Coulomb Blockade in Quantum Dots

Mean-field method

Ch. 6 from *Condensed Matter Field Theory - Altland and Simons*

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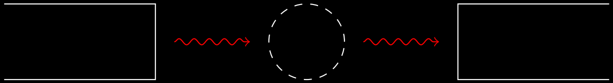
Quantum dots

What is a quantum dot?



Controlled tunneling of electrons through the dot:
- Input
- Coulomb blockade
- Spin (Pauli) blockade
- Readout

Contents

Quantum dot
- Coulomb blockade

Mean-field method for these and similar actions
- Decoupling through Hubbard-Stratonovich transformation
- Integrate over fermionic degrees of freedom
- Stationary phase approximation of auxiliary field
- Effective action

Coulomb blockade

Action

$$S[\bar{\Psi}_\alpha, \Psi_\alpha] = \sum_\alpha \bar{\psi}_\alpha(\partial_\tau + \epsilon_\alpha - \mu)\psi_\alpha + E_C(\sum_\alpha \bar{\Psi}_\alpha \Psi_\alpha - N_0)^2$$

Hubbard-Stratonovich transformation:

$$\psi^4 \to \psi^2 V + V^2 + V$$

In this case

$$e^{-E_C(\sum_\alpha \bar{\Psi}_\alpha \Psi_\alpha - N_0)^2} = \int DV \; e^{-\frac{V^2}{4E_C} + iN_0 V - i\sum_\alpha \bar{\Psi}_\alpha V \Psi_\alpha}$$

Coulomb blockade

This leads to

$$G_\alpha = \frac{1}{Z}\int D(\bar\psi,\psi) e^{-\underline{S_{free}[\bar\psi,\psi]}-\underline{S_{int}[\bar\psi,\psi]}}\bar\psi_\alpha\psi_\alpha$$

$$= \frac{1}{Z}\int D(\bar\psi,\psi) e^{-\underline{\sum_\alpha \bar\psi_\alpha(\partial_\tau+\epsilon_\alpha-\mu)\psi_\alpha}-\underline{E_C(\sum_\alpha \bar\Psi_\alpha\Psi_\alpha - N_0)^2}}\bar\psi_\alpha\psi_\alpha$$

$$= \frac{1}{Z}\int DV\ e^{-S[V]}\int D(\bar\psi,\psi) e^{-\underline{\sum_\alpha \bar\psi_\alpha(\partial_\tau+\epsilon_\alpha-\mu+iV)\psi_\alpha}}\bar\psi_\alpha\psi_\alpha$$

$$= \frac{1}{Z}\int DV\ e^{-S[V]}\int D(\bar\psi,\psi) e^{-S^V[\bar\psi,\psi]}\bar\psi_\alpha\psi_\alpha$$

$$S^V[\bar\psi,\psi] = S_{free}[\bar\psi,\psi]|_{\mu\to\mu-iV}$$

Coulomb blockade

Using a gauge transformation of the form

$$\psi(\tau) \to \psi(\tau) e^{-i \int^{\tau} d\tau' (V(\tau') - V_0)}$$

The action S^V can be reduced

$$S^V[\bar{\psi}, \psi] \to S^{V_0}[\bar{\psi}, \psi]$$

At the cost of an extra gauge factor

$$\frac{1}{Z} \int DV \; e^{-S[V]} \int D(\bar{\psi}, \psi) e^{-S^{V_0}[\bar{\psi}, \psi]} \; e^{-i \int^{\tau} d\tau' (V(\tau') - V_0)} \; \bar{\psi}_\alpha \psi_\alpha$$

Coulomb blockade

The integral over the auxiliary field can be split

$$\int DV = \prod_{n=0} \int dV_n = \int dV_0 \prod_{n=1} \int dV_n$$

The integration over de V_n components gives a Matsubara summation

$$-2E_C T \sum_{n\neq 0} \frac{1}{\omega_n^2}(1 - e^{-i\omega_n \tau}) = -E_C(|\tau| - T \cdot \tau^2)$$

Note: this is not a trivial calculation, but the answer is obtained by using

$$\sum_{k=1}^{\infty} \frac{cos(kx)}{k^2} \approx \frac{\pi^2}{6} - \frac{\pi|x|}{2} + \frac{x^2}{4}$$

Coulomb blockade

In our action, we can write

$$G(\tau) = \frac{1}{Z}\int DV\ e^{-S[V]}\int D(\bar{\psi},\psi)\ e^{-i\int^{\tau} d\tau'(V(\tau')-V_0)}\ e^{-S^{V_0}[\bar{\psi},\psi]}\ \bar{\psi}_\alpha\psi_\alpha$$

$$= \frac{1}{Z}\int DV\ e^{-S[V]}\ e^{-i\int^{\tau} d\tau'(V(\tau')-V_0)} Z^{V_0} G_\alpha^{V_0}$$

$$= \frac{F(\tau)}{Z}\int dV_0\ e^{-S[V_0]} Z^{V_0} G_\alpha^{V_0}$$

Where

$$F(\tau) \equiv e^{-E_C(|\tau|-T\cdot\tau^2)}$$

Coulomb blockade

In

$$G(\tau) = \frac{F(\tau)}{Z} \int dV_0 \; e^{-S[V_0]} Z^{V_0} G_\alpha^{V_0}$$

We can define a free energy $\mathcal{F}(\mu)$ as

$$Z^{V_0} = e^{-\beta \; \mathcal{F}^{V_0}(\mu)} = e^{-\beta \; \mathcal{F}(\mu - iV_0)}$$

So we have, writing out the $S[V_0]$:

$$G(\tau) = \frac{F(\tau)}{Z} \int dV_0 \; e^{-\frac{\beta}{4E_C} V_0^2 + i\beta N_0 V_0 - \beta \; \mathcal{F}(\mu - iV_0)} G_\alpha^{V_0}$$

which can be approximated using the stationary phase method

Stationary phase approximation

When given an integral of the form

$$I = \lim_{\lambda \to \infty} \int_{-\infty}^{-\infty} e^{-\lambda f(x)}$$

The relevant contributions occur around minimum x_0 of f:

$$\left. \frac{\partial f}{\partial x} \right|_{x=x_0} = 0$$

Which is known as the *saddle point equation*

Stationary phase approximation

In the case of our action, we have the exponent

$$-\frac{\beta}{4E_C}V_0^2 + i\beta N_0 V_0 - \beta\,\mathcal{F}(\mu - iV_0)$$

Which has a minimum (after some calculation)

$$0 = \frac{1}{2E_C}V_0 - iN_0 + i\left\langle \hat{N}\right\rangle_{\mu - iV_0}$$

where we used that

$$\frac{\partial}{\partial V_0}\mathcal{F}(\mu - iV_0) = -i\frac{\partial}{\partial \mu}\mathcal{F}(\mu - iV_0) = -i\left\langle \hat{N}\right\rangle_{\mu - iV_0}$$

where $\hat{N}$ is the number operator

$$\hat{N} \equiv \sum_{\alpha} a_\alpha^\dagger a_\alpha$$

Density of states

Filling in the solutions, we can calculate the tunneling density of states by

$$\nu(\epsilon) = -\frac{1}{\pi} Im\ tr\ G$$

After some calculations, we obtain

$$\nu(\epsilon) = \nu_0(\epsilon - E_C\ sgn(\epsilon))\ \Theta(|\epsilon| - E_C)$$

Where ν_0 is the density of states (shifted by the energy cost) of the non-interacting systems and Θ a step function.

This means that there is an energy gap of size $2E_C$ around the Fermi energy

Conclusions

The Coulomb blockade prevents electrons of certain energies to tunnel into the dot.

If it has energy $\epsilon > E_C$ it is allowed to tunnel