LIP6

# Automatic asymptotics for combinatorial series

*Sébastien JULLIOT*
Université Paris Diderot, Paris

Supervised by
Marc MEZZAROBBA
LIP6, Paris

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# Abstract

Enumerative combinatorics is interested in determining the number $a_n$ of objects of size $n$ in a class of combinatorial objects. Alternatively, one would like to obtain an asymptotic expansion of $a_n$. In analysis of algorithms, computing asymptotic expansions is used to compare performance, and therefore also a salient question.

Due to the fabulous diversity of combinatorial structures, such computations have long required intuition and specially crafted "tricks", only adapted to the problem at hand, or some close family.

Nowadays, powerful techniques have been developed, enabling one to study a vast amount of combinatorial constructions with standard procedures. In fact, most of the steps involved have now been implemented, for instance in the `ore_algebra` module of *SageMath*.

One very general case of application of these techniques is the *D-Finite* case.

This internship is devoted to understanding and effectively programming a complete combination of these techniques in *SageMath*, in the D-Finite case.

In the end, one should be able to type in a differential equation associated to $a_n$, and our code shall return an asymptotic expansion of $a_n$, up to any desired order.

# Contents

# Chapter 1

# Introduction

Pick your favourite combinatorial construction. Let $a_n$ the number of such structures of size $n$. We wish to be able to compute an asymptotic expansion of $a_n$ automatically. Let $f = \sum a_n z^n$ be the complex series associated to $(a_n)$.

We shall see that, if $f$ has a positive convergence radius, then one has asymptotically $a_n = A^n \theta(n)$ where $\theta$ has sub-exponential growth. Two principles, stated in [FS09], shall guide one's search :

- *First Principle of Coefficient Asymptotics* : The location of a function's singularities dictates the exponential growth $(A^n)$ of its coefficients.

- *Second Principle of Coefficient Asymptotics* : The nature of a function's singularities determines the associate subexponential factor $(\theta(n))$.

## 1.1 Mathematical sketch

**From a combinatorial problem to a differential equation** Powerful techniques exist to translate a combinatorial construction into a *D-finite* relation, namely a differential equation with polynomial coefficients. We will not cover those techniques here. If interested, one is referred to [FS09].

From now on, we will assume that a non trivial D-finite relation satisfied by $f$ is given, which is

$$y^{(r)} + \frac{p_{r-1}}{p_r} y^{(r-1)} + \cdots + \frac{p_0}{p_r} y = 0 \tag{1.1}$$

where $p_0, \ldots, p_r \in \mathbb{C}[X]$.

**Singularities location** We will first see that $f$ may only have singularities at roots of $p_r$. Thereafter, we define $\Xi := \{\text{roots of } p_r\}$. If $f$ has at least one singularity, minimal ones (by module) are called *dominant singularities*.

**Local basis structure theorems** Following the definition of *regular singular points*, where some technical condition is satisfied, we prove that, in a *slit* neighbourhood of any

such point $\zeta$, equation (1.1) admits a local basis of solutions of the form

$$(z - \zeta)^{\theta_j} \log^m(z - \zeta) H_j(z - \zeta)$$

with $H_j$ analytic at 0. This basis can be explicitly computed.

**Transfer theorems**   We then investigate *transfer theorems*. Assume $f$ has at least one singularity, and all dominant singularities are regular singular points. After expressing $f$ in the previous form around all dominant singularities, transfer theorems allow one to compute an asymptotic expansion of $f_n$.

## 1.2   Implementation overview

TODO

# Chapter 2

# Mathematical background

## 2.1 Reminders from complex analysis

TODO

## 2.2 Some differential equations theory

### 2.2.1 Scalar and system equations

**Scalar equations**  The equation

$$y^{(r)} = a_{r-1}(z)y^{(r-1)} + \cdots + a_0(r)y \tag{2.1}$$

where the $a_i$ are holomorphic is said to be a *scalar* (differential) equation.

**System equations**  The equation

$$Y' = A(z)Y \tag{2.2}$$

where $A(z)$ is an $n \times n$ matrix and $Y(z)$ is an $n$-dimensional vector is said to be a *system* (differential) equation.

**From scalar to system**  The following transformation is a classical trick to transform a scalar equation into a system one:
If $y$ is a solution to

$$y^{(r)} = a_{r-1}(z)y^{(r-1)} + \cdots + a_0(z)y$$

then $Y : z \mapsto \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$ is a solution to

$$Y' = A(z)Y$$

where

$$A(z) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ a_0(z) & & \dots & & a_{r-1}(z) \end{pmatrix}$$

We call $A$ the *companion matrix* of equation (2.2.1).

## 2.2.2   Solutions space

**Basis of solutions**   The following classical theorem is admitted.

Cauchy's existence and uniqueness theorem

**Theorem 1.** *Let $n$ an integer. Let also $A(z)$ an $n \times n$-matrix and $f(z)$ an $n$-dimensional vector, both holomorphic in some simply connected region $\Omega \subset \mathbb{C}$. Then the equation*

$$Y' = A(z)Y + f(z) \tag{2.3}$$

*has a unique solution such that*

$$y(z_0) = y_0$$

*where $z_0 \in \Omega$ and $y_0 \in \mathbb{C}^n$.*
*That solution is holomorphic on $\Omega$.*

It immediately follows

Basis of solutions for systems

**Corollary 1.** *Let $z_0 \in \mathbb{C}$. Suppose there exists a neighbourhood $\Omega$ of $z_0$ such that $A$ and $f$ are holomorphic on $\Omega$.*
*Then the set of solutions to equation (2.3) defined in $z_0$ forms an $n$-dimensional vector space.*

## 2.3    Singularities location

**Existence of a local basis of solutions**

**Theorem 2.** *Let $p_0, \ldots, p_r \in \mathbb{C}[X]$ and $z_0 \in \mathbb{C}$ such that $p_r(z_0) \neq 0$. Then, in some neighbourhood of $z_0$, the equation*

$$y^{(r)} + \frac{p_{r-1}}{p_r}y^{(r-1)} + \cdots + \frac{p_0}{p_r}y = 0 \qquad (2.4)$$

*admits a basis of $r$ analytic solutions.*

*Proof.* The polynomial $p_r$ has finite degree, therefore has a finite number of roots. Since $p_r(z_0) \neq 0$, there is some neighbourhood $\Omega$ of $z_0$ where $p_r$ does not vanish. It follows that all $\frac{p_i}{p_r}$ are analytic on $\Omega$.
Cauchy's theorem then applies and concludes the proof.  ∎

**Possible locations of singularities**

**Corollary 2.** *The only points where $f$* may *admit singularities are the zeros of $p_r$.*

## 2.4    Structure theorems

For further treatment of this section, one is referred to [Was65] (chapter II in particular).

### 2.4.1    Another transformation

Let $\zeta \in \mathbb{C}$, $f$ be a solution of

$$u^{(r)} + a_{r-1}(z)u^{(r-1)} + \cdots + a_0(z)u = 0$$

and define $Y : z \mapsto \begin{pmatrix} f(z) \\ \vdots \\ (z-\zeta)^{r-1}f^{(r-1)}(z) \end{pmatrix}$ (that is, $Y_i : z \mapsto (z-\zeta)^{i-1}f^{(i-1)}(z)$).

It is immediate that for all $i \leq r - 1$, we have

$$(z-\zeta)Y_i' = (i-1)Y_i + Y_{i+1}$$

and

$$(z-\zeta)Y_r' = (r-1)Y_r - (z-\zeta)a_{r-1}Y_{r-1} - \cdots - (z-\zeta)^r a_0 Y_1$$

Then, equation 1.1 is easily seen to be equivalent to

$$(z-\zeta)Y' = A_\zeta(z)Y \qquad (2.5)$$

where $A_\zeta(z)$ is an $r \times r$ matrix.

### 2.4.2   Regular singular points and indicial polynomials

**Regular singular points**

> **Definition 1.** *Let $f$ solution to equation* (1.1).
> *We say $\zeta$ is a* regular singular *point of $f$ is singular in $\zeta$, and $\zeta$ is a pole of $\frac{p_i}{p_r}$ of order at most $r - i$, for all $i \in [0, r-1]$.*
> *Equivalently, $\zeta$ is a regular singular point if $A_\zeta(z)$ is analytic in some neighbourhood of $\zeta$, when one writes $(z - \zeta)Y' = A_\zeta(z)Y$.*

**Indicial polynomial, $I_\zeta$**

> **Definition 2.** *The characteristic polynomial of $A_\zeta(\zeta)$ is named the* indicial polynomial *of equation 1.1 and 2.5 at $\zeta$, written $I_\zeta$.*

### 2.4.3   Results

**General structure theorem**

> **Theorem 3.** *Let $\zeta$ be a regular singular point of* (1.1). *No assumption is made on the roots of $I_\zeta$.*
> *Then, in a slit neighbourhood of $\zeta$, there exists a basis of solutions with functions of the form*
> $$(z - \zeta)^{\theta_j}(\log(z - \zeta))^m H_j(z - \zeta) \tag{2.6}$$
> *where $\theta_j$ are the roots of the indicial polynomial, each $H_j$ is analytic at 0, and $m$ is an integer.*

## 2.5   Transfer theorems

**Main transfer theorem**

> **Theorem 4.** *Let $f = \sum f_n z^n$ be analytic in some open disk around the origin except for a single dominant singularity $\zeta$ and points in the ray $R_\zeta = \{t\zeta : t \geq 1\}$. If $f$ has a convergent expansion of the form* (2.6) *in a disk centred at $\zeta$ minus $R_\zeta$ and $f(z) \sim C\left(1 - \frac{z}{\zeta}\right)^\alpha \left(\log\left(1 - \frac{z}{\zeta}\right)\right)^m$ with $\alpha \notin \mathbb{N}$, then*
> $$f_n \sim C\zeta^{-n} \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} (\log n)^m$$

# Chapter 3

# Implementation

TODO

# Bibliography

[Was65]  Wolfgang Wasow. *Asymptotic expansions for ordinary differential equations.* 1965.

[FS09]  Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics.* 2009, pp. 1–810. ISBN: 9780511801655. DOI: 10.1017/CBO9780511801655.