Textbook: M.F. Spotts *et al*., *Design of Machine Element*, 8th Ed.

Reference book: Ansel C. Ugural, Mechanical *Design of Machine Components*, 2nd Ed.

Mechanical Design

Introduction & Chapter 1

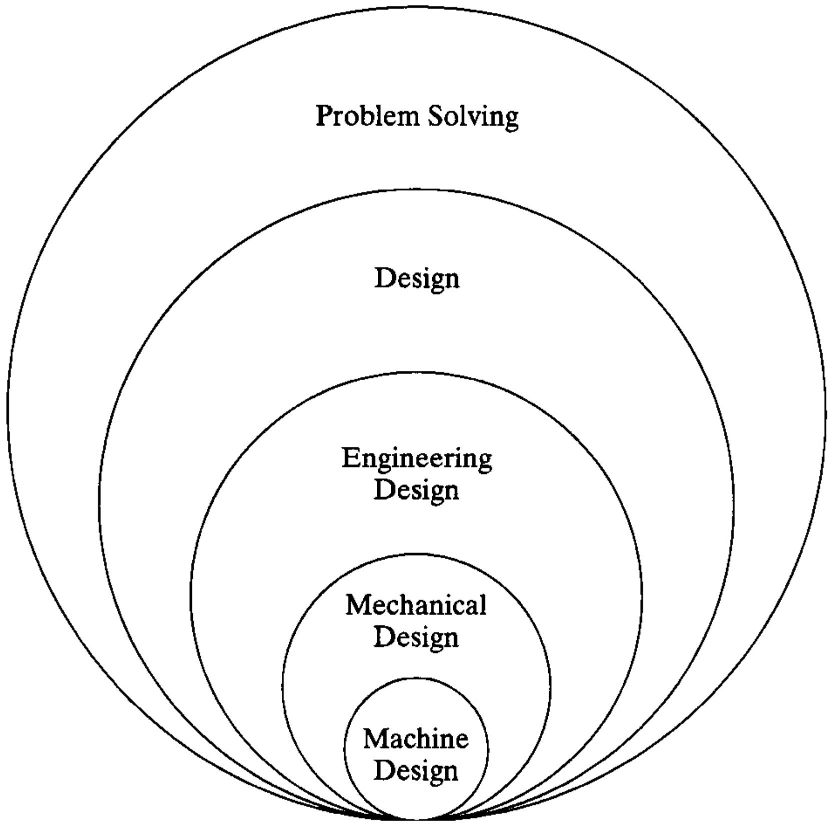
INTRODUCTION

What is DESIGN?

- **Engineering design is the process of devising a system, component, or process to meet desired needs.**
- **It is a decision-making process (often iterative), in which the basic science and mathematics and engineering sciences are applied to convert resources optimally to meet a stated objective.**
- **DESIGN is the process of problem solving.**
- **Problem solving is used by professionals from many different fields in the normal course of their work.**

INTRODUCTION

ENGINEERING DESIGN is the process of devising a system, component, or process to meet desire needs. It is a **decision-making process** (often **iterative**).



The hierarchy of problem solving

INTRODUCTION

Field of ***mechanical engineering*** is divided into two stems:

- Energy stem (heat transfer, thermodynamics, combustion, …)
- Structures and motion stem (strength of material, solid mechanics, kinematics, dynamics, …)

Mechanical Design

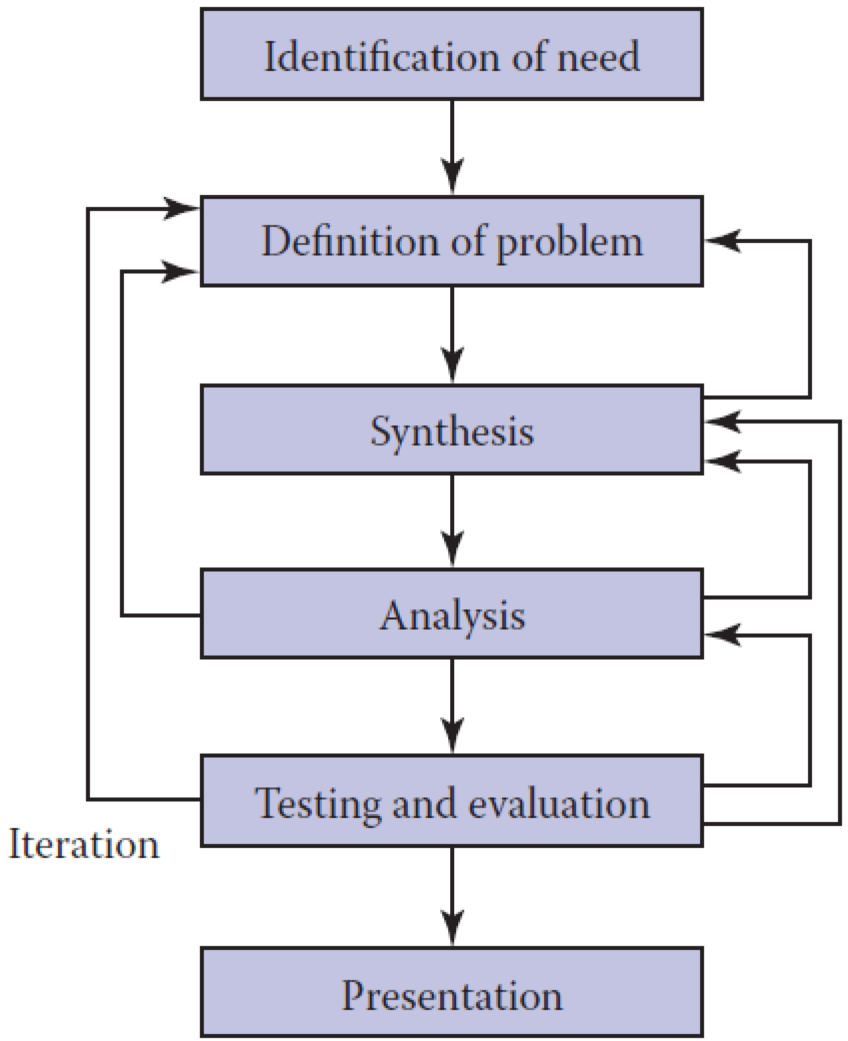
- Both stems of mechanical engineering can be involved.
- E.g., heat exchanger, air compressor, internal combustion engines, …

Machine Design

- **a subset of mechanical design in which the focus is on the structure and motion stem only**.
- e.g., gear box, V-belt driving system, machine structure, …

INTRODUCTION

THE DESIGN PROCESS:



The six-step process of design

Design Process

<u>Identification of Need:</u>

Ex. Present equipment may require improving durability, efficiency, weight, speed, or cost

Ex. New equipment may be needed to perform an automated function, such as computation, assembly, or servicing.

<u>Definition of the Problem:</u>

Ex. New equipment ordinarily consists of new members, with changes in size and material.

Ex. Standard and specification for parts, materials, or processes to achieve uniformity, efficiency, and a specific quality.

Ex. Feasibility study, alternative proposals.

Design Process

<u>Synthesis:</u>

***Putting together*, synthesis several components of a system, then do analysis and optimization, draw layouts, providing details, making supporting calculations.**

<u>Test and evaluation:</u>

Working design is fabricated as a prototype.

Test the prototype in a lab or on a computer to provide analysis database.

Subsequent to many Iterations.

<u>Test and evaluation:</u>

Describe a design graphically, verbally, and in writing.

Design Considerations

<u>Traditional considerations:</u>

Include strength, deflection, weight, size and shape, material properties, operating conditions, processing, cost, availability, usability, utility, and life for a mechanical component or the entire system.

<u>Miscellaneous considerations:</u>

***Include reliability, maintainability, ergonomics* (人體工學)*, and aesthetics* (美學).**

Design Analysis

Engineering modeling:

***Ex.* Geometric modeling is the method for obtaining data necessary for failure analysis early in design process.**

***Ex.* Theoretical solution, finite element modeling.**

Rational design procedure:

The rational design procedure to meet the strength requirements of a load-carrying member attempts to take the results of fundamental tests, such as tension, compression, and fatigue, and apply them to all complicated and involved situations encountered in present-day structures and machines.

Semi-rational or empirical approach may need.

Design Analysis

<u>Methods of analysis:</u>

Based on the mechanics of materials theory generally used.

The basic principles of analysis:

1. ***Statics.*** ***Equations of equilibrium***
2. ***Deformations.*** ***Stress-strain relations (Hooke's law)***
3. ***Geometry.*** ***Compatibility.***

THIS COURSE CONSISTS 3 PARTS

1. ***Basics (principles of mechanics).***

   design, analysis, notions, approaches

2. ***Failure theories and prevention.***

   why mechanical parts fail and how they can be designed

3. ***Design of mechanical elements.***

   the material of Parts 1 and 2 is applied to the analysis, selection, and design of specific mechanical elements

1. FUNDAMENTAL PRINCIPLE

LOAD AND STRESS ANALYSIS

Design methods for the various machine elements are founded on the theories of mechanics and strength of materials.

- The scope of such theories is very extensive.
- In some cases, these theories are based on simplified approximations, and attention should be directed to the limitations imposed by the assumptions that are made in arriving at working formulation.

SOLVING MECHANICAL COMPONENT PROBLEM

Design is the process of problem solving. It is very important to formulate a mechanical element problem and its solution accurately.

1. **Given**: Define the problem and known quantities.
2. **Find**: State consistently what is to be determined.
3. **Assumption**: List simplifying idealizations to be made.
4. **Solution**: Apply the appropriate equation to determine the unknowns.
5. **Comments**: Discuss the results briefly.

<u>1-1</u> STATIC EQUILIBRIUM

When a body is at rest, or in motion with constant velocity, the external forces acting upon it are <u>in equilibrium</u>.

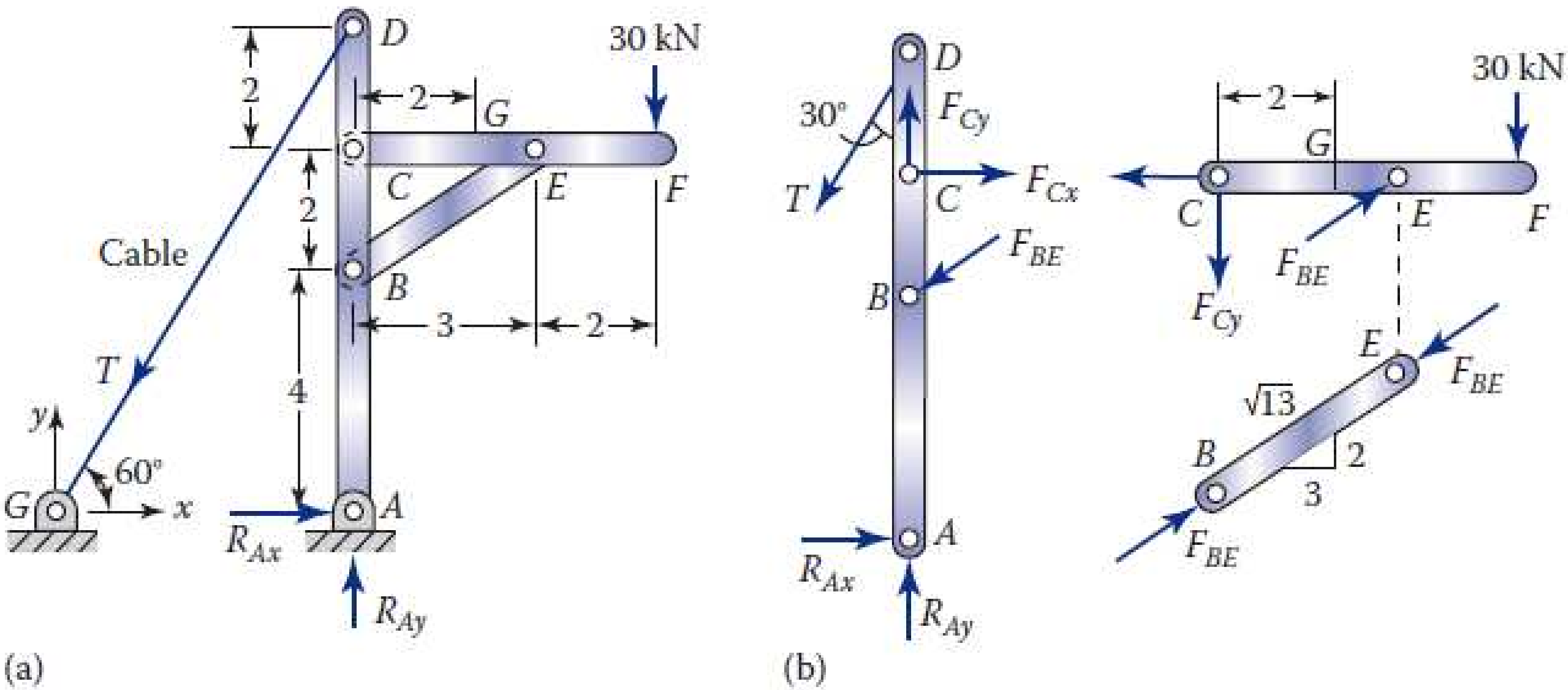
- This statement applies to the body as a whole or to any portion of it.
- Static equilibrium means that both ***<u>forces</u>*** and ***<u>moments</u>*** are in balance.

$$\sum \vec{F} = 0 \quad \longrightarrow \quad \sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$

$$\sum \vec{M} = 0 \quad \longrightarrow \quad \sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

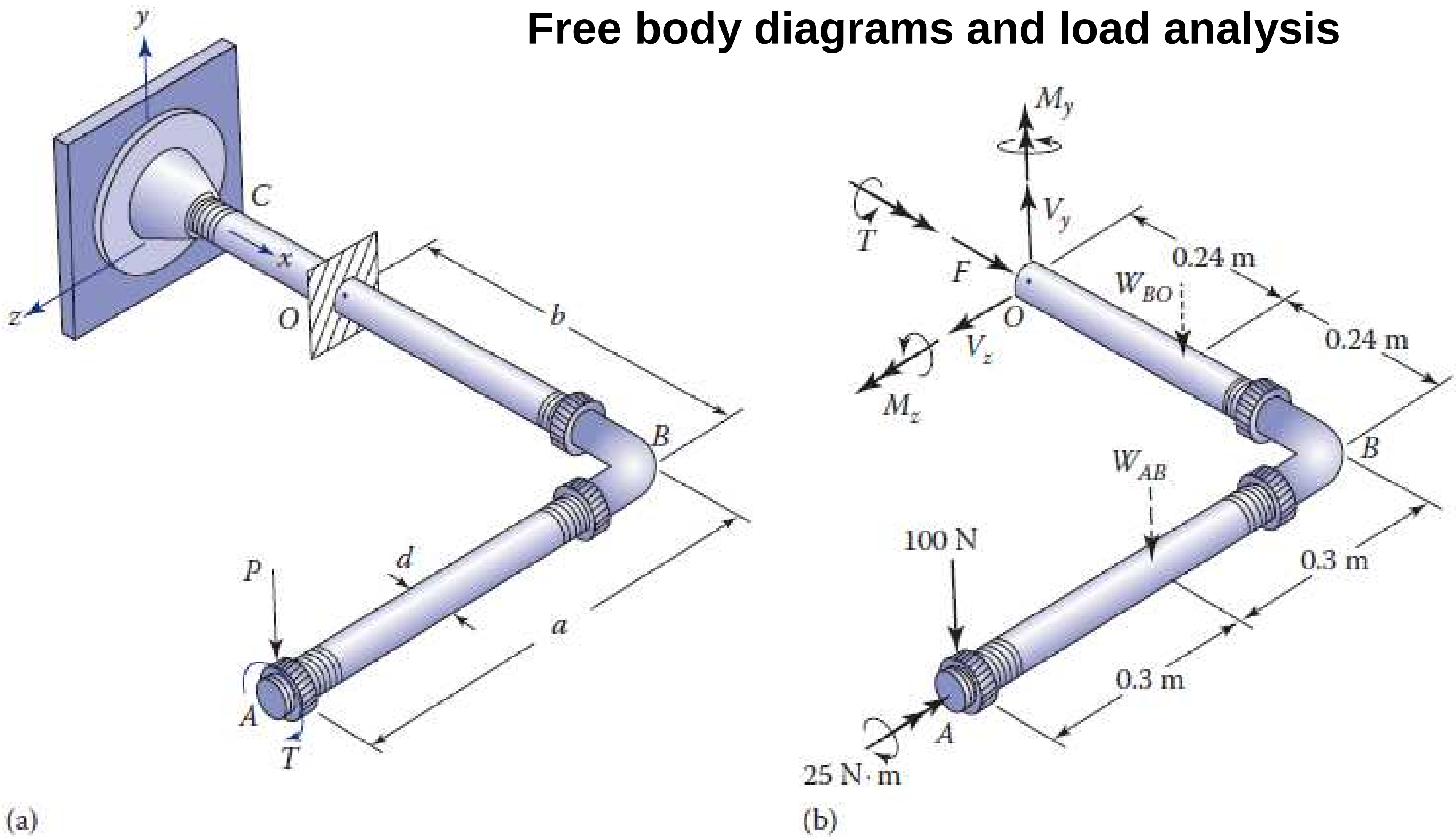
Example: Pin-Connected Frame

Free body diagrams and load analysis



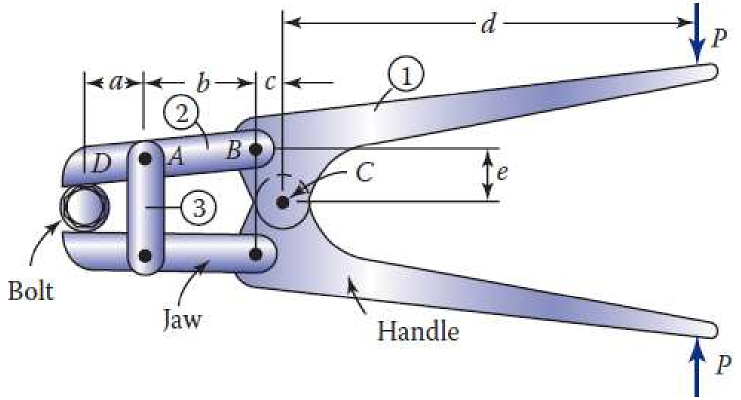
Example: Pipe

Free body diagrams and load analysis

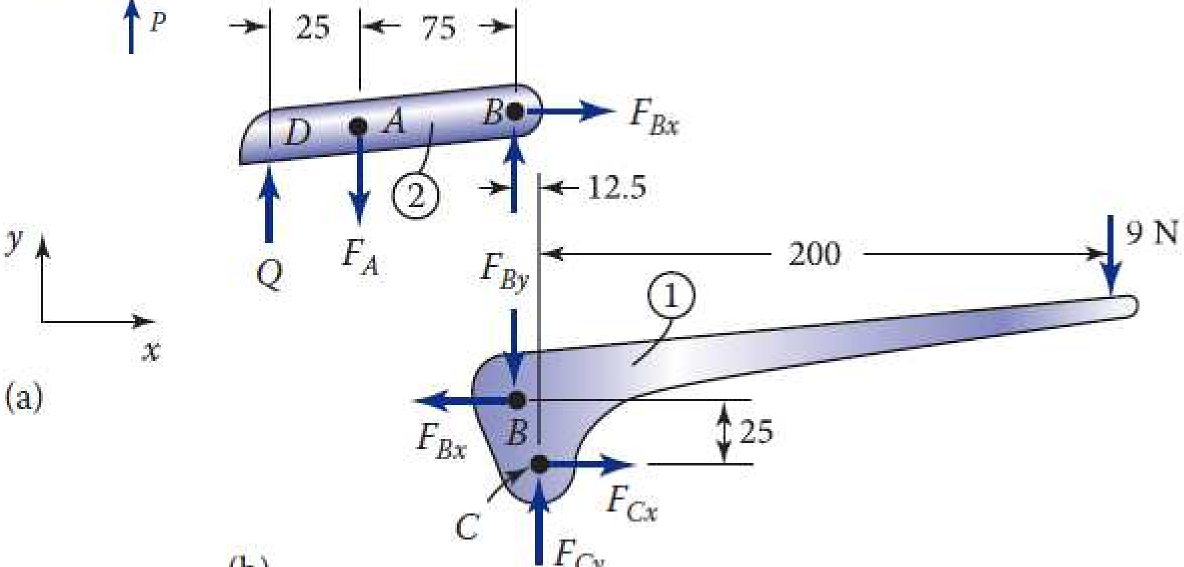


<u>Example:</u> Bolt Cutter

Free body diagrams and load analysis



(a)



(b)

1-2 ENGINEERING MATERIALS

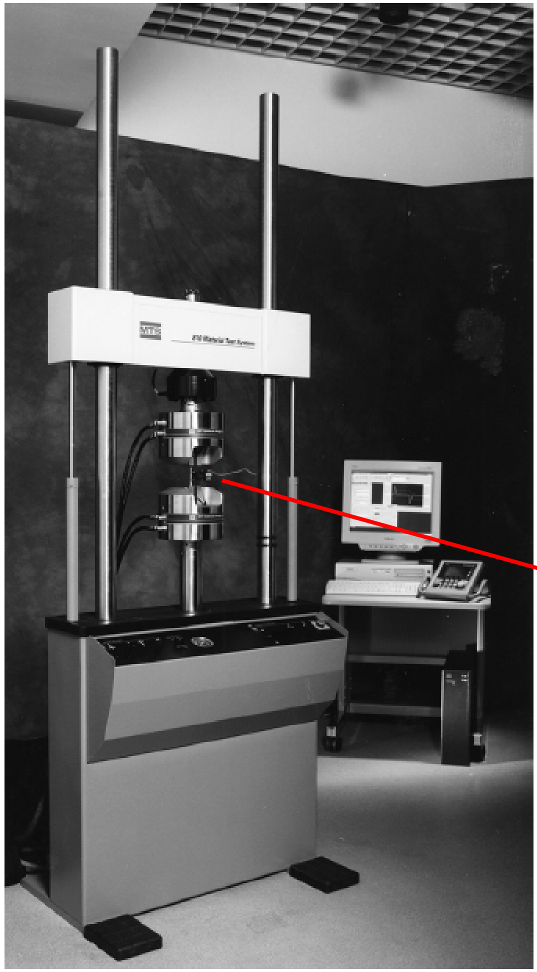
The mathematical equations used in designing are derived for an idealized material, which is assumed to have following properties:

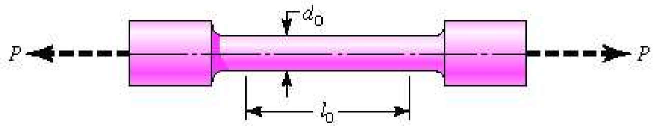
- **Perfect Elasticity** (linear, nonlinear): material that returns to its original form immediately upon removal of loads.

- **Homogeneous**: the same properties throughout the entire extend.

- **Isotropy**: elastic properties are the same in all direction.

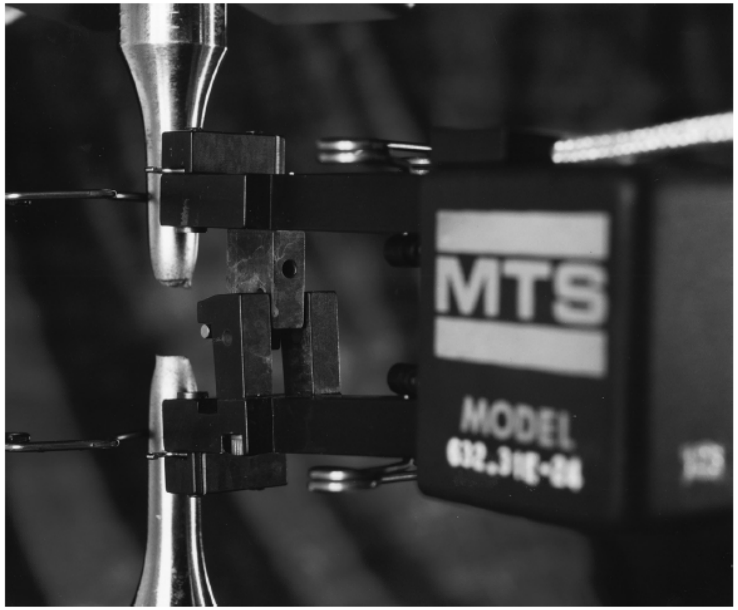
TENSILE TEST





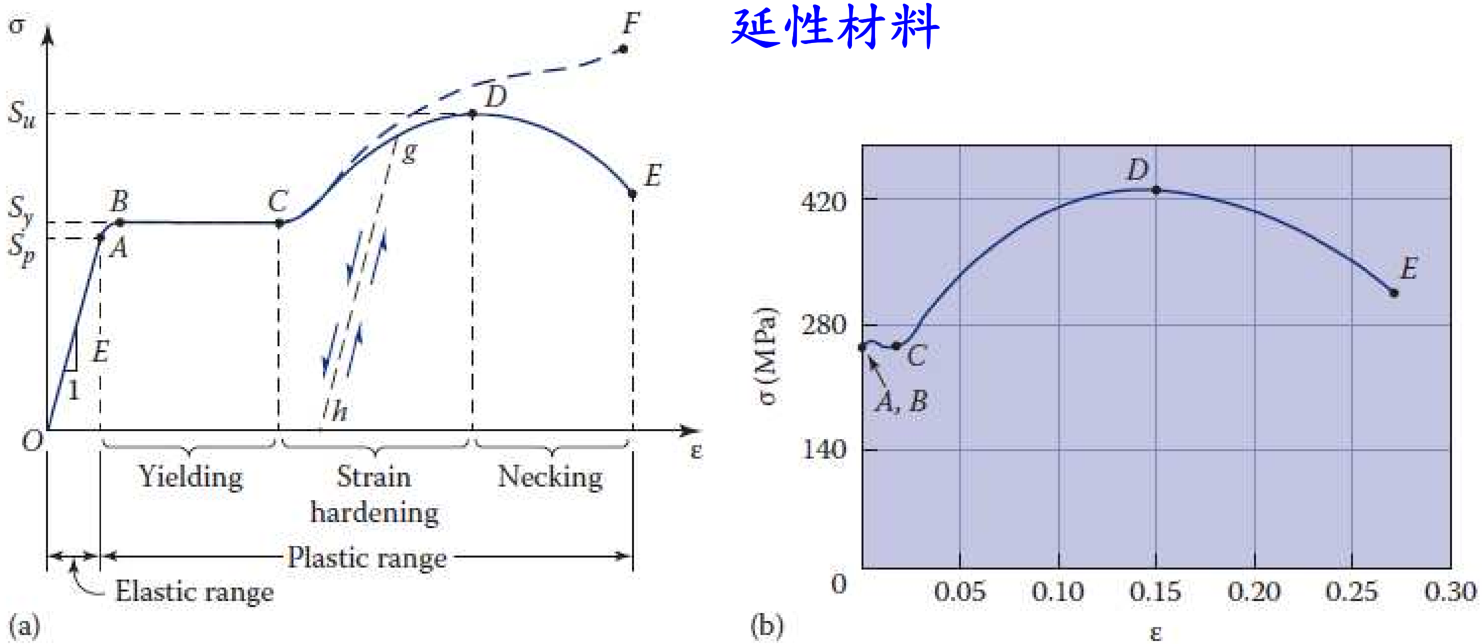
Figure

A typical tension-test specimen. Some of the standard dimensions used for d_0 are 2.5, 6.25, and 12.5 mm, but other sections and sizes are in use. Common gauge lengths l_0 used are 10, 25, and 50 mm.



Stress-strain diagrams in tensile test

延性材料



Stress–strain diagram for a typical structural steel in tension: (a) drawn not to scale and (b) drawn to scale.

Proportional limit

Elastic limit

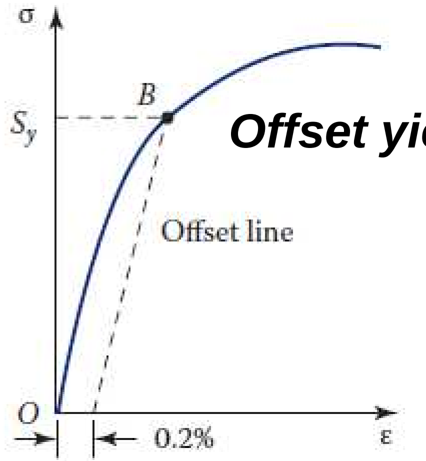
Yield point

Yield strength

Ultimate (tensile) strength

Fracture strength

脆性材料



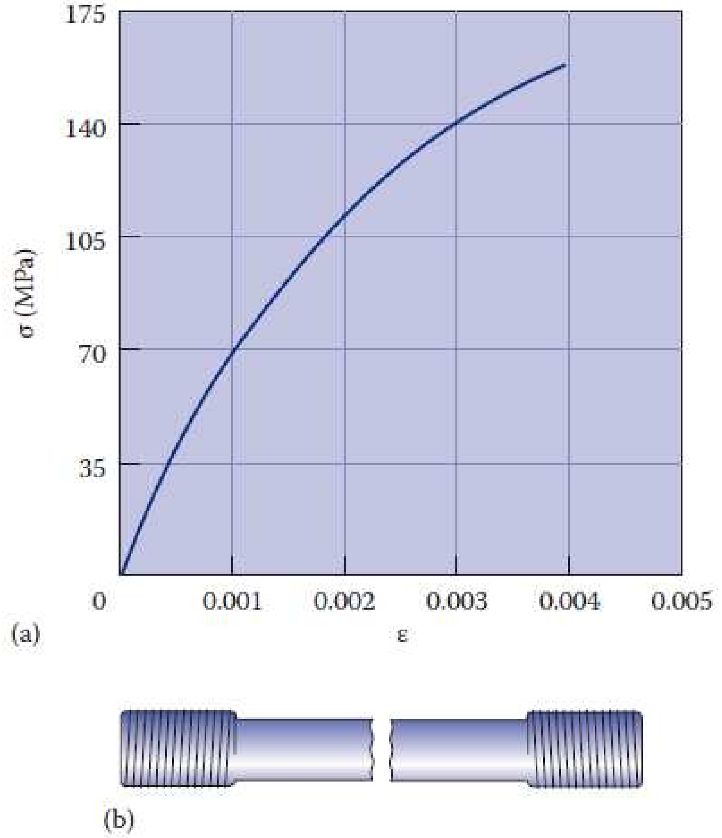
Offset yield strength

Determination of yield strength by the offset method.

Stress-strain diagrams in compressive test

Most *ductile materials* behave approximately the same in tension and compression over the elastic range. The yield strength is about the same in tension and compression (*even material*). But in the plastic range, the behavior is quite different.

For *brittle materials*, the entire compression stress-strain diagram has a shape similar to the shape of the tensile diagram. However, brittle materials usually have characteristic stresses in compression that are much greater than in tensions (uneven material).



Gray cast iron in tension: (a) stress–strain diagram and (b) fractured specimen.

1-3 TENSION & COMPRESSION STRESS

The average stress:

$$\sigma = \frac{P}{A}$$

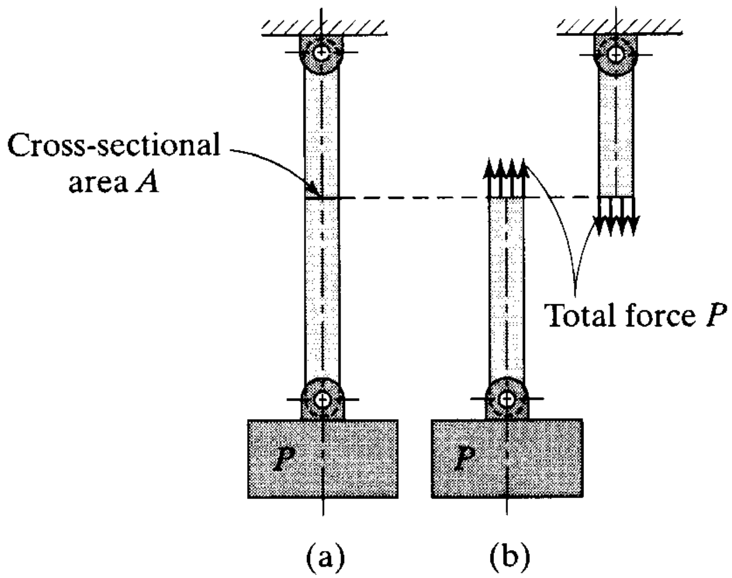
Strain:

$$\varepsilon = \frac{l' - l}{l} = \frac{\delta}{l}$$

Hooke's law:

$$\varepsilon = \frac{\sigma}{E}$$

$$\delta = \frac{Pl}{AE}$$



*The constant **E** is called the **modulus of elasticity**, or **Young's modulus**.*

EXAMPLE 1-1A

Let load P be equal to 22,500 N, and let the bar be 75 mm wide and 13 mm thick. The uniform portion of the bar is 1500 mm long. The material is **steel**.

(a) Find the stress in the uniform portion of the bar.

(b) Find the deformation of the uniform portion of the bar.

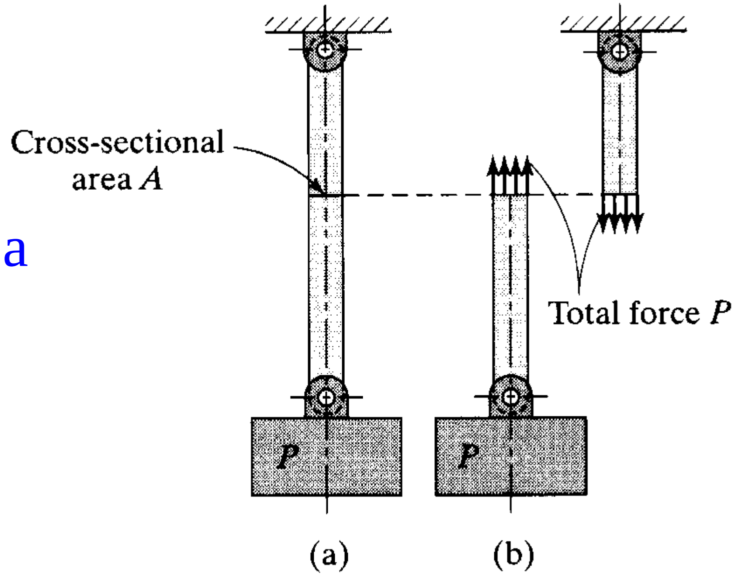
From Table 2-3A (p. 125), E=206 900 MPa

(a)

$$\sigma = \frac{P}{A} = \frac{22500}{75 \times 13} = 23.08 \ {}^{\text{N}}\!/_{\text{mm}^2} = 23.08 \text{ MPa}$$

(b)

$$\delta = \frac{Pl}{AE} = \frac{22500 \times 1500}{75 \times 13 \times 206900} = 0.167 \text{ mm}$$



1-4 FORCE AND MASS

Newton's second law:

force = mass $\times$ acceleration

	SI Unit	US Customary Unit
Mass	kg	slug
Force	N	lb
Length	m	ft
Gravity	9.8 m/s^2	32.2 ft/sec^2

What about weight? Is it mass or force?

工程上，也常習慣把公斤當成重量或力量的單位，把磅當成質量的單位。

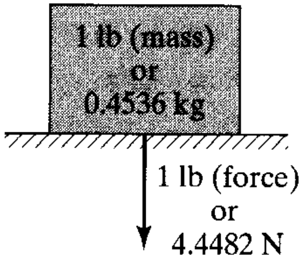
具有質量1公斤的物體，在地球上所受到的重力稱爲1公斤的力量。

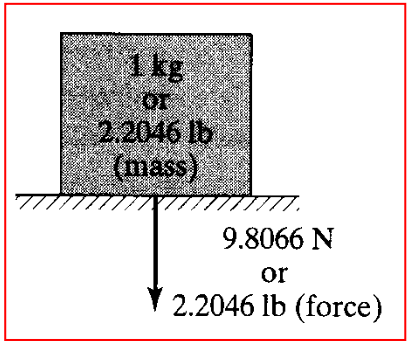
在地球上1磅重的物體，稱爲具有1磅的質量。

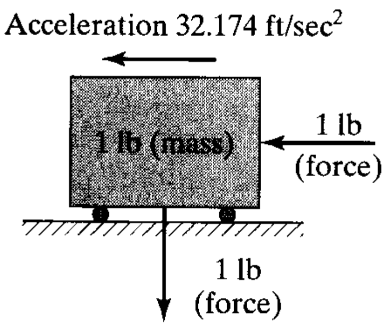
$$1\ \mathrm{kg} = 2.2\ \mathrm{lb}\left(\mathrm{mass}\right) \qquad 1\ \mathrm{lb}\left(\mathrm{force}\right) = 4.45\ \mathrm{N}$$

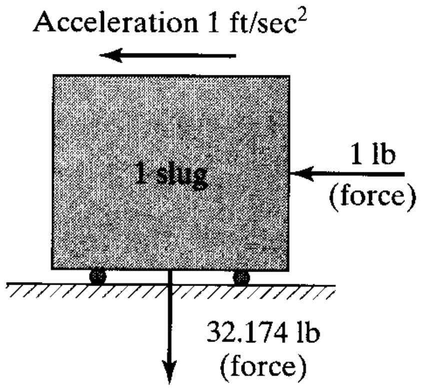
$$1\ \mathrm{lb}\left(\mathrm{mass}\right) = 0.454\ \mathrm{kg} \qquad 1\ \mathrm{N} = 0.225\ \mathrm{lb}\left(\mathrm{force}\right)$$

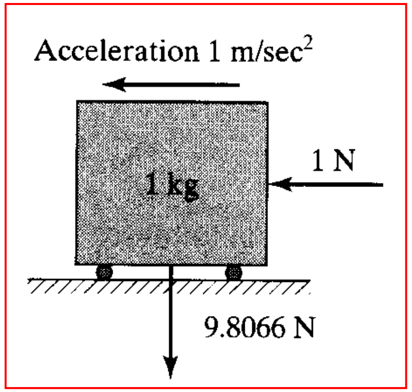
$$\boxed{W = mg}$$











1-6 STATICALLY INDETERMINATE

Machine parts are arranged in a manner whereby the forces cannot be determined by the equation of statics alone.

- Presence of more supports or members than minimum required for the equilibrium of the structure.
- For such problems, the deformations of the parts must be taken into consideration.

Static ➡

$$\boxed{\begin{aligned}\sum \vec{F} &= 0 \\ \sum \vec{M} &= 0\end{aligned}}$$

are not sufficient to solve the reactions

➕

Deformation conditions

靜定問題，用這些方程式就足夠解反力

EXAMPLE 1-2

Find the force in each of the vertical bars. The weight can be assumed to be rigid to **maintain the connections for the three vertical bars in a straight line**. Assume the support at the top to also be rigid.

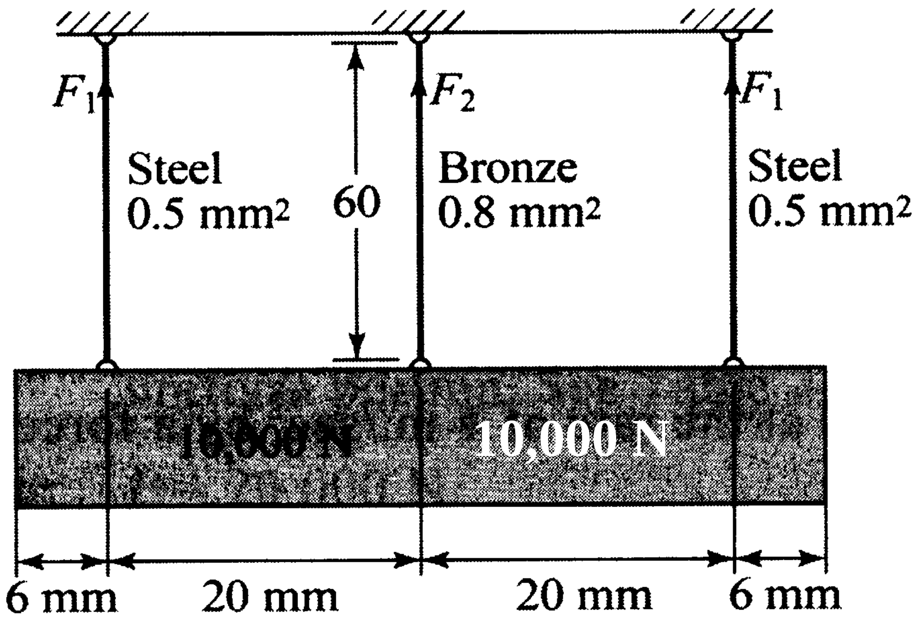


Figure 1-3

EXAMPLE 1-2

From Table 2-3A, E_{steel}= 206,900 MPa and E_{bronze}= 103,400 MPa

Statical equilibrium

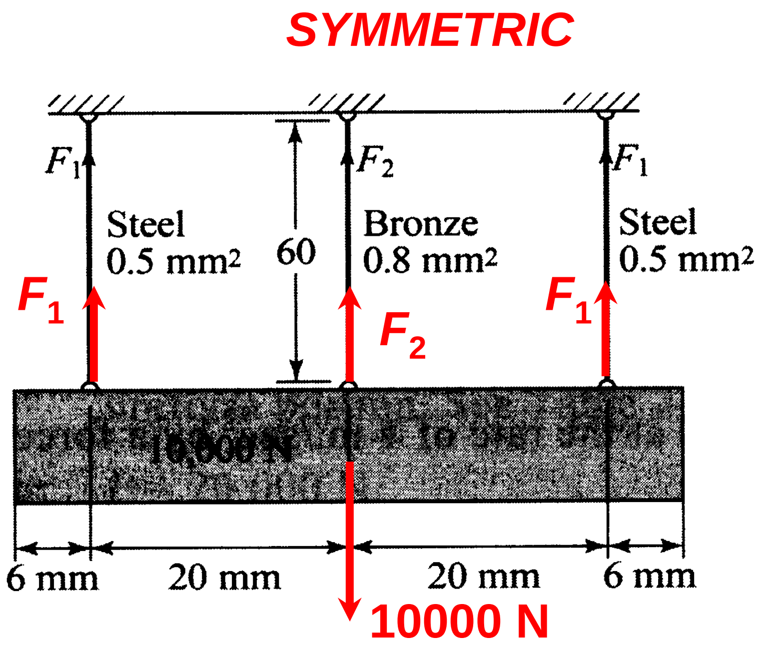
$$2F_1 + F_2 = 10000$$

Three vertical bars in a straight line

$$\delta_1 = \delta_2 \qquad \frac{F_1 l_1}{A_1 E_1} = \frac{F_2 l_2}{A_2 E_2}$$

$$\frac{F_1 \times 60}{0.5 \times 206900} = \frac{F_2 \times 60}{0.8 \times 103400}$$

$$F_1 = 1.25 F_2$$

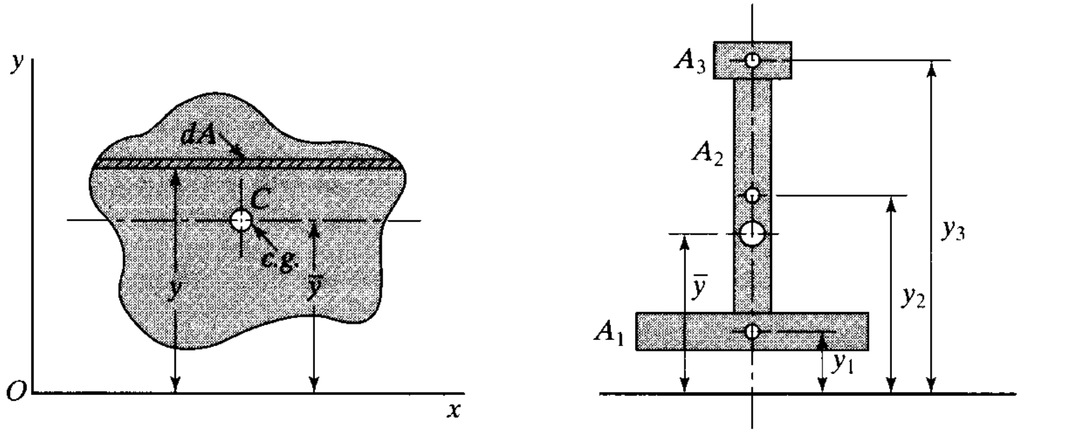


$$F_1 = 3571 \text{ N}$$
$$F_2 = 2857 \text{ N}$$

1-7 CENTER OF GRAVITY

$$\int (y-\overline{y})dA = 0 \qquad \cdots \qquad \overline{y} = \frac{\int y dA}{\int dA} = \frac{\int y dA}{A}$$

支撐在 **C** 點，則重力的合力矩等於零



(a) Center of gravity (b) Center of gravity of composite section

Figure 1-4

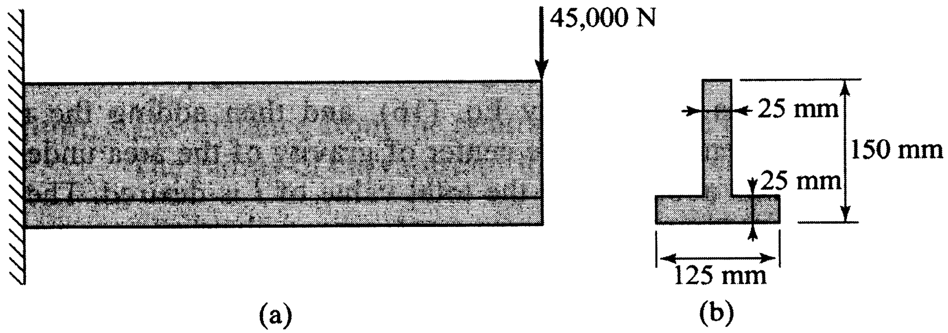
$$\boxed{\overline{y} = \frac{A_1\overline{y}_1 + A_2\overline{y}_2 + \cdots}{A_1 + A_2 + \cdots}}$$

EXAMPLE 1-3

Find the location of the center of gravity of the T-shape cross section.

$$\bar{y} = \frac{3750 \times 75 + 2500 \times 12.5}{3750 + 2500}$$

$$= 50 \text{ mm}$$

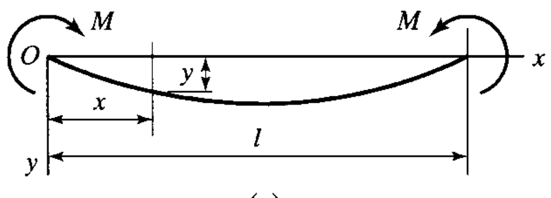


Center of gravity is 50 mm up from the bottom

Figure 1-21

1-8 BENDING OF BEAMS

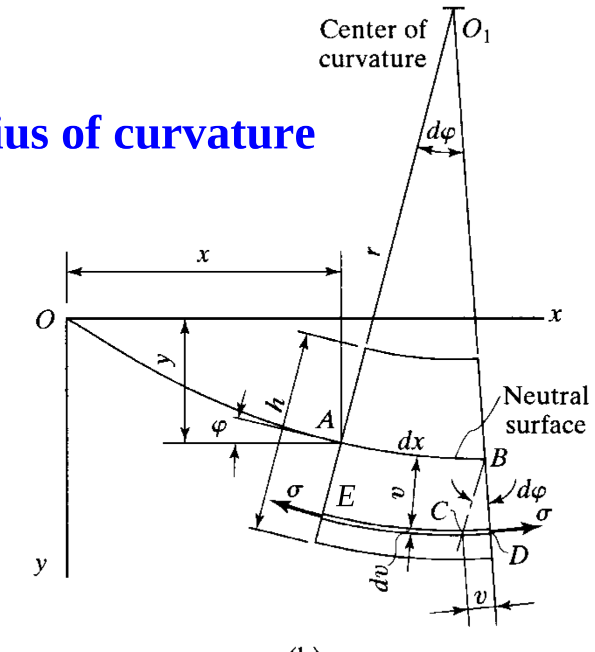
Suppose a long, thin, straight beam is bent into a curve by moments M applied at the ends.



$$\Delta O_1AB \sim \Delta O_1ED$$

$$\frac{r+\upsilon}{r} = \frac{dx+\upsilon d\varphi}{dx}$$

r = radius of curvature

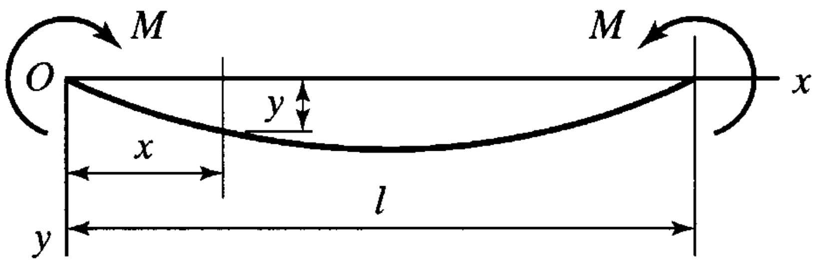


$$\frac{\upsilon}{r} = \frac{\upsilon d\varphi}{dx} = \varepsilon \qquad \frac{\sigma}{E} = \frac{\upsilon}{r}$$

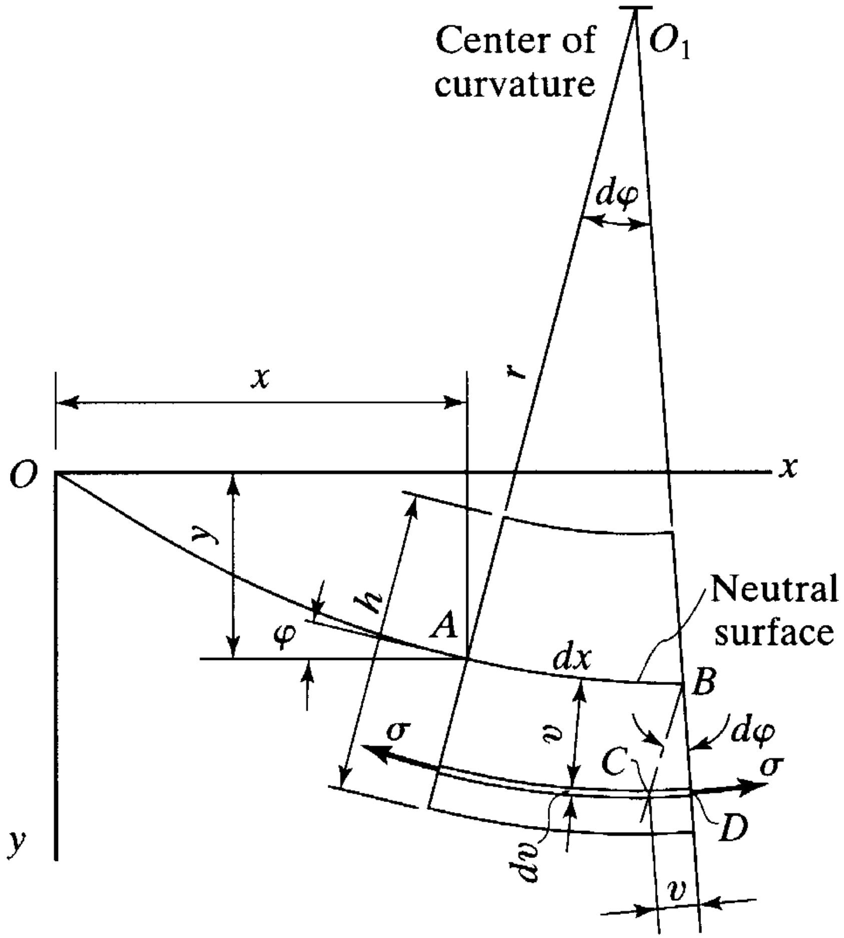
$$\boxed{\sigma = \frac{E}{r}\upsilon}$$

Figure 1-5

變形時，平面保持平面
存在一不變形的中性軸



(a)



(b)

$$\Delta O_1AB \sim \Delta O_1ED$$

$$\frac{r+\upsilon}{r}=\frac{dx+\upsilon d\varphi}{dx}$$

$$\frac{\upsilon}{r}=\frac{\upsilon d\varphi}{dx}=\varepsilon \qquad \frac{\sigma}{E}=\frac{\upsilon}{r}$$

$$\boxed{\sigma=\frac{E}{r}\upsilon}$$

變形時，平面保持平面
存在一不變形的中性軸

Balance of forces:

$$F = \int \sigma dA = \int \frac{E}{r} \upsilon dA = \frac{E}{r} \int \upsilon dA = 0$$

Pure bending

$$\int \upsilon dA = \bar{\upsilon} A = 0 \quad \Rightarrow \quad \bar{\upsilon} = 0$$

υ 從中性軸量起

Neutral axis passes through the center of gravity of the cross section.

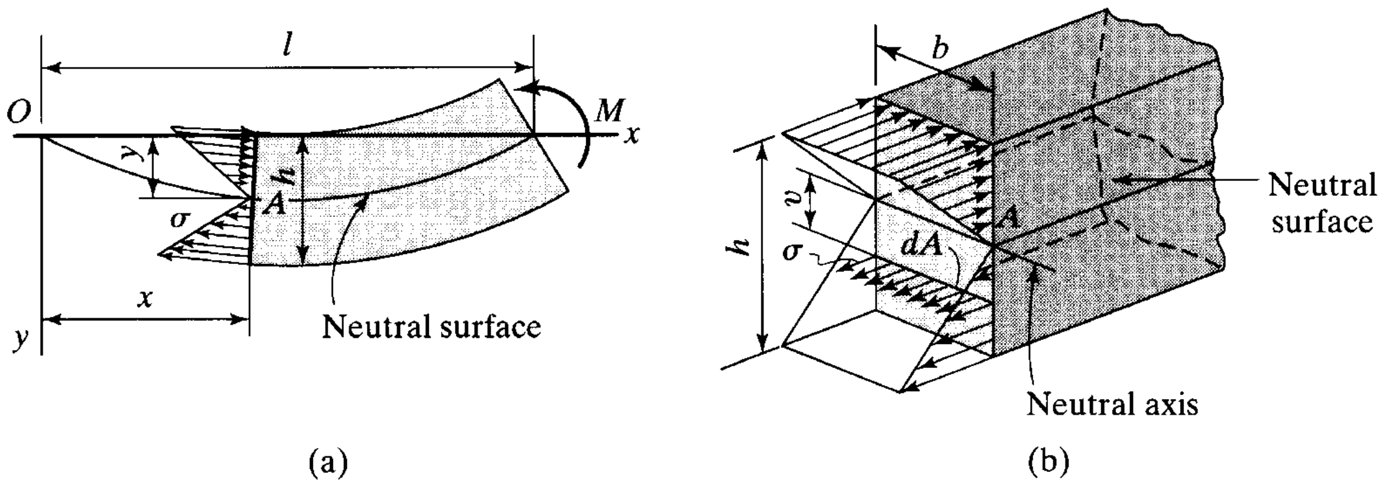


Figure 1-6

Balance of moments:

$$M = \int \sigma \upsilon dA = \int \frac{E}{r} \upsilon^2 dA = \frac{E}{r} \int \upsilon^2 dA = \frac{EI}{r}$$ **Bending moment**

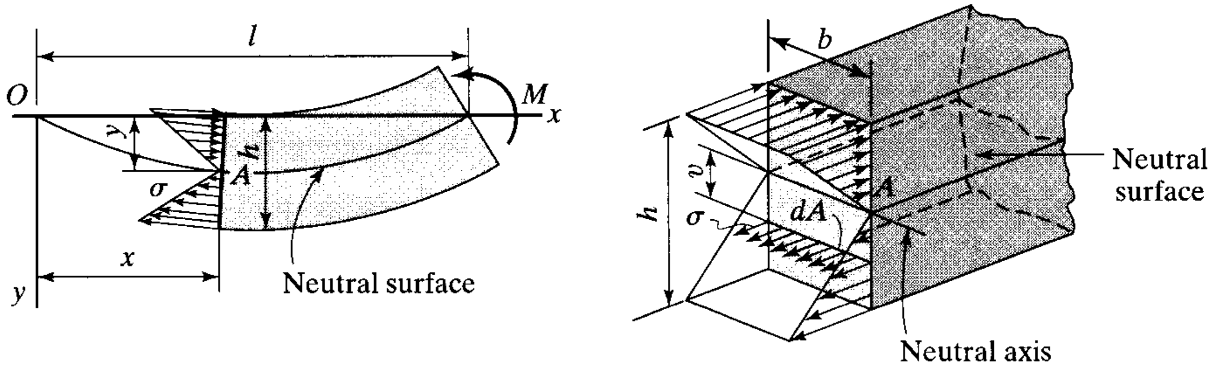
where $I = \int \upsilon^2 dA$: moment of inertia of the area (second moment of area)

對斷面中性軸的面積慣性矩

With $\sigma = \frac{E}{r} \upsilon$ and $M = \frac{EI}{r}$

$$\Rightarrow \quad \sigma = \frac{M\upsilon}{I} \quad \text{and} \quad \sigma_{\max} = \frac{Mc}{I}$$

Figure 1-6



(a) (b)

1-9 MOMENT OF INERTIA

Formula: $I = \int v^2 dA$

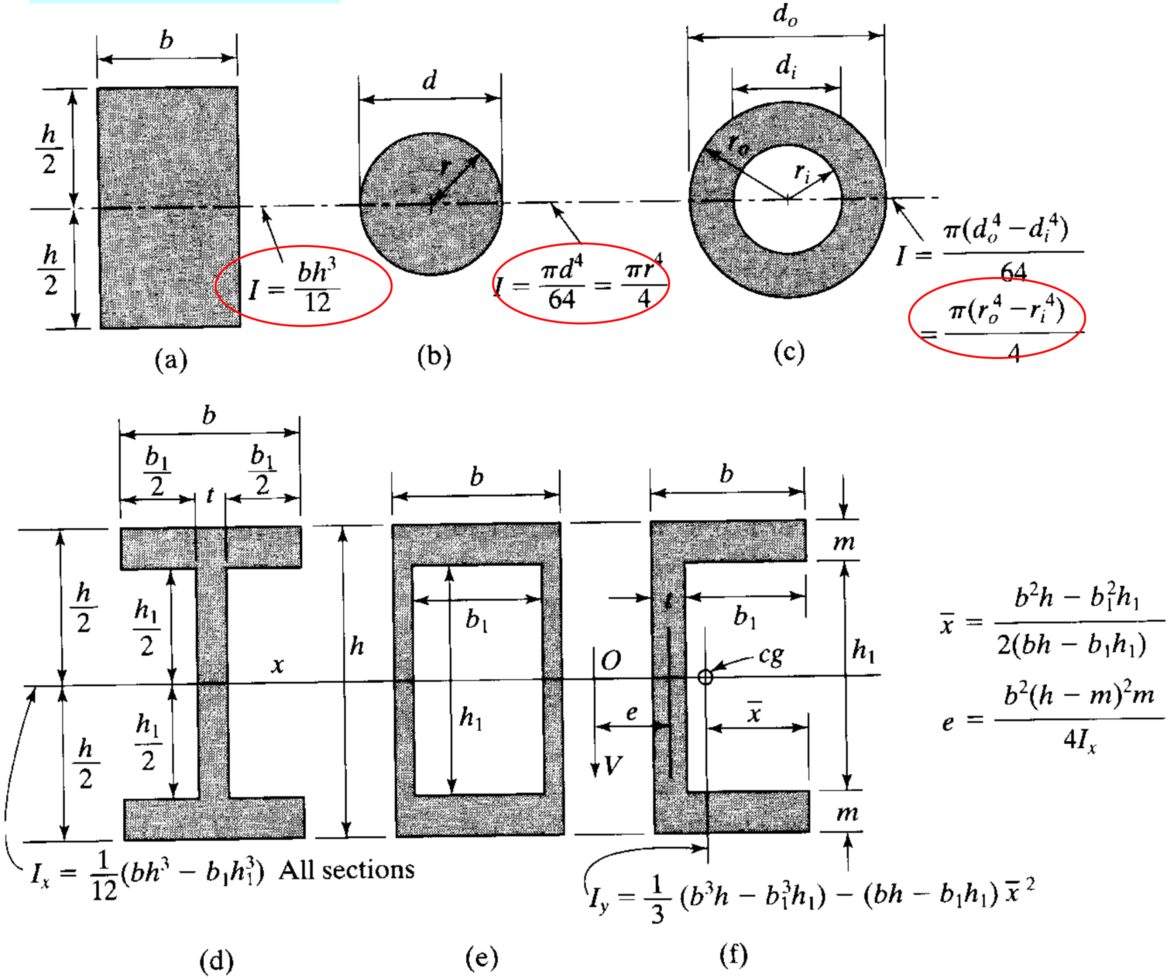
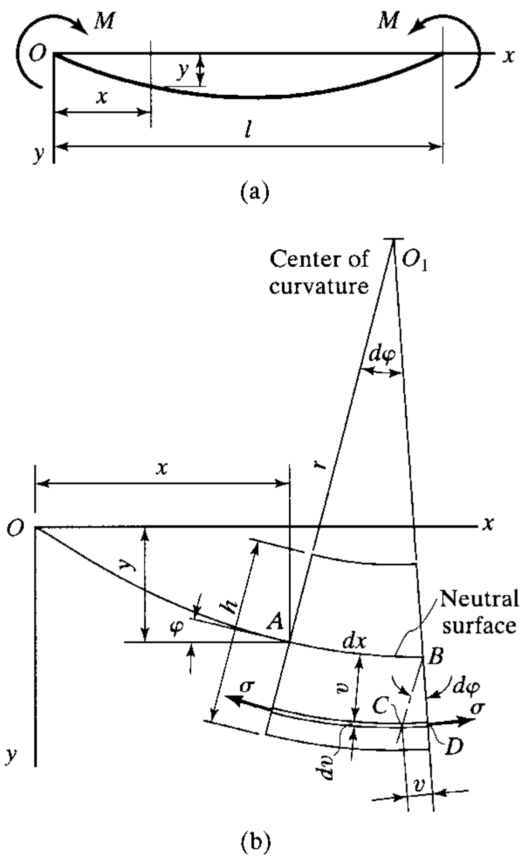


Figure 1-7

EXAMPLE 1-4

Let the beam in the figure be 50 mm wide and 72 mm deep. Let the bending moment *M* at each and be equal to 4,500,000 N-mm. Find the value of the maximum bending stress.



$$I = \frac{bh^3}{12} = \frac{50 \times 72^3}{12} = 1,555,200 \text{ mm}^4$$

$$\sigma_{\max} = \frac{Mc}{I} = \frac{4,500,000 \times 36}{1,555,200} = 104.2 \text{ MPa}$$

Special cases:

$$\sigma = \frac{6M}{bh^2}$$

Bending stress of a beam of RECTANGULAR cross section

$$\sigma = \frac{32M}{\pi d^3}$$

Bending stress of a beam of CIRCULAR cross section

1-10 TRANSFER AXIS FOR MOMENT OF INERTIA

About axis O: $$I_0 = \int \upsilon^2 dA$$

About axis 1: $$I_1 = \int (\upsilon + \bar{y})^2 dA = \int (\upsilon^2 + 2\upsilon\bar{y} + \bar{y}^2) dA$$

$$= I_0 + A\bar{y}^2$$

parallel-axis principle

$$\int \upsilon^2 dA = I_0$$

$$\int 2\upsilon\bar{y} dA = 2\bar{y}\int \upsilon dA = 0$$

$$\int \bar{y}^2 dA = \bar{y}^2 \int dA = A\bar{y}^2$$

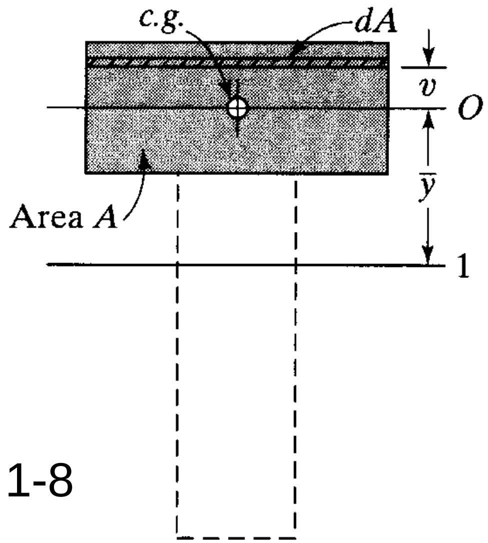


Figure 1-8

EXAMPLE 1-6

Find the value of the moment of inertia of the T-shaped cross section of Fig. 1-21(b).

For the stem

$$I_s = \frac{25 \times 150^3}{12} + 3750 \times 25^2$$

$$= 9{,}375{,}000 \text{ mm}^4$$

For the flange

$$I_f = \frac{100 \times 25^3}{12} + 2500 \times 37.5^2$$

$$= 3{,}646{,}000 \text{ mm}^4$$

Total I $\quad I = I_s + I_f = 13{,}021{,}000 \text{ mm}^4$

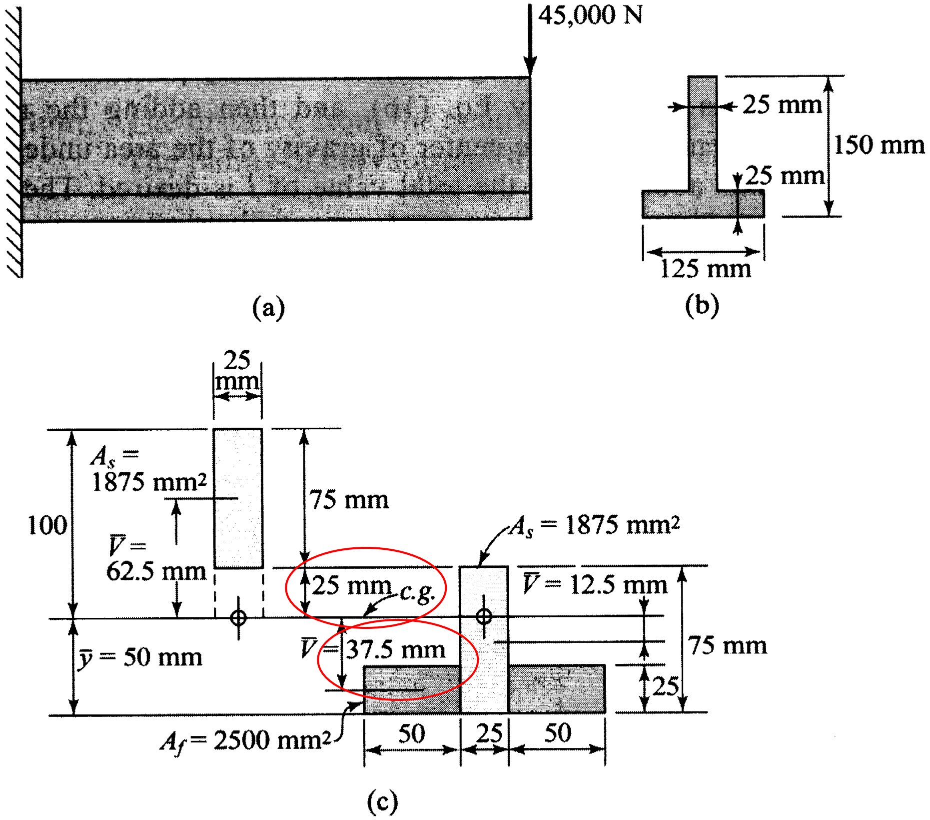


Figure 1-21

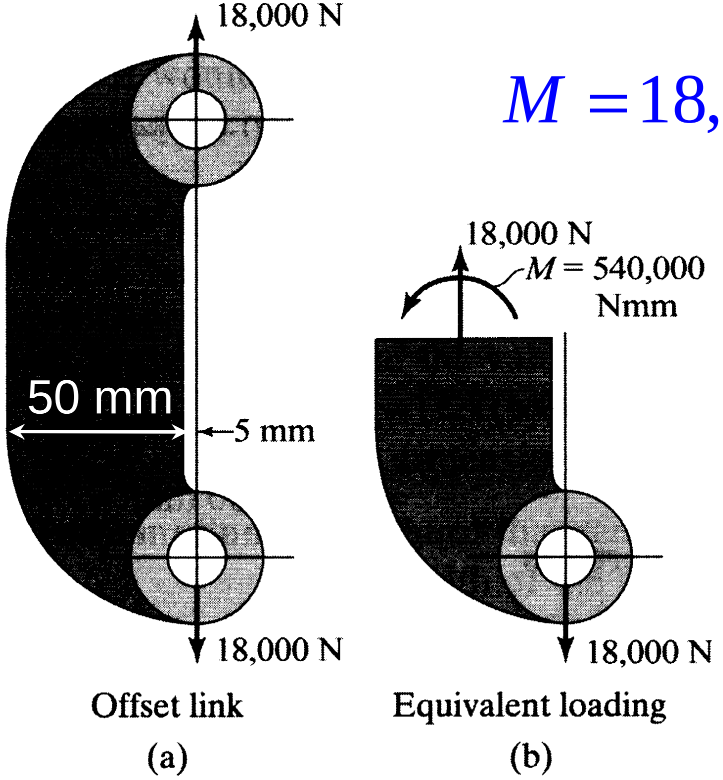
1-11 PRINCIPLE OF SUPERPOSITION

***Principle of superposition*: the resultant effect at any chosen point is the sum of the effects of various load.**

- It is valid for cases of loading only where the magnitude of the stress and deflection is directly proportional to the load (linear system).
- It cannot be applied if the loads produce deflection that are great that the basic configuration of the system is thereby changes (nonlinear).
- Not valid for slender member loaded in compression after the load reach the buckling load.

EXAMPLE 1-7

Calculate and plot the distribution of stress over cross section of the offset link. The main body of the link is straight and is ¾ in, thick.



$$M = 18,000 \times (25+5) = 540,000 \text{ Nmm}$$

Figure 1-9

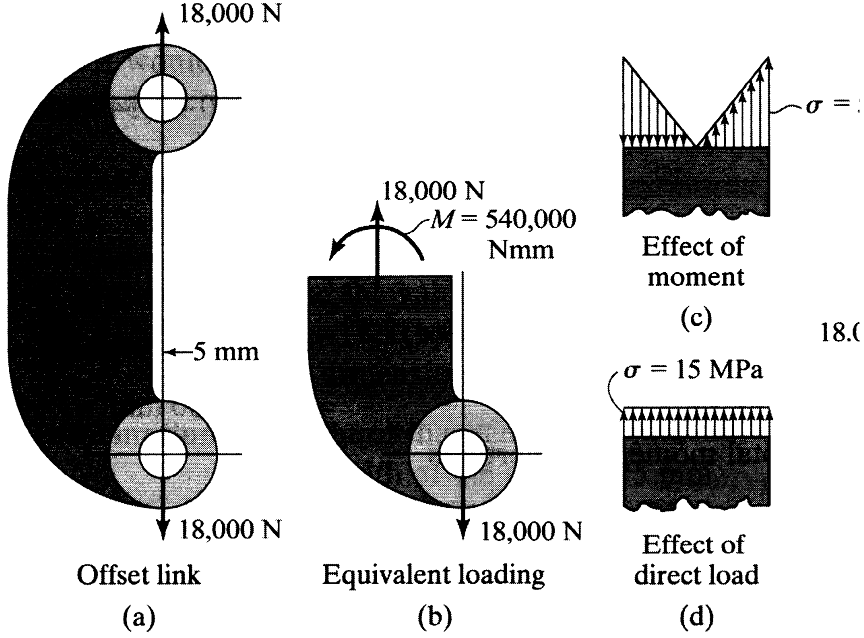
EXAMPLE 1-7

Direct stress

$$A = 24 \times 50 = 1200 \text{ mm}^2$$

tension

$$\sigma = \frac{P}{A} = \frac{18000}{1200} = 15 \text{ MPa}$$



Bending stress

$$I = \frac{bh^3}{12} = \frac{24 \times (50)^3}{12} = 250,000 \text{ mm}^4$$

$$\sigma = \frac{Mc}{I} = \frac{540,000 \times 25}{250,000} = 54 \text{ MPa}$$

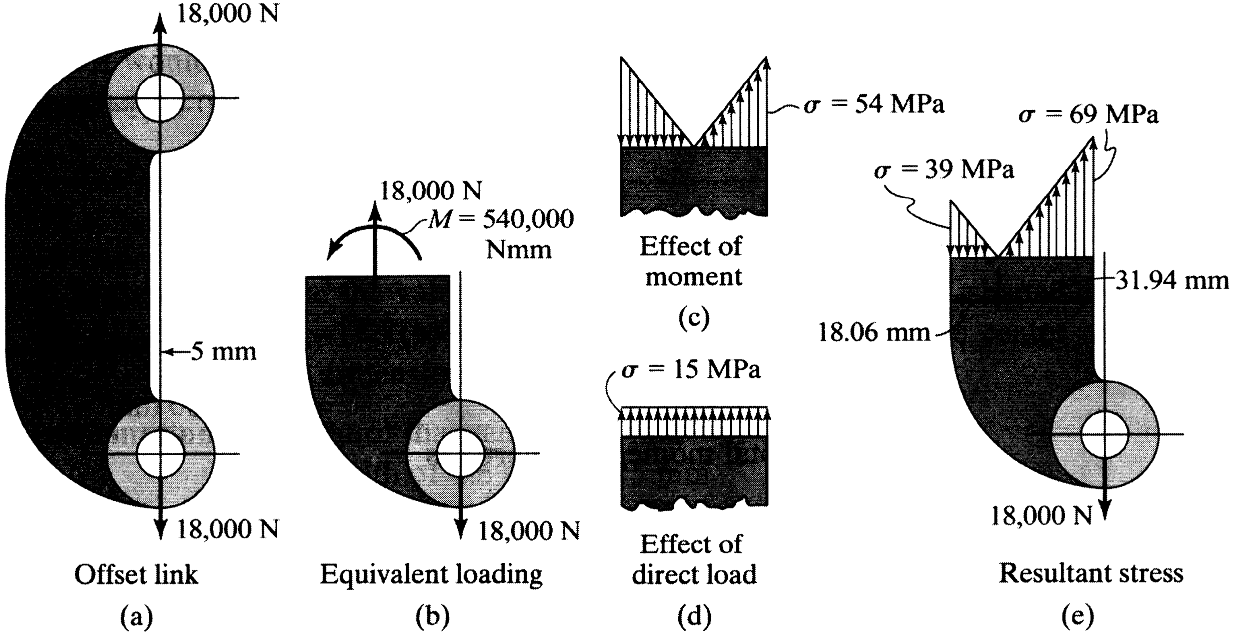
EXAMPLE 1-7

On the inside edge

$$\sigma = 15 + 54 = 69\ \text{MPa}$$ **tension**

On the outside edge

$$\sigma = 15 - 54 = -39\ \text{MPa}$$ **compression**



1-12 ADDITIONAL BEAM EQUATIONS

Fundamental equations of beam theory:

$$\frac{d^2 y}{dx^2} = -\frac{M}{EI}$$

$$\frac{dM}{dx} = V$$

$$\frac{dV}{dx} = \frac{d^2 M}{dx^2} = w$$

計算梁變形後的樣子

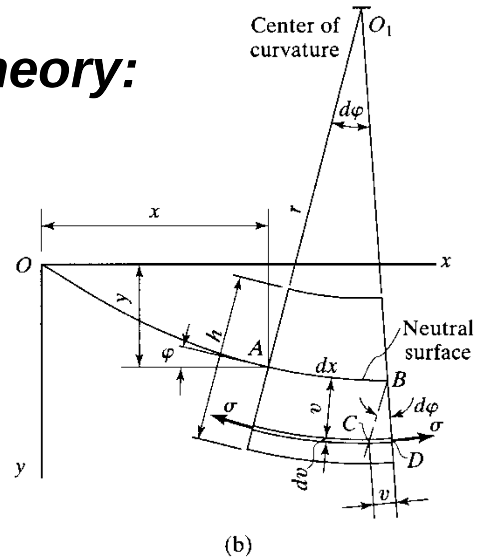


Figure 1-5(b)

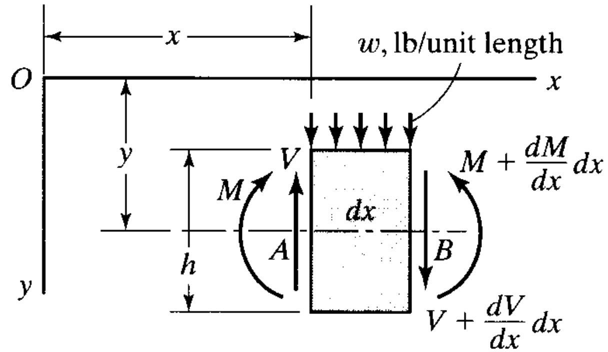


Figure 1-11

These equations can also be valid for the cases which are not pure bending.

DERIVE THE BEAM EQUATIONS (I)

For most beam used in engineering, $\varphi << 1 \quad \Rightarrow \quad \varphi \approx \tan\varphi = \dfrac{dy}{dx}$

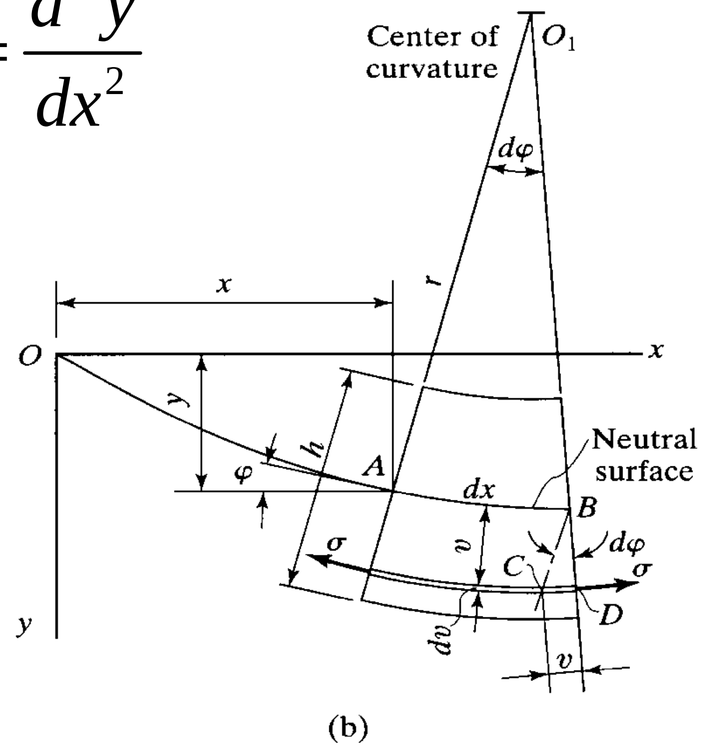
$$\varphi = \frac{dy}{dx} \qquad \text{and} \qquad \frac{d\varphi}{dx} = \frac{d^2y}{dx^2}$$

From the figure, $dx = -r\,d\varphi,$

負號：x 增加 $\Rightarrow$ φ 減少

so
$$\frac{d\varphi}{dx} = -\frac{1}{r} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{M}{EI}$$



$$M = \frac{EI}{r}$$

Figure 1-5(b)

DERIVE THE BEAM EQUATIONS (II)

When the bending moment of the beam is not constant, but varies with *x*, ***shearing forces***, as well as moments, exist.

From the figure on the right,

$$M + \frac{dM}{dx}dx - \left(V + \frac{dV}{dx}dx\right)dx - wdx\frac{dx}{2} - M = 0$$

When the higher-order terms are neglected,

$$\frac{dM}{dx} = V$$

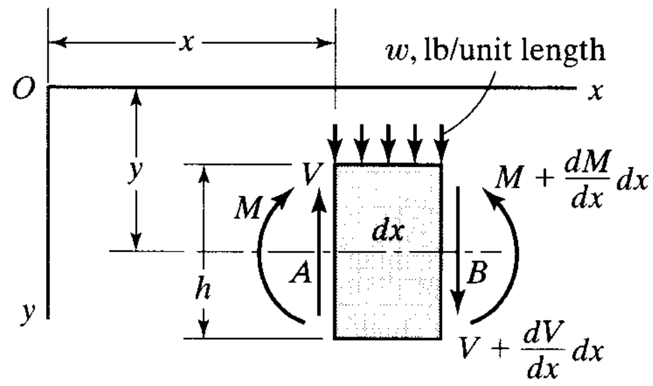


Figure 1-11

Max (or min) value of M occurs at where V=0.

DERIVE THE BEAM EQUATIONS (III)

Balance of force in the y direction,

$$-V + (V + \frac{dV}{dx}) + w = 0$$

$$\frac{dV}{dx} = w$$

$$w = \frac{dV}{dx} = \frac{d}{dx}\left(\frac{dM}{dx}\right) = \frac{d^2M}{dx^2}$$

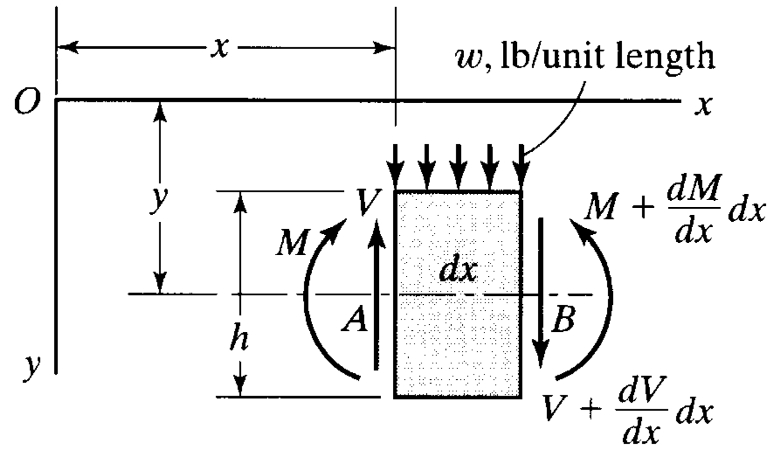


Figure 1-11

EXAMPLE 1-8

Derive the equation for the deflection *y* for the values of *x* located between the supports for the beam.

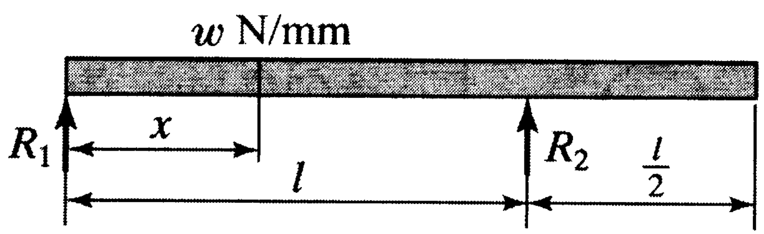


Figure 1-12

$$R_1 = \frac{3}{8} wl$$

$$R_2 = \frac{9}{8} wl$$

At location *x*

$$M(x) = \frac{3}{8} wlx - \frac{wx^2}{2}$$

$$EI \frac{d^2 y}{dx^2} = -M = -\frac{3}{8} wlx + \frac{wx^2}{2}$$

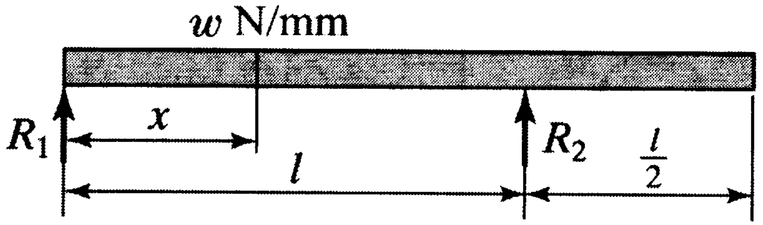
Integration

$$EI \frac{dy}{dx} = -\frac{3}{16} wlx^2 + \frac{wx^3}{6} + C_1$$

EXAMPLE 1-8

Integration again

$$EIy = -\frac{3}{48}wlx^3 + \frac{wx^4}{24} + C_1x + C_2$$



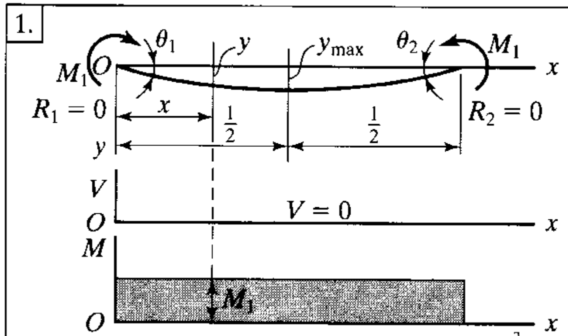
Boundary conditions

$$\begin{cases} x = 0 \text{ and } y = 0 \\ x = l \text{ and } y = 0 \end{cases} \Rightarrow \begin{cases} C_2 = 0 \\ C_1 = \dfrac{wl^3}{48} \end{cases}$$

Deflection

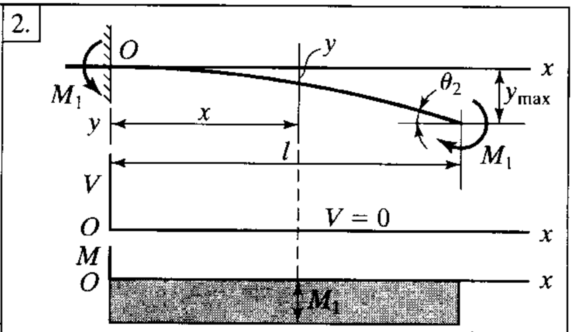
$$y = \frac{w}{48EI}\left(2x^4 - 3lx^3 + l^3x\right)$$

1.



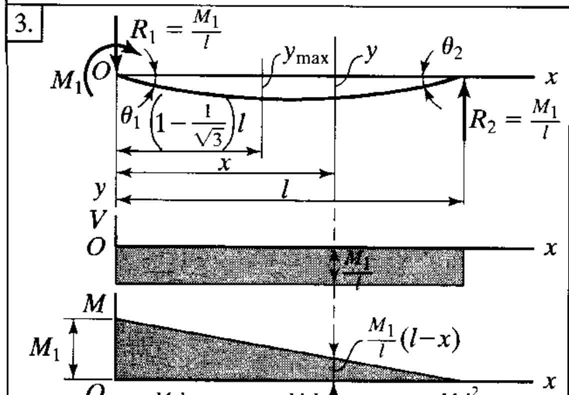
$\theta_1 = \theta_2 = \frac{M_1 l}{2EI}$; $y = \frac{M_1}{2EI}(lx - x^2)$; $y_{max} = \frac{M_1 l^2}{8EI}$

2.



$\theta_2 = \frac{M_1 l}{EI}$; $y = \frac{M_1 x^2}{2EI}$; $y_{max} = \frac{M_1 l^2}{2EI}$

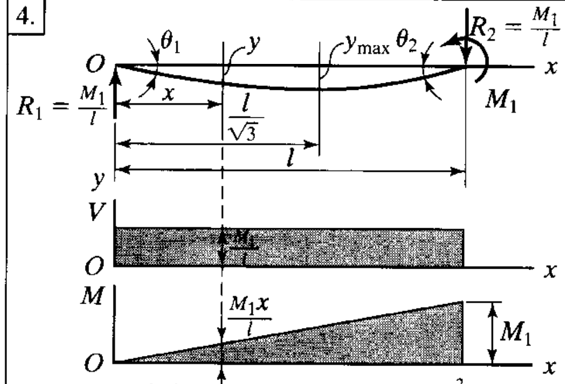
3.



$\theta_1 = \frac{M_1 l}{3EI}$; $\theta_2 = \frac{M_1 l}{6EI}$; $y_{max} = \frac{M_1 l^2}{9\sqrt{3}EI}$

$y = \frac{M_1}{6lEI}(2l^2 x - 3lx^2 + x^3)$

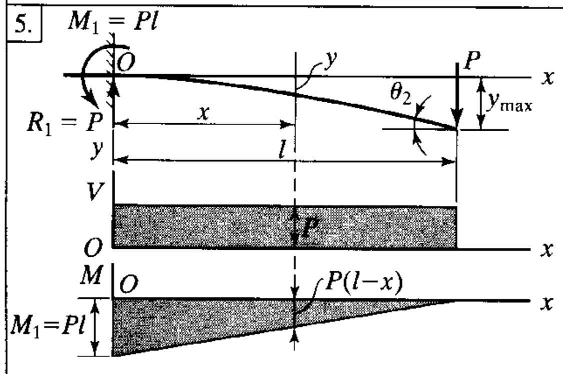
4.



$\theta_1 = \frac{M_1 l}{6EI}$; $\theta_2 = \frac{M_1 l}{3EI}$; $y_{max} = \frac{M_1 l^2}{9\sqrt{3}EI}$

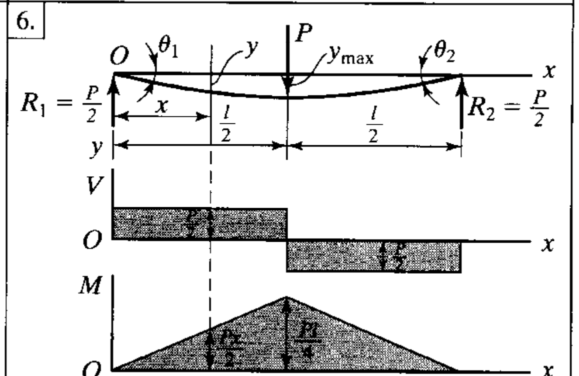
$y = \frac{M_1}{6lEI}(l^2 x - x^3)$

5.



$\theta_2 = \frac{Pl^2}{2EI}$; $y = \frac{P}{6EI}(3lx^2 - x^3)$; $y_{max} = \frac{Pl^3}{3EI}$

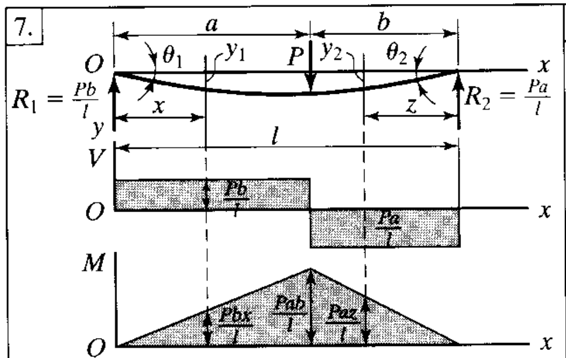
6.



$\theta_1 = \theta_2 = \frac{Pl^2}{16EI}$; $y_{max} = \frac{Pl^3}{48EI}$

For $O < x \leq \frac{l}{2}$; $y = \frac{P}{48EI}(3l^2 x - 4x^3)$

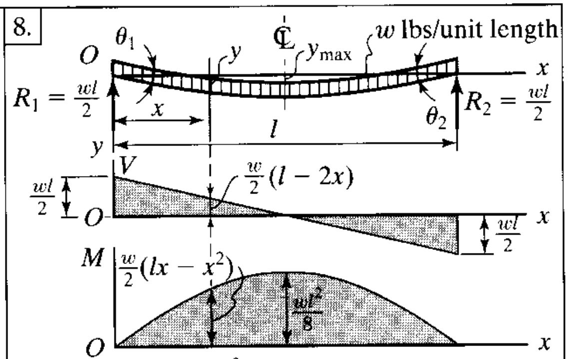
7.



$$\theta_1 = \frac{Pab(l+b)}{6lEI}; \quad y_1 = \frac{Pbx}{6lEI}(l^2 - b^2 - x^2)$$

$$\theta_2 = \frac{Pab(l+a)}{6lEI}; \quad y_2 = \frac{Paz}{6lEI}(l^2 - a^2 - z^2)$$

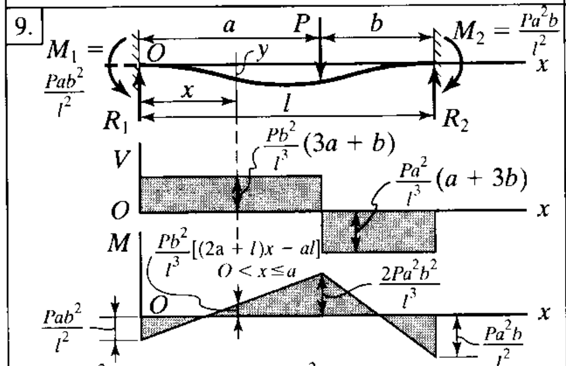
8.



$$\theta_1 = \theta_2 = \frac{wl^3}{24EI}; \quad y_{max} = \frac{5wl^4}{384EI}$$

$$y = \frac{wx}{24EI}(l^3 - 2lx^2 + x^3)$$

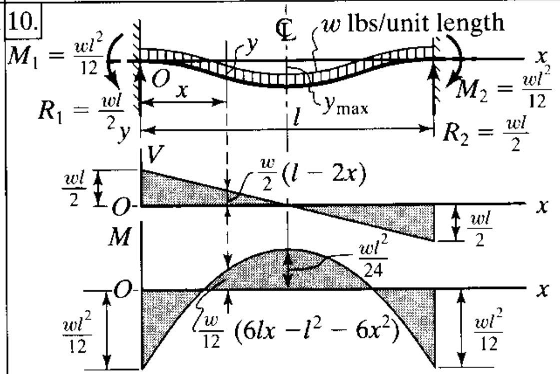
9.



$$R_1 = \frac{Pb^2}{l^3}(3a + b); \; R_2 = \frac{Pa^2}{l^3}(a + 3b); \; \theta_1 = \theta_2 = 0$$

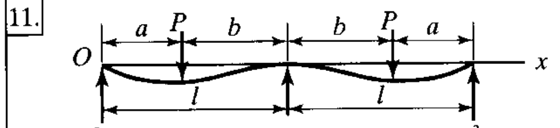
For $O < x \le a$; $y = \frac{Pb^2}{6l^3EI}[3alx^2 - (3a + b)x^3]$

10.

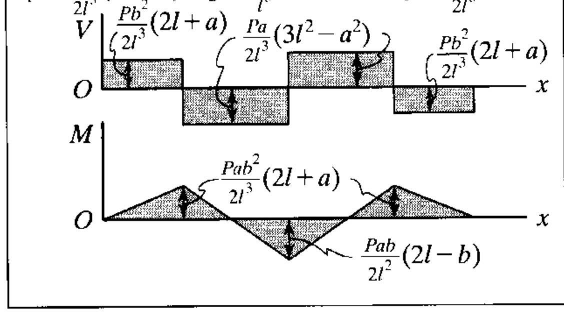


$$y = \frac{wx^2}{24EI}(l - x)^2; \; y_{max} = \frac{wl^4}{384EI}; \; \theta_1 = \theta_2 = 0$$

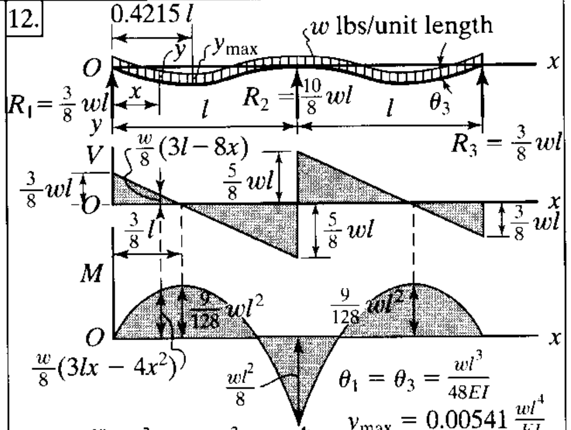
11.



$$R_1 = \frac{Pb^2}{2l^3}(2l + a) \; R_2 = \frac{Pa}{l^3}(3l^2 - a^2) \; R_3 = \frac{Pb^2}{2l^3}(2l + a)$$



12.

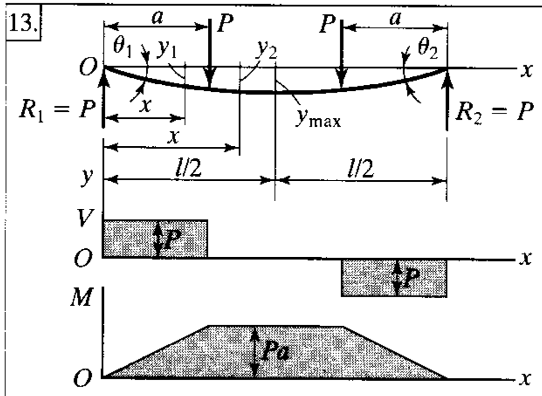


$$\theta_1 = \theta_3 = \frac{wl^3}{48EI}$$

$$y_{max} = 0.00541\frac{wl^4}{EI}$$

$$y = \frac{w}{48EI}(l^3x - 3lx^3 + 2x^4)$$

13.



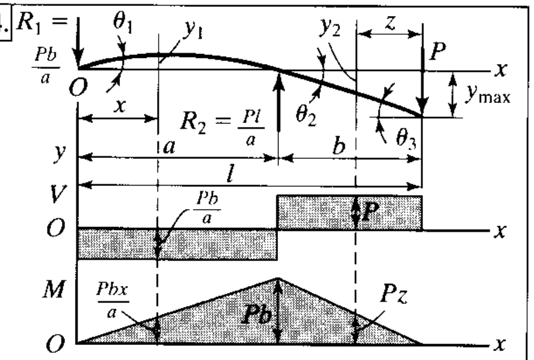
$$\theta_1 = \theta_2 = \frac{Pa(l-a)}{2EI}$$

For $O < x \leq a$, $\quad y_1 = \frac{Px}{6EI}\,[3a(l-a) - x^2]$

For $a \leq x \leq (l-a)$, $\quad y_2 = \frac{Pa}{6EI}\,[3x(l-x) - a^2]$

$$y_{max} = \frac{Pa}{24EI}\,(3l^2 - 4a^2)$$

14.



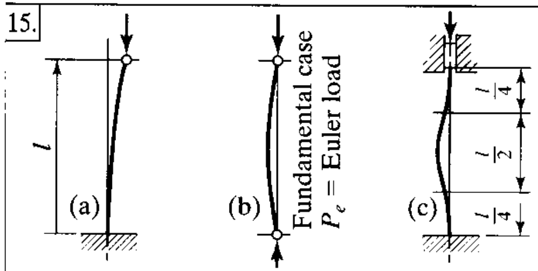
$$\theta_1 = \frac{Pab}{6EI}, \quad \theta_2 = \frac{Pab}{3EI}, \quad \theta_3 = \frac{Pb}{6EI}(2l + b)$$

For $O < x \leq a$; $\quad y_1 = \frac{Pbx}{6aEI}(x^2 - a^2)$

For $O < z \leq b$; $\quad y_2 = \frac{P}{6EI}\,[z^3 - b(2l+b)z + 2b^2l]$

$$y_{max} = \frac{Pb^2l}{3EI}$$

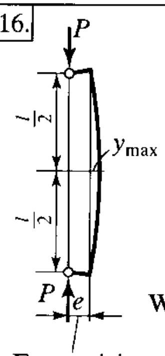
15.



$$P_{cr} = \frac{\pi^2 EI}{4l^2} \qquad P_e = \frac{\pi^2 EI}{l^2} \qquad P_{cr} = \frac{4\pi^2 EI}{l^2}$$

Critical or buckling loads for centrally loaded columns.

16.



Eccentricity

$$y_{max} = e\left(\sec \frac{l}{2}\sqrt{\frac{P}{EI}} - 1\right)$$

Maximum moment:

$$M_{max} = P(e + y_{max})$$

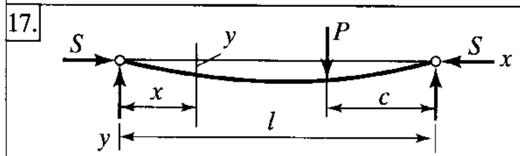
$$= Pe \sec \frac{l}{2}\sqrt{\frac{P}{EI}}$$

Maximum stress:

$$S_{max} = \frac{P}{A}\left(1 + \frac{ec}{i^2}\sec \frac{l}{2i}\sqrt{\frac{P}{AE}}\right)$$

Where A = area of cross section
$i = \sqrt{I/A}$, radius of gyration
c = distance from neutral axis to edge of section

17.

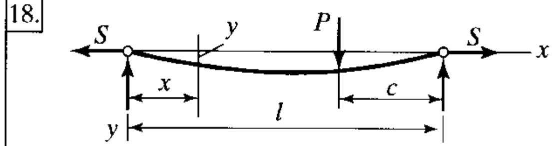


For $O < x \leq (l - c)$ $\qquad p^2 = \frac{S}{EI}$

$$y = \frac{P \sin pc}{Sp \sin pl}\sin px - \frac{Pc}{Sl}x$$

Equations for slope and moment can be found by differentiation.

18.

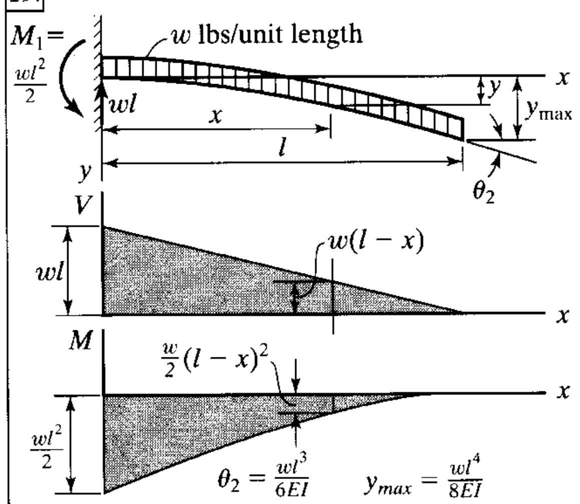


For $O < x \leq (l - c)$ $\qquad p^2 = \frac{S}{EI}$

$$y = -\frac{P \sinh pc}{Sp \sinh pl}\sinh px + \frac{Pc}{Sl}x$$

Equations for slope and moment can be found by differentiation.

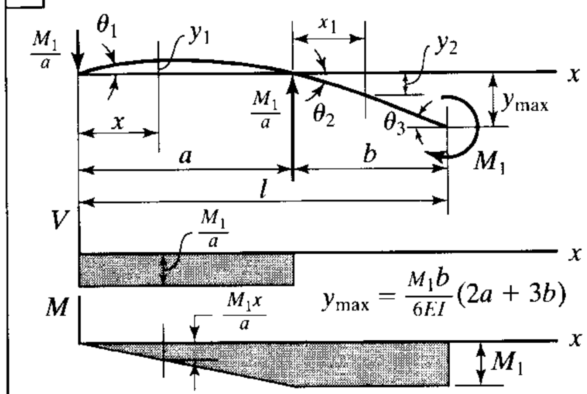
19.



$$\theta_2 = \frac{wl^3}{6EI} \qquad y_{max} = \frac{wl^4}{8EI}$$

$$y = \frac{wx^2}{24EI}(x^2 - 4lx + 6l^2)$$

20.



$$y_{max} = \frac{M_1 b}{6EI}(2a + 3b)$$

$$\theta_1 = \frac{M_1 a}{6EI}, \quad \theta_2 = \frac{M_1 a}{3EI}, \quad \theta_3 = \frac{M_1(a+3b)}{3EI}$$

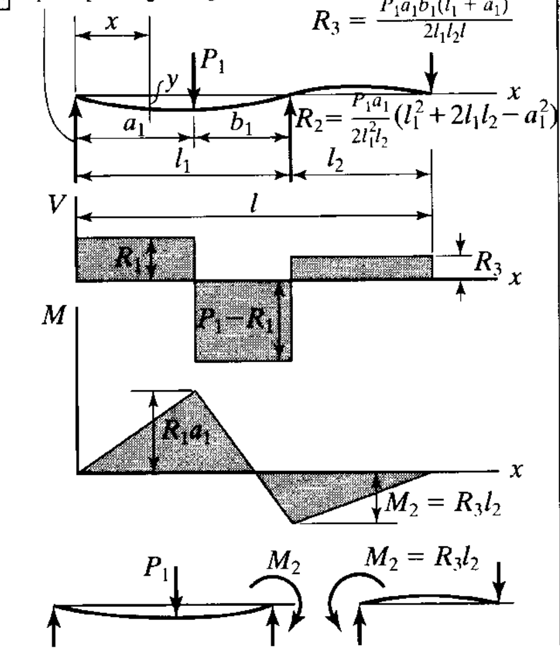
For $O \le x \le a$, $y_1 = -\frac{M_1 x}{6aEI}(a^2 - x^2)$

For $O \le x_1 \le b$, $y_2 = \frac{M_1}{6EI}(2ax_1 + 3x_1^2)$

21. $R_1 = P_1 - R_2 + R_3$

$$R_3 = \frac{P_1 a_1 b_1 (l_1 + a_1)}{2 l_1 l_2 l}$$

$$R_2 = \frac{P_1 a_1}{2 l_1^2 l_2}(l_1^2 + 2 l_1 l_2 - a_1^2)$$

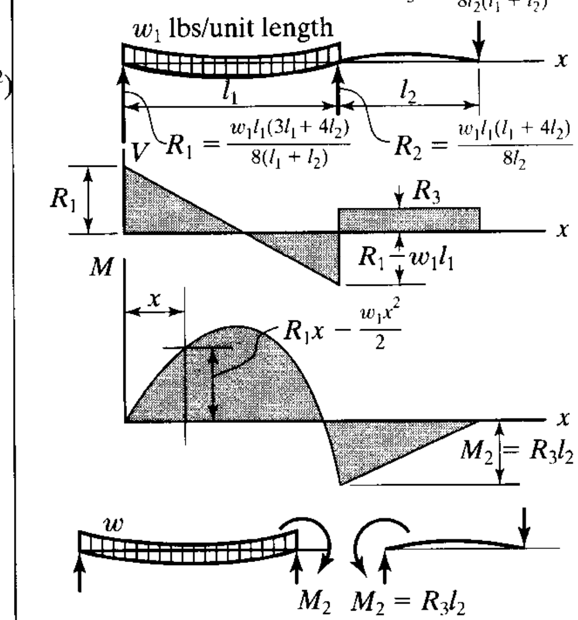


Deflections can be found by elementary equations and supcrpositions

22.

$$R_3 = \frac{w_1 l_1^3}{8 l_2 (l_1 + l_2)}$$

$$R_1 = \frac{w_1 l_1 (3 l_1 + 4 l_2)}{8(l_1 + l_2)} \qquad R_2 = \frac{w_1 l_1 (l_1 + 4 l_2)}{8 l_2}$$



Deflections can be found by elementary equations and superposition

EXAMPLE 1-9

Suppose it is specified that the deflection from its own weight at the center of a simply supported steel shaft should not exceed 0.30 mm per m of span.

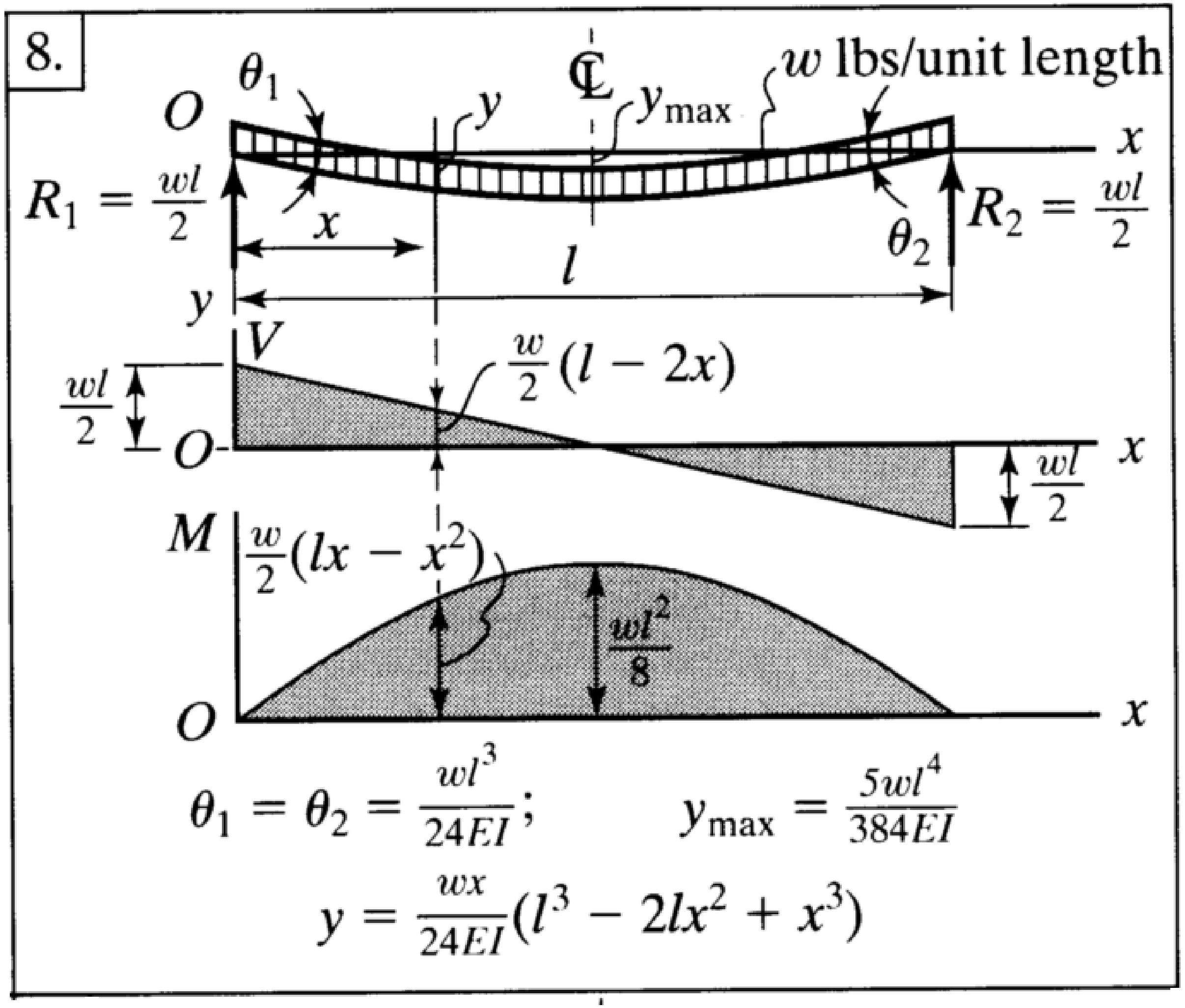
(a) Find the maximum permissible length for a 75-mm-diameter shaft.

(b) Find the stress caused by the weight of the shaft.

$$0.3\ \mathrm{mm/m} = 0.3\ \mathrm{mm}/1000\ \mathrm{mm} \quad \Rightarrow \quad y_{\mathrm{max}} \le \frac{0.3l}{1000}\ \mathrm{mm}$$

l **is the span in mm**

EXAMPLE 1-9



8.
θ_1
O
y
℄
y_{max}
w lbs/unit length
x
$R_1 = \frac{wl}{2}$
x
$R_2 = \frac{wl}{2}$
θ_2
l
y
V
$\frac{w}{2}(l - 2x)$
$\frac{wl}{2}$
O
x
$\frac{wl}{2}$
M
$\frac{w}{2}(lx - x^2)$
$\frac{wl^2}{8}$
O
x
$\theta_1 = \theta_2 = \frac{wl^3}{24EI}$; $y_{max} = \frac{5wl^4}{384EI}$
$y = \frac{wx}{24EI}(l^3 - 2lx^2 + x^3)$

EXAMPLE 1-9

(a)

From Table 2-3, *E*= 206,900 MPa and γ =0.0000768 N/mm^3

Deflection at the center

$$y_{\max} = \frac{5wl^4}{384EI} = \frac{0.3l}{1000} \quad \Rightarrow \quad l^3 = \frac{0.3\times 384EI}{1000\times 5w}$$

Circular shaft

$$I = \frac{\pi d^4}{64} = \frac{\pi \times 75^4}{64} = 1,553,159 \text{ mm}^4$$

$$w = \gamma A = 0.0000768 \times \frac{\pi \times 75^2}{4} = 0.339 \text{ N/mm}$$

$$l^3 = 2.184\times 10^{10} \quad \Rightarrow \quad l = 2795 \text{ mm} = 2.795 \text{ m}$$

EXAMPLE 1-9

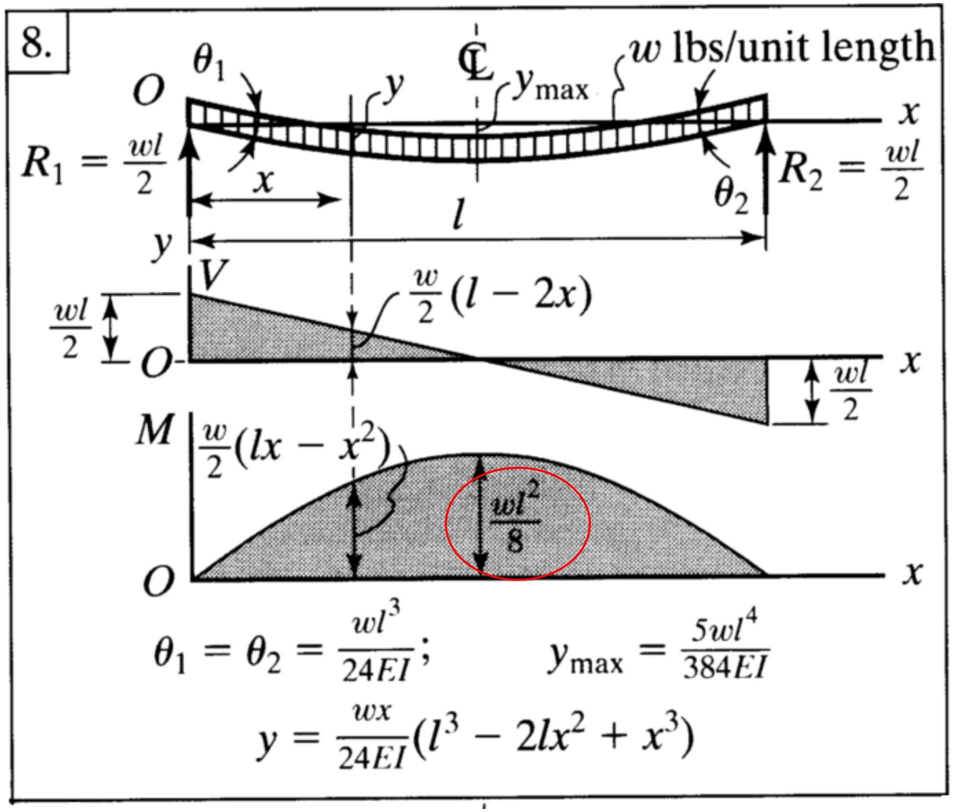
(b)

$$M_{\max} = \frac{wl^2}{8} = \frac{0.339 \times 2795^2}{8}$$

$$= 331035 \text{ Nmm}$$

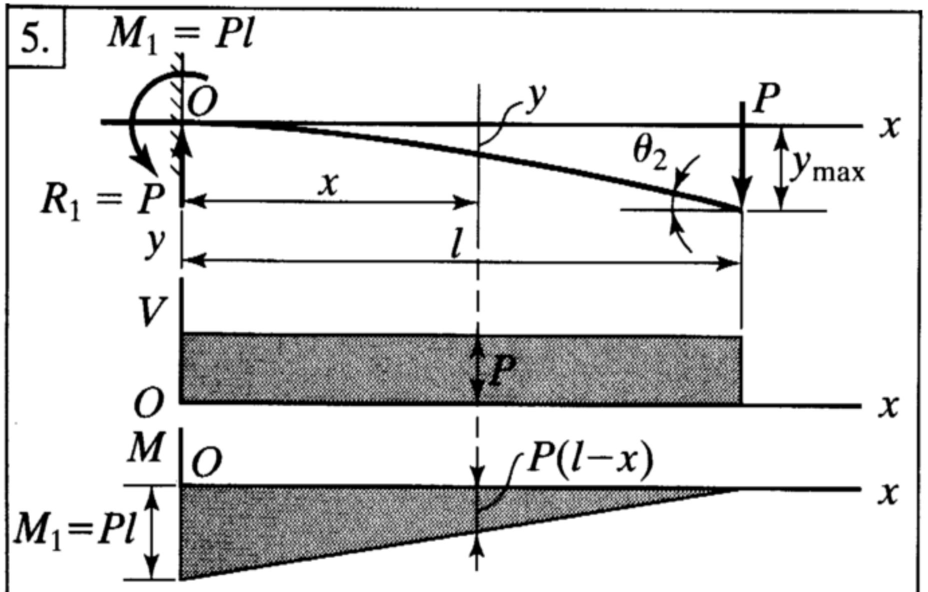
$$\sigma_{\max} = \frac{Mc}{I} = \frac{331035 \times 37.5}{1553159}$$

$$= 7.99 \text{ MPa}$$



EXAMPLE 1-10

Show that the spring constant, load for a unit deflection, for the two beams of the figure are the same.



$$\theta_2 = \frac{Pl^2}{2EI}; \quad y = \frac{P}{6EI}(3lx^2 - x^3); \quad y_{max} = \frac{Pl^3}{3EI}$$

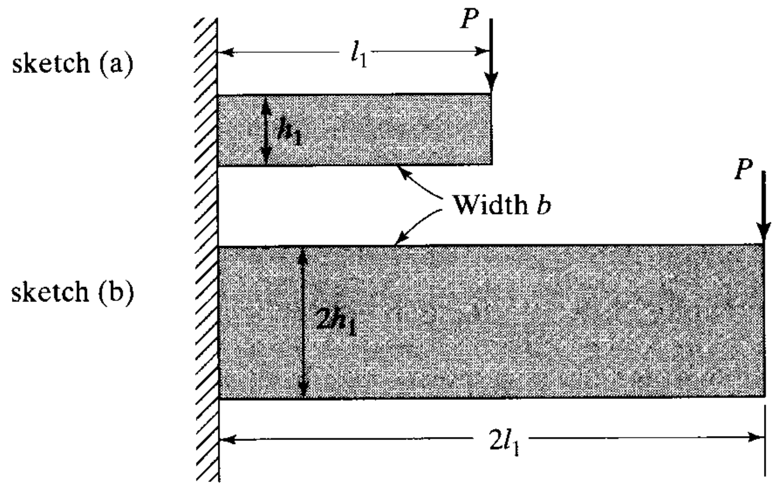
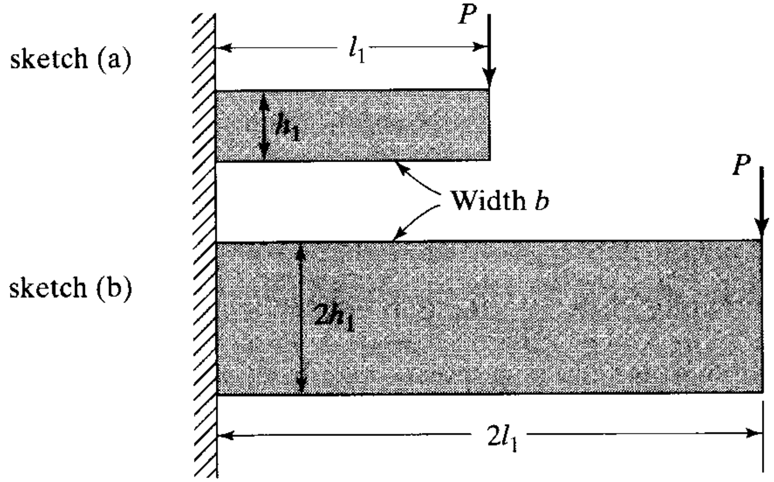


Figure 1-17

位移正比於作用力

EXAMPLE 1-10

$$k=\frac{P}{y_{\max}}=\frac{3EI}{l^3}=\frac{3E}{l^3}\frac{bh^3}{12}=\frac{Ebh^3}{4l^3}$$



For sketch (a)

$$k=\frac{Ebh_1^3}{4l_1^3}$$

For sketch (b)

$$k=\frac{Eb\left(2h_1\right)^3}{4\left(2l_1\right)^3}=\frac{Ebh_1^3}{4l_1^3}$$

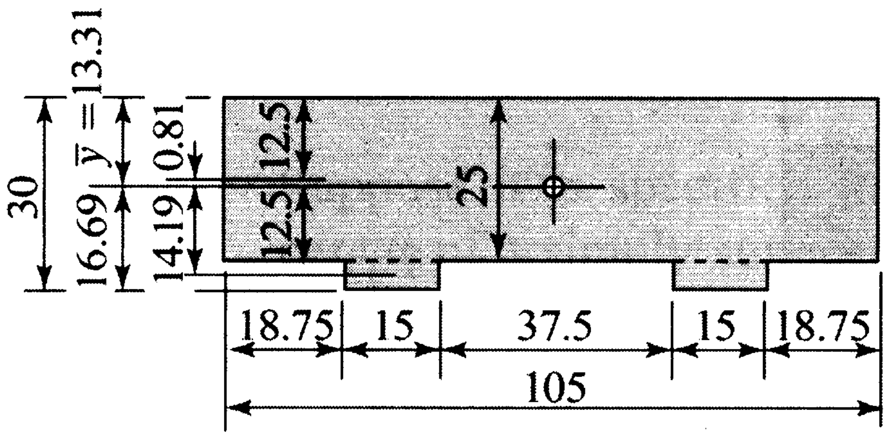
斷面高度和梁的長度比固定，
等效彈簧常數不變。

1-14 EFFECT OF RIBS ON CASTING

Ribs are sometimes added to the webs of castings to give greater strength and rigidity.

- The low rib gives a small increases in the moment of inertia (advantage).
- The distance from the neutral axis to the edge of the cross section becomes relatively greater, and the stress is accordingly increase (disadvantage).

$$\sigma_{\max} = \frac{Mc}{I}$$



EXAMPLE 1-11

Figure below represents the cross section through a **simply supported beam** 60 in. long that carries a 200 lb load at the center. E=15,000,000 psi.

(a) Find the value of the bending stress and deflection at the center if the ribs were omitted.

(b) Find the value of the bending stress and deflection at the center if the ribs are present.

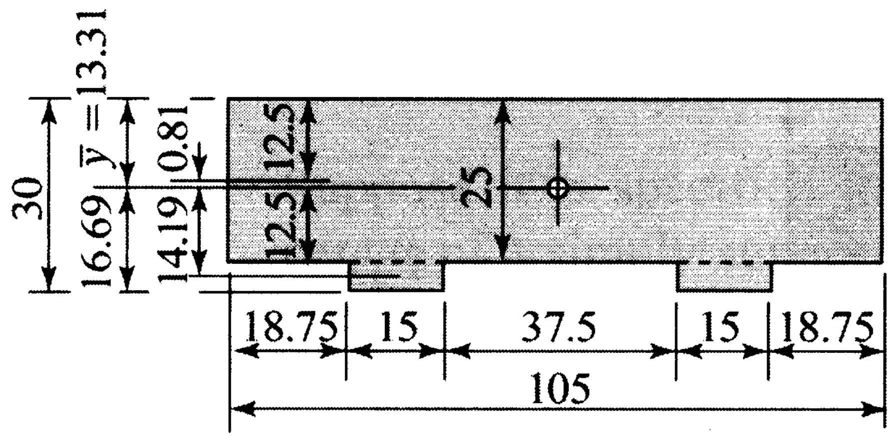
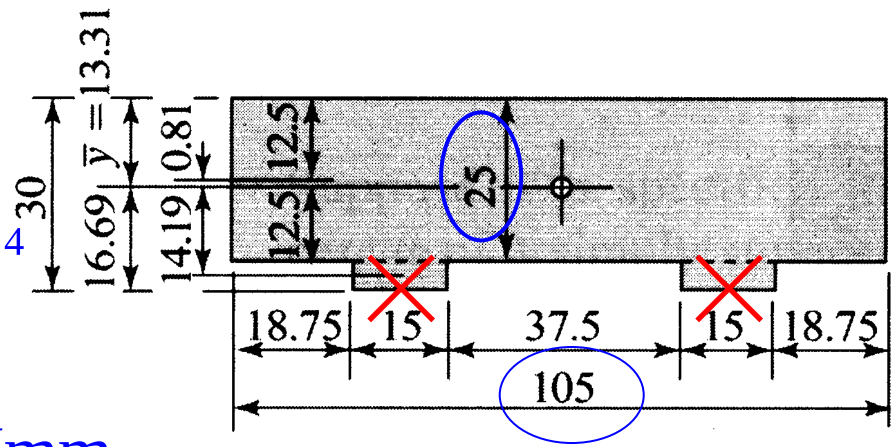


Figure 1-18

EXAMPLE 1-11

(a) ribs were omitted



$$I=\frac{bh^3}{12}=\frac{105\times 25^3}{12}=136719\ \text{mm}^4$$

$$M=\frac{Pl}{4}=\frac{900\times 1500}{4}=337500\ \text{Nmm}$$

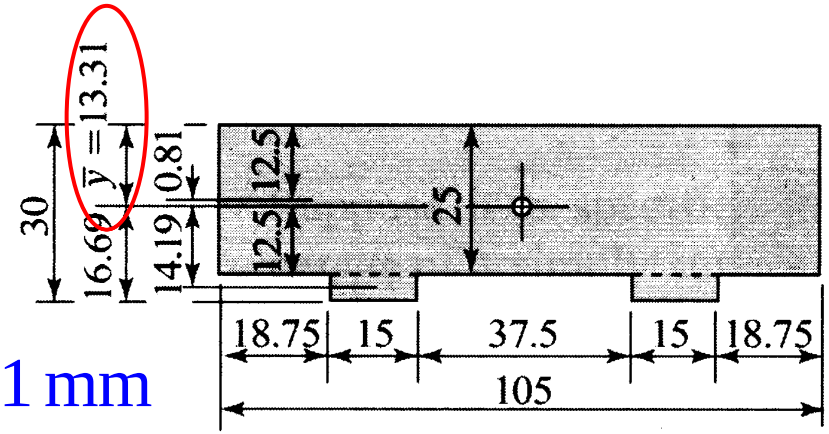
$$\sigma=\frac{Mc}{I}=\frac{337500\times 12.5}{136719}=30.85\ \text{MPa}$$

$$y=\frac{Pl^3}{48EI}=\frac{900\times 1500^3}{48\times 103400\times 136719}=4.476\ \text{mm}$$

Ref. Fig. 1-13, No. 6

EXAMPLE 1-11

(b) **ribs are present**



$$A = 105\times 25 + 2\times 5\times 15 = 2775\ \text{mm}^2$$

$$\bar{y} = \frac{105\times 25\times 12.5 + 150\times 27.5}{2775} = 13.31\ \text{mm}$$

For the main area

$$I = \frac{bh^3}{12} + A\bar{y}^2 = 136719 + 105\times 25\times (13.31 - 12.5)^2 = 139344\ \text{mm}^4$$

For the rib

$$I = 2\left(\frac{15\times 5^3}{12} + 15\times 5\times 14.19^2\right) = 30515\ \text{mm}^4$$

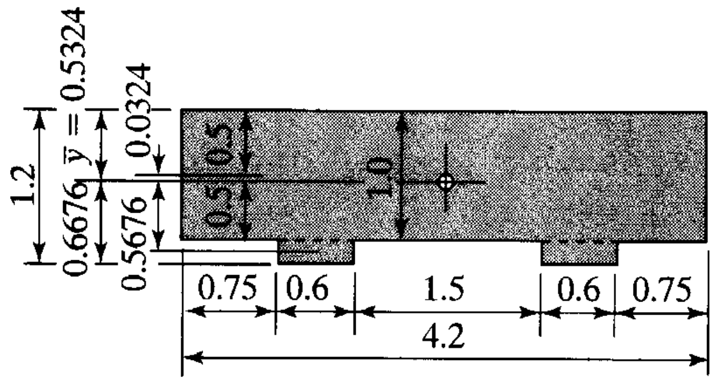
EXAMPLE 1-11

The total is

$$I = 139344 + 305159 = 169859 \text{ mm}^4 \quad （慣性矩提升）$$

$$\sigma = \frac{Mc}{I} = \frac{337500 \times 16.69}{169859} = 33.16 \text{ MPa} \quad （\text{ribs 承受應力}）$$

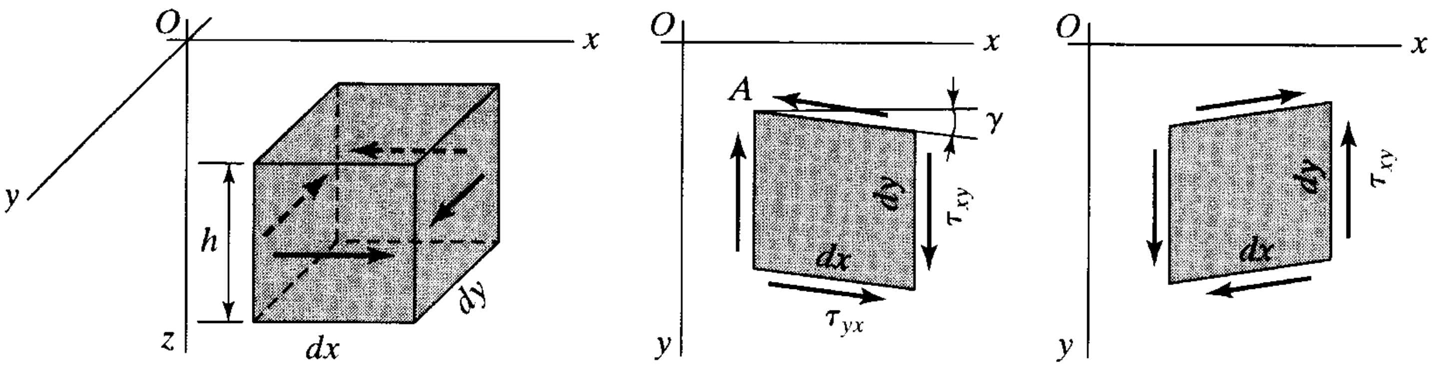
$$y = \frac{Pl^3}{48EI} = \frac{900 \times 1500^3}{48 \times 103400 \times 169859} = 3.60 \text{ mm (降低)}$$



1-15 SHEARING STRESS

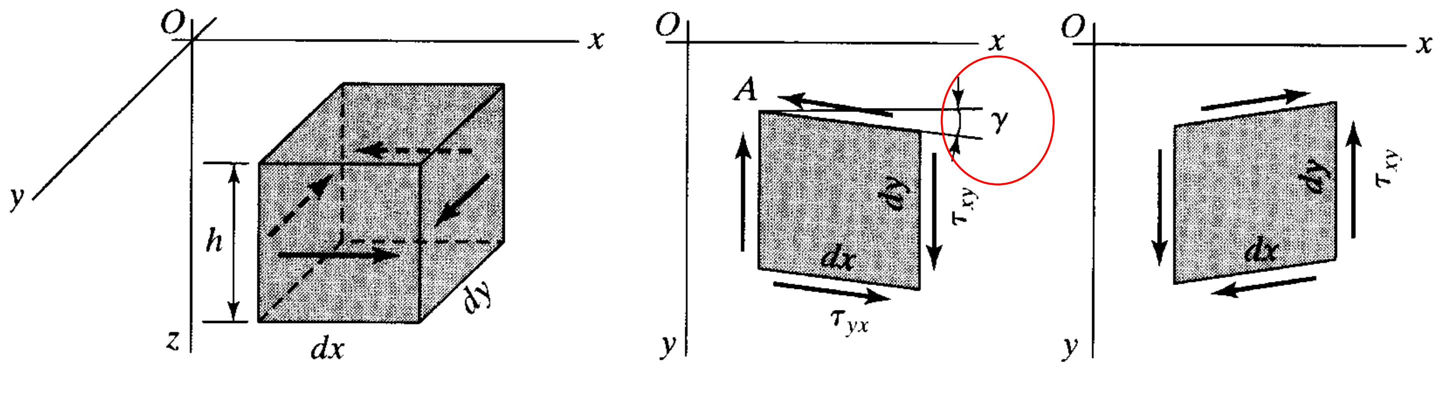
Shearing stresses act tangentially to element sides. Such loading causes no change in the length of the sides of the element, but produces distortion or change in angles in the corners.

- Balance of moments
- Balance of forces
- Hooke's law



(a) Element loaded in shear (b) Positive shear (c) Negative shear

Figure 1-19



(a) Element loaded in shear (b) Positive shear (c) Negative shear

Balance of moments

$$\tau_{yx}\left(dx\cdot h\right)dy-\tau_{xy}\left(dy\cdot h\right)dx=0$$

$$\Rightarrow \quad \tau_{yx}=\tau_{xy}$$

第一個下標代表面的法方向
第二個下標代表應力的指向

Hooke's law

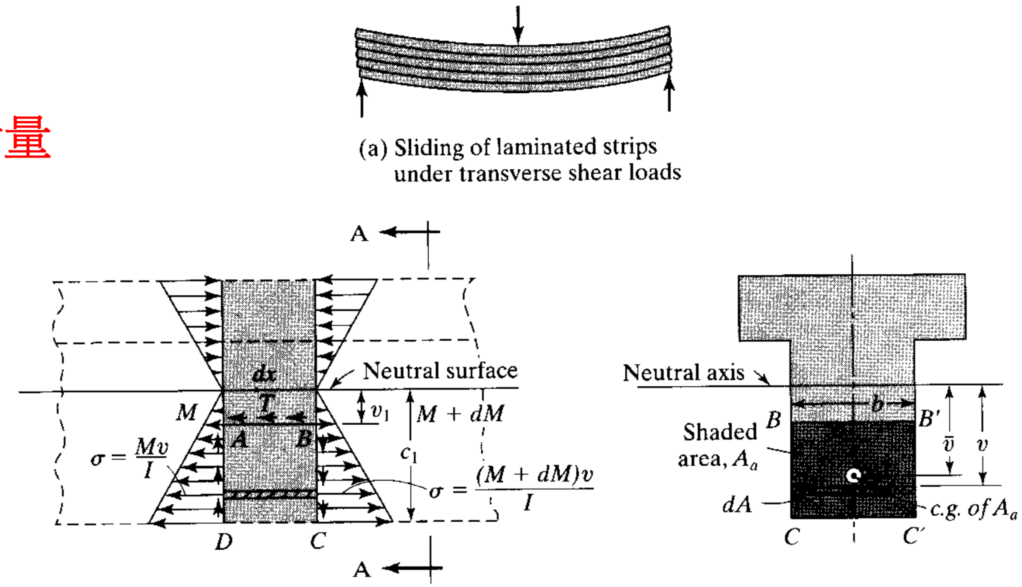
$$\tau_{xy}=\gamma G$$

$$G=\frac{E}{2\left(1+\mu\right)}$$

*The constant G is called the **shear modulus** of elasticity.*

1-16 TRANSVERSE SHEAR STRESS IN BEAMS

延性材料常以剪應力破壞爲設計考量



(a) Sliding of laminated strips under transverse shear loads

(b) Side view of beam

(c) Section A–A

Figure 1-20

Force on the left end:

$$\int_{v_1}^{c_1} \sigma dA = \int_{v_1}^{c_1} \frac{Mv}{I} dA$$

Force on the right end:

$$\int_{v_1}^{c_1} \sigma dA = \int_{v_1}^{c_1} \frac{(M + dM)v}{I} dA$$

Balance of force:

$$\tau b dx + \int_{v_1}^{c_1} \frac{Mv}{I} dA - \int_{v_1}^{c_1} \frac{(M + dM)v}{I} dA = 0$$

$$\boxed{\tau = \frac{1}{b}\int_{v_1}^{c_1} \frac{dM}{dx}\frac{v dA}{I} = \frac{V}{Ib}\int_{v_1}^{c_1} v dA = \frac{V}{Ib}\bar{v}A_a = \frac{V}{Ib}Q}$$

EXAMPLE 1-12

Find the transverse shear stress in the material 3 in. from the top surface for the beam.

$$\boxed{\tau = \frac{V}{Ib}\bar{\upsilon}A_a}$$

$$= \frac{45000}{13,021,000 \times 25} \times 62.5 \times 1875$$

$$= 16.2 \text{ MPa}$$

See example 1-6

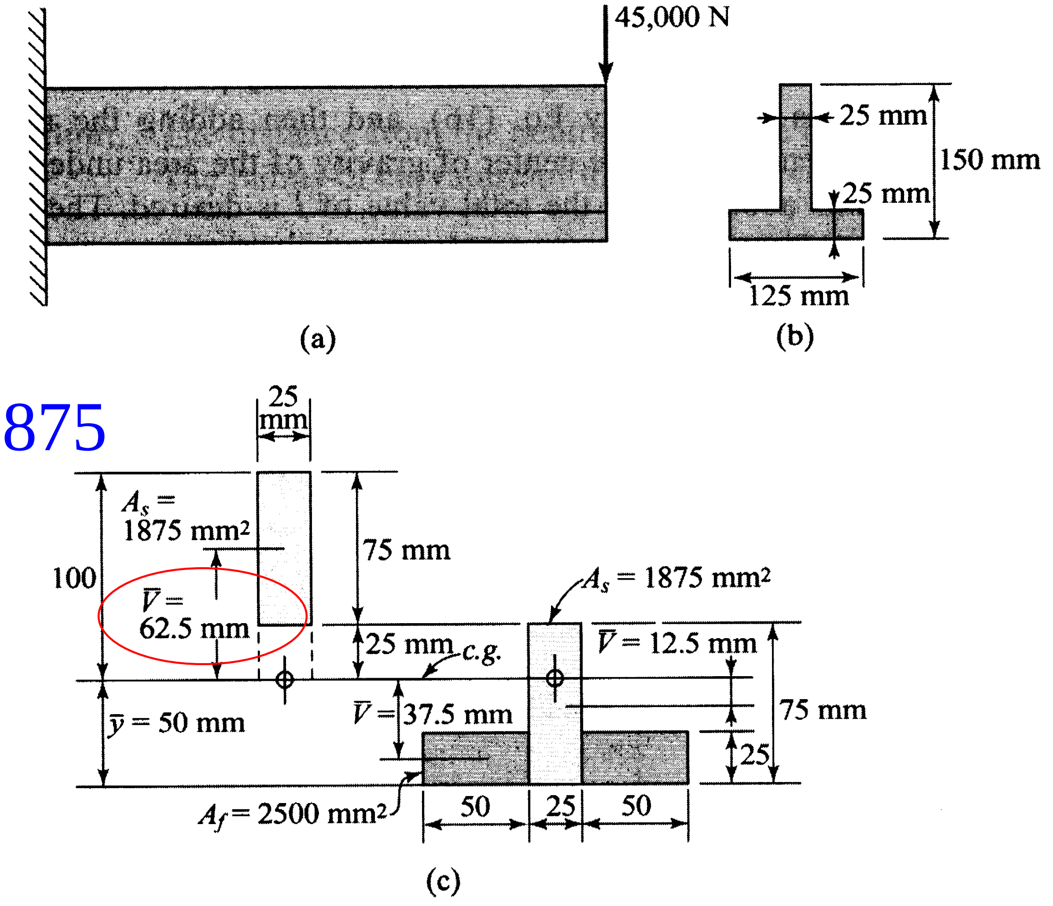


Figure 1-21

Special cases of τ :

$$\tau = \frac{V}{Ib}\bar{\upsilon}A_a$$

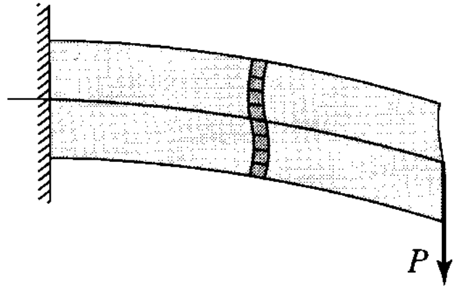
1. Rectangular cross section

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - \upsilon_1^2\right) = \frac{3V}{2A}\left(1 - \frac{4\upsilon_1^2}{h^2}\right)$$ 二次曲線

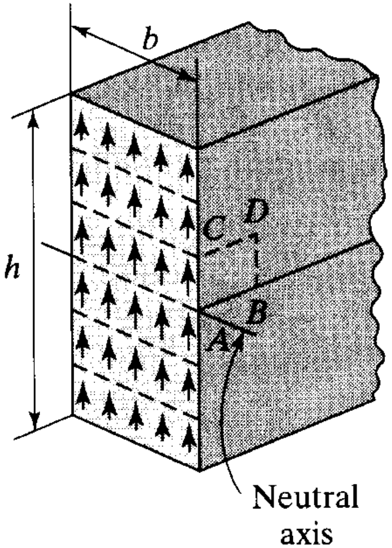
Maximum value occurs at the neutral axis: $\upsilon_1 = 0$

$$\boxed{\tau_{\max} = \frac{3V}{2A}}$$

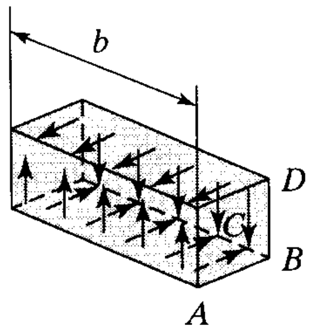
Distribution of shearing stress over cross section



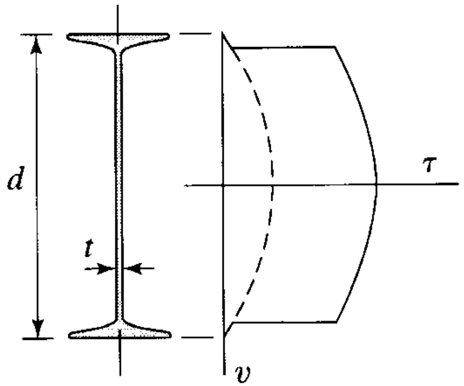
(c) Variation of shear deformation



(a)



(b) Shearing stress on element *ABCD* cut from beam



(d) Shear stress on web of I-beam

2. Circular cross section

$$\int_{\upsilon_1}^{c_1} \upsilon dA = \int_{\upsilon_1}^{r} 2\upsilon\sqrt{r^2 - \upsilon^2}\, d\upsilon = \frac{2}{3}\left(r^2 - \upsilon_1^2\right)^{3/2}$$

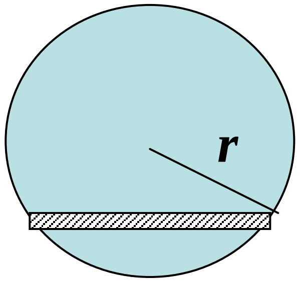
$$\Rightarrow \quad \tau = \frac{V}{Ib}\int_{\upsilon_1}^{c_1} \upsilon dA = \frac{4V}{3\pi r^4}\left(r^2 - \upsilon_1^2\right)$$

$$I = \pi r^4 / 4$$

$$b = 2\sqrt{r^2 - \upsilon_1^2}$$

Maximum value occurs at the neutral axis: $\upsilon_1 = 0$

$$\boxed{\tau_{\max} = \frac{4V}{3\pi r^2} = \frac{4V}{3A}}$$

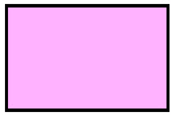
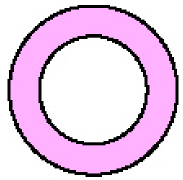
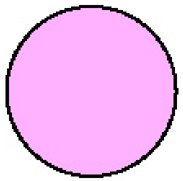
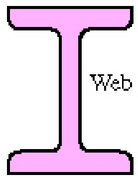


3. Circular tube with very thin wall

$$\tau_{\max} = \frac{2V}{A}$$

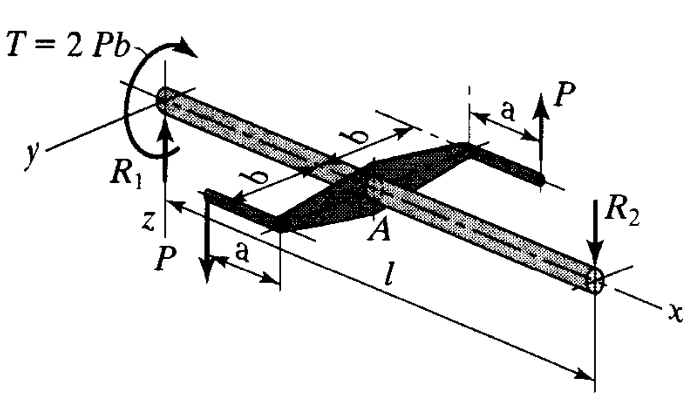
Table 3–2

Formulas for Maximum Shear Stress Due to Bending

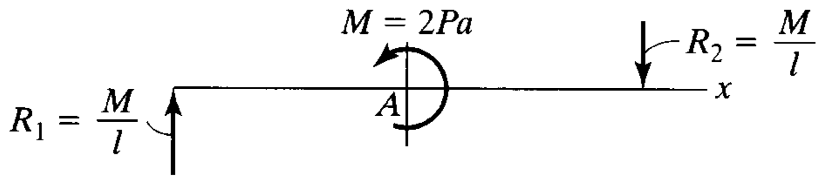
Beam Shape	Formula	Beam Shape	Formula
 Rectangular	$\tau_{max} = \frac{3V}{2A}$	 Hollow, thin-walled round	$\tau_{max} = \frac{2V}{A}$
 Circular	$\tau_{max} = \frac{4V}{3A}$	 Structural I beam (thin-walled)	$\tau_{max} = \frac{V}{A_{web}}$

1-17 SHEAR & BENDING MOMENT DIAGRAMS

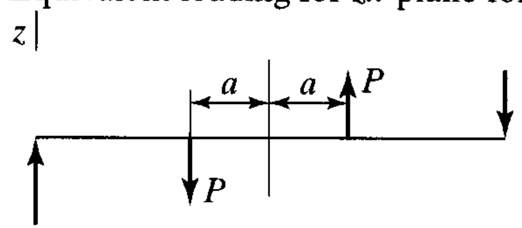
Actual loading and equivalent loading



(a) Actual loading on shaft.

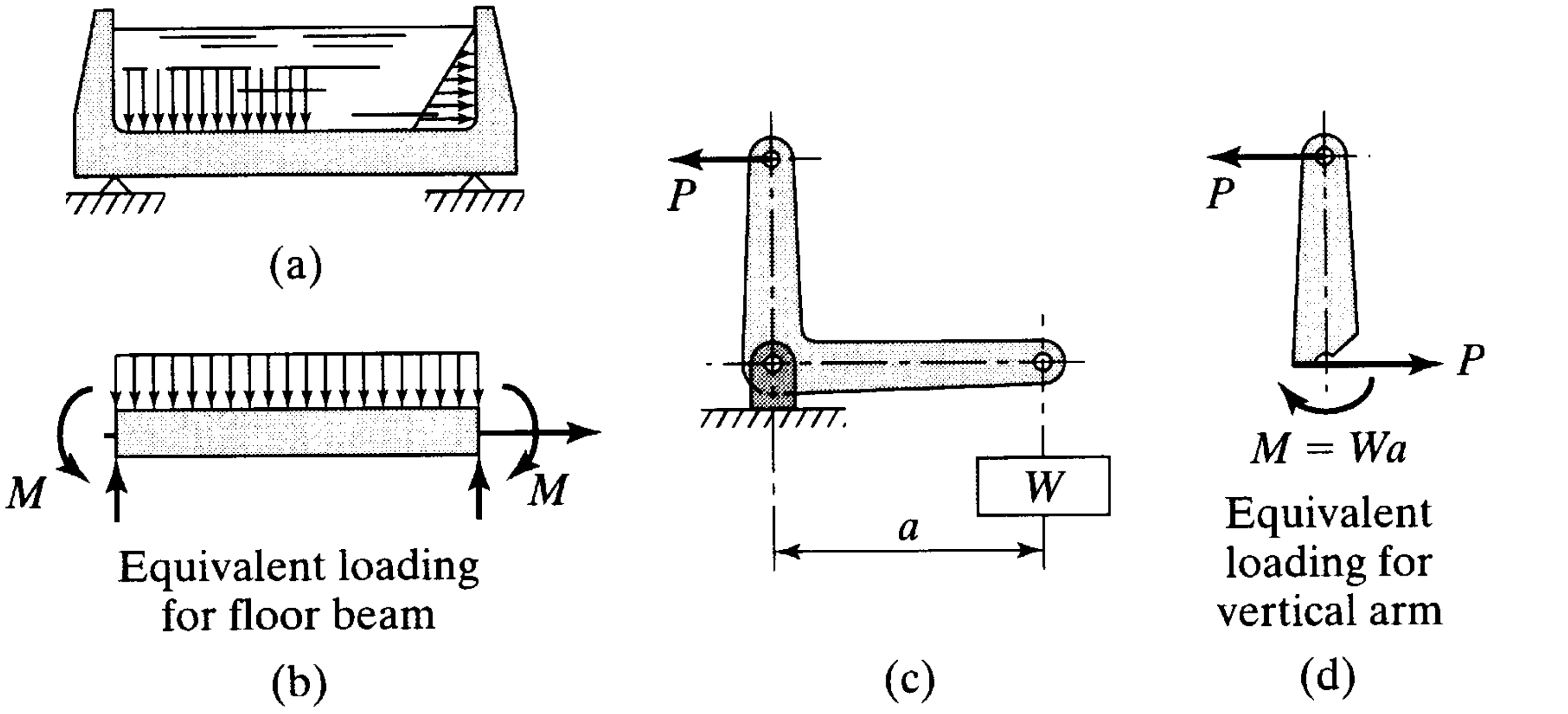


(b) Equivalent loading for zx-plane for (a).



(c) This is not equivalent loading because forces P are not in contact with shaft.

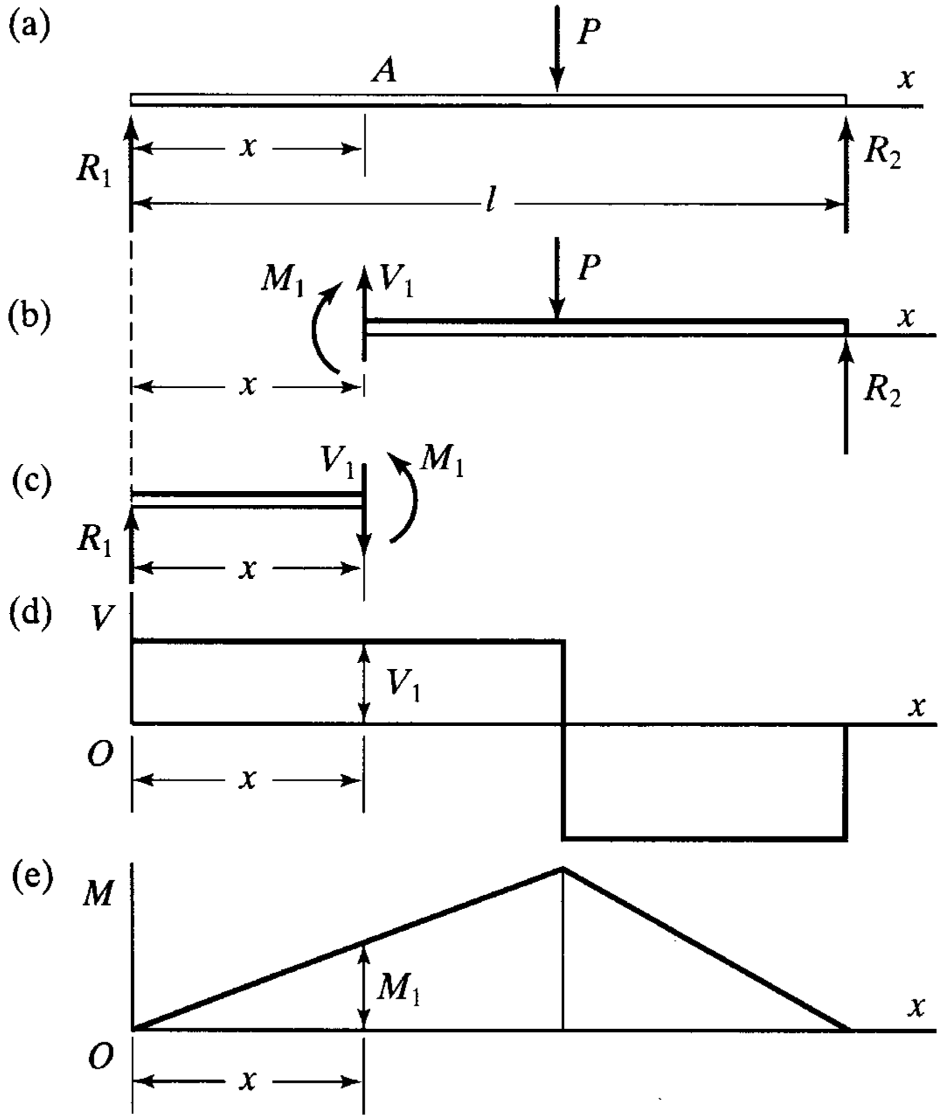
Actual loading and equivalent loading



$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0$$

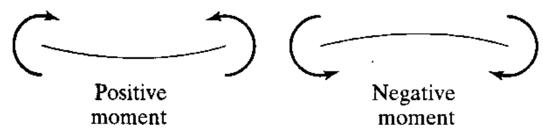
$$\sum M_x = 0, \quad \sum M_y = 0, \quad \sum M_z = 0$$

Shearing force and bending moment diagrams of simply supported beam



$$\frac{dM}{dx} = V$$

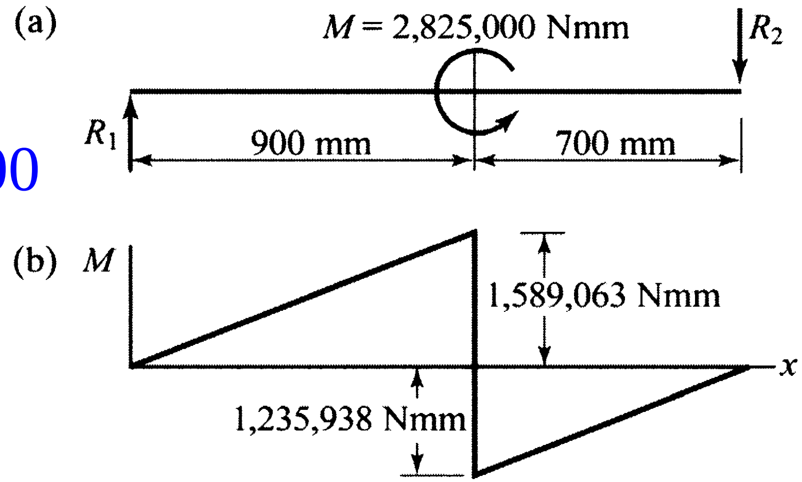
$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = w$$



EXAMPLE 1-13

A simply supported beam is loaded by a concentrated moment of 25000 in.lb as shown below. Find the values of reactions R1 and R2, and draw and dimension the bending moment diagram.

$$\sum F_y = 0 \quad \Rightarrow \quad R_1 = R_2$$



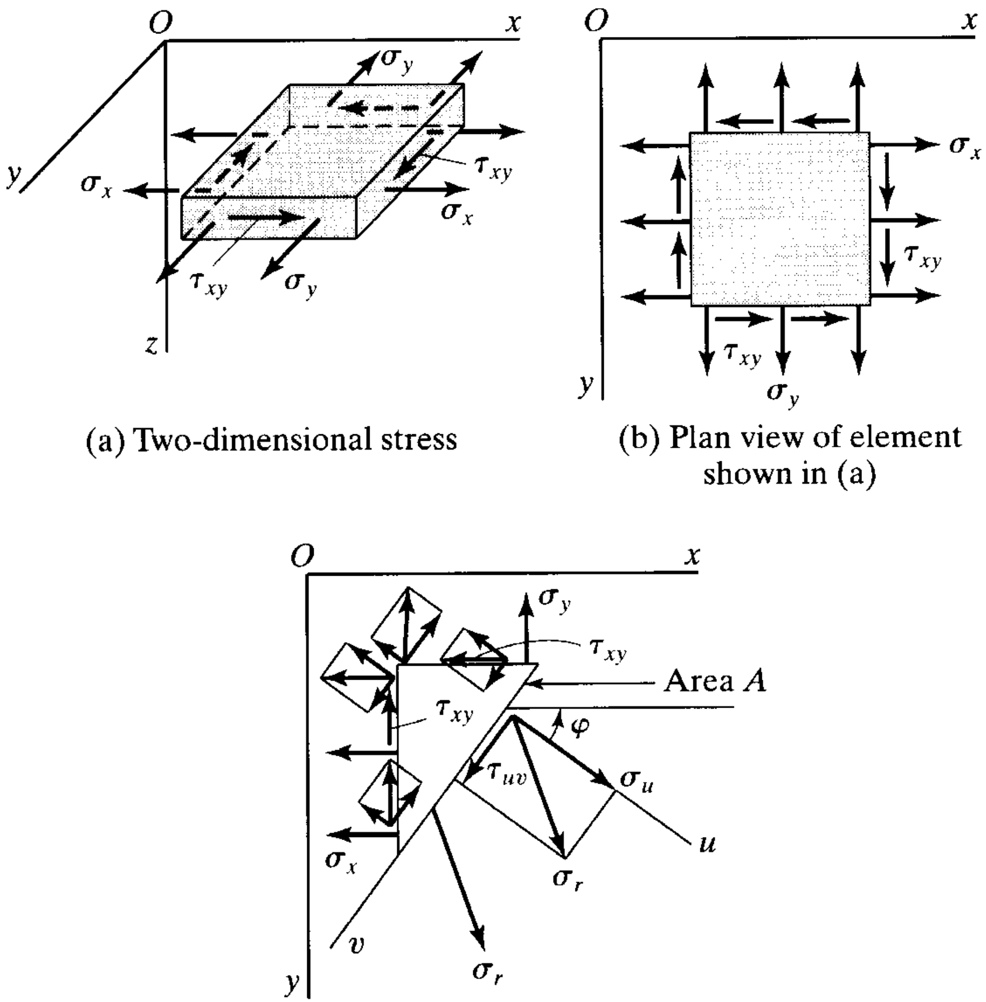
$$\sum M = 0 \quad \Rightarrow \quad (900+700)R_1 = 2,825,000$$

$$R_1 = \frac{2,825,000}{1600} = 1765.6 \ \ \mathrm{N}$$

$$M^- = 900R_1 = 900 \times 1765.6 = 1589063 \ \ \mathrm{Nm}$$

$$M^+ = 700R_2 = 700 \times 1765.6 = 1235938 \ \ \mathrm{Nm}$$

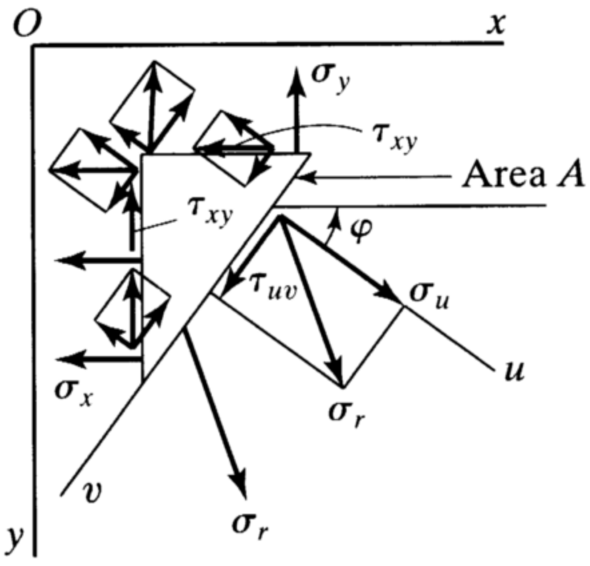
<u>1-19</u> STRESS IN ANY GIVEN DIRECTION



(a) Two-dimensional stress

(b) Plan view of element shown in (a)

(c) Components of stress in directions u and v



(c) Components of stress in directions u and v

$$\sigma_u = 2\tau_{xy}\sin\varphi\cos\varphi + \sigma_x\cos^2\varphi + \sigma_y\sin^2\varphi$$

$\sum F_u = 0$

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi + \tau_{xy}\sin 2\varphi$$

$$\sigma_v = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi - \tau_{xy}\sin 2\varphi$$

$$\tau_{uv} = \tau_{xy}\left(\cos^2\varphi - \sin^2\varphi\right) - \left(\sigma_x - \sigma_y\right)\sin\varphi\cos\varphi$$

$\sum F_v = 0$

$$\tau_{uv} = \tau_{xy}\cos 2\varphi - \frac{\sigma_x - \sigma_y}{2}\sin 2\varphi$$

$$\sigma_u + \sigma_v = \sigma_x + \sigma_y$$

φ **Positive when taken clockwise from the *x*-axis**

$$\boxed{\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi + \tau_{xy}\sin 2\varphi}$$

$$\begin{aligned}\sigma_u &= \frac{\sigma_x + \sigma_y}{2} + R\left(\cos 2\theta \cos 2\varphi + \sin 2\theta \sin 2\varphi\right) \\ &= \frac{\sigma_x + \sigma_y}{2} + R\cos\left(2\theta - 2\varphi\right)\end{aligned}$$

圓的參數方程式

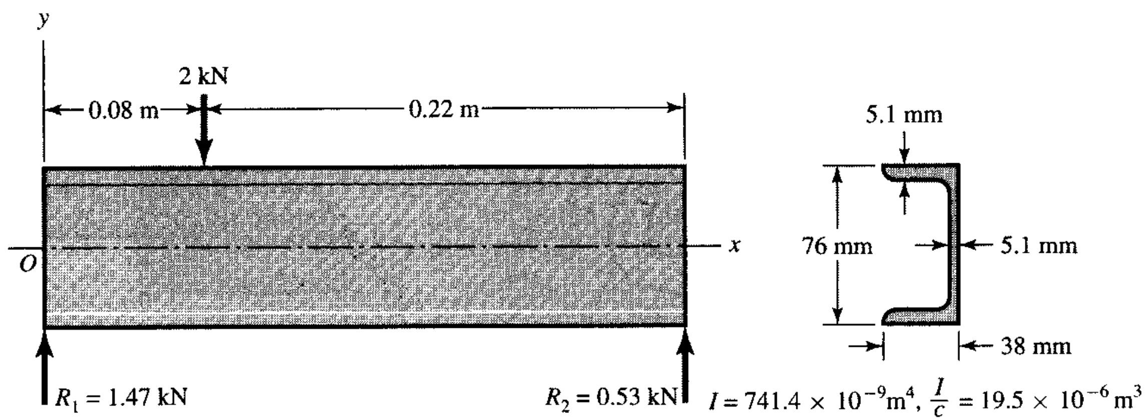
$$\boxed{\tau_{uv} = \tau_{xy}\cos 2\varphi - \frac{\sigma_x - \sigma_y}{2}\sin 2\varphi}$$

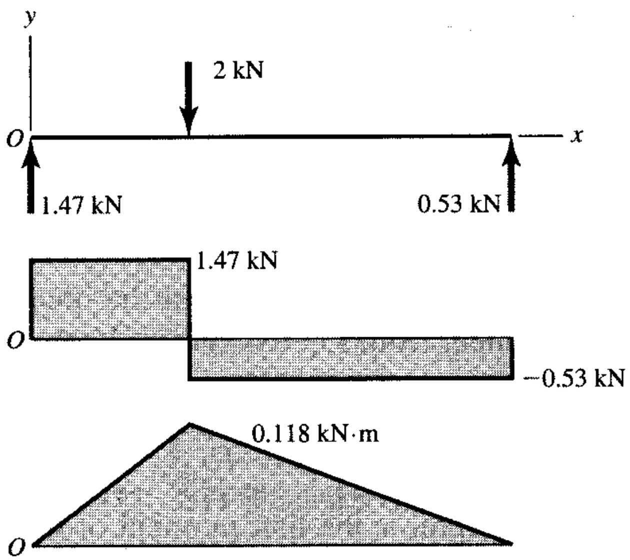
$$\begin{aligned}\tau_{uv} &= R\left(\sin 2\theta \cos 2\varphi - \cos 2\theta \sin 2\varphi\right) \\ &= R\sin\left(2\theta - 2\varphi\right)\end{aligned}$$

$$\begin{cases} \dfrac{\sigma_x - \sigma_y}{2} = R\cos 2\theta \\ \tau_{xy} = R\sin 2\theta \end{cases}, \quad \Rightarrow \quad \begin{cases} R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ \tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y} \end{cases}$$

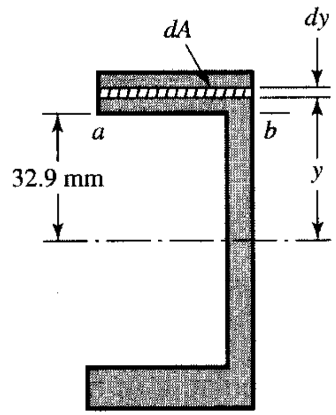
EXAMPLE

EXAMPLE 3–7 A beam 0.3 m long is to support a load of 2 kN acting 0.08 m from the left support, as shown in Fig. 3–19*a*. Basing the design only on bending stress, a designer has selected a 76-mm aluminum channel with the cross-sectional dimensions shown. If the direct shear is neglected, the stress in the beam may be actually higher than the designer thinks. Determine the principal stresses considering bending and direct shear and compare them with that considering bending only.





(*b*)



(*c*)

Solution The loading, shear-force, and bending-moment diagrams are shown in Fig. 3–19*b*. If the direct shear force is included in the analysis, the maximum stresses at the top and bottom of the beam will be the same as if only bending were considered. The maximum bending stresses are

$$\sigma = \pm\frac{Mc}{I} = \pm\frac{0.118(10^3)}{19.5 \times 10^{-6}} = \pm\, 6.05 \text{ MPa}$$

However, the maximum stress due to the combined bending and direct shear stresses may be maximum at the point $(76^-, 32.9)$ that is just to the left of the applied load, where the web joins the flange. To simplify the calculations we assume a cross section with square corners (Fig. 3–19*c*). The normal stress at section *ab*, with $x = 3$ in, is

$$\sigma = -\frac{My}{I} = -\frac{0.118(10^3)\,0.0329}{741.4 \times 10^{-9}} = -5.24 \text{ MPa}$$

For the shear stress at section *ab*, considering the area above *ab* and using Eq. (3–30) gives

$$Q = \bar{y}'A' = \left(0.0329 + \frac{0.0051}{2}\right)(0.038)(0.0051) = 6.87 \times 10^{-6}\,\text{m}^3$$

Using Eq. (3–31) with $V = 1.47$ kN, $I = 741.4 \times 10^{-9}$ m^4, $Q = 6.87 \times 10^{-6}$ m^3, and $b = 0.0051$ m yields

$$\tau_{xy} = -\frac{VQ}{Ib} = -\frac{1.47(6.87 \times 10^{-6})10^3}{741.4 \times 10^{-9}(0.0051)} = -2.67 \text{ MPa}$$

The negative sign comes from recognizing that the shear stress is down on an x face of a $dx\ dy$ element at the location being considered.

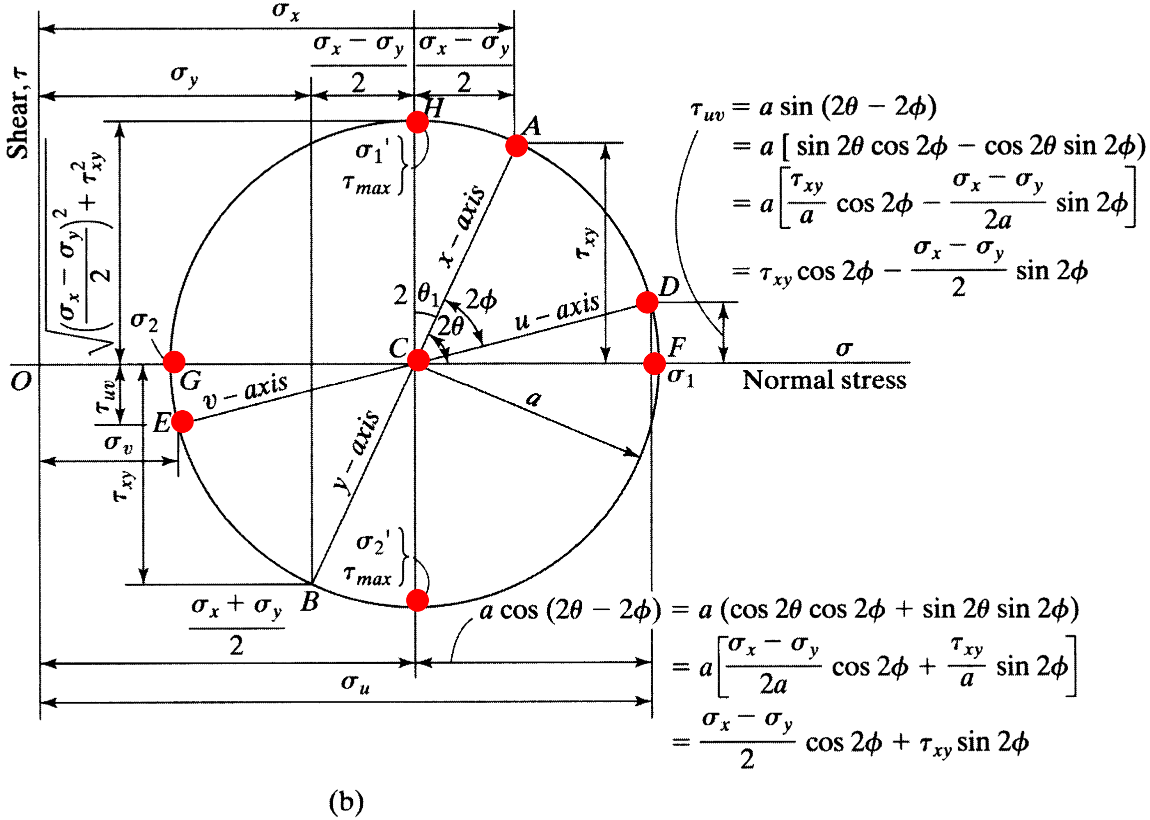
The principal stresses at the point can now be determined. Using Eq. (3–13), we find that at $x = 76^-$ mm, $y = 32.9$ mm,

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{-5.24 + 0}{2} \pm \sqrt{\left(\frac{-5.24 - 0}{2}\right)^2 + (-2.67)^2} = 1.12, -6.36 \text{ MPa}$$

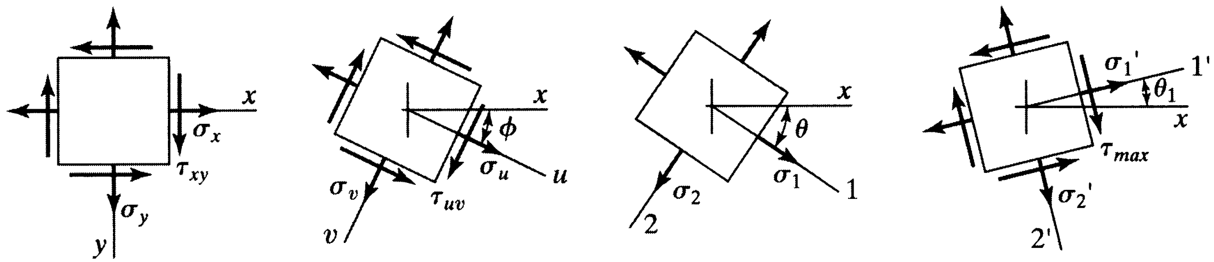
For a point at $x = 76^-$ mm, $y = -32.9$ mm, the principal stresses are $\sigma_1, \sigma_2 = 6.36$, -1.12 MPa. Thus we see that the maximum principal stresses are ± 1200 psi, 21 percent higher than thought by the designer.

1-20 MOHR'S CIRCLE



$$\tau_{uv} = a\sin(2\theta - 2\phi) = a\,[\sin 2\theta\cos 2\phi - \cos 2\theta \sin 2\phi] = a\left[\frac{\tau_{xy}}{a}\cos 2\phi - \frac{\sigma_x - \sigma_y}{2a}\sin 2\phi\right] = \tau_{xy}\cos 2\phi - \frac{\sigma_x - \sigma_y}{2}\sin 2\phi$$

$$a\cos(2\theta - 2\phi) = a\,(\cos 2\theta\cos 2\phi + \sin 2\theta\sin 2\phi) = a\left[\frac{\sigma_x - \sigma_y}{2a}\cos 2\phi + \frac{\tau_{xy}}{a}\sin 2\phi\right] = \frac{\sigma_x - \sigma_y}{2}\cos 2\phi + \tau_{xy}\sin 2\phi$$



(a) Given state of stress

(c) Stresses on element oriented at angle ϕ

(d) Principal stresses

(e) Maximum shear stress

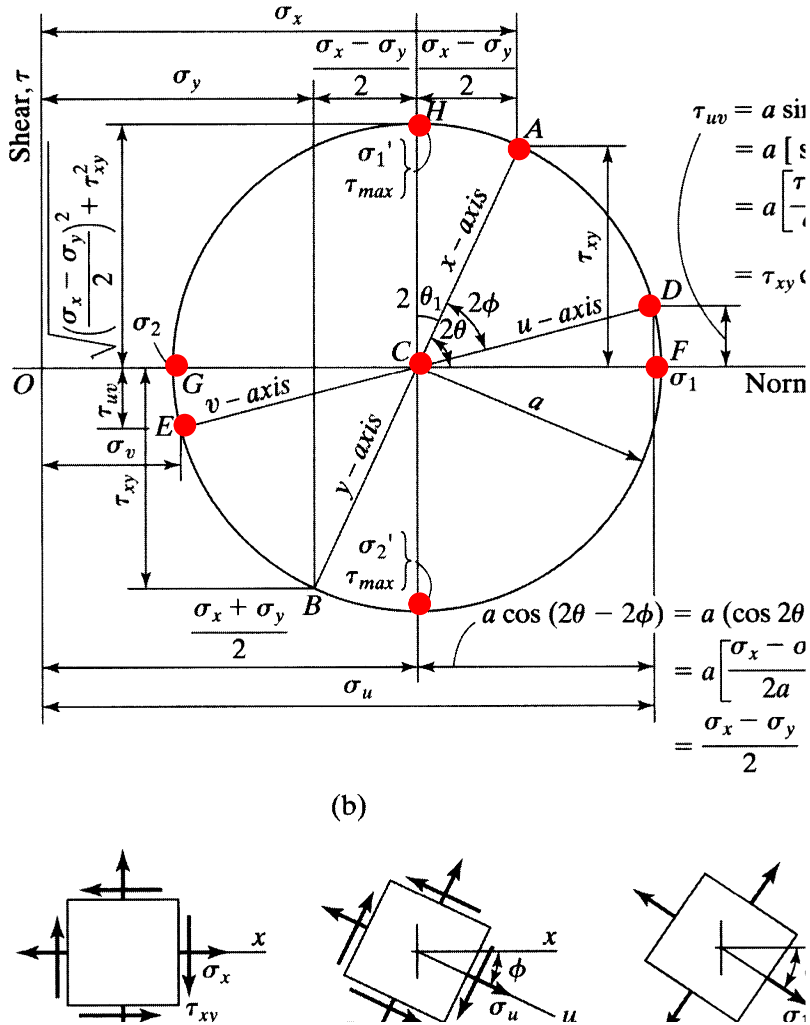
視覺化表示應力在不同方向上的大小

Maximum and minimum values of normal stresses

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}, \quad \text{for principal stresses}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



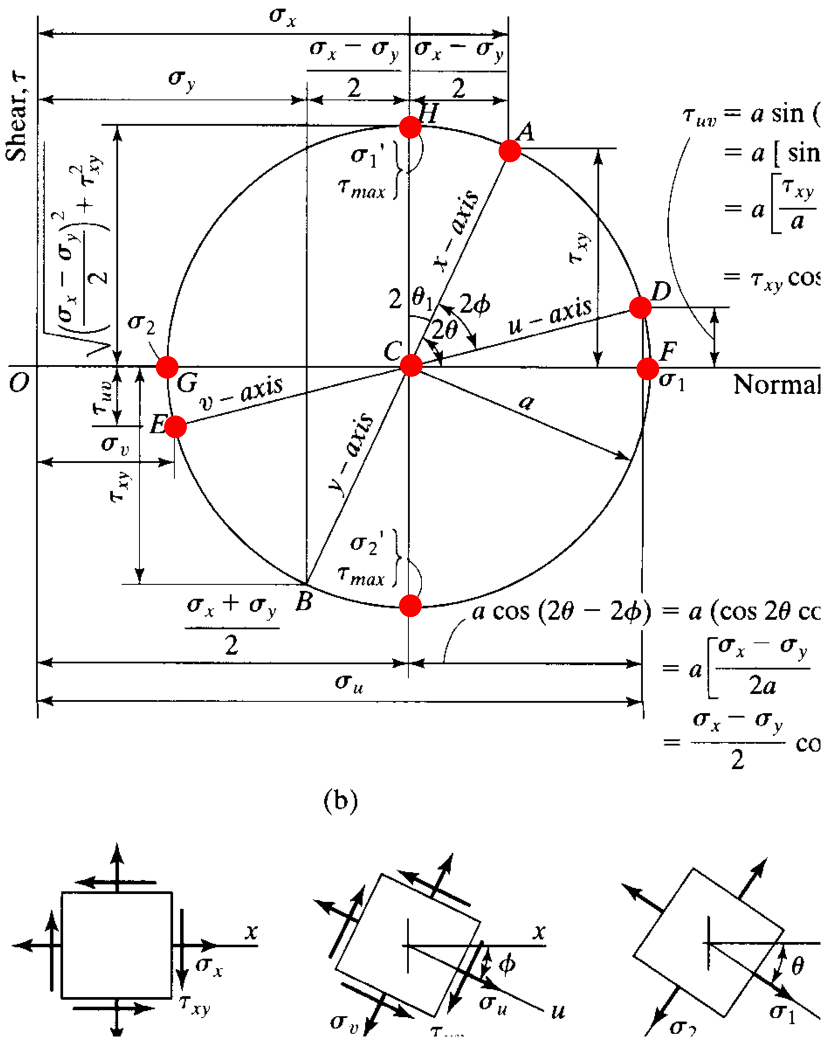
Maximum values of shear stress

$$\tan 2\theta_1 = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$2\theta_1 = \frac{\pi}{2} - 2\theta$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\sigma_1' = \sigma_2' = \frac{\sigma_x + \sigma_y}{2}$$



(b)

EXAMPLE 1-15

Problem Statement: Let the state of stress at some point in a body be defined as follows:

σ_x = 200 MPa, σ_y = -40 MPa, τ_{xy} = 50 MPa

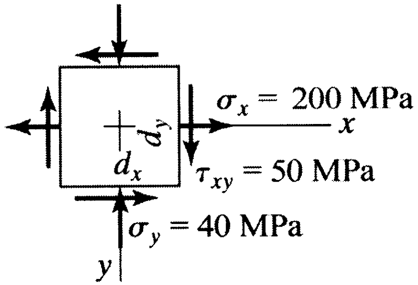
(a) Draw the view of the element for the given state of stress and mark values thereon.

(b) Draw the Mohr circle for the given state of stress and mark completely.

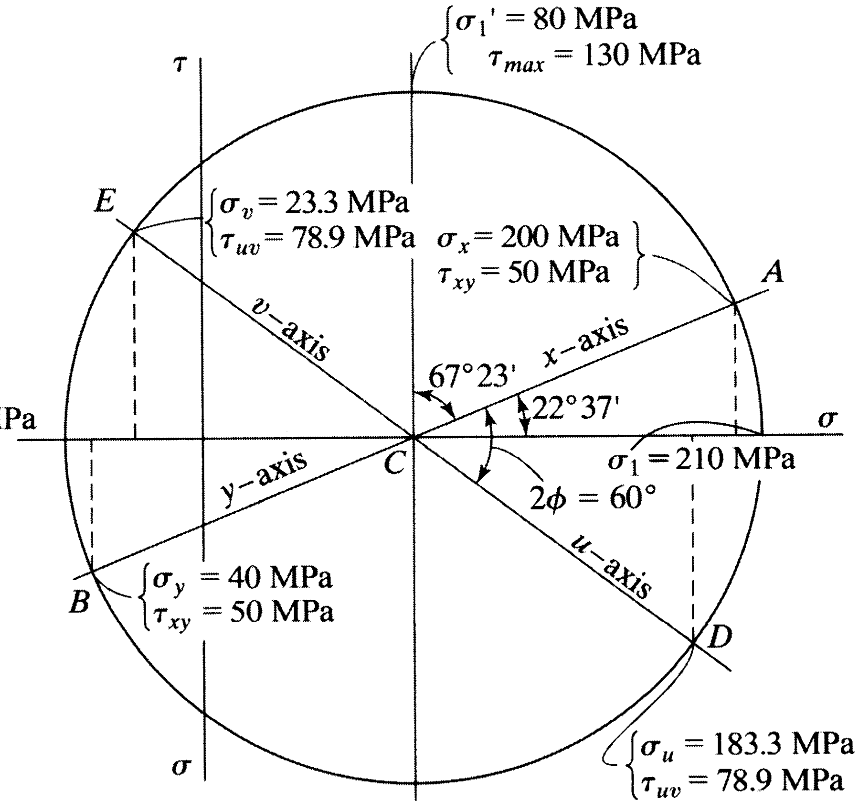
(c) Draw the element oriented 30° clockwise from the x-axis and show values of all stresses.

(d) Draw the element correctly oriented for principal stresses and show values.

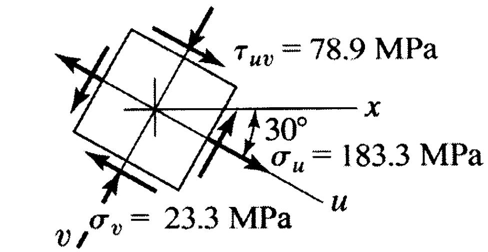
(e) Draw the element for maximum shear stress and mark values of all stresses



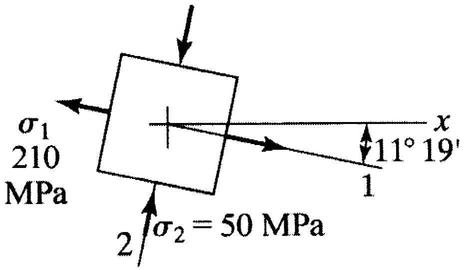
(a) Given state of stress



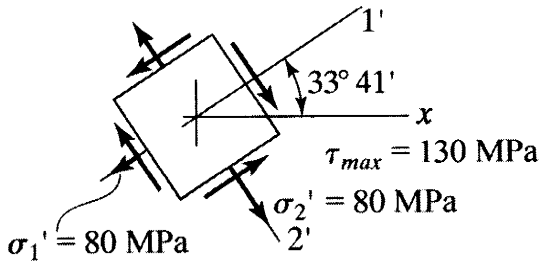
(b) Mohr circle for given stresses



(c) Stresses at 30° with x–axis



(d) Principal stresses



(e) Maximum shearing stress

EXAMPLE

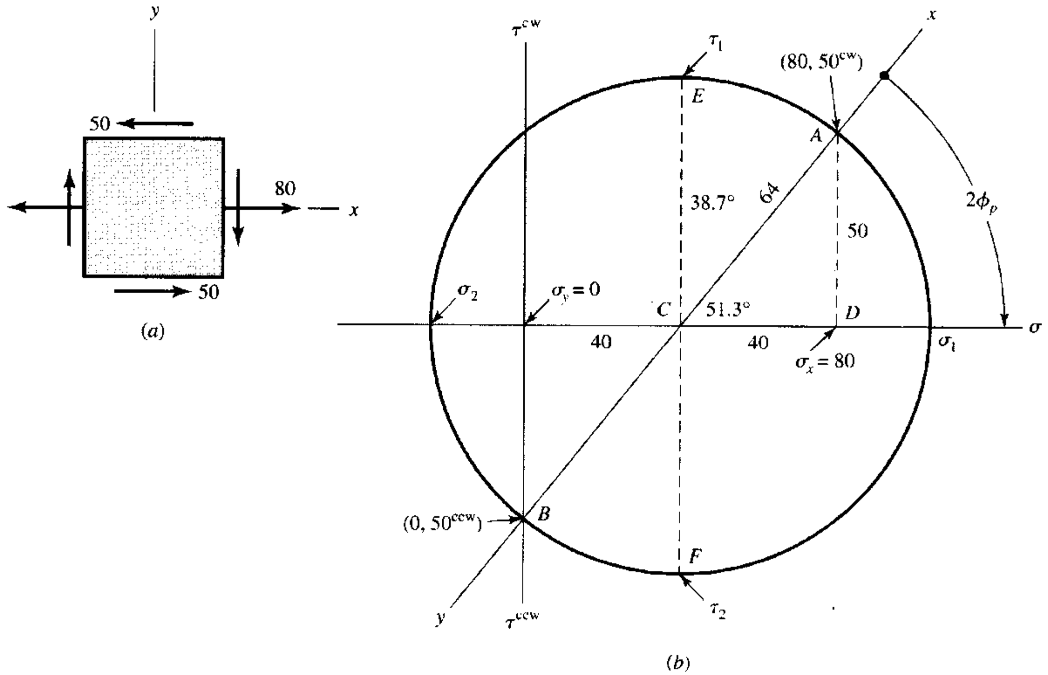
A stress element has $\sigma_x = 80$ MPa and $\tau_{xy} = 50$ MPa cw, as shown in Fig. 3–11*a*.

(*a*) Using Mohr's circle, find the principal stresses and directions, and show these on a stress element correctly aligned with respect to the *xy* coordinates. Draw another stress element to show τ_1 and τ_2, find the corresponding normal stresses, and label the drawing completely.

(*b*) Repeat part *a* using the transformation equations only.

'e 3–11

ses in MPa.



Solution (*a*) In the semigraphical approach used here, we first make an approximate freehand sketch of Mohr's circle and then use the geometry of the figure to obtain the desired information.

Draw the σ and τ axes first (Fig. 3–11*b*) and from the x face locate $\sigma_x = 80$ MPa along the σ axis. On the x face of the element, we see that the shear stress is 50 MPa in the cw direction. Thus, for the x face, this establishes point A $(80, 50^{cw})$ MPa. Corresponding to the y face, the stress is $\sigma = 0$ and $\tau = 50$ MPa in the ccw direction. This locates point B $(0, 50^{ccw})$ MPa. The line AB forms the diameter of the required circle, which can now be drawn. The intersection of the circle with the σ axis defines σ_1 and σ_2 as shown. Now, noting the triangle ACD, indicate on the sketch the length of the legs AD and CD as 50 and 40 MPa, respectively. The length of the hypotenuse AC is

Answer

$$\tau_1 = \sqrt{(50)^2 + (40)^2} = 64.0 \text{ MPa}$$

and this should be labeled on the sketch too. Since intersection C is 40 MPa from the origin, the principal stresses are now found to be

Answer

$$\sigma_1 = 40 + 64 = 104 \text{ MPa} \qquad \text{and} \qquad \sigma_2 = 40 - 64 = -24 \text{ MPa}$$

The angle 2ϕ from the x axis *cw* to σ_1 is

Answer

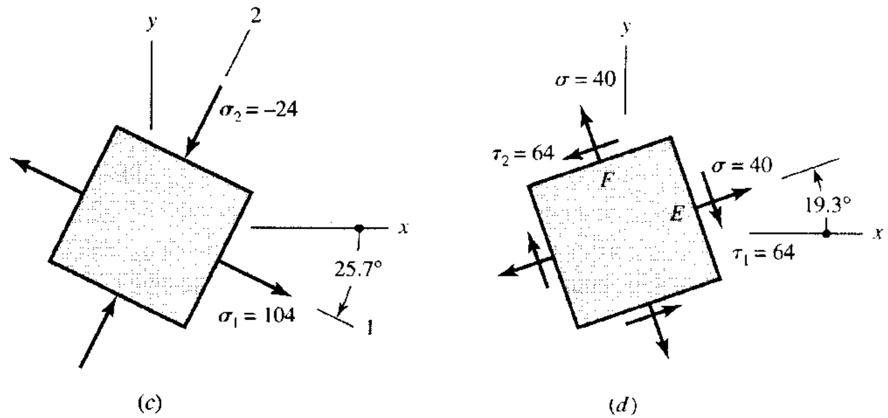
$$2\phi_p = \tan^{-1}\tfrac{50}{40} = 51.3^\circ$$

To draw the principal stress element (Fig. 3–11*c*), sketch the x and y axes parallel to the original axes. The angle ϕ_p on the stress element must be measured in the *same* direction as is the angle $2\phi_p$ on the Mohr circle. Thus, from x measure 25.7° (half of 51.3°) clockwise to locate the σ_1 axis. The σ_2 axis is 90° from the σ_1 axis and the stress element can now be completed and labeled as shown. Note that there are *no* shear stresses on this element.

The two maximum shear stresses occur at points E and F in Fig. 3–11*b*. The two normal stresses corresponding to these shear stresses are each 40 MPa, as indicated. Point E is 38.7° ccw from point A on Mohr's circle. Therefore, in Fig. 3–11*d*, draw a stress element oriented 19.3° (half of 38.7°) ccw from x. The element should then be labeled with magnitudes and directions as shown.

In constructing these stress elements it is important to indicate the x and y directions of the original reference system. This completes the link between the original machine element and the orientation of its principal stresses.

Answer



(*b*) The transformation equations are programmable. From Eq. (3–10),

$$\phi_p = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2(-50)}{80}\right) = -25.7^\circ, 64.3^\circ$$

From Eq. (3–8), for the first angle $\phi_p = -25.7^\circ$,

$$\sigma = \frac{80+0}{2} + \frac{80-0}{2}\cos[2(-25.7)] + (-50)\sin[2(-25.7)] = 104.03 \text{ MPa}$$

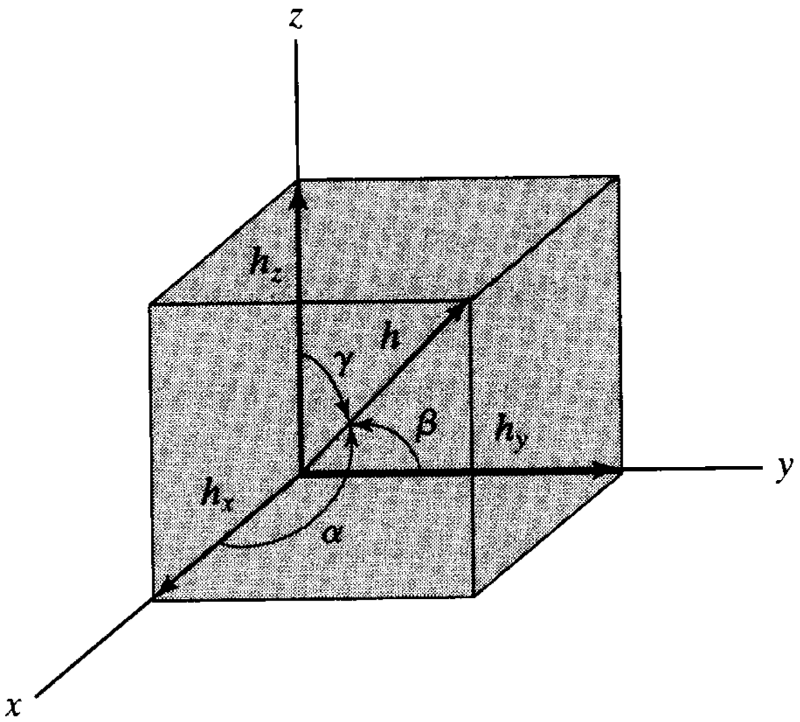
The shear on this surface is obtained from Eq. (3–9) as

$$\tau = -\frac{80-0}{2}\sin[2(-25.7)] + (-50)\cos[2(-25.7)] = 0 \text{ MPa}$$

which confirms that 104.03 MPa is a principal stress. From Eq. (3–8), for $\phi_p = 64.3^\circ$,

$$\sigma = \frac{80+0}{2} + \frac{80-0}{2}\cos[2(64.3)] + (-50)\sin[2(64.3)] = -24.03 \text{ MPa}$$

1-21 STRESS IN 3D



$$h^2 = h_x^2 + h_y^2 + h_z^2$$
$$= h^2\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right)$$

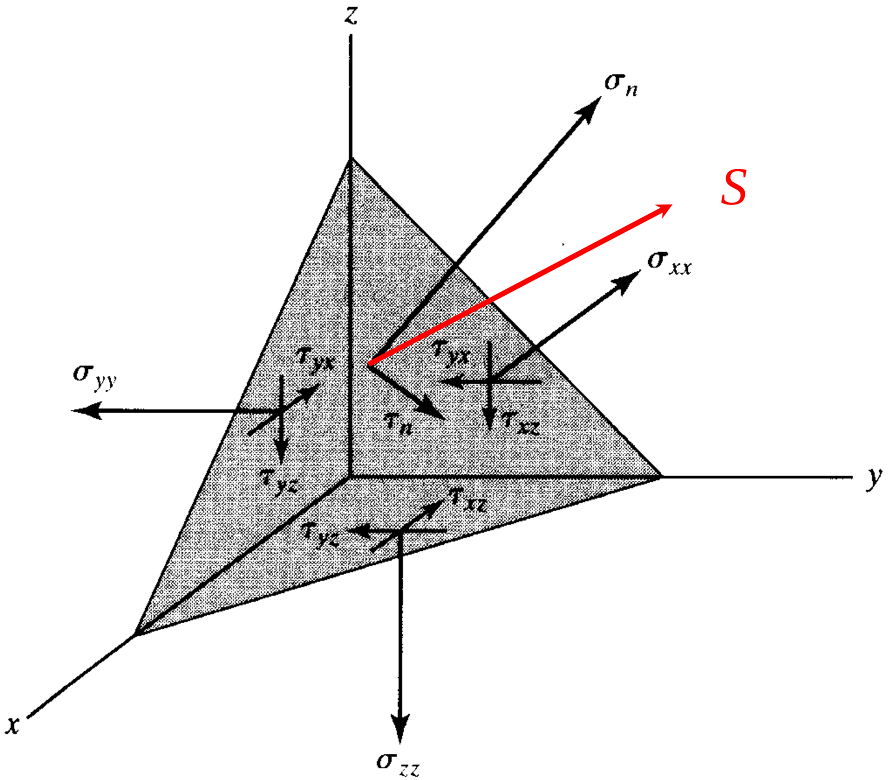
$$\Rightarrow \quad 1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

Let

$$\cos\alpha = l$$
$$\cos\beta = m$$
$$\cos\gamma = n$$

三維元素的廣義應力狀態？

$$\Rightarrow \quad 1 = l^2 + m^2 + n^2$$



$$A_x = Al$$
$$A_y = Am$$
$$A_z = An$$

Moment equilibrium results in

$$\tau_{xy} = \tau_{xy}$$
$$\tau_{xz} = \tau_{zx}$$
$$\tau_{yz} = \tau_{zy}$$

$$S^2 = \overbrace{S_x^2 + S_y^2 + S_z^2}^{x,\,y,\,z\text{ 方向分解}} = \overbrace{\sigma_n^2 + \tau_n^2}^{\text{切線、法線方向分解}}$$

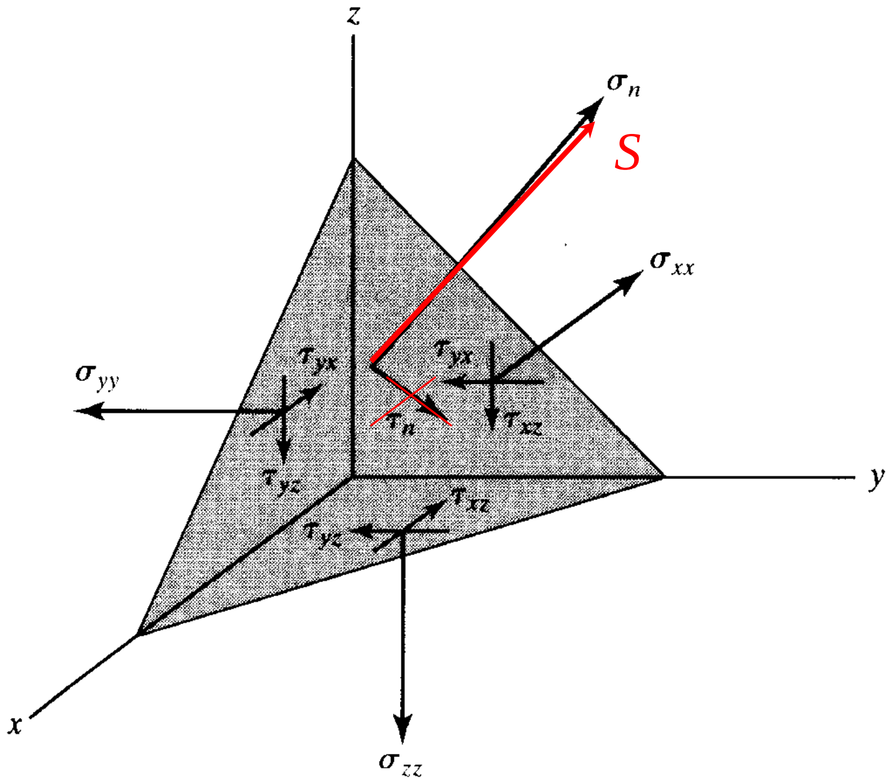
$$\tau_n = \sqrt{S^2 - \sigma_n^2}$$

S 斜面上總應力

斜面上總應力的分量：

$$S_x = S\cos\alpha = Sl$$
$$S_y = S\cos\beta = Sm$$
$$S_z = S\cos\gamma = Sn$$

若某個斜面上只有正應力：



Force equilibrium

$$\sum F_x = 0$$
$$\Rightarrow \quad AS_x = Al\sigma_{xx} + Am\tau_{xy} + An\tau_{xz}$$

$$\sum F_y = 0$$
$$\Rightarrow \quad AS_y = Al\tau_{xy} + Am\sigma_{yy} + An\tau_{yz}$$

$$\sum F_z = 0$$
$$\Rightarrow \quad AS_z = Al\tau_{xz} + Am\tau_{yz} + An\sigma_{zz}$$

Eigenvalue problem

$$\begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

主應力狀態：斜面上只有正應力存在（沒有剪應力）

$$
\begin{bmatrix}
\sigma_{xx}-S & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_{yy}-S & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_{zz}-S
\end{bmatrix}
\begin{pmatrix} l \\ m \\ n \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\begin{pmatrix} l \\ m \\ n \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\Rightarrow
\begin{vmatrix}
\sigma_{xx}-S & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_{yy}-S & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_{zz}-S
\end{vmatrix} = 0
$$

此行矩陣代表：
斜面的單位法向量

Eigenvalues are the principal stresses

$$
S = S_1 = \sigma_1
$$
$$
S = S_2 = \sigma_2
$$
$$
S = S_3 = \sigma_3
$$

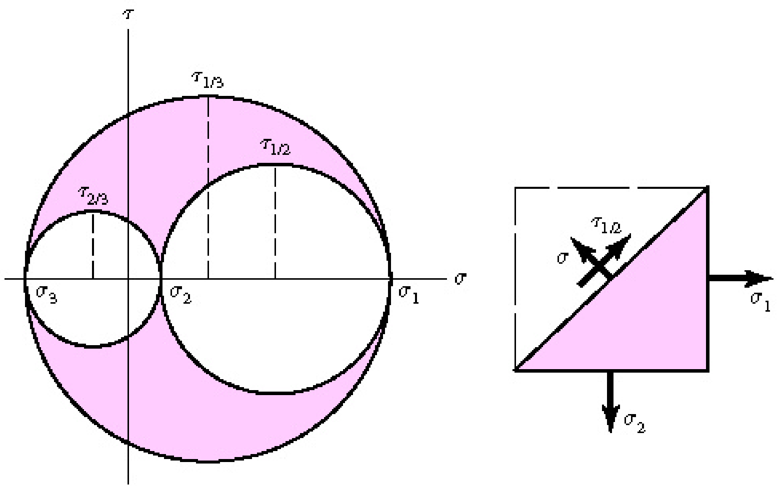
主應力態的的應力元素長什麼樣子？

Principal shear stresses

$$\tau_{1/2} = \frac{S_1 - S_2}{2} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\tau_{2/3} = \frac{S_2 - S_3}{2} = \frac{\sigma_2 - \sigma_3}{2}$$

$$\tau_{1/3} = \frac{S_1 - S_3}{2} = \frac{\sigma_1 - \sigma_3}{2}$$



(a) (b)

EXAMPLE 1-16

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 10 & 5 & 6 \\ 5 & 20 & 4 \\ 6 & 4 & 30 \end{bmatrix} \quad \text{MPa}$$

Find three principal stresses.

$$\begin{vmatrix} 10-S & 5 & 6 \\ 5 & 20-S & 4 \\ 6 & 4 & 30-S \end{vmatrix} = 0$$

$$S = S_1 = 7.1 \quad \text{MPa}$$
$$S = S_2 = 33.7 \quad \text{MPa}$$
$$S = S_3 = 19.1 \quad \text{MPa}$$

Two-dimensional state of stress:

$$\tau_{xz} = \tau_{zx} = \tau_{yz} = \tau_{zy} = \sigma_{zz} = 0$$

$$\begin{bmatrix} \sigma_{xx} - S & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} - S & 0 \\ 0 & 0 & -S \end{bmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$S\left[S^2 - \left(\sigma_{xx} + \sigma_{yy} \right) S + \left(\sigma_{xx}\sigma_{yy} - \tau_{xy} \right) \right] = 0$$

The roots are the principal normal stresses

More simply for the two-dimensional state of stress:

$$\begin{bmatrix} \sigma_{xx} - S & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} - S \end{bmatrix} \begin{pmatrix} l \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$S^2 - \left(\sigma_{xx} + \sigma_{yy}\right) S + \left(\sigma_{xx}\sigma_{yy} - \tau_{xy}\right) = 0$$

The roots are the principal normal stresses

EXAMPLE 1-17

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 5 & 20 \end{bmatrix} \quad \text{MPa}$$

$$\begin{vmatrix} 10-S & 5 \\ 5 & 20-S \end{vmatrix} = 0$$

$$S^2 - (10+20)S + (200-25) = 0$$

$$S^2 - 30S + 175 = 0$$

$$S = \frac{30 \pm \sqrt{30^2 - 4\times1\times175}}{2} \quad \Rightarrow \quad \begin{cases} S_1 = 22.1 \quad \text{MPa} \\ S_2 = 7.9 \quad \text{MPa} \end{cases}$$

1-22 STRESSES AND DEFORMATION

2D:

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \mu\sigma_{yy}\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \mu\sigma_{xx}\right)$$

$$\sigma_{xx} = \frac{1}{1-\mu^2}\left(\varepsilon_{xx} + \mu\varepsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{1}{1-\mu^2}\left(\varepsilon_{yy} + \mu\varepsilon_{xx}\right)$$

3D:

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \mu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \mu\left(\sigma_{xx} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \mu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$