LIP6

M2 memoir

Automatic asymptotics for combinatorial series

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September, 2020

Abstract

Enumerative combinatorics is interested in determining the number a_n of objects of size n in a class of combinatorial objects. Alternatively, rather than a complicated closed formula, one would like to obtain an asymptotic expansion of a_n. In analysis of algorithms, for instance, computing asymptotic expansions is used to compare performance, and therefore a salient question.

Due to the fabulous diversity of combinatorial structures, such computations have long required intuition and specially crafted “tricks”, only adapted to the problem at hand, or some close family.

Nowadays, powerful techniques have been developed, enabling one to study a vast amount of combinatorial constructions with standard procedures. In fact, most of the steps involved have now been separately implemented, for instance in the `ore_algebra` module of *SageMath*.

One frequent case of application of these techniques is the *D-Finite* case, where the generating series of (a_n) is characterized by a differential equation with polynomial coefficients, and initial values of the a_is.

This internship is devoted to understanding and effectively programming a combination of these techniques in *SageMath*, in the D-Finite case. In the end, one should be able to type in a differential equation associated to a_n, and our code shall return an asymptotic expansion of a_n, up to any desired order.

The novelty resides in the merging of several existing functions used in different parts of the analysis, associated to the computation of explicit constants.

Contents

Chapter 1

Introduction

Pick your favourite combinatorial construction. Let a_n the number of such structures of size n. We wish to be able to compute an asymptotic expansion of a_n automatically.
Let $f = \sum a_n z^n$ be the complex series associated to (a_n).
We shall see that, if f has a positive convergence radius, then one has asymptotically $a_n = A^n \theta(n)$ where θ has sub-exponential growth. Two principles, stated in [FS09], shall guide one's search :

- *First Principle of Coefficient Asymptotics* : The location of a function's singularities dictates the exponential growth (A^n) of its coefficients.
- *Second Principle of Coefficient Asymptotics* : The nature of a function's singularities determines the associate subexponential factor $(\theta(n))$.

1.1 State of the art

In 1990, Flajolet and Odlyzko [FO90] proved transfer theorems, that allow one to TODO

1.2 Mathematical sketch

From a combinatorial problem to a differential equation Powerful techniques exist to translate a combinatorial construction into a *D-finite* relation, namely a differential equation with polynomial coefficients. We will not cover those techniques here. If interested, one is referred to [FS09].
From now on, we will assume that a non trivial D-finite relation satisfied by f is given, which is a relation of the form

$$y^{(r)} + \frac{p_{r-1}}{p_r} y^{(r-1)} + \cdots + \frac{p_0}{p_r} y = 0 \tag{1.1}$$

where $p_0, \ldots, p_r \in \mathbb{C}[X]$.

D-finite (holonomic) function

Definition 1. *A function satisfying a D-finite relation will be said D-finite itself, or* holonomic.

Singularities location We will first see that f may only have singularities at roots of p_r. Thereafter, we define $\Xi := \{\text{roots of } p_r\}$. If f has at least one singularity, minimal ones (by module) are called *dominant singularities*.

Local basis structure theorems Following the definition of *regular singular points*, where some technical condition is satisfied, it can be proved that, in a *slit* neighbourhood of any such point ζ, equation (1.1) admits a local basis of solutions of the form

$$(z-\zeta)^{\theta_j} \log^m(z-\zeta) H_j(z-\zeta)$$

with H_j analytic at 0. This basis can be explicitly computed.

Transfer theorems We then investigate *transfer theorems*. Assume f has at least one singularity, and all dominant singularities are regular singular points. After expressing f in the previous form around all dominant singularities, transfer theorems allow one to compute an asymptotic expansion of f_n.

1.3 Implementation overview

The implementation is in SageMath, mostly (and vastly) relying on the `ore_algebra` and `AsymptoticRing` modules.

`extract_asymptotics` A function `extract_asymptotics` is implemented, with the following definition:

```
def extract_asymptotics(op,
                        first_coefficients,
                        z,
                        order=DEFAULT_ORDER,
                        precision=DEFAULT_PRECISION) -> expr
```

For a holonomic function f, `extract_asymptotics` takes

- A differential operator `op`, such that $op \cdot f = 0$
- A list `first_coefficients` of the first Taylor coefficients of f
- The variable `z` used in definition of `op`, only for technical reasons
- The desired `order` of the asymptotic expansion
- The desired certified `precision` for the constants

It returns an asymptotic expansion of the coefficients of f, up to the desired `order` and with constants certified at least with the given `precision`.

Global structure We first locate the roots of p_r, and group them by increasing module. Then, as long as no root has been *proved* to be a singularity of f, we iterate through the groups and sum their contributions. A root of p_r can indeed not always be proved to be a singularity of f only by computing the coefficients of f in the local basis: if one of these coefficients is precisely 0, successive approximations will never be able to distinguish it from 0, yet not proving either that it would be nil.

Then for each root, we make use of `local_basis_expansions`, defined in the `ore_algebra` module. That function allows one to compute a local basis of solutions to `op`, along with their expansion up to any desired order. A call to `numerical_transition_matrix` then allows us to determine the expression of f in that local basis, with constants certified to the desired `precision`.

Calling `SingularityAnalysis` (a specially crafted function from the `asymptotic_ring` module) on each term appearing and summing the results finally yields the desired expansion.

Chapter 2

Mathematical background

2.1 Notations

Definition 2. *Let f be a differentiable function. We note $f^{(k)}$ its k-th derivative.*

2.2 Reminders from complex analysis

Leibniz rule

Theorem 1.

$$(fg)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)} g^{(i)}$$

Analytic functions

Definition 3. *A function f is said to be* analytic *on Ω if, for all $z_0 \in \Omega$, it admits an expansion*

$$f(z) = \sum_{n \geq 0} f_n (z - z_0)^n$$

that converges on some neighbourhood of z_0.

Cauchy's integral formula

Theorem 2. *Let Ω be an open subset of $\mathbb{C}$.*
Let $\omega \in \Omega$ and $\rho > 0$ such that $\mathscr{B}_f(\omega, \rho) \subset \Omega$.
Let f be holomorphic on Ω.
Then for all $z_0 \in \mathscr{B}(\omega, \rho)$, we have

$$f(z_0) = \frac{1}{2i\pi} \int_{\mathscr{S}(\omega,r)} \frac{f(z)}{z - z_0} dz$$

Cauchy's coefficient formula

Corollary 1. *Let f be analytic on some neighbourhood Ω of $z_0 \in \mathbb{C}$, and $r > 0$ such that $\mathscr{B}_f(z_0, r) \subset \Omega$, then for all n, one has*

$$f_n = \frac{1}{2i\pi} \int_{\mathscr{C}(z_0,r)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Theorem 3. *Let f be analytic in some neighbourhood Ω of $z_0 \in \mathbb{C}$, and f_n such that on Ω one can write*

$$f(z) = \sum_{n \geq 0} f_n (z - z_0)^n$$

then for all n, we have

$$f_n = \frac{f^{(n)}(z_0)}{n!}$$

TODO

2.2.1 Complex logarithm

A few identities For a reference on the following definitions and identities, the reader is referred to [BC96].

From now on, except when explicitly stated otherwise, $\log z$ and $\arg z$ will stand for the principal determination of the complex logarithm and argument.

N_+, N_-

Definition 4. *Let $z_1, z_2 \in \mathbb{C}^*$. Define $N_+(z_1, z_2)$ and $N_-(z_1, z_2)$ with*

$$N_\pm = \begin{cases} -1 & \textit{if } \pi < \arg(z_1) \pm \arg(z_2) \\ 0 & \textit{if } -\pi < \arg(z_1) \pm \arg(z_2) \leq \pi \\ 1 & \textit{if } \arg(z_1) \pm \arg(z_2) \leq -\pi \end{cases}$$

Remark 1. *The previous definition is intended to have to following relations hold:*

$$\begin{cases} \arg(z_1 z_2) & = \arg(z_1) + \arg(z_2) + 2\pi N_+ \\ \arg\left(\frac{z_1}{z_2}\right) & = \arg(z_1) - \arg(z_2) + 2\pi N_- \end{cases}$$

Prop. 1. *Let $a, b, c \in \mathbb{C}^*$. Then*

$$\log(ab) = \log a + \log b + 2i\pi N_+(a, b)$$

$$\log\left(\frac{a}{b}\right) = \log a - \log b + 2i\pi N_-(a, b)$$

$$(ab)^c = a^c \times b^c \times e^{2i\pi c N_+(a,b)}$$

$$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c} e^{2i\pi c N_-(a,b)}$$

Corollary 2. *Let $x \in \mathbb{R}^{+*}$ and $z, t \in \mathbb{C}^*$. Then* $\arg(x) = 0$*, so all classical identities over the real numbers extend identically:*

$$\log(xz) = \log x + \log z$$
$$\log\left(\frac{x}{z}\right) = \log x - \log z$$
$$(xz)^t = x^t \times z^t$$
$$\left(\frac{x}{z}\right)^t = \frac{x^t}{z^t}$$

2.3 Some differential equations theory

In this section, we will present the results that enable one study a differential equation of the form (1.1).

2.3.1 Scalar and system equations

Differential equations The equation

$$y^{(r)} = a_{r-1}(z)y^{(r-1)} + \cdots + a_0(r)y \tag{2.1}$$

where the a_i are holomorphic is said to be a *scalar* (differential) equation.

Differential systems The equation

$$Y' = A(z)Y \tag{2.2}$$

where $A(z)$ is an $n \times n$ matrix and $Y(z)$ is an n-dimensional vector is said to be a (differential) *system of equations.*

From scalar to system The following transformation is a classical trick to transform a scalar equation into a system:
If y is a solution to

$$y^{(r)}(z) = a_{r-1}(z)y^{(r-1)}(z) + \cdots + a_0(z)y(z)$$

then $Y : z \mapsto \begin{pmatrix} y(z) \\ y'(z) \\ \vdots \\ y^{(n-1)}(z) \end{pmatrix}$ is a solution to

$$Y' = A(z)Y'(z)$$

where

$$A(z) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ 0 & & \ddots & & 1 \\ a_0(z) & & \dots & & a_{r-1}(z) \end{pmatrix}$$

We call A the *companion matrix* of equation (2.3.1).

2.3.2 Solutions space

Basis of solutions The following classical theorem is admitted.

Cauchy's existence and uniqueness theorem

Theorem 4. *Let n an integer. Let also $A(z)$ an $n \times n$-matrix and $f(z)$ an n-dimensional vector, both holomorphic in some simply connected region $\Omega \subset \mathbb{C}$. Then, for any $z_0 \in \Omega$ and $y_0 \in \mathbb{C}^n$, the equation*

$$Y' = A(z)Y \tag{2.3}$$

has a unique solution such that

$$y(z_0) = y_0$$

That solution is holomorphic on Ω.

It immediately follows

Basis of solutions for systems

Corollary 3. *Let $z_0 \in \mathbb{C}$. Suppose there exists a neighbourhood Ω of z_0 such that A and f are holomorphic on Ω.*
Then the set of solutions to equation (2.3) defined in z_0 forms an n-dimensional vector space.

2.4 Location of Singularities

Existence of a local basis of solutions

Theorem 5. *Let $p_0, \dots, p_r \in \mathbb{C}[X]$ and $z_0 \in \mathbb{C}$ such that $p_r(z_0) \neq 0$. Then, in some neighbourhood of z_0, the equation*

$$y^{(r)} + \frac{p_{r-1}}{p_r} y^{(r-1)} + \cdots + \frac{p_0}{p_r} y = 0 \tag{2.4}$$

admits a basis of r analytic solutions.

Proof. The polynomial p_r has finite degree, therefore has a finite number of roots. Since $p_r(z_0) \neq 0$, there is some neighbourhood Ω of z_0 where p_r does not vanish. It follows that all $\frac{p_i}{p_r}$ are analytic on Ω.
Cauchy's theorem then applies and concludes the proof. ■

Possible locations of singularities

Corollary 4. *The only points where f* may *admit singularities are the zeros of p_r.*

2.5 Structure theorems

For further treatment of this section, one is referred to [Was65] (chapter II in particular).

2.5.1 Another transformation

Let $\zeta \in \mathbb{C}$, y such that

$$y^{(r)} + a_{r-1}(z) y^{(r-1)} + \cdots + a_0(z) y = 0$$

and define $Y : z \mapsto \begin{pmatrix} y(z) \\ \vdots \\ (z-\zeta)^{r-1} y^{(r-1)}(z) \end{pmatrix}$ (that is, $Y_i : z \mapsto (z-\zeta)^{i-1} y^{(i-1)}(z)$).
For all $i \leq r-1$, we have

$$(z-\zeta) Y_i' = (i-1) Y_i + Y_{i+1}$$

and

$$(z-\zeta) Y_r' = (r-1) Y_r - (z-\zeta) a_{r-1} Y_{r-1} - \cdots - (z-\zeta)^r a_0 Y_1$$

Then, equation (1.1) is equivalent to

$$(z-\zeta) Y' = A_\zeta(z) Y \tag{2.5}$$

where $A_\zeta(z)$ is an $r \times r$ matrix.

2.5.2 Regular singular points and indicial polynomials

Regular singular points

Definition 5. *Consider a differential equation (E) of the form* (1.1).
We say that ζ is a regular singular *point of (E), and ζ is a pole of $\frac{p_i}{p_r}$ of order at most $r-i$, for all $i \in [0, r-1]$.*
Equivalently, ζ is a regular singular point if $A_\zeta(z)$ is analytic in some neighbourhood of ζ, when one writes $(z-\zeta)Y' = A_\zeta(z)Y$.

Indicial polynomial, I_ζ

Definition 6. *The characteristic polynomial of $A_\zeta(\zeta)$ is named the* indicial polynomial *of equation* (1.1) *and* (2.5) *at ζ, denoted I_ζ.*

2.5.3 Results

General structure theorem

Theorem 6. *Let ζ be a regular singular point of* (1.1). *No assumption is made on the roots of I_ζ.*
Then, in a slit neighbourhood of ζ, there exists a basis of solutions of the form

$$(z-\zeta)^{\theta_j}(\log(z-\zeta))^m H_j(z-\zeta) \tag{2.6}$$

where θ_j are the roots of the indicial polynomial, each H_j is analytic at 0, and $m \in \mathbb{N}$.

2.5.4 Special cases

G-functions

Definition 7. *A formal series $f = \sum f_n z^n \in \mathbb{Q}[[z]]$ is called a* G-function *if it is D-finite and there exists $C > 0$ such that for all n, we have*

$$\begin{cases} |f_n| < C^n \\ lcd(f_1, \dots, f_n) < C^n \end{cases}$$

where $lcd(f_1, \dots, f_n)$ is the least common denominator of $f_1, \dots, f_n$.

André-Chudnovsky-Katz Theorem

Theorem 7. *Let f be a G-function. Then a minimal order annihilating D-finite equation for f has only ordinary or regular singular points, and its indicial polynomial has only rational roots.*

Chapter 3

Singularity analysis and transfer theorems

In the previous sections, we have seen that a holonomic function f admits, in the neighbourhood of any singularity, an asymptotic expansion of the form (2.6).
In this section, we show how to compute an asymptotic expansion of the coefficients of these functions, and present a *transfer theorem* allowing one to directly deduce an asymptotic expansion of f.

3.1 Singularity analysis

Polynomial case

Theorem 8. *If* $\alpha \in \mathbb{N}$, *then* $[z^n]\left(1 - \frac{z}{\zeta}\right)^{\alpha} = 0$ *for* $n > k$ *because* $\left(1 - \frac{z}{\zeta}\right)^{\alpha}$ *is a polynomial. So that case can be completely ruled out in estimating asymptotic expansions.*

3.1.1 Basic scale transfer

We quote from [FS09]

Basic scale transfer

Theorem 9. *Let α be an arbitrary complex number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The coefficient of z^n in*

$$f(z) = (1-z)^{-\alpha}$$

admits for large n a complete asymptotic expansion in descending powers of n,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{k=1}^{\infty} \frac{e_k^{(\alpha)}}{n^k}\right)$$

where $e_k^{(\alpha)}$ is a polynomial in α of degree $2k$. More precisely,

$$e_k^{(\alpha)} = \sum_{i=k}^{2k} (-1)^i \lambda_{k,i} (\alpha+1)(\alpha+2)\dots(\alpha+i)$$

with $\sum\limits_{k,i \geq 0} \lambda_{k,i} v^k t^i = e^t (1+vt)^{-1-1/v}$.

3.1.2 Complete scale

In this section, we closely follow [Jun31].

Lemma 1. *Let $k \in \mathbb{N}^*$ and $f = \sum f_n z^n$ with*

$$f(z) = (1-z)^{-k}$$

Then for all n, we have

$$f_n = \frac{n^{k-1}}{\Gamma(k)} \left[1 + \frac{k(k-1)}{2n} + \dots + \frac{\Gamma(k)}{n^{k-1}}\right]$$

Proof. Start with

$$(1-z)^{-1} = \sum z^n$$

Now differentiating that relation $k-1$ times, we get

$$(k-1)!(1-z)^{-k} = \sum \frac{(n+k-1)!}{n!} z^n$$

Therefore $f_n = \frac{1}{(k-1)!}(n+1)(n+2)\dots(n+k-1)$, and the result follows by developing the product and grouping by powers of n. ∎

The following three results are quoted from [Jun31] without proof. For a reference, one may consult [KY28]. They shall be useful in the proof of theorem 10 to assert the domains of validity of the computed expansions.

Lemma 2. *Let $\phi(z)$ admit an asymptotic expansion*

$$\varphi(z) \sim c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots \tag{3.1}$$

as z goes to infinity following a half line d.
Then for every constant z_0, we also have, asymptotically along d,

$$\varphi(z_0 + z) \sim c_0 + \frac{c_1}{z} + \frac{-c_1 z_0 + c_2}{z^2} + \dots$$

Lemma 3. *Let $\varphi(z)$ be analytic in the form (3.1) on a half band*

$$\begin{cases} \Re(z) > a \\ \Im(z) \in]-b, b[\end{cases} \tag{3.2}$$

for a and b arbitrary positive real numbers.
Then $e^{\varphi(z)}$ can also be expanded, over the same band:

$$e^{\varphi(z)} \sim e^{c_0}\left(1 + \frac{c_1}{z} + \dots\right)$$

Lemma 4. *Let $\varphi(z)$ be analytic and expandable in the form (3.1) over a band (3.2), we also have*

$$\varphi'(z) \sim -\frac{c_1}{z^2} - \frac{2c_2}{z^2} - \dots$$

over any tighter band

$$\begin{cases} \Re(z) > a \\ \Im(z) \in]-b+\varepsilon, b+\varepsilon[\end{cases} \tag{3.3}$$

Lemma 5. *Let i an integer, n a natural number and $s \in \mathbb{C} \setminus \{-1, -2, \dots\}$.*
Then there exists functions $\psi_{i,j}$ that can be expanded asymptotically, such that

$$\frac{\Gamma^{(i)}(n+s)}{\Gamma(n+1)} = n^{s-1}[(\log n)^i \psi_{i,0}(n) + \dots + \psi_{i,i}(n)] \tag{3.4}$$

Proof. Start from Stirling's series for $\log \Gamma(z)$:

$$\begin{aligned} \log \Gamma(z) &\sim \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} \\ &\sim \frac{1}{2}\log(2\pi) + \left(z + \frac{1}{2}\right)\log(z) - z + \frac{1}{12z} - \frac{1}{360z^2} + \dots \end{aligned}$$

where B_n is the n-th Bernoulli number.
Then by lemma 3

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \cdot z^{-1/2} \cdot \varphi(z)$$

where $\varphi(z)$ can be expanded in asymptotic series over any half-band of type (3.2). Now, differentiating i times, we get

$$\Gamma^{(i)}(z) = \left(\frac{z}{e}\right)^z z^{-1/2}\left[(\log z)^i \varphi_{i,0}(z) + \cdots + \varphi_{i,i}(z)\right]$$

where the functions $\varphi_{i,j}$ can be expanded into asymptotic series, by lemmas 2 and 4. Therefore,

$$\frac{\Gamma^{(i)}(n+s)}{\Gamma(n+1)} = \left(\frac{n}{e}\right)^{s-1} \frac{\left(1+\frac{s}{n}\right)^{n+s-1/2}}{\left(1+\frac{1}{n}\right)^{n+1/2}} \cdot \frac{(\log(n+s))^i \varphi_{i,0}(n+s) + \cdots + \varphi_{i,i}(n+s)}{\varphi(n+1)}$$

and we may finally define the functions $\psi_{i,j}$ such that

$$\frac{\Gamma^{(i)}(n+s)}{\Gamma(n+1)} = n^{s-1}[(\log n)^i \psi_{i,0}(n) + \cdots + \psi_{i,i}(n)]$$

■

Expansion theorem in the log case

Theorem 10. *Let $a \in \mathbb{C}$ and $k \in \mathbb{N}^*$.*
Let

$$f : z \mapsto (1-z)^a \left(\log \frac{1}{1-z}\right)^k$$

then for large n one has

$$f_n = \begin{cases} \frac{n^{-a-1}}{\Gamma(-a)} \sum_{i=0}^{k} (\log n)^i \phi_i(n) & \text{if } a \notin \mathbb{N} \\ (-1)^a k\Gamma(1+a) n^{-a-1} \sum_{i=0}^{k} (\log n)^i \phi_i(n) & \text{if } a \in \mathbb{N} \end{cases}$$

where the functions ϕ_i admit asymptotic expansions of the form

$$\phi_i \sim c_{i,0} + \frac{c_{i,1}}{n} + \frac{c_{i,2}}{n^2} + \ldots$$

and the constants $c_{i,j}$ can be explicitly computed.

Proof. Assume $a \notin \mathbb{N}$. We can define

$$\phi_0 : z \mapsto \Gamma(-a)(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{\Gamma(n-a)}{n!} z^n$$

By differentiating i times with respect to a, we get $\frac{d^i}{da}\Gamma(-a) = (-1)^i \Gamma^{(i)}(-a)$, and $\frac{d^i}{da}(1-z)^{-a} = \frac{d^i}{da} e^{-a\log(1-z)} = \frac{d^i}{da} e^{a\left(\log \frac{1}{1-z}\right)} = \left(\log \frac{1}{1-z}\right)^i (1-z)^{-a}$ so by Leibniz' rule:

$$\phi_i := \frac{d^i}{da^i}\phi_0 = (1-z)^{-a} \sum_{j=0}^{i} \binom{i}{j} (-1)^j \Gamma^{(j)}(-a) \left(\log \frac{1}{1-z}\right)^{i-j}$$

Now, when i takes successively the values $0, \dots, k$, we get a triangular system of linear equations of unknowns the functions $(1-z)^a \left(\log \frac{1}{1-z}\right)^i$, the solution of which has the form

$$(1-z)^a \left(\log \frac{1}{1-z}\right)^k = \frac{1}{\Gamma(-a)} \left[\phi_k(z) + d_{k,k-1}\phi_{k-1}(z) + \dots + d_{k,0}\phi_0(z)\right] \tag{3.5}$$

where the coefficients $d_{i,j}$ are explicitly computable and only depend on i and j.
Now, by definition of the ϕ_is, we have for all i

$$\phi_i(z) = (-1)^i \sum_{n=0}^{\infty} \frac{\Gamma^{(i)}(n-a)}{n!} z^n$$

By expanding equation (3.5) into Taylor series, this leads to the following equality

$$f_n = \frac{1}{n!\Gamma(-a)} \left[\Gamma^{(k)}(n-a) + d_{k,k-1}\Gamma^{(k-1)}(n-a) + \dots + d_{k,0}\Gamma(n-a)\right]$$

We now use lemma 5 to conclude (recall that $n! = \Gamma(n+1)$).

***

To deal with the case $a \in \mathbb{N}$, use the relation

$$\begin{aligned}(1-z)^a \left(\log \frac{1}{1-z}\right)^k = & -a \int (1-z)^{a-1} \left(\log \frac{1}{1-z}\right)^k \\ & + k \int (1-z)^{a-1} \left(\log \frac{1}{1-z}\right)^{k-1}\end{aligned}$$

$a+1$ times to reduce to the first case. ■

3.2 Transfer theorems

The content of this section is from [FS09], section VI.

$\Delta(\phi, R)$

Definition 8. *For any positive numbers ϕ and R such that $R > 1$ and $\phi \in \left]0, \frac{\pi}{2}\right[$, we define*

$$\Delta(\phi, R) := \{z \mid |z| < R, z \neq 1 \text{ and } |\arg(z-1) > \phi\}$$

*A domain D such that there exists ϕ and R as above and $D = \Delta(\phi, R)$ will be called a Δ-*domain.
A function that is analytic on some Δ-domain will be called Δ-analytic.

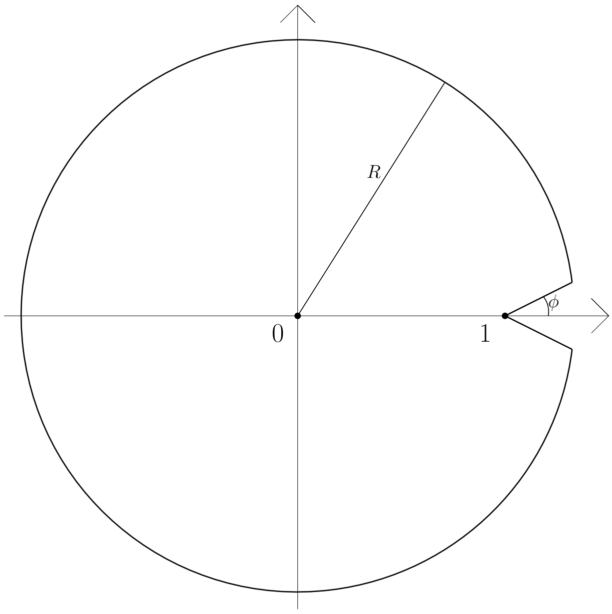

Figure 3.1: A Δ-domain

Transfer theorem for O and o

Theorem 11. *Let $\alpha, \beta \in \mathbb{R}$ and f be a Δ-analytic function.*

- *If f is such that, in the some neighbourhood of 1 and its Δ-domain, one has*

$$f(z) = O\left((1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}\right)$$

Then

$$[z^n]f(z) = O\left(n^{\alpha-1}(\log n)^{\beta}\right)$$

- *The same result holds, with O replaced by o.*

Chapter 4

Implementation

Section 4.1 presents the global structure of the algorithm, then details some parts. Section 4.2 briefly presents the tests. Finally, section 4.3 discusses performance aspects.

4.1 Structure

We first present here the main algorithm, and the important sub-algorithms. The implementation is mostly transparent, except for a few optimizations like computing the local solutions in 0 only once.

```
Algorithm 1: Main algorithm
Input: f defined by a a_n d^n/dz^n f + ... a_0 = 0, initial coefficients f_0, ..., f_n, order
       and precision
Output: Asymptotic expansion of f with at least order terms, and coefficients
        with given precision
begin
    Compute the roots of a_n and group them by increasing module.
    Initialise the sum S of contributions to the asymptotic expansion.
    while No contribution confirmed do
        Load next group G of roots
        foreach root ρ ∈ G do
            Compute the contribution of ρ.
        end
        Sum contributions of G and add to S
    end
    return S
end
```

To compute the contribution of a specific root, we use the following algorithm

Algorithm 2: Computing the contribution of a root

Input: All entries of the main algorithm, and a specific root ρ
Output: Contribution of ρ to the asymptotic expansion of f
begin
  Compute a basis $\mathscr{B}_0$ of solutions in 0, and decompose f in $\mathscr{B}_0$.
  Compute a basis $\mathscr{B}_\rho$ of solutions in ρ, and decompose f in $\mathscr{B}_\rho$.
  Initialize a sum S of terms contributions.
  foreach *term T in the local expansions* **do**
    Compute an asymptotic expansion of the Taylor coefficients of T.
    Add to S.
  end
end
return S

4.2 Tests

We give here a list of functions, along with an associated differential operator and an asymptotic expansion. We provide a name when the Taylor series is of combinatorial interest, and a reference for the not obvious/classical cases.

f	Differential operator	Asymptotic expansion	Sequence name
$\log(1-z)$	$(z-1)Dz^2+Dz$	$\frac{-1}{n}$	
$\log(1+z)$	$(z+1)Dz^2+Dz$	$\frac{(-1)^n}{n}$	
$\frac{1}{1-z}$	$(z-1)Dz+1$	1	
$\frac{1}{1+z}$	$(z-1)Dz+1$	$(-1)^n$	
$\frac{z}{1-2z}$	$z(1-2z)Dz-1$	2^{n-1}	
$\frac{1}{1-z^2}$	$(1-z^2)Dz-2z$	$\begin{cases}1 \text{ if } n \text{ is even}\\ 0 \text{ otherwise}\end{cases}$	
$(1-z)^{3/2}$	$2(z-1)Dz-3$	$\frac{1}{\sqrt{\pi n^5}}\left(\frac{3}{4}+\frac{45}{32n}+\frac{1155}{512n^2}+\dots\right)$	[FS09]
$\arctan(z)$	$(1+z^2)Dz^2+2zDz$	$\begin{cases}\frac{(-1)^n}{2n+1} \text{ if } n \text{ is odd}\\ 0 \text{ otherwise}\end{cases}$	
$\frac{1}{1-z}\log\left(\frac{1}{1-z}\right)$	$(1-z)^2Dz^2-3(1-z)Dz+1$	$\log(n)+\gamma+\frac{1}{2n}+o\left(\frac{1}{n}\right)$	Harmonic numbers
$\frac{z}{1-z-z^2}$	$(1-z-z^2)Dz^2-(2+4z)Dz-2$	$\frac{\varphi^n-(-\varphi)^{-n}}{\sqrt{5}}$	Fibonacci numbers
$\frac{1-\sqrt{1-4z}}{2z}$	$(4z^2-z)Dz^2+(10z-2)Dz+2$	$\frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{8n}+\frac{1}{128n^2}+\dots\right)$	Catalan numbers
$\frac{1-z-\sqrt{1-2z-3z^2}}{2z}$	$(3z^4+2z^3-z^2)Dz^2+(6z^3+3z^2-z)Dz+1$	$\frac{\sqrt{3}}{2\sqrt{\pi}n^{3/2}}3^n\left(1-\frac{15}{16n}+\frac{505}{512n^2}+\dots\right)$	Motzkin numbers [FS09]
	$(4z^2-z)Dz^2+(14z-2)Dz+6$	$\frac{4^n}{\pi n}\left(4-\frac{6}{n}+\frac{19-2(-1)^n}{2n^2}+\dots\right)$	Walks in $\mathbb{N}^2$ [Mel20]

Here is the return of a typical call to `tests.sage` (slightly formatted and with few digits for convenience):

```
-> sage tests.sage
log(1+z) -> [-1.00000*n^(-1)*(e^(I*arg(-1)))^n + O(n^(-3)*(e^(I*arg(-1)))^n)]

log(1-z) -> [([-1.00000 +/- 1e-10])*n^(-1) + O(n^(-3))]

1/(1-z) -> [1.00000]

1/(1+z) -> [1.00000*(e^(I*arg(-1)))^n]

1/(1 - z^2) -> [[0.500000 +/- 4.77e-7] + [0.500000 +/- 4.77e-7]*(e^(I*arg(-1)))^n + O(n^(-4)) + O(n^(-4)*(e^(I*arg(-1)))^n)]

1/(1-z) * log(1/(1-z)) (H_n) -> [([1.00000 +/- 1e-10] + [+/- 5.17e-26]*I)*log(n)
                                + ([1.00000 +/- 1e-10] + [+/- 5.17e-26]*I)*euler_gamma
                                + ([-1.00000 +/- 1e-10] + [+/- 5.17e-26]*I)*log(-1)
                                + [+/- 1.63e-25] + [3.14159 +/- 3.94e-6]*I
                                + ([0.500000 +/- 1e-11]
                                + [+/- 2.59e-26]*I)*n^(-1)
                                + O(n^(-2))]

z/(1-2z) -> [[0.500000 +/- 4.77e-7]*2^n]

Arctan -> [([+/- 5.57e-15] + [-0.500000 +/- 4.77e-7]*I)*n^(-1)*(e^(I*arg(1*I)))^n
          + ([+/- 5.57e-15] + [0.500000 +/- 4.77e-7]*I)*n^(-1)*(e^(I*arg(-1*I)))^n
          + O(n^(-3)*(e^(I*arg(-1*I)))^n) + O(n^(-3)*(e^(I*arg(1*I)))^n)]

random walks in Z*N -> [(([2.00000 +/- 1.91e-6])/sqrt(pi))*4^n*n^(-1/2)
```

```
                        + (([-1.25000 +/- 2.51e-6])/sqrt(pi))*4^n*n^(-3/2)
                        + (([1.14062 +/- 7.34e-6])/sqrt(pi))*4^n*n^(-5/2)
                        + (([-1.1230 +/- 5.61e-5])/sqrt(pi))*4^n*n^(-7/2)
                        + O(4^n*n^(-9/2))]

random walks in N^2 -> [([1.27324 +/- 1.98e-6] + [+/- 1.57e-7]*I)*4^n*n^(-1)
                        + ([-1.90986 +/- 2.96e-6] + [+/- 2.35e-7]*I)*4^n*n^(-2)
                        + ([0.318310 +/- 7.23e-7])*4^n*n^(-3)*(e^(I*arg(-1)))^n
                        + O(4^n*n^(-3))]

Fibonacci numbers -> [[0.44721 +/- 4.20e-6]*1.618033988749895?^n + O(1.618033988749895?^n*n^(-5))]

Catalan numbers -> [((([1.00000 +/- 9.54e-7])/sqrt(pi))*4^n*n^(-3/2)
                    + (([-1.12500 +/- 4.00e-6])/sqrt(pi))*4^n*n^(-5/2)
                    + (([1.1328 +/- 2.16e-5])/sqrt(pi))*4^n*n^(-7/2)
                    + (([-1.1279 +/- 9.19e-5])/sqrt(pi))*4^n*n^(-9/2)
                    + O(4^n*n^(-5))]

Motzkin numbers -> [((([0.86602 +/- 6.62e-6])/sqrt(pi))*3^n*n^(-3/2)
                    + (([-0.81190 +/- 6.64e-6])/sqrt(pi))*3^n*n^(-5/2)
                    + (([0.8542 +/- 2.92e-5])/sqrt(pi))*3^n*n^(-7/2)
                    + (([-0.855 +/- 3.66e-4])/sqrt(pi))*3^n*n^(-9/2)
                    + O(3^n*n^(-5))]

(1-z)^(3/2) -> [0.750000/sqrt(pi)*n^(-5/2)
                + 1.40625/sqrt(pi)*n^(-7/2)
                + (([2.25586 +/- 6.25e-7])/sqrt(pi))*n^(-9/2)
                + (([3.46069 +/- 3.36e-6])/sqrt(pi))*n^(-11/2)
                + (([5.22901 +/- 1.54e-6])/sqrt(pi))*n^(-13/2)
                + O(n^(-15/2))]

z(1-z) -> No singularity was found

(z - 1)^3 -> No singularity was found

1-z -> No singularity was found

exp(z) -> No singularity was found

sin(z) -> No singularity was found
```

4.3 Performance

Remark 2. *The following tests were only carried out through simple tools (`time` module from python, by averaging multiple calls, on my laptop with several other programs open, etc). Although the results seem in accordance with intuition and with the time spent on less expensive tests, one should keep in my they only give a vague idea of real performance. In particular, I did not attempt to compute the "real" complexity.*

On my laptop, it takes in average 14 seconds to compute the expansion for Motzkin numbers, with `order=20` and `precision=10^(-1000)`.

Chapter 5

Conclusion

The goal of this internship was to produce a piece of code, able to take in a function f, in the form of a differential equation of which f is solution, and the first Taylor coefficients of f in 0, and return an asymptotic expansion of f up to any desired order, with explicit constants also computed with certified digits up to any desired precision. That goal is achieved, as various tests indicate.
Although many steps involved had already been implemented in `Sagemath`, no function was available to perform the full analysis. Therefore, Marc and I intend to submit the code for acceptance to `ore_algebra`.
For the future, possible improvements include handling singularities in 0, entire series, and computing explicit bounds on the error made by the computed asymptotic expansion.

Acknowledgement

I wish to express here my gratitude towards Marc Mezzarobba for his trust and availability. This internship was only carried out to a term thanks to his constant attention and availability.
In a time of great uncertainty about my future, and despite the challenges of an entirely remote internship, it was always a pleasure to work under his benevolent guidance.

Bibliography

[KY28] K. Knopp and R.C. Young. *Theory and Application of Infinite Series*. Theory and Application of Infinite Series vol. 2. Blackie & Son, 1928.

[Jun31] R. Jungen. “Sur les séries de Taylor n’ayant que des singularités algébrico-logarithmiques sur leur cercle der convergence.” In: *Commentarii mathematici Helvetici* 3 (1931), pp. 266–306. URL: `http://eudml.org/doc/138566`.

[Was65] Wolfgang Wasow. *Asymptotic expansions for ordinary differential equations*. 1965.

[FO90] Philippe Flajolet and Andrew Odlyzko. “Singularity Analysis of Generating Functions”. In: *SIAM Journal on Discrete Mathematics* 3.2 (1990), pp. 216–240. ISSN: 0895-4801. DOI: `10.1137/0403019`.

[BC96] James Ward Brown and Ruel V Churchill. *Complex variables and applications*. McGrawHill, 1996.

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. 2009, pp. 1–810. ISBN: 9780511801655. DOI: `10.1017/CBO9780511801655`.

[Mel20] Stephen Melczer. “An Invitation to Analytic Combinatorics”. In: Cambridge, 2020.