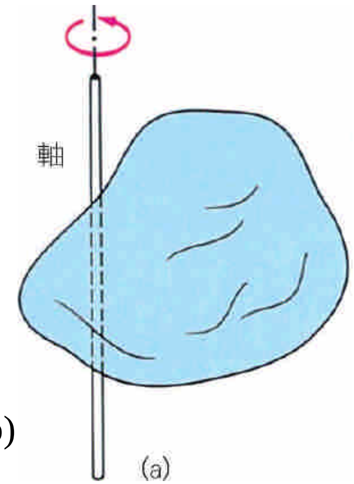
剛體的定軸轉動
(Rotation of a Rigid Body about a Fixed Axis)

- 剛體(Rigid body)⇒形狀及大小固定不變的物體。
- 定軸(Fixed axis) ⇒轉軸相對於物體及慣性座標系的方向皆為固定。
- 純轉動運動(Pure rotation motion)
  ⇒轉軸位置與方向固定。(如Fig.11.1a)
- 滾動運動(Rolling motion)
  ⇒轉軸位置改變，但方向固定。(如Fig.11.1b)
- 迴旋運動(Gyroscopic motion)－迴轉儀(或陀螺儀)運動
  ⇒轉軸位置與方向皆改變。(選擇性教材12.9,p.250)



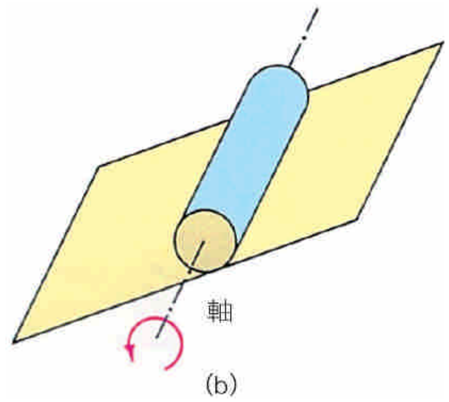
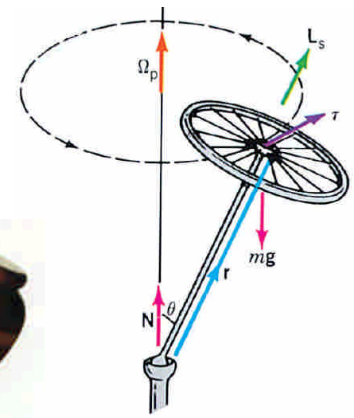
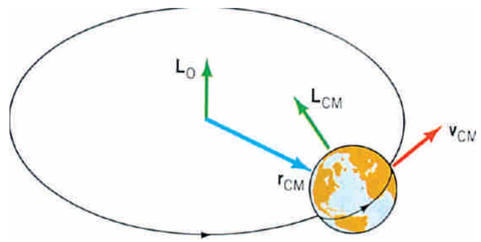


Fig.11.1





轉動運動學(rotational kinematics)

- 剛體以固定軸轉動一段時間，剛體上任一質點的位移皆不同，但角位移卻相同，故轉動運動須以角位移表示。

- 轉動角(或角位移)的定義：

$$\theta(radian) = s(\text{弧長}) / r(\text{半徑})$$

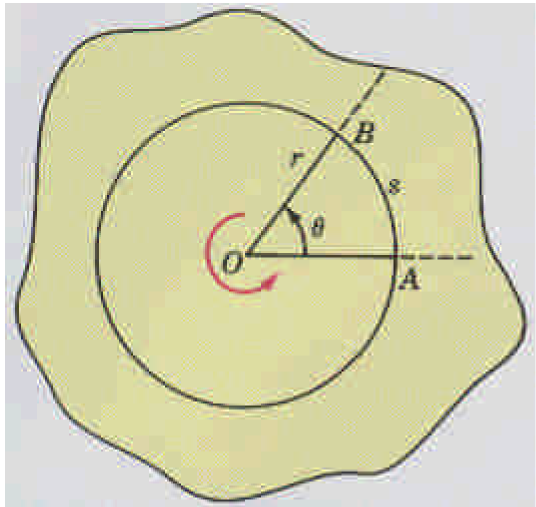


Fig.11.2

- 平均角速度的定義：

$$\Rightarrow \omega_{av} = \frac{\Delta\theta}{\Delta t} = \frac{\text{角位移}}{\text{時間差}}$$

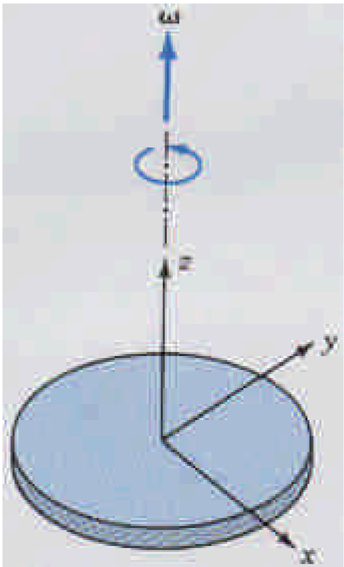


Fig.11.3

- 瞬間角速度的定義：

$$\Rightarrow \omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$ （具向量）

- 方向⇒由固定轉軸 z 軸往下看，逆時針為正，順時針為負。
- 線速度與角速度的關係：

$$\Rightarrow \omega = \frac{d\theta}{dt} = \frac{ds/r}{dt} = (\frac{ds}{dt})(\frac{1}{r}) \xrightarrow{\because \mathrm{v}=ds/dt} \mathrm{v} = \underline{r\omega}$$

- 平均角加速度的定義 $\Rightarrow \alpha_{av} = \frac{\Delta\omega}{\Delta t} = \frac{\text{角速度變化量}}{\text{時間變化量}}$

- 瞬間角加速度的定義 $\Rightarrow \alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$

- 切線加速度(tangential acceleration)與角加速度的關係：

$$\Rightarrow \alpha = \frac{d\omega}{dt} = \frac{d\mathrm{v}/r}{dt} = (\frac{d\mathrm{v}}{dt})(\frac{1}{r}) \xrightarrow{\because a_t=d\mathrm{v}/dt} a_t = \underline{r\alpha}$$

➢法線加速度(或向心加速度)

$$\Rightarrow a_r = \frac{\mathrm{v}^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

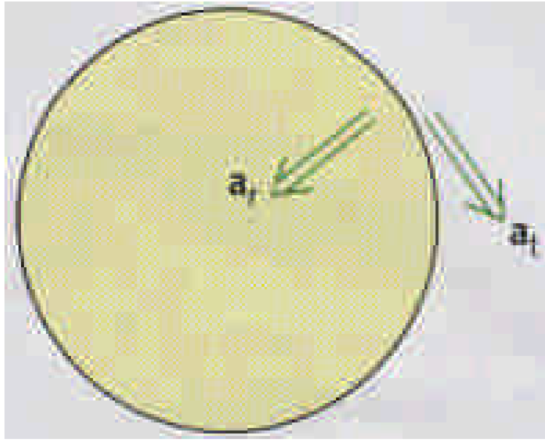


Fig.11.4

➢ 線加速度(linear acceleration)

$$\Rightarrow a = \sqrt{a_r^2 + a_t^2}$$

- 運動方程式(equations of kinematics)

直線平移運動方程式 ($a =$ 定值)	轉動運動方程式($\alpha =$ 定值)
$\mathrm{v} = \mathrm{v}_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + \mathrm{v}_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$\mathrm{v}^2 = \mathrm{v}_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

推導：

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \quad (\because \alpha = \text{const.})$$

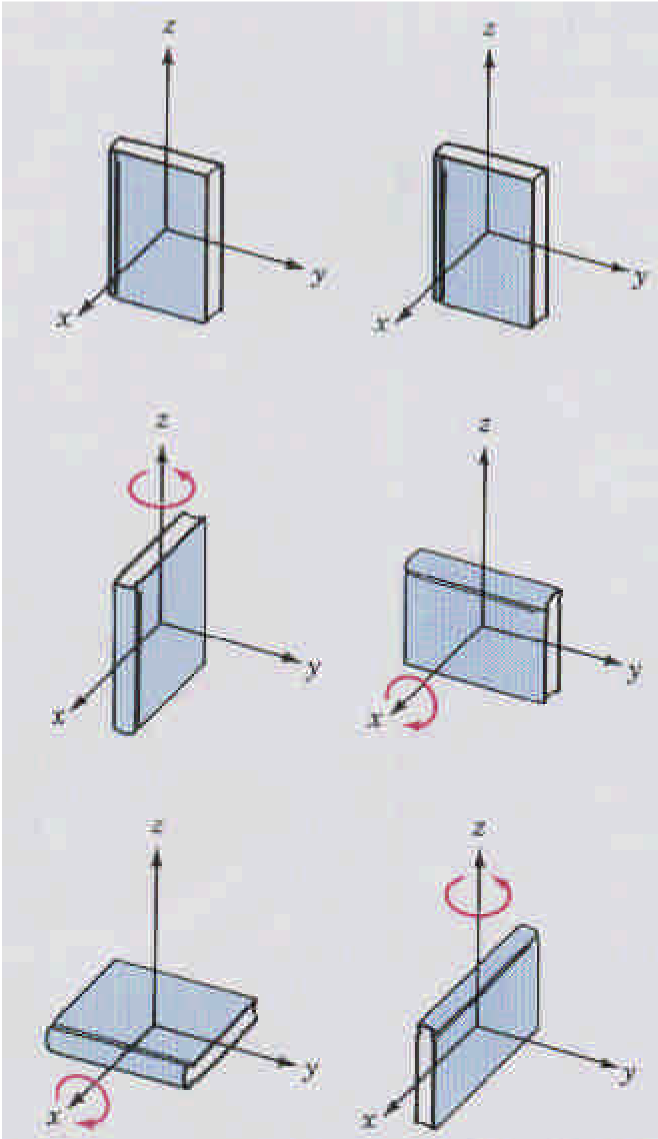
$$\left[\alpha = \frac{\omega - \omega_0}{t - 0} \quad \text{or} \quad \int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha dt\right] \Rightarrow \omega - \omega_0 = \alpha t \qquad (1)$$

$$\left[\begin{array}{l} \omega = \dfrac{d\theta}{dt} \Rightarrow d\theta = \omega dt = (\omega_0 + \alpha t)dt \Rightarrow \int_{\theta_0}^{\theta} d\theta = \int_0^t (\omega_0 + \alpha t)dt \\ \text{or} \quad \omega_{av} = \dfrac{1}{2}(\omega + \omega_0) = \dfrac{\Delta\theta}{\Delta t} \Rightarrow \dfrac{1}{2}(\omega_0 + \alpha t + \omega_0) = \dfrac{\theta - \theta_0}{t - 0} \end{array}\right]$$

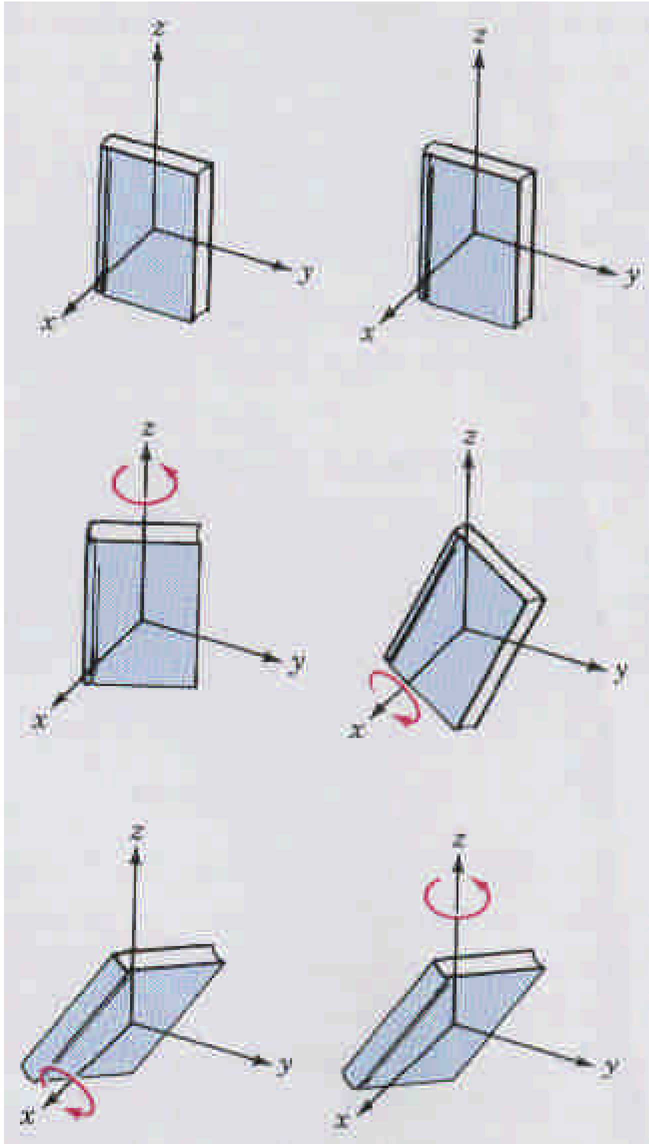
$$\Rightarrow \theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 \qquad (2)$$

$$\text{From (1)} \Rightarrow t = (\omega - \omega_0)/\alpha \quad \text{代入(2)}$$

$$\Rightarrow \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$



$$\Delta\theta_1 + \Delta\theta_2 \neq \Delta\theta_2 + \Delta\theta_1$$



$$d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$$

- 定軸及定方向轉動的物體，其角速度相對於物體上任一質點皆相同。(因角位移相同)

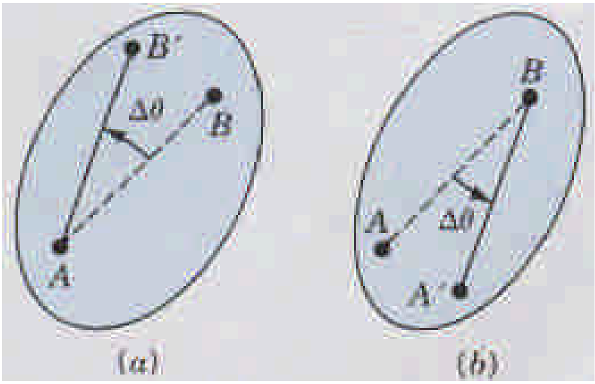


Fig.11.5

純滾動(Rolling) –只滾動不滑動

- 滾輪中心相對於輪邊點的速率 v_c 與輪邊點相對於輪心的速率v_t 相同。

$$v_c = v_t = R\omega$$

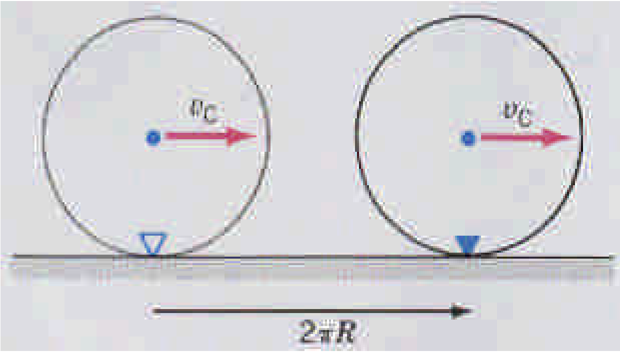
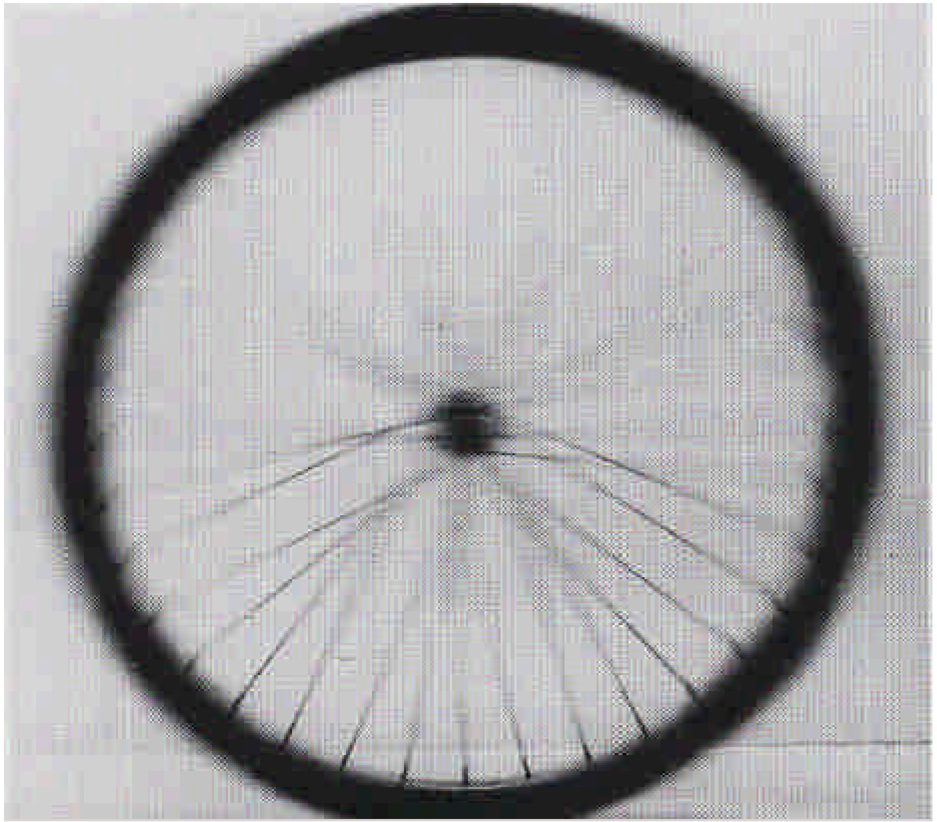


Fig.11.6

- 滾動為輪心平移運動與相對於輪心轉動的組合，故輪邊任一點的速度可表示成： $\vec{v} = \vec{v}_c + \vec{v}_t$

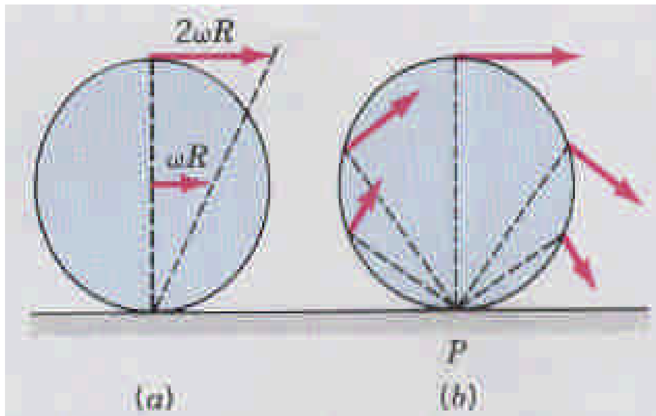
如：輪頂⇒ $\vec{v}_c$, $\vec{v}_t$ 同向， 故 $v = 2R\omega$

輪底⇒ $\vec{v}_c$, $\vec{v}_t$ 反向， 故 $v = 0$



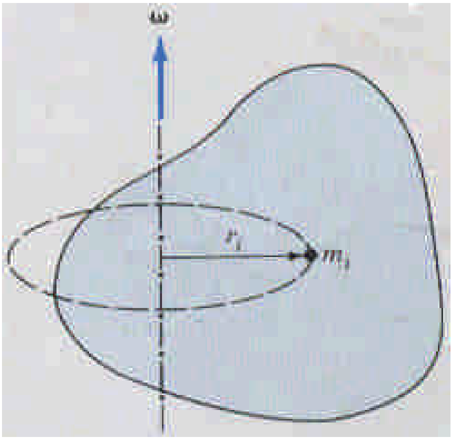
- 純滾動亦可視為純轉動

若將滾輪與地板的靜止接觸點視為轉動中心，則純滾動可利用轉動原理求取滾輪邊各點的線速度 $v = r\omega$，此線速度方向必垂直於該點至地板接觸點的連線。



轉動動能(rotational kinetic energy) $\Rightarrow K=\frac{1}{2}I\omega^2$

Fig.11.9



推導：

因剛體上各質點的角速度皆相同。

$$\therefore K=\sum K_i=\frac{1}{2}\sum m_i v_i^2=\frac{1}{2}\sum m_i r_i^2\omega^2$$

$$=\frac{1}{2}I\omega^2 \qquad (\text{where } I=\sum m_i r_i^2)$$

轉動慣量(moment of inertia) $\Rightarrow I=\sum m_i r_i^2$

⇒物體的轉動慣性，即反抗轉動角速度的改變。

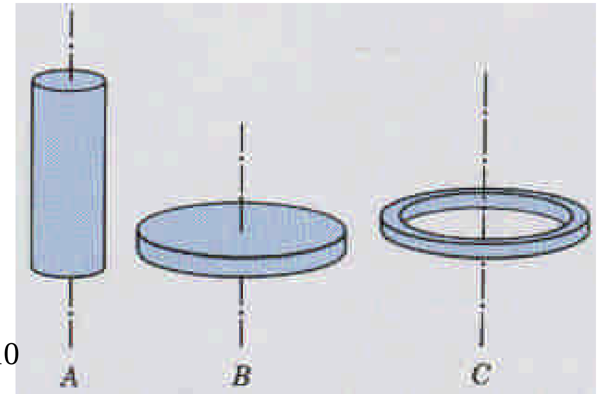


Fig.11.10

例(1)⇒同質量的圓柱體A, 圓盤B, 圓環C之轉動慣量比較：$I_C>I_B>I_A$

例(2)⇒不同握柄的鐵錘轉動慣量比較：$I_{(a)}>I_{(b)}$

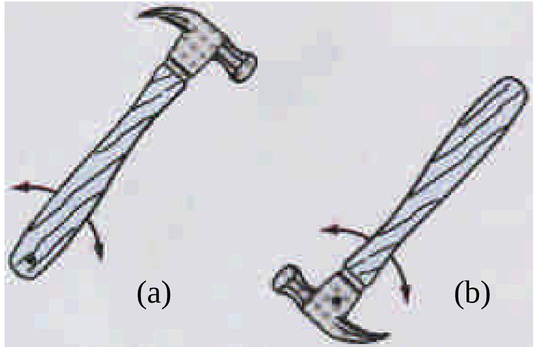


Fig.11.11

- 平行軸原理(parallel axis theorem)

$$\Rightarrow I = I_{CM} + Mh^2$$

其中 I_{CM} 係以質心為轉軸的轉動慣量。

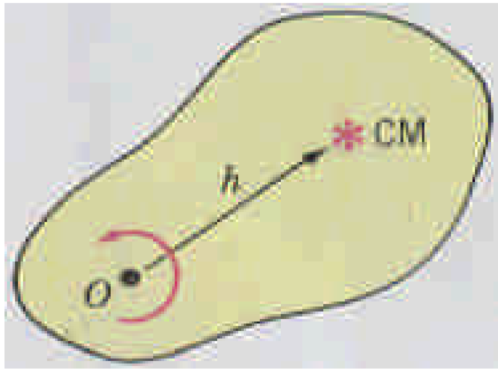


Fig.11.14

推導：

K_{CM} 為質心動能，K_{rel} 相對於質心的動能

$$K = K_{CM} + K_{rel} = \frac{1}{2}M\mathrm{v}_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

$$= \frac{1}{2}M(h\omega)^2 + \frac{1}{2}I_{CM}\omega^2 = \frac{1}{2}(Mh^2 + I_{CM})\omega^2$$

$$= \frac{1}{2}I\omega^2 \quad \Rightarrow \; I = I_{CM} + Mh^2$$

- 連續體(continuous body)的轉動慣量

$$\Rightarrow \; I = \int r^2 dm$$

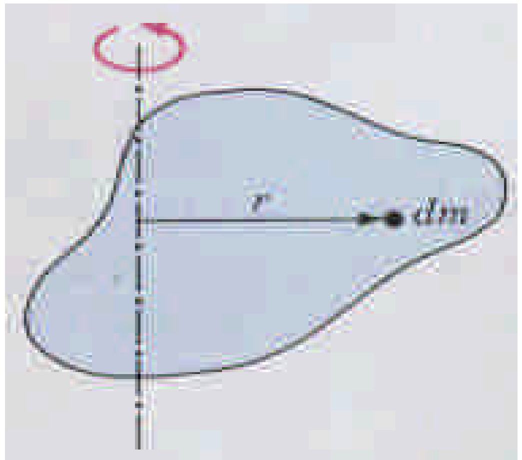
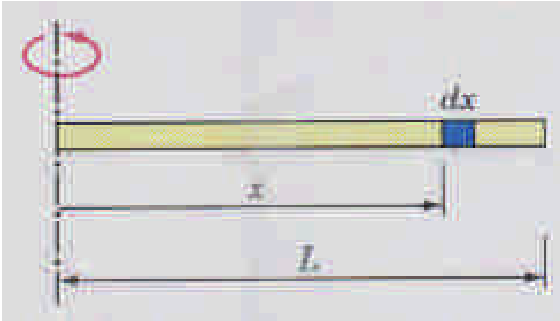


Fig.11.15

Example 11.5：



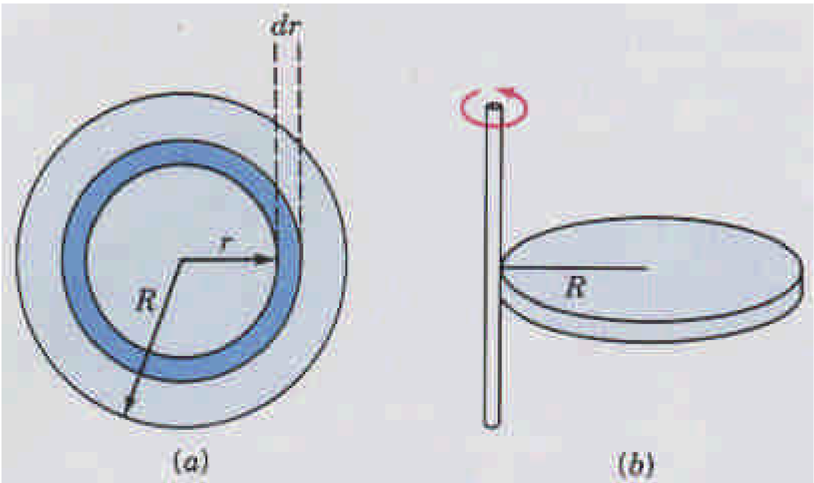
$$dI = r^2 dm = x^2(\lambda dx)$$

$$I_{END} = \int_0^L \lambda x^2 dx = \frac{\lambda L^3}{3} = \frac{ML^2}{3} \quad (\because M = \lambda L)$$

若以質心為旋轉中心，則： $I_{CM} = \int_{-L/2}^{L/2} \lambda x^2 dx = \frac{\lambda L^3}{12} = \frac{1}{12} ML^2$

若根據平行軸原理，則： $I_{END} = I_{CM} + M(\frac{L}{2})^2 = \frac{1}{3} ML^2$

Example 11.6：



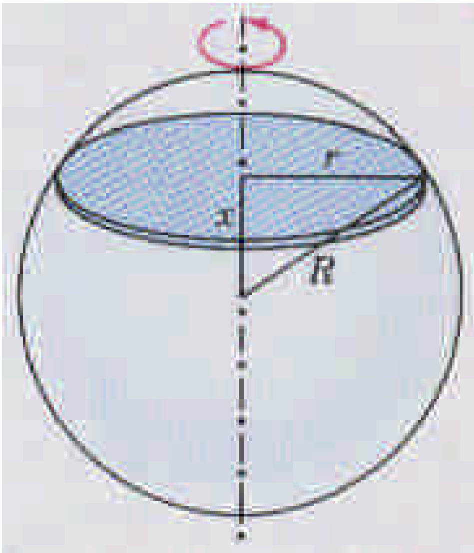
$$dI = r^2 dm = r^2(\sigma 2\pi r dr)$$

$$I_{CM} = 2\pi\sigma \int_0^R r^3 dr = \frac{1}{2}\pi\sigma R^4$$

$$= \frac{1}{2} MR^2 \quad (\because M = \sigma A = \sigma\pi R^2)$$

$$I_b = I_{CM} + MR^2 = \frac{3}{2} MR^2$$

Example 11.7：



$$dm=\rho\pi(R^2-x^2)dx$$

$$dI=\frac{1}{2}(dm)r^2=\frac{1}{2}\rho\pi(R^2-x^2)^2dx$$

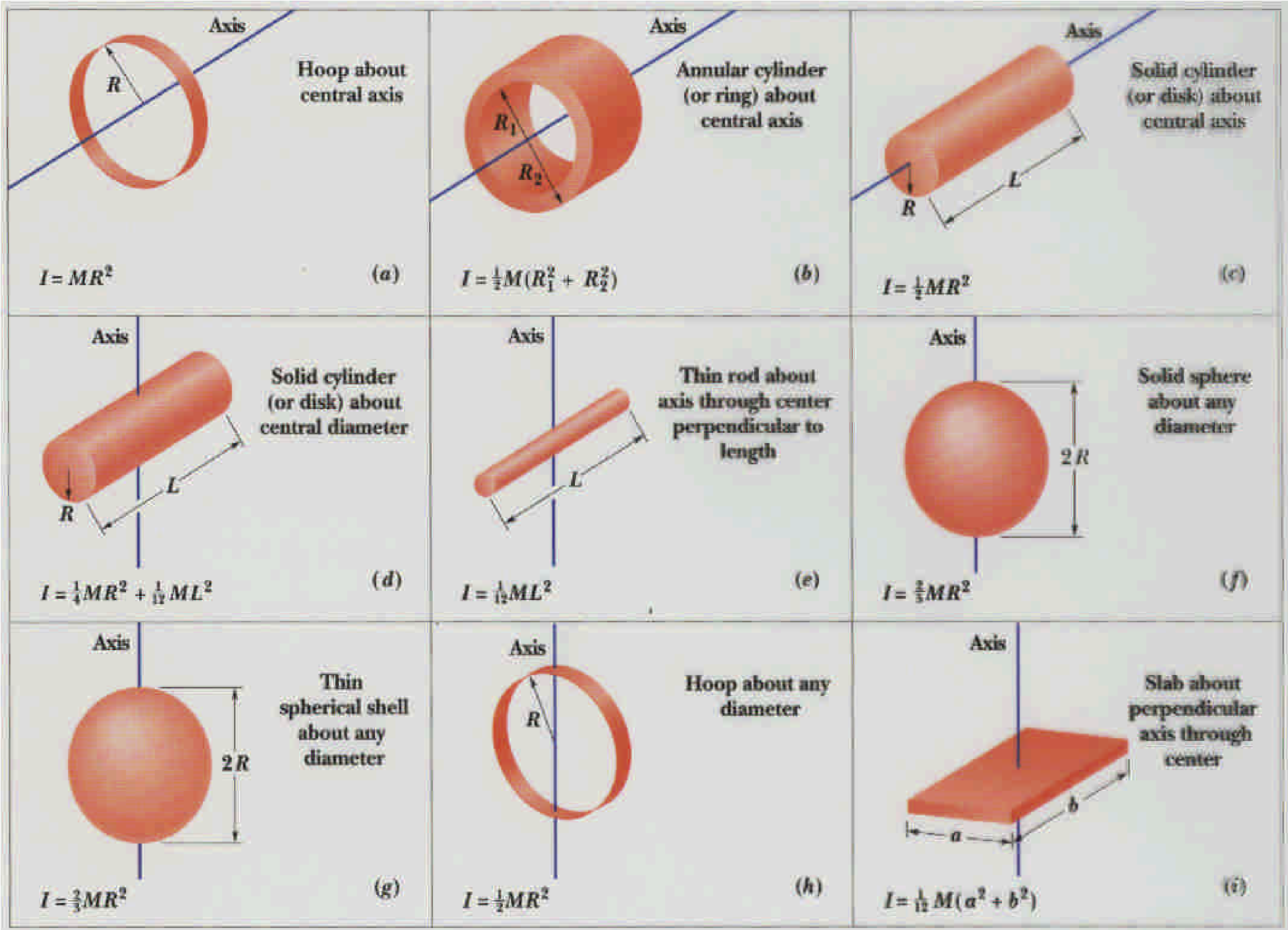
$$I=\frac{1}{2}\rho\pi\int_{-R}^{R}(R^4-2R^2x^2+x^4)dx$$

$$=\frac{1}{2}\rho\pi\left[R^4x-\frac{2}{3}R^2x^3+\frac{1}{5}x^5\right]_{-R}^{R}$$

$$=\frac{8}{15}\rho\pi R^5=\frac{2}{5}MR^2 \qquad \left(\because M=\rho\left(\frac{4}{3}\pi R^3\right)\right)$$

➢推導空心球殼的轉動慣量(problem 6)： $I=\frac{2}{3}MR^2$

[hint : $dI=r^2dm=(R\sin\theta)^2(\sigma 2\pi R\sin\theta\cdot Rd\theta)$]

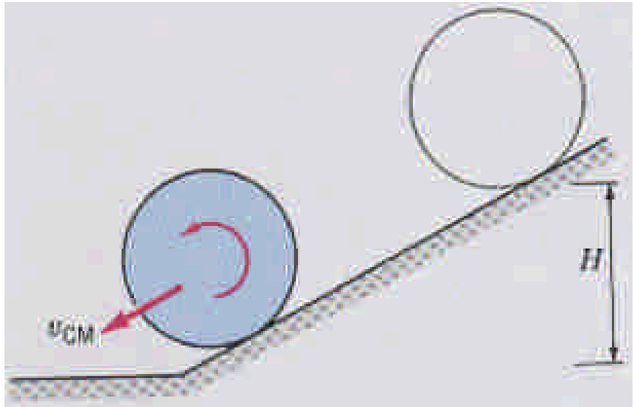


Axis
Hoop about central axis
R
$I = MR^2$
(a)
Axis
Annular cylinder (or ring) about central axis
R_1
R_2
$I = \frac{1}{2}M(R_1^2 + R_2^2)$
(b)
Axis
Solid cylinder (or disk) about central axis
L
R
$I = \frac{1}{2}MR^2$
(c)
Axis
Solid cylinder (or disk) about central diameter
L
R
$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
(d)
Axis
Thin rod about axis through center perpendicular to length
L
$I = \frac{1}{12}ML^2$
(e)
Axis
Solid sphere about any diameter
2R
$I = \frac{2}{5}MR^2$
(f)
Axis
Thin spherical shell about any diameter
2R
$I = \frac{2}{3}MR^2$
(g)
Axis
Hoop about any diameter
R
$I = \frac{1}{2}MR^2$
(h)
Axis
Slab about perpendicular axis through center
a
b
$I = \frac{1}{12}M(a^2 + b^2)$
(i)

✦ 涉及轉動運動的力學能守恆

$$\Rightarrow E = K + U \quad = (\underbrace{\tfrac{1}{2}Mv_{CM}^2}_{\text{平移動能}} + \underbrace{\tfrac{1}{2}I_{CM}\omega^2}_{\text{轉動動能}}) + \overbrace{U_g}^{\text{重力位能}} + \underbrace{U_{sp}}_{\text{彈力位能}}$$

Example 11.8：



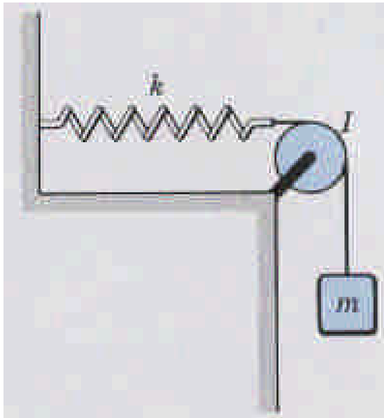
$$E_i = MgH \quad ; E_f = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

$$E_i = E_f \quad \Rightarrow v_{CM}^2 = \frac{2MgH}{M + I_{CM}/R^2}$$

$$I_{CM} = I_{sphere} = \frac{2}{5}MR^2 \Rightarrow v_{\text{sphere}} = \sqrt{10gH/7}$$

$$I_{CM} = I_{disk} = \frac{1}{2}MR^2 \Rightarrow v_{\text{disk}} = \sqrt{4gH/3}$$

Example 11.9：



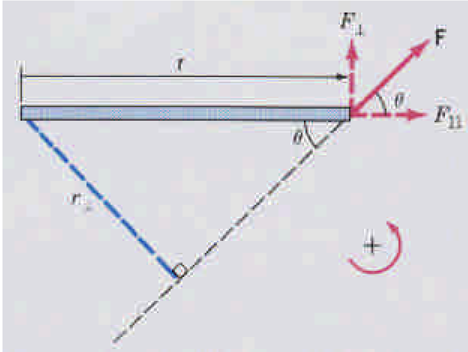
$$\Delta E = \Delta K + \Delta U = 0 \quad \Rightarrow \frac{1}{2}mv^2 + \frac{1}{2}I(\frac{v}{R})^2 + \frac{1}{2}kx^2 - mgx = 0$$

$$\xrightarrow{I=\frac{1}{2}MR^2} \frac{1}{2}(m + \frac{M}{2})v^2 + \frac{1}{2}kx^2 - mgx = 0 \quad \Rightarrow v = 2.4\text{m/s}$$

轉動動力學(rotational dynamics)

- 力矩(torque) — 產生轉動運動及角加速度。

  ➢ 定義：一力相對於轉軸或支點的轉動能力，即：



$$\tau = r_{\perp}F = rF_{\perp} = rF\sin\theta \qquad \text{or} \quad \vec{\tau} = \vec{r}\times\vec{F}$$

  ➢力矩為一向量，可利用轉向替代真正的指向(由右手定則判定)，逆時針轉向，力矩為正，順時針方向，力矩為負。

  ➢ 力矩式推導 $\Rightarrow r_2/r_1 = F_1/F_2 \ \Rightarrow r_1F_1 = r_2F_2$

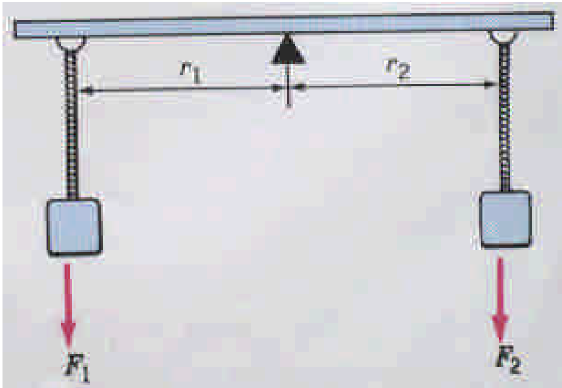


Fig.11.22

● 定軸轉動的剛體 $\Rightarrow\ \tau = I\alpha$ （相當於轉動運動第二定律）

推導：

假設定軸轉動剛體的某一質點受外力 $\vec{F}_i$ 作用

$$\vec{F}_i \ \Rightarrow \begin{cases} F_{i//} \Rightarrow \text{軸向分力(radial component)} \Rightarrow \text{不會產生轉動力矩} \\ F_{ir} \Rightarrow \text{徑向分力(tangential component)} \Rightarrow \text{不會產生轉動力矩} \\ F_{it} \Rightarrow \text{切線分力(tangential component)} \Rightarrow \text{可產生轉動力矩} \end{cases}$$

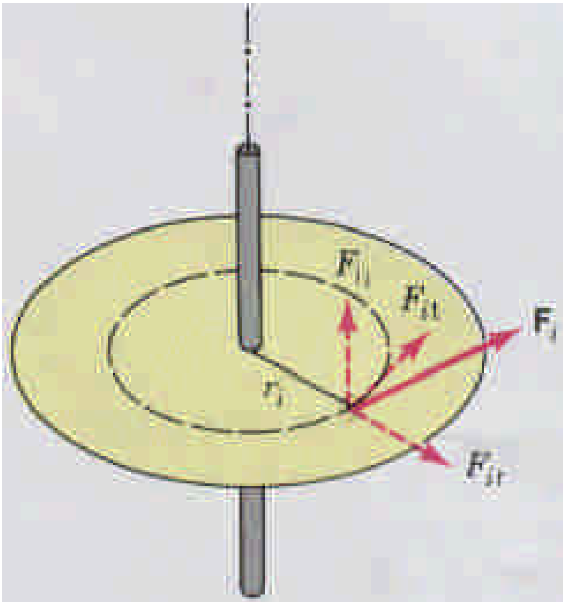


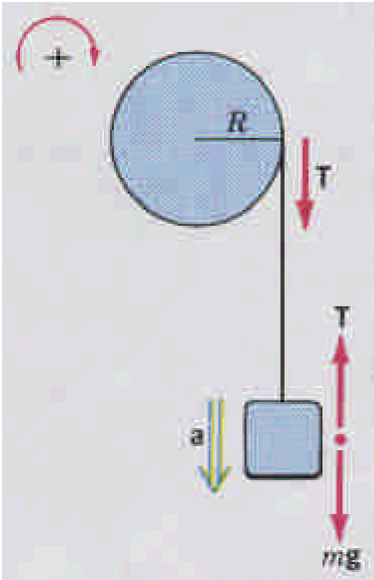
Fig.11.25

$$\tau = \sum \tau_i = \sum r_i F_{it} = (\sum m_i r_i^2)\alpha = I\alpha$$

$$(\because F_{it} = m_i a_{it} = m_i r_i \alpha)$$

- 轉軸位置與方向固定。
- 因轉軸通過質心且方向固定，即使質心做加速度運動亦成立。 $\tau_{CM} = I_{CM}\alpha_{CM}$

Example 11.12： Find (a) angular velocity after 3s ; (b) the speed of the block after it has fallen 1.6 m.



$$Block \qquad (F = ma) \qquad mg - T = ma \qquad (1)$$

$$Pulley \qquad (\tau = I\alpha) \qquad TR = (\frac{1}{2}MR^2)\alpha \qquad (2)$$

$$From \ (2) \xrightarrow{a=R\alpha} T = \frac{1}{2}Ma \qquad (3)$$

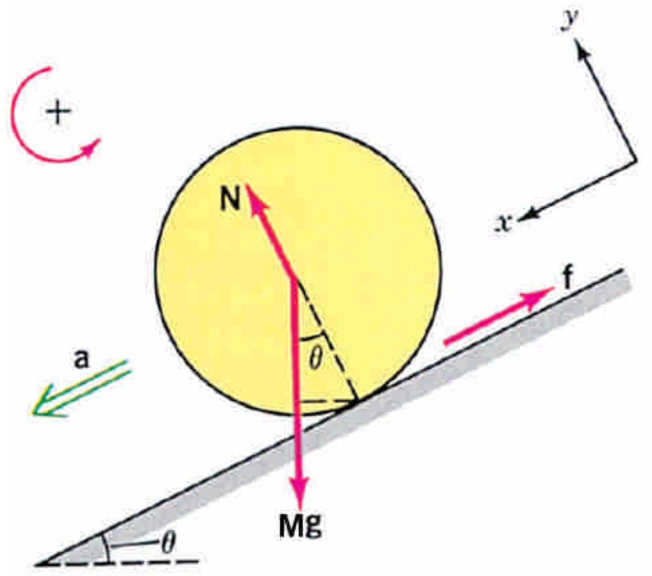
$$(1)+(3) \Rightarrow a = \frac{mg}{m+M/2} = 5m/s^2 \qquad (4)$$

$$\omega = \omega_0 + \alpha t = 0 + (\frac{a}{R})t = 30rad/s \quad \text{Ans (a)}$$

$$\mathrm{v}^2 = \mathrm{v}^2{}_0 + 2a\Delta y = 0 + 2(5m/s^2)(1.6m)$$

$$\mathrm{v} = 4m/s \qquad \text{Ans (b)}$$

Example 11.13： (a) Find the linear acceleration of the CM ; (b) What is the minimum coefficient of friction require for the sphere to roll without slipping.



$$(\sum F_x) \quad Mg\sin\theta - f = Ma \quad (1)$$

$$(\sum F_y) \quad N - Mg\cos\theta = 0 \quad (2)$$

$$(\sum \tau) \quad fR = I\alpha \quad (3)$$

$$(3) \xrightarrow{I=\frac{2}{5}MR^2} f = \frac{2}{5}Ma \quad (4)$$

$$(1)+(4) \Rightarrow a = \frac{5}{7}g\sin\theta \quad \text{Ans(a)} \quad (5)$$

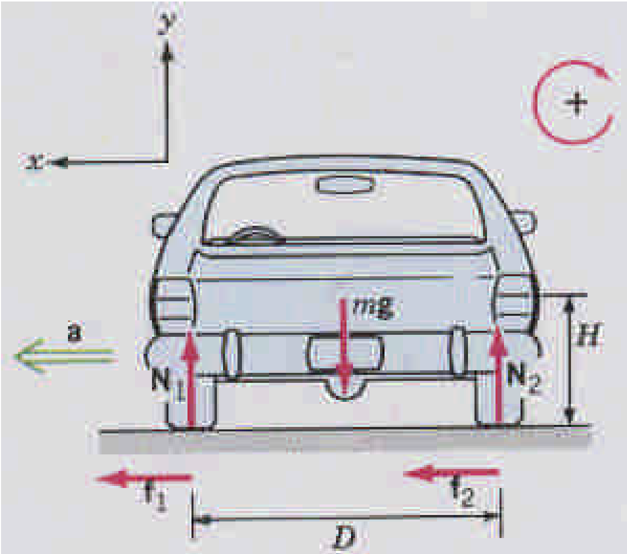
$$(5)代入(1) \quad f = \frac{2}{7}Mg\sin\theta \quad (6)$$

$$f = \mu N \xrightarrow{\text{From (2): } N=mg\cos\theta} f = \mu(Mg\cos\theta) \quad (7)$$

$$(6)代入(7)，\ f = \frac{2}{7}Mg\sin\theta = \mu(Mg\cos\theta) \Rightarrow \mu = \frac{2}{7}\tan\theta \quad \text{Ans(b)}$$

因純滾動無滑動的摩擦係數相當於靜摩擦係數μ_s，故 $\mu_s \geq \frac{2}{7}\tan\theta$

Example 11.14： A car goes around an unbanked curve of radius r at speed v. Find the critical speed at which it tends to overturn.



$$(\sum F_x) \qquad f_1 + f_2 = \frac{mv^2}{r} \qquad (1)$$

$$(\sum F_y) \qquad N_1 + N_2 - mg = 0 \qquad (2)$$

$$(\sum \tau) \qquad (f_1 + f_2)H + (N_1 - N_2)\frac{D}{2} = 0 \qquad (3)$$

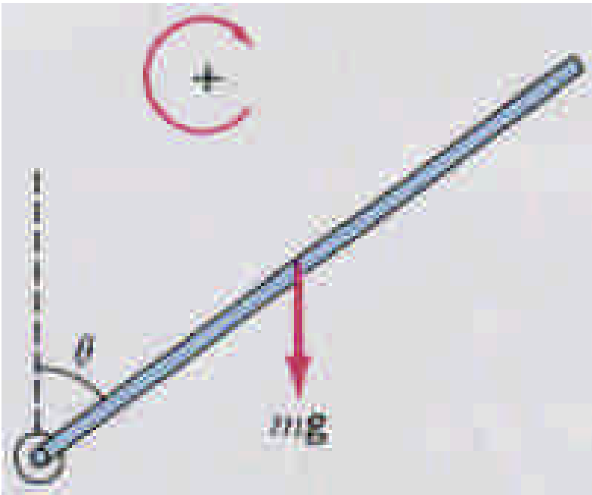
將(2)代入(3) $\Rightarrow N_1 = m(\frac{g}{2} - \frac{v^2 H}{rD}) \qquad (4)$

當N_1=0，汽車內側輪胎將離開地面，故

$$v^2_{max} = \frac{grD}{2H}$$

➢增加車身寬度(即輪距D)或降低質心高度H，可減小車反轉。

Example 11.15： Find (a)the angular acceleration of the rod? (b)the tangential linear acceleration when the rod is horizontal?



$$\tau = I\alpha \quad \Rightarrow \frac{mgL}{2}\sin\theta = \frac{ML^2}{3}\alpha$$

$$\Rightarrow \alpha = \frac{3g\sin\theta}{2L} \qquad \text{Ans (a)}$$

$$a_t = \alpha L = \frac{3g}{2} \qquad \text{Ans(b)}$$

•功(Work) $\Rightarrow dW = \tau\, d\theta$

功率(Power) $\Rightarrow P = \tau\omega$

➢ 推導：

$$dW = F_t ds = (F_t)(r d\theta) = \tau d\theta$$

$$P = \frac{dW}{dt} = \frac{\tau d\theta}{dt} = \tau\left(\frac{d\theta}{dt}\right) = \tau\omega$$

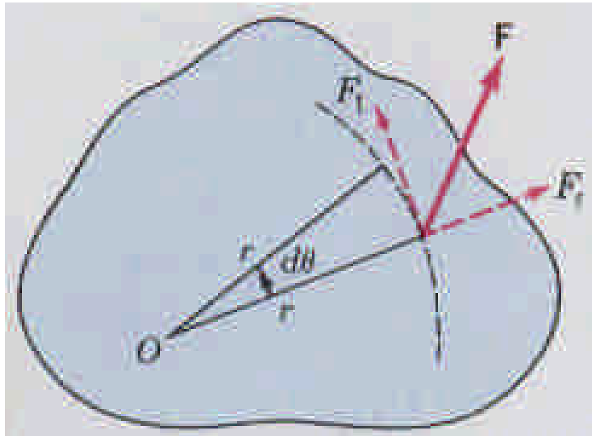


Fig.11.31

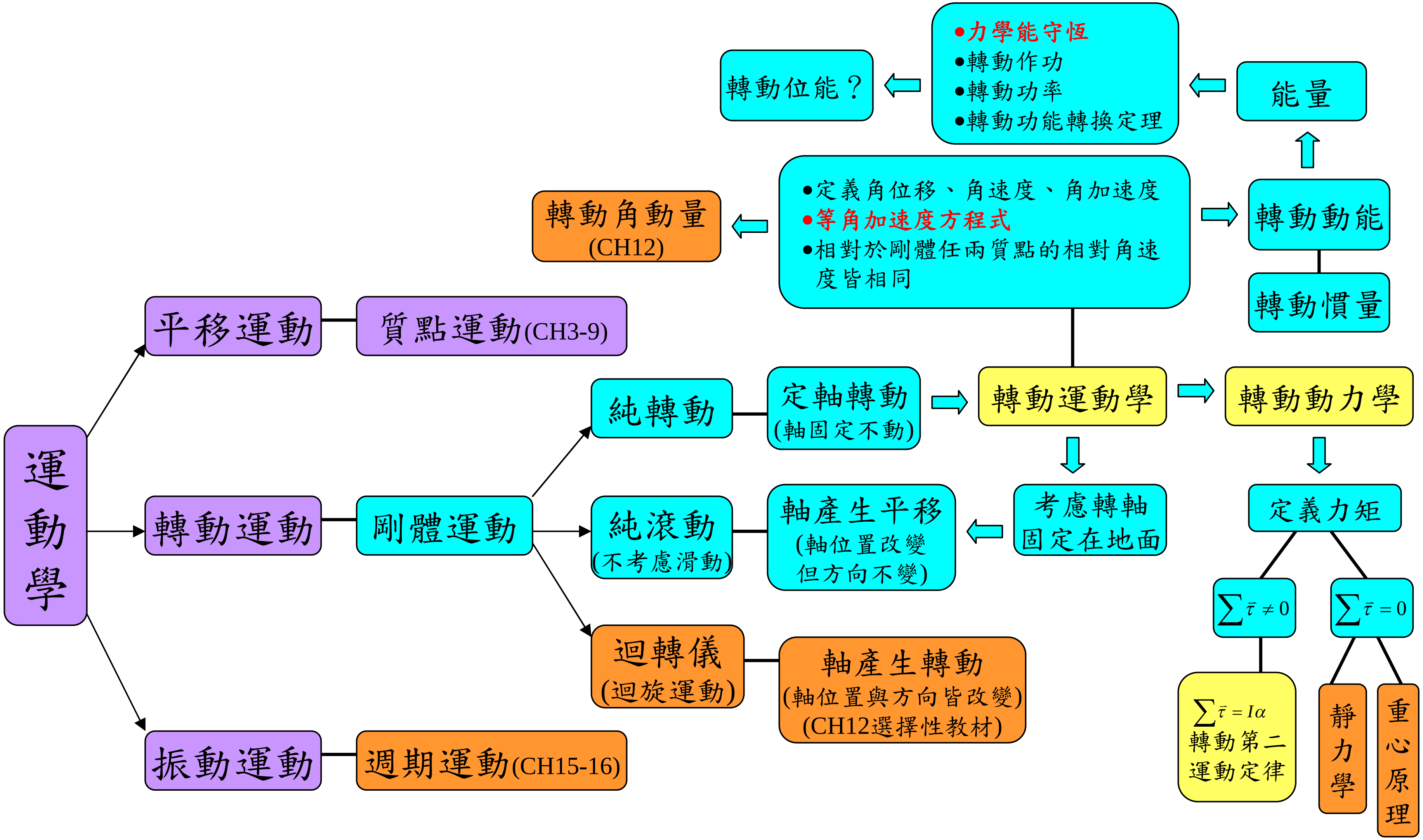
• 轉動運動的功能定理 $\Rightarrow W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$

➢推導：

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\omega \Rightarrow dW = \tau d\theta = I\omega d\omega$$

$$\Rightarrow \int dW = \int_{\omega_i}^{\omega_f} I\omega d\omega \Rightarrow W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

本章重要觀念發展脈絡彙整



習　題

- 教科書習題 (p.229~p.236)

  Exercise: 15,23,29,34,39,41,45,53,55,57,59,63

  Problem: 1,6

- 基本觀念問題：

  1.何謂轉動慣量(moment of inertia)？請說明其物理意義。

  2.何謂剛體？

  3. 請問一般剛體轉動運動的類型有哪些？如何區別？

- 延伸思考問題：

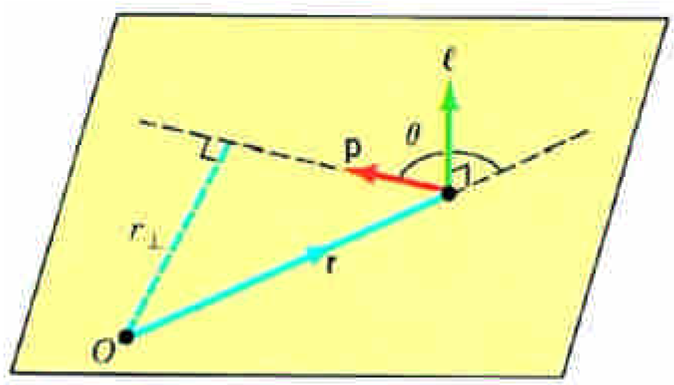
  1.請問是否存在轉動位能？請申述之。

角動量(Angular Momentum)

- 單一質點： $\vec{l}=\vec{r}\times\vec{p}\Rightarrow l=rp\sin\theta=r_{\perp}p$

(源自力矩向量的定義：$\vec{\tau}=\vec{r}\times\vec{F}$)

Fig.12.2



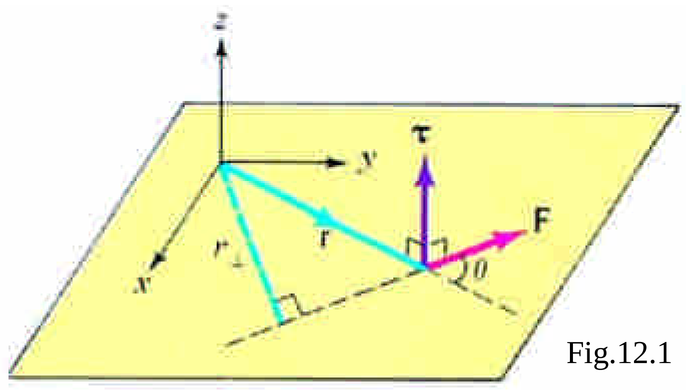


Fig.12.1

➢ 直線等速度運動

質點相對於其直線運動外一點所構成的角動量維持定值。證明如下：

$$l_A=r_A p\sin\theta_A \quad ; \quad l_B=r_B p\sin\theta_B$$

$$r_A\sin\theta_A=r_B\sin\theta_B=r_{\perp}\ \Rightarrow l_A=l_B=\text{const.}$$

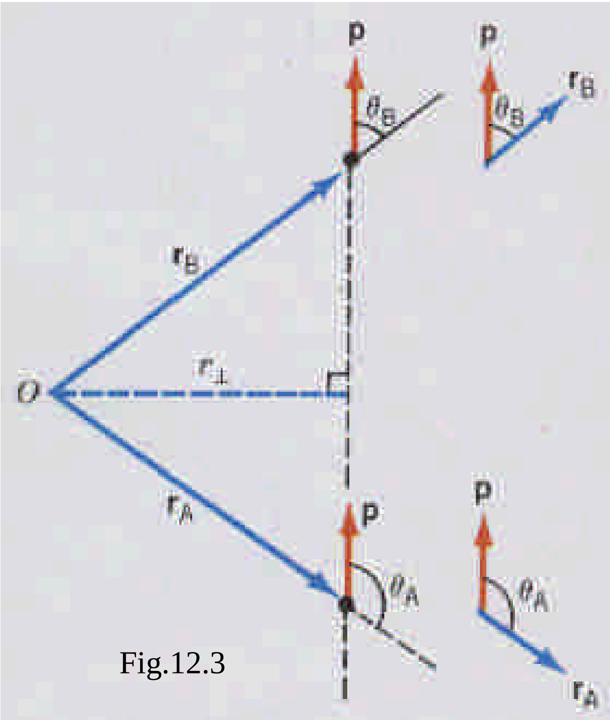


Fig.12.3

➢ 圓周運動 (motion in a circle)

♦ 相對於圓心

$$\Rightarrow \vec{l} = \vec{r} \times \vec{p} \xrightarrow{\theta=90^0 \text{ and } r=R} l = Rp = mvR = mR^2\omega$$

($\vec{l}$沿ω方向)

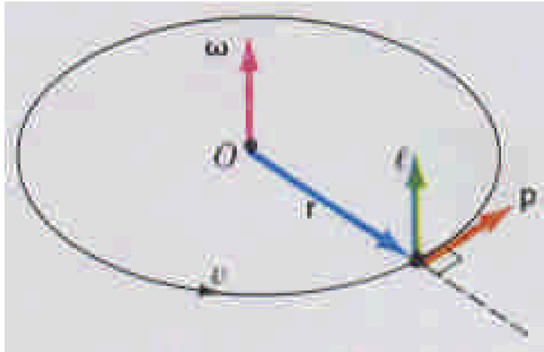


Fig.12.5

♦ 相對於通過圓心的軸上任一點：

$$\Rightarrow \vec{l} = \vec{r} \times \vec{p} \xrightarrow{\theta=90^0 \text{ but } r\neq R} l = mvr \quad (\vec{l}\text{非沿}\omega\text{方向})$$

$$\xrightarrow{\text{考慮}\omega\text{方向分量}} l_z = l\sin\phi = (mvr)(R/r) = mR^2\omega$$

♦ 若考慮ω方向的角動量，則其形式皆相同。

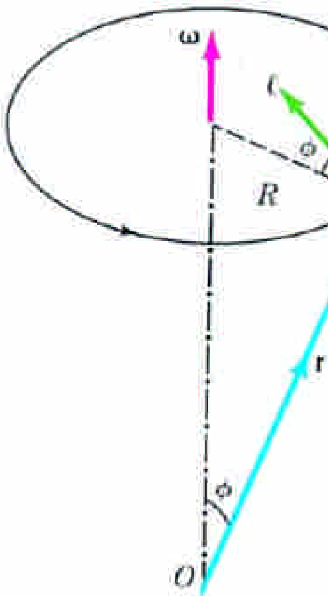


Fig.12.6

• 多質點系統 (system of particles)

$$\vec{L} = \sum \vec{l}_i = \sum \vec{r}_i \times \vec{p}_i$$

➢考慮定軸轉動的剛體 $\xrightarrow{\text{轉軸沿}\omega\text{方向}} L_z = I\omega$

推導 $\Rightarrow L_z = \sum l_{iz} = \underbrace{\sum m_i R_i^2}_{(I)}\omega = I\omega$

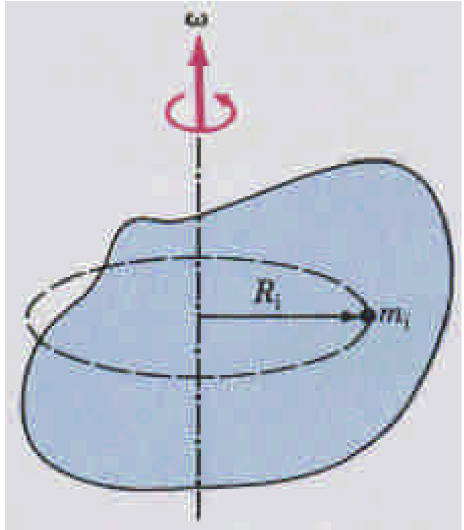


Fig.12.7

✦力矩與角動量的關係

- 單一質點 $\Rightarrow \vec{\tau} = \dfrac{d\vec{l}}{dt}$ (可對應線動量 $\Rightarrow \vec{F} = \dfrac{d\vec{p}}{dt}$)

$$\frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \vec{r}\times\frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt}\times\vec{p} = \vec{r}\times\vec{F} + \cancel{\vec{v}\times m\vec{v}} = \vec{r}\times\vec{F} = \vec{\tau}$$

- 多質點系統 $\Rightarrow \sum \vec{\tau}_i = \vec{\tau}_{EXT} = \dfrac{d\vec{L}}{dt}$

其中 $\vec{\tau}_{INT}$(淨內力矩)$=0$ ，證明如下：

$$\vec{\tau}_1 + \vec{\tau}_2 = \vec{r}_1\times\vec{F}_{12} + \vec{r}_2\times\vec{F}_{21} = (\vec{r}_1 - \vec{r}_2)\times\vec{F}_{12} = 0$$

$(\because \vec{r}_1 - \vec{r}_2$ and $\vec{F}_{12}$ are antiparallel)

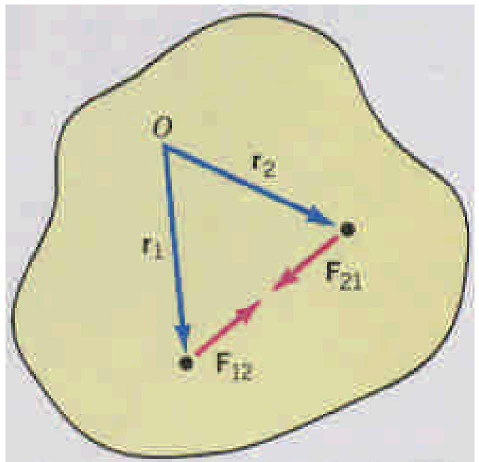


Fig.12.10

➢定軸轉動剛體 $\Rightarrow \tau_{EXT} = \dfrac{dL}{dt} = I\alpha$

(Rigid body, fixed axis)

$\boxed{推導}$ $\Rightarrow \dfrac{dL}{dt} = \dfrac{d}{dt}(I\omega) = I(\dfrac{d\omega}{dt}) = I\alpha \ = \tau$

Example 12.4： Find the linear acceleration of the blocks in Fig.12.11.

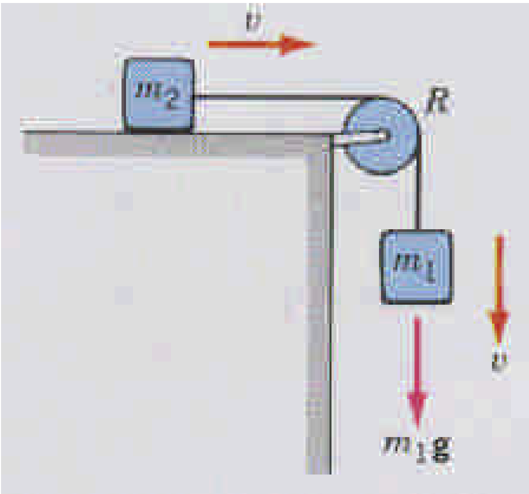


Fig.12.11

$$L = m_1 vR + m_2 vR + I\omega \quad ; \quad \tau_{EXT} = r_\perp F = R(m_1 g)$$

$$\tau_{EXT} = \frac{dL}{dt} = (m_1 + m_2)R \cdot \frac{dv}{dt} + I \cdot \frac{d\omega}{dt}$$

$$R(m_1 g) = (m_1 + m_2)Ra + I\frac{a}{R} \qquad (\because \frac{d\omega}{dt} = \alpha = a/R)$$

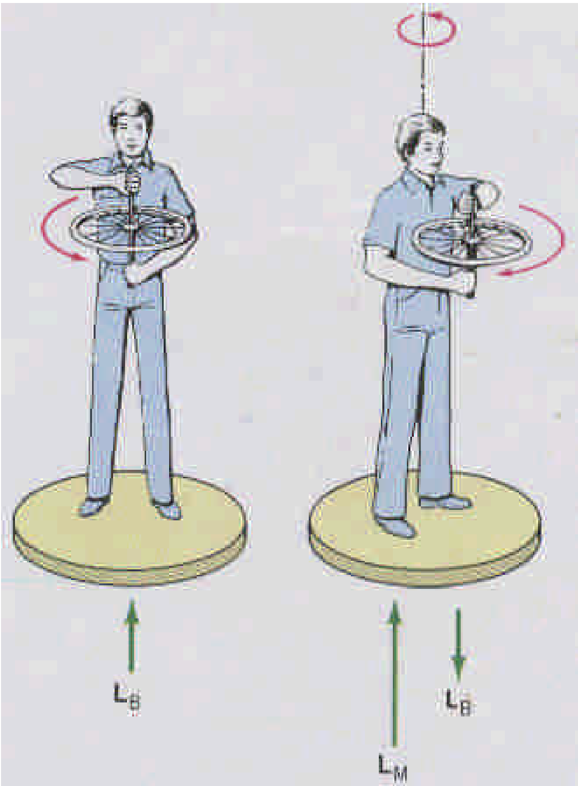
$$a = \frac{m_1 g}{m_1 + m_2 + M/2} = 4.9 \text{ m/s}^2$$

角動量守恆 (Conservation of Angular Momentum)

$\Rightarrow$ 若 $\vec{\tau}_{EXT} = 0$，則 $\vec{L}$ 維持不變（包括大小、方向）

- 定軸轉動剛體 $\Rightarrow L_f = L_i \Rightarrow I_f \omega_f = I_i \omega_i$

Example 12.7：



$$L_i = L_B = I_B \omega_B$$

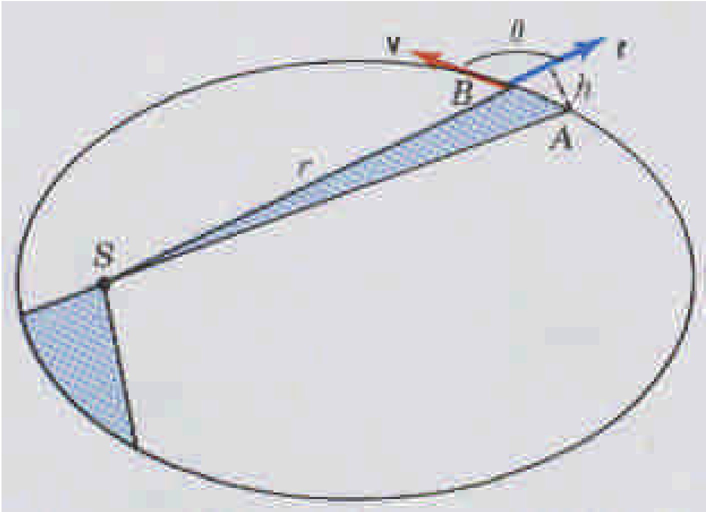
$$L_f = L_B + L_M = -I_B \omega_B + I_M \omega_M$$

$$\tau_{EXT} = 0 \;\Rightarrow\; L_i = L_f$$

$$\Rightarrow I_B \omega_B = -I_B \omega_B + I_M \omega_M$$

$$\Rightarrow \omega_M = \frac{2 I_B \omega_B}{I_M} = 5 \ (rad/s)$$

Example 12.8： Show the Kepler's second law of planetary motion that is a consequence of the conservation of angular momentum.



$$\Delta A = \frac{1}{2} rh = \frac{1}{2} r v \Delta t \sin\theta$$

$$(\because h = v\Delta t \sin(\pi - \theta) = v\Delta t \sin\theta)$$

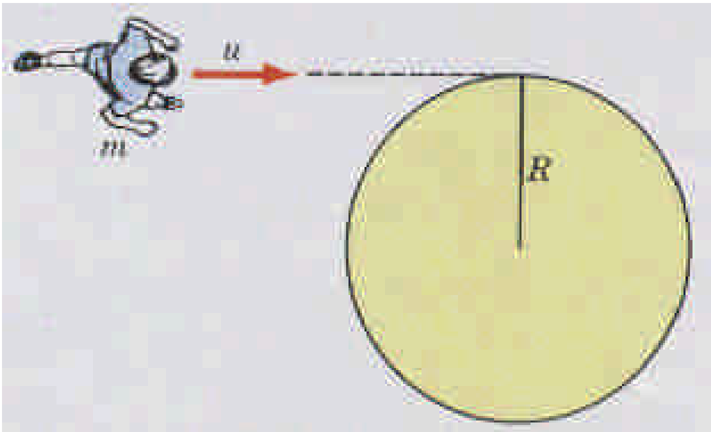
$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r v \sin\theta \qquad (1)$$

$$l = rp\sin\theta = mrv\sin\theta \qquad (2)$$

$$(1)代入(2) \Rightarrow \frac{\Delta A}{\Delta t} = \frac{l}{2m} = \text{const.}$$

$$(\because \vec{\tau}_{EXT} = \vec{r} \times \vec{F} = rF\sin 180^0 \hat{n} = 0 \ for \ planetary \ , \quad \therefore l = const.)$$

Example 12.9： Find the angular velocity of the platform (a) after the man jumps on or (b) walks to the center.



$$L_i = muR$$

$$L_f = (\frac{1}{2}MR^2 + mR^2)\omega_1$$

$$L_i = L_f$$

$$\Rightarrow \omega_1 = \frac{mu}{(M/2+m)R} = 1\ (rad/s) \quad \text{Ans(a)}$$

$$L_i = (\frac{1}{2}MR^2 + mR^2)\omega_1 = 640\ (kg \cdot m^2/s)$$

$$L_f = (\frac{1}{2}MR^2)\omega_2 = 320\omega_2$$

$$L_i = L_f \Rightarrow \omega_2 = 2\ (rad/s) \qquad \text{Ans(b)}$$

✦ 重心 (Center of Gravity)

- 若相對於物體上的某點，重力所造成的力矩和為零，則此點稱為重心。

  如右圖 $\Rightarrow w_1 l_1 = w_2 l_2$

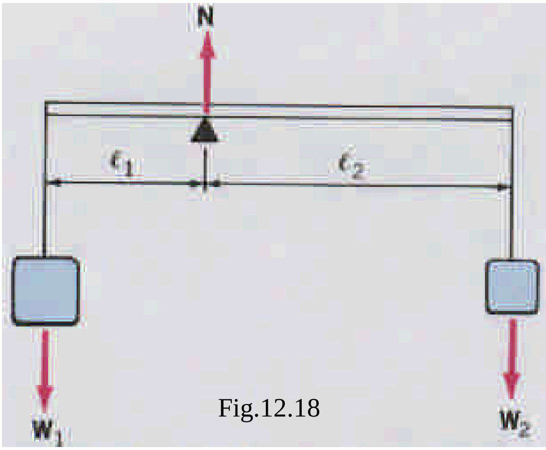


Fig.12.18

- 若相對某參考座標原點，則物體上各質點重力的力矩和相當於物體總重力在重心相對於原點所造成的力矩，即：

$$\sum \tau_i = w_1 x_1 + w_2 x_2 + \cdot\cdot\cdot + w_N x_N \left(= \sum w_i x_i\right)$$

$$= \left(\sum w_i\right) x_{CG}$$

$$\Rightarrow x_{CG} = \frac{\sum w_i x_i}{\sum w_i} = \frac{\sum m_i g_i x_i}{\sum m_i g_i}$$

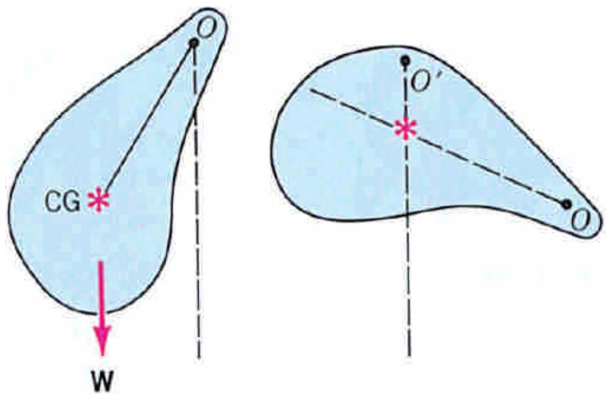


Fig.12.19

- 若 g_i 相同，則重心＝質心。 即 $x_{CG} = \dfrac{\sum m_i x_i}{\sum m_i} = \dfrac{\sum m_i x_i}{M} = x_{CM}$

- 重心必位於懸點的垂線上(因力矩平衡效應)，故兩懸點的垂線可決定平面物的重心。(如Fig.12.19所示)

靜力學 (Statics)

- 探討作用於靜止物體的外力與外力矩平衡。
- 靜力平衡(或靜態平衡，static equilibrium)的條件：

$$(a)\ \sum \vec{F} = 0 \quad (\text{即} a = 0\text{, 平移平衡})$$

$$(b)\ \sum \vec{\tau} = 0 \quad (\text{即} \alpha = 0\text{, 轉動平衡})$$

$$(c)\ \vec{\mathrm{v}} = 0 \quad (\text{表靜止狀態})$$

Note：

- 定軸轉動剛體若轉軸為z 軸，則 $\sum \tau_z = 0$
- 若 (c) 條件改為 $\vec{\mathrm{v}} \neq 0$，則表動態平衡，屬於動力學範疇。

靜力學解題指引

(1) 選定靜力平衡的物體，找出所有外力。

(2) 選定一座標系，劃出包含各分力的自由物體圖(free-body diagram)。

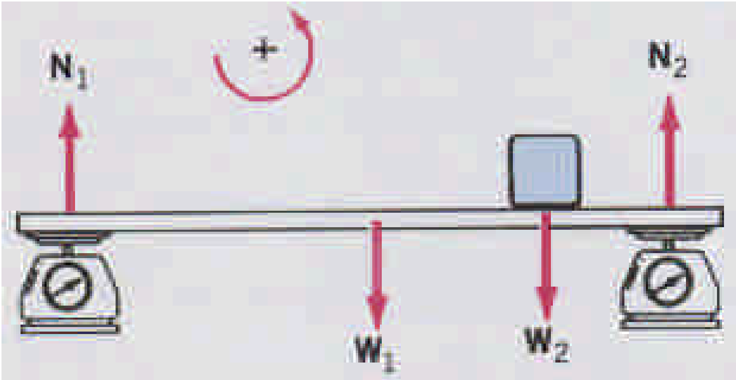
(3) 選定一轉軸來估算力矩，並以 標示正的力矩方向，但須注意作用於轉軸的力所造成的力矩為零。

(4) 寫出平衡方程式：

$$\sum F_x = 0\text{；}\quad \sum F_y = 0\text{；}\quad \sum \tau = 0$$

不同轉軸可產生不只一個 $\sum \tau=0$ 的力矩平衡式，基本上，未知數須與獨立方程式數目相等才能求得唯一的解。

Example 12.10： What are the forces exerted by the supports?

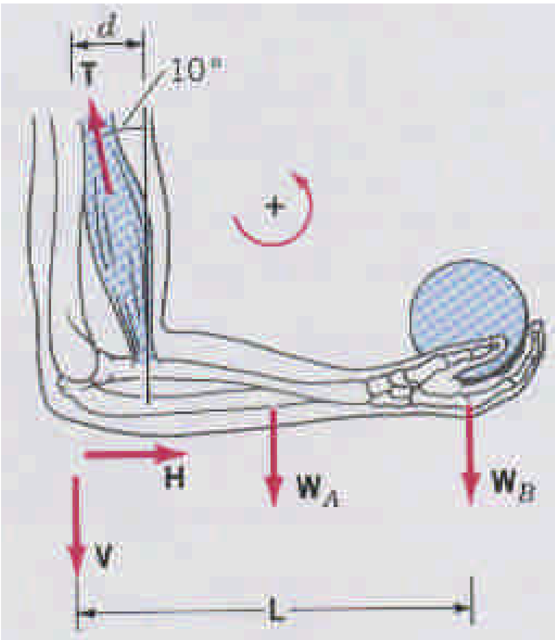


$$\sum F_y = N_1 + N_2 - W_1 - W_2 = 0 \qquad (1)$$

$$\sum \tau = -\frac{N_1 d}{2} - \frac{W_2 d}{4} + \frac{N_2 d}{2} = 0 \ \text{(轉軸在棒中央)}$$

$$\Rightarrow -N_1 + N_2 - \frac{W_2}{2} = 0 \qquad (2)$$

$(1)+(2) \Rightarrow N_2 = 25\text{N}$， 將$N_2$代入(2)式$\Rightarrow N_1 = 20\text{N}$

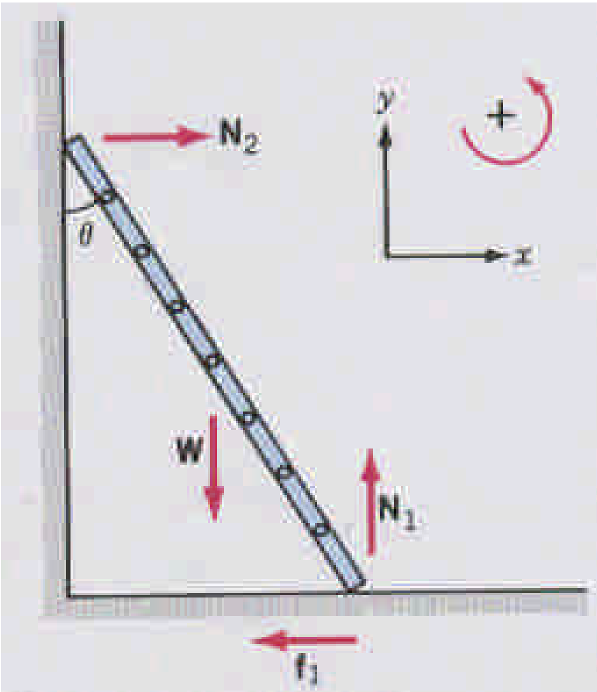


Example 12.11： What is the tension in the muscle?

$$\sum \tau = (T\cos\theta)d - \frac{W_A L}{2} - W_B L = 0$$

$$\Rightarrow T = 438N$$

Example 12.12：(a) Find the maximum angle θ to the wall such that the ladder does not slip, (b) the force (N_2)exerted by the wall ?



$$\sum F_x = N_2 - f_1 = 0 \qquad (1)$$

$$\sum F_y = N_1 - W = 0 \qquad (2)$$

$$\sum \tau = -WL/2\sin\theta - f_1 L\cos\theta + N_1 L\sin\theta = 0 \qquad (3)$$

（以牆壁接觸點為轉軸）

$$\sum \tau = +WL/2\sin\theta - N_2 L\cos\theta = 0 \qquad (4)$$

（以地面接觸點為轉軸）

From (2) $\Rightarrow W = N_1$ and $f_1 = \mu_s N_1$ $\Rightarrow$ 代入(3)

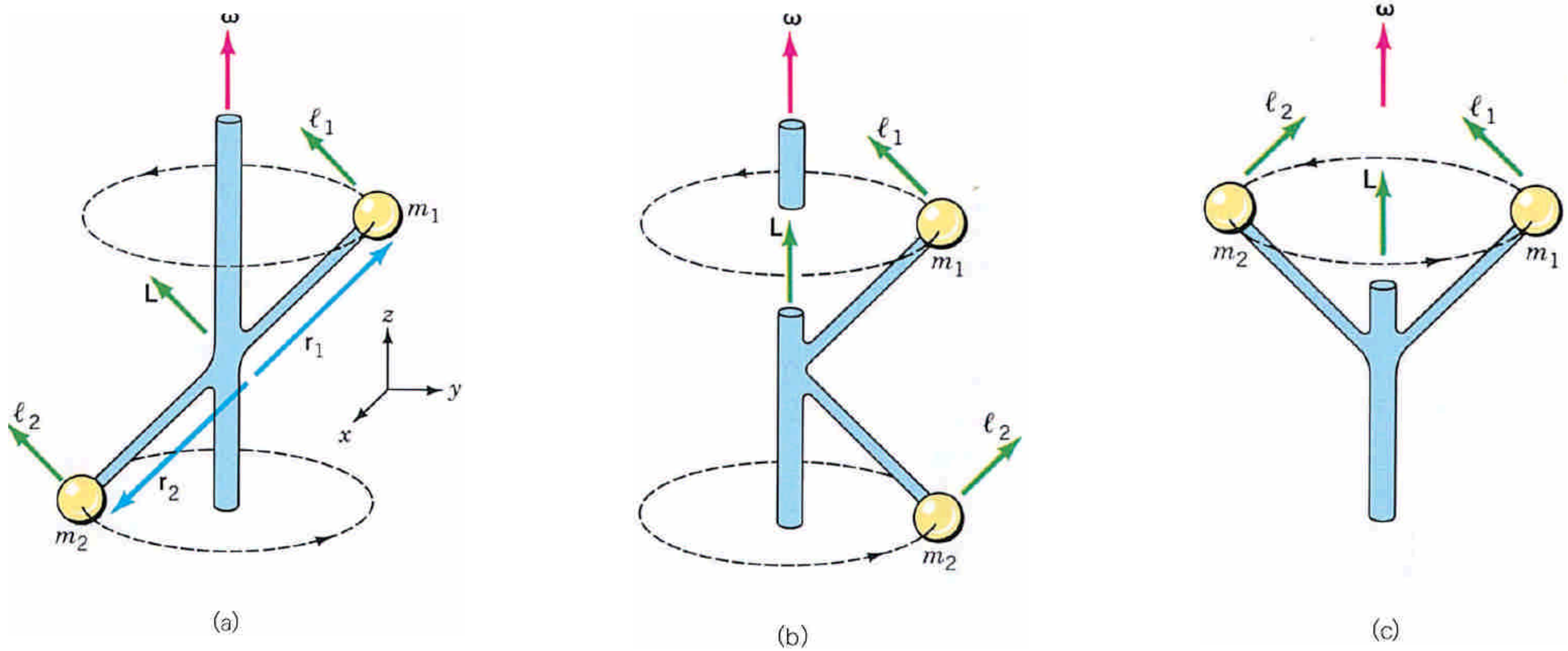
$$\Rightarrow \sin\theta - 2\mu_s\cos\theta = 0 \qquad \Rightarrow \tan\theta = 2\mu_s$$

$$\Rightarrow \theta = \tan^{-1}(2\mu_s) = 50.2^0$$ Ans (a)

$$N_2 = f_1 = \mu_s N_1 = \mu_s W = 0.6W$$ Ans (b)

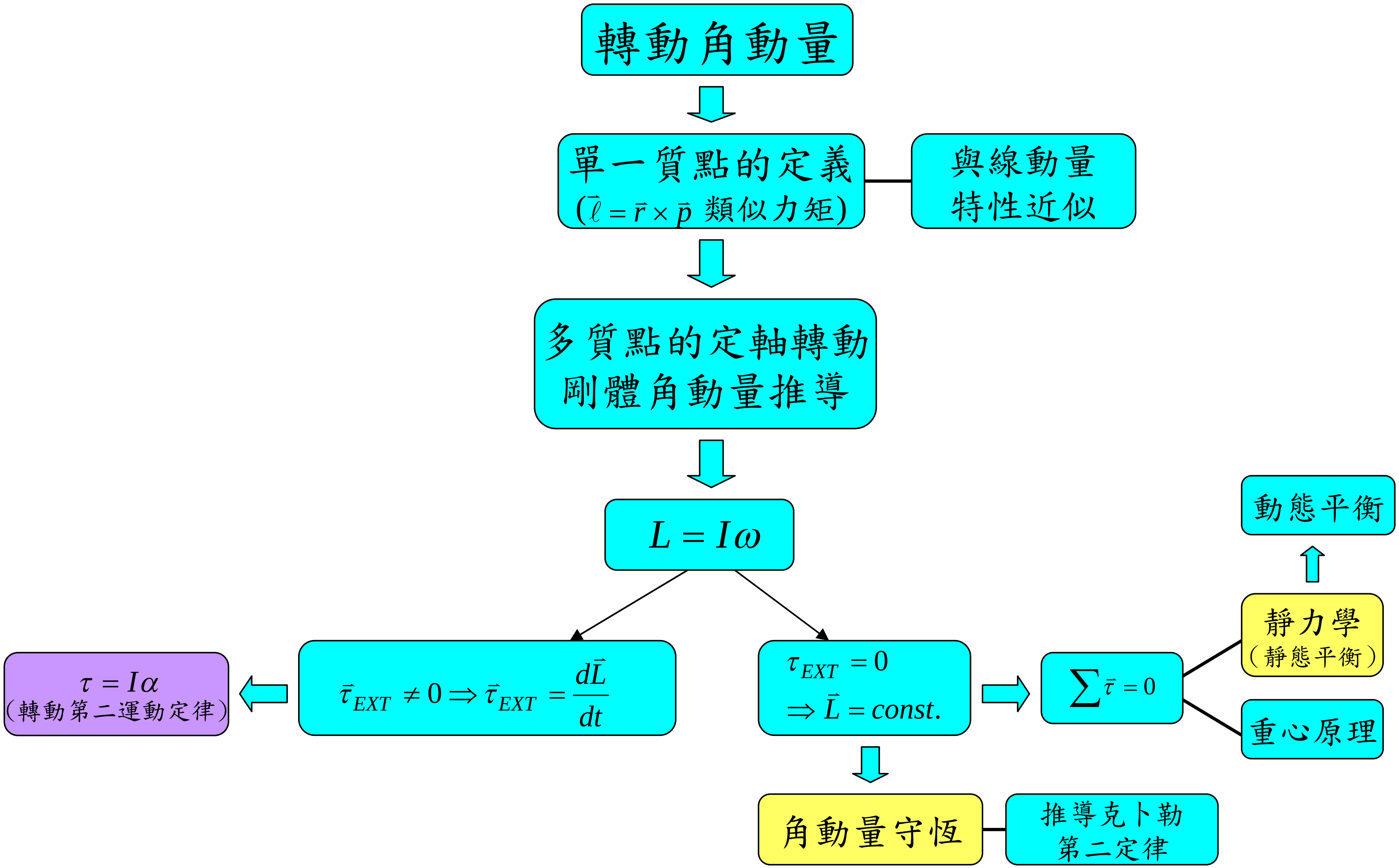
動力平衡(Dynamic Balance)

- 總角動量方向平行於轉軸方向(或轉動角速度方向)，稱為動力平衡(或動態平衡)。



(a)	(b)	(c)
非動態平衡 非靜態平衡($\sum \vec{\tau} \neq 0$)	動態平衡 非靜態平衡	動態平衡($\vec{L} \parallel \vec{\omega}$) 靜態平衡($\sum \vec{\tau} = 0$)

本章重要觀念發展脈絡彙整



習　題

- 教科書習題 (p.257~p.264)

  Exercise: 9,17,21,29,31,33,35,39,43

  Problem: 10,15,19,21

- 基本觀念問題：

  1.請寫出剛體靜力平衡(static equilibrium)的條件？

- 延伸思考問題：

  1.剛體定軸轉動運動是否具有描述質點平移運動之類似牛頓三大運動定律？請由力矩或角動量申述說明之。