Population Modeling of Dragons Inspired by the Game of Thrones Series Utilizing a Modified Lotka-Volterra Model

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Abstract

Dragons, depicted as apex predators in the fictional television series Game of Thrones, provide a unique lens to study predator-prey interactions in ecosystems. This project examines the ecological impact of dragons if introduced into modern ecosystems, using mathematical models to simulate their effects. We adapt the Lotka-Volterra predator-prey equations to represent dragon-prey dynamics while incorporating environmental factors such as habitat size, prey availability, and human activity. Our findings indicate that supporting even a small population of dragons requires vast resources and a delicate balance of ecosystem components. Additionally, the introduction of dragons results in significant shifts in population dynamics, emphasizing the fragile equilibrium of ecological systems. These results provide insight into the challenges posed by introducing large, apex predators into existing ecosystems.

1 Introduction

In the television series Game of Thrones, based on A Song of Ice and Fire by George R.R. Martin, dragons are portrayed as powerful apex predators with unique traits like fire-breathing and long lifespans[1]. While fictional, dragons can be studied by basing many of our assumptions on principles of evolution, real-world biology, and ecology. As top predators, their presence would have significant impacts on any ecosystem they inhabit, requiring vast resources and potentially disrupting existing ecological balances. This project examines what might happen if dragons existed in today's world. Using mathematical models, we explore how dragons could potentially interact with prey, their environment, and humans. We aim to understand the balance needed to sustain dragons and how their introduction might affect biodiversity and ecosystem stability.

Mathematical models are important tools for understanding ecological systems. The Lotka-Volterra equations, a key model for predator-prey dynamics, describe how predators and prey interact over time through growth and consumption rates [6]. We adapted the Lotka-Volterra model to include seasonal changes, mortality due to prey scarcity, and a logistic growth term for dragon growth. These changes make the model closer to lore-accurate dragon dynamics and illuminate how dragons could affect modern ecosystems.

To simplify our model, we employed many assumptions. We assumed dragons are assumed to share biological traits with large, cold-blooded reptiles like komodo dragons and caimans, enabling the application of established principles such as metabolic rates, energy efficiency, and predatory behaviors to their ecological modeling. Westeros' environment is considered stable during the study, with no large-scale disasters like meteor impacts or volcanic eruptions, and existing ecosystems are assumed to be in equilibrium, simplifying the model by providing a baseline for measuring the dragons' impact. Dragons are positioned at the top of the food chain due to their immense size, strength, and predatory dominance, with no natural predators and the ability to influence all other species. Activities such as flying and fire-breathing are presumed to require twice their normal caloric intake, significantly shaping their predatory behavior and ecosystem energy demands. Dragons are also modeled to grow logistically, starting at 10kg at birth, reaching 40kg after one year, and continuing to grow until a maximum mass of 120,000kg, with their development modeled using the Von Bertalanffy Growth Equation.

2 Background

The introduction of dragons into modern ecosystems can be modeled mathematically by integrating key principles from population dynamics, ecological carrying capacity, and growth models. This section details the derivation of the equations and their underlying assumptions.

2.1 Von Bertalanffy Growth Model

To model the growth of the dragon we evaluated the work of Von Bertalanffy's growth rate equation [2] [7], and used the process outlined in the article to inform changes to our model. Operating under the assumption that a dragon's volume is proportional to two thirds the power of the mass of the dragon, we make use of the following adaptation of the VB equation measured with time in months. The term $3kM^{2/3}$ is the mass scaling factor, determined by the dragon's mass to the two thirds power and the growth rate k. The 3 term simply exists to establish the scaling behavior of mass in the model.Integrating Equation (1) yields Equation (2), the mass function with respect to time. Using the initial conditions for mass denoted in the prompt, in conjunction with our assumed maximum mass of $1.2 \cdot 10^5$ kg, we can find key values for our analysis using the $M(t)$ function. It follows that $M(0) = 10\text{kg}, M(12) = 40\text{kg}$, and $\lim_{t\to\infty} M(t) = 1.2 \cdot 10^5\text{kg}$. Using these values and solving equation (2) for k yields a value of $2.27 \cdot 10^{-3}$, which is our growth rate of the dragon. The relationship between the dragons mass M and time t is modeled in Figure 1, with the line showing the dragons mass at the given time, and the color below indicating the growth rate at that time, darker colors indicate a faster growth rate.

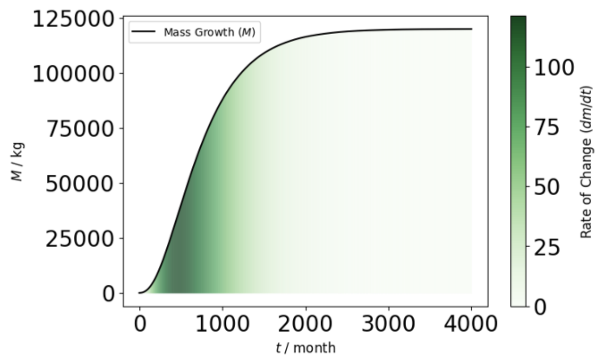

Figure 1: This graph highlights the relationship between a dragon's mass M and time t. The color indicates the growth rate at that time, darker colors indicate a faster growth rate.

$$\frac{dM}{dt} = 3kM^{2/3}(M_\infty^{1/3} - M^{1/3}) \tag{1}$$

$$M(t) = [(1 - e^{-kt})(M_\infty^{1/3} - M_0^{1/3})]^3 \tag{2}$$

2.2 Carrying Capacity for Dragons

As with most creatures, we assume that larger dragons have higher caloric needs, hence we develop a carrying capacity function $N_{max}(A, P, M)$ to determine the maximum number of dragons the environment can support at any given time. This function is dependent on 2 factors, the number of dragons the current prey population can support, and the number of dragons the available area can support. To do this we first defined the area required to support a dragon of mass M, defined by equation (3)

Equation (3) denotes that the area required per dragon is proportional to two thirds is mass and inversely proportional to the prey population[2]. The area constraint A_{Max} and equation (3) allows us to determine the number of dragons the available area can support as equation (4). The number of dragons supported by the current prey population is determined by equation (5). Equation (5) operates under the assumption that

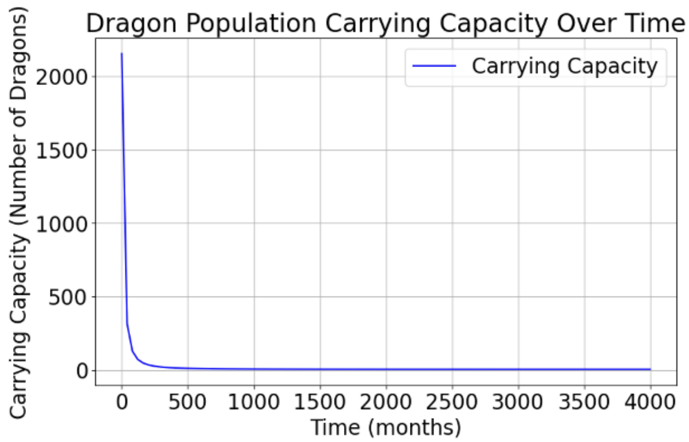

Figure 2: This graph highlights the change in the dragon population carrying capacity over time.

the food required by a dragon scales proportionally to its mass. Subtracting N_{Prey} from N_{Land} allows us to define their difference Δ as equation (6). Now that we have Δ we are able to write our function for N_{Max}. The denominator of equation (7) includes a smoothing factor s=10 applied to the function. This changes the transition between the dominant parameters to be smooth, allowing us to calculate a single curve for our carrying capacity function [4][6]. It modifies the behavior of N_{Max} such that when Δ is very positive the (7) converges to the value of N_{Prey}, when Δ is very negative (7) converges to the value of N_{Land}, and when Δ is an intermediate value (7) will evaluate as a weighted average between N_{Land} and N_{Prey}. Equation (8) ensures that the carrying capacity function does not evaluate to be negative. The carrying capacity as a function of time can be seen in Figure 2 below.

$$A_d = M^{2/3}(\frac{A_{\text{Max}}}{P}) \tag{3}$$

$$N_{\text{Land}} = (\frac{A_{\text{Max}}}{A_d}) \tag{4}$$

$$N_{\text{Prey}} = (\frac{P}{M}) \tag{5}$$

$$\Delta = N_{\text{Land}} - N_{\text{Prey}} \tag{6}$$

$$N_{\text{Max}} = N_{\text{Max}} + \frac{\Delta}{1 + e^{-s\Delta}} \tag{7}$$

$$N_{\text{Max}} = \max(N_{\text{Max}}, 0) \tag{8}$$

3 Our Model

To examine the interactions between dragon and prey populations, we adopted a basic version of the Lotka-Volterra model [9][10]. Equation (10) represents change in the prey population and equation (11) defines change in the dragon population. Our equations are parameterized by the following: P (prey population measured in Gcal), D (dragon population measured in number of dragons), $c = 0.15$ (prey reproduction rate), $e = 0.01$ (predation rate of dragons hunting prey), $a = 0.02$ (predator conversion efficiency, prey to dragon growth), $M = 0.001$ (dragon mortality rate). To add complexity to the model, we made modifications 3 specific modifications to the dragon population model.

$$\dot{P} = P\,(c - e \cdot D) \tag{9}$$

$$\dot{D} = D(a \cdot P - m) \tag{10}$$

3.1 Seasonal Efficiency Factor

Operating under our assumption that dragons are cold-blooded creatures, we note that they are more effective predators in warmer months of the year. To incorporate this we utilized a sine function scaled by a parameter for seasonal dependence[3]. With this in mind, equation 12 describes our season variation parameter for dragons within our model. It is parameterized by r, which scales the impact of seasonal changes on the dragon population, and the period $T = 12$ allows us to define 2 seasons in a year, a 6 month summer followed by a 6 month winter. The sine term encapsulates the inefficiencies of hunting in the winter and promoting hunting in the summer.

$$r \cdot \sin\left(\frac{2\pi t}{T}\right) \tag{11}$$

3.2 Mortality Due to Prey Scarcity

The population of dragons is heavily dependent on the available caloric resources, so in order to model the decline in dragon population that would inevitably result from a lower prey population, we develop equation (13), where P_{Max} is the maximum prey population the environment can support. This term allows us to scale the impact of a declining prey population on dragons.

3.3 Logistic Growth Term

To ensure the dragon population does not exceed the constraints posed by the environment, we also include a logistic growth term that limits the dragon population's growth based on equation (7).

$$1 - \frac{D}{N_{\mathrm{Max}}} \tag{12}$$

The adoption of equations (12), (13), and (14) into equation (11) gives us our final model of the dragon-prey interaction as equation (10) representing the change in the prey population and equation (15) representing the changing dragon population. These combine to form equation (16), our final 2D population model.

$$\begin{cases} \dot{D} = D \cdot \left((r \cdot \sin(2\pi \cdot \frac{t}{T}) + a \cdot P - m) \cdot (1 - \frac{D}{N_{\max}(M(t),P,A)}) - (1 - \frac{P}{P_{\max}}) \right) \\ \dot{P} = P \cdot (c - e \cdot D) \end{cases} \tag{13}$$

4 Results

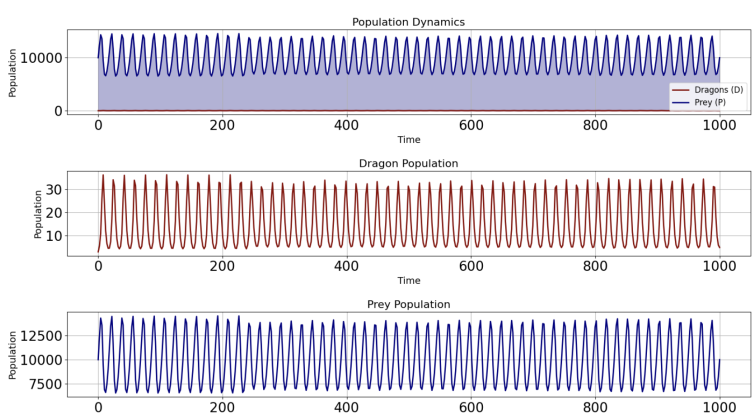

Figure 3: This graph highlights the change of the dragon and prey populations with time. The initial conditions for this run were 3 drgaons and 10,000 gigacalories of available prey.

For a worked example of our model, we chose an initial dragon population of 3, and an initial prey population of 10,000 gigacalories of available prey. We found that our dragon population was very oscillatory and constantly changed with time. This heavily impacts our prey population, and ultimately leads to oscillations in our prey population with time as well. This is an interesting result given our model formulation since our prey population had very limited complexity, leading to the growth rate of the prey population being solely dependent on a fixed birth rate.

Figure 3 highlights the population change over time for the dragon and prey populations. In this example, we can clearly see the initial population of 3 dragons and 10,000 gigacalories of prey immediately increasing to approximately 35 and 14,000 respectively. They then oscillated between these values and 5 dragons and 7500 gigacalories respectively. The code for this example was modified from examples in the scientific python doumentation [5].

With this same worked example we also analyzed the phase portrait in an attempt to understand the long-term behavior of the dragon and prey populations. This is highlighted in Figure 4, where our later time steps are slowly approaching a lower dragon and prey population along an elliptical path. This elliptical path is expected given the oscillations between the populations. Furthermore, the later time steps appear to remain along the same elliptical trajectories, hinting that the long term population dynamics of our dragon and prey populations will be along closed curves and not settle to a fixed point.

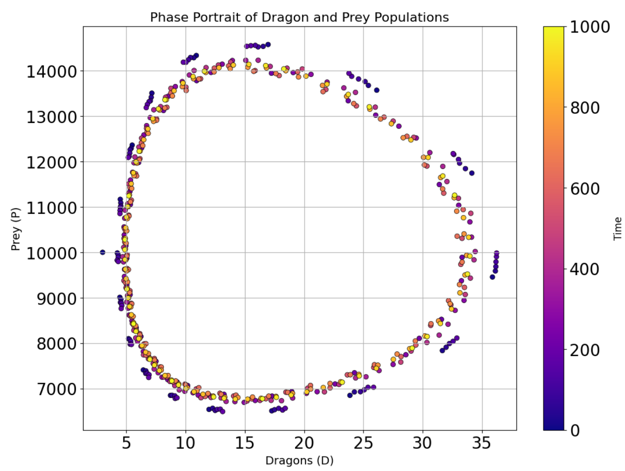

Figure 4: This graph showcases the phase portrait for the worked example of the dragon population model. The initial conditions were 3 dragons and 10,000 gigacalories of available prey.

5 Conclusions and Discussion

Our model successfully implemented a modified Lotka-Volterra model to simulate dragon-prey dynamics in a stable ecosystem. The incorporation of seasonal variations resulted in variation in the oscillations, rather than oscillating to the same populations, maxima and minima were influenced by the effect of seasonal changes on dragon's hunting efficiency. The carrying capacity model highlighted the effect of the increasing size of the dragons over time, modeled by the Von Bertalanffy Growth Model for mass, and successfully prevented unrealistic growth of the dragon population by accounting for prey availability and land availability constraints. All of these complications resulted in stable oscillatory behavior in the dragon-prey model; the dragon population oscillates between 5-35 individuals, while the prey population oscillates between 7,000-13,000 Gcal. The stable oscillations in our model indicate that natural regulatory mechanisms could prevent both prey extinction and predator overpopulation. The current model, while effective at stimulating basic dragon-prey dynamics, has several inherent limitations in its mathematical framework.The base Lotka-Volterra model used for the prey population assumes a stable, uniform prey population, and does not account for intricate interactions between dragons and different types of prey. Additionally our use of sinusoidal seasonal effects, while illuminating, does present an oversimplified view of climate effects. We make a number of simplifying assumptions about the environment and biological factors of the dragon population that are not reflective of a real world ecosystem, not limited to a stable ecosystem, equal spatial distribution of resources, and constant dragon mortality rates. In order to improve upon these limitations, the model could be enhanced by integrating multiple prey species with different characteristics, developing a more complex climate interaction model, adding an age structure to dragon and prey populations to improve realism of mortality rates, and incorporating stochastic elements to model random environmental events.

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A Group Contributions

Mike: Assisted with code examples, specifically in creating the 3-panel population dynamics graph. Developed the inclusion of seasonal variations, including research into the modeling techniques and implementation of the term into our model and coded solution. Wrote about the seasonal parameter and results, and compiled the final Latex document.

Luca: Assisted with construction of the Dragon-Prey model and helped test code to determine the parameter values that netted realistic results. Developed the Growth Model and Carrying Capacity functions and integrated them into the Lotka-Volterra Model. Wrote code to visualize dragon growth over time and solve for the dragons' growth rate k. Contributed sections 2.1, 2.2, 3.1, 3.2, 3.3 and 5 to the writeup.

Derek: Conducted research to develop most of the assumptions for the project, including dragon biology, energy requirements, and environmental interactions, and presented them to the group. Conducted additional research on apex predator ecology to estimate spatial and dietary needs for dragons. Assisted with constructing the Dragon-Prey model by incorporating climate influences, dragon-specific characteristics, and carrying capacities for prey. Helped define environmental scenarios with seasonal variations and prey distributions to reflect realistic ecological conditions. Contributed to the findings summary and helped prepare the presentation.

B Code Availability

Here is a link to our open-access jupyter notebook for creating all of our worked examples https://github.com/mlundqui/mlundqui.github.io/blob/master/Math142Project.ipynb.