STAR COLORING OF HYPERGRAPHS

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Abstract.

1. Introduction

Hypergraph coloring is widely studied, such as in [3–6, 11–13, 17, 19, 20]. Erdős and Lovász [7] proved that the vertex set of any $(r+1)$-uniform hypergraph with maximum degree Δ can be colored with $c\Delta^{1/r}$, for some constant c, such that there is no monochromatic hyperedge. This result is proved by an application of Lovász Local Lemma, which is also introduced in the same paper.

The *star-coloring* of a hypergraph is defined as the vertex coloring without a bicolored (2-colored) star, where a *star* in a hypergraph is a collection of edges with a nonempty common intersection. The star coloring of graphs is introduced by Grünbaum [15], who proved that a graph with maximum degree 3 has an acyclic coloring with 4 colors. In the same paper, acyclic coloring of graphs is introduced, defined as the vertex coloring without bicolored cycles. Both, the star coloring and acyclic coloring problems are shown to be NP-complete by Albertson et al. [1] and Kostochka [18], respectively.

Alon, Mcdiarmid, and Reed [2] proved that there exist graphs with maximum degree Δ that have acyclic coloring using $\Omega((\Delta^{\frac{4}{3}})/(log\Delta)^{\frac{1}{3}})$ colors. They also show that every graph has an acyclic coloring with $O(\Delta^{\frac{4}{3}})$ colors. Recently, there have been some improvements in the constant factor of the upper bound in [9, 14, 21] by using the entropy compression method. Similar bounds for the star coloring are obtained by Fertin et al. [10], showing every graph with maximum degree Δ has a star coloring using at most $\lceil 20\Delta^{3/2}\rceil$ colors and that there exist graphs that need $\Omega(\Delta^{\frac{3}{2}}/(log\Delta)^{\frac{1}{2}})$ colors in a star coloring. We study here the star coloring of hypergraphs and aim to generalize the known bounds for graphs to hypergraphs.

- Our assumptions: linear, k-intersecting hypergraphs...

1.1. **Notation.** Let H be a hypergraph with vertex-set $V(H)$ and edge-set $E(H) \subseteq 2^{V(G)}$. We define $e(H) = |E(H)|$. H is called *r-uniform* if every edge has size r. For $v \in V(H)$ we denote by

$$\deg(v) \;=\; \big|\{\, e \in E(H) : v \in e\}\big|$$

the number of edges containing v, and by

$$\Delta(H) \;=\; \max_{v \in V(H)} \deg(v)$$

the maximum degree of H.
A vertex coloring of a Hypergraph H is called a *star coloring* if

(1) It contains no monochromatic hyperedges, i.e. the coloring is proper

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(2) It contains no bicolored 3-paths of length greater than 3, except if the 3-path is a star-path, that is if the three hyperedges have at least one vertex in common.

The *star chromatic number* of an r-uniform hypergraph H, denoted $\chi_s^r(H)$, is the minimum k for which H admits such a coloring. We let $\chi_s^r(d) = \max\{\chi_s^r(H) : \Delta(H) = d\}$.

1.2. **Probabilistic tools.** We will make use of the following standard probabilistic tools.

Theorem 1 (General Lovasz Local Lemma). *[8] Suppose that $G = (V, E)$ is a dependency graph for the events $A_1, A_2, ..., A_n$ and suppose there are real numbers $y_1, y_2, ..., y_n$ such that $0 \leq y_i < 1$ and*

$$Pr[A_i] \leq y_i \prod_{(i,j)\in E} (1 - y_j) \qquad \text{for all } 1 \leq i \leq n. \tag{1}$$

Then $Pr[\bigwedge_{i=1}^n \bar{A}_i] \geq \prod_{i=1}^n (1 - y_i)$. In particular, with positive probability no event A_i holds.

Lemma 1 (Chernoff bound, Eq.2.5 in [16]). *Let $X \sim \mathrm{Bin}(n, p)$. For every $t \geq 0$,*

$$\Pr\big[X \geq np + t\big] \;\leq\; \exp\Big(-\frac{t^2}{2\,(np + t/3)}\Big). \tag{2}$$

2. Upper Bound

Theorem 2. *Let H be a r-uniform hypergraph of maximum degree Δ, with vertex-set $V = V(H)$ and edge-set $E = E(H)$. Then star chromatic number $\chi_s^r(H) \leq \lceil (2^{3r-3} \cdot 81 \cdot r^4 \cdot \alpha \cdot \Delta^3)^{\frac{2}{3r-4}} \rceil$ for $r \geq 2$ and $\alpha = 3$.*

Proof. Let $x = \lceil (2^{3r-3} \cdot 81 \cdot r^4 \cdot \alpha \cdot \Delta^3)^{\frac{2}{3r-4}} \rceil$, and let us color V with these x colors, by assigning each vertex a color chosen uniformly at random from a set of colors $C = \{1, 2, 3, \ldots, x\}$. Now we will show that with nonzero probability, a coloring so obtained is a star coloring of hypergraph H.

We first have to define 'bad' event types :

- **Type M:** For a given edge $e \in E$, bad event $A_{(M,e)}$ is the event that e is monochromatic.
- **Type i:** For a given vertex set $S \subseteq V$ of $3r - i$ vertices, $A_{(i,S)}$ is the event that there exists edges $e_1 \subset S$, $e_2 \subset S$, $e_3 \subset S$, which constitute either a non-star, non-triangle 3 path or a triangle, that is bicolored or monochromatic.

Where a type i event type is defined for all $2 \leq i \leq \frac{3r}{2}$. We note that within this range of bad events all possible non-star triangles and 3-paths are accounted for.

If none of these events occur in its construction, then a coloring of H is a star coloring.

We construct now a dependency graph D of these events, where each event of any type is a node in this graph. We define an edge in the dependency graph as follows:

- Between events $A_{(M,e)}$ and $A_{(i,S)}$ if $e \cap S \neq \emptyset$.
- Between events $A_{(i,S_1)}$ and $A_{(i,S_2)}$ if $S_1 \cap S_2 \neq \emptyset$.

Observation 1.

(1) A vertex $v \in V$ belongs to at most Δ edges in E.
(2) A vertex $v \in V$ belongs to fewer than $\frac{3}{2}r^2\Delta^3$ non-star, non-triangle paths of length 3 in E.
(3) A vertex $v \in V$ belongs to fewer than $\Delta^3 \cdot r^2$ triangles.

Proof.

(1) Δ is the maximum degree in H.

(2) Suppose v is a vertex in one of the terminal-edges of a linear non-star, non-triangle path. A first edge may be selected in Δ ways, a second edge in $r(\Delta-1)$ ways, and a third edge in $(r-1)(\Delta-1)$ ways, where the vertex connecting the first and the second edges is banned to avoid forming a star-path. Now, note that the starting vertex may be incident to both the first and the middle edges here (and possibly to the third edge as well, but not all of them at the same time, which would create a star-path).

Now, consider the case where our vertex is incident only to the middle edge. There can be Δ such middle edges, with $\binom{r-1}{2}\cdot(\Delta-1)\cdot(\Delta-1)$ maximum terminal edges per middle edge. In total, this gives an upper bound of fewer than $\frac{3}{2}r^2\Delta^3$ non-star paths of length 3.

Note that any non-linear 3-paths, that is 3-paths that include edges connected with several vertices, instead of just one, are counted multiple times using this method.

(3) A vertex $v \in V$ belongs to fewer than $\Delta^2 \cdot r$ 2-paths. The number of ways these two edges may be connected to form a triangle is at most $\binom{r-1}{1}\cdot\min\{\Delta,\binom{r-1}{1}\}$. For simplicity, we settle on the upper bound of $r^2\Delta^3$ maximum triangles for $v \in V$. Now note that any non-linear triangles are counted several times with this method.

□

Lemma 2. *An upper bound for the number of adjacent vertices in D a type i event has with type j events, denoted $D_{i,j}$ is given by the entry in the row i and column j of the following table:*

	M	*2*	*3*	$\dots$	$\frac{3r}{2}$
M	$r\Delta$	$5/2r^3\Delta^3$	$5/2r^3\Delta^3$	$\dots$	$5/2r^3\Delta^3$
2	$3r\Delta$	$15/2r^3\Delta^3$	$15/2r^3\Delta^3$	$\dots$	$15/2r^3\Delta^3$
3	$3r\Delta$	$15/2r^3\Delta^3$	$15/2r^3\Delta^3$	$\dots$	$15/2r^3\Delta^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\frac{3r}{2}$	$3r\Delta$	$15/2r^3\Delta^3$	$15/2r^3\Delta^3$	$\dots$	$15/2r^3\Delta^3$

The upper bound

$$D_{i,j} \leq \frac{15}{2}r^3\Delta^3.$$

holds for general $D_{i,j}$, and is used for simplicity whenever $i \neq M$ and $j \neq M$. While more specifically, the number of adjacent vertices in the dependency graph of a type M event with events of type i, denoted $D_{M,i}$ is at most

$$D_{M,i} \leq \frac{5}{2}r^3\Delta^3.$$

The number of adjacent vertices in the dependency graph of a type i event with events of type M, denoted $D_{i,M}$ is at most

$$D_{i,M} \leq 3r\Delta.$$

Proof. We establish the bound by using the results from

Observation 2.

(1) *The number of 3-paths of $3r-j$ vertices a vertex can participate in is bounded by $\frac{3}{2}r^2\Delta^3$.*
(2) *The number of triangles with $3r-j$ vertices a vertex can participate in is at most $r^2\Delta^3$.*
(3) *Since each event involves at most $3r-2$ vertices, the total number of adjacent events in the dependency graph due to 3-paths and triangles can not exceed:*

$$\frac{9}{2}r^3\Delta^3 + 3r^3\Delta^3 = \frac{15}{2}r^3\Delta^3.$$

□

Observation 3.

(1) *For each event $A_{(M,e)}$ of type M,*

$$Pr[A_{(M,e)}] = \frac{1}{x^{r-1}}, \quad \text{where } x \text{ is the number of colors available.}$$

(2) *For each event $A_{(i,S)}$ of type i for $2 \leq i \leq \frac{3r}{2}$,*

$$Pr[A_{(i,S)}] = \frac{\binom{x}{2}\cdot 2^{3r-i} - x\cdot(x-2)}{x^{3r-i}} < \frac{2^{3r-1-i}}{x^{3r-2-i}} < \frac{2^{3r-3}}{x^{(3r/2)-2}}.$$

In order to apply the Lovasz's Local Lemma, we choose $y_M = Pr[A_{(M,e)}]\cdot c$ for all events $A_{(M,e)}$, and again $y_i = Pr[A_{(i,S)}]\cdot c$ for all events $A_{(i,S)}$ of type i. And we let $c = 2$ for $r \geq 2$ and require that y_M, y_i lie in the interval $[0, 1)$:

Now, we verify that they do :

$$y_M = \frac{c}{x^{r-1}} > 0$$

Now we investigate the upper bound, and substitute x:

$$\begin{aligned} y_M &\leq \frac{c}{\left(2^{3r-3}\cdot 81\cdot r^4\cdot\alpha\cdot\Delta^3\right)^{\frac{2}{3}\left(1+\frac{1}{3r-4}\right)}} \\ y_M &\leq \frac{c}{2^{2r}(3r)^{8/3}\alpha^{2/3}\Delta^2\left(2^{3r-4}\cdot 81\cdot r^4\cdot\alpha\cdot\Delta^3\right)^{\frac{2}{9r-12}}} \\ &< \frac{c}{\left(2^{2r}(3r)^{8/3}\alpha^{2/3}\Delta^2\right)\cdot 1^{\frac{2}{9r-12}}} < 1 \end{aligned} \tag{3}$$

Thus, y_M satisfies the required condition.

Next, we verify y_i:

$$y_i = c\cdot\frac{\binom{x}{2}\cdot 2^{3r-i} - x\cdot(x-2)}{x^{3r-i}} > 0$$

$$y_i \leq c\cdot\frac{2^{3r-3}}{x^{\frac{3r-4}{2}}}$$

Substituting x:

$$y_i \leq \frac{c}{81\cdot r^4\cdot\alpha\cdot\Delta^3} < 1 \tag{4}$$

Thus, y_i also satisfies the required condition. Note that, (4) is satisfied by all y_i.

Now we require :

$$\mathbb{P}[A_{(M,e)}] \leq y_M \cdot (1-y_M)^{r\Delta} \cdot \prod_{i=2}^{\frac{3r}{2}} (1-y_i)^{\frac{5}{2}r^3\Delta^3}, \tag{5}$$

$$\mathbb{P}[A_{(i,S)}] \leq y_i \cdot (1-y_M)^{3r\Delta} \cdot \prod_{i=2}^{\frac{3r}{2}} (1-y_i)^{\frac{15}{2}r^3\Delta^3}, \tag{6}$$

which reduces to proving:

$$\frac{1}{c} \leq (1-y_M)^{3r\Delta} \cdot \prod_{i=2}^{\frac{3r}{2}} (1-y_i)^{\frac{15}{2}r^3\Delta^3}. \tag{7}$$

Note that:

$$(1-y_M)^{3r\Delta} \cdot \prod_{i=2}^{\frac{3r}{2}} (1-y_i)^{\frac{15}{2}r^3\Delta^3} \leq (1-y_M)^{r\Delta} \cdot \prod_{i=2}^{\frac{3r}{2}} (1-y_i)^{\frac{5}{2}r^3\Delta^3},$$

So in order to prove inequality (5), it suffices to show that the following expression holds:

$$\frac{1}{c} < \left(1 - \frac{c}{x^{r-1}}\right)^{3r\Delta} \cdot \prod_{i=2}^{\frac{3r}{2}} \left(1 - c \cdot \frac{2^{3r-3}}{x^{(3r-4)/2}}\right)^{\frac{15}{2}r^3\Delta^3} \tag{8}$$

As we have $0 \leq y_i < 1$ for all $i \in \{M, 2, 3 \ldots, \frac{3r}{2}\}$, and all exponents are greater than 1, we may apply Bernoulli's inequality:

$$(1+x)^r \geq 1 + r \cdot x \quad \forall x \geq -1, \forall r > 1.$$

To do so repeatedly, we further require $y_M \cdot 3r \cdot r \cdot \Delta < 1$ and $y_i \cdot \frac{15}{2} \cdot r^3 \cdot \Delta^3 \cdot \frac{3r}{2} < 1$. The latter can be easily derived from (4), and the former from (3),

Finally, we want to show:

$$\frac{1}{c} < \left(1 - 3r\Delta \cdot \frac{c}{x^{r-1}}\right) \cdot \left(1 - \frac{15}{2}r^3\Delta^3 \cdot c \cdot \frac{2^{3r-3}}{x^{(3r-4)/2}} \cdot \frac{3r}{2}\right) \tag{9}$$

Substituting

$$x = \left\lceil \left(2^{3r-3} \cdot 81 \cdot r^4 \cdot \alpha \cdot \Delta^3\right)^{\frac{2}{3r-4}} \right\rceil > \left(2^{3r-3} \cdot 81 \cdot r^4 \cdot \alpha \cdot \Delta^3\right)^{\frac{2}{3r-4}}, \tag{10}$$

we derive an upper bound for $y_M \cdot 3r\Delta$ from (3):

$$\begin{aligned} y_M \cdot 3r\Delta &< \frac{c \cdot 3r\Delta}{\left(2^{2r}(3r)^{8/3}\alpha^{2/3}\Delta^2\right) \cdot 1^{\frac{1}{9r-12}}} \\ &\leq \frac{c}{25 \cdot 19 \cdot \Delta} \end{aligned} \tag{11}$$

Here, we have used the lower bounds on $\alpha \geq 2$, and $r \geq 2$.

Now, we apply the bound from (11) and substitute x to establish (9):

$$\frac{1}{c} \leq \left(1 - \frac{c}{475\Delta}\right) \cdot \left(1 - \frac{5c}{36\alpha}\right) \leq (1 - 3r\Delta \cdot y_M) \cdot \left(1 - \frac{15}{2} r^3 \Delta^3 \cdot c \cdot \frac{2^{3r-3}}{x^{(3r-4)/2}} \cdot \frac{3r}{2}\right)$$

which is satisfied for $\alpha = 3$, $\Delta \geq 2$ and $c = 2$. We've used the lower bound for x in (10).

□

3. Lower Bound

Let $G_r(n,p)$ be a r-uniform Erdős-Renyi *random hypergraph* on n vertices where each possible hyperedge exists with probability p. We let $\alpha(G_r(n,p))$ denote the size of a maximum independent set in $G_r(n,p)$. Let C_m^r denote an r-uniform cycle on m edges, which is defined by a sequence edges $E_1, \ldots, E_m$ such that $E_i \cap E_{i+1} \neq \emptyset$ for $1 \leq i \leq m-1$, and also $E_m \cap E_0 \neq \emptyset$.

Theorem 3. *If* $\frac{d}{D} \geq \frac{2}{r}$, *then* $\chi_s^r(d) \geq \frac{r-1}{r} d^{\frac{1}{r-1}}$.

We show a general bound on star chromatic number for all hypergraphs. With Lemma 3, we aim to show that with high probability, there is a bichromatic triangle in $G_r(n,p)$. In general, the bad events that violate star coloring include all kinds of non-star subhypergraphs with three or more edges.

Lemma 3. *For a real number* $\left(\frac{r^2 \ln(n)}{n} \cdot \frac{2^{r+3}}{3^{r-4}}\right)^{\frac{1}{3}} \leq p \leq 1$, *let* $m \leq \alpha(G_r(n,p))$, *that is the size of the maximum independent set in* $G_r(n,p)$. *Then, asymptotically almost surely and uniformly over* p, *any coloring of* $G_r(n,p)$ *with* $k \leq \frac{n}{r}$ *colors, where each color class has at most* $\alpha(G_r(n,p)) \leq \left(\frac{r^r \ln n}{p}\right)^{\frac{1}{r-1}}$ *vertices, contains a bichromatic triangle* C_3^r.

Proof. Considering a particular coloring of $G_r(n,p)$ with k colors, from all possible k^n colorings, we show that the probability of the event E of $G_r(n,p)$ not having a bichromatic triangle C_3^r is $o(n^{-n})$.

Let $\mathcal{K}_1, \ldots, \mathcal{K}_k$ denote color classes. These color classes are partitioned into subsets containing $\frac{3r}{4}$ vertices each. We then consider the triangles formed by unique pairs of of these subsets, where each hyperedge in a triangle shares exactly half of it's vertices with one hyperedge, and the remaining half with another. We also require that these triangles have no edge in common.

Let E be the event of $G_r(n,p)$ not having any of the triangles as defined above. Let E' be the event of $G_r(n,p)$ not having any bicolored non-star subhypergraph. Let a be the maximum number of edge-disjoint linear C_3^r's on a set of $\frac{3r}{2}$ vertices and b be the number of distinct pairs of subsets with $\frac{3r}{4}$ vertices. Then, since $E' \subseteq E$, we have

$$\Pr(E') \leq \Pr(E) \leq (1-p^3)^{ab}. \tag{12}$$

Let us establish a lower bound on a. First, we consider the number of triangles in $\frac{3r}{2}$ vertices. We observe that an edge of r vertices appears in exactly $\frac{1}{2}\binom{r}{r/2}$ triangles, that is the number of ways a second hyperedge in the triangle can be picked, with each choice of second hyperedge counted twice, as two hyperedges completely determine the third hyperedge. We compute the total number of triangles by summing over all distinct first-hyperedges, note that each triangle is counted three times in this arrangement:

$$\frac{1}{3}\binom{3r/2}{r}\binom{r}{r/2}\frac{1}{2}$$

We now enforce a further requirement on these triangles, that no hyperedge should appear in more than one triangle, so that their probabilities in (12) may be independent. This is achieved by applying the following simple greedy algorithm :

Algorithm: Greedy Algorithm for Triangle Selection

Input: T : set of all $(r/2)$-simple triangles in a $(3r/2)$ vertex set.
Output: S : Subset of triangles with no shared hyperedges

```
S ← ∅;
while |T| > 0 do
    Pick a triangle t = {E_1, E_2, E_3} from T;
    S ← S ∪ {t};
    Remove all triangles from T containing E_1, E_2, or E_3;
end
return S
```

Observe that each iteration of the while loop removes at most $\frac{3}{2} \cdot \binom{r}{r/2} + 1$ vertices from T, and of these 1 is recovered in the output. Therefore, the output has at least

$$a \geq \frac{\frac{1}{6}\binom{3r/2}{r}\binom{r}{r/2}}{\frac{3}{2}\binom{r}{r/2}+1} \geq \frac{1}{18}\frac{\binom{3r/2}{r}\binom{r}{r/2}}{\binom{r}{r/2}} = \frac{1}{18}\binom{3r/2}{r} \geq \frac{1}{18}\cdot\left(\frac{3}{2}\right)^r. \tag{13}$$

triangles.

We now derive a lower bound on b. Let us partition each color class $\mathcal{K}_i$ into sets of $\frac{3r}{4}$ vertices. To do so, we remove $|\mathcal{K}_i| \mod \frac{3r}{4}$ vertices from each color class $\mathcal{K}_i$, that is less than $\frac{3r}{4}$ vertices per color. Allowing monochromatic triangles, we count the number of unique 2-combinations of the $\frac{3r}{4}$ sized components. With at most $k \cdot \frac{3r}{4}$ vertices removed, we have at least

$$\frac{n - k\frac{3r}{4}}{\frac{3r}{4}}$$

components with $\frac{3r}{4}$ vertices. Then;

$$b \geq \binom{\frac{n-k\frac{3r}{4}}{\frac{3r}{4}}}{2} \geq \frac{\left(\frac{n-k\frac{3r}{4}}{\frac{3r}{4}}\right)^2}{4} \geq \frac{n^2}{36r^2} \quad \text{for} \quad k \leq \frac{n}{r}. \tag{14}$$

Following from (12), and substituting lower bounds (13) and (14) for a and b, we have:

$$\begin{aligned}(1-p^3)^{ab} &\leq e^{-p^3ab} \leq \exp\left\{-p^3 \cdot \frac{n^2}{36r^2} \cdot \frac{1}{18} \cdot \left(\frac{3}{2}\right)^r\right\} \\ &\leq \exp\left\{-p^3 \cdot \frac{n^2}{r^2} \cdot \frac{3^{r-4}}{2^{r+3}}\right\} \leq \exp\{-n\ln(n)\} = o(n^{-n}) \text{ for } p \geq \left(\frac{r^2\ln(n)}{n} \cdot \frac{2^{r+3}}{3^{r-4}}\right)^{\frac{1}{3}}.\end{aligned}$$

as claimed. □

In the following lemma, we provide an upper bound for the size of a maximum independent set (α) in r-uniform random hypergraphs.

Lemma 4. *Asymptotically almost surely,* $\alpha(G_r(n,p)) \leq \left(\frac{r^r \ln n}{p}\right)^{\frac{1}{r-1}}$.

Proof. Let $G = G_r(n,p)$. For any number t,

$$\begin{aligned}\Pr(\alpha(H) \geq t) &\leq \binom{n}{t}(1-p)^{\binom{t}{r}} \leq \left(\frac{en}{t}\right)^t e^{-p\binom{t}{r}} \\ &\leq \exp\left\{t \cdot \left(1 + \ln n - \ln t - p \cdot \frac{t^{r-1}}{r^r}\right)\right\} = \exp\{t \cdot (1-\ln t)\}.\end{aligned}$$

Note that t grows without bound for constant r and increasing n.
By letting, $t = \left(\frac{r^r \ln n}{p}\right)^{\frac{1}{r-1}}$, we have $Pr(\alpha(H) \geq t) = o(1)$, for sufficiently large n. □

Proof of Theorem 3. We choose n sufficiently large with respect to d as:

$$\frac{2}{r}D \leq d \leq D, \tag{15}$$

where $D = \binom{n-1}{r-1}$. Let $p = \frac{d-s}{D}$, where $s = \sqrt{2d\ln(Dn)}$.

By Chernoff Bound in (2), we have

$$\begin{aligned}\Pr(\Delta(G(n,p)) > d) &\leq n \cdot \Pr(\text{BIN}(D,p) \geq d) \\ &= n \cdot \Pr\left(\text{BIN}\left(D, \frac{d-s}{D}\right) \geq d\right) \leq n \cdot \exp\left\{-\frac{s^2}{2(Dp+s/3)}\right\} \\ &\leq n \cdot \exp\left\{-\frac{s^2}{2d}\right\} = o(1).\end{aligned}$$

□

In (15), we made choice of d as $\frac{2}{r}D \leq d \leq D$. We now show p values derived from this range of d to be above the minimum required for the application of Lemma 3.

We require $0 \leq p \leq 1$:

$$0 \leq \frac{d - \sqrt{2d\ln(Dn)}}{D} \leq 1.$$

The right inequality holds by our assumption. The left inequality follows because $d \geq \sqrt{2d\ln(Dn)}$, or equivalently, $d \geq 2\ln(Dn)$, for sufficiently large n.
Now we consider whether p values generated by this choice of lower bound on d fall within the range needed in Lemma 3:

$$\begin{aligned} p &\geq \frac{\frac{2D}{r} - \sqrt{Dr\ln(Dn)}}{D} = \frac{2}{r} - \sqrt{\frac{r\ln(Dn)}{D}} \\ &\geq \frac{2}{r} - \sqrt{\frac{r\ln(\frac{e^{r-1}(n-1)^{r-1}}{(r-1)^{r-1}})}{D}} = \frac{2}{r} - \sqrt{\frac{r(r-1)\left(1+\ln(n-1)-\ln(r-1)\right)}{D}} \\ &\geq \frac{2}{r} - \sqrt{\frac{r(r-1)\left(n-1\right)}{D}} \geq \frac{2}{r} - \sqrt{\frac{r^{r+1}}{\left(n-1\right)^{r-2}}} \geq \left(\frac{\ln(n)\cdot r^2\cdot 2^{r+3}}{n\cdot 3^{r-4}}\right)^{\frac{1}{3}}. \end{aligned} \tag{16}$$

for $r \geq 3$, and sufficiently large n.

Let $H_{(n,p)}$ represent a hypergraph. By Lemma 4, any coloring with more than m vertices in a single color introduces a monochromatic edge. Furthermore, if we let

$$p \geq \frac{2}{r} - \sqrt{\frac{r^{r+1}}{\left(n-1\right)^{r-2}}},$$

then any coloring with at most k colors and each color containing at most m vertices inevitably creates either a bichromatic or monochromatic cycle, by (16) and Lemma 3.

Therefore, for sufficiently large n,

$$\Pr\left(\chi_s^r(H_{(n,p)}) \geq k\right) \geq \frac{n-1}{n},$$

we may say. Under the additional constraint $\Delta \leq d$, applying the result from (**??**) gives:

$$\Pr\left(\chi_s^r(H_{(n,p)}) \geq k, \Delta \leq d\right) \geq \frac{n-1}{n} - \left(\frac{r-1}{n-1}\right)^{r-1} > 0. \tag{17}$$

which ensures the existance of a hypergraph satisfying these conditions, for $r \geq 3$ and sufficiently large n. We may then use the upper bound on d established in (15) to conclude:

$$\chi_r(d) \geq k \geq \frac{n}{r} > \frac{d^{\frac{1}{r-1}}(r-1)}{r}.$$

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