LIP6

M2 memoir

Automatic asymptotic for combinatorial series

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Abstract

Enumerative combinatorics is interested in determining the number a_n of objects of size n in a class of combinatorial objects. Alternatively, one would like to obtain an asymptotic expansion of a_n. In algorithms analysis, computing asymptotic expansions is used to compare performances, and therefore also a salient question.

Due to the fabulous diversity of combinatorial structures, such computations have long required intuition and specially crafted "tricks", only adapted to the problem at hand, or some close family.

Nowadays, powerful techniques have been developed, that are able to enumerate precisely or compute an asymptotic expansion of a vast amount of combinatorial constructions. In fact, such generality has been attained that most of the steps involved may now be carried out automatically in seconds by any modern computer.

One very general case of application of these techniques is the *D-Finite* case.

This internship is devoted to understanding and effectively programming a complete combination of these techniques in *SageMath*, in the D-Finite case, relying in particular on the `ore_algebra` module.

In the end, one should be able to type in a D-finite relation associated to a_n, and our code shall return an asymptotic expansion of a_n, up to any desired order.

Contents

Chapter 1

Introduction

Pick your favourite combinatorial construction. Let a_n the number of such structures of size n. We wish to be able to compute an asymptotic expansion of a_n automatically.

Let $f = \sum a_n z^n$ the complex series associated to (a_n).
We shall see that, if f has at least one singularity, then asymptotically $a_n = A^n \theta(n)$ where θ has sub-exponential growth. Furthermore, two principles shall guide one's search :

- *First Principle of Coefficient Asymptotics* : The location of a function's singularities dictates the exponential growth (A^n) of its coefficients.
- *Second Principle of Coefficient Asymptotics* : The nature of a function's singularities determines the associate subexponential factor $(\theta(n))$.

1.1 Mathematical sketch

From a combinatorial problem to a differential equation Powerful techniques exist to translate a combinatorial construction into a *D-finite* relation, namely a differential equation with polynomial coefficients. We will not cover those techniques here. If interested, one is referred to [FS09].

From now on, we will assume that a D-finite relation satisfied by f is given, which is

$$y^{(r)} + \frac{p_{r-1}}{p_r} y^{(r-1)} + \cdots + \frac{p_0}{p_r} y = 0 \tag{1.1}$$

where $p_0, \ldots, p_r \in \mathbb{C}[X]$.

Singularities location We will first see that f may only have singularities at roots of p_r. Thereafter, we define $\Xi := \{\text{roots of } p_r\}$. If f has at least one singularity, minimal ones (by module) are called *dominant singularities*.

Local basis structure theorems Following the definition of *regular singular points*, where some technical condition is satisfied, we prove that, in a *slit* neighbourhood of any such point ζ, equation 1.1 admits a local basis of solutions of the form

$$(z-\zeta)^{\theta_j} \log^m(z-\zeta) H_j(z-\zeta)$$

with H_j analytic at 0. This basis can be explicitly computed.

Transfer theorems We then investigate *transfer theorems*. Assume f has at least one singularity, and all dominant singularities are regular singular points. After expressing f in the previous form around all dominant singularities, transfer theorems allow to compute an asymptotic expansion f_n.

1.2 Implementation overview

TODO

Chapter 2

Mathematical background

2.1 Reminders from complex analysis

TODO

2.2 Singularities location

Existence of a local basis of solutions

Theorem 1. *Let $z_0 \in \mathbb{C}$ such that $p_r(z_0) \neq 0$.*
Then, in some neighbourhood of z_0, equation 1.1 admits a basis of r analytic solutions.

Proof. We will show that a formal solution a uniquely defined by initial values of f_i. Then, that the thereby defined formal sum is analytic in some neighbourhood of z_0.

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Let $g = \sum g_n z^n \in \mathbb{C}[[X]]$.
Then g is solution of equation 1.1 if and only if its coefficients satisfy

$$p_r(z) \sum_n \frac{(n+r)!}{n!} g_{n+r} z^n + \cdots + p_0(z) \sum_n g_n z^n = 0$$

which is again equivalent to

$$\sum_n \left[p_r(z) \frac{(n+r)!}{n!} g_{n+r} + \cdots + p_0(z) g_n \right] z^n = 0$$

Now, by formal sum equality, we get a necessary and sufficient condition :

$$\forall n, p_r(z_0) \frac{(n+r)!}{n!} g_{n+r} + \cdots + p_0(z_0) g_n = 0 \tag{2.1}$$

Assuming $p_r(z_0) \neq 0$, a formal solution is therefore determined by its r first coefficients, and we get a basis of r formal solutions.

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Let $C_0 := \max_{k \in [0, r-1]} |g_k|$ and $M := \max \left| \frac{p_k(z_0)}{p_r(z_0)} \right|$.
By equation 2.1 and triangle inequality, one gets

$$|g_r| \leq M r C_0$$

An immediate recurrence then yields $|g_{n+r}| \leq C_0 (Mr)^n$ for all n.
So any formal solution's coefficients have at most exponential growth ; it follows that any formal solution is actually analytic in some neighbourhood of z_0. ■

Possible locations of singularities

Corollary 1. *The only points where f* may *admit singularities are the zeros of p_r.*

2.3 Structure theorems

For further treatment of this section, one is referred to [Was65] (chapter II in particular).

2.3.1 Transformation into a matrix system

Recall equation 1.1 :

$$u^{(r)} + \frac{p_{r-1}}{p_r} u^{(r-1)} + \cdots + \frac{p_0}{p_r} u = 0$$

Let us rename $a_i = \frac{p_i}{p_r}$. Now define $Y : z \mapsto \begin{pmatrix} u \\ xu' \\ \vdots \\ x^{r-1}u^{(n-1)} \end{pmatrix}$ (that is, $y_i = x^{i-1}u^{(i-1)}$).

It is immediate that for all $i \leq r-1$, we have

$$zy_i' = (j-1)y_j + y_{j+1}$$

and

$$zy_r' = (r-1)y_r - za_{r-1}y_{r-1} - \cdots - z^r a_0 y_1$$

Then, equation 1.1 is easily seen to be equivalent to

$$zY' = A(z)Y \tag{2.2}$$

where $A(z)$ is an $r \times r$ matrix.

2.3.2 Regular singular points and indicial polynomials

Regular singular points and Fuchsian condition

Definition 1. *Let $f : z \mapsto \sum a_n z^n$ solution to 1.1.*
We say ζ is a regular singular *point of f if ζ is a pole of $\frac{p_i}{p_r}$ of order at most $r-i$, for all $i \in [0, r-1]$.*

Equivalently, ζ is a regular singular point if $A(z)$ is analytic in some neighbourhood of ζ, when one writes $(z-\zeta)Y' = A(z)Y$.

A linear differential equation with only regular singular points is called Fuchsian.

Indicial polynomial, I_ζ

Definition 2. *The characteristic polynomial of $A(\zeta)$ is named the* indicial polynomial *of equation 1.1 and 2.2 at ζ, written I_ζ.*

2.3.3 Results

Structure theorem without congruent roots

Theorem 2. *Let ζ be a singular regular point of 1.1. Suppose I_ζ is such that no two roots differ by an integer (in particular, all roots are distinct).*
Then, in a slit neighbourhood of ζ, there exists a basis of solutions with functions of the form

$$(z-\zeta)^{\theta_j} H_j(z-\zeta) \tag{2.3}$$

where θ_j are the roots of the indicial polynomial and each H_j is analytic at 0.

General structure theorem

Theorem 3. *Let ζ be a singular regular point of 1.1. No assumption is made on the roots of I_ζ.*
Then, in a slit neighbourhood of ζ, there exists a basis of solutions with functions of the form

$$(z-\zeta)^{\theta_j} \log^m(z-\zeta) H_j(z-\zeta) \tag{2.4}$$

where θ_j are the roots of the indicial polynomial, each H_j is analytic at 0, and m is an integer.

2.4 Transfer theorems

Main transfer theorem

Theorem 4. *Let $f = \sum f_n z^n$ analytic in some open disk around the origin except for a single dominant singularity ζ and points in the ray $R_\zeta = \{t\zeta : t \geq 1\}$.*

If f has a convergent expansion of the form 2.4 in a disk centred at ζ minus R_ζ and $f(z) \sim C\left(1-\frac{z}{\zeta}\right)^\alpha \log^m\left(1-\frac{z}{\zeta}\right)$ with $\alpha \notin \mathbb{N}$, then

$$f_n \sim C\zeta^{-n}\frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\log^m(n)$$

Chapter 3

Implementation

TODO

Bibliography

[Was65] Wolfgang Wasow. *Asymptotic expansions for ordinary differential equations.* 1965.

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics.* 2009, pp. 1–810. ISBN: 9780511801655. DOI: 10.1017/CBO9780511801655.