# Title
Subtitle

Article *article title*

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# Contents

Discussion of mean-field methods for large parameter systems

- Coulomb blockade

- Decoupling through Hubbard-Stratonovich transformation
- Integrate over fermionic degrees of freedom
- Stationary phase approximation of auxiliary field
- Effective action

# Coulomb blockade

Action

$$S[\bar{\Psi}_\alpha, \Psi_\alpha] = \sum_\alpha \bar{\psi}_\alpha(\partial_\tau + \epsilon_\alpha - \mu)\psi_\alpha + E_C(\sum_\alpha \bar{\Psi}_\alpha \Psi_\alpha - N_0)^2$$

Hubbard-Stratonovich transformation:

$$\psi^4 \to \psi^2 V + V^2 + V$$

In this case

$$e^{-E_C(\sum_\alpha \bar{\Psi}_\alpha \Psi_\alpha - N_0)^2} = \int DV \ e^{-\frac{V^2}{4E_C} + iN_0 V - i\sum_\alpha \bar{\Psi}_\alpha V \Psi_\alpha}$$

# Coulomb blockade

This leads to

$$G_\alpha = \frac{1}{Z} \int D(\bar{\psi}, \psi) e^{-S_{free}[\bar{\psi}, \psi] - S_{int}[\bar{\psi}, \psi]} \bar{\psi}_\alpha \psi_\alpha$$

$$= \frac{1}{Z} \int D(\bar{\psi}, \psi) e^{-\sum_\alpha \bar{\psi}_\alpha (\partial_\tau + \epsilon_\alpha - \mu)\psi_\alpha - E_C(\sum_\alpha \bar{\Psi}_\alpha \Psi_\alpha - N_0)^2} \bar{\psi}_\alpha \psi$$

$$= \frac{1}{Z} \int DV \ e^{-S[V]} \int D(\bar{\psi}, \psi) e^{-\sum_\alpha \bar{\psi}_\alpha (\partial_\tau + \epsilon_\alpha - \mu + iV)\psi_\alpha} \bar{\psi}_\alpha \psi_\alpha$$

$$= \frac{1}{Z} \int DV \ e^{-S[V]} \int D(\bar{\psi}, \psi) e^{-S^V[\bar{\psi}, \psi]} \bar{\psi}_\alpha \psi_\alpha$$

$$S^V[\bar{\psi}, \psi] = S_{free}[\bar{\psi}, \psi]|_{\mu \to \mu - iV}$$

# Coulomb blockade

Using a gauge transformation of the form

$$\psi(\tau) \to \psi(\tau) e^{-i \int^{\tau} d\tau' (V(\tau') - V_0)}$$

The action $S^V$ can be reduced

$$S^V[\bar{\psi}, \psi] \to S^{V_0}[\bar{\psi}, \psi]$$

At the cost of an extra gauge factor

$$\frac{1}{Z} \int DV \; e^{-S[V]} \int D(\bar{\psi}, \psi) e^{-S^{V_0}[\bar{\psi}, \psi]} \; e^{-i \int^{\tau} d\tau' (V(\tau') - V_0)} \; \bar{\psi}_\alpha \psi_\alpha$$

# Coulomb blockade

The integral over the auxiliary field can be split

$$\int DV = \prod_{n=0} \int dV_n = \int dV_0 \prod_{n=1} \int dV_n$$

The integration over de $V_n$ components gives a Matsubara summation

$$-2E_C T \sum_{n \neq 0} \frac{1}{\omega_n^2}(1 - e^{-i\omega_n \tau}) = -E_C(|\tau| - T \cdot \tau^2)$$

*Note:* this is not a trivial calculation, but the answer is obtained by using

$$\sum_{k=1}^{\infty} \frac{cos(kx)}{k^2} \approx \frac{\pi^2}{6} - \frac{\pi|x|}{2} + \frac{x^2}{4}$$

## Coulomb blockade

In our action, we can write

$$G(\tau) = \frac{1}{Z} \int DV \; e^{-S[V]} \int D(\bar{\psi}, \psi) \; e^{-i \int^\tau d\tau' (V(\tau') - V_0)} \; e^{-S^{V_0}[\bar{\psi}, \psi]} \; \bar{\psi}_\alpha \psi_\alpha$$

$$= \frac{1}{Z} \int DV \; e^{-S[V]} \int D(\bar{\psi}, \psi) \; e^{-i \int^\tau d\tau' (V(\tau') - V_0)} Z^{V_0} \, G_\alpha^{V_0}$$

$$= \frac{F(\tau)}{Z} \int dV_0 \; e^{-S[V_0]} Z^{V_0} \, G_\alpha^{V_0}$$

Where

$$F(\tau) \equiv e^{-E_C(|\tau| - T \cdot \tau^2)}$$

# Coulomb blockade

In

$$G(\tau) = \frac{F(\tau)}{Z} \int dV_0 \ e^{-S[V_0]} Z^{V_0} G_\alpha^{V_0}$$

We can define a free energy $\mathcal{F}(\mu)$ as

$$Z^{V_0} = e^{-\beta \ \mathcal{F}^{V_0}(\mu)} = e^{-\beta \ \mathcal{F}(\mu - iV_0)}$$

So we have, writing out the $S[V_0]$:

$$G(\tau) = \frac{F(\tau)}{Z} \int dV_0 \ e^{-\frac{\beta}{4E_C}V_0^2 + i\beta N_0 V_0 - \beta \ \mathcal{F}(\mu - iV_0)} G_\alpha^{V_0}$$

which can be approximated using the stationary phase method

# Stationary phase approximation

When given an integral of the form

$$I = \lim_{\lambda \to \infty} \int_{-\infty}^{-\infty} e^{-\lambda f(x)}$$

The relevant contributions occur around minimum $x_0$ of $f$:

$$\left. \frac{\partial f}{\partial x} \right|_{x=x_0} = 0$$

Which is known as the *saddle point equation*

## Stationary phase approximation

In the case of our action, we have the exponent

$$-\frac{\beta}{4E_C}V_0^2 + i\beta N_0 V_0 - \beta \, \mathcal{F}(\mu - iV_0)$$

Which has a minimum (after some calculation)

$$0 = \frac{1}{2E_C}V_0 - iN_0 + i\left\langle \hat{N} \right\rangle_{\mu - iV_0}$$

where we used that

$$\frac{\partial}{\partial V_0}\mathcal{F}(\mu - iV_0) = -i\frac{\partial}{\partial \mu}\mathcal{F}(\mu - iV_0) = -i\left\langle \hat{N} \right\rangle_{\mu - iV_0}$$

where $\hat{N}$ is the number operator

$$\hat{N} \equiv \sum_\alpha a_\alpha^\dagger a_\alpha$$