Automata & Logic HW12 26 Jan 22 FABIAN SCHAUB

1) $FL(\phi)$ is the smallest set containing $\phi$, which fulfills:

- $\phi_1 \vee \phi_2 \in FL(\phi) \Rightarrow \{\phi_1, \phi_2\} \subseteq FL(\phi)$
- $\phi_1 \wedge \phi_2 \in FL(\phi) \Rightarrow \{\phi_1, \phi_2\} \subseteq FL(\phi)$
- $X\phi_1 \in FL(\phi) \Rightarrow \{\phi_1\} \subseteq FL(\phi)$
- $\phi_1 U \phi_2 \in FL(\phi) \Rightarrow \{\phi_1, \phi_2, X(\phi_1 U \phi_2), \phi_1 \wedge X(\phi_1 U \phi_2), \phi_2 \vee (\phi_1 \wedge X(\phi_1 U \phi_2))\} \subseteq FL(\phi)$
- $\phi_1 R \phi_2 \in FL(\phi) \Rightarrow \{\phi_1, \phi_2, X(\phi_1 R \phi_2), \phi_1 \vee X(\phi_1 R \phi_2), \phi_2 \wedge (\phi_1 \vee X(\phi_1 R \phi_2))\} \subseteq FL(\phi)$

Prove: $|FL(\phi)| \overset{?}{\in} O(|\phi|)$

Induction on $\phi$:

<u>Base</u>: $\phi := p$: $FL(\phi) = \{p\} \Rightarrow |FL(\phi)| = 1$
$|\phi| = 1$

$\phi := \neg p$: $FL(\phi) = \{\neg p\} \Rightarrow |FL(\phi)| = 2$
$|\phi| = 2$

<u>Step</u>: $FL(\phi_1 \vee \phi_2) \supseteq \{\phi_1, \phi_2\} \Rightarrow |FL(\phi_1 \vee \phi_2)| = |FL(\phi_1)| + |FL(\phi_2)| + \underset{\phi_1 \vee \phi_2}{1}$

~~$\overset{IH}{\Rightarrow} |FL(\phi)| \in O(|FL(\phi_1)| + |FL(\phi_2)|) \subseteq O(|FL(\phi)|)$~~

$\overset{IH}{\Rightarrow} |FL(\phi)| \in O(|\phi_1| + |\phi_2| + 1) \subseteq O(2|\phi| + 1) = O(|\phi|)$

Same for $\phi_1 \wedge \phi_2$

- $FL(\phi_1 U \phi_2) \supseteq \{\phi_1, \phi_2, X(\phi_1 U \phi_2), \phi_1 \wedge X(\phi_1 U \phi_2), \phi_2 \vee (\phi_1 \wedge X(\phi_1 U \phi_2))\} =: N$

$\Rightarrow FL(\phi_1 U \phi_2) = N \cup \bigcup_{n \in N} FL(n) = N \cup FL(\phi_1) \cup FL(\phi_2)$

$\Rightarrow |FL(\phi)| = |N| + \underset{\phi_1 U \phi_2}{1} + |FL(\phi_1)| + |FL(\phi_2)| \overset{IH}{\Rightarrow} FL(\phi) \in O(6 + |\phi_1| + |\phi_2|) = O(|\phi|)$

Same for $\phi_1 R \phi_2$ $\square$

2) $P := \emptyset$, $Q := \{p\}$, $R := \{q\}$, $S := \{p, q\}$

a) $P^* Q^* R^* \Sigma^\omega \equiv \top U (p U (q U ((p \vee q))))$

b) $(P^+ Q^+ R^+ S^+)^\omega \equiv G((I \wedge X(I U (p \wedge X(p U (q \wedge X(q U I))))))$

$\vee (p \wedge X(p U (q \wedge X(q U (I \wedge X(I U p))))))$

$\vee (q \wedge X(q U (I \wedge X(I U (p \wedge X(p U q))))))$

$) \wedge T$

c) $\{w \in \Sigma^\omega \mid |w|_P = \infty \Rightarrow |w|_Q = \infty\}$

$\equiv G(\bot \vee F(p U T))$

d) $(\Sigma^* P \Sigma^* Q \Sigma^* R)^\omega$

$\cancel{\equiv G(((p \vee q) U T) \vee (T U (p \vee q)) \vee ((p \vee q) U p) \vee (p U (p \vee q)) \vee (p \vee q) U}$

$\equiv ((p \vee q) U T) \wedge (G((p \vee q) U (T \wedge X((p \vee q) U (p \wedge X((p \vee q) U q))))))$

$\vee G(((p \vee q) U p) \vee ((p \vee q) U q)))$

e) $\{w \in \Sigma^{\omega} \mid \text{between every two } S\text{, there is at least one } R\}$

$\equiv G\big((p \wedge q) \rightarrow X(\cancel{\;}\neg(p \wedge q)\, U (q \wedge$

$\equiv G\Big(\big((p \wedge q) \wedge XF(p \wedge q)\big) \rightarrow X\big(\neg(p \wedge q)\, U q\big)\Big)$

3) $G\phi \overset{?}{\equiv} \phi \wedge XG\phi$

$G\phi \equiv \neg(\top\, U \neg\phi) \equiv$

$\equiv \neg\big(\neg\phi \vee (\top \wedge X(\top\, U \neg\phi))\big)$

$\equiv \neg\big(\neg\phi \vee X(\top\, U \neg\phi)\big)$

$\equiv \phi \wedge \neg X(\top\, U \neg\phi)$

$\equiv \phi \wedge X\neg(\top\, U \neg\phi)$

$\equiv \phi \wedge XG\phi \quad \square$