

LIP6

M2 MEMOIR

# Automatic asymptotics for combinatorial series

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## Abstract

Enumerative combinatorics is interested in determining the number  $a_n$  of objects of size  $n$  in a class of combinatorial objects. Alternatively, rather than a complicated closed formula, one would like to obtain an asymptotic expansion of  $a_n$ . In analysis of algorithms, for instance, computing asymptotic expansions is used to compare performance, and therefore a salient question.

Due to the fabulous diversity of combinatorial structures, such computations have long required intuition and specially crafted “tricks”, only adapted to the problem at hand, or some close family.

Nowadays, powerful techniques have been developed, enabling one to study a vast amount of combinatorial constructions with standard procedures. In fact, most of the steps involved have now been separately implemented, for instance in the `ore_algebra` module of *SageMath*.

One frequent case of application of these techniques is the *D-Finite* case, where the generating series of  $(a_n)$  is characterized by a differential equation with polynomial coefficients, and initial values of the  $a_i$ s.

This internship is devoted to understanding and effectively programming a combination of these techniques in *SageMath*, in the D-Finite case. In the end, one should be able to type in a differential equation associated to  $a_n$ , and our code shall return an asymptotic expansion of  $a_n$ , up to any desired order.

The novelty resides in the merging of several existing functions used in different parts of the analysis, associated to the computation of explicit constants.

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# Chapter 1

## Introduction

Pick your favourite combinatorial construction. Let  $a_n$  the number of such structures of size  $n$ . We wish to be able to compute an asymptotic expansion of  $a_n$  automatically.

Let  $f = \sum a_n z^n$  be the complex series associated to  $(a_n)$ .

We shall see that, if  $f$  has a positive convergence radius, then one has asymptotically  $a_n = A^n \theta(n)$  where  $\theta$  has sub-exponential growth. Two principles, stated in [FS09], shall guide one's search :

- *First Principle of Coefficient Asymptotics* : The location of a function's singularities dictates the exponential growth ( $A^n$ ) of its coefficients.
- *Second Principle of Coefficient Asymptotics* : The nature of a function's singularities determines the associate subexponential factor ( $\theta(n)$ ).

### 1.1 State of the art

In 1990, Flajolet and Odlyzko [FO90] proved transfer theorems, that allow one to TODO

### 1.2 Mathematical sketch

**From a combinatorial problem to a differential equation** Powerful techniques exist to translate a combinatorial construction into a *D-finite* relation, namely a differential equation with polynomial coefficients. We will not cover those techniques here. If interested, one is referred to [FS09].

From now on, we will assume that a non trivial D-finite relation satisfied by  $f$  is given, which is a relation of the form

$$y^{(r)} + \frac{p_{r-1}}{p_r} y^{(r-1)} + \dots + \frac{p_0}{p_r} y = 0 \quad (1.1)$$

where  $p_0, \dots, p_r \in \mathbb{C}[X]$ .

D-finite (holonomic) function

**Definition 1.** A function satisfying a D-finite relation will be said D-finite itself, or holonomic.

**Singularities location** We will first see that  $f$  may only have singularities at roots of  $p_r$ . Thereafter, we define  $\Xi := \{\text{roots of } p_r\}$ . If  $f$  has at least one singularity, minimal ones (by module) are called *dominant singularities*.

**Local basis structure theorems** Following the definition of *regular singular points*, where some technical condition is satisfied, it can be proved that, in a *slit* neighbourhood of any such point  $\zeta$ , equation (1.1) admits a local basis of solutions of the form

$$(z - \zeta)^{\theta_j} \log^m(z - \zeta) H_j(z - \zeta)$$

with  $H_j$  analytic at 0. This basis can be explicitly computed.

**Transfer theorems** We then investigate *transfer theorems*. Assume  $f$  has at least one singularity, and all dominant singularities are regular singular points. After expressing  $f$  in the previous form around all dominant singularities, transfer theorems allow one to compute an asymptotic expansion of  $f_n$ .

## 1.3 Implementation overview

The implementation is in SageMath, mostly (and vastly) relying on the `ore_algebra` and `AsymptoticRing` modules.

**extract\_asymptotics** A function `extract_asymptotics` is implemented, with the following definition:

```
1 def extract_asymptotics(op,
2                           first_coefficients,
3                           z,
4                           order=DEFAULT_ORDER,
5                           precision=DEFAULT_PRECISION) -> expr
```

For a holonomic function  $f$ , `extract_asymptotics` takes

- A differential operator `op`, such that  $op \cdot f = 0$
- A list `first_coefficients` of the first Taylor coefficients of  $f$
- The variable `z` used in definition of `op`, only for technical reasons
- The desired `order` of the asymptotic expansion
- The desired certified `precision` for the constants

It returns an asymptotic expansion of the coefficients of  $f$ , up to the desired `order` and with constants certified at least with the given `precision`.

**Global structure** We first locate the roots of  $p_r$ , and group them by increasing module. Then, as long as no root has been *proved* to be a singularity of  $f$ , we iterate through the groups and sum their contributions. A root of  $p_r$  can indeed not always be proved to be a singularity of  $f$  only by computing the coefficients of  $f$  in the local basis: if one of these coefficients is precisely 0, successive approximations will never be able to distinguish it from 0, yet not proving either that it would be nil.

Then for each root, we make use of `local_basis_expansions`, defined in the `ore_algebra` module. That function allows one to compute a local basis of solutions to `op`, along with their expansion up to any desired order. A call to `numerical_transition_matrix` then allows us to determine the expression of  $f$  in that local basis, with constants certified to the desired `precision`.

Calling `SingularityAnalysis` (a specially crafted function from the `asymptotic_ring` module) on each term appearing and summing the results finally yields the desired expansion.

# Chapter 2

## Mathematical background

### 2.1 Notations

**Definition 2.** Let  $f$  be a differentiable function. We note  $f^{(k)}$  its  $k$ -th derivative.

### 2.2 Reminders from complex analysis

#### Leibniz rule

**Theorem 1.**

$$(fg)^{(n)} = \sum_{i=0}^n \binom{n}{i} f^{(n-i)} g^{(i)}$$

#### Analytic functions

**Definition 3.** A function  $f$  is said to be analytic on  $\Omega$  if, for all  $z_0 \in \Omega$ , it admits an expansion

$$f(z) = \sum_{n \geq 0} f_n(z - z_0)^n$$

that converges on some neighbourhood of  $z_0$ .

#### Cauchy's integral formula

**Theorem 2.** Let  $\Omega$  be an open subset of  $\mathbb{C}$ .

Let  $\omega \in \Omega$  and  $\rho > 0$  such that  $\mathcal{B}_f(\omega, \rho) \subset \Omega$ .

Let  $f$  be holomorphic on  $\Omega$ .

Then for all  $z_0 \in \mathcal{B}(\omega, \rho)$ , we have

$$f(z_0) = \frac{1}{2i\pi} \int_{\mathcal{S}(\omega, r)} \frac{f(z)}{z - z_0} dz$$

## Cauchy's coefficient formula

**Corollary 1.** *Let  $f$  be analytic on some neighbourhood  $\Omega$  of  $z_0 \in \mathbb{C}$ , and  $r > 0$  such that  $\mathcal{B}_f(z_0, r) \subset \Omega$ , then for all  $n$ , one has*

$$f_n = \frac{1}{2i\pi} \int_{\mathcal{C}(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

**Theorem 3.** *Let  $f$  be analytic in some neighbourhood  $\Omega$  of  $z_0 \in \mathbb{C}$ , and  $f_n$  such that on  $\Omega$  one can write*

$$f(z) = \sum_{n \geq 0} f_n (z - z_0)^n$$

*then for all  $n$ , we have*

$$f_n = \frac{f^{(n)}(z_0)}{n!}$$

TODO

### 2.2.1 Complex logarithm

**A few identities** For a reference on the following definitions and identities, the reader is referred to [BC96].

From now on, except when explicitly stated otherwise,  $\log z$  and  $\arg z$  will stand for the principal determination of the complex logarithm and argument.

$N_+, N_-$

**Definition 4.** *Let  $z_1, z_2 \in \mathbb{C}^*$ . Define  $N_+(z_1, z_2)$  and  $N_-(z_1, z_2)$  with*

$$N_{\pm} = \begin{cases} -1 & \text{if } \pi < \arg(z_1) \pm \arg(z_2) \\ 0 & \text{if } -\pi < \arg(z_1) \pm \arg(z_2) \leq \pi \\ 1 & \text{if } \arg(z_1) \pm \arg(z_2) \leq -\pi \end{cases}$$

**Remark 1.** *The previous definition is intended to have to following relations hold:*

$$\begin{cases} \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) + 2\pi N_+ \\ \arg\left(\frac{z_1}{z_2}\right) &= \arg(z_1) - \arg(z_2) + 2\pi N_- \end{cases}$$

**Prop. 1.** *Let  $a, b, c \in \mathbb{C}^*$ . Then*

$$\begin{aligned} \log(ab) &= \log a + \log b + 2i\pi N_+(a, b) \\ \log\left(\frac{a}{b}\right) &= \log a - \log b + 2i\pi N_-(a, b) \\ (ab)^c &= a^c \times b^c \times e^{2i\pi c N_+(a, b)} \\ \left(\frac{a}{b}\right)^c &= \frac{a^c}{b^c} e^{2i\pi c N_-(a, b)} \end{aligned}$$



**Corollary 2.** *Let  $x \in \mathbb{R}^{+*}$  and  $z, t \in \mathbb{C}^*$ . Then  $\arg(x) = 0$ , so all classical identities over the real numbers extend identically:*

$$\begin{aligned}\log(xz) &= \log x + \log z \\ \log\left(\frac{x}{z}\right) &= \log x - \log z \\ (xz)^t &= x^t \times z^t \\ \left(\frac{x}{z}\right)^t &= \frac{x^t}{z^t}\end{aligned}$$

## 2.3 Some differential equations theory

In this section, we will present the results that enable one study a differential equation of the form (1.1).

### 2.3.1 Scalar and system equations

**Differential equations** The equation

$$y^{(r)} = a_{r-1}(z)y^{(r-1)} + \cdots + a_0(z)y \quad (2.1)$$

where the  $a_i$  are holomorphic is said to be a *scalar* (differential) equation.

**Differential systems** The equation

$$Y' = A(z)Y \quad (2.2)$$

where  $A(z)$  is an  $n \times n$  matrix and  $Y(z)$  is an  $n$ -dimensional vector is said to be a (differential) *system of equations*.

**From scalar to system** The following transformation is a classical trick to transform a scalar equation into a system:

If  $y$  is a solution to

$$y^{(r)}(z) = a_{r-1}(z)y^{(r-1)}(z) + \cdots + a_0(z)y(z)$$

then  $Y : z \mapsto \begin{pmatrix} y(z) \\ y'(z) \\ \vdots \\ y^{(n-1)}(z) \end{pmatrix}$  is a solution to

$$Y' = A(z)Y'(z)$$

where

$$A(z) = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ 0 & & \ddots & 1 \\ a_0(z) & \dots & a_{r-1}(z) & \end{pmatrix}$$

We call  $A$  the *companion matrix* of equation (2.3.1).

### 2.3.2 Solutions space

**Basis of solutions** The following classical theorem is admitted.

#### Cauchy's existence and uniqueness theorem

**Theorem 4.** *Let  $n$  an integer. Let also  $A(z)$  an  $n \times n$ -matrix and  $f(z)$  an  $n$ -dimensional vector, both holomorphic in some simply connected region  $\Omega \subset \mathbb{C}$ .*

*Then, for any  $z_0 \in \Omega$  and  $y_0 \in \mathbb{C}^n$ , the equation*

$$Y' = A(z)Y \tag{2.3}$$

*has a unique solution such that*

$$y(z_0) = y_0$$

*That solution is holomorphic on  $\Omega$ .*

It immediately follows

#### Basis of solutions for systems

**Corollary 3.** *Let  $z_0 \in \mathbb{C}$ . Suppose there exists a neighbourhood  $\Omega$  of  $z_0$  such that  $A$  and  $f$  are holomorphic on  $\Omega$ .*

*Then the set of solutions to equation (2.3) defined in  $\Omega$  forms an  $n$ -dimensional vector space.*

## 2.4 Location of Singularities

### Existence of a local basis of solutions

**Theorem 5.** *Let  $p_0, \dots, p_r \in \mathbb{C}[X]$  and  $z_0 \in \mathbb{C}$  such that  $p_r(z_0) \neq 0$ . Then, in some neighbourhood of  $z_0$ , the equation*

$$y^{(r)} + \frac{p_{r-1}}{p_r} y^{(r-1)} + \dots + \frac{p_0}{p_r} y = 0 \quad (2.4)$$

*admits a basis of  $r$  analytic solutions.*

*Proof.* The polynomial  $p_r$  has finite degree, therefore has a finite number of roots. Since  $p_r(z_0) \neq 0$ , there is some neighbourhood  $\Omega$  of  $z_0$  where  $p_r$  does not vanish. It follows that all  $\frac{p_i}{p_r}$  are analytic on  $\Omega$ . Cauchy's theorem then applies and concludes the proof. ■

### Possible locations of singularities

**Corollary 4.** *The only points where  $f$  may admit singularities are the zeros of  $p_r$ .*

## 2.5 Structure theorems

For further treatment of this section, one is referred to [Was65] (chapter II in particular).

### 2.5.1 Another transformation

Let  $\zeta \in \mathbb{C}$ ,  $y$  such that

$$y^{(r)} + a_{r-1}(z)y^{(r-1)} + \dots + a_0(z)y = 0$$

and define  $Y : z \mapsto \begin{pmatrix} y(z) \\ \vdots \\ (z - \zeta)^{r-1} y^{(r-1)}(z) \end{pmatrix}$  (that is,  $Y_i : z \mapsto (z - \zeta)^{i-1} y^{(i-1)}(z)$ ).

For all  $i \leq r - 1$ , we have

$$(z - \zeta)Y'_i = (i - 1)Y_i + Y_{i+1}$$

and

$$(z - \zeta)Y'_r = (r - 1)Y_r - (z - \zeta)a_{r-1}Y_{r-1} - \dots - (z - \zeta)^r a_0 Y_1$$

Then, equation (1.1) is equivalent to

$$(z - \zeta)Y' = A_\zeta(z)Y \quad (2.5)$$

where  $A_\zeta(z)$  is an  $r \times r$  matrix.

## 2.5.2 Regular singular points and indicial polynomials

### Regular singular points

**Definition 5.** Consider a differential equation  $(E)$  of the form (1.1).

We say that  $\zeta$  is a regular singular point of  $(E)$ , and  $\zeta$  is a pole of  $\frac{p_i}{p_r}$  of order at most  $r - i$ , for all  $i \in [0, r - 1]$ .

Equivalently,  $\zeta$  is a regular singular point if  $A_\zeta(z)$  is analytic in some neighbourhood of  $\zeta$ , when one writes  $(z - \zeta)Y' = A_\zeta(z)Y$ .

### Indicial polynomial, $I_\zeta$

**Definition 6.** The characteristic polynomial of  $A_\zeta(\zeta)$  is named the indicial polynomial of equation (1.1) and (2.5) at  $\zeta$ , denoted  $I_\zeta$ .

## 2.5.3 Results

### General structure theorem

**Theorem 6.** Let  $\zeta$  be a regular singular point of (1.1). No assumption is made on the roots of  $I_\zeta$ .

Then, in a slit neighbourhood of  $\zeta$ , there exists a basis of solutions of the form

$$(z - \zeta)^{\theta_j} (\log(z - \zeta))^m H_j(z - \zeta) \quad (2.6)$$

where  $\theta_j$  are the roots of the indicial polynomial, each  $H_j$  is analytic at 0, and  $m \in \mathbb{N}$ .

## 2.5.4 Special cases

### G-functions

**Definition 7.** A formal series  $f = \sum f_n z^n \in \mathbb{Q}[[z]]$  is called a G-function if it is D-finite and there exists  $C > 0$  such that for all  $n$ , we have

$$\begin{cases} |f_n| < C^n \\ \text{lcd}(f_1, \dots, f_n) < C^n \end{cases}$$

where  $\text{lcd}(f_1, \dots, f_n)$  is the least common denominator of  $f_1, \dots, f_n$ .

### André-Chudnovsky-Katz Theorem

**Theorem 7.** Let  $f$  be a G-function. Then a minimal order annihilating D-finite equation for  $f$  has only ordinary or regular singular points, and its indicial polynomial has only rational roots.

# Chapter 3

## Singularity analysis and transfer theorems

In the previous sections, we have seen that a holonomic function  $f$  admits, in the neighbourhood of any singularity, an asymptotic expansion of the form (2.6).

In this section, we show how to compute an asymptotic expansion of the coefficients of these functions, and present a *transfer theorem* allowing one to directly deduce an asymptotic expansion of  $f$ .

### 3.1 Singularity analysis

#### Polynomial case

**Theorem 8.** *If  $\alpha \in \mathbb{N}$ , then  $[z^n](1 - \frac{z}{\xi})^\alpha = 0$  for  $n > k$  because  $(1 - \frac{z}{\xi})^\alpha$  is a polynomial. So that case can be completely ruled out in estimating asymptotic expansions.*

#### 3.1.1 Basic scale transfer

We quote from [FS09]

## Basic scale transfer

**Theorem 9.** *Let  $\alpha$  be an arbitrary complex number in  $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ . The coefficient of  $z^n$  in*

$$f(z) = (1 - z)^{-\alpha}$$

*admits for large  $n$  a complete asymptotic expansion in descending powers of  $n$ ,*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left( 1 + \sum_{k=1}^{\infty} \frac{e_k^{(\alpha)}}{n^k} \right)$$

*where  $e_k^{(\alpha)}$  is a polynomial in  $\alpha$  of degree  $2k$ . More precisely,*

$$e_k^{(\alpha)} = \sum_{i=k}^{2k} (-1)^i \lambda_{k,i} (\alpha + 1)(\alpha + 2) \dots (\alpha + i)$$

*with  $\sum_{k,i \geq 0} \lambda_{k,i} v^k t^i = e^t (1 + vt)^{-1-1/v}$ .*

### 3.1.2 Complete scale

In this section, we closely follow [Jun31].

**Lemma 1.** *Let  $k \in \mathbb{N}^*$  and  $f = \sum f_n z^n$  with*

$$f(z) = (1 - z)^{-k}$$

*Then for all  $n$ , we have*

$$f_n = \frac{n^{k-1}}{\Gamma(k)} \left[ 1 + \frac{k(k-1)}{2n} + \dots + \frac{\Gamma(k)}{n^{k-1}} \right]$$

*Proof.* Start with

$$(1 - z)^{-1} = \sum z^n$$

Now differentiating that relation  $k - 1$  times, we get

$$(k-1)!(1-z)^{-k} = \sum \frac{(n+k-1)!}{n!} z^n$$

Therefore  $f_n = \frac{1}{(k-1)!} (n+1)(n+2) \dots (n+k-1)$ , and the result follows by developing the product and grouping by powers of  $n$ . ■

The following three results are quoted from [Jun31] without proof. For a reference, one may consult [KY28]. They shall be useful in the proof of theorem 10 to assert the domains of validity of the computed expansions.

**Lemma 2.** *Let  $\phi(z)$  admit an asymptotic expansion*

$$\varphi(z) \sim c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots \quad (3.1)$$

*as  $z$  goes to infinity following a half line  $d$ .*

*Then for every constant  $z_0$ , we also have, asymptotically along  $d$ ,*

$$\varphi(z_0 + z) \sim c_0 + \frac{c_1}{z} + \frac{-c_1 z_0 + c_2}{z^2} + \dots$$

**Lemma 3.** *Let  $\varphi(z)$  be analytic in the form (3.1) on a half band*

$$\begin{cases} \Re(z) > a \\ \Im(z) \in ]-b, b[ \end{cases} \quad (3.2)$$

*for  $a$  and  $b$  arbitrary positive real numbers.*

*Then  $e^{\varphi(z)}$  can also be expanded, over the same band:*

$$e^{\varphi(z)} \sim e^{c_0} \left( 1 + \frac{c_1}{z} + \dots \right)$$

**Lemma 4.** *Let  $\varphi(z)$  be analytic and expandable in the form (3.1) over a band (3.2), we also have*

$$\varphi'(z) \sim -\frac{c_1}{z^2} - \frac{2c_2}{z^3} - \dots$$

*over any tighter band*

$$\begin{cases} \Re(z) > a \\ \Im(z) \in ]-b + \varepsilon, b + \varepsilon[ \end{cases} \quad (3.3)$$

**Lemma 5.** *Let  $i$  an integer,  $n$  a natural number and  $s \in \mathbb{C} \setminus \{-1, -2, \dots\}$ .*

*Then there exists functions  $\psi_{i,j}$  that can be expanded asymptotically, such that*

$$\frac{\Gamma^{(i)}(n+s)}{\Gamma(n+1)} = n^{s-1} [(\log n)^i \psi_{i,0}(n) + \dots + \psi_{i,i}(n)] \quad (3.4)$$

*Proof.* Start from Stirling's series for  $\log \Gamma(z)$ :

$$\begin{aligned} \log \Gamma(z) &\sim \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} \\ &\sim \frac{1}{2} \log(2\pi) + \left(z + \frac{1}{2}\right) \log(z) - z + \frac{1}{12z} - \frac{1}{360z^3} + \dots \end{aligned}$$

where  $B_n$  is the  $n$ -th Bernoulli number.

Then by lemma 3

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \cdot z^{-1/2} \cdot \varphi(z)$$

where  $\varphi(z)$  can be expanded in asymptotic series over any half-band of type (3.2). Now, differentiating  $i$  times, we get

$$\Gamma^{(i)}(z) = \left(\frac{z}{e}\right)^z z^{-1/2} \left[ (\log z)^i \varphi_{i,0}(z) + \cdots + \varphi_{i,i}(z) \right]$$

where the functions  $\varphi_{i,j}$  can be expanded into asymptotic series, by lemmas 2 and 4. Therefore,

$$\frac{\Gamma^{(i)}(n+s)}{\Gamma(n+1)} = \left(\frac{n}{e}\right)^{s-1} \frac{\left(1 + \frac{s}{n}\right)^{n+s-1/2}}{\left(1 + \frac{1}{n}\right)^{n+1/2}} \cdot \frac{(\log(n+s))^i \varphi_{i,0}(n+s) + \cdots + \varphi_{i,i}(n+s)}{\varphi(n+1)}$$

and we may finally define the functions  $\psi_{i,j}$  such that

$$\frac{\Gamma^{(i)}(n+s)}{\Gamma(n+1)} = n^{s-1} [(\log n)^i \psi_{i,0}(n) + \cdots + \psi_{i,i}(n)]$$

■

#### Expansion theorem in the log case

**Theorem 10.** *Let  $a \in \mathbb{C}$  and  $k \in \mathbb{N}^*$ .*

*Let*

$$f : z \mapsto (1-z)^a \left( \log \frac{1}{1-z} \right)^k$$

*then for large  $n$  one has*

$$f_n = \begin{cases} \frac{n^{-a-1}}{\Gamma(-a)} \sum_{i=0}^k (\log n)^i \phi_i(n) & \text{if } a \notin \mathbb{N} \\ (-1)^a k \Gamma(1+a) n^{-a-1} \sum_{i=0}^k (\log n)^i \phi_i(n) & \text{if } a \in \mathbb{N} \end{cases}$$

*where the functions  $\phi_i$  admit asymptotic expansions of the form*

$$\phi_i \sim c_{i,0} + \frac{c_{i,1}}{n} + \frac{c_{i,2}}{n^2} + \cdots$$

*and the constants  $c_{i,j}$  can be explicitly computed.*

*Proof.* Assume  $a \notin \mathbb{N}$ . We can define

$$\phi_0 : z \mapsto \Gamma(-a)(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{\Gamma(n-a)}{n!} z^n$$

By differentiating  $i$  times with respect to  $a$ , we get  $\frac{d^i}{da} \Gamma(-a) = (-1)^i \Gamma^{(i)}(-a)$ , and  $\frac{d^i}{da} (1-z)^{-a} = \frac{d^i}{da} e^{-a \log(1-z)} = \frac{d^i}{da} e^{a \log \frac{1}{1-z}} = \left( \log \frac{1}{1-z} \right)^i (1-z)^{-a}$  so by Leibniz' rule:

$$\phi_i := \frac{d^i}{da^i} \phi_0 = (1-z)^{-a} \sum_{j=0}^i \binom{i}{j} (-1)^j \Gamma^{(j)}(-a) \left( \log \frac{1}{1-z} \right)^{i-j}$$



Now, when  $i$  takes successively the values  $0, \dots, k$ , we get a triangular system of linear equations of unknowns the functions  $(1-z)^a \left(\log \frac{1}{1-z}\right)^i$ , the solution of which has the form

$$(1-z)^a \left(\log \frac{1}{1-z}\right)^k = \frac{1}{\Gamma(-a)} [\phi_k(z) + d_{k,k-1}\phi_{k-1}(z) + \dots + d_{k,0}\phi_0(z)] \quad (3.5)$$

where the coefficients  $d_{i,j}$  are explicitly computable and only depend on  $i$  and  $j$ . Now, by definition of the  $\phi_i$ s, we have for all  $i$

$$\phi_i(z) = (-1)^i \sum_{n=0}^{\infty} \frac{\Gamma^{(i)}(n-a)}{n!} z^n$$

By expanding equation (3.5) into Taylor series, this leads to the following equality

$$f_n = \frac{1}{n! \Gamma(-a)} [\Gamma^{(k)}(n-a) + d_{k,k-1} \Gamma^{(k-1)}(n-a) + \dots + d_{k,0} \Gamma(n-a)]$$

We now use lemma 5 to conclude (recall that  $n! = \Gamma(n+1)$ ).

\*\*\*

To deal with the case  $a \in \mathbb{N}$ , use the relation

$$\begin{aligned} (1-z)^a \left(\log \frac{1}{1-z}\right)^k &= -a \int (1-z)^{a-1} \left(\log \frac{1}{1-z}\right)^k \\ &\quad + k \int (1-z)^{a-1} \left(\log \frac{1}{1-z}\right)^{k-1} \end{aligned}$$

$a+1$  times to reduce to the first case. ■

## 3.2 Transfer theorems

The content of this section is from [FS09], section VI.

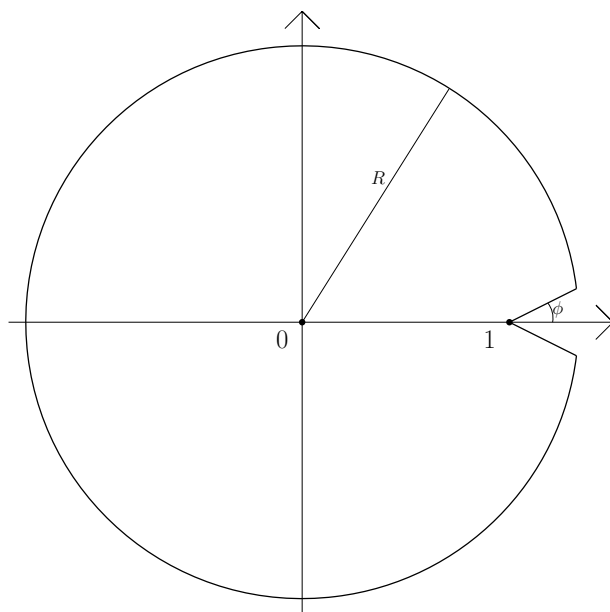
$\Delta(\phi, R)$

**Definition 8.** For any positive numbers  $\phi$  and  $R$  such that  $R > 1$  and  $\phi \in ]0, \frac{\pi}{2}[$ , we define

$$\Delta(\phi, R) := \{z \mid |z| < R, z \neq 1 \text{ and } |\arg(z-1)| > \phi\}$$

A domain  $D$  such that there exists  $\phi$  and  $R$  as above and  $D = \Delta(\phi, R)$  will be called a  $\Delta$ -domain.

A function that is analytic on some  $\Delta$ -domain will be called  $\Delta$ -analytic.

Figure 3.1: A  $\Delta$ -domainTransfer theorem for  $O$  and  $o$ 

**Theorem 11.** *Let  $\alpha, \beta \in \mathbb{R}$  and  $f$  be a  $\Delta$ -analytic function.*

- *If  $f$  is such that, in the some neighbourhood of 1 and its  $\Delta$ -domain, one has*

$$f(z) = O\left((1-z)^{-\alpha}\left(\log \frac{1}{1-z}\right)^\beta\right)$$

*Then*

$$[z^n]f(z) = O\left(n^{\alpha-1}(\log n)^\beta\right)$$

- *The same result holds, with  $O$  replaced by  $o$ .*

# Chapter 4

## Implementation

Section 4.1 presents the global structure of the algorithm, then details some parts. Section 4.2 briefly presents the tests. Finally, section 4.3 discusses performance aspects.

### 4.1 Structure

We first present here the main algorithm, and the important sub-algorithms. The implementation is mostly transparent, except for a few optimizations like computing the local solutions in 0 only once.

**Algorithm 1:** Main algorithm

**Input:**  $f$  defined by a  $a_n \frac{d^n}{dz^n} f + \dots a_0 = 0$ , initial coefficients  $f_0, \dots, f_n$ , order and precision

**Output:** Asymptotic expansion of  $f$  with at least order terms, and coefficients with given precision

**begin**

    Compute the roots of  $a_n$  and group them by increasing module.

    Initialise the sum  $S$  of contributions to the asymptotic expansion.

**while** *No contribution confirmed* **do**

        Load next group  $G$  of roots

**foreach** *root*  $\rho \in G$  **do**

            | Compute the contribution of  $\rho$ .

**end**

        Sum contributions of  $G$  and add to  $S$

**end**

**return**  $S$

**end**

To compute the contribution of a specific root, we use the following algorithm

---

**Algorithm 2:** Computing the contribution of a root

---

**Input:** All entries of the main algorithm, and a specific root  $\rho$   
**Output:** Contribution of  $\rho$  to the asymptotic expansion of  $f$

```

begin
  Compute a basis  $\mathcal{B}_0$  of solutions in 0, and decompose  $f$  in  $\mathcal{B}_0$ .
  Compute a basis  $\mathcal{B}_\rho$  of solutions in  $\rho$ , and decompose  $f$  in  $\mathcal{B}_\rho$ .
  Initialize a sum  $S$  of terms contributions.
  foreach term  $T$  in the local expansions do
    Compute an asymptotic expansion of the Taylor coefficients of  $T$ .
    Add to  $S$ .
  end
end
return  $S$ 

```

---

## 4.2 Tests

We give here a list of functions, along with an associated differential operator and an asymptotic expansion. We provide a name when the Taylor series is of combinatorial interest, and a reference for the not obvious/classical cases.

$f$	Differential operator	Asymptotic expansion	Sequence name
$\log(1 - z)$	$(z - 1)Dz^2 + Dz$	$\frac{-1}{n}$	
$\log(1 + z)$	$(z + 1)Dz^2 + Dz$	$\frac{(-1)^n}{n}$	
$\frac{1}{1-z}$	$(z - 1)Dz + 1$	1	
$\frac{1}{1+z}$	$(z - 1)Dz + 1$	$(-1)^n$	
$\frac{z}{1-2z}$	$z(1 - 2z)Dz - 1$	$2^{n-1}$	
$\frac{1}{1-z^2}$	$(1 - z^2)Dz - 2z$	$\begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$	
$(1 - z)^{3/2}$	$2(z - 1)Dz - 3$	$\frac{1}{\sqrt{\pi n^5}} \left( \frac{3}{4} + \frac{45}{32n} + \frac{1155}{512n^2} + \dots \right)$	[FS09]
$\arctan(z)$	$(1 + z^2)Dz^2 + 2zDz$	$\begin{cases} \frac{(-1)^n}{2n+1} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$	
$\frac{1}{1-z} \log\left(\frac{1}{1-z}\right)$	$(1 - z)^2 Dz^2 - 3(1 - z)Dz + 1$	$\log(n) + \gamma + \frac{1}{2n} + o\left(\frac{1}{n}\right)$	Harmonic numbers
$\frac{z}{1-z-z^2}$	$(1 - z - z^2)Dz^2 - (2 + 4z)Dz - 2$	$\frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$	Fibonacci numbers
$\frac{1 - \sqrt{1-4z}}{2z}$	$(4z^2 - z)Dz^2 + (10z - 2)Dz + 2$	$\frac{4^n}{\sqrt{\pi n}} \left( 1 - \frac{1}{8n} + \frac{1}{128n^2} + \dots \right)$	Catalan numbers
$\frac{1-z-\sqrt{1-2z-3z^2}}{2z}$	$(3z^4 + 2z^3 - z^2)Dz^2 + (6z^3 + 3z^2 - z)Dz + 1$	$\frac{\sqrt{3}}{2\sqrt{\pi n} 3^{n/2}} 3^n \left( 1 - \frac{15}{16n} + \frac{505}{512n^2} + \dots \right)$	Motzkin numbers [FS09]
	$(4z^2 - z)Dz^2 + (14z - 2)Dz + 6$	$\frac{4^n}{\pi n} \left( 4 - \frac{6}{n} + \frac{19-2(-1)^n}{2n^2} + \dots \right)$	Walks in $\mathbb{N}^2$ [Mel20]

## 4.3 Performance

TODO

# Chapter 5

## Conclusion

The goal of this internship was to produce a piece of code, able to take in a function  $f$ , in the form of a differential equation of which  $f$  is solution, and the first Taylor coefficients of  $f$  in 0, and return an asymptotic expansion of  $f$  up to any desired order, with explicit constants also computed with certified digits up to any desired precision. That goal is achieved, as various tests indicate.

Although many steps involved had already been implemented in **Sagemath**, no function was available to perform the full analysis. Therefore, Marc and I intend to submit the code for acceptance to **ore\_algebra**.

For the future, possible improvements include handling singularities in 0, entire series, and computing explicit bounds on the error made by the computed asymptotic expansion.

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