Fabry-Perot wavelength calibration

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The goal of this document is to to derive the equations needed to create a phase map from a calibration interferogram. The phase map contains the number (float) of the channel with a given wavelength (a priori the scanning wavelength) for each pixel.

1 Notations

We recall the Fabry-Perot equation:

$$\lambda p = 2ne\cos\theta$$

where λ is the wavelength, p is the order (integer), n is the optical index (usually 1), e is the physical spacing between the plates and θ is the incidence angle of the beam.

We use in the following the additional notations:

- λ_r: the reference wavelength for which the order (integer) is known;
- p_r: the reference order (integer) associated to λ_r;
- λ_c: the calibration wavelength;
- p_c: the calibration order (integer) associated to λ_c;
- λ_s: the scanning wavelength;
- p_s: the scanning order (integer) associated to λ_s.

A priori, for a mean spacing e_m of the Fabry-Perot plates, λ_r is observed at the optical centre of the etalon ($\theta = 0$):

$$\lambda p = \lambda_r p_r = 2ne_m$$

2 Calibration interfrogram with two rings

2.1 From rings to wavelength variation at order p_c

A calibration interferogram is observed for a given spacing e_i. Usually, only one line passes through the instrument at the calibration wavelength (but it could happen that several calibration lines are used).

We note λ_{c0} the wavelength transmitted at order p_c in the centre:

$$2ne_i = \lambda_{c0} p_c \Leftrightarrow \lambda p = \lambda_{c0} p_c \cos\theta$$

Due to the Fabry-Perot, the line draws concentric rings. Let's consider the case with two rings of radius r_1 and r_2 ($r_1 < r_2$). Each ring correspond to the same wavelength λ_c transmitted in two different orders $p_1 = p_c$ and $p_2 = p_c - 1$:

$$\lambda_c p_1 = \lambda_c p_c = \lambda_{c0} p_c \cos\theta_1$$

$$\lambda_c p_2 = \lambda_c(p_c - 1) = \lambda_{c0} p_c \cos\theta_2$$

θ and r are linked using the camera focal length f_c and the physical pixel size s_{pix}. Indeed, usually r is given in pixels and we have at first approximation:

$$\theta \sim \tan\theta \sim \frac{s_{pix} r}{f_c} = br$$

where b is a coefficient that should be equal to $b = \frac{s_{pix}}{f_c}$. In addition, θ is usually small so we can write

$$\cos\theta = 1 - \theta^2/2 = 1 - b^2 r^2/2$$

Using the two rings, we can determine the b parameter since we have:

$$\lambda_{c0} p_c = \frac{2\lambda_c p_c}{2 - b^2 r_1^2} = \frac{2\lambda_c(p_c - 1)}{2 - b^2 r_2^2} \Rightarrow p_c(2 - b^2 r_2^2) = (p_c - 1)(2 - b^2 r_1^2)$$

This leads to

$$b^2 = \frac{2}{p_c(r_2^2 - r_1^2) + r_1^2}$$

Therefore, the Fabry-Perot equation can be written as follows:

$$\lambda p = \lambda_{c0} p_c - \frac{\lambda_c r^2}{r_2^2 - r_1^2}$$

The next step is to derive the wavelength variation across the field of view if the order is fixed to p_c. We therefore write:

$$\lambda p_c = \lambda_{c0} p_c - \frac{\lambda_c r^2}{r_2^2 - r_1^2} \Leftrightarrow \lambda - \lambda_{c0} = -\frac{\lambda_c}{p_c} \frac{r^2}{r_2^2 - r_1^2}$$

2.2 Wavelength variation without parabola approximation

We have seen previously that $\tan\theta \sim \frac{s_{pix} r}{f_c} = br$. The approximation here is only the optical geometry, therefore we can almost perfectly match $\tan\theta$ and br. In that case, we can write

$$\cos\theta = \cos\left(\arctan\left(br\right)\right) = \frac{1}{\sqrt{1 + b^2 r^2}}$$

For a single interferogram, the thickness e_i remains constant as well as the wavelength, which leads to:

$$\frac{\lambda_c p_c}{\cos\theta_1} = \frac{\lambda_c(p_c - 1)}{\cos\theta_2} \Rightarrow p_c^2(1 + b^2 r_1^2) = (p_c - 1)^2(1 + b^2 r_2^2)$$

$$\Leftrightarrow b^2 = \frac{p_c^2 - (p_c - 1)^2}{r_2^2(p_c - 1)^2 - r_1^2 p_c^2} = \frac{2p_c - 1}{p_c^2(r_2^2 - r_1^2) - 2p_c r_2^2 + r_2^2}$$

Then, the wavelength variation in the field can be written as follows:

$$\lambda p = \frac{\lambda_{c0} p_c}{\sqrt{1 + b^2 r^2}}$$

Since at $r = r_1$ we observe λ_c at order p_c, we therefore obtain

$$\lambda_{c0} = \lambda_c \sqrt{1 + b^2 r_1^2}$$

This leads to the wavelength variation in the order p_c:

$$\lambda = \frac{\lambda_{c0}}{\sqrt{1 + b^2 r^2}} = \lambda_c \sqrt{\frac{1 + b^2 r_1^2}{1 + b^2 r^2}}$$

2.3 Wavelength variation: from order p_c to order p_s

Since the spacing is fixed to e_i, we can write the wavelength variation when order is fixed to p_s:

$$\lambda p_s = \lambda_{c0} p_c \cos\theta \Leftrightarrow \lambda = \frac{\lambda_{c0} p_c}{p_s} \cos\theta = \lambda_{s0} \cos\theta$$

where $\lambda_{s0} = \frac{\lambda_{c0} p_c}{p_s}$ is the central wavelength for order p_s.

In the case of the parabola approximation, we obtain:

$$\lambda = \frac{\lambda_{c0} p_c}{p_s} \left(1 - \frac{b^2 r^2}{2}\right) \Leftrightarrow \lambda - \lambda_{s0} = -\frac{\lambda_c}{p_s} \frac{r^2}{r_2^2 - r_1^2}$$

We clearly see that the wavelength variation is not the same for orders p_c and p_s.

2.4 Determination of θ

Let's consider that we have determined λ_1, the central wavelength at an order p_0 for a spacing e_1. Consecutive rings (angle θ_i) for the same wavelength correspond to different orders $p_i = p_0 - i$:

$$\lambda_1 (p_0 - i) = \lambda_1 p_0 \cos\theta_i$$

Therefore, we can easily deduced that

$$\cos\theta_i = \frac{p_0 - i}{p_0}$$

This depends on the central order p_0, and is independent of the wavelength.

For $p = 798$, we obtain $\theta_1 = 2.8°$, $\theta_2 = 4.1°$, $\theta_3 = 5.0°$, $\theta_4 = 5.7°$, $\theta_5 = 6.4°$, $\theta_6 = 7.0°$, ...

2.5 Free spectral range in the centre vs off axis

The free spectral range can be defined as the difference between the wavelengths ($\Delta\lambda = \lambda_2 - \lambda_1$) transmitted in two adjacent orders p_0 and $p_1 = p_0 + 1$ for a given thickness e and a given incidence on the plates θ, usually chosen as $\theta = 0$. We can also define the free spectral range thanks to the angle difference $\Delta\theta = \theta_2 - \theta_1$ between two rings transmitting the same wavelength λ in two adjacent orders p_0 and $p_1 = p_0 + 1$ for a given thickness e. It can also be defined as the thickness displacement $\Delta e = e_2 - e_1$ needed to transmit a given wavelength λ in two adjacent orders p_0 and $p_1 = p_0 + 1$ for a given incidence on the plates θ, usually chosen as $\theta = 0$. In the case of a scanning Fabry-Perot, the latter definition is used since we change the thickness to scan the free spectral range.

Let's consider that λ_1 is transmitted for $\theta = 0$ at order p_0 when the thickness is e_1:

$$\lambda_1 p_0 = 2ne_1$$

For a thickness e_2, λ_1 is still transmitted for $\theta = 0$ but at order $p_1 = p_0 + 1$:

$$\lambda_1 p_1 = \lambda_1 p_0 + \lambda_1 = 2ne_2 = 2ne_1 + 2n\Delta e \Leftrightarrow \lambda_1 = 2n\Delta e$$

The free spectral range is the wavelength variation in order p_0 between the two plates separations:

$$\lambda_2 p_0 = \lambda_1 p_1 = \lambda_1 p_0 + \lambda_1 \Rightarrow \Delta\lambda = \frac{\lambda_1}{p_0}$$

Now, we consider the incidence θ_i where λ_1 is transmitted in order $p_i = p_0 - i$ for a spacing between the plate e_1. We have shown that:

$$\cos\theta_i = \frac{p_0 - i}{p_0}$$

We want to see what is the difference between the wavelength transmitted at order $p_0 - i$ for both spacings e_1 and e_2. For spacing e_2, we can write:

$$\begin{aligned} 2ne_2 \cos\theta_i &= 2ne_1 \cos\theta_i + \lambda_1 \cos\theta_i \\ &= \lambda_1(p_0 - i) + \lambda_1 \tfrac{p_0 - i}{p_0} \\ &= \lambda_2 p_0 \cos\theta_i = \lambda_2(p_0 - i) \end{aligned}$$

Hence

$$\Delta\lambda = \frac{\lambda_1}{p_0}$$

The interval scanned has not changed, however, we see that at this angle where λ_1 was transmitted in order p_0, this wavelength is no longer transmitted. This is due to the change of order at this θ. Indeed, at order $p_0 - i$, the free spectral might be $\lambda_1/(p_0 - i)$. Therefore, we miss some part of the spectrum.

The error made is therefore:

$$\delta\lambda = \lambda_1/(p_0 - i) - \lambda_1/p_0 = \frac{\lambda_1 i}{p_0(p_0 - i)} \Leftrightarrow \frac{\delta\lambda}{\Delta\lambda} = \frac{i}{p_0 - i}$$

To know the error made compared to the spectral resolution, we need to multiply the previous result by twice the finesse. In the case $i = 2$ and with a finesse $F = 16$, the error is 8% the FWHM, if $i = 5$ and $F = 16$, the error is around 20% the FWHM.

2.6 From wavelength variation to phase map

In this step, we need to know what is the physical displacement δe of the plates between two consecutive channels in order to derive the variation of wavelength for one channel $\delta\lambda = \frac{\partial\lambda}{\partial ch}$.

We have seen in the previous section that the wavelength range is the same everywhere in the field of view. Therefore, the knowledge of the step in wavelength between two channels can be evaluated in the centre (we only consider the wavelength range between λ_1 and λ_2):

$$\delta\lambda p = 2n\delta e$$

If the spacing step does not change, the elementary wavelength variation for the calibration is:

$$\delta\lambda_c = \frac{2n\delta e}{p_c} \Leftrightarrow \delta e = \frac{\delta\lambda_c p_c}{2n}$$

and for the observation:

$$\delta\lambda_s = \frac{2n\delta e}{p_s} \Leftrightarrow \delta\lambda_s = \frac{\delta\lambda_c p_c}{p_s} \Rightarrow \frac{\delta\lambda_s}{\delta\lambda_c} = \frac{p_c}{p_s}$$

In that case, the phase maps that is $\frac{\lambda - \lambda_0}{\delta\lambda}$ is the same for the calibration and for the observation:

$$\frac{\lambda_s - \lambda_{s0}}{\delta\lambda_s} = -\frac{\lambda_c}{p_s \delta\lambda_s} \frac{r^2}{r_2^2 - r_1^2} = -\frac{\lambda_c}{p_c \delta\lambda_c} \frac{r^2}{r_2^2 - r_1^2} = \frac{\lambda_c - \lambda_{c0}}{\delta\lambda_c}$$

If one scans a full spectral range ($\Delta\lambda$) using n_{ch} channels, we have:

$$\Delta\lambda p = 2n\Delta e = 2n\delta e \times n_{ch} = \frac{\lambda p}{p} = \lambda$$

therefore, we see that the spacing has to be adjusted (it is proportional to λ). In that case,

$$\delta\lambda = \frac{\Delta\lambda}{n_{ch}} = \frac{\lambda}{p n_{ch}} = \frac{2n\delta e}{p}$$

and the phase maps for the observation is:

$$\frac{\lambda_s - \lambda_{s0}}{\delta\lambda_s} = -\frac{\lambda_c}{p_s \delta\lambda_s}\frac{r^2}{r_2^2 - r_1^2} = -\frac{n_{ch}\lambda_c}{\lambda_s}\frac{r^2}{r_2^2 - r_1^2}$$

while the phase map for the calibration would be:

$$\frac{\lambda_c - \lambda_{c0}}{\delta\lambda_c} = -\frac{\lambda_c}{p_c \delta\lambda_c}\frac{r^2}{r_2^2 - r_1^2} = -n_{ch}\frac{r^2}{r_2^2 - r_1^2}$$

Hence

$$\frac{\lambda_s - \lambda_{s0}}{\delta\lambda_s} = \frac{\lambda_c - \lambda_{c0}}{\delta\lambda_c}\frac{\lambda_c}{\lambda_s}$$

In GHASP observations, the spacing between the plates is adjusted so that a full free spectral range is explored both for calibration and observation. We can quantify the accuracy of such an adjustment. The parameters corresponding to the plate spacing is *Qgval* (in .adt files). Hereafter is an example where the observations have 32 channels. One would need 33 channels to have the same wavelength in the centre as in the first channel.
At $\lambda_c = 6598.950\mathring{A}$, the *Qgval* parameter is -358 for channel 1 and +336 for channel 32.
At $\lambda_s = 6582.788\mathring{A}$, the *Qgval* parameter is -357 for channel 1 and +335 for channel 32.
At $\lambda_s = 6610.218\mathring{A}$, the *Qgval* parameter is -359 for channel 1 and +336 for channel 32.
From these measurements, one can estimate how much is missed if no correction is applied from calibration to observing wavelengths. In the case no spacing correction is applied, the spacing variation for the calibration would be $\Delta e = \frac{\lambda_c}{2n}$ to scan the full free spectral range. Such a spacing would induce at the order of scan a wavelength variation $\Delta\lambda = \frac{\lambda_c}{p_s}$ whereas the true free spectral range may be $\Delta\lambda_s = \frac{\lambda_s}{p_s}$. In the above example, using $\lambda_s = 6610.218$, the relative error is

$$\frac{\Delta\lambda - \Delta\lambda_s}{\Delta\lambda_s} = \frac{\lambda_c - \lambda_s}{\lambda_s} = 0.17\%$$

This is in agreement with the spacing variations observed (0.15%). This remains very small.

In particular, if the order is not the expected order, e.g. $p_s + 1$ instead of p_s, the free spectral range would be $\Delta\lambda_{s+1} = \frac{\lambda_s}{p_s+1}$ and the relative error would be

$$\frac{\Delta\lambda_s - \Delta\lambda_{s+1}}{\Delta\lambda_s} = \frac{1}{p_s + 1} = 0.12\%$$

if $p_s = 798$. The uncertainty in that case is of the same order.

2.7 Adjustment of the central channel wavelength

An additional correction is needed in order to select correctly the wavelength of a given first channel (it can be the first channel or the middle channel) after the phase map correction. Indeed, the phase map derived previously enables to know the shift in spectral channels needed to put $\lambda_{s0} = \frac{\lambda_{c0} p_c}{p_s}$ in the first channel at any position in the field (λ_{c0} is the central wavelength in the first calibration channel).

In the case we use a calibration cube, one derives the phase map directly by finding the position of the line λ_c in each pixel. This gives the channel where λ_c is transmitted. Shifting the spectrum by this amount ensures λ_c to be in the first channel everywhere in the field of view. At this stage, to be correct, the phase map might be this map minus the value of the map at the centre. This would ensure that the wavelength for each calibrated channel is the wavelength of the centre of the rings (before calibration). The central wavelength λ_{ci} in channel i is then easily computed using the channel p where λ_c is transmitted in the centre:

$$\lambda_{ci} = \delta\lambda_c(p - i) + \lambda_c$$

Therefore, once λ_{ci} is known, one can compute the corresponding wavelength transmitted at order p_s:

$$\lambda_{si} = \frac{\lambda_{ci} p_c}{p_s}$$

and derive the phase map:

$$\frac{\lambda_s - \lambda_{si}}{\delta\lambda_s} = \frac{\lambda_c - \lambda_{ci}}{\delta\lambda_c}\frac{\lambda_c}{\lambda_s}$$

Let's consider λ_{si} being the central wavelength for the central channel (where the spacing is constant between calibration and observation). After the wavelength calibration, one would prefer that the spectrum is centred around λ_s. To do so, it is sufficient to compute

$$\frac{\lambda_s - \lambda_{si}}{\delta\lambda_s}$$

and add (subtract?) this value to the phase map previously computed.

In the case spacings are not adjusted, we only have to skip the step where we multiply the phase map by $\frac{\lambda_c}{\lambda_s}$.

Remark: by unwrapping the phase map, one makes some small errors by considering the wavelength is always transmitted in the same order, which is not true depending on the position in the field.

A priori, the reference wavelength is not the calibration wavelength λ_c but λ_r transmitted at order p_r. From the knowledge of λ_r and p_r one can easily compute p_c, the order in which λ_c is transmitted:

$$\lambda_r p_r = (\lambda_c + \delta\lambda_c)p_c = \lambda_c p_c \pm \delta\lambda_c p_c = \lambda_c \left(p_c \pm p_c \frac{\delta\lambda_c}{\lambda_c}\right)$$

Since $2\delta\lambda_c$ has to be lower than one free spectral range, the quantity $p_c\frac{\delta\lambda_c}{\lambda_c}$ should be lower than 0.5. Hence p_c is the rounded value of $\lambda_r p_r/\lambda_c$.

2.8 Link with 2D interferogram

The procedure discussed above only uses one interferogram where we have determined the centre and the radius of the calibration rings. However, one can use the Airy function in order to adjust the interferograms:

$$T = T_0 \frac{1}{1 + \frac{4F^2}{\pi^2}\sin^2(\phi/2)} \tag{1}$$

where $\phi = 2\pi\frac{2ne\cos\theta}{\lambda}$, T_0 is the maximum transmission (it depends on the reflectivity and transmission of the FP mirrors) and F is the finesse.

In such a case, the parameters than can be adjusted are in principle T_0, F, ne, the centre of the rings and b ($\theta = br$). The wavelength λ_c is the only fixed parameter. The spacing e depends on the order p_c and has to be adjusted in order to match the set of rings. The radius of the rings can therefore be computed using e, b, λ_c, and the consecutive orders $p_c - i$.

2.9 Case with several lines

If several lines are present, we might observe several set of rings associated to each wavelength, a priori at different orders. One can take advantage of these multiple lines to produce a nice phase map.