Star Coloring of Hypergraphs

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Introduction

- A ***star coloring*** of an undirected graph G is a proper vertex-coloring (no two adjacent vertices share a color) such that no path of length 3 in G is bicolored.
- The ***star chromatic number*** of G, denoted $\chi_s(G)$, is the smallest k for which G admits a star coloring with k colors.

Figure 1: In a star coloring, a subgraph induced by vertices of any two colors will reveal star-like structures as seen in the figure.

- A ***hypergraph*** $H = (V, E)$ is defined by a set V of vertices, and a family E of subsets of V, which are called hyperedges defined on V.
- H is an r-uniform hypergraph, if every hyperedge has exactly r vertices.
- A vertex coloring of a Hypergraph H is called a ***star coloring*** if
  1. It contains no monochromatic hyperedges, i.e. the coloring is proper
  2. It contains no bicolored 3-paths of length greater than 3, except if the 3-path is a star-path, with the three edges having at least one vertex in common.
- The ***star chromatic number*** of an r-uniform hypergraph H, denoted $\chi_s(H)$, is the minimum k for which H admits such a coloring.
- Our aim is to derive upper and lower bounds on the star chromatic number of hypergraphs, specifically in terms of **maximum degree** Δ
- We will make use of the Lovasz Local lemma in it's general form[2] : Given a dependency graph G of events $A_1, A_2, ..., A_n$ and real numbers $y_1, y_2, ..., y_n$ such that $0 \le y_i < 1$ and

$$Pr[A_i] \le y_i \prod_{(i,j)\in E} (1 - y_j)$$

for all $1 \le i \le n$. Then

$$Pr[\bigwedge_{i=1}^{n} \bar{A}_i] \ge \prod_{i=1}^{n} (1 - y_i).$$

In particular, with positive probability no event A_i holds.

1 Upper Bound

Given an r-uniform hypergraph H with maximum degree Δ, we show that with non-zero probability, a star-coloring exists with x colors, where

$$x = \lceil (2^{3r-3} \cdot 243 \cdot r^4 \cdot \Delta^3)^{\frac{2}{3r-4}} \rceil$$

Proof sketch.

- *Random coloring:* Assign each vertex one of x colors uniformly.
- *We define bad event types:*
  - $\mathbf{A_M}$: some edge is monochromatic.
  - $\mathbf{A_i}$: a bicolored non-star 3-path of "size" $3r - i$, defined for $i : 2 \to \frac{3r}{2}$.
- *Dependency graph:* We construct a dependency graph of bad events of these types, where every bad event is a vertex, and shares an edge with another bad-event if their associated vertex sets share a vertex
- For example, two bad events of type A_M would share an edge if the two hyperedges share a vertex

	M	2	3	...	$\frac{3k}{2}$
M	$k\Delta$	$\frac{5}{2}k^3\Delta^3$	$\frac{5}{2}k^3\Delta^3$	...	$\frac{5}{2}k^3\Delta^3$
2	$3k\Delta$	$\frac{15}{2}k^3\Delta^3$	$\frac{15}{2}k^3\Delta^3$	...	$\frac{15}{2}k^3\Delta^3$
3	$3k\Delta$	$\frac{15}{2}k^3\Delta^3$	$\frac{15}{2}k^3\Delta^3$	...	$\frac{15}{2}k^3\Delta^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\frac{3k}{2}$	$3k\Delta$	$\frac{15}{2}k^3\Delta^3$	$\frac{15}{2}k^3\Delta^3$	...	$\frac{15}{2}k^3\Delta^3$

Table 1: Dependency table of bad events. Entry $[i, j]$ gives an upper bound on the number of bad events of type j that a bad event of type i is not independent with.

- *Apply Lovasz Local Lemma in it's general form:* Probabiltiy of bad events is small enough that none of them happen with non-zero probability, i.e. a proper star-coloring exists with positive probability.

2 Lower Bound

- **Random hypergraphs:** We consider r-uniform random hypergraphs $H_{(n,p)}$ of n vertices with each possible hyperedge existing with probability p.
- We provide an upper bound for the size of a maximum independent set in r-uniform random hypergraphs, $\alpha(H_{(n,p)})$.
- We consider random colorings of $H_{(n,p)}$, and establish a lower bound on the star-chromatic number in three steps :
  1. **Every k-coloring either creates a monochromatic edge or a bichromatic triangle:** For $p > \left(\left(\frac{r^2 \ln n}{n} \cdot \frac{2^{r+3}}{3^{r-4}}\right)^{1/3}\right)$ and $k \le n/r$ all of the k^n possible k-colorings of $H_{(n,p)}$ violate, asymptotically almost surely, the star-coloring requirements. Specifically:
     - If there exists at least one color that is assigned to more than $\alpha(H_{(n,p)})$ vertices, then there exists a monochromatic hyperedge in $H_{(n,p)}$.
     - In the absence of such color classes, we demonstrate by a method adapted from Lemma 4 in [1] that there exists a bichromatic triangle with high probability.
  2. **Upper bound on the maximum degree Δ:** We show that by our choice of p the maximum degree $\Delta(H(n, p))$ remains below a prescribed threshold d with probability tending to 1.
  3. **Intersection argument.** Since "having a bichromatic triangle" and "having $\Delta \le d$" each hold with probability $\to 1$, there exists a hypergraph with both properties. Such a hypergraph can't be colored with k colors, as long as n and by extension d is large enough. This yields

$$\chi_s\big(H_{(n,p)}\big) \ \ge \ k \ = \ \frac{n}{r} \ > \ \frac{d^{1/(r-1)}(r-1)}{r} \ge \ \frac{\Delta^{1/(r-1)}(r-1)}{r},$$

hence $\chi_r(d) \ge d^{\frac{1}{r-1}}\frac{r-1}{r}$ for all large d.

3 Conclusion

We established upper and lower bounds on the star chromatic number in terms of Δ; future work may improve the upper bound, and may relax the requirement where n and Δ have to be asymptotically larger in comparison to r, which is currently the case, especially for small values of r (Δ can be significantly closer to r after about $r = 21$).

References

[1] Louigi Addario-Berry, Louis Esperet, Ross J Kang, Colin JH McDiarmid, and Alexandre Pinlou. Acyclic improper colourings of graphs with bounded maximum degree. *Discrete mathematics*, 310(2):223–229, 2010.

[2] Paul Erdős and László Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *in Infinite and Finite Sets (A. Hajnal et al., eds)*. North-Holland, Amsterdam, 1975.