Physics 311 Final Project

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1 Learning Objectives

1.1 Motivations and Goals

One of the most intriguing topics in Classical Mechanics is the discussion of rigid-body rotation. In Zero-G environment, the behaviour of rotating rigid bodies can be even more bizarre and unintuitive, yet, most amazingly, they can all be classically analysed. I believe these problems encourage critical thinking (not relying on everyday intuition) and make Physics more fun. This exercise discusses external-torque-free, energy-dissipating motion of a body with two equal moments. The exercise requires familiarity with conservation of angular momentum, kinetic energy of rotating bodies, and Euler Angles.

1.2 List of Specific Learning Objectives

After completing these exercises, students should have a better understanding of how to ...

1. *Identify conserved and non-conserved quantities*
2. *Identify "body frame" and "space frame" and their corresponding fixed axis*
3. *Express the quantity of interest as a function of known quantities*
4. *Determine stability of a system based on the rate of change of a quantity*

2 Exercises

2.1 Water Bottle in Space

Consider, in Zero-G, a massless, thin cylindrical bottle fully filled with water. The water is incompressible but flows freely in the bottle. Thus, the bottle can be considered a uniform-mass cylinder with principle moments $I_3 < I_1 = I_2$. The bottle is given an initial rotation (but the water inside it is initially stationary) with the angular velocity $\vec{\omega}_0$ misaligned with the principle axis $\hat{e}_3$ by a small angle.

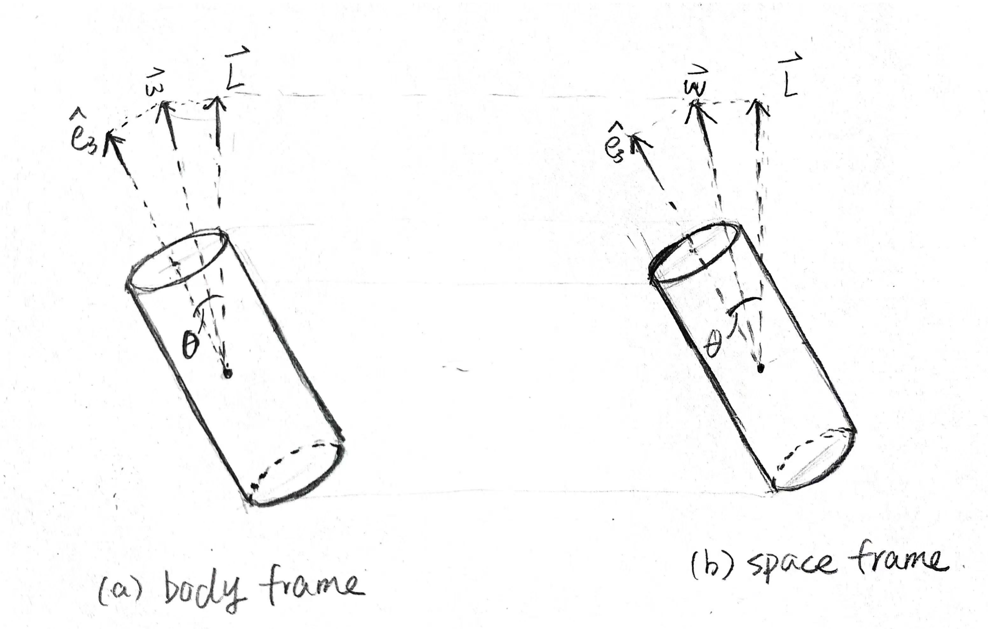


Figure 1: draw on me

(a) First, assuming the kinetic energy is conserved, sketch in Figure 1 the "body cone" (traced out by $\vec{\omega}$ in the body frame) and the "space cone" (traced out by $\vec{\omega}$ in the space frame). Specify the fixed vector in each frame. Sketching the paths the vector head trace is sufficient.

(b) With $T = \frac{1}{2}\vec{\omega} \cdot \vec{L}$ write down the angular momentum and kinetic energy in terms of the principle moments (I_1, I_2, and I_3) and the three components of $\vec{\omega}$ in the body frame (ω_1, ω_2, and ω_3).

(c) Given the angle between the angular momentum vector $\vec{L}$ and the principle axis $\hat{e}_3$, express the component of angular momentum $\vec{L}$ on the $\hat{e}_3$ direction L_3 in terms of the constant L. Then express ω_3 as a function of θ

(d) Now, in the real world, the initially stationary water is not rigid. Although the principle moments are not changed, the water flows and dissipates kinetic energy due to complicated internal forces. This energy dissipation affects the motion of bottle. To predict the effect, find out the relationship between θ and T, using small angle approximation $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Find out how θ changes with decreasing kinetic energy, i.e., $\dot{T} < 0$. Is the initial rotation stable? Hint: combine the equations of T and $\vec{L}$

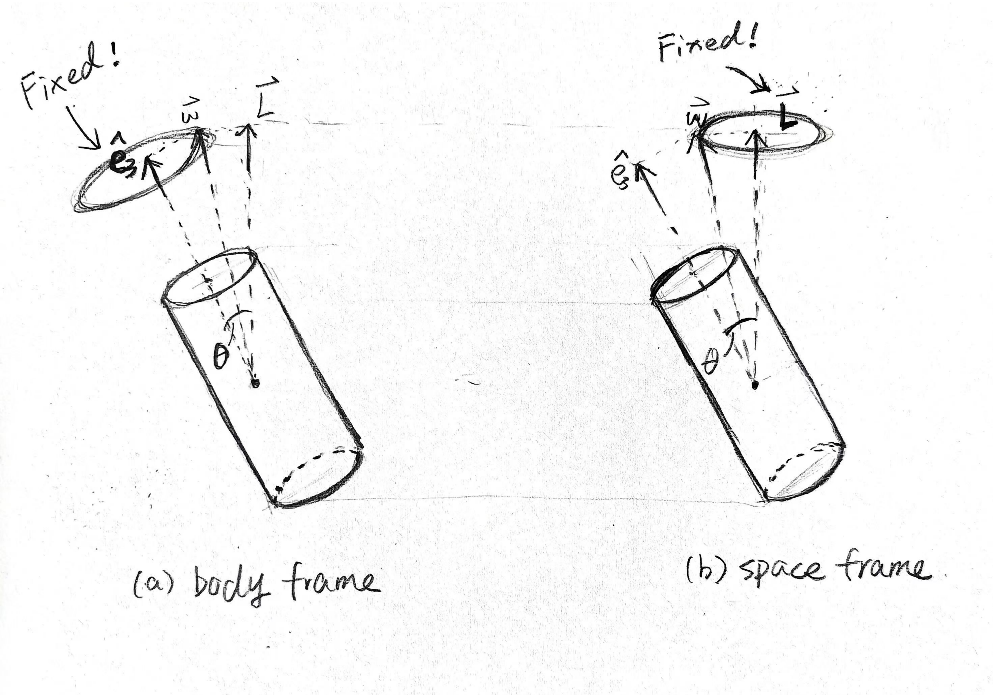
(e) Now imagine the initial angular velocity is closer to $\hat{e}_1$ or $\hat{e}_2$, what does energy dissipation contribute to the bottle's behaviour?

(f) Explain qualitatively why the water bottle's rotation is stable/unstable.

(g) Can you derive the same conclusion using the Lagrangian formalism?

SOLUTION:

(a)



(b) The angular momentum in body frame:

$$\vec{L} = \boldsymbol{I}\vec{\omega} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) \tag{1}$$

The kinetic energy:

$$\begin{aligned} T &= \frac{1}{2}\vec{\omega}\cdot\vec{L} \\ &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \end{aligned} \tag{2}$$

(c)

$$L_3 = \vec{L}\cdot\hat{e}_3 = L\cos\theta = I_3\omega_3 \tag{3}$$

$$\omega_3 = \frac{L}{I_3}\cos\theta \tag{4}$$

(d) The angular momentum gives us:

$$L^2 = \vec{L}\cdot\vec{L} = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \tag{5}$$

Given $I_1 = I_2$, we can rewrite T and L as the following:

$$2T = I_1(\omega_1^2 + \omega_2^2) + I_3\omega_3^2 \tag{6}$$

$$L^2 = I_1^2(\omega_1^2 + \omega_2^2) + I_3^2\omega_3^2 \tag{7}$$

Rearranging (7) we get:

$$\omega_1^2 + \omega_2^2 = \frac{L^2 - I_3^2\omega_3^2}{I_1^2} \tag{8}$$

Substituting into (6) to get:

$$2T = \frac{L^2}{I_1} + \omega_3^2 I_3 (1 - \frac{I_3}{I_1}) \tag{9}$$

With (4),

$$2T = \frac{L^2}{I_1} + \frac{L^2}{I_3}\cos^2\theta(1 - \frac{I_3}{I_1}) \tag{10}$$

Then, take the time derivative on both sides to find:

$$\dot{T} = \frac{L^2}{I_3}\sin\theta\cos\theta(\frac{I_3}{I_1} - 1)\dot{\theta} \tag{11}$$

Using small-angle approximation,

$$\dot{T} \approx \frac{L^2}{I_3}(\frac{I_3}{I_1} - 1)\theta\dot{\theta} \tag{12}$$

Rearrange to find:

$$\dot{\theta} \approx \frac{I_3 I_1}{L^2\theta(I_3 - I_1)}\dot{T} \tag{13}$$

With $\theta > 0$ and $I_3 < I_1$, we see that when $\dot{T} < 0$, we have $\dot{\theta} > 0$, i.e., the angle between $\hat{e}_3$ increases as the kinetic energy is being dissipated.

(e) In this case, $I_3 > I_1$, then we have an inverse relationship between $\dot{\theta}$ and $\dot{T}$, and θ tends to diminish and return to zero as the kinetic energy gets smaller. Thus, the rotation is stable as the energy is being dissipated.

(f) Without diving into the mess of frictions and other internal forces inside the bottle, the energy is being dissipated while the total angular momentum is being conserved, forcing the system to rotate about the principle axis with the largest moment of inertia, which in this case is any axis perpendicular to $\hat{e}_3$. From equation (13) we can see that when the kinetic energy stops changing ($\dot{T} = 0$; no relative motion between any parts of the body, hence no energy dissipation), θ becomes constant, and the bottle enters a state of stable rotation about the largest-moment-of-inertia axis.

(g) I can't. I tried so hard for this but ended up getting a total mess. I think the challenging part is determining the number of degrees of freedom, algebra, and playing with the generalized force.

3 References

Muller, Derek. https://www.youtube.com/watch?v=1VPfZ˙XzisU.
Peter W. Likins. *Effects of Energy Dissipation on the Free Body Motions of Spacecraft*