flexible modes?

2. Study robust stability.

Material

1. Slides `Flexible_Link_ADL.pdf` and paper ICRA 1990 (just for a complete reference).

2. Matlab files `EulerBeam_Modeling2Modes.m` which gives the quantities ω_i, $\phi_i'(0)$, $\phi(\ell)$ and calls other files `r2reig.m`, `r2rfind.m`, `r2rmod.m`, `r2rnorm.m`.

3. A LiveScript Matlab file `Beam_Control.mlx` as a generic reference (you can use it or not)

Feel free to ask any question.