LIP6

M2 MEMOIR

Automatic asymptotics for combinatorial series

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Abstract

Enumerative combinatorics is interested in determining the number a_n of objects of size n in a class of combinatorial objects. Alternatively, rather than a complicated closed formula, one would like to obtain an asymptotic expansion of a_n. In analysis of algorithms, for instance, computing asymptotic expansions is used to compare performance, and therefore a salient question.

Due to the fabulous diversity of combinatorial structures, such computations have long required intuition and specially crafted "tricks", only adapted to the problem at hand, or some close family.

Nowadays, powerful techniques have been developed, enabling one to study a vast amount of combinatorial constructions with standard procedures. In fact, most of the steps involved have now been separately implemented, for instance in the `ore_algebra` module of *SageMath*.

One very general case of application of these techniques is the *D-Finite* case.

This internship is devoted to understanding and effectively programming a complete combination of these techniques in *SageMath*, in the D-Finite case. In the end, one should be able to type in a differential equation associated to a_n, and our code shall return an asymptotic expansion of a_n, up to any desired order.

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Chapter 1

Introduction

Pick your favourite combinatorial construction. Let a_n the number of such structures of size n. We wish to be able to compute an asymptotic expansion of a_n automatically.
Let $f = \sum a_n z^n$ be the complex series associated to (a_n).
We shall see that, if f has a positive convergence radius, then one has asymptotically $a_n = A^n \theta(n)$ where θ has sub-exponential growth. Two principles, stated in [FS09], shall guide one's search :

- *First Principle of Coefficient Asymptotics* : The location of a function's singularities dictates the exponential growth (A^n) of its coefficients.
- *Second Principle of Coefficient Asymptotics* : The nature of a function's singularities determines the associate subexponential factor $(\theta(n))$.

1.1 Mathematical sketch

From a combinatorial problem to a differential equation Powerful techniques exist to translate a combinatorial construction into a *D-finite* relation, namely a differential equation with polynomial coefficients. We will not cover those techniques here. If interested, one is referred to [FS09].
From now on, we will assume that a non trivial D-finite relation satisfied by f is given, which is

$$y^{(r)} + \frac{p_{r-1}}{p_r} y^{(r-1)} + \cdots + \frac{p_0}{p_r} y = 0 \tag{1.1}$$

where $p_0, \ldots, p_r \in \mathbb{C}[X]$.

Singularities location We will first see that f may only have singularities at roots of p_r. Thereafter, we define $\Xi := \{\text{roots of } p_r\}$. If f has at least one singularity, minimal ones (by module) are called *dominant singularities*.

Local basis structure theorems Following the definition of *regular singular points*, where some technical condition is satisfied, we prove that, in a *slit* neighbourhood of any such point ζ, equation (1.1) admits a local basis of solutions of the form

$$(z-\zeta)^{\theta_j} \log^m(z-\zeta) H_j(z-\zeta)$$

with H_j analytic at 0. This basis can be explicitly computed.

Transfer theorems We then investigate *transfer theorems*. Assume f has at least one singularity, and all dominant singularities are regular singular points. After expressing f in the previous form around all dominant singularities, transfer theorems allow one to compute an asymptotic expansion of f_n.

1.2 Implementation overview

TODO

Chapter 2

Mathematical background

2.1 Notations

Definition 1. *Let f be a differentiable function. We note $f^{(k)}$ its k-th derivative.*

Open balls, closed balls, circles

Definition 2. *For $z_0 \in \mathbb{C}$ and $r > 0$ we define*

$\mathscr{B}(z_0, r)$ *the open ball of center z_0 and radius r as*

$$\mathscr{B}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$$

$\mathscr{B}_f(z_0, r)$ *the closed ball of center z_0 and radius r as*

$$\mathscr{B}_f(z_0, r) := \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

$\mathscr{C}(z_0, r)$ *the circle of center z_0 and radius r as*

$$\mathscr{C}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| = r\}$$

2.2 Reminders from complex analysis

Leibniz rule

Theorem 1.

$$(fg)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)} g^{(i)}$$

Analytic functions

Definition 3. *A function f is said to be* analytic *on Ω if, for all $z_0 \in \Omega$, it admits an expansion*

$$f(z) = \sum_{n \geq 0} f_n (z - z_0)^n$$

that converges on some neighbourhood of z_0.

Cauchy's integral formula

Theorem 2. *Let Ω be an open subset of $\mathbb{C}$.*
Let $\omega \in \Omega$ and $\rho > 0$ such that $\mathscr{B}_f(\omega, \rho) \subset \Omega$.
Let f be holomorphic on Ω.
Then for all $z_0 \in \mathscr{B}(\omega, \rho)$, we have

$$f(z_0) = \frac{1}{2i\pi} \int_{\mathscr{S}(\omega,r)} \frac{f(z)}{z - z_0} dz$$

Cauchy's coefficient formula

Corollary 1. *Let f be analytic on some neighbourhood Ω of $z_0 \in \mathbb{C}$, and $r > 0$ such that $\mathscr{B}_f(z_0, r) \subset \Omega$, then for all n, one has*

$$f_n = \frac{1}{2i\pi} \int_{\mathscr{C}(z_0,r)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Theorem 3. *Let f be analytic in some neighbourhood Ω of $z_0 \in \mathbb{C}$, and f_n such that on Ω one can write*

$$f(z) = \sum_{n \geq 0} f_n (z - z_0)^n$$

then for all n, we have

$$f_n = \frac{f^{(n)}(z_0)}{n!}$$

TODO

2.2.1 Complex logarithm

A few identities For a reference over the following definitions and identities, the reader is referred to [BC96].
From now on, except explicitly stated otherwise, $\log z$ and $\arg z$ will stand for the principal determination of the complex logarithm and argument.

N_+, N_-

Definition 4. *Let $z_1, z_2 \in \mathbb{C}^*$. Define $N_+(z_1, z_2)$ and $N_-(z_1, z_2)$ with*

$$N_\pm = \begin{cases} -1 & \text{if } \pi < \arg(z_1) \pm \arg(z_2) \\ 0 & \text{if } -\pi < \arg(z_1) \pm \arg(z_2) \leq \pi \\ 1 & \text{if } \arg(z_1) \pm \arg(z_2) \leq -\pi \end{cases}$$

Remark 1. *The previous definition is intended to have to following relations hold:*

$$\begin{cases} \arg(z_1 z_2) & = \arg(z_1) + \arg(z_2) + 2\pi N_+ \\ \arg\left(\frac{z_1}{z_2}\right) & = \arg(z_1) - \arg(z_2) + 2\pi N_- \end{cases}$$

Prop. 1. *Let $a, b, c \in \mathbb{C}^*$. Then*

$$\log(ab) = \log a + \log b + 2i\pi N_+(a, b)$$
$$\log\left(\frac{a}{b}\right) = \log a - \log b + 2i\pi N_-(a, b)$$
$$(ab)^c = a^c \times b^c \times e^{2i\pi c N_+(a,b)}$$
$$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c} e^{2i\pi c N_-(a,b)}$$

Corollary 2. *Let $x \in \mathbb{R}^{+*}$ and $z, t \in \mathbb{C}^*$. Then $\arg(x) = 0$, so all classical identities over real numbers extend identically:*

$$\log(xz) = \log x + \log z$$
$$\log\left(\frac{x}{z}\right) = \log x - \log z$$
$$(xz)^t = x^t \times z^t$$
$$\left(\frac{x}{z}\right)^t = \frac{x^t}{z^t}$$

2.3 Some differential equations theory

2.3.1 Scalar and system equations

Scalar equations The equation

$$y^{(r)} = a_{r-1}(z) y^{(r-1)} + \cdots + a_0(r) y \tag{2.1}$$

where the a_i are holomorphic is said to be a *scalar* (differential) equation.

System equations The equation

$$Y' = A(z)Y \tag{2.2}$$

where $A(z)$ is an $n \times n$ matrix and $Y(z)$ is an n-dimensional vector is said to be a *system* (differential) equation.

From scalar to system The following transformation is a classical trick to transform a scalar equation into a system one:
If y is a solution to

$$y^{(r)} = a_{r-1}(z)y^{(r-1)} + \cdots + a_0(z)y$$

then $Y : z \mapsto \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$ is a solution to

$$Y' = A(z)Y$$

where

$$A(z) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & \text{\Large 0} & \\ \text{\Large 0} & & \ddots & 1 & \\ a_0(z) & & \dots & a_{r-1}(z) & \end{pmatrix}$$

We call A the *companion matrix* of equation (2.3.1).

2.3.2 Solutions space

Basis of solutions The following classical theorem is admitted.

Cauchy's existence and uniqueness theorem

Theorem 4. *Let n an integer. Let also $A(z)$ an $n \times n$-matrix and $f(z)$ an n-dimensional vector, both holomorphic in some simply connected region $\Omega \subset \mathbb{C}$.*
Then the equation

$$Y' = A(z)Y + f(z) \tag{2.3}$$

has a unique solution such that

$$y(z_0) = y_0$$

where $z_0 \in \Omega$ and $y_0 \in \mathbb{C}^n$.
That solution is holomorphic on Ω.

It immediately follows

Basis of solutions for systems

Corollary 3. *Let $z_0 \in \mathbb{C}$. Suppose there exists a neighbourhood Ω of z_0 such that A and f are holomorphic on Ω.*
Then the set of solutions to equation (2.3) defined in z_0 forms an n-dimensional vector space.

2.4 Singularities location

Existence of a local basis of solutions

Theorem 5. *Let $p_0, \dots, p_r \in \mathbb{C}[X]$ and $z_0 \in \mathbb{C}$ such that $p_r(z_0) \neq 0$.*
Then, in some neighbourhood of z_0, the equation

$$y^{(r)} + \frac{p_{r-1}}{p_r} y^{(r-1)} + \cdots + \frac{p_0}{p_r} y = 0 \tag{2.4}$$

admits a basis of r analytic solutions.

Proof. The polynomial p_r has finite degree, therefore has a finite number of roots. Since $p_r(z_0) \neq 0$, there is some neighbourhood Ω of z_0 where p_r does not vanish. It follows that all $\frac{p_i}{p_r}$ are analytic on Ω.
Cauchy's theorem then applies and concludes the proof. ∎

Possible locations of singularities

Corollary 4. *The only points where f* may *admit singularities are the zeros of p_r.*

2.5 Structure theorems

For further treatment of this section, one is referred to [Was65] (chapter II in particular).

2.5.1 Another transformation

Let $\zeta \in \mathbb{C}$, f be a solution of

$$u^{(r)} + a_{r-1}(z) u^{(r-1)} + \cdots + a_0(z) u = 0$$

and define $Y : z \mapsto \begin{pmatrix} f(z) \\ \vdots \\ (z-\zeta)^{r-1} f^{(r-1)}(z) \end{pmatrix}$ (that is, $Y_i : z \mapsto (z-\zeta)^{i-1} f^{(i-1)}(z)$).
It is immediate that for all $i \leq r-1$, we have

$$(z-\zeta) Y_i' = (i-1) Y_i + Y_{i+1}$$

and

$$(z-\zeta)Y_r' = (r-1)Y_r - (z-\zeta)a_{r-1}Y_{r-1} - \cdots - (z-\zeta)^r a_0 Y_1$$

Then, equation (1.1) is easily seen to be equivalent to

$$(z-\zeta)Y' = A_\zeta(z)Y \tag{2.5}$$

where $A_\zeta(z)$ is an $r \times r$ matrix.

2.5.2 Regular singular points and indicial polynomials

Regular singular points

Definition 5. *Let f solution to equation* (1.1)*.*
We say ζ is a regular singular *point of f is singular in ζ, and ζ is a pole of $\frac{p_i}{p_r}$ of order at most $r-i$, for all $i \in [0, r-1]$.*
Equivalently, ζ is a regular singular point if $A_\zeta(z)$ is analytic in some neighbourhood of ζ, when one writes $(z-\zeta)Y' = A_\zeta(z)Y$.

Indicial polynomial, I_ζ

Definition 6. *The characteristic polynomial of $A_\zeta(\zeta)$ is named the* indicial polynomial *of equation* (1.1) *and* (2.5) *at ζ, written I_ζ.*

2.5.3 Results

General structure theorem

Theorem 6. *Let ζ be a regular singular point of* (1.1)*. No assumption is made on the roots of I_ζ.*
Then, in a slit neighbourhood of ζ, there exists a basis of solutions with functions of the form

$$(z-\zeta)^{\theta_j}(\log(z-\zeta))^m H_j(z-\zeta) \tag{2.6}$$

where θ_j are the roots of the indicial polynomial, each H_j is analytic at 0, and $m \in \mathbb{N}$.

2.5.4 Special cases

G-functions

Definition 7. *A formal series $f = \sum f_n z^n \in \mathbb{Q}[[z]]$ is called a* G-function *if it is D-finite and there exists $C > 0$ such that for all n, we have*

$$\begin{cases} |f_n| < C^n \\ lcd(f_1, \ldots, f_n) < C^n \end{cases}$$

André-Chudnovsky-Katz Theorem

Theorem 7. *Let f be a G-function. Then a minimal order annihilating D-finite equation for f has only ordinary or regular singular points, and its indicial polynomial has only rational roots.*

2.6 Transfer theorems

All the following results come, directly or not, from [FO90].

Main transfer theorem

Theorem 8. *Let $f = \sum f_n z^n$ be analytic in some open disk around the origin except for a single dominant singularity ζ and points in the ray $R_\zeta = \{t\zeta : t \geq 1\}$. If f has a convergent expansion of the form (2.6) in a disk centred at ζ minus R_ζ and $f(z) \sim C\left(1 - \frac{z}{\zeta}\right)^\alpha \left(\log\left(1 - \frac{z}{\zeta}\right)\right)^m$ with $\alpha \notin \mathbb{N}$, then*

$$f_n \sim C\zeta^{-n} \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} (\log n)^m$$

Polynomial case

Theorem 9. *If $\alpha \in \mathbb{N}$ and $k = 0$, then $[z^n]\left(1 - \frac{z}{\zeta}\right)^\alpha = 0$ for $n > k$ because $\left(1 - \frac{z}{\zeta}\right)^\alpha$ is a polynomial. So that case can be completely ruled out in estimating asymptotic expansions.*

2.6.1 Basic scale transfer

We quote from [FS09]

Basic scale transfer

Theorem 10. *Let α be an arbitrary complex number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The coefficient of z^n in*

$$f(z) = (1-z)^{-\alpha}$$

admits for large n a complete asymptotic expansion in descending powers of n,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \sum_{k=1}^{\infty} \frac{e_k^{(\alpha)}}{n^k}\right)$$

where $e_k^{(\alpha)}$ is a polynomial in α of degree $2k$. More precisely,

$$e_k^{(\alpha)} = \sum_{i=k}^{2k} (-1)^i \lambda_{k,i} (\alpha+1)(\alpha+2)\ldots(\alpha+i)$$

with $\sum_{k,i\geq 0} \lambda_{k,i} v^k t^i = e^t(1+vt)^{-1-1/v}$.

Let $\mathscr{H}$ denote the contour

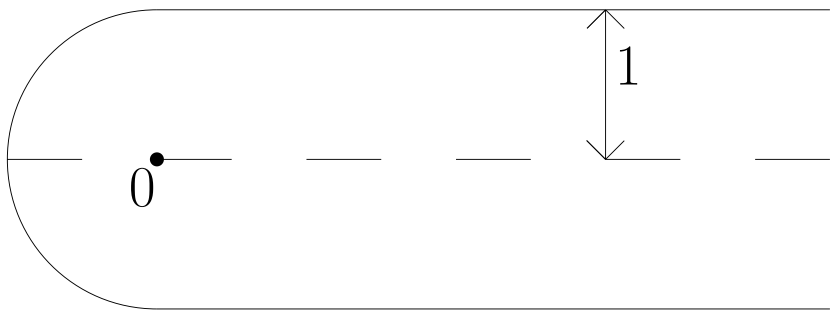

Figure 2.1: $\mathscr{H}$

Lemma 1. *Let $n \in \mathbb{N}$. Then*

$$\int_H (-t)^n e^{-t} dt = 0$$

Proof. The map $t \mapsto (-t)^n e^{-t}$ is an entire function. Thus, with $H_{\leq r} := \{z \in H \mid \Re(z) \leq r\}$ for $r > 0$, we have

$$\int_{H_{\leq r}} (-t)^n e^{-t} dt = -\int_{\{r+iu \ \mid \ u \in [-1,1]\}} (-t)^n e^{-t} dt$$

and the modulus of the latter integral tends to 0 as r tends to infinity. ∎

Lemma 2. *Let $a \notin \mathbb{N}$. Then*

$$\int_H (-t)^a e^{-t} dt = \frac{2i\pi}{\Gamma(-a)}$$

Remark 2. *Lemma 1 can be viewed as a limit case of lemma 2, since Γ has poles at negative integers.*

Theorem 11. *Let $a \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$ and $b \in \mathbb{N}^*$.*
Let

$$f : z \mapsto (1-z)^{-a}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^b$$

then for large n one has

$$f_n = [z^n]f(z) \sim \frac{n^{-a-1}}{\Gamma(-a)}(\log n)^b\left(1 + \sum_{k\geq 1}\frac{e_k^{(a,b)}}{\log^k n}\right)$$

with

$$e_k^{(a,b)} = (-1)^k\binom{b}{k}\Gamma(-a)\frac{d^k}{ds^k}\left[\frac{1}{\Gamma(-s)}\right](a)$$

Proof. Note that f is analytic in the plane slit $[1, +\infty]$.
By Cauchy's coefficients formula (1) in 0:

$$f_n = \frac{1}{2i\pi}\int_C \frac{f(z)}{z^{n+1}}dz$$

Where C is a deformed circle with a notch coming back to the left of the complex 1.

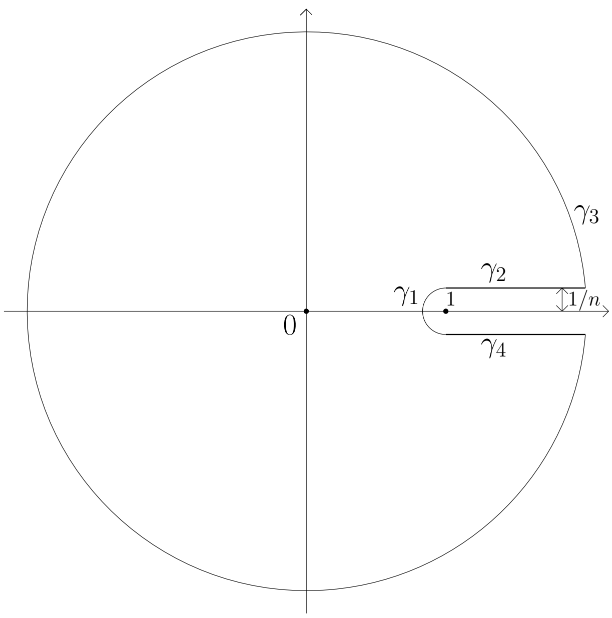

Figure 2.2: C

More precisely, $C := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\begin{aligned}
\gamma_1 &:= \left\{ z = 1 - \frac{u}{n};\ u = e^{i\theta}, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\} \\
\gamma_2, \gamma_4 &:= \left\{ z = 1 + \frac{u \pm i}{n};\ u \in [0, n] \right\} \\
\gamma_3 &:= \left\{ z;\ |z| = \left(4 + \frac{1}{n^2}\right)^{1/2}, \Re(z) \leq 2 \right\}
\end{aligned}$$

Outer circle Let $r = \left(4 + \frac{1}{n^2}\right)^{1/2}$. Then

$$\begin{aligned}
\left| \frac{1}{2i\pi} \int_{\gamma_3} \frac{f(z)}{z^{n+1}} dz \right| &\leq \frac{1}{2\pi} \int_{\gamma_3} \left| \frac{f(z)}{z^{n+1}} \right| dz \\
&\leq \frac{1}{2i\pi} \int_{\mathscr{C}(0,r)} \left| \frac{f(z)}{z^{n+1}} \right| dz \\
&\leq \frac{1}{2\pi r^{n+1}} (r+1)^{-a} \log(r+1)^b
\end{aligned}$$

which is a $O(r^{-n})$, so the main part of our expansion is expected to come from the integral over $\gamma_1 \cup \gamma_2 \cup \gamma_4$.

Change of variable Now introduce the change of variable $z \mapsto 1 + \frac{t}{n}$. Then (using corollary 2 and the fact that b is a positive integer)

$$\begin{aligned}
\frac{f(z)}{z^{n+1}} dz &= (-t)^{-a} n^a \left(\log\left(-\frac{n}{t}\right) \right)^b \left(1 + \frac{t}{n}\right)^{-n-1} \frac{dt}{n} \\
&= (-t)^{-a} n^{a-1} (\log n)^b \left(1 - \frac{\log(-t)}{\log n} \right)^b \left(1 + \frac{t}{n}\right)^{-n-1} dt \\
&= (-t)^{-a} n^{a-1} (\log n)^b \left(\sum_{i=0}^{b} \binom{b}{i} \left(-\frac{\log(-t)}{\log n} \right)^i \right) \left(1 + \frac{t}{n}\right)^{-n-1} dt
\end{aligned}$$

so

$$f_n = \frac{1}{2i\pi} n^{a-1} (\log n)^b \sum_{i=0}^{b} \binom{b}{i} \int_{\mathscr{H}_n} (-t)^{-a} \left(-\frac{\log(-t)}{\log n} \right)^i \left(1 + \frac{t}{n}\right)^{-n-1} dt$$

where $\mathscr{H}_n := \{z \in \mathscr{H} \mid \Re(z) \leq n\}$.

Away from the origin Let $\hat{\mathscr{H}} := \{t \in \mathscr{H} \mid \Re(t) \geq (\log n)^2\}$.

Study of $\left(1 + \frac{t}{n}\right)^{n+1}$ Let us first recall the following inequality, from basic real analysis

Remark 3. *For any $y \in \mathbb{R}^+$, one has $\log(1+y) \geq y - \frac{y^2}{2}$.*

Let $x \in \mathbb{R}^+$. Then

$$\begin{aligned}
\left(1 + \frac{x}{n}\right)^{-n} &= e^{-n \log\left(1 + \frac{x}{n}\right)} \\
&\leq e^{-x} \cdot e^{\frac{x^2}{2n}}
\end{aligned}$$

Thus, for $\Re(t) \geq (\log n)^2$, one has

$$\begin{aligned}\left|1+\frac{t}{n}\right|^{-n-1} &\leq \left|1+\frac{t}{n}\right|^{-n} \\ &\leq \left(1+\frac{\Re(t)}{n}\right)^{-n} \\ &\leq \left(1+\frac{(\log n)^2}{n}\right)^{-n} \\ &\leq e^{-(\log n)^2} \cdot e^{\frac{(\log n)^4}{2n}}\end{aligned}$$

***

On $\hat{\mathscr{H}}^+$, we have

$$\begin{aligned}arg\left(-\frac{n}{t}\right) &= arg(-n) - arg(t) \\ &\in [\pi - Arg(\log^2(n)+i), \pi]\end{aligned}$$

So

$$\left|\log\left(\frac{n}{t}\right)^{\beta}\right| \leq \left(\log\left(\left|\frac{n}{t}\right|\right) + \left|arg\left(\frac{n}{t}\right)\right|\right)^{\beta}$$

$$\begin{aligned}\left|\int_{\hat{\mathscr{H}}} (-t)^{-\alpha}\left(1-\frac{\log(-t)}{\log n}\right)^{\beta}\left(1+\frac{t}{n}\right)^{-n-1} dt\right| &\leq e^{-(\log n)^2} \cdot e^{\frac{(\log n)^4}{2n}} \int_{\hat{\mathscr{H}}} \left|(-t)^{-\alpha}\left(1-\frac{\log(-t)}{\log n}\right)^{\beta}\right| dt \\ &\leq\end{aligned}$$

■

Chapter 3

Implementation

TODO

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