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The assumption of local realism imposes constraints, such as Bell inequalities, on quantities obtained from measurements. In recent years, various tests of local realism have gained popularity in undergraduate laboratories, giving students the exciting opportunity to experimentally contradict this philosophical assumption. The standard test of the CHSH (Clauser–Horne–Shimony–Holt) Bell inequality requires 16 measurements, whereas a test of Freedman's inequality requires only three measurements. The calculations required to test Freedman's inequality are correspondingly simpler and the theory is less abstract. We suggest that students may benefit from testing Freedman's inequality before proceeding to the CHSH inequality and other more complicated experiments. Our measured data violated Freedman's inequality by more than six standard deviations.
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I. INTRODUCTION

Correlated-photon experiments have generated tremendous excitement in the advanced laboratory community.[1–11] In fact, a major function of the Advanced Laboratory Physics Association is to offer deeply discounted single-photon detectors for correlated-photon experiments.[12] Through these experiments, quantum mechanics can be taught as a laboratory course,[13] and even Bell's theorem can be tested in the undergraduate laboratory.[1]

Quantum entanglement fascinates the general public as well as physicists. Unfortunately, this popular intrigue makes quantum entanglement a favorite of purveyors of pseudoscience.[14] If students have laboratory experience in the rigorous experimental foundations of quantum mechanics, they will be able to satisfy their own curiosity. Perhaps more importantly, students also will be able to answer questions raised by the public, thereby combatting misinformation.

The standard version of Bell's inequality for optical experiments is the Clauser–Horne–Shimony–Holt (CHSH) inequality.[1,10,15] A test of the CHSH inequality requires measurements at 16 combinations of polarizer angles, and calculation of an abstract quantity often called S.[1] This is an excellent laboratory exercise in the advanced undergraduate laboratory, partly due to the very complexity of the calculations and the abstraction of the theory. However, we believe students may find the whole topic more comprehensible if they test a simpler Bell inequality first.

In 1972, the first Bell inequality ever tested[16] was the simple version derived in Stuart Freedman's Ph.D. thesis.[17] Perhaps Freedman's inequality is not common knowledge in the advanced laboratory community; we were surprised when we came across it. Yet, Alain Aspect's first paper on the topic in 1981 also used the Freedman inequality.[18] Because Freedman's inequality requires only three measurements, and the corresponding derivation and calculations are simpler and less abstract, it is an excellent introduction to Bell inequalities. Just as we teach the infinite square well before the finite square well, and we teach Newtonian mechanics before Lagrangian mechanics, the Freedman inequality logically precedes the CHSH inequality in a pedagogical sequence.

The Freedman inequality is not the only comparatively simple alternative to the CHSH inequality. However, the relatively simple version of Hardy's test requires four single-photon detectors,[5] as does the measurement of entanglement witnesses.[11] The Freedman inequality, like the CHSH inequality, requires only two single-photon detectors. As the detectors are the most expensive components of the apparatus, some departments (like ours) may choose to purchase only two detectors.

Many resources are available to help spread correlated-photon experiments to more colleges and universities. Besides the many publications over the past 16 years, detailed instructions are available online.[19] Additionally, every summer there are two-and-a-half-day immersions in which instructors learn how to set up and teach these experiments "with confidence."[12] (One of us can say from experience that an immersion can even give you the confidence to eventually write a paper about a topic you previously knew nothing about.) Even more amazing, participants in the immersions can apply to the Jonathan F. Reichert Foundation for partial funding of the equipment.[20] The opportunity to learn experimental quantum mechanics is expanding to more students, and the repertoire of instructional experiments likewise continues to grow. A test of the Freedman inequality merits inclusion among pedagogically valuable experiments due to its relative simplicity as well as its historical significance.

II. THEORY: FREEDMAN'S INEQUALITY

Freedman's inequality is a special case of the Clauser-Horne inequality.[21] Clauser and Horne derive the inequality clearly and completely, but we reproduce it for the reader's convenience.

We will need an algebraic lemma whose derivation is straightforward and easily accessible to undergraduate students. If real numbers x_1, x_2, y_1, y_2, X, and Y satisfy

$$0 \le x_1 < X, \tag{1a}$$

$$0 \le x_2 \le X, \tag{1b}$$

$$0 \le y_1 \le Y, \tag{1c}$$

$$0 \le y_2 \le Y, \tag{1d}$$

and

$$U \equiv x_1 y_1 - x_1 y_2 + x_2 y_1 + x_2 y_2 - Y x_2 - X y_1, \tag{2}$$

then it can be shown that

$$-XY \le U \le 0. \tag{3}$$

To demonstrate the upper limit on U, we look at two cases. When $x_1 \ge x_2$, we factor

$$U = (x_1 - X)y_1 + (y_1 - Y)x_2 + (x_2 - x_1)y_2, \tag{4}$$

where all the terms in parentheses are nonpositive due to Eq. (1) and $x_1 \ge x_2$, so that $U \le 0$. In the case $x_1 < x_2$, we factor

$$\begin{aligned} U &= x_1(y_1 - y_2) + (x_2 - X)y_1 - x_2(Y - y_2) \\ &\le x_1(y_1 - y_2) + (x_2 - X)y_1 - x_1(Y - y_2). \end{aligned} \tag{5}$$

The inequality in Eq. (5) is valid because x_1 replaces x_2 in the final term on the right side; a smaller positive term or 0 is subtracted on the right side. The right side simplifies to $(x_2–X)y_1 + x_1(y_1–Y)$, which must be nonpositive via Eq. (1). So $U \le 0$ in all cases.

To demonstrate the lower limit on U, we factor $U+XY$ three ways

$$U + XY = (X - x_2)(Y - y_1) + x_1 y_1 + (x_2 - x_1)y_2, \tag{6a}$$

$$U + XY = (X - x_2)(Y - y_1) + x_2 y_2 + x_1(y_1 - y_2), \tag{6b}$$

$$U + XY = (X - x_2)(Y - y_1) + x_2 y_1 + (x_2 - x_1)(y_2 - y_1). \tag{6c}$$

We examine three cases. If $x_2 \ge x_1$, all the terms on the right side of Eq. (6a) are nonnegative, so $U + XY \ge 0$. If $y_1 \ge y_2$, all the terms on the right side of Eq. (6b) are nonnegative, so again $U + XY \ge 0$. Finally, if neither previous case is true so that $x_1 < x_2$ and $y_1 < y_2$, all the terms on the right side of Eq. (6c) are nonnegative. Since $U + XY \ge 0$ in all cases, the proof of Eq. (3) is now complete.

Now we proceed to the physics. We consider a source of photon pairs. The two photons travel in different directions, so two analyzers are used. Following Clauser and Horne, we refer to the combination of a polarizer and detector as an "analyzer." Let N_{tot} be the total number of photon pairs arriving at the analyzers in a certain time interval. $N(a,b)$ represents the number of measured coincidences (simultaneous detection of two photons) within the same time interval, where a and b represent the angles of the two polarizers, which we will label A and B.

For a sufficiently long time interval, the fraction of photon pairs detected (as coincidences) is the detection probability

$$P(a,b) = \frac{N(a,b)}{N_{tot}}. \tag{7}$$

We assume, for now, that each photon pair has a probability $p_{12}(\lambda,a,b)$ of detection, where λ is called a hidden variable (which may represent multiple variables), and the subscript refers to the two photons, labeled 1 and 2 (not "twelve"). λ and therefore the probability of detection may vary from one photon pair to another; Eq. (7) gives the average probability of detection over an ensemble of photon pairs. λ predetermines whether a particular photon pair is likely to be detected. We could consider the special case in which $p_{12}(\lambda,a,b)$ is always 0 or 1, such that each photon pair's fate (coincidence detection or nondetection) is predetermined with complete certainty: the measurable quantity has a definite value regardless of whether anyone knows what it is. This assumption is called *realism*.[1] Freedman's inequality actually applies to a much broader class of hidden-variables theories in which $p_{12}(\lambda,a,b)$ may have intermediate values.

$\rho(\lambda)$ is the probability distribution of λ, so that

$$P(a,b) = \int \rho(\lambda) p_{12}(\lambda,a,b)\, d\lambda. \tag{8}$$

We specify that photon 1 travels to polarizer A, and photon 2 travels to polarizer B. If we assume that photon 1 is unaffected by polarizer B, and photon 2 is unaffected by polarizer A, then $p_{12}(\lambda,a,b)$ may be factored as

$$p_{12}(\lambda,a,b) = p_1(\lambda,a)\, p_2(\lambda,b). \tag{9}$$

Equation (9) assumes *locality*, which means, in this case, that the measurement of one photon is unaffected by the other photon's polarizer. Equation (9) relies also on the hidden-variables assumption. In fact, the dependence on a common variable λ is what allows the outcomes of the two measurements to be written as independent events.[22] λ characterizes both photons. In a local hidden-variables theory, λ is the (only) explanation for any correlations in the measurements of the photon pairs.

Let $p_1(\lambda,\infty)$ represent the probability of detecting photon 1 when the polarizer in front of it is removed. There is no known mechanism through which the presence of the polarizer can increase the number of detected photons. (Clauser and Horne call this the no-enhancement assumption.) Therefore,

$$0 \le p_1(\lambda,a) \le p_1(\lambda,\infty). \tag{10a}$$

Similarly, for any other angle a' of polarizer A,

$$0 \le p_1(\lambda,a') \le p_1(\lambda,\infty). \tag{10b}$$

The same inequalities apply to polarizer B

$$0 \le p_2(\lambda,b) \le p_2(\lambda,\infty), \tag{10c}$$

$$0 \le p_2(\lambda,b') \le p_2(\lambda,\infty). \tag{10d}$$

Applying the lemma of Eqs. (1)–(3)

$$\begin{aligned} -p_1(\lambda,\infty)p_2(\lambda,\infty) \le\; & p_1(\lambda,a)p_2(\lambda,b) - p_1(\lambda,a)p_2(\lambda,b') \\ & + p_1(\lambda,a')p_2(\lambda,b) \\ & + p_1(\lambda,a')p_2(\lambda,b') \\ & - p_1(\lambda,a')p_2(\lambda,\infty) \\ & - p_1(\lambda,\infty)p_2(\lambda,b) \le 0. \end{aligned} \tag{11}$$

When Eq. (11) is multiplied by $\rho(\lambda)d\lambda$ and integrated, and each term is simplified through Eq. (8), then

$$\begin{aligned} -P(\infty,\infty) \le\; & P(a,b) - P(a,b') + P(a',b) \\ & + P(a',b') - P(a',\infty) - P(\infty,b) \le 0. \end{aligned} \tag{12}$$

We next assume rotational invariance in the photon pairs so that the measured coincidence count depends only on the angle $\phi = |a\text{-}b|$ between the polarizers: $P(a,b) = P(\phi)$. Then choosing a, a', b, and b' to satisfy

$$|a - b| = |a' - b| = |a' - b'| = |a - b'|/3 = \phi, \qquad (13)$$

Eq. (12) simplifies to

$$-P(\infty,\infty) \le 3P(\phi) - P(3\phi) - P(a',\infty) - P(\infty,b) \le 0. \qquad (14)$$

Freedman derived an ingenious simplification of Eq. (14).[17] Recognizing that nothing changes when a polarizer is rotated 180°, $P(\phi) = P(\phi+180°)$. Using $P(202.5°) = P(22.5°)$ in Eq. (14) with $\phi = 67.5°$,

$$-P(\infty,\infty) \le 3P(67.5°) - P(22.5°) - P(a',\infty) - P(\infty,b) \le 0. \qquad (15)$$

With $\phi = 22.5°$, Eq. (14) immediately becomes

$$-P(\infty,\infty) \le 3P(22.5°) - P(67.5°) - P(a',\infty) - P(\infty,b) \le 0. \qquad (16)$$

Multiplying Eq. (15) by -1 and adding it to Eq. (16) yields

$$-P(\infty,\infty) \le 4P(22.5°) - 4P(67.5°) \le P(\infty,\infty). \qquad (17)$$

Equation (17) is expressed in terms of measurable quantities when multiplied by N_{tot} and combined with Eq. (7), rewriting $N(a,b)$ as $N(\phi)$

$$-N_0 \le 4N(22.5°) - 4N(67.5°) \le N_0, \qquad (18)$$

where $N_0 \equiv N(\infty,\infty)$ is the number of measured coincidences when both polarizers are removed, and $N(\phi)$ is the number of coincidences when the angle between polarizers is ϕ. Equation (18) simplifies to

$$\delta = \left| \frac{N(22.5°) - (67.5°)}{N_0} \right| - \frac{1}{4} \le 0, \qquad (19)$$

where δ is a quantity easily computed from measured data. This is Freedman's inequality.

Although the algebraic lemma is a bit laborious, students can be led to understand every step in the derivation of Eq. (19). It is critical that they understand every assumption so that they can appreciate the significance of a violation of the inequality. While the derivation of the CHSH inequality is comparable in complexity, the quantity in the CHSH inequality depends on 16 measurements and cannot be conveniently expressed in a single equation. In contrast, the quantity δ in Eq. (19) is written simply and is therefore less of an abstraction.

If valid assumptions are made in the derivation of Freedman's inequality, δ must be nonpositive. If measurement contradicts this requirement, then one or more of the assumptions must be incorrect. It is easy to empirically test the no-enhancement assumption. Similarly, it is easy to test the assumption of rotational invariance (coincidence counts depending only on ϕ, not on absolute angles a or b). Instructors and students who do not wish to assume rotational invariance may return to Eq. (12) and set $a = 0°$, $b = 22.5°$, $a' = 45°$, and $b' = 67.5°$. The resulting inequality requires seven measurements, which is still less than the 16 required for the CHSH inequality, and the derivation is still simpler.

The two remaining assumptions are locality, and an assumption that includes realism as a special case. One or both of these must be wrong if experiment shows that δ is positive. Realism is a special case of the assumption that a hidden variable λ determines the probability $p_{12}(\lambda,a,b)$ of a coincidence. If $\lambda = 0$ for all photon pairs, and $p_{12}(\lambda,a,b) = (\lambda +$ the quantum mechanical prediction), then quantum mechanics is a special case of the same assumption of which realism is also a special case. A violation of the Freedman inequality cannot contradict realism without also contradicting quantum mechanics. On this basis, we can understand David Griffiths's statement, "It is a curious twist of fate that the EPR paradox, which *assumed* locality to *prove* realism, led finally to the repudiation of locality and left the issue of realism undecided."[23]

A violation of Freedman's inequality thus must contradict Eq. (9), which expresses locality in the context of a hidden-variables theory. The most obvious conclusion is that nonlocality is a fact of nature, and the two entangled photons maintain some kind of connection. However, there are many interpretations of quantum mechanics, and not all are nonlocal.[22] Comprehensive discussions of philosophical interpretations fill entire books.[24]

III. THEORY: QUANTUM MECHANICS

The ideal wavefunction of the photon pair is[1]

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1|H\rangle_2 + |V\rangle_1|V\rangle_2\right). \qquad (20)$$

Assuming ideal polarizers with 100% transmittance for light polarized in the direction of the polarizer axis, the probability of detecting a coincidence is

$$P_{ideal}(a,b) = |\langle a|_1 \langle b|_2 |\Psi\rangle|^2, \qquad (21)$$

where

$$|a\rangle_1 = |H\rangle_1 \sin a + |V\rangle_1 \cos a \qquad (22a)$$

and

$$|b\rangle_2 = |H\rangle_2 \sin b + |V\rangle_2 \cos b, \qquad (22b)$$

with angles measured from the vertical. Equation (21) then reduces to $P_{ideal}(a,b) = \frac{1}{2}(\sin a \ \sin b + \cos a \ \cos b)^2 = \frac{1}{2}\cos^2(a\text{-}b) = \frac{1}{2}\cos^2\phi = P_{ideal}(\phi)$. If ε_1 and ε_2 are the transmittances of the polarizers for light polarized in the direction of the polarizer axes, then the quantum prediction is reduced by a factor of $\varepsilon_1\varepsilon_2$[16]

$$\frac{N(\phi)}{N_0} = P_{actual}(\phi) = \frac{1}{2}\varepsilon_1\varepsilon_2\cos^2\phi, \qquad (23)$$

assuming negligible transmission of light polarized perpendicular to the polarizer axis.

We can carefully distinguish among P in Eq. (7), P_{ideal} in Eq. (21), and P_{actual} in Eq. (23). P is the ratio of measured coincidences to the total number of photon pairs arriving at the analyzers (which is not directly measured because detector efficiencies are not ideal). P_{ideal} is the ratio of coincidences that would be measured if the polarizers and detectors were both ideal, to the total number of photon pairs arriving at the analyzers. P_{actual} is the ratio of measured coincidences (when polarizers are present) to the measured coincidences when the polarizers are removed.

Using Eq. (23), we can make a quantum mechanical prediction of δ: $\delta = |P(22.5°) - P(67.5°)| - 1/4 = (\varepsilon_1)(\varepsilon_2)/2\sqrt{2} - (1/4)$. A violation of the Freedman inequality requires the geometric mean of the transmittances to be at least 0.84. This requirement is not imposed on the CHSH inequality.

IV. EXPERIMENT

Our apparatus, shown in Fig. 1, is based on Galvez,[19] who gives many additional details. Our light source is a 405 nm laser (advertised as 20 mW, though we measure 5 mW). A half-wave plate is used to adjust the polarization to 45° from the vertical. The light encounters a pair of beta barium borate (BBO) crystals, in which some of the 405 nm photons split into two 810 nm photons, in a process called spontaneous parametric downconversion. One of the two crystals produces horizontally polarized infrared photons, and the other produces vertically polarized infrared photons. A compensating quartz crystal is placed before the crystal pair to eliminate a phase shift between the $|H\rangle_1|H\rangle_2$ and $|V\rangle_1|V\rangle_2$ components.

The photons in a pair travel along opposite lines on a cone with a 3° half angle. We place two collimators to intercept infrared photons at opposite positions on the cone in the horizontal plane. A polarizer and bandpass filter are placed in front of each collimator, and a fiber optic cable is connected behind it, leading to a single-photon detector. The output voltages of the single-photon detectors are connected to a counting circuit[25] on a FPGA (field programmable gate array) board, which communicates the results to a computer.

Over 50 s intervals, we measured the following coincidences: $N(\phi)$ for ϕ from 0° to 90° at multiples of 11.25°; N_0 (coincidences with both polarizers removed); and coincidences N_1 and N_2 with one polarizer removed at a time. We can calculate transmittances as $\varepsilon_1 = 2N_1/N_0$ and $\varepsilon_2 = 2N_2/N_0$,[16] assuming negligible transmission of light polarized perpendicular to the polarizer axis. This assumption is justified by the advertised transmission of 0.05% for the perpendicular polarization.[26]

Uncertainties are of critical importance because a positive δ is inconclusive if the uncertainty in δ is larger than δ itself. Assuming Poisson statistics, the uncertainty in each coincidence count is the square root of the count itself.[1] The uncertainty in δ, σ_δ, is calculated as

$$\sigma_\delta^2 = \left[\sigma_{N(22.5°)} \frac{\partial \delta}{\partial N(22.5°)}\right]^2 + \left[\sigma_{N(67.5°)} \frac{\partial \delta}{\partial N(67.5°)}\right]^2 + \left[\sigma_{N_0} \frac{\partial \delta}{\partial N_0}\right]^2 = \frac{N(22.5°) + N(67.5°)}{N_0^2} + \frac{[N(22.5°) - N(67.5°)]^2}{N_0^3}. \tag{24}$$

We obtained $N_0 = 4474$, $N(22.5°) = 1821$, and $N(67.5°) = 377$. We then calculate $\delta = 0.073 \pm 0.011$, which violates the Freedman inequality by more than six standard deviations.

From our measurements of $N_0 = 4474$, $N_1 = 2183$, and $N_2 = 2154$, we calculate $\varepsilon_1 = 0.98 \pm 0.03$ and $\varepsilon_2 = 0.96 \pm 0.03$. (Curiously, the data sheet for the polarizers shows a transmittance of only 0.84.[26] Perhaps this is a worst case result.) Using Eq. (23), the quantum mechanical prediction for δ is 0.083 ± 0.015, where the uncertainty is due to the uncertainty in the transmittances. The theoretical result agrees well with our measured result of $\delta = 0.073 \pm 0.011$.

Measured data are compared with quantum theory in Fig. 2. The trend is correct, but the measured results are consistently higher than theory. We suspect that this is caused by accidental coincidences, partly due to ambient light. We note that accidental coincidences only make it harder to violate Freedman's equality: δ contains the difference, $N(22.5°) - N(67.5°)$, which is mostly unaffected by accidental coincidences since a similar number of accidental coincidences is added to each term. The other measured term in δ is N_0, in the denominator, so that the addition of accidental coincidences reduces δ.

V. CONCLUSIONS

Correlated photon experiments have been strongly promoted by the Advanced Lab Physics Association. A test of Freedman's inequality may usefully supplement or precede a test of the CHSH inequality and other more advanced experiments. Freedman's inequality requires only three measurements, and two of them are part of the CHSH test; adding a simple measurement of N_0 requires less than a minute of extra time. Thus, a test of Freedman's inequality can be easily combined with a test of the CHSH inequality. Alternatively, Freedman's inequality could be a valuable stand-alone experiment in which students

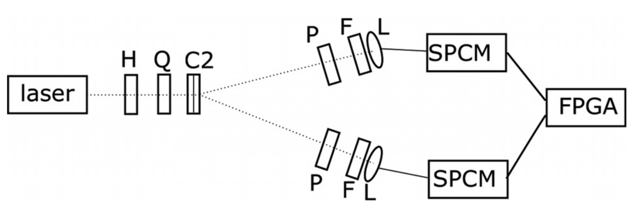

Fig. 1. Schematic diagram of the apparatus (not to scale). The beam from a 405 nm laser travels through a half-wave plate (H), a compensating quartz crystal (Q), and a pair of BBO crystals (C2). Entangled pairs of 810 nm photons go through polarizers (P), infrared bandpass filters (F), collimating lenses (L), and fiber optic cables to single-photon counting modules (SPCM). The SPCM outputs are connected to an FPGA counting circuit. Many additional details about similar apparatuses are in Refs. 2 and 19.

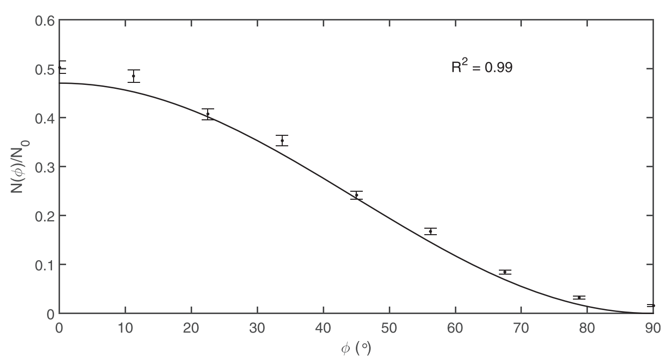

Fig. 2. Measured data (circles) and quantum theory (solid line). There are no fitting parameters.

can exhaustively master every step of the derivations and calculations. Furthermore, because a test of Freedman's inequality requires only two single-photon detectors, it may be more financially feasible for some departments than alternative tests that require more detectors.

Despite these advantages, Freedman's inequality is subject to limitations, which is why the CHSH inequality is preferred in research. The CHSH inequality is derived without the no-enhancement assumption or the assumption of rotational invariance. Thus, the CHSH inequality allows for a stronger test of local realism. Freedman's inequality is based partially on the properties of polarizers (no enhancement, and invariance under 180° rotation), and on the properties of the entangled photons used (rotational invariance). Thus, a violation of Freedman's inequality does not rule out local realism under other experimental conditions. Although Freedman's inequality is a weaker test of local realism, we feel that its simpler derivation makes it a valuable pedagogical experiment. In fact, a careful analysis of its limitations can lead to a discussion of the subtler "loopholes" that have only recently been closed.[27–29]

Many aspects of physics attract students. One such aspect is the investigation of the ultimate nature of reality. Within physics, tests of Bell inequalities may be the deepest investigations of reality's ultimate nature.

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