BBM408 First Assignment

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Question 1

Solve the following recurrence using telescoping:

$$\boldsymbol{na_n = (n-3)a_{n-1} + 1}$$

for $n \geq 3$ with $a_0 = 0$, $a_1 = 1$, $a_2 = 0$.

Solution:

$na_n = (n-3)a_{n-1} + 1$
$na_n = (n-1)a_{n-1} - 2a_{n-1} + 1$

The substitution

$na_n = b_n$ is applied. (eq-1)

From this substitution these equations are obtained:

$(n-1)a_{n-1} = b_{n-1}$
$a_{n-1} = \frac{b_{n-1}}{n-1}$

These equations yield new recurrence:

$b_0 = 0,\ b_1 = 1,\ b_2 = 0$

$b_n = b_{n-1} - \frac{2b_{n-1}}{n-1} + 1$
$(n-1)b_n = (n-1)b_{n-1} - 2b_{n-1} + (n-1)$
$(n-1)b_n = (n-2)b_{n-1} - b_{n-1} + (n-1)$

The substitution

$(n-1)b_n = c_n$ is applied. (eq-2)

From this substitution these equations are obtained:

$(n-2)b_{n-1} = c_{n-1}$
$b_{n-1} = \frac{c_{n-1}}{n-2}$

These equations yield new recurrence:

$c_0 = 0,\ c_1 = 0,\ c_2 = 0$

$c_n = c_{n-1} - \frac{c_{n-1}}{n-2} + (n-1)$
$(n-2)c_n = (n-2)c_{n-1} - c_{n-1} + (n-1)(n-2)$
$(n-2)c_n = (n-3)c_{n-1} + (n-1)(n-2)$

The substitution

$(n-2)c_n = d_n$ is applied. (eq-3)

From this substitution this equation are obtained:

$(n-3)c_{n-1} = d_{n-1}$

This equation yields new recurrence:

$d_0 = 0,\ d_1 = 0,\ d_2 = 0$

$d_n = d_{n-1} + (n-1)(n-2)$

This last recurrence is a recurrence with constant coefficients. So, it can be solved with using telescoping. Writing some equations of the recurrence:

$d_3 = d_2 + (1 * 2)$
$d_4 = d_3 + (2 * 3)$
$d_5 = d_4 + (3 * 4)$
.
.
.
$d_{n-2} = d_{n-3} + (n-3)(n-4)$
$d_{n-1} = d_{n-2} + (n-2)(n-3)$
$d_n = d_{n-1} + (n-1)(n-2)$

Sum up these equations side by side yields this equation:

$d_n = d_2 + (1 * 2) + (2 * 3) + ... + (n-3)(n-2) + (n-2)(n-1)$ (eq-4)
where $d_2 = 0$.

After substitute $k = n - 2$, right hand side of the (eq-4) it will be like this:

$(-1 * 0) + (0 * 1) + (1 * 2) + ... + (k-2)(k-1) + k(k-1) + k(k+1)$

Which equals to:

$(1 + 1^2) + (2 + 2^2) + (3 + 3^2) + ... + ((k-1) + (k-1)^2) + (k + k^2)$

This equation could be written like this:

$(1 + 2 + 3 + ... + (k-1) + k) + (1^2 + 2^2 + 3^2 + ... + (k-1)^2 + k^2)$

$\frac{k(k+1)}{2} + \frac{k(k+1)(2k+1)}{6}$

$\frac{k(k+1)(k+2)}{3}$

Backsubstitute $n = k + 2$, final version of right hand side of the (eq-4) like this:

$\frac{(n-2)(n-1)n}{3}$

So, full equation of (eq-4) is this:

$d_n = \frac{(n-2)(n-1)n}{3}$

Backsubstitute $d_n = (n-2)c_n$, (eq-3), this equations are obtained:

$(n-2)c_n = \frac{(n-2)(n-1)n}{3}$
$c_n = \frac{(n-1)n}{3}$

Backsubstitute $c_n = (n-1)b_n$, (eq-2), this equations are obtained:

$(n-1)b_n = \frac{(n-1)n}{3}$
$b_n = \frac{n}{3}$

Backsubstitute $b_n = na_n$, (eq-1), this final equations are obtained:

$na_n = \frac{n}{3}$
$\boldsymbol{a_n = \frac{1}{3}}$

Checking some values:

for n = 3	$3a_3 = 0a_2 + 1$	$a_3 = 1/3$
for n = 4	$4a_4 = 1a_3 + 1$	$a_4 = 1/3$
for n = 5	$5a_5 = 2a_4 + 1$	$a_5 = 1/3$

Question 2

Solve the recurrence $\boldsymbol{a_n = 2a_{n-1} - a_{n-2}}$ for $n \geq 2$ with $a_0 = 0$ and $a_1 = 1$ using the characteristic equation method.

Solution:

$a_n = 2a_{n-1} - a_{n-2}$ (eq-5)

Substitute $a_n = r^n$, and (eq-5) will be like this:

$r^n = 2r^{n-1} - r^{n-2}$

Divide both side by r^{n-2}

$r^2 = 2r - 1$
$r^2 - 2r + 1 = 0$
$(r-1)^2 = 0$ (eq-6)
$r_0 = 1$

There is one root for (eq-6). So, the characteristic equation of (eq-5) is:

$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ (eq-7)

Put r_0 to (eq-7):

$a_n = \alpha_1 + \alpha_2 n$

It is known that $a_0 = 0$ and $a_1 = 1$. So,

for n = 0 $\alpha_1 = 0$
for n = 1 $\alpha_2 = 1$

Hence, solution of the recurrence is:

$\boldsymbol{a_n = n}$

Checking some values:

for n=2 $a_2 = 2a_1 - a_0$ $a_2 = 2$
for n=3 $a_3 = 2a_2 - a_1$ $a_3 = 3$
for n=4 $a_4 = 2a_3 - a_2$ $a_4 = 4$

Question 3

Solve the recurrence $\boldsymbol{a_n = 3a_{n/2} + n^2}$ using the master method.

Solution:

The Master Theorem is:

$T(n) = aT(\frac{n}{b}) + f(n)$
where $a \geq 1$, $b > 1$, and $f(n) = n^k + \log^p n$

At the equation above

$a = 3$, $b = 2$, and $f(n) = n^2$ where $k = 2$ and $p = 0$.

Since $\log_b a < k$ $(\log_2 3 < 2)$ and $p \geq 0$,

$\boldsymbol{a_n = \theta(n^2)}$

Question 4

Solve the recurrence

$\boldsymbol{2a(n) = -3a(n-1) + 3a(n-2) + 2a(n-3)}$

for $n \geq 3$ with $a(0) = 1$, $a(1) = -2$, $a(2) = 4$ using generating functions.

Solution:

Ordinary Generating Function is known:

$A(z) = \sum_{k \geq 0} a_k z^k$

$\sum_{N \geq 0} z^N = \frac{1}{1-z}$

Make given recurrence valid for all n:

$2a(n) = -3a(n-1) + 3a(n-2) + 2a(n-3) + 2\delta_{n_0} - \delta_{n_1} - \delta_{n_2}$

$2A(z) = -3zA(z) + 3z^2A(z) + 2z^3A(z) + 2 - z - z^2$

$A(z) = \frac{-z^2-z+2}{-2z^3-3z^2+3z+2}$

$A(z) = \frac{(-z+1)(z+2)}{(-2z-1)(z-1)(z+2)}$

$A(z) = \frac{1}{1+2z}$

Expand the recurrence relation:

$\boldsymbol{a(n) = (-2)^n}$

Checking some values:

for n = 3:	$2a_3 = -3a_2 + 3a_1 + 2a_0$	$a_3 = -8$
for n = 4:	$2a_4 = -3a_3 + 3a_2 + 2a_1$	$a_4 = 16$
for n = 5:	$2a_5 = -3a_4 + 3a_3 + 2a_2$	$a_5 = -32$