NGUYÊN HÀM VÀ TÍCH PHÂN

BÀI 1. BÀI TẬP SỬ DỤNG CÔNG THỨC NGUYÊN HÀM, TÍCH PHÂN

I. Bảng công thức nguyên hàm mở rộng

$\int (ax+b)^{\alpha}\,dx = \frac{1}{a}\left(\frac{ax+b}{\alpha+1}\right)^{\alpha+1} + c, \alpha \neq -1$	$\int cos(ax+b)\,dx = \frac{1}{a} sin(ax+b) + c$
$\int \frac{dx}{ax+b} = \frac{1}{a} ln\lvert ax+b\rvert + c\ + c$	$\int sin(ax+b)\,dx = \frac{-1}{a} cos(ax+b) + c$
$\int e^{ax+b}\,dx = \frac{1}{a} e^{ax+b} + c$	$\int tg(ax+b)\,dx = -\frac{1}{a} ln\lvert cos(ax+b)\rvert + c$
$\int m^{ax+b}\,dx = \frac{1}{a\,ln\,m} m^{ax+b} + c$	$\int cotg(ax+b)\,dx = \frac{1}{a} ln\lvert sin(ax+b)\rvert + c$
$\int \frac{dx}{a^2+x^2} = \frac{1}{a} arctg\frac{x}{a} + c$	$\int \frac{dx}{sin^2(ax+b)} = \frac{-1}{a} cotg(ax+b) + c$
$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} ln\left\lvert\frac{a+x}{a-x}\right\rvert + c$	$\int \frac{dx}{cos^2(ax+b)} = \frac{1}{a} tg(ax+b) + c$
$\int \frac{dx}{\sqrt{x^2+a^2}} = ln\left(x+\sqrt{x^2+a^2}\right) + c$	$\int arcsin\frac{x}{a}\,dx = x\,arcsin\frac{x}{a} + \sqrt{a^2-x^2} + c$
$\int \frac{dx}{\sqrt{a^2-x^2}} = arcsin\frac{x}{\lvert a\rvert} + c$	$\int arccos\frac{x}{a}\,dx = x\,arccos\frac{x}{a} - \sqrt{a^2-x^2} + c$
$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} arccos\left\lvert\frac{x}{a}\right\rvert + c$	$\int arctg\frac{x}{a}\,dx = x\,arctg\frac{x}{a} - \frac{a}{2} ln\left(a^2+x^2\right) + c$
$\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a} ln\left\lvert\frac{a+\sqrt{x^2+a^2}}{x}\right\rvert + c$	$\int arc\,cotg\frac{x}{a}\,dx = x\,arc\,cotg\frac{x}{a} + \frac{a}{2} ln\left(a^2+x^2\right) + c$
$\int ln(ax+b)\,dx = \left(x+\frac{b}{a}\right) ln(ax+b) - x + c$	$\int \frac{dx}{sin(ax+b)} = \frac{1}{a} ln\left\lvert tg\frac{ax+b}{2}\right\rvert + c$
$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} arcsin\frac{x}{a} + c$	$\int \frac{dx}{sin(ax+b)} = \frac{1}{a} ln\left\lvert tg\frac{ax+b}{2}\right\rvert + c$
$\int e^{ax} sin\,bx\,dx = \frac{e^{ax}(a\,sin\,bx - b\,cos\,bx)}{a^2+b^2} + c$	$\int e^{ax} cos\,bx\,dx = \frac{e^{ax}(a\,cos\,bx + b\,sin\,bx)}{a^2+b^2} + c$

II. NHỮNG CHÚ Ý KHI SỬ DỤNG CÔNG THỨC KHÔNG CÓ TRONG SGK 12

Các công thức có mặt trong II. mà không có trong SGK 12 khi sử dụng phải chứng minh lại bằng cách trình bày dưới dạng bổ đề. Có nhiều cách chứng minh bổ đề nhưng cách đơn giản nhất là chứng minh bằng cách lấy đạo hàm

1. Ví dụ 1: Chứng minh: $\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$; $\int \frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c$

Chứng minh: $\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx=\frac{1}{2a}\left(\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right)=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$

$$\int \frac{dx}{a^2-x^2}=\frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx=\frac{1}{2a}\left(\int\frac{dx}{a+x}-\int\frac{d(a-x)}{a-x}\right)=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c$$

2. Ví dụ 2: Chứng minh rằng: $\int \frac{dx}{\sqrt{x^2+a^2}}=\ln\left(x+\sqrt{x^2+a^2}\right)+c$

Chứng minh: Lấy đạo hàm ta có: $\left[\ln\left(x+\sqrt{x^2+a^2}\right)+c\right]'=\frac{1+\left(\sqrt{x^2+a^2}\right)'}{x+\sqrt{x^2+a^2}}=$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\left(1+\frac{x}{\sqrt{x^2+a^2}}\right)=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}=\frac{1}{\sqrt{x^2+a^2}}$$

3. Ví dụ 3: Chứng minh rằng: $\int \frac{dx}{a^2+x^2}=\frac{1}{a}u+c$ (với $\operatorname{tg} u=\frac{x}{a}$)

Đặt $\operatorname{tg} u=\frac{x}{a},\ u\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\Rightarrow \int\frac{dx}{a^2+x^2}=\int\frac{d(a\operatorname{tg} u)}{a^2\left(1+\operatorname{tg}^2 u\right)}=\frac{1}{a}\int du=\frac{1}{a}u+c$

4. Ví dụ 4: Chứng minh rằng: $\int\frac{dx}{\sqrt{a^2-x^2}}=u+c$ (với $\sin u=\frac{x}{a}$, $a>0$)

Đặt $\sin u=\frac{x}{a}, u\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\Rightarrow\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{d(a\sin u)}{\sqrt{a^2\left(1-\sin^2 u\right)}}=\int du=u+c$

Bình luận: Trước năm 2001, SGK12 có cho sử dụng công thức nguyên hàm $\int\frac{dx}{a^2+x^2}=\frac{1}{a}\operatorname{arctg}\frac{x}{a}+c$ và $\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+c$ ($a>0$) nhưng sau đó không giống bất cứ nước nào trên thế giới, họ lại cấm không cho sử dụng khái niệm hàm ngược arctg x, arcsin x. Cách trình bày trên để khắc phục lệnh cấm này.

III. CÁC DẠNG TÍCH PHÂN ĐƠN GIẢN

III.1. CÁC KỸ NĂNG CƠ BẢN:

1. Biểu diễn luỹ thừa dạng chính tắc:

$$\sqrt[n]{x} = x^{\frac{1}{n}} \ ; \ \sqrt[n]{x^m} = x^{\frac{m}{n}} \ ; \ \sqrt[n]{\sqrt[k]{x^m}} = x^{\frac{m}{nk}}$$

$$\frac{1}{x^n} = x^{-n} \ ; \ \frac{1}{\sqrt[n]{x}} = x^{-\frac{1}{n}} \ ; \ \frac{1}{\sqrt[n]{x^m}} = x^{-\frac{m}{n}} \ ; \ \frac{1}{\sqrt[n]{\sqrt[k]{x^m}}} = x^{-\frac{m}{nk}}$$

2. Biến đổi vi phân:

$$dx = d(x \pm 1) = d(x \pm 2) = \ldots = d(x \pm p)$$

$$adx = d(ax \pm 1) = d(ax \pm 2) = \ldots = d(ax \pm p)$$

$$\frac{1}{a}dx = d\left(\frac{x \pm 1}{a}\right) = d\left(\frac{x \pm 2}{a}\right) = \cdots = d\left(\frac{x \pm p}{a}\right)$$

III.2. CÁC BÀI TẬP MẪU MINH HOẠ

1. $\int \frac{x^3}{x-1}\mathrm{dx} = \int \frac{(x^3-1)+1}{x-1}\mathrm{dx} = \int \left(x^2 + x + 1 + \frac{1}{x-1}\right)\mathrm{dx}$

$$= \int (x^2 + x + 1)\mathrm{dx} + \int \frac{d(x-1)}{x-1} = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln|x-1| + c$$

2. $\int x\sqrt{4x+7}\,\mathrm{dx} = \frac{1}{4}\int \left[(4x+7)-7\right]\sqrt{4x+7}\,\mathrm{dx}$

$$= \frac{1}{16}\int \left[(4x+7)^{\frac{3}{2}} - 7(4x+7)^{\frac{1}{2}}\right] d(4x+7) = \frac{1}{16}\left[\frac{2}{5}(4x+7)^{\frac{5}{2}} - 7\cdot\frac{2}{3}(4x+7)^{\frac{3}{2}}\right] + c$$

3. $I_{17} = \int \frac{\mathrm{d}x}{2x^2+5} = \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}(\sqrt{2}\,x)}{(\sqrt{2}\,x)^2 + (\sqrt{5})^2} = \frac{1}{\sqrt{10}}\mathrm{arctg}\left(\frac{\sqrt{10}}{5}x\right) + c$

4. $\int \frac{\mathrm{dx}}{2^x+5} = \frac{1}{\ln 2}\int \frac{d(2^x)}{2^x(2^x+5)} = \frac{1}{5\ln 2}\int \left(\frac{1}{2^x} - \frac{1}{2^x+5}\right) d(2^x) = \frac{1}{5\ln 2}\ln\left|\frac{2^x}{2^x+5}\right| + c$

5. $\int \frac{\cos^5 x}{1-\sin x}dx = \int \cos^3 x(1+\sin x)\,dx = \int \left[(1-\sin^2 x)\cos x + \cos^3 x \sin x\right]\mathrm{dx}$

$$= \int (1-\sin^2 x)\,d(\sin x) - \int \cos^3 x\,d(\cos x) = \sin x - \frac{\sin^3 x}{3} - \frac{\cos^4 x}{4} + c$$

III.3. CÁC BÀI TẬP DÀNH CHO BẠN ĐỌC TỰ GIẢI

$\mathrm{J}_1 = \int \frac{(\mathrm{x}+1)(\mathrm{x}+2)(\mathrm{x}+3)(\mathrm{x}+4)}{\mathrm{x}\sqrt{\mathrm{x}}}\mathrm{dx}$; $\mathrm{J}_2 = \int \frac{7\mathrm{x}-3}{2\mathrm{x}+5}\mathrm{dx}$; $\mathrm{J}_3 = \int \frac{3\mathrm{x}^2-7\mathrm{x}+5}{\mathrm{x}-2}\mathrm{dx}$

$\mathrm{J}_4 = \int \frac{2\mathrm{x}^3-5\mathrm{x}^2+7\mathrm{x}-10}{\mathrm{x}-1}\mathrm{dx}$; $\mathrm{J}_5 = \int \frac{4\mathrm{x}^2-9\mathrm{x}+10}{2\mathrm{x}-1}\mathrm{dx}$; $\mathrm{J}_6 = \int \frac{2\mathrm{x}^2-3\mathrm{x}+9}{(\mathrm{x}-1)^{10}}\mathrm{dx}$

$\mathrm{J}_7 = \int \frac{\mathrm{x}^3-3\mathrm{x}^2+4\mathrm{x}-9}{(\mathrm{x}-2)^{15}}\mathrm{dx}$; $\mathrm{J}_8 = \int \frac{2\mathrm{x}^3+5\mathrm{x}^2-11\mathrm{x}+4}{(\mathrm{x}+1)^{30}}\mathrm{dx}$

$\mathrm{J}_9 = \int (\mathrm{x}+3)^{100}(\mathrm{x}-1)^3\mathrm{dx}$; $\mathrm{J}_{10} = \int (\mathrm{x}-1)^2(5\mathrm{x}+2)^{15}\mathrm{dx}$; $\mathrm{J}_{11} = \int (\mathrm{x}^2+3\mathrm{x}-5)(2\mathrm{x}-1)^{33}\mathrm{dx}$

$$J_{12}=\int\left(2x^2+3\right).\sqrt[5]{\left(x-1\right)^3}\,dx\ ;\ J_{13}=\int\frac{x^2-3x+5}{\sqrt[7]{\left(2x+1\right)^4}}dx\ ;\ J_{14}=\int x^4.\sqrt[9]{\left(2x^5+3\right)^4}\,dx$$

$$J_{15}=\int\frac{x^9}{\sqrt[5]{\left(2-3x^{10}\right)^4}}dx\ ;\ J_{16}=\int\frac{x}{x+\sqrt{x^2-1}}dx\ ;\ J_{17}=\int\frac{x^3}{x-\sqrt{x^2-1}}dx$$

$$J_{18}=\int\frac{dx}{\left(x-2\right)\left(x+5\right)}\ ;\ J_{19}=\int\frac{dx}{\left(x^2+2\right)\left(x^2+6\right)}\ ;\ J_{20}=\int\frac{dx}{\left(x^2-2\right)\left(x^2+3\right)}$$

$$J_{21}=\int\frac{x\,dx}{\left(x^2-3\right)\left(x^2-7\right)}\ ;\ J_{22}=\int\frac{dx}{\left(3x^2+7\right)\left(x^2+2\right)}\ ;\ J_{23}=\int\frac{dx}{\left(2x^2+5\right)\left(x^2-3\right)}$$

$$J_{24}=\int_1^{\ln 2}\frac{dx}{\sqrt{e^x-1}}\ ;\ J_{25}=\int_0^{\ln 2}\frac{e^{2x}\,dx}{\sqrt{e^x+1}}\ ;\ J_{26}=\int_0^{\ln 2}\sqrt{e^x+1}\,dx\ ;\ J_{27}=\int_0^{\ln 2}\frac{1-e^x}{1+e^x}dx$$

$$J_{28}=\int_0^1\frac{e^{-x}\,dx}{1+e^{-x}}\ ;\ J_{29}=\int_0^1\frac{\left(1+e^x\right)^2dx}{1+e^{2x}}\ ;\ J_{30}=\int_0^1\frac{dx}{e^{2x}+e^x}\ ;\ J_{31}=\int_0^1\frac{\left(1+e^x\right)^2}{e^{3x}}dx$$

$$J_{32}=\int_0^{\ln 2}\frac{dx}{e^{x+3}}\ ;\ J_{33}=\int_0^{\ln 4}\frac{dx}{e^x-4e^{-x}}\ ;\ J_{34}=\int_0^1\frac{e^{-3x}\,dx}{1+e^{-x}}\ ;\ J_{35}=\int_1^e\frac{\sqrt{1+\ln x}}{x}dx$$

$$J_{36}=\int_0^{\sqrt{3}}x^5\sqrt{1+x^2}\,dx\ ;\ J_{37}=\int_0^1x^5\left(1-x^3\right)^6dx\ ;\ J_{38}=\int_0^1x^3\sqrt{1-x^2}\,dx$$

$$J_{39}=\int_0^1\frac{dx}{4^x+3}\ ;\ J_{40}=\int_0^1\frac{dx}{4^x+2^{-x}}\ ;\ J_{41}=\int_0^1\frac{\left(2^x+1\right)^2dx}{4^{-x}}\ ;\ J_{42}=\int_0^1e^{2x}\sqrt{1+e^x}\,dx$$

BÀI 2. TÍCH PHÂN CÁC HÀM SỐ CÓ MẪU SỐ CHỨA TAM THỨC BẬC 2

A. CÔNG THỨC SỬ DỤNG VÀ KỸ NĂNG BIẾN ĐỔI

1. $\displaystyle\int\frac{du}{u^2+a^2}=\frac{1}{a}arctg\frac{u}{a}+c$

2. $\displaystyle\int\frac{du}{u^2-a^2}=\frac{1}{2a}ln\left|\frac{u-a}{u+a}\right|+c$

3. $\displaystyle\int\frac{du}{a^2-u^2}=\frac{1}{2a}ln\left|\frac{a+u}{a-u}\right|+c$

4. $\displaystyle\int\frac{du}{\sqrt{u}}=2\sqrt{u}+c$

5. $\displaystyle\int\frac{du}{\sqrt{a^2-u^2}}=arcsin\frac{u}{a}+c\ \left(a>0\right)$

6. $\displaystyle\int\frac{du}{\sqrt{u^2\pm p}}=ln\left|u+\sqrt{u^2\pm p}\right|+c$

Kỹ năng biến đổi tam thức bậc 2:

1. $ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$ **2.** $ax^2+bx+c=\pm\left(mx+n\right)^2\pm p^2$

B. CÁC DẠNG TÍCH PHÂN

I. Dạng 1: $\boldsymbol{A = \int \frac{dx}{ax^2 + bx + c}}$

1. Phương pháp: $\int \frac{dx}{ax^2 + bx + c} = \int \frac{dx}{(mx+n)^2 + p^2} = \frac{1}{mp} arctg \frac{mx+n}{p} + c$

$$\int \frac{dx}{ax^2 + bx + c} = \int \frac{dx}{(mx+n)^2 - p^2} = \frac{1}{2mp} \ln\left|\frac{mx+n-p}{mx+n+p}\right| + c$$

2. Các bài tập mẫu minh họa

• $A_1 = \int \frac{dx}{4x^2 + 8x + 1} = \int \frac{dx}{(2x+2)^2 - 3} = \frac{1}{2}\int \frac{d(2x+2)}{(2x+2)^2 - (\sqrt{3})^2} = \frac{1}{4\sqrt{3}} \ln\left|\frac{2x+2-\sqrt{3}}{2x+2+\sqrt{3}}\right| + c$

3. Các bài tập dành cho bạn đọc tự giải:

$A_1 = \int \frac{dx}{3x^2 - 4x - 2}$; $A_2 = \int \frac{dx}{-4x^2 + 6x + 1}$; $A_3 = \int \frac{dx}{5x^2 - 8x + 6}$;

$A_4 = \int_1^2 \frac{dx}{7x^2 - 4x + 3}$; $A_5 = \int_0^1 \frac{dx}{6 - 3x + 2x^2}$; $A_6 = \int_0^1 \frac{dx}{4x^2 - 6x + 3}$

II. Dạng 2: $\boldsymbol{B = \int \frac{(mx+n)}{ax^2 + bx + c} dx}$

1. Phương pháp: $B = \int \frac{(mx+n)}{ax^2 + bx + c} dx = \int \frac{\frac{m}{2a}(2ax+b) + \left(n - \frac{mb}{2a}\right)}{ax^2 + bx + c} dx =$

$$= \frac{m}{2a} \int \frac{d\left(ax^2 + bx + c\right)}{ax^2 + bx + c} + \left(n - \frac{mb}{2a}\right) A = \frac{m}{2a} ln\left|ax^2 + bx + c\right| + \left(n - \frac{mb}{2a}\right) A$$

Cách 2: Phương pháp hệ số bất định (sử dụng khi mẫu có nghiệm)

• Nếu mẫu có nghiệm kép $x = x_0$ tức là $ax^2 + bx + c = a(x - x_0)^2$

thì ta giả sử: $\frac{mx+n}{ax^2 + bx + c} = \frac{\alpha}{x - x_0} + \frac{\beta}{(x - x_0)^2} \quad \forall x$

Quy đồng vế phải và đồng nhất hệ số ở hai vế để tìm α, β.

Với α, β vừa tìm ta có: $B = \int \frac{(mx+n)}{ax^2 + bx + c} dx = \alpha \ln|x - x_0| - \frac{\beta}{x - x_0} + c$

• Nếu mẫu có 2 nghiệm phân biệt x_1, x_2: $ax^2 + bx + c = a(x - x_1)(x - x_2)$ thì ta giả sử

$\frac{mx+n}{ax^2 + bx + c} = \frac{\alpha}{x - x_1} + \frac{\beta}{x - x_2} \quad \forall x$

Quy đồng vế phải và đồng nhất hệ số ở hai vế để tìm α, β.

Với α, β vừa tìm ta có: $B=\int \frac{(mx+n)}{ax^2+bx+c}\,\mathrm{dx} = \alpha\ln|x-x_1|+\beta\ln|x-x_2|+c$

2. Các bài tập mẫu minh họa:

• $B_1=\int \frac{2x+3}{9x^2-6x+1}dx = \int \frac{\frac{1}{9}(18x-6)+\frac{11}{3}}{9x^2-6x+1}\,d\,x = \frac{1}{9}\int \frac{(18x-6)\,d\,x}{9x^2-6x+1}+\frac{11}{3}\int \frac{d\,x}{9x^2-6x+1}$

$=\frac{1}{9}\int \frac{d\left(9x^2-6x+1\right)}{9x^2-6x+1}+\frac{11}{9}\int \frac{d(3x-1)}{(3x-1)^2}=\frac{2}{9}\ln|3x-1|-\frac{11}{9(3x-1)}+c$

3. Các bài tập dành cho bạn đọc tự giải:

$B_1=\int \frac{(7-3x)dx}{4x^2-6x-1}$; $B_2=\int \frac{(3x-4)dx}{2x^2-7x+9}$; $B_3=\int \frac{(2-7x)dx}{5x^2-8x-4}$;

III. Dạng 3: $C=\int \frac{dx}{\sqrt{ax^2+bx+c}}$

1. Phương pháp: Bổ đề: $\int \frac{du}{\sqrt{u^2+k}}=\ln\left|u+\sqrt{u^2+k}\right|+c$

Biến đổi nguyên hàm về 1 trong 2 dạng sau:

$$C=\int \frac{\mathrm{dx}}{\sqrt{ax^2+bx+c}}=\int \frac{\mathrm{dx}}{\sqrt{(mx+n)^2+k}}=\frac{1}{m}\ln\left|(mx+n)+\sqrt{(mx+n)^2+k}\right|+c$$

$$C=\int \frac{\mathrm{dx}}{\sqrt{ax^2+bx+c}}=\int \frac{\mathrm{dx}}{\sqrt{p^2-(mx+n)^2}}=\frac{1}{m}\arcsin\frac{mx+n}{p}\quad (p>0)$$

2. Các bài tập mẫu minh họa:

• $C_3=\int \frac{d\,x}{\sqrt{4x^2-10x-5}}=\frac{1}{2}\int \frac{d\,x}{\sqrt{\left(x-\frac{5}{4}\right)^2-\frac{45}{16}}}=\ln\left|x-\frac{5}{4}+\sqrt{\left(x-\frac{5}{4}\right)^2-\frac{45}{16}}\right|+c$

3. Các bài tập dành cho bạn đọc tự giải:

$C_1=\int \frac{dx}{\sqrt{3x^2-8x+1}}$; $C_2=\int \frac{dx}{\sqrt{7-8x-10x^2}}$; $C_3=\int \frac{dx}{\sqrt{5-12x-4\sqrt{2}\,x^2}}$

IV. Dạng 4: $D=\int \frac{(mx+n)dx}{\sqrt{ax^2+bx+c}}$

1. Phương pháp:

$$D=\frac{m}{2a}\int \frac{(2ax+b)\mathrm{dx}}{\sqrt{ax^2+bx+c}}-\frac{mb}{2a}\int \frac{\mathrm{dx}}{\sqrt{ax^2+bx+c}}=\frac{m}{2a}\int \frac{d\left(ax^2+bx+c\right)}{\sqrt{ax^2+bx+c}}-\frac{mb}{2a}\cdot C$$

2. Các bài tập mẫu minh họa:

• $D_1=\int_0^1\frac{(x+4)\,dx}{\sqrt{x^2+4x+5}}=\int_0^1\frac{(x+2)\,dx}{\sqrt{x^2+4x+5}}+2\int_0^1\frac{dx}{\sqrt{x^2+4x+5}}$

$$=\frac{1}{2}\int_0^1\frac{d\left(x^2+4x+5\right)}{\sqrt{x^2+4x+5}}+2\int_0^1\frac{dx}{\sqrt{(x+2)^2+1}}=\left(\sqrt{x^2+4x+5}+2\ln\left|(x+2)+\sqrt{x^2+4x+5}\right|\right)\Bigg|_0^1$$

$$=\sqrt{10}-\sqrt{5}+2\ln\left(3+\sqrt{10}\right)-2\ln\left(2+\sqrt{5}\right)=\sqrt{10}-\sqrt{5}+2\ln\frac{3+\sqrt{10}}{2+\sqrt{5}}$$

3. Các bài tập dành cho bạn đọc tự giải:

$$D_1=\int\frac{(5-4x)dx}{\sqrt{3x^2-2x+1}}\ ;\ D_2=\int\frac{(3x+7)dx}{\sqrt{2x^2-5x-1}}\ ;\ D_3=\int\frac{(8x-11)dx}{\sqrt{9-6x-4x^2}}$$

V. Dạng 5: $\boldsymbol{E=\int\frac{dx}{(px+q)\sqrt{ax^2+bx+c}}}$

1. Phương pháp: Đặt $px+q=\frac{1}{t}\Rightarrow p\,dx=\frac{-dt}{t^2};x=\frac{1}{p}\left(\frac{1}{t}-q\right)$. Khi đó:

$$E=\int\frac{dx}{(px+q)\sqrt{ax^2+bx+c}}=\int\frac{-dt/pt^2}{\frac{1}{t}\sqrt{\frac{a}{p^2}\left(\frac{1}{t}-q\right)^2+\frac{b}{p}\left(\frac{1}{t}-q\right)+c}}=\pm\int\frac{dt}{\sqrt{\alpha t^2+\beta t+\gamma}}$$

2. Các bài tập mẫu minh họa:

• $\boldsymbol{E_1=\int_2^3\frac{dx}{(x-1)\sqrt{x^2-2x+2}}}$. Đặt $x-1=\frac{1}{t}\Rightarrow x=\frac{t+1}{t};\begin{cases}x=2\Rightarrow t=1\\ x=3\Rightarrow t=\frac{1}{2}\\ dx=\frac{-dt}{t^2}\end{cases}$

Khi đó: $E_1=\int_2^3\frac{dx}{(x-1)\sqrt{x^2-2x+2}}=\int_1^{1/2}\frac{-dt/t^2}{\frac{1}{t}\sqrt{\left(\frac{t+1}{t}\right)^2-2\left(\frac{t+1}{t}\right)+2}}$

$$=\int_{1/2}^1\frac{dt}{\sqrt{t^2+1}}=\ln\left|t+\sqrt{t^2+1}\right|\Bigg|_{1/2}^1=\ln\left(1+\sqrt{2}\right)-\ln\frac{1+\sqrt{5}}{2}=\ln\frac{2+2\sqrt{2}}{1+\sqrt{5}}$$

3. Các bài tập dành cho bạn đọc tự giải:

$$E_1=\int_1^2\frac{dx}{(2x+3)\sqrt{x^2+3x-1}}\ ;\ E_2=\int_2^3\frac{dx}{(3x-4)\sqrt{2x^2+3x+7}}\ ;\ E_3=\int_2^3\frac{dx}{(x-1)\sqrt{x^2+1}}$$

VI. Dạng 6: $\boldsymbol{F=\int\frac{(mx+n)dx}{(px+q)\sqrt{ax^2+bx+c}}}$

1. Phương pháp: $F=\int\frac{(mx+n)\,\mathrm{dx}}{(px+q)\sqrt{ax^2+bx+c}}=\int\frac{\frac{m}{p}(px+q)+\left(n-\frac{mq}{p}\right)}{(px+q)\sqrt{ax^2+bx+c}}\,\mathrm{dx}$

$$F=\frac{m}{p}\int\frac{\mathrm{dx}}{\sqrt{ax^2+bx+c}}+\left(n-\frac{mq}{p}\right)\int\frac{\mathrm{dx}}{(px+q)\sqrt{ax^2+bx+c}}=\frac{m}{p}C+\left(n-\frac{mq}{p}\right)E$$

2. Các bài tập mẫu minh họa:

$$F_1=\int_0^1\frac{(2x+3)\,\mathrm{d}x}{(x+1)\sqrt{x^2+2x+2}}=2\int_0^1\frac{\mathrm{dx}}{\sqrt{\mathrm{x}^2+2\mathrm{x}+2}}+\int_0^1\frac{\mathrm{dx}}{(\mathrm{x}+1)\sqrt{\mathrm{x}^2+2\mathrm{x}+2}}=2\mathrm{I}+\mathrm{J}$$

$$I=\int_0^1\frac{\mathrm{dx}}{\sqrt{x^2+2x+2}}=\int_0^1\frac{\mathrm{dx}}{\sqrt{(x+1)^2+1}}=\ln\left|(x+1)+\sqrt{(x+1)^2+1}\right|\Big|_0^1=\ln\frac{2+\sqrt{5}}{1+\sqrt{2}}$$

$J=\int_0^1\frac{dx}{(x+1)\sqrt{x^2+2x+2}}$. Đặt $x+1=\frac{1}{t}\Rightarrow\begin{cases}x=0\Rightarrow t=1\\ x=1\Rightarrow t=\frac{1}{2}\\ \mathrm{dx}=\frac{-dt}{t^2}\end{cases}$. Khi đó:

$$\mathrm{J}=\int_1^{1/2}\frac{-\mathrm{dt}/\mathrm{t}^2}{\frac{1}{\mathrm{t}}\sqrt{\left(\frac{1}{\mathrm{t}}-1\right)^2+2\left(\frac{1}{\mathrm{t}}-1\right)+2}}=\int_{1/2}^1\frac{\mathrm{dt}}{\sqrt{\mathrm{t}^2+1}}=\ln\left|\mathrm{t}+\sqrt{\mathrm{t}^2+1}\right|\Big|_{1/2}^1=\ln\frac{2+2\sqrt{2}}{1+\sqrt{5}}$$

$$\Rightarrow \mathrm{F}_1=2\mathrm{I}+\mathrm{J}=2\ln\frac{2+\sqrt{5}}{1+\sqrt{2}}+\ln\frac{2+2\sqrt{2}}{1+\sqrt{5}}=\ln\frac{2\left(9+4\sqrt{5}\right)}{\left(1+\sqrt{2}\right)\left(1+\sqrt{5}\right)}$$

• $$\boldsymbol{F_2=\int_{-2}^{-3/2}\frac{(x+3)\,dx}{(2x+1)\sqrt{-x^2-4x-3}}}=\int_{-2}^{-3/2}\frac{\frac{1}{2}(2x+1)+\frac{5}{2}}{(2x+1)\sqrt{-x^2-4x-3}}\,dx$$

$$=\frac{1}{2}\int_{-2}^{-3/2}\frac{\mathrm{dx}}{\sqrt{-\mathrm{x}^2-4\mathrm{x}-3}}+\frac{5}{2}\int_{-2}^{-3/2}\frac{\mathrm{dx}}{(2\mathrm{x}+1)\sqrt{-\mathrm{x}^2-4\mathrm{x}-3}}=\frac{1}{2}\mathrm{I}+\frac{5}{2}\mathrm{J}$$

$$I=\int_{-2}^{-3/2}\frac{dx}{\sqrt{-x^2-4x-3}}=\int_{-2}^{-3/2}\frac{\mathrm{dx}}{\sqrt{1-(\mathrm{x}+2)^2}}=\arcsin(\mathrm{x}+2)\Big|_{-2}^{-3/2}=\frac{\pi}{6}$$

$J=\int_{-2}^{-3/2}\frac{dx}{(2x+1)\sqrt{-x^2-4x-3}}$. Đặt $2x+1=\frac{1}{t}\Rightarrow x=\frac{1-t}{2t};\begin{cases}x=-2\Rightarrow t=\frac{-1}{3}\\ x=\frac{-3}{2}\Rightarrow t=\frac{-1}{2}\\ 2\,dx=\frac{-dt}{t^2}\end{cases}$

$$\mathrm{J}=\int_{-1/3}^{-1/2}\frac{-\mathrm{dt}/2\mathrm{t}^2}{\frac{1}{\mathrm{t}}\sqrt{\frac{-1}{4}\left(1-\frac{1}{\mathrm{t}}\right)^2-2\left(\frac{1}{\mathrm{t}}-1\right)-3}}=\int_{-1/2}^{-1/3}\frac{\mathrm{dt}}{\sqrt{-5\mathrm{t}^2-6\mathrm{t}-1}}$$

$$=\frac{1}{\sqrt{5}}\int_{-1/2}^{-1/3}\frac{\mathrm{dt}}{\sqrt{\left(\frac{2}{5}\right)^2-\left(\mathrm{t}+\frac{3}{5}\right)^2}}=\frac{1}{\sqrt{5}}\arcsin\frac{5\mathrm{t}+3}{2}\Bigg|_{-1/2}^{-1/3}=\frac{1}{\sqrt{5}}\left(\arcsin\frac{2}{3}-\arcsin\frac{1}{4}\right)$$

Vậy $F_2 = \frac{1}{2}I + \frac{5}{2}J = \frac{\pi}{12} + \frac{\sqrt{5}}{2}\left(\arcsin\frac{2}{3} - \arcsin\frac{1}{4}\right)$

3. Các bài tập dành cho bạn đọc tự giải:

$$F_1 = \int_0^1 \frac{(4x+7)\,dx}{(8-5x)\sqrt{3x^2-4x+2}};\ F_2 = \int_0^1 \frac{(6-7x)\,dx}{(2x+5)\sqrt{x^2-x+4}};\ F_3 = \int_0^1 \frac{(7-9x)\,dx}{(4x+3)\sqrt{2x^2+x+1}}$$

VII. Dạng 7: $G = \int \frac{xdx}{(ax^2+b)\sqrt{cx^2+d}}$

1. Phương pháp: Đặt $t = \sqrt{cx^2+d} \Rightarrow t^2 = cx^2 + d \Rightarrow x^2 = \frac{t^2-d}{c};\ x\,dx = \frac{t\,dt}{c}$

Khi đó: $G = \frac{1}{c}\int \frac{t\,dt}{\left[\frac{a(t^2-d)}{c} + b\right]t} = \frac{1}{c^2}\int \frac{dt}{at^2 + (bc-ad)} = \frac{1}{c^2}\cdot A$

2. Các bài tập mẫu minh họa:

• $G_1 = \int_0^1 \frac{xdx}{(5-2x^2)\sqrt{6x^2+1}}$. Đặt $t = \sqrt{6x^2+1} \Rightarrow \begin{cases} x=0 \Rightarrow t=1 \\ x=1 \Rightarrow t=\sqrt{7} \\ 6x\,dx = t\,dt \end{cases}$. Khi đó:

$$G_1 = \frac{1}{6}\int_1^{\sqrt{7}} \frac{t\,dt}{\left(\frac{16-t^2}{3}\right)t} = \frac{1}{2}\int_1^{\sqrt{7}} \frac{dt}{4^2-t^2} = \frac{1}{2}\left(\frac{1}{8}\ln\frac{4+t}{4-t}\right)\Bigg|_1^{\sqrt{7}} = \frac{1}{16}\ln\frac{3\left(4+\sqrt{7}\right)}{5\left(4-\sqrt{7}\right)}$$

3. Các bài tập dành cho bạn đọc tự giải:

$$G_1 = \int_1^2 \frac{x\,dx}{(4x^2-3)\sqrt{5-x^2}};\ G_2 = \int_1^{\sqrt{2}} \frac{x\,dx}{(5x^2-11)\sqrt{7-3x^2}};\ G_3 = \int_0^1 \frac{x\,dx}{(8-7x^2)\sqrt{2x^2+1}}$$

VIII. Dạng 8: $H = \int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$

1. Phương pháp:

Đặt $xt = \sqrt{cx^2+d} \Rightarrow x^2t^2 = cx^2 + d \Rightarrow x^2 = \frac{d}{t^2-c} \Rightarrow xdx = \frac{-td.dt}{(t^2-c)^2}$

$\Rightarrow \frac{dx}{\sqrt{cx^2+d}} = \frac{xdx}{x(xt)} = \frac{-td.dt/(t^2-c)^2}{td/(t^2-c)} = \frac{-dt}{t^2-c}$. Khi đó ta có:

$$H = \int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}} = \int \frac{-dt}{\left(\frac{ad}{t^2-c} + b\right)(t^2-c)} = \int \frac{-dt}{bt^2 + (ad-bc)} = A$$

2. Các bài tập mẫu minh họa:

• $\boldsymbol{H_1 = \int_2^3 \frac{dx}{\left(x^2 - 2\right)\sqrt{x^2 + 3}}}$. Đặt $xt = \sqrt{x^2+3} \Rightarrow t = \frac{\sqrt{x^2+3}}{x} \Rightarrow \begin{cases} x = 3 \Rightarrow t = \frac{2}{3} \\ x = 2 \Rightarrow t = \frac{\sqrt{7}}{2} \end{cases}$

và $x^2t^2 = x^2 + 3 \Rightarrow \left(t^2-1\right)x^2 = 3 \Rightarrow x^2 = \frac{3}{t^2-1} \Rightarrow x\,dx = \frac{-3t\,dt}{\left(t^2-1\right)^2}$

$\frac{dx}{\sqrt{x^2+3}} = \frac{x\,dx}{x\left(xt\right)} = \frac{-3t\,dt\big/\left(t^2-1\right)^2}{3t\big/\left(t^2-1\right)} = \frac{-dt}{t^2-1}$. Khi đó ta có:

$$H_1 = \int_{\sqrt{7}/2}^{2/\sqrt{3}} \frac{dt}{2t^2-5} = \frac{1}{2\sqrt{10}}\ln\left|\frac{t\sqrt{2}-\sqrt{5}}{t\sqrt{2}+\sqrt{5}}\right|\Bigg|_{\sqrt{7}/2}^{2/\sqrt{3}} = \frac{1}{2\sqrt{10}}\ln\frac{\left(2\sqrt{2}-\sqrt{15}\right)\left(\sqrt{14}+2\sqrt{5}\right)}{\left(2\sqrt{2}+\sqrt{15}\right)\left(\sqrt{14}-2\sqrt{5}\right)}$$

3. Các bài tập dành cho bạn đọc tự giải:

$$H_1 = \int_1^2 \frac{dx}{\left(3x^2-1\right)\sqrt{5x^2-2}}\ ;\ H_2 = \int_1^2 \frac{dx}{\left(x^2+3x+2\right)\sqrt{x^2+3x-1}}\ ;\ H_3 = \int_1^2 \frac{\sqrt{x^2+5}}{x^2+2}\,dx$$

IX. Dạng 9: $\boldsymbol{I = \int \frac{\left(mx+n\right)dx}{\left(ax^2+b\right)\sqrt{cx^2+d}}}$

1. Phương pháp: $I = m\int \frac{xdx}{\left(ax^2+b\right)\sqrt{cx^2+d}} + n\int \frac{dx}{\left(ax^2+b\right)\sqrt{cx^2+d}} = mG + nH$

2. Các bài tập mẫu minh họa:

• $\boldsymbol{I_1 = \int_2^3 \frac{\left(4x+3\right)dx}{\left(x^2 - 2x - 4\right)\sqrt{3x^2 - 6x + 5}}} = \int_2^3 \frac{\left[4\left(x-1\right)+7\right]dx}{\left[\left(x-1\right)^2-5\right]\sqrt{3\left(x-1\right)^2+2}}$

$$= \int_1^2 \frac{\left(4u+7\right)du}{\left(u^2-5\right)\sqrt{3u^2+2}} = 4\int_1^2 \frac{udu}{\left(u^2-5\right)\sqrt{3u^2+2}} + 7\int_1^2 \frac{du}{\left(u^2-5\right)\sqrt{3u^2+2}} = 4J - 7L$$

Xét $J = \int_1^2 \frac{udu}{\left(u^2-5\right)\sqrt{3u^2+2}}$. Đặt $t = \sqrt{3u^2+2} \Rightarrow u^2 = \frac{t^2-2}{3} \Rightarrow udu = \frac{tdt}{3}$

$$J = \int_1^2 \frac{udu}{\left(u^2-5\right)\sqrt{3u^2+2}} = \int_{\sqrt{5}}^{\sqrt{14}} \frac{tdt}{\left(t^2-17\right)t} = \int_{\sqrt{5}}^{\sqrt{14}} \frac{dt}{t^2-17} = \frac{1}{2\sqrt{17}}\ln\left|\frac{t-\sqrt{17}}{t+\sqrt{17}}\right|\Bigg|_{\sqrt{5}}^{\sqrt{14}}$$

$$= \frac{1}{2\sqrt{17}}\left(\ln\frac{\sqrt{17}-\sqrt{14}}{\sqrt{17}+\sqrt{14}} - \ln\frac{\sqrt{17}-\sqrt{5}}{\sqrt{17}+\sqrt{5}}\right) = \frac{1}{2\sqrt{17}}\ln\frac{\left(\sqrt{17}-\sqrt{14}\right)\left(\sqrt{17}+\sqrt{5}\right)}{\left(\sqrt{17}+\sqrt{14}\right)\left(\sqrt{17}-\sqrt{5}\right)}$$

Xét $L = \int_1^2 \frac{du}{\left(u^2-5\right)\sqrt{3u^2+2}}$. Đặt $ut = \sqrt{3u^2+2} \Rightarrow u^2t^2 = 3u^2+2 \Rightarrow u^2 = \frac{2}{t^2-3}$

$\Rightarrow\ udu = \frac{-2tdt}{\left(t^2-3\right)^2} \Rightarrow \frac{du}{\sqrt{3u^2+2}} = \frac{udu}{u\left(ut\right)} = \frac{-2tdt\big/\left(t^2-3\right)^2}{2t\big/\left(t^2-3\right)} = \frac{dt}{t^2-3}$. Khi đó:

$$L=\int_1^2\frac{du}{(u^2-5)\sqrt{3u^2+2}}=\int_2^{\sqrt{14}/2}\frac{dt}{\left(\frac{2}{t^2-3}-5\right)(t^2-3)}=\int_2^{\sqrt{14}/2}\frac{dt}{17-5t^2}=$$

$$=\frac{1}{\sqrt{5}}\cdot\frac{1}{2\sqrt{17}}\ln\left|\frac{\sqrt{17}+t\sqrt{5}}{\sqrt{17}-t\sqrt{5}}\right|\Bigg|_2^{\sqrt{14}/2}=\frac{1}{2\sqrt{85}}\ln\frac{\left(\sqrt{70}+2\sqrt{17}\right)\left(2\sqrt{5}-\sqrt{17}\right)}{\left(\sqrt{70}-2\sqrt{17}\right)\left(2\sqrt{5}+\sqrt{17}\right)}$$

$$\Rightarrow\ I_1=4J-7L=\frac{4}{2\sqrt{17}}\ln\frac{\left(\sqrt{17}-\sqrt{14}\right)\left(\sqrt{17}+\sqrt{5}\right)}{\left(\sqrt{17}+\sqrt{14}\right)\left(\sqrt{17}-\sqrt{5}\right)}-\frac{7}{2\sqrt{85}}\ln\frac{\left(\sqrt{70}+2\sqrt{17}\right)\left(2\sqrt{5}-\sqrt{17}\right)}{\left(\sqrt{70}-2\sqrt{17}\right)\left(2\sqrt{5}+\sqrt{17}\right)}$$

• $$\boldsymbol{I_2=\int_{\sqrt{2}-1}^{\sqrt{6}-1}\frac{(2x+1)dx}{(x^2+2x+6)\sqrt{2x^2+4x-1}}}=\int_{\sqrt{2}-1}^{\sqrt{6}-1}\frac{[2(x+1)-1]dx}{\left[(x+1)^2+5\right]\sqrt{2(x+1)^2-3}}$$

$$=\int_{\sqrt{2}}^{\sqrt{6}}\frac{(2u-1)du}{(u^2+5)\sqrt{2u^2-3}}=2\int_{\sqrt{2}}^{\sqrt{6}}\frac{udu}{(u^2+5)\sqrt{2u^2-3}}-\int_{\sqrt{2}}^{\sqrt{6}}\frac{du}{(u^2+5)\sqrt{2u^2-3}}=2J-L$$

Xét $J=\int_{\sqrt{2}}^{\sqrt{6}}\frac{udu}{(u^2+5)\sqrt{2u^2-3}}$. Đặt $t=\sqrt{2u^2-3}\Rightarrow u^2=\frac{t^2+3}{2}\Rightarrow udu=\frac{tdt}{2}$

$$J=\int_{\sqrt{2}}^{\sqrt{6}}\frac{udu}{(u^2+5)\sqrt{2u^2-3}}=\int_1^3\frac{tdt}{(t^2+13)t}=\int_1^3\frac{dt}{t^2+13}=\frac{2}{\sqrt{13}}\left(\operatorname{arctg}\frac{3}{\sqrt{13}}-\operatorname{arctg}\frac{1}{\sqrt{13}}\right)$$

Xét $L=\int_{\sqrt{2}}^{\sqrt{6}}\frac{du}{(u^2+5)\sqrt{2u^2-3}}$. Đặt $ut=\sqrt{2u^2-3}\Rightarrow u^2t^2=2u^2-3\Rightarrow u^2=\frac{3}{2-t^2}$

$\Rightarrow\ udu=\frac{3tdt}{(2-t^2)^2}\Rightarrow\ \frac{du}{\sqrt{2u^2-3}}=\frac{udu}{u(ut)}=\frac{3tdt/(2-t^2)^2}{3t/(2-t^2)}=\frac{dt}{2-t^2}$. Khi đó:

$$L=\int_{\sqrt{2}}^{\sqrt{6}}\frac{du}{(u^2+5)\sqrt{2u^2-3}}=\int_{1/\sqrt{2}}^{3/\sqrt{6}}\frac{dt}{\left(\frac{3}{2-t^2}+5\right)(2-t^2)}=\int_{1/\sqrt{2}}^{3/\sqrt{6}}\frac{dt}{13-5t^2}=\frac{1}{5}\int_{1/\sqrt{2}}^{3/\sqrt{6}}\frac{dt}{\frac{13}{5}-t^2}$$

$$=\frac{1}{5}\cdot\frac{1}{2\sqrt{13/5}}\ln\left|\frac{\sqrt{13/5}+t}{\sqrt{13/5}-t}\right|\Bigg|_{1/\sqrt{2}}^{3/\sqrt{6}}=\frac{1}{2\sqrt{65}}\left(\ln\frac{\sqrt{78}+3\sqrt{5}}{\sqrt{78}-3\sqrt{5}}-\ln\frac{\sqrt{26}-\sqrt{5}}{\sqrt{26}+\sqrt{5}}\right)$$

$$I_2=2J-L=\frac{4}{\sqrt{13}}\left(\operatorname{arctg}\frac{3}{\sqrt{13}}-\operatorname{arctg}\frac{1}{\sqrt{13}}\right)-\frac{1}{2\sqrt{65}}\ln\frac{\left(\sqrt{78}+3\sqrt{5}\right)\left(\sqrt{26}+\sqrt{5}\right)}{\left(\sqrt{78}-3\sqrt{5}\right)\left(\sqrt{26}-\sqrt{5}\right)}$$

BÀI 3. BIẾN ĐỔI VÀ ĐỔI BIẾN NÂNG CAO
TÍCH PHÂN HÀM PHÂN THỨC HỮU TỈ

I. DẠNG 1: TÁCH CÁC MẪU SỐ CHỨA CÁC NHÂN TỬ ĐỒNG BẬC

Các bài tập mẫu minh họa:

• $\boldsymbol{A_1 = \int \frac{dx}{(x-2)(x+5)}} = \frac{1}{7}\int \frac{(x+5)-(x-2)}{(x-2)(x+5)}dx = \frac{1}{7}\int\left(\frac{1}{x-5}-\frac{1}{x+5}\right)dx = \frac{1}{7}\ln\left|\frac{x-2}{x+5}\right| + c$

•$\boldsymbol{A_2 = \int \frac{dx}{(x-5)(x+2)(x+4)}} = \frac{1}{9}\int \frac{(x+4)-(x-5)}{(x-5)(x+2)(x+4)}dx$

$= \frac{1}{9}\int\left[\frac{1}{(x-5)(x+2)} - \frac{1}{(x+2)(x+4)}\right]dx = \frac{1}{63}\int \frac{(x+2)-(x-5)}{(x-5)(x+2)}dx - \frac{1}{18}\int \frac{(x+4)-(x+2)}{(x+2)(x+4)}dx$

$= \frac{1}{63}\int\left(\frac{1}{x-5}-\frac{1}{x+2}\right)dx + \frac{1}{18}\int\left(\frac{1}{x+4}-\frac{1}{x+2}\right)dx = \frac{1}{63}\ln\left|\frac{x-5}{x+2}\right| + \frac{1}{18}\ln\left|\frac{x+4}{x+2}\right| + c$

II. DẠNG 2: TÁCH CÁC MẪU SỐ CHỨA CÁC NHÂN TỬ KHÔNG ĐỒNG BẬC

1. Các bài tập mẫu minh họa:

•$\boldsymbol{B_1 = \int \frac{dx}{x^3-3x}} = \int \frac{dx}{x(x^2-3)} = \frac{1}{3}\int \frac{x^2-(x^2-3)}{x(x^2-3)}dx = \frac{1}{3}\left(\int \frac{xdx}{x^2-3} - \int \frac{dx}{x}\right)$

$= \frac{1}{3}\left[\frac{1}{2}\int \frac{d(x^2-3)}{x^2-3} - \int \frac{dx}{x}\right] = \frac{1}{3}\left(\frac{1}{2}\ln|x^2-3| - \ln|x|\right) + c = \frac{1}{6}\ln\left|\frac{x^2-3}{x^2}\right| + c$

• $\boldsymbol{B_2 = \int \frac{dx}{x^7-10x^3}} = \int \frac{dx}{x^3(x^4-10)} = \frac{1}{10}\int \frac{x^4-(x^4-10)}{x^3(x^4-10)}dx = \frac{1}{10}\left(\int \frac{xdx}{x^4-10} - \int \frac{dx}{x^3}\right)$

$= \frac{1}{10}\left(\frac{1}{2}\int \frac{d(x^2)}{(x^2)^2-10} - \int \frac{dx}{x^3}\right) = \frac{1}{20}\left(\frac{1}{\sqrt{10}}\ln\left|\frac{x^2-\sqrt{10}}{x^2+\sqrt{10}}\right| + \frac{1}{x^2}\right) + c$

2. Các bài tập dành cho bạn đọc tự giải:

$B_1 = \int \frac{dx}{x^3+5x}$; $B_2 = \int \frac{dx}{x^9-7x^4}$; $B_3 = \int \frac{dx}{x^{11}-8x^5}$; $B_4 = \int \frac{dx}{x^6+9x}$; $B_5 = \int \frac{dx}{x^7+13x}$

$B_6 = \int \frac{dx}{x^3+6x^2+19x+22}$; $B_7 = \int \frac{dx}{x^3-3x^2+14x-12}$; $B_8 = \int \frac{dx}{x^4+4x^3+6x^2+7x+4}$

III. DẠNG 3: KĨ THUẬT NHẢY TẦNG LẦU KHI MẪU SỐ LÀ HÀM ĐA THỨC BẬC 4

•$\boldsymbol{C_1 = \int \frac{dx}{x^4-1}} = \int \frac{dx}{(x^2-1)(x^2+1)} = \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{(x^2-1)(x^2+1)}dx = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\text{arctg}x + c$

•$\boldsymbol{C_2 = \int \frac{xdx}{x^4-1}} = \frac{1}{2}\int \frac{d(x^2)}{(x^2-1)(x^2+1)} = \frac{1}{4}\int\left(\frac{1}{x^2-1}-\frac{1}{x^2+1}\right)d(x^2) = \frac{1}{4}\ln\left|\frac{x^2-1}{x^2+1}\right| + c$

•$\boldsymbol{C_3 = \int \frac{x^2dx}{x^4-1}} = \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)}{(x^2+1)(x^2-1)}dx = \frac{1}{2}\int\left(\frac{1}{x^2-1}+\frac{1}{x^2+1}\right)dx$

$= \frac{1}{2}\int \frac{dx}{x^2-1} + \frac{1}{2}\int \frac{dx}{x^2+1} = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\text{arctg}x + c$

$$\bullet\ \boldsymbol{C_4} = \int \frac{\boldsymbol{x^3 dx}}{\boldsymbol{x^4 - 1}} = \frac{1}{4}\int \frac{d\left(x^4 - 1\right)}{x^4 - 1} = \frac{1}{4}\ln\left|x^4 - 1\right| + c$$

$$\bullet \boldsymbol{C_5} = \int \frac{\boldsymbol{x^4 dx}}{\boldsymbol{x^4 - 1}} = \int \frac{\left(x^4 - 1\right) + 1}{x^4 - 1}dx = \int dx + \int \frac{dx}{x^4 - 1} = x + C_1 = x + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\operatorname{arctg}x + c$$

$$\bullet \boldsymbol{C_6} = \int \frac{\boldsymbol{xdx}}{\boldsymbol{x^4 + 1}} = \frac{1}{2}\int \frac{d\left(x^2\right)}{\left(x^2\right)^2 + 1} = \frac{1}{2}\operatorname{arctg}\left(x^2\right) + c$$

$$\bullet \boldsymbol{C_7} = \int \frac{\boldsymbol{x^3 dx}}{\boldsymbol{x^4 + 1}} = \frac{1}{4}\int \frac{d\left(x^4 + 1\right)}{x^4 + 1} = \frac{1}{4}\ln\left|x^4 + 1\right| + c$$

$$\bullet \boldsymbol{C_8} = \int \frac{\boldsymbol{x^2 - 1}}{\boldsymbol{x^4 + 1}}\boldsymbol{dx} = \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}dx = \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 - \left(\sqrt{2}\right)^2} = \frac{1}{2\sqrt{2}}\ln\left|\frac{\left(x + \frac{1}{x}\right) - \sqrt{2}}{\left(x + \frac{1}{x}\right) + \sqrt{2}}\right| + c$$

$$\bullet\ \boldsymbol{C_9} = \int \frac{\boldsymbol{x^2 + 1}}{\boldsymbol{x^4 + 1}}\boldsymbol{dx} = \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}dx = \int \frac{d\left(x - \frac{1}{x}\right)}{\left(x - \frac{1}{x}\right)^2 + \left(\sqrt{2}\right)^2} = \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x^2 - 1}{x\sqrt{2}} + c$$

$$\bullet \boldsymbol{C_{10}} = \int \frac{\boldsymbol{dx}}{\boldsymbol{x^4 + 1}} = \frac{1}{2}\int \frac{\left(x^2 + 1\right) - \left(x^2 - 1\right)}{x^4 + 1}dx = \frac{1}{2}\left(\int \frac{x^2 + 1}{x^4 + 1}dx - \int \frac{x^2 - 1}{x^4 + 1}dx\right)$$

$$= \frac{1}{2}\left(C_9 - C_8\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x^2 - 1}{x\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}\right|\right) + c$$

$$\bullet \boldsymbol{C_{11}} = \int \frac{\boldsymbol{x^2 dx}}{\boldsymbol{x^4 + 1}} = \frac{1}{2}\int \frac{\left(x^2 + 1\right) + \left(x^2 - 1\right)}{x^4 + 1}dx = \frac{1}{2}\left(\int \frac{x^2 + 1}{x^4 + 1}dx + \int \frac{x^2 - 1}{x^4 + 1}dx\right)$$

$$= \frac{1}{2}\left(C_9 + C_8\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x^2 - 1}{x\sqrt{2}} + \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}\right|\right) + c$$

$$\bullet \boldsymbol{C_{12}} = \int \frac{\boldsymbol{x^4 dx}}{\boldsymbol{x^4 + 1}} = \int \frac{\left(x^4 + 1\right) - 1}{x^4 + 1}dx = x - \frac{1}{2}\left(\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x^2 - 1}{x\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}\right|\right) + c$$

$$\bullet \boldsymbol{C_{13}} = \int \frac{\boldsymbol{\left(x^2 - 1\right)dx}}{\boldsymbol{x^4 - 5x^3 - 4x^2 - 5x + 1}} = \int \frac{\left(1 - \frac{1}{x^2}\right)dx}{x^2 + \frac{1}{x^2} - 5\left(x + \frac{1}{x}\right) - 4} = \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 - 5\left(x + \frac{1}{x}\right) - 6}$$

$$= \int \frac{du}{u^2 - 5u - 6} = \int \frac{du}{(u-6)(u+1)} = \frac{1}{7}\int\left(\frac{1}{u-6} - \frac{1}{u+1}\right)du = \frac{1}{7}\ln\left|\frac{x^2 - 6x + 1}{x^2 + x + 1}\right| + c$$

$$\bullet\ \boldsymbol{C_{14}} = \int \frac{\boldsymbol{dx}}{\boldsymbol{x^4 + x^2 + 1}} = \frac{1}{2}\int \frac{\left(x^2 + 1\right) - \left(x^2 - 1\right)}{x^4 + x^2 + 1}dx = \frac{1}{2}\left[\int \frac{x^2 + 1}{x^4 + x^2 + 1}dx - \int \frac{x^2 - 1}{x^4 + x^2 + 1}dx\right]$$

$$=\frac{1}{2}\left[\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x^2+\frac{1}{x^2}\right)+1}-\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x^2+\frac{1}{x^2}\right)+1}\right]=\frac{1}{4}\left[\int\frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+3}-\int\frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-1}\right]$$

$$=\frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{x-\frac{1}{x}}{\sqrt{3}}-\frac{1}{4}\ln\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right|+c=\frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{x^2-1}{x\sqrt{3}}-\frac{1}{4}\ln\left|\frac{x^2-x+1}{x^2+x+1}\right|+c$$

IV. DẠNG 4: KĨ THUẬT NHẢY TẦNG LẦU KHI MẪU SỐ LÀ HÀM ĐA THỨC BẬC 3

• $D_1=\int\frac{dx}{x^3-1}=\int\frac{dx}{(x-1)(x^2+x+1)}=\int\frac{d(x-1)}{(x-1)\left[(x-1)^2+3(x-1)+3\right]}$

$$=\int\frac{dt}{t(t^2+3t+3)}=\frac{1}{3}\int\frac{(t^2+3t+3)-(t^2+3t)}{t(t^2+3t+3)}dt=\frac{1}{3}\left(\int\frac{dt}{t}-\int\frac{(t+3)dt}{t^2+3t+3}\right)$$

$$=\frac{1}{3}\left(\int\frac{dt}{t}-\frac{1}{2}\int\frac{(2t+3)dt}{t^2+3t+3}-\frac{3}{2}\int\frac{dt}{t^2+3t+3}\right)=\frac{1}{6}\ln\left|\frac{x^2-2x+1}{x^2+x+1}\right|-\frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+c$$

• $D_2=\int\frac{dx}{x^3+1}=\int\frac{dx}{(x+1)(x^2-x+1)}=\int\frac{d(x+1)}{(x+1)\left[(x+1)^2-3(x+1)+3\right]}$

$$=\int\frac{dt}{t(t^2-3t+3)}=\frac{1}{3}\int\frac{(t^2-3t+3)-(t^2-3t)}{t(t^2-3t+3)}dt=\frac{1}{3}\left(\int\frac{dt}{t}-\int\frac{(t-3)dt}{t^2-3t+3}\right)$$

$$=\frac{1}{3}\left(\int\frac{dt}{t}-\frac{1}{2}\int\frac{d(t^2-3t+3)}{t^2-3t+3}+\frac{3}{2}\int\frac{dt}{\left(t-\frac{3}{2}\right)^2+\frac{3}{4}}\right)=$$

$$\frac{1}{3}\left(\frac{1}{2}\ln\left|\frac{t^2}{t^2-3t+3}\right|+\sqrt{3}\operatorname{arctg}\frac{2t-3}{\sqrt{3}}\right)+c=\frac{1}{6}\ln\left|\frac{x^2+2x+1}{x^2-x+1}\right|+\frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}+c$$

• $D_3=\int\frac{xdx}{x^3-1}=\int\frac{xdx}{(x-1)(x^2+x+1)}=\frac{1}{3}\int\frac{(x^2+x+1)-(x-1)^2}{(x-1)(x^2+x+1)}dx$

$$=\frac{1}{3}\int\left(\frac{1}{x-1}-\frac{x-1}{x^2+x+1}\right)dx=\frac{1}{3}\left[\int\frac{dx}{x-1}-\frac{1}{2}\int\frac{(2x+1)dx}{x^2+x+1}+\frac{3}{2}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$=\frac{1}{3}\left[\ln|x-1|-\frac{1}{2}\ln|x^2+x+1|+\sqrt{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}\right]+c$$

• $D_4=\int\frac{xdx}{x^3+1}=\int\frac{xdx}{(x+1)(x^2-x+1)}=\frac{-1}{3}\int\frac{(x^2-x+1)-(x+1)^2}{(x+1)(x^2-x+1)}dx$

$$=\frac{-1}{3}\int\left(\frac{1}{x+1}-\frac{x+1}{x^2-x+1}\right)dx=\frac{-1}{3}\left[\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{(2x-1)dx}{x^2-x+1}-\frac{3}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$=\frac{-1}{3}\left[\ln|x+1|-\frac{1}{2}\ln|x^2-x+1|-\sqrt{3}\text{arctg}\frac{2x-1}{\sqrt{3}}\right]+c=\frac{-1}{6}\ln\left|\frac{x^2+2x+1}{x^2-x+1}\right|-\frac{1}{\sqrt{3}}\text{arctg}\frac{2x-1}{\sqrt{3}}+c$$

V. DẠNG 5: KĨ THUẬT NHẢY TẦNG LẦU KHI MẪU LÀ HÀM ĐA THỨC BẬC 6

• $\boldsymbol{E_1=\int\frac{dx}{x^6-1}}=\int\frac{dx}{(x^3-1)(x^3+1)}=\frac{1}{2}\left[\int\frac{dx}{x^3-1}-\int\frac{dx}{x^3+1}\right]=\frac{1}{2}(D_1-D_2)$

$$=\frac{1}{2}\left[\left(\frac{1}{6}\ln\left|\frac{x^2-2x+1}{x^2+x+1}\right|-\frac{1}{2\sqrt{3}}\text{arctg}\frac{2x+1}{\sqrt{3}}\right)-\left(\frac{1}{6}\ln\left|\frac{x^2+2x+1}{x^2-x+1}\right|+\frac{1}{2\sqrt{3}}\text{arctg}\frac{2x-1}{\sqrt{3}}\right)\right]$$

$$=\frac{1}{12}\ln\left|\frac{(x^2-2x+1)(x^2-x+1)}{(x^2+2x+1)(x^2+x+1)}\right|-\frac{1}{4\sqrt{3}}\left(\text{arctg}\frac{2x+1}{\sqrt{3}}+\text{arctg}\frac{2x-1}{\sqrt{3}}\right)+c$$

• $\boldsymbol{E_2=\int\frac{xdx}{x^6-1}}=\frac{1}{2}\int\frac{d(x^2)}{(x^2)^3-1}=\frac{1}{2}\int\frac{du}{u^3-1}=\frac{1}{2}D_1$

$$=\frac{1}{2}\left[\frac{1}{6}\ln\left|\frac{u^2-2u+1}{u^2+u+1}\right|-\frac{1}{2\sqrt{3}}\text{arctg}\frac{2u+1}{\sqrt{3}}\right]+c=\frac{1}{12}\ln\left|\frac{x^4-2x^2+1}{x^4+x^2+1}\right|-\frac{1}{2\sqrt{3}}\text{arctg}\frac{2x^2+1}{\sqrt{3}}+c$$

•$\boldsymbol{E_3=\int\frac{x^2dx}{x^6-1}}=\frac{1}{3}\int\frac{d(x^3)}{x^6-1}=\frac{1}{3}\cdot\frac{1}{2}\ln\left|\frac{x^3-1}{x^3+1}\right|+c=\frac{1}{6}\ln\left|\frac{x^3-1}{x^3+1}\right|+c$

• $\boldsymbol{E_4=\int\frac{x^3dx}{x^6-1}}=\frac{1}{2}\int\frac{x^2d(x^2)}{x^6-1}=\frac{1}{2}\int\frac{udu}{u^3-1}=\frac{1}{2}\int\frac{udu}{(u-1)(u^2+u+1)}=$

$$=\frac{1}{12}\ln\left|\frac{(u-1)^2}{u^2+u+1}\right|+\frac{1}{2\sqrt{3}}\text{arctg}\frac{2u+1}{\sqrt{3}}+c=\frac{1}{12}\ln\left|\frac{x^4-2x^2+1}{x^4+x^2+1}\right|+\frac{1}{2\sqrt{3}}\text{arctg}\frac{2x^2+1}{\sqrt{3}}+c$$

•$\boldsymbol{E_5=\int\frac{x^4dx}{x^6-1}}=\int\frac{(x^4+x^2+1)-(x^2-1)-2}{(x^2-1)(x^4+x^2+1)}dx=\int\frac{dx}{x^2-1}-\int\frac{dx}{x^4+x^2+1}-2\int\frac{dx}{x^6-1}$

$$=\frac{1}{12}\ln\left|\frac{(x^2-2x+1)(x^2-x+1)}{(x^2+2x+1)(x^2+x+1)}\right|+\frac{1}{2\sqrt{3}}\left(\text{arctg}\frac{2x+1}{\sqrt{3}}+\text{arctg}\frac{2x-1}{\sqrt{3}}-\text{arctg}\frac{x^2-1}{x\sqrt{3}}\right)+c$$

• $\boldsymbol{E_6=\int\frac{x^5dx}{x^6-1}}=\frac{1}{6}\int\frac{d(x^6)}{x^6-1}=\frac{1}{6}\ln|x^6-1|+c$

• $\boldsymbol{E_7=\int\frac{x^6dx}{x^6-1}}=\int\frac{(x^6-1)+1}{x^6-1}dx=\int dx+\int\frac{dx}{x^6-1}=x+E_1$

$$=x+\frac{1}{12}\ln\left|\frac{(x^2-2x+1)(x^2-x+1)}{(x^2+2x+1)(x^2+x+1)}\right|-\frac{1}{4\sqrt{3}}\left(\text{arctg}\frac{2x+1}{\sqrt{3}}+\text{arctg}\frac{2x-1}{\sqrt{3}}\right)+c$$

• $\boldsymbol{E_8=\int\frac{x^4-1}{x^6+1}dx}=\int\frac{(x^2+1)(x^2-1)dx}{(x^2+1)(x^4-x^2+1)}=\int\frac{(x^2-1)dx}{x^4-x^2+1}=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x^2+\frac{1}{x^2}\right)-1}$

$$= \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2 - \left(\sqrt{3}\right)^2} = \frac{1}{2\sqrt{3}} \ln\left|\frac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right| + c = \frac{1}{2\sqrt{3}} \ln\left|\frac{x^2 - x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right| + c$$

• $\boldsymbol{E_9 = \int \frac{x^4+1}{x^6+1}dx} = \int \frac{\left(x^4-x^2+1\right)+x^2}{\left(x^2+1\right)\left(x^4-x^2+1\right)}dx = \int \frac{dx}{x^2+1} + \int \frac{x^2dx}{x^6+1}$

$$= \int \frac{dx}{x^2+1} + \frac{1}{3}\int \frac{d\left(x^3\right)}{x^6+1} = \text{arctg}x + \frac{1}{3}\text{arctg}\left(x^3\right) + c$$

• $\boldsymbol{E_{10} = \int \frac{dx}{x^6+1}} = \frac{1}{2}\int \frac{\left(x^4+1\right)-\left(x^4-1\right)}{x^6+1}dx = \frac{1}{2}\left(E_9 - E_8\right) =$

$$= \frac{1}{2}\left(\text{arctg}x + \frac{1}{3}\text{arctg}\left(x^3\right) - \frac{1}{2\sqrt{3}}\ln\left|\frac{x^2-x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right|\right) + c$$

• $\boldsymbol{E_{11} = \int \frac{x^2+x}{x^6+1}dx} = \frac{1}{3}\int \frac{d\left(x^3\right)}{x^6+1} + \frac{1}{2}\int \frac{d\left(x^2\right)}{x^6+1} = \frac{1}{3}\int \frac{d\left(x^3\right)}{x^6+1} + \frac{1}{2}D_2$ (thay x^2 vào D_2)

$$= \frac{1}{3}\text{arctg}\left(x^3\right) + \frac{1}{2}\left(\frac{1}{6}\ln\left|\frac{x^4+2x^2+1}{x^4-x^2+1}\right| + \frac{1}{2\sqrt{3}}\text{arctg}\frac{2x^2-1}{\sqrt{3}}\right) + c$$

VI. DẠNG 6: SỬ DỤNG KHAI TRIỂN TAYLOR

• Đa thức $P_n(x)$ bậc n có khai triển Taylor tại điểm x = a là:

$$\boxed{P_n(x) = P_n(a) + \frac{P_n'(a)}{1!}(x-a) + \frac{P_n''(a)}{2!}(x-a)^2 + \cdots + \frac{P_n^{(n)}(a)}{n!}(x-a)^n}$$

1. Các bài tập mẫu minh họa:

• $\boldsymbol{F_1 = \int \frac{3x^4 - 5x^3 + 7x - 8}{(x+2)^{50}}dx}$. Đặt $P_4(x) = 3x^4 - 5x^3 + 7x - 8$

$$\Leftrightarrow P_4(x) = P_4(-2) + \frac{P_4'(-2)}{1!}(x+2) + \frac{P_4''(-2)}{2!}(x+2)^2 + \frac{P_4^{(3)}(-2)}{3!}(x+2)^3 + \frac{P_4^{(4)}(-2)}{4!}(x+2)^4$$

$$\Leftrightarrow P_4(x) = 66 - 149(x+2) + 48(x+2)^2 - 29(x+2)^3 + 3(x+2)^4$$

$$\Rightarrow F_1 = \int \frac{66 - 149(x+2) + 48(x+2)^2 - 29(x+2)^3 + 3(x+2)^4}{(x+2)^{50}}dx$$

$$= \int \left[66(x+2)^{-50} - 149(x+2)^{-49} + 48(x+2)^{-48} - 29(x+2)^{-47} + 3(x+2)^{-46}\right]dx$$

$$= \frac{-66}{49(x+2)^{49}} + \frac{149}{48(x+2)^{48}} - \frac{48}{47(x+2)^{47}} + \frac{29}{46(x+2)^{46}} - \frac{3}{45(x+2)^{45}} + c$$

VII. DẠNG 7: KĨ THUẬT NHẢY TẦNG LẦU KHI MẪU LÀ HÀM ĐA THỨC BẬC CAO

1. Các bài tập mẫu minh họa:

$$\bullet \boldsymbol{G_1} = \int \frac{\boldsymbol{dx}}{\boldsymbol{3x^{100} + 5x}} = \int \frac{dx}{x\left(3x^{99}+5\right)} = \frac{1}{5}\int \frac{\left(3x^{99}+5\right)-3x^{99}}{x\left(3x^{99}+5\right)}dx = \frac{1}{5}\left[\int \frac{dx}{x} - \int \frac{3x^{98}dx}{3x^{99}+5}\right]$$

$$= \frac{1}{5}\left[\int \frac{dx}{x} - \frac{1}{99}\int \frac{d\left(3x^{99}+5\right)}{3x^{99}+5}\right] = \frac{1}{5}\left[\ln|x| - \frac{1}{99}\ln\left|3x^{99}+5\right|\right] + c = \frac{1}{495}\ln\left|\frac{x^{99}}{3x^{99}+5}\right| + c$$

$$\bullet \boldsymbol{G_2} = \int \frac{\boldsymbol{dx}}{\boldsymbol{x\left(2x^{50}+7\right)^2}} = \frac{1}{7}\int \frac{\left(2x^{50}+7\right)-2x^{50}}{x\left(2x^{50}+7\right)^2}dx = \frac{1}{7}\left[\int \frac{dx}{x\left(2x^{50}+7\right)} - \int \frac{2x^{49}dx}{\left(2x^{50}+7\right)^2}\right]$$

$$= \frac{1}{7}\left[\frac{1}{7}\int \frac{\left(2x^{50}+7\right)-2x^{50}}{x\left(2x^{50}+7\right)}dx - \int \frac{2x^{49}dx}{\left(2x^{50}+7\right)^2}\right] = \frac{1}{49}\left[\int \frac{dx}{x} - \int \frac{2x^{49}dx}{2x^{50}+7}\right] - \frac{1}{7}\int \frac{2x^{49}dx}{\left(2x^{50}+7\right)^2}$$

$$= \frac{1}{49}\left[\int \frac{dx}{x} - \frac{1}{50}\int \frac{d\left(2x^{50}+7\right)}{2x^{50}+7}\right] - \frac{1}{350}\int \frac{d\left(2x^{50}+7\right)}{\left(2x^{50}+7\right)^2}$$

$$= \frac{1}{49}\ln|x| - \frac{1}{49.50}\ln\left|2x^{50}+7\right| + \frac{1}{350\left(2x^{50}+7\right)} = \frac{1}{49.50}\ln\left|\frac{x^{50}}{2x^{50}+7}\right| + \frac{1}{350\left(2x^{50}+7\right)} + c$$

$$\bullet \boldsymbol{G_3} = \int \frac{\boldsymbol{dx}}{\boldsymbol{x\left(ax^n+b\right)^k}} = \frac{1}{b}\int \frac{\left(ax^n+b\right)-ax^n}{x\left(ax^n+b\right)^k}dx = \frac{1}{b}\int \frac{dx}{x\left(ax^n+b\right)^{k-1}} - \frac{1}{nb}\int \frac{d\left(ax^n+b\right)}{\left(ax^n+b\right)^k}$$

$$= \frac{1}{b^2}\int \frac{dx}{x\left(ax^n+b\right)^{k-2}} - \frac{1}{nb^2}\int \frac{d\left(ax^n+b\right)}{\left(ax^n+b\right)^{k-1}} - \frac{1}{nb}\int \frac{d\left(ax^n+b\right)}{\left(ax^n+b\right)^k} = \cdots$$

$$= \frac{1}{b^k}\ln|x| + \frac{1}{n}\left[\frac{1}{b(k-1)\left(ax^n+b\right)^{k-1}} + \cdots + \frac{1}{b^{k-1}\left(ax^n+b\right)} - \frac{1}{b^k}\ln\left|ax^n+b\right|\right] + c$$

$$= \frac{1}{nb^k}\ln\left|\frac{x^n}{ax^n+b}\right| + \frac{1}{n}\left[\frac{1}{b(k-1)\left(ax^n+b\right)^{k-1}} + \cdots + \frac{1}{b^{k-1}\left(ax^n+b\right)}\right] + c$$

$$\bullet \boldsymbol{G_4} = \int \frac{\boldsymbol{\left(1-x^{2000}\right)dx}}{\boldsymbol{x\left(1+x^{2000}\right)}} = \int \frac{\left(1+x^{2000}\right)-2x^{2000}}{x\left(1+x^{2000}\right)}dx = \int \frac{dx}{x} - \int \frac{2x^{1999}dx}{\left(1+x^{2000}\right)}$$

$$= \int \frac{dx}{x} - \frac{1}{1000}\int \frac{d\left(1+x^{2000}\right)}{\left(1+x^{2000}\right)} = \ln|x| - \frac{1}{1000}\ln\left|1+x^{2000}\right| + c = \ln\left|\frac{x^{1000}}{1+x^{2000}}\right| + c$$

$$\bullet \boldsymbol{G_5} = \int \frac{\boldsymbol{x^{19}dx}}{\boldsymbol{\left(3+x^{10}\right)^2}} = \frac{1}{10}\int \frac{x^{10}.10x^9dx}{\left(3+x^{10}\right)^2} = \frac{1}{10}\int \frac{x^{10}d\left(x^{10}\right)}{\left(3+x^{10}\right)^2} = \frac{1}{10}\int \frac{\left(x^{10}+3\right)-3}{\left(3+x^{10}\right)^2}d\left(x^{10}+3\right)$$

$$= \frac{1}{10}\left[\int \frac{d\left(x^{10}+3\right)}{3+x^{10}} - 3\int \frac{d\left(x^{10}+3\right)}{\left(3+x^{10}\right)^2}\right] = \frac{1}{10}\ln\left|3+x^{10}\right| + \frac{3}{10\left(3+x^{10}\right)} + c$$

$$\bullet \boldsymbol{G_6} = \int \frac{\boldsymbol{x^{99}dx}}{\boldsymbol{\left(2x^{50}-3\right)^7}} = \int \frac{x^{50}.x^{49}dx}{\left(2x^{50}-3\right)^7} = \frac{1}{200}\int \frac{\left(2x^{50}-3\right)+3}{\left(2x^{50}-3\right)^7}d\left(2x^{50}-3\right)$$

$$= \frac{1}{200}\left[\int \frac{d\left(2x^{50}-3\right)}{\left(2x^{50}-3\right)^6} + 3\int \frac{d\left(2x^{50}-3\right)}{\left(2x^{50}-3\right)^7}\right] = \frac{-1}{200}\left[\frac{1}{5\left(2x^{50}-3\right)^5} + \frac{1}{2\left(2x^{50}-3\right)^6}\right] + c$$

$$= \frac{-1}{200}\cdot\frac{2\left(2x^{50}-3\right)+5}{10\left(2x^{50}-3\right)^6} + c = \frac{1-4x^{50}}{2000\left(2x^{50}-3\right)^6} + c$$

$$\bullet\, G_7 = \int \frac{x^{2n-1}dx}{\left(ax^n+b\right)^k} = \int \frac{x^n x^{n-1}dx}{\left(ax^n+b\right)^k} = \frac{1}{na^2}\int \frac{\left(ax^n+b\right)-b}{\left(ax^n+b\right)^k}d\left(ax^n+b\right)$$

$$= \frac{1}{na^2}\left[\int \frac{d\left(ax^n+b\right)}{\left(ax^n+b\right)^{k-1}} - b\int \frac{d\left(ax^n+b\right)}{\left(ax^n+b\right)^k}\right] = \frac{1}{na^2}\left[\frac{-1}{(k-2)\left(ax^n+b\right)^{k-2}} + \frac{b}{(k-1)\left(ax^n+b\right)^{k-1}}\right] + c$$

$$= \frac{1}{na^2}\cdot\frac{b(k-2)-(k-1)\left(ax^n+b\right)}{(k-1)(k-2)\left(ax^n+b\right)^{k-1}} + c = \frac{-kax^n-b}{na^2\,(k-1)(k-2)\left(ax^n+b\right)^{k-1}} + c$$

2. Các bài tập dành cho bạn đọc tự giải:

$$G_1 = \int \frac{xdx}{x^8-1};\, G_2 = \int \frac{x^5-x}{x^8+1}dx;\, G_3 = \int \frac{dx}{x^8-1};\, G_4 = \int \frac{xdx}{x^8+1};\, G_5^{****} = \int \frac{dx}{x^8+1}$$

VIII. DẠNG 8: KĨ THUẬT CHỒNG NHỊ THỨC

$$\bullet\, H_1 = \int \frac{(3x-5)^{10}}{(x+2)^{12}}dx = \int \left(\frac{3x-5}{x+2}\right)^{10}\frac{dx}{(x+2)^2}$$

$$= \frac{1}{11}\int \left(\frac{3x-5}{x+2}\right)^{10} d\left(\frac{3x-5}{x+2}\right) = \frac{1}{121}\left(\frac{3x-5}{x+2}\right)^{11} + c$$

$$\bullet\, H_2 = \int \frac{(7x-1)^{99}}{(2x+1)^{101}}dx = \int \left(\frac{7x-1}{2x+1}\right)^{99}\frac{dx}{(2x+1)^2} = \frac{1}{9}\int \left(\frac{7x-1}{2x+1}\right)^{99} d\left(\frac{7x-1}{2x+1}\right)$$

$$= \frac{1}{9}\cdot\frac{1}{100}\left(\frac{7x-1}{2x+1}\right)^{100} + c = \frac{1}{900}\left(\frac{7x-1}{2x+1}\right)^{100} + c$$

$$\bullet\, H_3 = \int \frac{dx}{(x+3)^5(x+5)^3} = \int \frac{dx}{\left(\frac{x+3}{x+5}\right)^5(x+5)^8} = \int \frac{1}{\left(\frac{x+3}{x+5}\right)^5}\cdot\frac{1}{(x+5)^6}\cdot\frac{dx}{(x+5)^2}$$

$$= \frac{1}{2^7}\int \frac{1}{\left(\frac{x+3}{x+5}\right)^5}\cdot\left[\frac{(x+3)-(x+5)}{x+5}\right]^6 d\left(\frac{x+3}{x+5}\right) = \frac{1}{2^7}\int \frac{1}{u^5}\cdot(u-1)^6\,du$$

$$= \frac{1}{2^7}\int \frac{u^6-6u^5+15u^4-20u^3+15u^2-6u+1}{u^5}du$$

$$= \frac{1}{2^7}\int \left(u-6+\frac{15}{u}-\frac{20}{u^2}+\frac{15}{u^3}-\frac{6}{u^4}+\frac{1}{u^5}\right)du$$

$$= \frac{1}{2^7}\left(\frac{u^2}{2}-6u+15\ln|u|+\frac{20}{u}-\frac{15}{2u^2}+\frac{2}{u^3}-\frac{1}{4u^4}\right)+c$$

$$= \frac{1}{2^7}\left[\frac{1}{2}\left(\frac{x+3}{x+5}\right)^2-6\left(\frac{x+3}{x+5}\right)+15\ln\left|\frac{x+3}{x+5}\right|\right]+$$

$$+\frac{1}{2^7}\left[20\left(\frac{x+5}{x+3}\right)-\frac{15}{2}\left(\frac{x+5}{x+3}\right)^2+2\left(\frac{x+5}{x+3}\right)^3-\frac{1}{4}\left(\frac{x+5}{x+3}\right)^4\right]+c$$

Các bài tập dành cho bạn đọc tự giải:

• $H_1 = \int \dfrac{dx}{(3x-2)^7(3x+4)^3}$; $H_2 = \int \dfrac{dx}{(2x-1)^3(3x-1)^4}$; $H_3 = \int \dfrac{dx}{(3x+2)^5(4x-1)^4}$

BÀI 4. TÍCH PHÂN CƠ BẢN CỦA CÁC HÀM SỐ LƯỢNG GIÁC

A. CÔNG THỨC SỬ DỤNG

1. KHAI TRIỂN NHỊ THỨC NEWTON

$$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1}b + ... + C_n^k a^{n-k}b^k + ... + C_n^{n-1}ab^{n-1} + C_n^n b^n$$

trong đó $C_n^k = \dfrac{n!}{k!(n-k)!}$ và $m! = 1.2....(m-1)m$ với qui ước 0! = 1

2. CÁC CÔNG THỨC NGUYÊN HÀM LƯỢNG GIÁC

$$\int cos(ax+b)\,dx = \frac{1}{a}sin(ax+b)+c \qquad \int sin(ax+b)\,dx = -\frac{1}{a}cos(ax+b)+c$$

$$\int \frac{dx}{cos^2(ax+b)} = \frac{1}{a}tg(ax+b)+c \qquad \int \frac{dx}{sin^2(ax+b)} = -\frac{1}{a}cotg(ax+b)+c$$

B. CÁC DẠNG TÍCH PHÂN

I. Dạng 1: $A_{1.1} = \int (sinx)^n\,dx$; $A_{1.2}\int (cosx)^n\,dx$

1. Công thức hạ bậc

$$sin^2 x = \frac{1-cos\,2x}{2}\ ;\ cos^2 x = \frac{1+cos\,2x}{2}\ ;\ sin^3 x = \frac{-sin\,3x+3\,sin\,x}{4}\ ;\ cos^3 x = \frac{cos\,3x+3\,cos\,x}{4}$$

2. Phương pháp

2.1. Nếu n chẵn thì sử dụng công thức hạ bậc

2.2. Nếu n = 3 thì sử dụng công thức hạ bậc hoặc biến đổi theo 2.3.

2.3. Nếu 3 ≤ n lẻ (n = 2p +1) thì thực hiện biến đổi:

$$A_{1.1} = \int (sinx)^n\,dx = \int (sinx)^{2p+1}\,dx = \int (sin\,x)^{2p}\,sin\,xdx = -\int \left(1-cos^2 x\right)^p d(cos\,x)$$

$$= -\int \left[C_p^0 - C_p^1 \cos^2 x + ... + (-1)^k C_p^k \left(\cos^2 x\right)^k + ... + (-1)^p C_p^p \left(\cos^2 x\right)^p\right] d(\cos x)$$

$$= -\left[C_p^0 \cos x - \frac{1}{3}C_p^1 \cos^3 x + ... + \frac{(-1)^k}{2k+1}C_p^k (\cos x)^{2k+1} + ... + \frac{(-1)^p}{2p+1}C_p^p (\cos x)^{2p+1}\right] + c$$

$$A_{1.2} = \int (cosx)^n\,dx = \int (cosx)^{2p+1}\,dx = \int (cos\,x)^{2p}\,cos\,xdx = \int \left(1-sin^2 x\right)^p d(sin\,x)$$

$$= \int \left[C_p^0 - C_p^1 \sin^2 x + ... + (-1)^k C_p^k \left(\sin^2 x\right)^k + ... + (-1)^p C_p^p \left(\sin^2 x\right)^p \right] d(\sin x)$$

$$= \left[C_p^0 \sin x - \frac{1}{3} C_p^1 \sin^3 x + ... + \frac{(-1)^k}{2k+1} C_p^k (\sin x)^{2k+1} + ... + \frac{(-1)^p}{2p+1} C_p^p (\sin x)^{2p+1} \right] + c$$

• $\boldsymbol{A_1 = \int \cos^6 x dx} = \int \left(\cos^2 x\right)^3 dx = \int \left(\frac{1+\cos 2x}{2}\right)^3 dx$

$$= \frac{1}{4} \int (1+\cos 2x)^3 dx = \frac{1}{4} \int \left(1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x\right) dx$$

$$= \frac{1}{4} \int \left(1 + 3\cos 2x + \frac{3(1+2\cos 4x)}{2} + \frac{\cos 3x + 3\cos x}{4}\right) dx$$

$$= \frac{1}{16} \left(7x + 6\sin 2x + 3\sin 4x + \frac{1}{3}\sin 3x + 3\sin x\right) + c$$

• $\boldsymbol{A_2 = \int (\sin 5x)^9 dx} = \int (\sin 5x)^8 (\sin 5x) dx = -\frac{1}{5} \int \left(1 - \cos^2 5x\right)^4 d(\cos 5x)$

$$= -\frac{1}{5} \int \left[1 - 4\cos^2 5x + 6\cos^4 5x - 4\cos^6 5x + \cos^8 5x\right] d(\cos 5x)$$

$$= -\frac{1}{5} \left(\cos 5x - \frac{4}{3}\cos^3 5x + \frac{6}{5}\cos^5 5x - \frac{4}{7}\cos^7 5x + \frac{1}{9}\cos^9 5x\right) + c$$

II. Dạng 2: $\boldsymbol{B = \int \sin^m x \cos^n x \, dx}$ (m, n$\in$N)

1. Phương pháp:

1.1. Trường hợp 1: m, n là các số nguyên

a. Nếu m chẵn, n chẵn thì sử dụng công thức hạ bậc, biến đổi tích thành tổng.

b. Nếu m chẵn, n lẻ (n =2p +1) thì biến đổi:

$$\boldsymbol{B = \int (\sin x)^m (\cos x)^{2p+1} dx} = \int (\sin x)^m (\cos x)^{2p} \cos x dx = \int (\sin x)^m \left(1 - \sin^2 x\right)^p d(\sin x)$$

$$= \int (\sin x)^m \left[C_p^0 - C_p^1 \sin^2 x + ... + (-1)^k C_p^k \left(\sin^2 x\right)^k + ... + (-1)^p C_p^p \left(\sin^2 x\right)^p \right] d(\sin x) =$$

$$\left[C_p^0 \frac{(\sin x)^{m+1}}{m+1} - C_p^1 \frac{(\sin x)^{m+3}}{m+3} + ... + (-1)^k C_p^k \frac{(\sin x)^{2k+1+m}}{2k+1+m} + ... + (-1)^p C_p^p \frac{(\sin x)^{2p+1+m}}{2p+1+m} \right] + c$$

c. Nếu m chẵn, n lẻ (n =2p +1) thì biến đổi:

$$\boldsymbol{B = \int (\sin x)^{2p+1} (\cos x)^n dx} = \int (\cos x)^n (\sin x)^{2p} \sin x dx = -\int (\cos x)^n \left(1 - \cos^2 x\right)^p d(\cos x)$$

$$= -\int (\cos x)^n \left[C_p^0 - C_p^1 \cos^2 x + ... + (-1)^k C_p^k \left(\cos^2 x\right)^k + ... + (-1)^p C_p^p \left(\cos^2 x\right)^p \right] d(\cos x) =$$

$$-\left[C_p^0 \frac{(\cos x)^{n+1}}{n+1} - C_p^1 \frac{(\cos x)^{n+3}}{n+3} + ... + (-1)^k C_p^k \frac{(\cos x)^{2k+1+n}}{2k+1+n} + ... + (-1)^p C_p^p \frac{(\cos x)^{2p+1+n}}{2p+1+n} \right] + c$$

d. Nếu m lẻ, n lẻ thì sử dụng biến đổi 1.2. hoặc 1.3. cho số mũ lẻ bé hơn.

1.2. Nếu m, n là các số hữu tỉ thì biến đổi và đặt u = sinx ta có:

$$B=\int sin^m x\,cos^n x dx=\int (sin\,x)^m \left(cos^2 x\right)^{\frac{n-1}{2}} cos\,x dx=\int u^m \left(1-u^2\right)^{\frac{m-1}{2}} du \quad (*)$$

• Tích phân (*) tính được ⇔ 1 trong 3 số $\frac{m+1}{2}$; $\frac{n-1}{2}$; $\frac{m+k}{2}$ là số nguyên

2. Các bài tập mẫu minh họa

• $\boldsymbol{B_1=\int (sinx)^2 (cosx)^4 dx}=\frac{1}{4}\int (sin\,2x)^2 (cos\,x)^2 dx$

$$=\frac{1}{16}\int (1-\cos 4x)(1+\cos 2x)dx=\frac{1}{16}\int (1+\cos 2x-\cos 4x-\cos 2x\cos 4x)dx$$

$$=\frac{1}{16}\int \left[1+\cos 2x-\cos 4x-\frac{1}{2}(\cos 6x+\cos 2x)\right]dx$$

$$=\frac{1}{32}\int (2+\cos 2x-2\cos 4x-\cos 6x)dx=\frac{1}{32}\left(2x+\frac{\sin 2x}{2}-\frac{\sin 4x}{2}-\frac{\sin 6x}{6}\right)+c$$

• $\boldsymbol{B_2=\int (sin5x)^9 (cos5x)^{111} dx}=\int (cos\,5x)^{111} (sin\,5x)^8 sin\,5x\,dx$

$$=\frac{-1}{5}\int (\cos 5x)^{111} \left(1-\cos^2 5x\right)^4 d(\cos 5x)$$

$$=-\frac{1}{5}\int (\cos 5x)^{111} \left(1-4\cos^2 5x+6\cos^4 5x-4\cos^6 5x+\cos^8 5x\right)d(\cos 5x)$$

$$=-\frac{1}{5}\left[\frac{(\cos 5x)^{112}}{112}-\frac{4(\cos 5x)^{114}}{114}+\frac{6(\cos 5x)^{116}}{116}-\frac{4(\cos 5x)^{118}}{118}+\frac{(\cos 5x)^{120}}{120}\right]+c$$

• $\boldsymbol{B_3=\int \frac{(sin3x)^7}{\sqrt[5]{cos^4 3x}}dx}=\int (\cos 3x)^{\frac{-4}{5}} (\sin 3x)^6 \sin 3x dx=\frac{-1}{3}\int (\cos 3x)^{\frac{-4}{5}} \left(1-\cos^2 3x\right)^3 d(\cos 3x)$

$$=\frac{-1}{3}\int (\cos 3x)^{\frac{-4}{5}} \left(1-3\cos^2 3x+3\cos^4 3x-\cos^6 3x\right)d(\cos 3x)$$

$$=\frac{-1}{3}\left[5(\cos 3x)^{\frac{1}{5}}-\frac{15}{11}(\cos 3x)^{\frac{11}{5}}+\frac{15}{21}(\cos 3x)^{\frac{21}{5}}-\frac{5}{31}(\cos 3x)^{\frac{31}{5}}\right]+c$$

• $\boldsymbol{B_4=\int \frac{dx}{(sinx)^3 (cosx)^5}}=\int \frac{dx}{\left(\frac{sin\,x}{cos\,x}\right)^3 cos^8 x}=\int \frac{1}{tg^3 x}\left(\frac{1}{cos^2 x}\right)^3 \frac{dx}{cos^2 x}$

$$=\int \frac{\left(1+tg^2 x\right)^3}{(tg\,x)^3}d(tg\,x)=\int \frac{1+3tg^2 x+3tg^4 x+tg^6 x}{tg^3 x}d(tg\,x)$$

$$=\int \left[(tg\,x)^{-3}+\frac{3}{tg\,x}+3tg\,x+tg^3 x\right]d(tg\,x)=\frac{-1}{2tg^2 x}+3\ln|tg\,x|+\frac{3}{2}tg^2 x+\frac{1}{4}tg^4 x+c$$

• $\boldsymbol{B_5=\int \frac{dx}{sin^4 xcosx}}=\int \frac{\cos x dx}{\sin^4 x\cos^2 x}=\int \frac{d(\sin x)}{\sin^4 x\left(1-\sin^2 x\right)}=\int \frac{\left(1-\sin^4 x\right)+\sin^4 x}{\sin^4 x\left(1-\sin^2 x\right)}d(\sin x)$

$$=\int \frac{1+\sin^2 x}{\sin^4 x}d(\sin x)+\int \frac{d(\sin x)}{1-\sin^2 x}=\frac{-1}{3(\sin x)^3}-\frac{1}{\sin x}+\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+c$$

• $\boldsymbol{B_6} = \int \frac{dx}{\sqrt[3]{\sin^5 x\cos x}} = \int (\sin x)^{\frac{-5}{3}}(\cos x)^{\frac{-1}{3}}\,dx = \int (\sin x)^{\frac{-5}{3}}(\cos x)^{\frac{-4}{3}}\cos x\,dx$

$$= \int (\sin x)^{\frac{-5}{3}}(\cos x)^{\frac{-4}{3}}\,d(\sin x) = \int u^{\frac{-5}{3}}\left(1-u^2\right)^{\frac{-2}{3}}du = \int u^{-3}\left(\frac{1-u^2}{u^2}\right)^{\frac{-2}{3}}du$$

Đặt $\frac{1-u^2}{u^2} = v^3 \Rightarrow -2u^{-3}du = 3v^2dv$; $v = \left(\frac{1-u^2}{u^2}\right)^{1/3} = \left(\frac{\cos^2 x}{\sin^2 x}\right)^{1/3} = (\mathrm{tg}\,x)^{\frac{-2}{3}}$

$$\Rightarrow B_6 = \int u^{-3}\left(\frac{1-u^2}{u^2}\right)^{\frac{-2}{3}}du = \frac{-3}{2}\int dv = -\frac{3}{2}v + c = -\frac{3}{2}(\mathrm{tg}\,x)^{\frac{-2}{3}} + c$$

Cách 2: $B_7 = \int \frac{1}{\sqrt[3]{\left(\frac{\sin x}{\cos x}\right)^5}}\cdot\frac{dx}{\cos^2 x} = \int (\mathrm{tg}\,x)^{\frac{-5}{3}}d(\mathrm{tg}\,x) = -\frac{3}{2}(\mathrm{tg}\,x)^{\frac{-2}{3}} + c$

III. Dạng 3: $\boldsymbol{C_{3.1} = \int (tg\ x)^n\,dx\ ;\ C_{3.2} = \int (cotg\ x)^n\,dx}$ $(n \in N)$

1. Công thức sử dụng

- $\int \left(1+tg^2 x\right)dx = \int \frac{dx}{cos^2 x} = \int d\left(tg\ x\right) = tg\ x + c$
- $\int \left(1+cotg^2 x\right)dx = -\int \frac{dx}{sin^2 x} = -\int d\left(cotg\ x\right) = -cotg\ x + c$
- $\int tg\ xdx = \int \frac{\sin x}{\cos x}dx = -\int \frac{d(\cos x)}{\cos x} = -\ln|\cos x| + c$
- $\int cotg\ xdx = \int \frac{\cos x}{\sin x}dx = \int \frac{d(\sin x)}{\sin x} = \ln|\sin x| + c$

2. Các bài tập mẫu minh họa

• $\boldsymbol{C_1 = \int (tgx)^{2k}dx} = \int \Big[(\mathrm{tg}\,x)^{2k-2}\left(1+\mathrm{tg}^2 x\right) - (\mathrm{tg}\,x)^{2k-4}\left(1+\mathrm{tg}^2 x\right) + (\mathrm{tg}\,x)^{2k-6}\left(1+\mathrm{tg}^2 x\right) -$

$$-(\mathrm{tg}\,x)^{2k-8}\left(1+\mathrm{tg}^2 x\right) + \ldots + (-1)^{k-1}(\mathrm{tg}\,x)^0\left(1+\mathrm{tg}^2 x\right) + (-1)^k\Big]dx$$

$$= \int\left[(\mathrm{tg}\,x)^{2k-2} - (\mathrm{tg}\,x)^{2k-4} + (\mathrm{tg}\,x)^{2k-6} - \ldots + (-1)^{k-1}(\mathrm{tg}\,x)^0\right]d(\mathrm{tg}\,x) + (-1)^k\int dx$$

$$= \frac{(\mathrm{tg}\,x)^{2k-1}}{2k-1} - \frac{(\mathrm{tg}\,x)^{2k-3}}{2k-3} + \frac{(\mathrm{tg}\,x)^{2k-5}}{2k-5} - \cdots + (-1)^{k-1}\frac{\mathrm{tg}\,x}{1} + (-1)^k x + c$$

• $\boldsymbol{C_2 = \int (tgx)^{2k+1}dx} = \int \Big[(\mathrm{tg}\,x)^{2k-1}\left(1+\mathrm{tg}^2 x\right) - (\mathrm{tg}\,x)^{2k-3}\left(1+\mathrm{tg}^2 x\right) +$

$$+(\mathrm{tg}\,x)^{2k-5}\left(1+\mathrm{tg}^2 x\right) - \ldots + (-1)^{k-1}(\mathrm{tg}\,x)\left(1+\mathrm{tg}^2 x\right) + (-1)^k\,\mathrm{tg}\,x\Big]dx$$

$$= \int\left[(\mathrm{tg}\,x)^{2k-1} - (\mathrm{tg}\,x)^{2k-3} + (\mathrm{tg}\,x)^{2k-5} - \ldots + (-1)^{k-1}(\mathrm{tg}\,x)\right]d(\mathrm{tg}\,x) + (-1)^k\int \mathrm{tg}\,xdx$$

$$= \frac{(\mathrm{tg}\,x)^{2k}}{2k} - \frac{(\mathrm{tg}\,x)^{2k-2}}{2k-2} + \frac{(\mathrm{tg}\,x)^{2k-4}}{2k-4} - \cdots + (-1)^{k-1}\frac{(\mathrm{tg}\,x)^2}{2} - (-1)^k\ln|\cos x| + c$$

$$\bullet\, \boldsymbol{C_3 = \int (cotgx)^{2k}\, dx} = \int \left[(\text{cotg}\, x)^{2k-2}\left(1+co\,\text{tg}^2\, x\right) - (\text{cotg}\, x)^{2k-4}\left(1+co\,\text{tg}^2\, x\right) + \right.$$

$$\left. + (\text{cotg x})^{2k-6}\left(1+\text{co tg}^2\text{ x}\right) - ... + (-1)^{k-1}(\text{cotg x})^0\left(1+\text{co tg}^2\text{ x}\right) + (-1)^k\right]\text{dx}$$

$$= -\int \left[(\text{cotg x})^{2k-2} - (\text{cotg x})^{2k-4} + ... + (-1)^{k-1}(\text{cotg x})^0\right]\text{d}(\text{cotg x}) + (-1)^k\int \text{dx}$$

$$= -\left[\frac{(\text{cotg x})^{2k-1}}{2k-1} - \frac{(\text{cotg x})^{2k-3}}{2k-3} + \frac{(\text{cotg x})^{2k-5}}{2k-5} - \cdots + (-1)^{k-1}\frac{\text{cotg x}}{1}\right] + (-1)^k\text{ x} + \text{c}$$

$$\bullet\, \boldsymbol{C_4 = \int (cotgx)^{2k+1}\, dx} = \int \left[(\text{cotg}\, x)^{2k-1}\left(1+co\,\text{tg}^2\, x\right) - (\text{cotg}\, x)^{2k-3}\left(1+co\,\text{tg}^2\, x\right) + \right.$$

$$\left. + (\text{cotg x})^{2k-5}\left(1+\text{co tg}^2\text{ x}\right) - ... + (-1)^{k-1}(\text{cotg x})^1\left(1+\text{co tg}^2\text{ x}\right) + (-1)^k\text{ cotg x}\right]\text{dx}$$

$$= -\int \left[(\text{cotg x})^{2k-1} - (\text{cotg x})^{2k-3} + ... + (-1)^{k-1}(\text{cotg x})\right]\text{d}(\text{cotg x}) + (-1)^k\int \text{cotg x dx}$$

$$= -\left[\frac{(\text{cotg x})^{2k}}{2k} - \frac{(\text{cotg x})^{2k-2}}{2k-2} + \cdots + (-1)^{k-1}\frac{(\text{cotg x})^2}{2}\right] + (-1)^k\ln|\sin\text{ x}| + \text{c}$$

$$\bullet\, \boldsymbol{C_5 = \int (tgx + cotgx)^5\, dx} = \int \left[(\text{tg}\, x)^5 + 5(\text{tg}\, x)^4\,\text{cotg}\, x + 10(\text{tg}\, x)^3(\text{cotg}\, x)^2 + \right.$$

$$\left. + 10(\text{tg x})^2(\text{cotg x})^3 + 5\,\text{tg x}(\text{cotg x})^4 + (\text{cotg x})^5\right]\text{dx}$$

$$= \int \left[(\text{tg x})^5 + (\text{cotg x})^5 + 5(\text{tg x})^3 + 5(\text{cotg x})^3 + 10\,\text{tg x} + 10\,\text{cotg x}\right]\text{dx}$$

$$= \int \left[(\text{tg x})^5 + 5(\text{tg x})^3 + 10\,\text{tg x}\right]\text{dx} + \int \left[(\text{cotg x})^5 + 5(\text{cotg x})^3 + 10\,\text{cotg x}\right]\text{dx}$$

$$= \int \left[(\text{tg x})^3\left(1+\text{tg}^2\text{ x}\right) + 4\,\text{tg x}\left(1+\text{tg}^2\text{ x}\right) + 6\,\text{tg x}\right]\text{dx}$$

$$+ \int \left[(\text{cotg x})^3\left(1+\text{cotg}^2\text{ x}\right) + 4\,\text{cotg x}\left(1+\text{cotg}^2\text{ x}\right) + 6\,\text{cotg x}\right]\text{dx}$$

$$= \int \left[(\text{tg x})^3 + 4\,\text{tg x}\right]\text{d}(\text{tg x}) + 6\int \text{tg x dx} - \int \left[(\text{cotg x})^3 + 4\,\text{cotg x}\right]\text{d}(\text{cotg x}) + 6\int \text{cotg x dx}$$

$$= \frac{(\text{tg x})^4}{4} + 2\,\text{tg}^2\text{ x} - 6\ln|\cos\text{ x}| - \frac{(\text{cotg x})^4}{4} - 2\,\text{cotg}^2\text{ x} + 6\ln|\sin\text{ x}| + \text{c}$$

IV. Dạng 4: $\boldsymbol{D_{4.1} = \int \frac{(tg\ x)^m}{(cos\ x)^n}\, dx\ ;\ D_{4.2} = \int \frac{(cotg\ x)^m}{(sin\ x)^n}\, dx}$

1. Phương pháp: Xét đại diện $D_{4.1} = \int \frac{(tg\ x)^m}{(cos\ x)^n}\, dx$

1.1. Nếu n chẵn (n = 2k) thì biến đổi:

$$\boldsymbol{D_{4.1} = \int \frac{(tgx)^m}{(cosx)^{2k}}\, dx} = \int (tg\ x)^m\left(\frac{1}{cos^2\ x}\right)^{k-1}\frac{dx}{cos^2\ x} = \int (tg\ x)^m\left(1+tg^2\ x\right)^{k-1} d\,(tg\ x)$$

$$= \int (\mathrm{tg}\, x)^m \left[C_{k-1}^0 + C_{k-1}^1 (\mathrm{tg}^2\, x)^1 + \ldots + C_{k-1}^p (\mathrm{tg}^2\, x)^p + \ldots + C_{k-1}^{k-1} (\mathrm{tg}^2\, x)^{k-1} \right] d(\mathrm{tg}\, x)$$

$$= C_{k-1}^0 \frac{(\mathrm{tg}\, x)^{m+1}}{m+1} + C_{k-1}^1 \frac{(\mathrm{tg}\, x)^{m+3}}{m+3} + \ldots + C_{k-1}^p \frac{(\mathrm{tg}\, x)^{m+2p+1}}{m+2p+1} + \ldots + C_{k-1}^{k-1} \frac{(\mathrm{tg}\, x)^{m+2k-1}}{m+2k-1} + c$$

1.2. Nếu m lẻ, n lẻ (m = 2k +1, n = 2h +1) thì biến đổi:

$$\boldsymbol{D_{4.1}} = \int \frac{(tgx)^{2k+1}}{(cosx)^{2h+1}} dx = \int (tg\, x)^{2k} \left(\frac{1}{cos\, x} \right)^{2h} \frac{tg\, x}{cosx} dx = \int (tg^2\, x)^k \left(\frac{1}{cos\, x} \right)^{2h} \frac{sin\, x}{cos^2\, x} dx$$

$$= \int \left(\frac{1}{\cos^2 x} - 1 \right)^k \left(\frac{1}{\cos x} \right)^{2h} d\left(\frac{1}{\cos x} \right) = \int (u^2 - 1)^k\, u^{2h} du \qquad (\text{ở đây } u = \frac{1}{\cos x})$$

$$= \int u^{2h} \left[C_k^0 (u^2)^k - C_k^1 (u^2)^{k-1} + \ldots + (-1)^p C_k^p (u^2)^{k-p} + \ldots + (-1)^k C_k^k \right] du$$

$$= C_k^0 \frac{u^{2k+2h+1}}{2k+2h+1} - C_k^1 \frac{u^{2k+2h-1}}{2k+2h-1} + \ldots + (-1)^p C_k^p \frac{u^{2k+2h-2p+1}}{2k+2h-2p+1} + \ldots + (-1)^k C_k^k \frac{u^{2h+1}}{2h+1} + c$$

1.3. Nếu m chẵn, n lẻ (m = 2k, n = 2h + 1) thì sử dụng biến đổi:

$$D_{4.1} = \int \frac{(\mathrm{tg}\, x)^{2k}}{(\cos x)^{2h+1}} dx = \int \frac{(\sin x)^{2k} \cos x}{(\cos x)^{2(k+h+1)}} dx = \int \frac{(\sin x)^{2k}}{(1 - \sin^2 x)^{k+h+1}} d(\sin x); (u = \sin x)$$

$$D_{4.1} = \int \frac{u^{2k} du}{(1-u^2)^{k+h+1}} = \int \frac{u^{2k-2}\left[1 - (1-u^2)\right]}{(1-u^2)^{k+h+1}} du = \int \frac{u^{2k-2} du}{(1-u^2)^{k+h+1}} - \int \frac{u^{2k-2} du}{(1-u^2)^{k+h}}$$

Hệ thức trên là hệ thức truy hồi, kết hợp với bài tích phân hàm phân thức hữu tỉ ta có thể tính được $D_{4.1}$.

2. Các bài tập mẫu minh họa:

• $\boldsymbol{D_1} = \int \frac{(tg3x)^7}{(cos3x)^6} dx = \int (\mathrm{tg}\, 3x)^7 \left[\frac{1}{(\cos 3x)^2} \right]^2 \frac{dx}{(\cos 3x)^2} = \frac{1}{3} \int (\mathrm{tg}\, 3x)^7 (1 + \mathrm{tg}^2\, 3x)^2\, d(\mathrm{tg}\, 3x)$

$$= \frac{1}{3} \int (\mathrm{tg} 3x)^7 \left[1 + 2(\mathrm{tg} 3x)^2 + (\mathrm{tg} 3x)^4 \right] d(\mathrm{tg} 3x) = \frac{1}{3} \left[\frac{(\mathrm{tg} 3x)^8}{8} + 2\frac{(\mathrm{tg} 3x)^{10}}{10} + \frac{(\mathrm{tg} 3x)^{12}}{12} \right] + c$$

• $\boldsymbol{D_2} = \int \frac{(cotg5x)^{10}}{(sin5x)^8} dx = \int (cotg\, 5x)^{10} \left[\frac{1}{(sin\, 5x)^2} \right]^3 \frac{dx}{(sin\, 5x)^2}$

$$= -\frac{1}{5} \int (\mathrm{cotg}\, 5x)^{10} \left[1 + \mathrm{cotg}^2\, 5x \right]^3 d(\mathrm{cotg}\, 5x)$$

$$= -\frac{1}{5} \left[\frac{(\mathrm{cotg}\, 5x)^{11}}{11} + 3\frac{(\mathrm{cotg}\, 5x)^{13}}{13} + 3\frac{(\mathrm{cotg}\, 5x)^{15}}{15} + \frac{(\mathrm{cotg}\, 5x)^{17}}{17} \right] + c$$

• $\boldsymbol{D_3} = \int \frac{(tg4x)^7}{(cos4x)^{95}} dx = \int (tg\, 4x)^6 \left(\frac{1}{cos\, 4x} \right)^{94} \frac{tg\, 4x}{cos\, 4x} dx$

$$=\frac{1}{4}\int\left[\frac{1}{(\cos 4x)^2}-1\right]^3\left(\frac{1}{\cos 4x}\right)^{94}d\left(\frac{1}{\cos 4x}\right)=\frac{1}{4}\int u^{94}\left(u^2-1\right)^3du$$

$$=\frac{1}{4}\int u^{94}\left(u^6-3u^4+3u^2-1\right)du=\frac{1}{4}\left[\frac{u^{101}}{101}-3\frac{u^{99}}{99}+3\frac{u^{97}}{97}-\frac{u^{95}}{95}\right]+c$$

$$=\frac{1}{4}\left[\frac{1}{101(\cos 4x)^{101}}-\frac{1}{33(\cos 4x)^{99}}+\frac{3}{97(\cos 4x)^{97}}-\frac{1}{95(\cos 4x)^{95}}\right]+c$$

• $\boldsymbol{D_4=\int\frac{(cotg3x)^9}{(sin3x)^{41}}dx}=\int(cotg\,3x)^8\left(\frac{1}{sin\,3x}\right)^{40}\frac{cotg\,3x}{sin\,3x}dx$

$$=-\frac{1}{3}\int\left(\frac{1}{\sin^2 x}-1\right)^4\left(\frac{1}{\sin 3x}\right)^{40}d\left(\frac{1}{\sin 3x}\right)=-\frac{1}{3}\int u^{40}\left(u^2-1\right)^4du$$

$$=-\frac{1}{3}\int u^{40}\left(u^8-4u^6+6u^4-4u^2+1\right)^4du=-\frac{1}{3}\left[\frac{u^{49}}{49}-4\frac{u^{47}}{47}+6\frac{u^{45}}{45}-4\frac{u^{43}}{43}+\frac{u^{41}}{41}\right]+c$$

$$=-\frac{1}{3}\left[\frac{1}{49(\sin 3x)^{49}}-\frac{4}{47(\sin 3x)^{47}}+\frac{2}{15(\sin 3x)^{45}}-\frac{4}{43(\sin 3x)^{43}}+\frac{1}{41(\sin 3x)^{41}}\right]+c$$

• $\boldsymbol{D_5=\int\frac{(tgx)^2\,dx}{cosx}}=\int\frac{(sin\,x)^2}{(cos\,x)^2}\cdot\frac{cos\,xdx}{(cos\,x)^2}=\int\left(\frac{sin\,x}{1-sin^2\,x}\right)^2d\,(sin\,x)$

$$=\int\left[\frac{(1+\sin x)-(1-\sin x)}{(1+\sin x)(1-\sin x)}\right]^2d(\sin x)=\int\left(\frac{1}{1-\sin x}-\frac{1}{1+\sin x}\right)^2d(\sin x)$$

$$=\int\left[\frac{1}{(1-\sin x)^2}+\frac{1}{(1+\sin x)^2}-\frac{2}{1-\sin^2 x}\right]d(\sin x)=\frac{1}{1-\sin x}-\frac{1}{1+\sin x}-\ln\left|\frac{1+\sin x}{1-\sin x}\right|+c$$

• $\boldsymbol{D_6=\int\frac{(tgx)^4}{cosx}dx}=\int\frac{(sin\,x)^4}{(cos\,x)^4}\cdot\frac{cos\,xdx}{(cos\,x)^2}=\int\frac{(sin\,x)^4}{\left(1-sin^2\,x\right)^3}d\,(sin\,x)$

$$=\int\frac{u^4du}{\left(1-u^2\right)^3}=\int\frac{1-\left(1-u^4\right)}{\left(1-u^2\right)^3}du=\int\frac{du}{\left(1-u^2\right)^3}-\int\frac{1+u^2}{\left(1-u^2\right)^2}du=I_2-I_1$$

$$I_1=\int\frac{\left(1+u^2\right)du}{\left(1-u^2\right)^2}=\int\frac{\left(1+\frac{1}{u^2}\right)du}{\left(u-\frac{1}{u}\right)^2}=\int\frac{d\left(u-\frac{1}{u}\right)}{\left(u-\frac{1}{u}\right)^2}=-\frac{1}{u-\frac{1}{u}}+c=\frac{u}{1-u^2}+c$$

$$I_2=\int\frac{du}{\left(1-u^2\right)^3}=\frac{1}{8}\int\left[\frac{(1+u)+(1-u)}{(1+u)(1-u)}\right]^3du=\frac{1}{8}\int\left[\frac{1}{1-u}+\frac{1}{1+u}\right]^3du$$

$$=\frac{1}{8}\int\left[\frac{1}{(1-u)^3}+\frac{1}{(1+u)^3}+\frac{3}{\left(1-u^2\right)}\left(\frac{1}{1-u}+\frac{1}{1+u}\right)\right]du$$

$$=\frac{1}{8}\left[\frac{1}{2(1-u)^2}-\frac{1}{2(1+u)^2}+6\int\frac{du}{\left(1-u^2\right)^2}\right]=\frac{1}{8}\left[\frac{(1+u)^2-(1-u)^2}{2\left(1-u^2\right)^2}+3\int\frac{\left(1+u^2\right)+\left(1-u^2\right)}{\left(1-u^2\right)^2}du\right]$$

$$=\frac{u}{4\left(1-u^2\right)^2}+\frac{3}{8}\int\frac{\left(1+u^2\right)du}{\left(1-u^2\right)^2}+\frac{3}{8}\int\frac{du}{1-u^2}=\frac{u}{4\left(1-u^2\right)^2}+\frac{3}{8}I_1+\frac{3}{16}\ln\left|\frac{1+u}{1-u}\right|+c$$

$$\Rightarrow D_6 = I_2 - I_1 = \frac{u}{4\left(1-u^2\right)^2} + \frac{3}{8}I_1 + \frac{3}{16}\ln\left|\frac{1+u}{1-u}\right| - I_1$$

$$= \frac{u}{4\left(1-u^2\right)^2} - \frac{5}{8}\cdot\frac{u}{1-u^2} + \frac{3}{16}\ln\left|\frac{1+u}{1-u}\right| + c = \frac{2u-5u\left(1-u^2\right)}{8\left(1-u^2\right)^2} + \frac{3}{16}\ln\left|\frac{1+u}{1-u}\right| + c$$

$$= \frac{5u^3-3u}{8\left(1-u^2\right)^2} + \frac{3}{16}\ln\left|\frac{1+u}{1-u}\right| + c = \frac{5(\sin x)^3 - 3\sin x}{8(\cos x)^4} + \frac{3}{16}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + c$$

3. Các bài tập dành cho bạn đọc tự giải:

$$D_1 = \int \frac{(\mathrm{tg}\,6x)^{20}}{(\cos 6x)^8}dx\ ;\ D_2 = \int \frac{(\mathrm{cotg}\,3x)^{11}}{(\sin 3x)^{21}}dx;\ \ D_3 = \int \frac{(\mathrm{tg}\,x)^4}{(\cos x)^3}dx\ ;\ D_4 = \int \frac{(\mathrm{cotg}\,2x)^6}{(\cos 2x)^5}dx$$

V. Dạng 5: Sử dụng công thức biến đổi tích thành tổng

1. Phương pháp:

$$E_{5.1} = \int (\cos mx)(\cos nx)\,dx = \frac{1}{2}\int\left[\cos(m-n)x + \cos(m+n)x\right]dx$$

$$E_{5.2} = \int (\sin mx)(\sin nx)\,dx = \frac{1}{2}\int\left[\cos(m-n)x - \cos(m+n)x\right]dx$$

$$E_{5.3} = \int (\sin mx)(\cos nx)\,dx = \frac{1}{2}\int\left[\sin(m+n)x + \sin(m-n)x\right]dx$$

$$E_{5.4} = \int (\cos mx)(\sin nx)\,dx = \frac{1}{2}\int\left[\sin(m+n)x - \sin(m-n)x\right]dx$$

2. Các bài tập mẫu minh họa:

• $\boldsymbol{E_1 = \int \cos2x\,.\cos5x\,.\cos9x\,dx} = \frac{1}{2}\int \cos 2x(\cos 14x + \cos 4x)$

$$= \frac{1}{4}\int\left[(\cos 16x + \cos 12x) + (\cos 6x + \cos 2x)\right]dx = \frac{1}{4}\left(\frac{\sin 16x}{16} + \frac{\sin 12x}{12} + \frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right) + c$$

• $\boldsymbol{E_2 = \int (\cos x)^3\ \sin8x\,dx} = \int \frac{(3\cos x + \cos 3x)}{4}\sin 8x\,dx$

$$= \frac{1}{4}\int(3\cos x\sin 8x + \cos 3x\sin 8x)\,dx = \frac{1}{4}\int\left[\frac{3}{2}(\sin 9x + \sin 7x) + \frac{1}{2}(\sin 11x + \sin 5x)\right]dx$$

$$= -\frac{1}{8}\left(\frac{3}{9}\cos 9x + \frac{3}{7}\cos 7x + \frac{1}{11}\cos 11x + \frac{1}{5}\cos 5x\right) + c$$

• $\boldsymbol{E_3 = \int (\sin x)^4\ (\sin3x)(\cos10x)\,dx} = \frac{1}{8}\int (1-\cos 2x)^2\,(\sin 13x + \sin 7x)\,dx$

$$=\frac{1}{8}\int\left(1-2\cos 2x+\cos^2 2x\right)(\sin 13x+\sin 7x)dx$$

$$=\frac{1}{8}\int\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)(\sin 13x+\sin 7x)dx$$

$$=\frac{1}{16}\int(3-4\cos 2x+\cos 4x)(\sin 13x+\sin 7x)dx$$

$$=\frac{1}{16}\int\left[3(\sin 13x+\sin 7x)-4\cos 2x(\sin 13x+\sin 7x)+\cos 4x(\sin 13x+\sin 7x)\right]dx$$

$$=\frac{1}{16}\int\left[3(\sin 13x+\sin 7x)-2(\sin 15x+\sin 11x+\sin 9x+\sin 5x)+\right.$$

$$\left.+\frac{1}{2}(\sin 17x+\sin 9x+\sin 11x+\sin 3x)\right]dx$$

$$=\frac{1}{32}\int(\sin 17x-4\sin 15x+6\sin 13x-3\sin 11x-3\sin 9x+6\sin 7x-4\sin 5x+\sin 3x)dx$$

$$=\frac{-1}{32}\left(\frac{\cos 17x}{17}-\frac{4\cos 15x}{15}+\frac{6\cos 13x}{13}-\frac{3\cos 11x}{11}-\frac{\cos 9x}{3}+\frac{6\cos 7x}{7}-\frac{4\cos 5x}{5}+\frac{\cos 3x}{3}\right)+c$$

• $\boldsymbol{E_4=\int(\cos x)^5(\sin 5x)dx}=\int(\cos x)^3(\cos x)^2(\sin 5x)dx$

$$=\int\frac{\cos 3x+3\cos x}{4}\cdot\frac{1+\cos 2x}{2}\cdot\sin 5x\,dx$$

$$=\frac{1}{8}\int\left[(\cos 3x+3\cos x)\sin 5x+(\cos 3x+3\cos x)\cos 2x\sin 5x\right]dx$$

$$=\frac{1}{8}\int\left[(\cos 3x+3\cos x)\sin 5x+(\cos 3x+3\cos x)\frac{\sin 7x+\sin 3x}{2}\right]dx$$

$$=\frac{1}{16}\int\left[2\sin 5x(\cos 3x+3\cos x)+(\cos 3x+3\cos x)(\sin 7x+\sin 3x)\right]dx$$

$$=\frac{1}{32}\int\left[2(\sin 8x+\sin 2x)+6(\sin 6x+\sin 4x)+(\sin 10x+\sin 4x)+\right.$$

$$\left.+3(\sin 8x+\sin 6x)+\sin 6x+3(\sin 4x+\sin 2x)\right]dx$$

$$=\frac{1}{32}\int(\sin 10x+5\sin 8x+10\sin 6x+10\sin 4x+5\sin 2x)dx$$

$$=\frac{-1}{32}\left(\frac{\cos 10x}{10}+\frac{5\cos 8x}{8}+\frac{5\cos 6x}{3}+\frac{5\cos 4x}{2}+\frac{5\cos 2x}{2}\right)+c$$

• $\boldsymbol{E_5=\int\frac{(\sin 3x)(\sin 4x)}{tgx+cotg2x}dx}=\int\frac{(\sin 3x)(\sin 4x)}{\frac{\sin x}{\cos x}+\frac{\cos 2x}{\sin 2x}}dx=\int\frac{(\sin 3x)(\sin 4x)}{\frac{\cos(2x-x)}{\cos x.\sin 2x}}dx$

$$=\int(\sin 2x)(\sin 3x)(\sin 4x)dx=\frac{1}{2}\int(\cos 2x-\cos 6x)\sin 3x\,dx$$

$$=\frac{1}{4}\int\left[\left(\sin 5x+\sin x\right)-\left(\sin 9x-\sin 3x\right)\right]dx=\frac{-1}{4}\left(\frac{\cos 5x}{5}+\frac{\cos x}{1}-\frac{\cos 9x}{9}+\frac{\cos 3x}{3}\right)+c$$

3. Các bài tập dành cho bạn đọc tự giải:

$$E_1=\int\left(\sin 3x\right)^4\left(\cos 2x\right)^3 dx\ ;\ E_2=\int\left(\sin x\right)^5\left(\cos 5x\right)^2 dx\ ;\ E_3=\int\frac{\left(\sin 8x\right)^5 dx}{\left(\operatorname{tg} 3x+\operatorname{tg} 5x\right)^2}$$