

# Multiple basic hypergeometric transformation formulas arising from the balanced duality transformation

Yasushi KAJIHARA

## Abstract

Some multiple hypergeometric transformation formulas arising from the balanced duality transformation formula are discussed through the symmetry. Derivations of some transformation formulas with different dimensions are given by taking certain limits of the balanced duality transformation. By combining some of them, some transformation formulas for  $A_n$  basic hypergeometric series is given. They include some generalizations of Watson, Sears and  ${}_8W_7$  transformations.

Keywords: basic hypergeometric series, multivariate basic hypergeometric series

## 1 Introduction

This paper can be considered as a continuation of our paper [13]. Namely, we discuss some multiple hypergeometric transformation formulas arising from the balanced duality transformation formula through the symmetry in this paper. We obtain some transformation formulas with different dimensions by taking certain limits of the balanced duality transformation. By combining some of them, we give some transformation formulas for  $A_n$  basic hypergeometric series. They include some generalizations of Watson, Sears and nonterminating and terminating  ${}_8W_7$  transformations.

The hypergeometric series  ${}_{r+1}F_r$  is defined by

$${}_{r+1}F_r \left[ \begin{matrix} a_0, & a_1, & a_2, & \cdots, & a_r \\ & b_1, & b_2, & \cdots, & b_r \end{matrix} ; z \right] := \sum_{k \in \mathbb{N}} \frac{[a_0, a_1, \dots, a_r]_k}{k! [b_1, \dots, b_r]_k} z^k, \quad (1.1)$$

where  $[c]_k = c(c+1) \cdots (c+k-1)$  is Pochhammer symbol and  $[d_1, \dots, d_r]_k = [d_1]_k \cdots [d_r]_k$ .

The very well-poised hypergeometric series have nice properties such as the existence of various kinds of summation and transformation formulas which contains more parameters than other hypergeometric series and they have reciprocal structure (For precise see an excellent exposition by G.E. Andrews [1]). In [2], W.N.Bailey derived the following transformation formula for terminating balanced and very well-poised  ${}_9F_8$  series which is nowadays called the Bailey transformation formula:

$$\begin{aligned} & {}_9F_8 \left[ \begin{matrix} a, & a/2+1, & b, & c, & d, \\ & a/2, & 1+a-b, & 1+a-c, & 1+a-d, \end{matrix} \right. \\ & \quad \left. \begin{matrix} e, & f, & g, & -N \\ & 1+a-e, & 1+a-f, & 1+a-g, & 1+a+N \end{matrix} ; 1 \right] \\ &= \frac{[1+a, 1+\lambda-e, 1+\lambda-f, 1+\lambda-g]_N}{[1+\lambda, 1+a-e, 1+a-f, 1+a-g]_N} \\ &\times {}_9F_8 \left[ \begin{matrix} \lambda, & \lambda/2+1, & \lambda+b-a, & \lambda+c-a, & \lambda+d-a, \\ & \lambda/2, & 1+a-b, & 1+a-c, & 1+a-d, \end{matrix} \right] \end{aligned} \quad (1.2)$$

$$\left[ \begin{array}{cccc} e, & f, & g, & -N \\ 1 + \lambda - e, & 1 + \lambda - f, & 1 + \lambda - g, & 1 + \lambda + N \end{array} ; 1 \right],$$

where  $\lambda = 1 + 2a - b - c - d$ , and the parameters are subject to the restriction (called as balancing condition) that

$$2 + 3a = b + c + d + e + f + g - N. \quad (1.3)$$

Among various hypergeometric transformation formulas, the Bailey transformation itself (1.2) and its special and limiting cases, including their basic and elliptic analogues, have many significant applications in various branches of mathematics and mathematical physics.

On the other hand, Holman, Biedenharn and Louck [9], [10] introduced a class of multiple generalization of hypergeometric series in need of the explicit expressions of the Clebsch-Gordan coefficients of the tensor product of certain irreducible representations of the Lie group  $SU(n+1)$ . In the series of papers, S. Milne [19] introduced its basic analogue and investigated further. In the course of works (see the expository paper by Milne [22] and references therein), he and his collaborators succeeded to obtain some multiple hypergeometric transformation and summation formulas by using a certain rational function identity which is nowadays referred as Milne's fundamental lemma. After that, many methods of the derivation of multiple hypergeometric identities have been worked out such as ingenious uses of certain matrix techniques by Krattenthaler, Schlosser, Milne, Lilly and Newcomb, see [4], [21], [23], [24].

In the previous work [13], we derived a certain generalization of the Euler transformation formula for multiple (basic) hypergeometric series with different dimensions by using the techniques in the theory of Macdonald polynomials and Macdonald's  $q$ -difference operators. By interpreting our multiple Euler transformation formula as the generating series, we further obtained several types of multiple hypergeometric summations and transformations.

Among these, we consider the (*balanced*) *duality transformation formula* (2.12) which generalize the following  ${}_9F_8$  transformation (see Bailey's book [3]):

$$\begin{aligned} & {}_9F_8 \left[ \begin{array}{cccccc} a, & a/2 + 1, & b, & c, & d, & \\ & a/2, & 1 + a - b, & 1 + a - c, & 1 + a - d, & \end{array} \right. \\ & \quad \left. \begin{array}{cccccc} e, & f, & g, & -N \\ 1 + a - e, & 1 + a - f, & 1 + a - g, & 1 + a + N \end{array} ; 1 \right] \\ &= \frac{[1 + a, 1 + a - b - c, 1 + a - b - d, 1 + a - b - e, 1 + a - b - f, g]_N}{[1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a - f, g - b]_N} \\ &\times {}_9F_8 \left[ \begin{array}{cccccc} b - g - N, & (b - g - N)/2 + 1, & b, & 1 + a - c - g, & 1 + a - d - g, & \\ & (b - g - N)/2, & 1 - g - N, & b + c - a - N, & b + d - a - N, & \\ & & 1 + a - e - g, & 1 + a - f - g, & b - a - N, & -N \\ & & b + e - a - N, & b + f - a - N, & 1 + a - g, & 1 + b - g \end{array} ; 1 \right], \end{aligned} \quad (1.4)$$

with the balancing condition (1.3), as the formula of particular importance. (1.4) is different from (1.2), but (1.4) can be obtained by duplicating (1.2). In the joint work with M. Noumi [15], we derived an analogous formula ((3.17) of [15]) for elliptic hypergeometric series introduced by Frenkel and Turaev [5] by starting from the Frobenius determinant and proposed the notion of (*balanced*) *duality transformation* there.

In this paper, we present an alternative approach by starting from the balanced duality transformation formula for multiple hypergeometric series of type  $A$ . In Section 3, we present some transformation formula for basic hypergeometric series of type  $A$  with different dimension. They include most of results in our previous work [13]. What is remarkable is that from the balanced duality transformation formula, one can obtain multiple Euler transformation formula itself.

By iterating twice a special case of our Sears transformation formula (see section 7 in [13]), we verified an  $A_n$  Sears transformation formula in [12]. Later by the same idea as above, we obtained in [15] two types of  $A_n$  Bailey transformation formulas: one of which is previously

known by Milne and Newcomb [24] in basic case and Rosengren [27] in elliptic case and another has appeared to be new in both cases. In Section 4, we further employ this idea to obtain several  $A_n$  basic hypergeometric transformations which generalize Watson, Sears and nonterminating  ${}_8W_7$  transformations. They includes known ones due to Milne and his collaborators. We will see here that our class of multiple hypergeometric transformations shed a light to the structure of some  $A_n$  hypergeometric transformation formulas.

## 2 Preliminaries

In this section, we give some notations for multiple basic hypergeometric series and present the balanced duality transformation formula. We basically refer the notations of  $q$ -series and basic hypergeometric series from the book by Gasper and Rahman [7]. Throughout of this paper, we assume that  $q$  is a complex number under the condition  $0 < |q| < 1$ . We define  $q$ -shifted factorial as

$$(a)_\infty := (a; q)_\infty = \prod_{n \in \mathbb{N}} (1 - aq^n), \quad (a)_k := (a; q)_k = \frac{(a)_\infty}{(aq^k)_\infty} \quad \text{for } k \in \mathbb{C}. \quad (2.1)$$

where, unless stated otherwise, we omit the basis  $q$ . In this paper we employ the notation as

$$(a_1)_k \cdot (a_2)_k \cdots (a_n)_k = (a_1, a_2, \dots, a_n)_k. \quad (2.2)$$

The basic hypergeometric series  ${}_{r+1}\phi_r$  is defined by

$$\begin{aligned} {}_{r+1}\phi_r \left[ \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ c_1, \dots, c_r \end{matrix}; q; u \right] &= {}_{r+1}\phi_r \left[ \begin{matrix} a_0, \{a_i\}_r \\ \{c_i\}_r \end{matrix}; q; u \right] \\ &:= \sum_{k \in \mathbb{N}} \frac{(a_0, a_1, \dots, a_{r+1})_k}{(c_1, \dots, c_r, q)_k} u^k \end{aligned} \quad (2.3)$$

with  $r+1$  numerator parameters  $a_0, a_1, \dots, a_{r+1}$  and  $r$  denominator parameters  $c_1, \dots, c_r$ . We call  ${}_{r+1}\phi_r$  series  $k$ -balanced if  $q^k a_1 a_2 \cdots a_{r+1} = c_1 \cdots c_r$  and  $u = q$ : a 1-balanced series is called balanced (or Saalschützian). An  ${}_{r+1}\phi_r$  series is called well-poised if  $a_0 q = a_1 c_1 = \cdots = a_r c_r$ . In addition, if  $a_1 = q\sqrt{a_0}$  and  $a_2 = -q\sqrt{a_0}$ , then the  ${}_{r+1}\phi_r$  is called very well-poised. We denote the very well-poised basic hypergeometric series  ${}_{r+3}\phi_{r+2}$  as  ${}_{r+3}W_{r+2}$  series defined by the following:

$$\begin{aligned} {}_{r+3}W_{r+2} [s; \{a_i\}_r; q; u] &:= {}_{r+3}\phi_{r+2} \left[ \begin{matrix} s, & q\sqrt{s}, & -q\sqrt{s}, & \{a_i\}_r \\ \sqrt{s}, & -\sqrt{s}, & \{sq/a_i\}_r \end{matrix}; q; u \right] \\ &= \sum_{k \in \mathbb{N}} \frac{1 - sq^{2k}}{1 - s} \frac{(s, a_1 \cdots a_r)_k}{(q, sq/a_1, \dots, sq/a_r)_k} u^k. \end{aligned} \quad (2.4)$$

Furthermore, all of the very well-poised basic hypergeometric series  ${}_{r+3}W_{r+2}$  in this paper, the parameters  $s, a_1, \dots, a_r$  and the argument satisfy the very-well-poised-balancing condition

$$a_1 \cdots a_r u = \left( \pm (sq)^{\frac{1}{2}} \right)^{r-1} \quad (2.5)$$

with either the plus and minus sign. We call a  ${}_{r+3}W_{r+2}$  series very-well-poised-balanced if (2.5) holds. Note that a very-well-poised-balanced  ${}_{r+3}W_{r+2}$  series is (1-)balanced if

$$a_1 \cdots a_r = s^{\frac{r-1}{2}} q^{\frac{r-3}{2}} \quad (2.6)$$

and  $u = q$ .

Now, we note the conventions for naming series as  $A_n$  basic hypergeometric series (or basic hypergeometric series in  $SU(n+1)$ ). Let  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$  be a multi-index. We denote

$$\Delta(x) := \prod_{1 \leq i < j \leq n} (x_i - x_j) \quad \text{and} \quad \Delta(xq^\gamma) := \prod_{1 \leq i < j \leq n} (x_i q^{\gamma_i} - x_j q^{\gamma_j}), \quad (2.7)$$

as the Vandermonde determinant for the sets of variables  $x = (x_1, \dots, x_n)$  and  $xq^\gamma = (x_1 q^{\gamma_1}, \dots, x_n q^{\gamma_n})$  respectively. In this paper we refer multiple series of the form

$$\sum_{\gamma \in \mathbb{N}^n} \frac{\Delta(xq^\gamma)}{\Delta(x)} H(\gamma) \quad (2.8)$$

which reduce to basic hypergeometric series  ${}_r+1\phi_r$  for a nonnegative integer  $r$  when  $n = 1$  and symmetric with respect to the subscript  $1 \leq i \leq n$  as  $A_n$  basic hypergeometric series. We call such a series balanced if it reduces to a balanced series when  $n = 1$ . Very well-poised and so on are defined similarly. The subscript  $n$  in the label  $A_n$  attached to the series is the dimension of the multiple series (2.8).

In our previous work [13], we derived a hypergeometric transformation formula for multiple basic hypergeometric series of type  $A$  generalizing the following transformation for terminating balanced  ${}_{10}W_9$  series in the one dimensional case:

$$\begin{aligned} {}_{10}W_9 [a; b, c, d, e, f, \mu f q^N, q^{-N}; q; q] &= \frac{(\mu b f / a, \mu c f / a, \mu d f / a, \mu e f / a, a q, f)_N}{(a q / b, a q / c, a q / d, a q / e, \mu q, \mu f / a)_N} \\ &\quad \times {}_{10}W_9 [\mu; a q / b f, a q / c f, a q / d f, a q / e f, \mu f / a, \mu f q^N, q^{-N}; q; q], \end{aligned} \quad (2.9)$$

where  $\mu = a^3 q^2 / b c d e f^2$ . Note that (2.9) is a basic analogue of (1.4). The transformation (2.9) can be obtained by iterating the Bailey transformation formula for  ${}_{10}W_9$  series

$$\begin{aligned} {}_{10}W_9 [a; b, c, d, e, f, \lambda a q^{N+1} / e f, q^{-N}; q; q] &= \frac{(a q, a q / e f, \lambda q / e, \lambda q / f)_N}{(a q / e, a q / f, \lambda q, \lambda q / e f)_N} \\ &\quad \times {}_{10}W_9 [\lambda; \lambda b / a, \lambda c / a, \lambda d / a, e, f, \lambda a q^{N+1} / e f, q^{-N}; q; q] \quad (\lambda = a^2 q / b c d), \end{aligned} \quad (2.10)$$

twice. Note also that (2.10) is a basic analogue of (1.2).

To simplify the expressions for multiple very well-poised series, we introduce the notation of  $W^{n,m}$  series as the following:

$$\begin{aligned} W^{n,m} \left( \begin{matrix} \{a_i\}_n \\ \{x_i\}_n \end{matrix} \middle| s; \{u_k\}_m; \{v_k\}_m; q; z \right) & \\ := \sum_{\gamma \in \mathbb{N}^n} z^{|\gamma|} \prod_{1 \leq i < j \leq n} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - s q^{|\gamma| + \gamma_i} x_i}{1 - s x_i} & \\ \times \prod_{1 \leq j \leq n} \frac{(s x_j)_{|\gamma|}}{((s q / a_j) x_j)_{|\gamma|}} \left( \prod_{1 \leq i \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \right) & \\ \times \prod_{1 \leq k \leq m} \frac{(v_k)_{|\gamma|}}{(s q / u_k)_{|\gamma|}} \left( \prod_{1 \leq i \leq n} \frac{(u_k x_i)_{\gamma_i}}{((s q / v_k) x_i)_{\gamma_i}} \right), & \end{aligned} \quad (2.11)$$

where  $|\gamma| = \gamma_1 + \dots + \gamma_n$  is the length of a multi-index  $\gamma$ .

The very starting point of all the discussions of the present paper is the *balanced duality transformation formula* (Corollary 6.3 of [13] with a different notation) between the  $W^{n,m+2}$  series ( $A_{n+2m+8} W_{2m+7}$  series) and  $W^{m,n+2}$  series ( $A_{m+2n+8} W_{2n+7}$  series):

$$W^{n,m+2} \left( \begin{matrix} \{b_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; \{c_k y_k\}_m, d, e; \{f y_k^{-1}\}_m, \mu f q^N, q^{-N}; q; q \right) \quad (2.12)$$

$$\begin{aligned}
&= \frac{(\mu df/a, \mu ef/a)_N}{(aq/d, aq/e)_N} \prod_{1 \leq k \leq m} \frac{((\mu c_k f/a)y_k, f y_k^{-1})_N}{(\mu q y_k, (aq/c_k)y_k^{-1})_N} \prod_{1 \leq i \leq n} \frac{(aq x_i, (\mu b_i f/a)x_i^{-1})_N}{((aq/b_i)x_i, (\mu f/a)x_i^{-1})_N} \\
&\times W^{m,n+2} \left( \begin{matrix} \{aq/c_k f\}_m \\ \{y_k\}_m \end{matrix} \middle| \mu; \{(aq/b_i f)x_i\}_n, aq/df, aq/ef; \right. \\
&\quad \left. \{(\mu f/a)x_i^{-1}\}_n, \mu f q^N, q^{-N}; q; q \right),
\end{aligned}$$

where  $\mu = a^{m+2}q^{m+1}/BCdef^{m+1}$ . Here we denote  $B = b_1 \cdots b_n$  and  $C = c_1 \cdots c_m$ . In this paper, we frequently use such notations.

In the case when  $m = 1$  and  $y_1 = 1$ , (2.12) reduces to

$$\begin{aligned}
&W^{n,3} \left( \begin{matrix} \{b_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; c, d, e; f, \mu f q^N, q^{-N}; q; q \right) \\
&= \frac{(\mu c f/a, \mu d f/a, \mu e f/a, f)_N}{(aq/c, aq/d, aq/e, \mu q)_N} \prod_{1 \leq i \leq n} \frac{(aq x_i, (\mu b_i f/a)x_i^{-1})_N}{((aq/b_i)x_i, (\mu f/a)x_i^{-1})_N} \\
&\times {}_{2n+8}W_{2n+7} [\mu; \{(aq/b_i f)x_i\}_n, aq/cf, aq/df, aq/ef, \\
&\quad \{(\mu f/a)x_i/x_i\}_n, \mu f q^N, q^{-N}; q; q], \quad (\mu = a^3 q^2 / Bcdef^2).
\end{aligned} \tag{2.13}$$

Note that  $m = n = 1$  and  $x_1 = y_1 = 1$  case of the balanced duality transformation formula (2.12) is terminating balanced  ${}_{10}W_9$  transformation (2.9).

In [13], (2.12) was obtained by taking the coefficients of  $u^N$  in both sides of "0-balanced" case of the multiple Euler transformation formula for multiple basic hypergeometric series of type  $A$  with different dimensions (Theorem 1.1 of [13])

$$\begin{aligned}
&\sum_{\gamma \in \mathbb{N}^n} u^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{(b_k x_i y_k)_{\gamma_i}}{(c x_i y_k)_{\gamma_i}} \\
&= \frac{(ABu/c^m)_\infty}{(u)_\infty} \sum_{\delta \in \mathbb{N}^m} \left( \frac{ABu}{c^m} \right)^{|\delta|} \frac{\Delta(yq^\delta)}{\Delta(y)} \\
&\times \prod_{1 \leq k, l \leq m} \frac{((c/b_l)y_k/y_l)_{\delta_k}}{(q y_k / y_l)_{\delta_k}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{((c/a_i)x_i y_k)_{\delta_k}}{(c x_i y_k)_{\delta_k}}.
\end{aligned} \tag{2.14}$$

From this point of view, we can state the balanced duality transformation (2.12) in more general form: (Proposition 6.2 in [13])

$$\begin{aligned}
&\sum_{\gamma \in \mathbb{N}^n, |\gamma|=N} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{(b_k x_i y_k)_{\gamma_i}}{(c x_i y_k)_{\gamma_i}} \\
&= \sum_{\delta \in \mathbb{N}^m, |\delta|=N} \frac{\Delta(yq^\delta)}{\Delta(y)} \prod_{1 \leq k, l \leq m} \frac{((c/b_l)y_k/y_l)_{\delta_k}}{(q y_k / y_l)_{\delta_k}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{((c/a_i)x_i y_k)_{\delta_k}}{(c x_i y_k)_{\delta_k}},
\end{aligned} \tag{2.15}$$

when  $AB = c^m$ .

The balanced duality transformation formula (2.12) corresponds to the case when  $m, n \geq 2$  of (2.15) by a rearrangement of parameters in the multiple basic hypergeometric series. Note that, in the case when  $m = n = 1$ , (2.15) becomes tautological. We shall also remark that the remaining case ( $m = 1, n \geq 2$ ) corresponds to the  $A_n$  Jackson summation formula for terminating balanced  $W^{n,2}$  series

$$\begin{aligned}
&W^{n,2} \left( \begin{matrix} \{b_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; c, e; d, q^{-N}; q; q \right) \\
&= \frac{(aq/Bc, aq/cd)_N}{(aq/Bcd, aq/c)_N} \prod_{1 \leq i \leq n} \frac{((aq/b_i d)x_i, aq x_i)_N}{((aq/b_i)x_i, (aq/d)x_i)_N}
\end{aligned} \tag{2.16}$$

provided  $a^2q^{N+1} = Bcde$ , which is originally due to S.Milne [20] in a different notation. For these facts, the informed readers might see [13] for ordinary and basic case and Noumi and the author [15] for elliptic case.

*Remark 2.1.* In the case when  $n = 1$  and  $x_1 = 1$ , (2.16) reduces to the Jackson summation formula for terminating balanced  ${}_8W_7$  series:

$${}_8W_7[a; b, c, d, e, q^{-N}; q; q] = \frac{(aq, aq/bc, aq/bd, aq/cd)_N}{(aq/bcd, aq/b, aq/c, aq/d)_N}, \quad (a^2q^{N+1} = bcde). \quad (2.17)$$

We also mention that (2.16) can be obtained by letting  $aq = ef$  in (2.13) and by relabeling the parameters.

### 3 Limit cases

In this section, we shall show that several transformation formulas with different dimension can be obtained from the balanced duality transformation formula (2.12) by taking certain limits. We see that most of principal transformation formulas in [13] which have been obtained by taking a certain coefficient in the multiple Euler transformation formula (2.14) can be recovered and we find some new transformation formulas. Furthermore we show that (2.14) itself can be acquired in this manner.

In addition, we shall write down the cases when the dimension of summand in each side of the transformation is one with particular attention. We consider them as particularly significant ones since they have an extra symmetry. We will explore some  $A_n$  hypergeometric transformations by using some of them in the next section.

#### 3.1 (Non-balanced) Duality transformation formula and its inverse

##### (Non-balanced) Duality transformation formula

**Proposition 3.1.**

$$\begin{aligned} W^{n,m+1} \left( \begin{matrix} \{b_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; c, \{d_k y_k\}_m; q^{-N}, \{e y_k^{-1}\}_m; q; \frac{a^{m+1} q^{N+m+1}}{BcDe^m} \right) \\ = \frac{(a^{m+1} q^{m+1} / BcDe^m)_N}{(aq/c)_N} \prod_{1 \leq i \leq n} \frac{(aqx_i)_N}{((aq/b_i)x_i)_N} \prod_{1 \leq k \leq m} \frac{(ey_k^{-1})_N}{((aq/d_k)y_k^{-1})_N} \\ \times \sum_{\delta \in \mathbb{N}^m} q^{|\gamma|} \frac{\Delta(yq^\delta)}{\Delta(y)} \frac{(q^{-N})_{|\gamma|}}{(a^{m+1} q^{m+1} / BcDe^m)_{|\gamma|}} \prod_{1 \leq k, l \leq m} \frac{((aq/d_l e)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}} \\ \times \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{((aq/b_i e)x_i y_k)_{\delta_k}}{((aq/e)x_i y_k)_{\delta_k}} \prod_{1 \leq k \leq m} \frac{((aq/ce)y_k)_{\delta_k}}{(q^{1-N} e^{-1} y_k)_{\delta_k}}, \end{aligned} \quad (3.1)$$

where  $B = b_1 \cdots b_m$  and  $D = d_1 \cdots d_n$ .

*Proof.* Take the limit  $e \rightarrow \infty$  in (2.12). By rearranging the parameter as  $f \rightarrow e$ , we arrive at the desired identity.  $\square$

*Remark 3.1.* The transformation formula (3.1) has already appeared in Section 6.1 of our previous work [13] with a different notation. Later Rosengren has also obtained in [26] by using his reduction formula of Karlsson-Minton type. In contract to that we see here, as is mentioned in [26], the balanced duality transformation formula (2.12) can also be considered as a special case of (3.1). Indeed, in [13] though we have started the derivation of (3.1) from the multiple Euler transformation (2.14) in general case, (2.12) has been obtained from the "zero"-balanced case of (2.14).

In the case when  $m = n = 1$  and  $x_1 = y_1 = 1$ , (3.1) reduces to

$$\begin{aligned} {}_8W_7 \left[ a; b, c, d, e, q^{-N}; q; \frac{a^2 q^{N+2}}{bcde} \right] \\ = \frac{(a^2 q^2 / bcde, e, aq)_N}{(aq/b, aq/c, aq/d)_N} {}_4\phi_3 \left[ \begin{matrix} q^{-N}, aq/be, aq/ce, aq/de \\ q^{1-N}/e, a^2 q^2 / bcde, aq/e \end{matrix}; q; q \right]. \end{aligned} \quad (3.2)$$

In [13], (3.1) is referred as Watson type transformation formula. But, hereafter we shall propose to call (3.1) (non balanced) duality transformation formula.

In the case when  $m = 1$  and  $y_1$ , (3.1) reduces to

$$\begin{aligned} W^{n,2} \left( \begin{matrix} \{b_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; c, d; q^{-N}, e; q; \frac{a^2 q^{N+2}}{Bcde} \right) &= \frac{(a^2 q^2 / Bcde, e)_N}{(aq/c, aq/d)_N} \prod_{1 \leq i \leq n} \frac{(aqx_i)_N}{((aq/b_i)x_i)_N} \\ &\times {}_{n+3}\phi_{n+2} \left[ \begin{matrix} q^{-N}, \{(aq/b_i e)x_i\}_n, aq/ce, aq/de \\ q^{1-N}/e, \{(aq/e)x_i\}_n, a^2 q^2 / Bcde \end{matrix}; q; q \right]. \end{aligned} \quad (3.3)$$

In the case of  $n = 1$  and  $x_1 = 1$ , (3.1) reduces to

$$\begin{aligned} {}_{2m+6}W_{2m+5} \left[ a; b, \{c_k y_k\}_m, d, \{e y_k^{-1}\}_m, q^{-N}; q; \frac{a^{m+1} q^{N+m+1}}{bCde^m} \right] \\ = \frac{(a^{m+1} q^{m+1} / bCde^m, aq)_N}{(aq/b, aq/d)_N} \prod_{1 \leq k \leq m} \frac{(e y_k^{-1})_N}{((aq/c_k) y_k^{-1})_N} \\ \times \sum_{\delta \in \mathbb{N}^m} q^{|\delta|} \frac{\Delta(y q^\delta)}{\Delta(y)} \prod_{1 \leq k, l \leq m} \frac{((aq/c_l e) y_k / y_l)_{\delta_k}}{(q y_k / y_l)_{\delta_k}} \\ \times \frac{(q^{-N})_{|\delta|}}{(a^{m+1} q^{m+1} / bCde^m)_{|\delta|}} \prod_{1 \leq k \leq m} \frac{((aq/be) y_k, (aq/de) y_k)_{\delta_k}}{((aq/e) y_k, (q^{1-N}/e) y_k)_{\delta_k}}. \end{aligned} \quad (3.4)$$

Further, by letting  $aq = de$  in (3.3), we obtain Milne's  $A_n$  generalization of Rogers' summation formula for terminating very well-poised  ${}_6W_5$  series:

$$W^{n,1} \left( \begin{matrix} \{b_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; c; q^{-N}; q; \frac{aq^{N+1}}{Bc} \right) = \frac{(aq/Bc)_N}{(aq/c)_N} \prod_{1 \leq i \leq n} \frac{(aqx_i)_N}{((aq/b_i)x_i)_N}. \quad (3.5)$$

*Remark 3.2.* In the case when  $n = 1$  and  $x_1 = 1$ , (3.5) reduces to the Rogers' summation formula ( (2.4.2) in [7] ):

$${}_6W_5 \left[ a; b, c, q^{-N}; q; \frac{aq^{N+1}}{bc} \right] = \frac{(aq, aq/bc)_N}{(aq/b, aq/c)_N}. \quad (3.6)$$

## The inverse of the duality transformation

**Proposition 3.2.**

$$\begin{aligned} \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(x q^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{(b_k x_i y_k)_{\gamma_i}}{(e x_i y_k)_{\gamma_i}} \\ \times \frac{(q^{-N})_{|\gamma|}}{(d)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(c x_i)_{\gamma_i}}{((ABc q^{1-N} / de^m) x_i)_{\gamma_i}} \\ = \frac{(de^m / AB)_N}{(d)_N} \prod_{1 \leq k \leq m} \frac{((de^m b_k / ABc) y_k)_N}{((de^{m+1} / ABc) y_k)_N} \prod_{1 \leq i \leq n} \frac{((de^m a_i / ABc) x_i^{-1})_N}{((de^m / ABc) x_i^{-1})_N} \end{aligned} \quad (3.7)$$

$$\times W^{m,n+1} \left( \begin{matrix} \{e/b_k\}_m \\ \{y_k\}_m \end{matrix} \middle| de^{m+1}q^{-1}/ABc; e/c, \{(e/a_i)x_i\}_n; \right. \\ \left. q^{-N}, \{(de^m/ABc)x_i^{-1}\}_n; q; dq^N \right).$$

*Proof.* Substitute the parameters  $e$  and  $f$  as  $aq/e$  and  $aq/f$  respectively in (2.12). Then take the limit  $a \rightarrow \infty$ . Finally, by rearranging the parameters as  $b_i \rightarrow a_i$  ( $1 \leq i \leq n$ ),  $c_k \rightarrow b_k$  ( $1 \leq k \leq m$ ),  $d \rightarrow c$ ,  $e \rightarrow d$ ,  $f \rightarrow e$ , we have the desired result.  $\square$

In the case when  $m = 1$  and  $y_1 = 1$ , (3.7) reduces to

$$\sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(b x_i, c x_i)_{\gamma_i}}{(e x_i, (Abc q^{1-N}/de) x_i)_{\gamma_i}} \\ \times \frac{(q^{-N})_{|\gamma|}}{(d)_{|\gamma|}} = \frac{(de/Ac, de/Ab)_N}{(de^2/Abc, d)_N} \prod_{1 \leq i \leq n} \frac{((dea_i/Abc)x_i^{-1})_N}{((de/Abc)x_i^{-1})_N} \\ \times {}_{2n+6}W_{2n+5} [de^2 q^{-1}/Abc; \{(e/a_i)x_i\}_n, \{(deq^{-1}/Abc)x_i^{-1}\}_n, e/b, e/c, q^{-N}; q; dq^N]. \quad (3.8)$$

In the case when  $n = 1$  and  $x_1 = 1$ , (3.7) reduces to

$${}_{m+3}\phi_{m+2} \left[ \begin{matrix} q^{-N}, a, \{b_k y_k\}_m, c \\ d, \{e y_k\}_m, aBc q^{1-N}/de^m; q; q \end{matrix} \right] \\ = \frac{(de^m/aB, de^m/Bc)_N}{(d, de^m/aBc)_N} \prod_{1 \leq k \leq m} \frac{((de^m b_k/aBc)y_k)_N}{((de^{m+1}/aBc)y_k)_N} \\ \times W^{m,2} \left( \begin{matrix} \{e/b_k\}_m \\ \{y_k\}_m \end{matrix} \middle| de^{m+1}q^{-1}/aBc; e/c, e/a; q^{-N}, de^m/ABc; q; dq^N \right). \quad (3.9)$$

*Remark 3.3.* The reason why we call (3.7) as the inverse of the (non-balanced) duality transformation (3.1) is that the transformations which one obtain by applying (3.1) and (3.7) in both order turn out to be identical as a (multiple) hypergeometric series. Note also that (3.7) can be obtained by relabeling parameters in (3.1) appropriately. However, we will use special cases of these transformations to establish an  $A_n$  generalization of basic hypergeometric transformation formulas in the next section.

In the case when  $m = n = 1$ , (3.7) reduces to

$${}_4\phi_3 \left[ \begin{matrix} q^{-N}, a, b, c \\ d, e, abc q^{1-N}/de; q; q \end{matrix} \right] = \frac{(de/bc, de/ac, de/ab)_N}{(de^2/abc, de/abc, d)_N} \\ \times {}_8W_7 [de^2 q^{-1}/abc; e/a, deq^{-1}/abc, e/b, e/c, q^{-N}; q; dq^N]. \quad (3.10)$$

We also mention that by letting  $e = c$  in (3.8) and then rearranging the parameter  $d$  as  $Abq^{1-N}/c$ , we recover an  $A_n$  generalization of Pfaff-Saalschütz summation formula for terminating balanced  ${}_3\phi_2$  series due to Milne (Theorem 4.15 in [21])

$$\sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(b x_i)_{\gamma_i}}{(c x_i)_{\gamma_i}} \\ \times \frac{(q^{-N})_{|\gamma|}}{(Abq^{1-N}/c)_{|\gamma|}} = \frac{(c/b)_N}{(c/Ab)_N} \prod_{1 \leq i \leq n} \frac{((c/a_i)x_i)_N}{(c x_i)_N}. \quad (3.11)$$

*Remark 3.4.* In the case when  $n = 1$ , (3.11) reduces to Jackson's Pfaff-Saalschütz summation formula for terminating balanced  ${}_3\phi_2$  series (the formula (1.7.2) in [7])

$${}_3\phi_2 \left[ \begin{matrix} a, b, q^{-N} \\ c, abq^{1-N}/c; q; q \end{matrix} \right] = \frac{(c/a, c/b)_N}{(c, c/ab)_N}. \quad (3.12)$$



*Remark 3.5.* Similarly to the expression (2.15) for the balanced duality transformation formula (2.12), (3.7) can be stated in more general form:

$$\begin{aligned} & \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{(b_k x_i y_k)_{\gamma_i}}{(c x_i y_k)_{\gamma_i}} \\ & \times \frac{(q^{-N})_{|\gamma|}}{(ABq^{1-N}/c^m)_{|\gamma|}} = \frac{(q)_N}{(c^m/AB)_N} \sum_{\delta \in \mathbb{N}^m, |\delta|=N} \frac{\Delta(yq^\delta)}{\Delta(y)} \\ & \times \prod_{1 \leq k, l \leq m} \frac{((c/b_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{((c/a_i)x_i y_k)_{\delta_k}}{(c x_i y_k)_{\delta_k}}. \end{aligned} \quad (3.13)$$

Note that (3.13) can be obtained by taking the coefficient of  $u^N$  in the multiple basic Euler transformation formula (2.14). We remark that (3.7) corresponds to the  $m \geq 2$  case of (3.13) by rearrangement of parameters and in the case when  $m = 1$ , (3.13) reduces to  $A_n$  Pfaff-Saalschütz summation formula (3.11).

### 3.2 ${}_4\phi_3$ transformation formulas of type A

#### Sears transformation of type A

**Proposition 3.3.**

$$\begin{aligned} & \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{(c_k x_i y_k)_{\gamma_i}}{(d x_i y_k)_{\gamma_i}} \\ & \times \frac{(q^{-N}, a)_{|\gamma|}}{(e, aBCq^{1-N}/d^m e)_{|\gamma|}} = \frac{(e/a, d^m e/BC)_N}{(e, d^m e/aBC)_N} \sum_{\delta \in \mathbb{N}^m} q^{|\delta|} \frac{\Delta(yq^\delta)}{\Delta(y)} \\ & \times \frac{(q^{-N}, a)_{|\delta|}}{(q^{1-N} a/e, d^m e/BC)_{|\delta|}} \prod_{1 \leq k, l \leq m} \frac{((d/c_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{((d/b_i)x_i y_k)_{\delta_k}}{(d x_i y_k)_{\delta_k}}. \end{aligned} \quad (3.14)$$

*Proof.* Replace the parameters in (2.12)  $d, e$  and  $f$  with  $aq/d, aq/e$  and  $aq/f$  respectively. Then put  $a = 0$ . Then rearranging parameters  $f \rightarrow d$  and  $def^m q^{N-1}/BC \rightarrow a$  leads to the desired identity.  $\square$

In the case when  $m = 1$  and  $y_1 = 1$ , (3.14) reduces to

$$\begin{aligned} & \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \frac{(q^{-N}, a)_{|\gamma|}}{(e, aBcq^{1-N}/de)_{|\gamma|}} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(c x_i)_{\gamma_i}}{(d x_i)_{\gamma_i}} \\ & = \frac{(e/a, de/Bc)_N}{(e, de/aBc)_N} {}_{n+3}\phi_{n+2} \left[ \begin{matrix} q^{-N}, a, \{(d/b_i)x_i\}_n, d/c \\ q^{1-N} a/e, \{d x_i\}_n, de/Bc \end{matrix}; q, q \right]. \end{aligned} \quad (3.15)$$

*Remark 3.6.* The multiple Sears transformation formula (3.14) was originally established in [13] (the formula (7.1) in [13]). In the case when  $m = n = 1$  and  $x_1 = y_1 = 1$ , (3.14) reduces to the Sears transformation formula for terminating balanced  ${}_4\phi_3$  series:

$${}_4\phi_3 \left[ \begin{matrix} q^{-N}, a, b, c \\ d, e, abcq^{1-N}/de \end{matrix}; q, q \right] = \frac{(e/a, de/bc)_N}{(e, de/abc)_N} {}_4\phi_3 \left[ \begin{matrix} q^{-N}, a, d/b, d/c \\ d, aq^{1-N}/e, de/bc \end{matrix}; q, q \right]. \quad (3.16)$$

Further informations for multiple Sears transformation (3.14) can also be found in [12].

We shall add a few remarks that we have missed in our previous works in [13] and [12].

Let  $N$  tend to infinity in (3.14), we get the following transformation formula for multiple nonterminating  ${}_3\phi_2$  series of type A:

**Corollary 3.1.**

$$\begin{aligned} & \sum_{\gamma \in \mathbb{N}^n} \left( \frac{d^m e}{aBC} \right)^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{(c_k x_i y_k)_{\gamma_i}}{(dx_i y_k)_{\gamma_i}} \\ & \times \frac{(a)_{|\gamma|}}{(e)_{|\gamma|}} = \frac{(e/a, d^m e/aBC)_\infty}{(e, d^m e/aBC)_\infty} \sum_{\delta \in \mathbb{N}^m} \left( \frac{e}{a} \right)^{|\delta|} \frac{\Delta(yq^\delta)}{\Delta(y)} \\ & \times \frac{(a)_{|\delta|}}{(d^m e/aBC)_{|\delta|}} \prod_{1 \leq k, l \leq m} \frac{((d/c_l) y_k / y_l)_{\delta_k}}{(qy_k / y_l)_{\delta_k}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{((d/b_i) x_i y_k)_{\delta_k}}{(dx_i y_k)_{\delta_k}} \end{aligned} \quad (3.17)$$

holds if it satisfy the convergence condition  $\max(|d^m e/aBC|, |e/a|) < 1$ .

*Remark 3.7.* (3.17) was already established as the equation (3.2) in [12]. In the case when  $m = n = 1$  and  $x_1 = y_1 = 1$ , (3.17) reduces to the following nonterminating  ${}_3\phi_2$  transformation formula

$${}_3\phi_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc} \right] = \frac{(e/a, de/bc)_\infty}{(e, de/abc)_\infty} {}_3\phi_2 \left[ \begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, \frac{e}{a} \right]. \quad (3.18)$$

In the case when  $m = 1$  and  $y_1 = 1$ , (3.17) reduces to

$$\begin{aligned} & \sum_{\gamma \in \mathbb{N}^n} \left( \frac{de}{aBc} \right)^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \frac{(a)_{|\gamma|}}{(e)_{|\gamma|}} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{(dx_i)_{\gamma_i}} \\ & = \frac{(e/a, de/Bc)_\infty}{(e, de/aBc)_\infty} {}_{n+2}\phi_{n+1} \left[ \begin{matrix} d/c, a, \{(d/b_i) x_i\}_n \\ de/Bc, \{dx_i\}_n \end{matrix}; q, \frac{e}{a} \right] \end{aligned} \quad (3.19)$$

holds if it satisfy the convergence condition  $\max(|d^m e/aBC|, |e/a|) < 1$ .

Here we give a remark on the convergence of multiple series.

*Remark 3.8. (Convergence of the multiple series)* In the course of obtaining an infinite multiple sum identity such as (3.17), one needs to ensure the limiting procedure. To justify the process, one is required to use the dominated convergence theorem. Furthermore, to find the convergence condition of the dominated series such as the series in (3.17), one can quote the ratio test for multiple power series in classical work by J.Horn [11]. Since the details of such justifications in this paper are all in the same line as the corresponding discussions in Milne-Newcomb [24] (see also Milne [21] and Milne-Newcomb [25]), we shall omit in this paper.

We shall propose the following special case, namely a transformation formula between  $A_n$   ${}_{m+2}\phi_{m+1}$  series with integer parameter differences and  $A_m$  terminating balanced  ${}_{n+2}\phi_{n+1}$  series:

$$\begin{aligned} & \sum_{\gamma \in \mathbb{N}^n} \left( q^{1-|M|/A} \right)^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \\ & \times \frac{(b)_{|\gamma|}}{(bq)_{|\gamma|}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{(cq^{m_k} x_i y_k)_{\gamma_i}}{(cx_i y_k)_{\gamma_i}} \\ & = \frac{(q, q^{1-|M|} b/A)_\infty}{(b, q^{1-|M|} /A)_\infty} \sum_{\delta \in \mathbb{N}^m} q^{|\delta|} \frac{\Delta(yq^\delta)}{\Delta(y)} \prod_{1 \leq k, l \leq m} \frac{(q^{-m_l} y_k / y_l)_{\delta_k}}{(qy_k / y_l)_{\delta_k}} \\ & \times \frac{(b)_{|\delta|}}{(q^{1-|M|} b/A)_{|\delta|}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{((c/a_i) x_i y_k)_{\delta_k}}{(cx_i y_k)_{\delta_k}}. \end{aligned} \quad (3.20)$$

In the case when  $n = 1$  and  $x_1 = 1$ , the multiple series in the right hand side in (3.20) can be summed by the following  $A_n$  generalization of  $q$ -Pfaff-Saalschütz summation formula

$$\sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(bx_i)_{\gamma_i}}{(cx_i)_{\gamma_i}} \quad (3.21)$$

$$\times \frac{(a)_{|\gamma|}}{(abq^{1-|M|}/c)_{|\gamma|}} = \frac{(c/b)_{|M|}}{(c/ab)_{|M|}} \prod_{1 \leq i \leq n} \frac{((c/a)x_i)_{m_i}}{(cx_i)_{m_i}},$$

which can be obtained from (3.11) by elementary polynomial argument (see the proof of Corollary 4.1), to recover Gasper's  $q$ -analogue of Minton-Karlssohn summation formula for  $_{n+2}\phi_{n+1}$  series [6]:

$$_{n+2}\phi_{n+1} \left[ \begin{matrix} a, b, \{cq^{m_i}x_i\}_n \\ bq, \{cx_i\}_n \end{matrix}; q; a^{-1}q^{1-|M|} \right] = b^{|M|} \frac{(q, bq/a)_\infty}{(bq, q/a)_\infty} \prod_{1 \leq i \leq n} \frac{((c/b)x_i)_{m_i}}{(cx_i)_{m_i}}. \quad (3.22)$$

In the case when  $m = 1$  and  $y_1 = 1$ , (3.20) reduces to:

$$\begin{aligned} \sum_{\gamma \in \mathbb{N}^n} (q^{1-m}/A)^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cq^m x_i)_{\gamma_i}}{(cx_i)_{\gamma_i}} \\ \times \frac{(b)_{|\gamma|}}{(bq)_{|\gamma|}} = \frac{(q, q^{1-M}b/A)_\infty}{(b, q^{1-M}/A)_\infty} {}_{n+2}\phi_{n+1} \left[ \begin{matrix} q^{-M}, b, \{(c/a_i)x_i\}_n \\ q^{1-M}b/A, \{cx_i\}_n \end{matrix}; q; q \right]. \end{aligned} \quad (3.23)$$

In the case when  $m = n = 1$  and  $x_1 = y_1 = 1$ , (3.20) reduces to:

$${}_3\phi_2 \left[ \begin{matrix} a, b, cq^M \\ bq, c \end{matrix}; q; q^{1-M}/a \right] = b^M \frac{(q, bq/a, c/b)_\infty}{(bq, q/a, c)_\infty}. \quad (3.24)$$

It may be of worth to note that, furthermore, if we replace  $e \rightarrow aBCu/d^m$  and take limit  $a \rightarrow \infty$  in (3.17), we recover multiple basic Euler transformation formula of type  $A$  (2.14)

$$\begin{aligned} \sum_{\gamma \in \mathbb{N}^n} u^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{(b_k x_i y_k)_{\gamma_i}}{(cx_i y_k)_{\gamma_i}} \\ = \frac{(ABu/c^m)_\infty}{(u)_\infty} \sum_{\delta \in \mathbb{N}^m} \left( \frac{ABu}{c^m} \right)^{|\delta|} \frac{\Delta(yq^\delta)}{\Delta(y)} \\ \times \prod_{1 \leq k, l \leq m} \frac{((c/b_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{((c/a_i)x_i y_k)_{\delta_k}}{(cx_i y_k)_{\delta_k}}, \end{aligned}$$

when  $\max(|u|, |ABu/c^m|) < 1$ .

## Reversing version

**Proposition 3.4.** *Under the balancing condition*

$$a^m Bc q^{1-N} = dEf, \quad (3.25)$$

*we have the following:*

$$\begin{aligned} \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{(dx_i)_{\gamma_i}} \prod_{1 \leq k \leq m} \frac{(ay_k)_{|\gamma|}}{(e_k y_k)_{|\gamma|}} \\ \times \frac{(q^{-N})_{|\gamma|}}{(f)_{|\gamma|}} = \frac{(Ef/a^m B)_N}{(f)_N} \prod_{1 \leq k \leq m} \frac{(ay_k)_N}{(e_k y_k)_N} \prod_{1 \leq i \leq n} \frac{((Ef/a^m c)z_i)_N}{((Ef/a^m Bc)z_i)_N} \end{aligned} \quad (3.26)$$

$$\begin{aligned} & \times \sum_{\delta \in \mathbb{N}^m} q^{|\delta|} \frac{\Delta(wq^\delta)}{\Delta(w)} \prod_{1 \leq k, l \leq m} \frac{((e_l/a)w_k/w_l)_{\delta_k}}{(qw_k/w_l)_{\delta_k}} \prod_{1 \leq k \leq m} \frac{((f/a)w_k)_{\delta_k}}{((dEf/a^{m+1}Bc)w_k)_{\delta_k}} \\ & \times \frac{(q^{-N})_{|\delta|}}{(Ef/a^m B)_{|\delta|}} \prod_{1 \leq i \leq n} \frac{((Ef/a^m b_i c)z_i)_{|\delta|}}{((Ef/a^m c)z_i)_{|\delta|}} \end{aligned}$$

where  $z_i = b_i/Bx_i$ ,  $(1 \leq i \leq n)$  and  $w_k = y_k^{-1}$ ,  $(1 \leq k \leq m)$  respectively.

*Proof.* Rewrite the parameters  $c_k \rightarrow aq/c_k$   $(1 \leq k \leq m)$  and  $e \rightarrow aq/e$  in (2.12). Then put  $a = 0$ . Finally by rearranging the parameters  $f \rightarrow a$ ,  $d \rightarrow c$ ,  $Bdf^m q^{1-N}/Ce \rightarrow d$ ,  $c_k \rightarrow e_k$   $(1 \leq k \leq m)$ ,  $e \rightarrow f$ , we arrive at the desired identity.  $\square$

In the case when  $m = 1$  and  $y_1 = 1$ , (3.26) reduces to

$$\begin{aligned} & \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{(dx_i)_{\gamma_i}} \\ & \times \frac{(q^{-N}, a)_{|\gamma|}}{(e, f)_{|\gamma|}} = \frac{(ef/aB, a)_N}{(e, f)_N} \prod_{1 \leq i \leq n} \frac{((ef/ac)z_i)_N}{((ef/aBc)z_i)_N} \\ & \times {}_{n+3}\phi_{n+2} \left[ \begin{matrix} q^{-N}, e/a, f/a, \{(ef/ab_i c)z_i\}_n \\ def/a^2 Bc, ef/aB, \{(ef/ac)z_i\}_n \end{matrix} ; q; q \right], \end{aligned} \quad (3.27)$$

when the following balancing condition holds:

$$aBcq^{1-N} = def, \quad (3.28)$$

and where  $z_i = b_i/Bx_i$ ,  $i = 1, 2, \dots, n$ . By letting  $f = a$  in (3.27) and relabeling the parameters appropriately, we also recover the  $A_n$  Pfaff-Saalschütz summation (3.11).

*Remark 3.9.* (3.26) has already appeared as the 2nd Sears transformation formula (Proposition 7.2.) in [13], up to relabeling parameters. In the case when  $m = n = 1$  and  $x_1 = y_1 = 1$ , (3.26) reduces to a transformation formula for terminating balanced  ${}_4\phi_3$  series

$$\begin{aligned} {}_4\phi_3 \left[ \begin{matrix} q^{-N}, a, b, c \\ d, e, f \end{matrix} ; q; q \right] &= \frac{(ef/ab, ef/ac, a)_N}{(e, f, ef/abc)_N} \\ &\times {}_4\phi_3 \left[ \begin{matrix} q^{-N}, ef/abc, e/a, f/a \\ def/a^2 bc, ef/ab, ef/ac \end{matrix} ; q; q \right] \end{aligned} \quad (3.29)$$

when

$$abcq^{1-N} = def. \quad (3.30)$$

Note that this is obtained by reversing the order of the summation of the Sears transformation (3.16) and is also verified by iterating Sears transformation twice in a proper fashion.

Let  $N \rightarrow \infty$  in (3.26). Then we have

$$\begin{aligned} & \sum_{\gamma \in \mathbb{N}^n} x_1^{-\gamma_1} \dots x_n^{-\gamma_n} \left( \frac{Ef}{a^m Bc} \right)^{|\gamma|} q^{e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} (cx_i)_{\gamma_i} \\ & \times [(f)_{|\gamma|}]^{-1} \prod_{1 \leq k \leq m} \frac{(ay_k)_{|\gamma|}}{(e_k y_k)_{|\gamma|}} \\ & = \frac{(Ef/a^m B)_\infty}{(f)_\infty} \prod_{1 \leq k \leq m} \frac{(ay_k)_\infty}{(e_k y_k)_\infty} \prod_{1 \leq i \leq n} \frac{((Ef/a^m c)z_i)_\infty}{((Ef/a^m Bc)z_i)_\infty} \\ & \times \sum_{\delta \in \mathbb{N}^m} w_1^{-\delta_1} \dots w_m^{-\delta_m} a^{|\delta|} q^{e_2(\delta)} \frac{\Delta(wq^\delta)}{\Delta(w)} \prod_{1 \leq k, l \leq m} \frac{((e_l/a)w_k/w_l)_{\delta_k}}{(qw_k/w_l)_{\delta_k}} \prod_{1 \leq k \leq m} ((f/a)w_k)_{\delta_k} \end{aligned} \quad (3.31)$$

$$\times [(Ef/a^m B)_{|\delta|}]^{-1} \prod_{1 \leq i \leq n} \frac{((Ef/a^m b_i c) z_i)_{|\delta|}}{((Ef/a^m c) z_i)_{|\delta|}},$$

where  $z_i = b_i/Bx_i$ , ( $1 \leq i \leq n$ ),  $w_k = y_k^{-1}$ , ( $1 \leq k \leq m$ ) respectively and  $e_2(\gamma)$  is the second elementary symmetric function of the variable  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ .

The case  $m = 1$  and  $y_1 = 1$  of (3.31) is

$$\begin{aligned} \sum_{\gamma \in \mathbb{N}^n} x_1^{-\gamma_1} \cdots x_n^{-\gamma_n} \left( \frac{ef}{aBc} \right)^{|\gamma|} q^{e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \\ \times \frac{(a)_{|\gamma|}}{(e, f)_{|\gamma|}} \prod_{1 \leq i \leq n} (c x_i)_{\gamma_i} = \frac{(ef/aB, a)_\infty}{(e, f)_\infty} \prod_{1 \leq i \leq n} \frac{((ef/ac) z_i)_\infty}{((ef/aBc) z_i)_\infty} \\ \times {}_{n+2}\phi_{n+1} \left[ \begin{matrix} e/a, f/a, \{(ef/aBc) z_i\}_n \\ ef/aB, \{(ef/ac) z_i\}_n \end{matrix}; q; a \right], \end{aligned} \quad (3.32)$$

where  $z_i = b_i/Bx_i$  for  $i = 1, \dots, n$ . We mention that one yield an  $A_n$  Gauss summation theorem by putting  $f = a$  and relabeling the parameters:

$$\begin{aligned} \sum_{\gamma \in \mathbb{N}^n} x_1^{-\gamma_1} \cdots x_n^{-\gamma_n} \left( \frac{ef}{aBc} \right)^{|\gamma|} q^{e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \\ \times (c)_{|\gamma|}^{-1} \prod_{1 \leq i \leq n} (a x_i)_{\gamma_i} = \frac{(c/B)_\infty}{(c)_\infty} \prod_{1 \leq i \leq n} \frac{((cB/ab_i) x_i^{-1})_\infty}{((c/ab_i) x_i^{-1})_\infty}. \end{aligned} \quad (3.33)$$

*Remark 3.10.* In the case when  $m = n = 1$  and  $x_1 = y_1 = 1$ , (3.31) reduces to  ${}_3\phi_2$  transformation formula

$${}_3\phi_2 \left[ \begin{matrix} a, b, c \\ e, f \end{matrix}; q; \frac{ef}{abc} \right] = \frac{(ef/ab, ef/ac, a)_\infty}{(e, f, ef/abc)_\infty} {}_3\phi_2 \left[ \begin{matrix} e/a, f/a, ef/abc \\ ef/ab, ef/ac \end{matrix}; q; a \right].$$

In the case when  $mn = 1$  and  $x_1 = 1$ , (3.33) reduces to the basic Gauss summation formation formula for  ${}_2\phi_1$

$${}_2\phi_1 \left[ \begin{matrix} a, b \\ c \end{matrix}; q; \frac{c}{ab} \right] = \frac{(c/a, c/b)_\infty}{(c, c/ab)_\infty}.$$

We also remark that (3.31) and (3.33) have already appeared in our previous work [14]. But they contain errors, so we restate them here.

### 3.3 ${}_8W_7$ transformations

#### Nonterminating ${}_8W_7$ transformation formula

By taking the limit  $N \rightarrow \infty$  in (2.12), we have the following:

**Proposition 3.5.** Assume that  $\left| \frac{a^{m+1} q^{m+1}}{BCdef^m} x_i^{-1} \right| < 1$  for all  $i = 1, \dots, n$  and  $|f y_k^{-1}| < 1$  for all  $k = 1, \dots, m$ . Then we have

$$\begin{aligned} \sum_{\gamma \in \mathbb{N}^n} x_1^{-\gamma_1} \cdots x_n^{-\gamma_n} \left( \frac{a^{m+1} q^{m+1}}{BCdef^m} \right)^{|\gamma|} q^{e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - a q^{|\gamma| + \gamma_i} x_i}{1 - a x_i} \\ \times \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{(c_k x_i y_k)_{\gamma_i}}{((aq/f) x_i y_k)_{\gamma_i}} \\ \times \prod_{1 \leq i \leq n} \frac{(a x_i)_{|\gamma|}}{((aq/b_i) x_i)_{|\gamma|}} \prod_{1 \leq k \leq m} \frac{(f y_k^{-1})_{|\gamma|}}{((aq/c_k) y_k^{-1})_{|\gamma|}} \end{aligned} \quad (3.34)$$

$$\begin{aligned}
& \times \frac{1}{(aq/d, aq/e)_{|\gamma|}} \prod_{1 \leq i \leq n} ((dx_i, ex_i)_{\gamma_i}) \\
= & \frac{(\mu df/a, \mu ef/a)_{\infty}}{(aq/d, aq/e)_{\infty}} \prod_{1 \leq k \leq m} \frac{((\mu c_k f/a)y_k, f y_k^{-1})_{\infty}}{(\mu q y_k, (aq/c_k) y_k^{-1})_{\infty}} \\
& \times \prod_{1 \leq i \leq n} \frac{(aq x_i, (\mu b_i f/a) x_i^{-1})_{\infty}}{((aq/b_i) x_i, (\mu f/a) x_i^{-1})_{\infty}} \\
& \times \sum_{\delta \in \mathbb{N}^m} y_1^{-\delta_1} \dots y_m^{-\delta_m} f^{|\delta|} q^{e_2(\delta)} \frac{\Delta(y q^{\delta})}{\Delta(y)} \prod_{1 \leq k \leq m} \frac{1 - \mu q^{|\delta| + \delta_k} y_k}{1 - \mu y_k} \\
& \times \prod_{1 \leq k, l \leq m} \frac{((aq/c_l f) y_k / y_l)_{\delta_k}}{(q y_k / y_l)_{\delta_k}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{((aq/b_i f) x_i y_k)_{\delta_k}}{((aq/f) x_i y_k)_{\delta_k}} \\
& \times \prod_{1 \leq k \leq m} \frac{(\mu y_k)_{|\delta|}}{((\mu c_k f/a) y_k)_{|\delta|}} \prod_{1 \leq i \leq n} \frac{((\mu f/a) x_i^{-1})_{|\delta|}}{((\mu b_i f/a) x_i^{-1})_{|\delta|}} \\
& \times \frac{1}{(\mu df/a, \mu ef/a)_{|\delta|}} \prod_{1 \leq k \leq m} ((aq/df) y_k, (aq/ef) y_k)_{\delta_k},
\end{aligned}$$

where  $\mu = a^{m+2} q^{m+1} / BCdef^{m+1}$ .

For the justification of this limiting procedure for (3.34), see Remark 3.7.

In the case when  $m = 1$  and  $y_1 = 1$ , (3.34) reduces to

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{N}^n} x_1^{-\gamma_1} \dots x_n^{-\gamma_n} \left( \frac{a^2 q^2}{Bcdef} \right)^{|\gamma|} q^{e_2(\gamma)} \frac{\Delta(x q^{\gamma})}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - a q^{|\gamma| + \gamma_i} x_i}{1 - a x_i} \\
& \times \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(c x_i, d x_i, e x_i)_{\gamma_i}}{((aq/f) x_i)_{\gamma_i}} \\
& \times \frac{(f)_{|\gamma|}}{(aq/c, aq/d, aq/e)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(a x_i)_{|\gamma|}}{((aq/b_i) x_i)_{|\gamma|}} \\
= & \frac{(\mu c f/a, \mu d f/a, \mu e f/a, f)_{\infty}}{(aq/c, aq/d, aq/e, \mu q)_{\infty}} \prod_{1 \leq i \leq n} \frac{(aq x_i, (\mu b_i f/a) x_i^{-1})_{\infty}}{((aq/b_i) x_i, (\mu f/a) x_i^{-1})_{\infty}} \\
& \times {}_{2n+4}W_{2n+3} [\mu; \{(aq/b_i f) x_i\}_n aq/cf, aq/df, aq/ef, \{(\mu f/a) x_i^{-1}\}_n; q; f],
\end{aligned} \tag{3.35}$$

where  $\mu = a^3 q^2 / Bcdef^2$ .

*Remark 3.11.* In the case when  $m = n = 1$  and  $x_1 = y_1 = 1$ , (3.34) reduces to the following transformation formula for nonterminating  ${}_8W_7$  series

$$\begin{aligned}
{}_8W_7 \left[ a; b, c, d, e, f; q; \frac{a^2 q^2}{bcdef} \right] &= \frac{(\mu b f/a, \mu c f/a, \mu d f/a, \mu e f/a, aq, f)_{\infty}}{(aq/b, aq/c, aq/d, aq/e, \mu q, \mu f/a)_{\infty}} \\
&\times {}_8W_7 [\mu; aq/bf, aq/cf, aq/df, aq/ef, \mu f/a; q; f],
\end{aligned} \tag{3.36}$$

where  $\mu = a^3 q^2 / bcdef^2$ , under the convergence condition  $\max(|a^2 q^2 / bcdef|, |f|) < 1$ .

We mention that by assuming  $e = aq/f$  in (3.35), we get the following:

**Corollary 3.2.** Assume that  $\left| \frac{aq}{Bcd} x_i^{-1} \right| < 1$  for all  $i = 1, \dots, n$ . Then

$$\sum_{\gamma \in \mathbb{N}^n} x_1^{-\gamma_1} \dots x_n^{-\gamma_n} \left( \frac{aq}{Bcd} \right)^{|\gamma|} q^{e_2(\gamma)} \frac{\Delta(x q^{\gamma})}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - a q^{|\gamma| + \gamma_i} x_i}{1 - a x_i} \tag{3.37}$$

$$\begin{aligned}
& \times \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} (c x_i, d x_i)_{\gamma_i} \\
& \times [(aq/c, aq/d)_{|\gamma|}]^{-1} \prod_{1 \leq i \leq n} \frac{(a x_i)_{|\gamma|}}{((aq/b_i) x_i)_{|\gamma|}} \\
& = \frac{(aq/Bc, aq/Bd)_{\infty}}{(aq/c, aq/d)_{\infty}} \prod_{1 \leq i \leq n} \frac{(aq x_i, (aq b_i / Bcd) x_i^{-1})_{\infty}}{((aq/b_i) x_i, (aq/Bcd) x_i^{-1})_{\infty}}.
\end{aligned}$$

*Remark 3.12.* In the case when  $n = 1$  and  $x_1 = 1$ , (3.37) reduces to the Bailey summation formula for nonterminating  ${}_6W_5$  series (See [7])

$${}_6W_5 \left[ a; b, c, d, q; \frac{aq}{bcd} \right] = \frac{(aq, aq/cd, aq/bd, aq/bc)_{\infty}}{(aq/bcd, aq/b, aq/c, aq/d)_{\infty}}. \quad (3.38)$$

$A_n$  nonterminating  ${}_6W_5$  summation formula (3.37) is due to S.C. Milne and has appeared as Theorem 4.27 in [20] with a different expression (See also Theorem A.4 in Milne-Newcomb [25]). Note that (3.37) can also be obtained by taking the limit  $N \rightarrow \infty$  in  $A_n$  Jackson summation formula (2.16).

### Terminating ${}_8W_7$ transformation

**Proposition 3.6.**

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{N}^n} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \left( \frac{a^{m+1} q^{m+N+1}}{BCd^m e} \right)^{|\gamma|} q^{-e_2(\gamma)} \frac{\Delta(xq^{\gamma})}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - aq^{|\gamma| + \gamma_i} x_i}{1 - ax_i} \\
& \times \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{(c_k x_i y_k)_{\gamma_i}}{((aq/d) x_i y_k)_{\gamma_i}} \\
& \times \prod_{1 \leq i \leq n} \frac{1}{(aq^{N+1} x_i, (aq/e) x_i)_{\gamma_i}} \\
& \times (q^{-N}, e)_{|\gamma|} \prod_{1 \leq i \leq n} \frac{(a x_i)_{|\gamma|}}{((aq/b_i) x_i)_{|\gamma|}} \prod_{1 \leq k \leq m} \frac{(dy_k^{-1})_{|\gamma|}}{((aq/c_k) y_k^{-1})_{|\gamma|}} \\
& = \prod_{1 \leq k \leq m} \frac{((aq/c_k e) y_k^{-1}, dy_k^{-1})_N}{((aq/c_k) y_k^{-1}, (d/e) y_k^{-1})_N} \prod_{1 \leq i \leq n} \frac{((aq/b_i e) x_i, aq x_i)_N}{((aq/b_i) x_i, (aq/e) x_i)_N} \\
& \times \sum_{\delta \in \mathbb{N}^m} y_1^{\delta_1} \cdots y_m^{\delta_m} \left( \frac{BCd^{m-1}}{a^m} \right)^{|\delta|} q^{-e_2(\delta)} \frac{\Delta(yq^{\delta})}{\Delta(y)} \prod_{1 \leq k \leq m} \frac{1 - (q^{-N} e/d) q^{|\delta| + \delta_k} y_k}{1 - (q^{-N} e/d) y_k} \\
& \times \prod_{1 \leq k, l \leq m} \frac{((aq/c_l d) y_k / y_l)_{\delta_k}}{(q y_k / y_l)_{\delta_k}} \prod_{1 \leq i \leq n, 1 \leq k \leq m} \frac{((aq/b_i d) x_i y_k)_{\delta_k}}{((aq/d) x_i y_k)_{\delta_k}} \\
& \times \prod_{1 \leq k \leq m} \frac{1}{((e/d) y_k, (q^{1-N}/d) y_k)_{\delta_k}} \\
& \times (q^{-N}, e)_{|\delta|} \prod_{1 \leq k \leq m} \frac{((q^{-N} e/d) y_k)_{|\delta|}}{((q^{-N} c_k e/a) y_k)_{|\delta|}} \prod_{1 \leq i \leq n} \frac{((q^{-N} e/a) x_i^{-1})_{|\delta|}}{((q^{-N} b_i e/a) x_i^{-1})_{|\delta|}}.
\end{aligned} \quad (3.39)$$

*Proof.* Relabel  $e$  as  $\mu f q^N$ , i.e. interchange  $e \leftrightarrow \mu f q^N$ . Then let  $d$  tend to infinity. Finally, by relabeling  $f$  as  $d$ , we arrive at (3.39).  $\square$

In the case when  $m = 1$  and  $y_1 = 1$ , (3.39) reduces to

$$\sum_{\gamma \in \mathbb{N}^n} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \left( \frac{a^2 q^{2+N}}{Bcde} \right)^{|\gamma|} q^{-e_2(\gamma)} \frac{\Delta(xq^{\gamma})}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - aq^{|\gamma| + \gamma_i} x_i}{1 - ax_i} \quad (3.40)$$

$$\begin{aligned}
& \times \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(c x_i)_{\gamma_i}}{((a q / d) x_i, (a q / e) x_i, a q^{N+1} x_i)_{\gamma_i}} \\
& \times \frac{(q^{-N}, d, e)_{|\gamma|}}{(a q / c)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(a x_i)_{|\gamma|}}{((a q / b_i) x_i)_{|\gamma|}} \\
& = \frac{(a q / c e, d)_N}{(a q / c, d / e)_N} \prod_{1 \leq i \leq n} \frac{((a q / b_i e) x_i, a q x_i)_N}{((a q / b_i) x_i, (a q / e) x_i)_N} \\
& \times {}_{2n+6}W_{2n+5} \left[ q^{-N} e / d; \{(a q / b_i d) x_i\}_n, \{(q^{-N} e / a) x_i^{-1}\}_n, a q / c d, e, q^{-N}; q; \frac{B c}{a} \right].
\end{aligned}$$

*Remark 3.13.* In the case when  $m = n = 1$  and  $x_1 = y_1 = 1$ , (3.39) reduces to transformation formula for terminating  ${}_8W_7$  series

$$\begin{aligned}
& {}_8W_7 \left[ a; b, c, d, e, q^{-N}; q; \frac{a^2 q^{N+2}}{b c d e} \right] \\
& = \frac{(a q / b e, a q / c e, a q, d)_N}{(a q / b, a q / c, a q / e, d / e)_N} {}_8W_7 \left[ q^{-N} e / d; a q / b d, q^{-N} e / a, a q / c d, e, q^{-N}; q; \frac{b c}{a} \right].
\end{aligned} \tag{3.41}$$

Let  $a q = c d$  in (3.40). Then, by rearranging parameters, we have

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{N}^n} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \left( \frac{a q^{1+N}}{B c} \right)^{|\gamma|} q^{-e_2(\gamma)} \frac{\Delta(x q^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - a q^{|\gamma| + \gamma_i} x_i}{1 - a x_i} \\
& \times \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} [((a q / c) x_i, a q^{1+N} x_i)_{\gamma_i}]^{-1} \\
& \times (q^{-N}, c)_{|\gamma|} \prod_{1 \leq i \leq n} \frac{(a x_i)_{|\gamma|}}{((a q / b_i) x_i)_{|\gamma|}} = \prod_{1 \leq i \leq n} \frac{((a q / b_i c) x_i, a q x_i)_N}{((a q / b_i) x_i, (a q / c) x_i)_N}.
\end{aligned} \tag{3.42}$$

*Remark 3.14.* In the case when  $n = 1$  and  $x_1 = 1$ , (3.42) reduces to the Rogers' summation formula for terminating  ${}_6W_5$  series (3.6).

As we have seen in this section, one can recover our previous results in [13] from the balanced duality transformation formula (2.12) by limiting procedures. Combining with results in [13] and Rosengren [26], one can consider the master formula for multiple basic hypergeometric transformations of type  $A$  with different dimensions presented in this section to either of multiple basic Euler transformation (2.14), Sears transformation (3.14), (non balanced) duality transformation (3.1) and balanced duality transformation formula (2.12).

## 4 $A_n$ hypergeometric transformations

In this section, we present several hypergeometric transformation formulas with same dimension  $n$  (for multiple basic hypergeometric series of type  $A_n$ ) by combining some special cases of hypergeometric transformation with different dimensions which we have obtained in the previous section. It contains new transformation formulas and some of these are previously known by Milne and his collaborators (see [23], [24], and [22]). However, our proofs of them are completely different from theirs and seem to be simpler.

### 4.1 $A_n$ Watson transformations

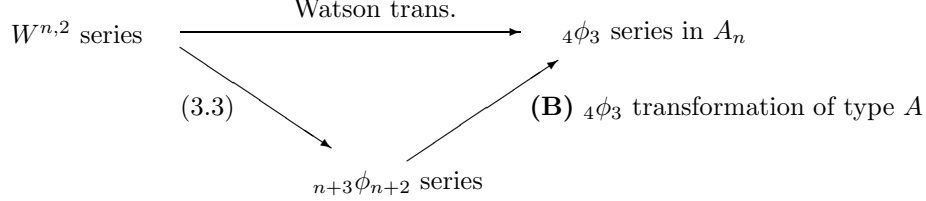
In this subsection and next, we derive several  $A_n$  generalization of the Watson transformation formula between terminating  ${}_8W_7$  series and terminating balanced  ${}_4\phi_3$  series ( (2.5.1) in [7] ):

$${}_8W_7 \left[ a; b, c, d, e, q^{-N}; q; \frac{a^2 q^{2+N}}{b c d e} \right] \tag{4.1}$$



$$= \frac{(aq, aq/de)_N}{(aq/d, aq/e)_N} {}_4\phi_3 \left[ \begin{matrix} q^{-N}, d, e, aq/bc \\ aq/b, aq/c, deq^{-N}/a \end{matrix}; q; q \right].$$

Especially, we give two types of  $A_n$  Watson transformation formula whose series in the left hand side are expressible in terms of  $W^{n,2}$  series here. We will use a special case ( $m = 1$  (3.3)) of the (non-balanced) duality transformation formula (3.1) and special cases of  ${}_4\phi_3$  series of type  $A$  to the identities below. To be precise, we produce them according to the following diagram.



## The 1st one

**Proposition 4.1.**

$$\begin{aligned}
 W^{n,2} \left( \begin{matrix} \{b_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; c, d; e, q^{-N}; q; \frac{a^2 q^{N+2}}{Bcde} \right) & \quad (4.2) \\
 = \frac{(aq/Bd)_N}{(aq/d)_N} \prod_{1 \leq i \leq n} \frac{(ax_i)_N}{((aq/b_i)x_i)_N} \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \\
 \times \frac{(q^{-N}, aq/ce)_{|\gamma|}}{(Bdq^{-N}/a, aq/c)_{|\gamma|}} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(dx_i)_{\gamma_i}}{((aq/e)x_i)_{\gamma_i}}.
 \end{aligned}$$

*Proof.* We combine (3.3) and  $n = 1$  and  $m \rightarrow n$  case of (3.14)

$$\begin{aligned}
 n+3\phi_{n+2} \left[ \begin{matrix} q^{-N}, a, c, \{u_i\}_n \\ e, \{v_i\}_n, acUq^{1-N}/eV \end{matrix}; q, q \right] & \quad (4.3) \\
 = \frac{(e, eV/acU)_N}{(e/a, eV/cU)_N} \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(vq^\gamma)}{\Delta(v)} \\
 \times \frac{(q^{-N}, a)_{|\gamma|}}{(q^{1-N}a/e, eV/cU)_{|\gamma|}} \prod_{1 \leq i, j \leq n} \frac{(v_i/u_j)_{\gamma_i}}{(qv_i/v_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(v_i/c)_{\gamma_i}}{(v_i)_{\gamma_i}},
 \end{aligned}$$

here we give in the useful form for the present proof and latter uses in this section, as **(B)** in the above diagram. Note that both of the series in the right hand side of (3.3) and that in the left hand side of (4.3) satisfy same condition as basic hypergeometric series: namely they are terminating balanced  $n+3\phi_{n+2}$  series. On the set of variables

$$(\{(aq/b_i e)x_i\}_n, aq/ce, aq/de, a^2 q^2/Bcde, \{(aq/e)x_i\}_n), \quad (4.4)$$

we consider the following change of variables:

$$\begin{aligned}
 \tilde{a} &= aq/ce, & \tilde{c} &= aq/de, & \tilde{e} &= a^2 q^2/Bcde \\
 \tilde{u}_i &= (aq/b_i e)x_i, & \tilde{v}_i &= (aq/e)x_i & (i &= 1, \dots, n).
 \end{aligned} \quad (4.5)$$

For given function  $\psi$ , we denote by  $\tilde{\psi} = \psi(a, c, e, \{u_i\}_n, \{v_i\}_n)$  the function that is obtained by replacing the variables  $(a, c, e, \{u_i\}_n, \{v_i\}_n)$  by  $(\tilde{a}, \tilde{c}, \dots)$ . In this case, the change of variables

(4.5) is a transposition inside of each sets of numerator parameters in  $_{n+3}\phi_{n+2}$  series and of denominator parameters. Hence, the right hand side of (3.3) is invariant under this change of variables. By applying (4.3) to the series the right hand side in (3.3), this invariance implies (4.2).  $\square$

We also give similar transformation formula for multiple series which terminates with respect to a certain multi-index. In this paper, we call such transformation formulas as *rectangular* and transformations for the multiple series which terminates with respect to the length of multi-indices as *triangular*.

**Corollary 4.1.**

$$\begin{aligned} W^{n,2} \left( \begin{matrix} \{q^{-m_i}\}_n \\ \{x_i\}_n \end{matrix} \middle| a; c, d; b, e; q; \frac{a^2 q^{|M|+2}}{bcde} \right) \\ = \frac{(aq/bd)_{|M|}}{(aq/d)_{|M|}} \prod_{1 \leq i \leq n} \frac{(aqx_i)_{m_i}}{((aq/b)x_i)_{m_i}} \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \\ \times \frac{(b, aq/ce)_{|\gamma|}}{(bdq^{-|M|}/a, aq/c)_{|\gamma|}} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(dx_i)_{\gamma_i}}{((aq/e)x_i)_{\gamma_i}}. \end{aligned} \quad (4.6)$$

*Proof.* We first write the product factor in the right hand side of (4.2) as a quotient of infinite products using (2.1). Set  $b_i = q^{-m_i}$  in (4.2), and notice that (4.6) is true for  $b = q^{-N}$  for all nonnegative integer  $N$ . Clear the denominators in (4.6). Then we find that it is a polynomial identity in  $b^{-1}$  with an infinite number of roots. Thus, (4.6) is true for arbitrary  $b$ .  $\square$

All the corollaries in this section can be proved by similar arguments from the formulas in the preceding propositions. So, hereafter we will not repeat this procedure in the rest of this paper.

*Remark 4.1.* (4.6) has appeared in Theorem 6.1 of Milne and Lilly [23].

## The 2nd one

**Proposition 4.2.**

$$\begin{aligned} W^{n,2} \left( \begin{matrix} \{b_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; c, d; e, q^{-N}; q; \frac{a^2 q^{N+2}}{Bcde} \right) \\ = \prod_{1 \leq i \leq n} \frac{(ax_i, (aq/b_i e)x_i)_N}{((aq/b_i)x_i, (aq/e)x_i)_N} \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(zq^\gamma)}{\Delta(z)} \\ \times \frac{(q^{-N}, e)_{|\gamma|}}{(aq/c, aq/d)_{|\gamma|}} \prod_{1 \leq i, j \leq n} \frac{(b_j z_i/z_j)_{\gamma_i}}{(qz_i/z_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{((aq/cd)z_i)_{\gamma_i}}{((Beq^{-N}/a)z_i)_{\gamma_i}}, \end{aligned} \quad (4.7)$$

where  $z_i = b_i/Bx_i$  for  $1 \leq i \leq n$ .

*Proof.* We combine (3.3) and  $n = 1$  and  $m \rightarrow n$  case of (3.26)

$$\begin{aligned} {}_{n+3}\phi_{n+2} \left[ \begin{matrix} q^{-N}, a, c, \{u_i\}_n \\ e, \{v_i\}_n, acUq^{1-N}/eV; q, q \end{matrix} \right] \\ = \frac{(eV/aU, eV/cU)_N}{(eV/acU, e)_{|N|}} \prod_{1 \leq i \leq n} \frac{(u_i)_N}{(v_i)_N} \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(u^{-1}q^\gamma)}{\Delta(u^{-1})} \\ \times \frac{(q^{-N}, eV/acU)_{|\gamma|}}{(eV/aU, eV/cU)_{|\gamma|}} \prod_{1 \leq i, j \leq n} \frac{(v_j/u_i)_{\gamma_i}}{(qu_j/u_i)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(e/u_i)_{\gamma_i}}{(q^{1-N}/u_i)_{\gamma_i}}, \end{aligned} \quad (4.8)$$

here we present in a modified form, as **(B)**. Notice that both of the series in the right hand side of (3.3) and that in the left hand side of (4.8) are terminating balanced  $_{n+3}\phi_{n+2}$  series. In this case, we consider the following change of variables

$$\begin{aligned} \tilde{a} &= aq/ce, & \tilde{c} &= aq/de, & \tilde{e} &= a^2q^2/Bcde \\ \tilde{u}_i &= (aq/b_ie)x_i, & \tilde{v}_i &= (aq/e)x_i & (i &= 1, \dots, n). \end{aligned} \quad (4.9)$$

Since this change of variables is same as (4.5) in the proof of Proposition 4.1. one can obtain the desired identity (4.7) by plugging (4.8) to the series in the right hand side of (3.3) according to this change of variables.  $\square$

## Rectangular version

### Corollary 4.2.

$$\begin{aligned} W^{n,2} \left( \begin{matrix} \{q^{-m_i}\}_n \\ \{x_i\}_n \end{matrix} \middle| a; c, d; b, e; q; \frac{a^2q^{|M|+2}}{bcde} \right) \\ = \prod_{1 \leq i \leq n} \frac{(aqx_i, (aq/be)x_i)_{m_i}}{((aq/b)x_i, (aq/e)x_i)_{m_i}} \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(zq^\gamma)}{\Delta(z)} \\ \times \frac{(b, e)_{|\gamma|}}{(aq/c, aq/d)_{|\gamma|}} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j}z_i/z_j)_{\gamma_i}}{(qz_i/z_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{((aq/ce)x_i)_{\gamma_i}}{((beq^{-|M|}/a)x_i)_{\gamma_i}}, \end{aligned} \quad (4.10)$$

where  $z_i = q^{-m_i+|M|}x_i^{-1}$  for  $i = 1, \dots, n$ .

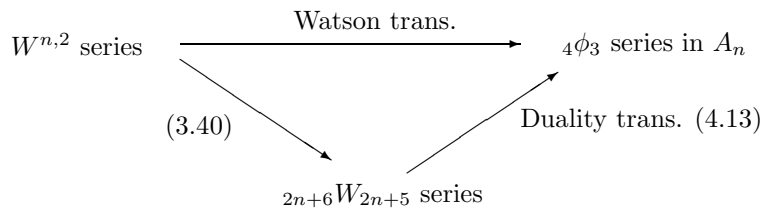
*Remark 4.2.* In the case when  $n = 1$  and  $x_1 = 1$ , all of (4.2), (4.6), (4.7) and (4.10) reduce to the Watson transformation formula (4.1). (4.7) can be proved by a similar limiting procedure as the previous section from the following  $A_n$  Bailey transformation formula for terminating balanced  $_{10}W_9$  series:

$$\begin{aligned} W^{n,3} \left( \begin{matrix} \{e_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; b, c, d; q^{-N}, f, a\lambda q^{1+N}/Ef; q \right) \\ = \prod_{1 \leq i \leq n} \frac{(aqx_i, (aq/e_i f)x_i, (\lambda q/e_i)z_i, (\lambda q/f)z_i)_N}{((aq/e_i)x_i, (aq/f)x_i, \lambda qz_i, (\lambda q/e_i f)z_i)_N} \\ \times W^{n,3} \left( \begin{matrix} \{e_i\}_n \\ \{z_i\}_n \end{matrix} \middle| \lambda; aq/cd, aq/bd, aq/bc; q^{-N}, f, a\lambda q^{1+N}/Ef; q \right) \end{aligned} \quad (4.11)$$

where  $\lambda = a^2q/bcd$  and  $z_i = e_i/Ex_i$  for  $1 \leq i \leq n$ , which has first appeared as (4.36) in [15], and by rearranging the parameters.

## 4.2 Another type of $A_n$ Watson transformation

Here, we present a yet another  $A_n$  Watson transformation with a different form in the series in both sides from those in the previous subsection. We will use the  $m = 1$  case of the terminating  ${}_8W_7$  transformation (3.40) of type  $A$  in Section 3.3. and the  $n = 1$  case of the duality transformation formula in a modified form to produce. We construct it according to the following procedure:



**Proposition 4.3.**

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{N}^n} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \left( \frac{a^2 q^{2+N}}{Bcde} \right)^{|\gamma|} q^{-e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - aq^{|\gamma| + \gamma_i} x_i}{1 - ax_i} \\
& \quad \times \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{((aq/d)x_i, (aq/e)x_i, aq^{N+1}x_i)_{\gamma_i}} \\
& \quad \times \frac{(q^{-N}, d, e)_{|\gamma|}}{(aq/c)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(ax_i)_{|\gamma|}}{((aq/b_i)x_i)_{|\gamma|}} \\
& = \frac{(aq/Bc)_N}{(aq/c)_N} \prod_{1 \leq i \leq n} \frac{(aqx_i)_N}{((aq/b_i)x_i)_N} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(xq^\delta)}{\Delta(x)} \\
& \quad \times \frac{(q^{-N})_{|\delta|}}{(q^{-N}Bc/a)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\delta_i}}{(qx_i / x_j)_{\delta_i}} \prod_{1 \leq i \leq n} \frac{((aq/de)x_i, cx_i)_{\delta_i}}{((aq/d)x_i, (aq/e)x_i)_{\delta_i}}.
\end{aligned} \tag{4.12}$$

*Proof.* We use (3.40) and (3.4) with a modified form

$$\begin{aligned}
& {}_{2n+6}W_{2n+5} \left[ a; b, \{u_i\}_n, d, \{v_i\}_n, q^{-N}; q; \frac{a^{n+1}q^{N+n+1}}{bdUV} \right] \\
& = \frac{(a^{n+1}q^{n+1}/bdUV, aq)_N}{(aq/b, aq/d)_N} \prod_{1 \leq i \leq n} \frac{(v_i)_N}{(aq/u_i)_N} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(v^{-1}q^\delta)}{\Delta(v^{-1})} \\
& \quad \times \frac{(q^{-N})_{|\delta|}}{(a^{n+1}q^{n+1}/bdUV)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(aq/u_j v_i)_{\delta_i}}{(qv_j/v_i)_{\delta_i}} \prod_{1 \leq i \leq n} \frac{(aq/bv_i, aq/dv_i)_{\delta_i}}{(aq/v_i, q^{1-N}/v_i)_{\delta_i}}.
\end{aligned} \tag{4.13}$$

to obtain. It is not hard to see that both of the series in the right hand side of (3.40) and in the left hand side of (4.13) satisfy the same condition: they are  ${}_{2n+6}W_{2n+5}$  series and very-well-poised-balanced. We consider the following change of variables:

$$\begin{aligned}
\tilde{a} &= q^{-N}e/d, & \tilde{b} &= aq/cd, & \tilde{d} &= e, \\
\tilde{u}_i &= (aq/b_i d)x_i, & \tilde{v}_i &= (q^{-N}e/a)x_i^{-1} & (i &= 1, \dots, n),
\end{aligned} \tag{4.14}$$

which is a transposition of the variables in  ${}_{2n+6}W_{2n+5}$  series in (3.40). Note that  ${}_{r+3}W_{r+2}$  series is symmetric with respect to the variables  $a_1, \dots, a_r$ . So the series in the right hand side of (3.40) is invariant under this change of variables. This invariance implies (4.12) by applying (4.13) according to the change of variables (4.14).  $\square$

## Rectangular version

**Corollary 4.3.**

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{N}^n} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \left( \frac{a^2 q^{2+|M|}}{bcde} \right)^{|\gamma|} q^{-e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - aq^{|\gamma| + \gamma_i} x_i}{1 - ax_i} \\
& \quad \times \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{((aq/b)x_i, (aq/d)x_i, (aq/e)x_i)_{\gamma_i}} \\
& \quad \times \frac{(b, d, e)_{|\gamma|}}{(aq/c)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(ax_i)_{|\gamma|}}{((aq^{m_i}x_i)_{|\gamma|}} \\
& = \frac{(aq/bc)_{|M|}}{(aq/c)_{|M|}} \prod_{1 \leq i \leq n} \frac{(aqx_i)_{m_i}}{((aq/b)x_i)_{m_i}} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(xq^\delta)}{\Delta(x)}
\end{aligned} \tag{4.15}$$

$$\times \frac{(b)_{|\delta|}}{(q^{-|M|}bc/a)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_i/x_j)_{\delta_i}}{(qx_i/x_j)_{\delta_i}} \prod_{1 \leq i \leq n} \frac{((aq/de)x_i, cx_i)_{\delta_i}}{((aq/d)x_i, (aq/e)x_i)_{\delta_i}}$$

*Remark 4.3.* In the case when  $n = 1$  and  $x_1 = 1$ , (4.12) and (4.15) reduce to the Watson transformation formula (4.1).

*Remark 4.4.* In a similar fashion as we yield (4.12), we also obtain the following transformation formula between  $A_n {}_8W_7$  series and  $A_n$  terminating balanced  ${}_4\phi_3$  series:

$$\begin{aligned} & \sum_{\gamma \in \mathbb{N}^n} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \left( \frac{a^2 q^{2+N}}{Bcde} \right)^{|\gamma|} q^{-e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - aq^{|\gamma| + \gamma_i} x_i}{1 - ax_i} \\ & \times \prod_{1 \leq i, j \leq n} \frac{(b_j x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{((aq/d)x_i, (aq/e)x_i, aq^{N+1}x_i)_{\gamma_i}} \\ & \times \frac{(q^{-N}, d, e)_{|\gamma|}}{(aq/c)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(ax_i)_{|\gamma|}}{((aq/b_i)x_i)_{|\gamma|}} \\ & = B^N \frac{(aq/Bc)_N}{(aq/c)_N} \prod_{1 \leq i \leq n} \frac{(aqx_i, (aq/b_i d)x_i, (aq/b_i e)x_i)_N}{((aq/b_i, (aq/d)x_i, (aq/e)x_i)x_i)_N} \\ & \times \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(x^{-1}q^\delta)}{\Delta(x^{-1})} \prod_{1 \leq i, j \leq n} \frac{(b_j x_j/x_i)_{\delta_i}}{(qx_j/x_i)_{\delta_i}} \\ & \times \frac{(q^{-N})_{|\delta|}}{(q^{-N}Bc/a)_{|\delta|}} \prod_{1 \leq i \leq n} \frac{((q^{-1-N}b_i cde/a^2)x_i^{-1}, (q^{-N}b_i/a)x_i^{-1})_{\delta_i}}{((q^{-N}b_i d/a)x_i^{-1}, (q^{-N}b_i e/a)x_i^{-1})_{\delta_i}}. \end{aligned} \quad (4.16)$$

We obtain it by applying the change of variables:

$$\begin{aligned} \tilde{a} &= q^{-N}e/d, & \tilde{b} &= aq/cd, & \tilde{d} &= e, \\ \tilde{u}_i &= (q^{-N}e/a)x_i^{-1}, & \tilde{v}_i &= (aq/b_i d)x_i, & (i &= 1, \dots, n). \end{aligned} \quad (4.17)$$

The rectangular version of (4.16) is given by

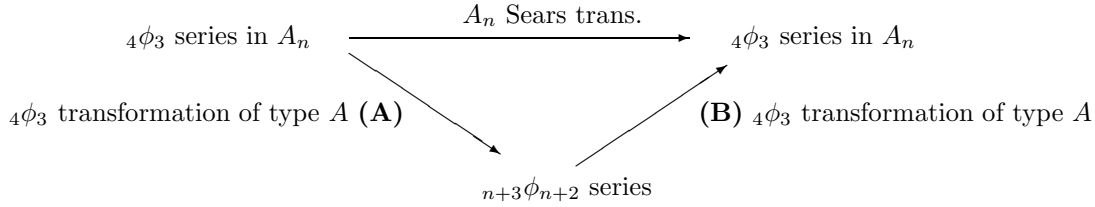
$$\begin{aligned} & \sum_{\gamma \in \mathbb{N}^n} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \left( \frac{a^2 q^{2+|M|}}{bcde} \right)^{|\gamma|} q^{-e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - aq^{|\gamma| + \gamma_i} x_i}{1 - ax_i} \\ & \times \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{((aq/b)x_i, (aq/d)x_i, (aq/e)x_i)_{\gamma_i}} \\ & \times \frac{(b, d, e)_{|\gamma|}}{(aq/c)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(ax_i)_{|\gamma|}}{((aq^{1+m_i})x_i)_{|\gamma|}} \\ & = b^{|M|} \frac{(aq/bc)_{|M|}}{(aq/c)_{|M|}} \prod_{1 \leq i \leq n} \frac{(aqx_i, (aq/bd)x_i, (aq/be)x_i)_{m_i}}{((aq/b)x_i, (aq/d)x_i, (aq/e)x_i)x_i)_{m_i}} \\ & \times \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(x^{-1}q^\delta)}{\Delta(x^{-1})} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_j/x_i)_{\delta_i}}{(qx_j/x_i)_{\delta_i}} \\ & \times \frac{(b)_{|\delta|}}{(q^{-|M|}bc/a)_{|\delta|}} \prod_{1 \leq i \leq n} \frac{((q^{-1-m_i}bcde/a^2)x_i^{-1}, (q^{-m_i}b/a)x_i^{-1})_{\delta_i}}{((q^{-m_i}bd/a)x_i^{-1}, (q^{-m_i}be/a)x_i^{-1})_{\delta_i}}. \end{aligned} \quad (4.18)$$

In the case when  $n = 1$  and  $x_1 = 1$ , the transformation (4.16) and (4.18) reduces to the following transformation formula between terminating  ${}_8W_7$  series and terminating balanced  ${}_4\phi_3$  series:

$$\begin{aligned} & {}_8W_7 \left[ a; b, c, d, e, q^{-N}; q; \frac{a^2 q^{2+N}}{bcde} \right] \\ & = b^N \frac{(aq, aq/bc, aq/bd, aq/be)_N}{(aq/b, aq/c, aq/d, aq/e)_N} {}_4\phi_3 \left[ \begin{matrix} q^{-N}, b, q^{-1-N}bcde/a^2, q^{-N}b/a \\ bcq^{-N}/a, bdq^{-N}/a, beq^{-N}/a \end{matrix}; q; q \right]. \end{aligned} \quad (4.19)$$

### 4.3 $A_n$ Sears transformations

In this and next subsection, we present some  $A_n$  generalizations of the Sears transformation formula for terminating balanced  ${}_4\phi_3$  series (3.16). In particular, we will prove two  $A_n$  Sears transformations whose form of the series in both sides are same as that in the right hand side of the  $A_n$  Watson transformation formulas in Section 4.1. We produce these identities by combining certain special cases of  ${}_4\phi_3$  series of type A in Section 3.2. Our way to prove them is figured as the following diagram:



**The 1st one**

**Proposition 4.4.**

$$\begin{aligned}
 & \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(c x_i)_{\gamma_i}}{(d x_i)_{\gamma_i}} \\
 & \times \frac{(q^{-N}, a)_{|\gamma|}}{(e, a B c q^{1-N} / d e)_{|\gamma|}} = \frac{(e / B, d e / a c)_N}{(e, d e / a B c)_N} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(xq^\delta)}{\Delta(x)} \\
 & \times \frac{(q^{-N}, d / c)_{|\delta|}}{(d e / a c, q^{1-N} B / e)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\delta_i}}{(q x_i / x_j)_{\delta_i}} \prod_{1 \leq i \leq n} \frac{((d / a) x_i)_{\delta_i}}{(d x_i)_{\delta_i}}.
 \end{aligned} \tag{4.20}$$

*Proof.* We use (3.27) as **(A)** and (4.8) as **(B)** in the above diagram. It is not hard to see that both of the series in the right hand side of (3.27) and in the left hand side in (4.8) are terminating balanced  ${}_{n+3}\phi_{n+2}$  series. We consider the following change of variables:

$$\begin{aligned}
 \tilde{a} &= e/a, & \tilde{c} &= f/a, & \tilde{e} &= d e f / a^2 B c. \\
 \tilde{u}_i &= \frac{e f}{a b_i c} z_i, & \tilde{v}_i &= \frac{e f}{a c} z_i & (1 \leq i \leq n).
 \end{aligned} \tag{4.21}$$

Note that the series in the right hand side of (3.27) is invariant under this change of variables. Applying (4.8) to the  ${}_{n+3}\phi_{n+2}$  series in the right hand side in (3.27) leads to the desired result (4.20).  $\square$

In [12], we have already shown that (4.20) can also be obtained by combining (3.15) as **(A)** and (4.3) as **(B)**. In this case, The change of variables is given by

$$\begin{aligned}
 \tilde{a} &= d/c, & \tilde{c} &= a, & \tilde{e} &= d e / B c. \\
 \tilde{u}_i &= \frac{d}{b_i} x_i, & \tilde{v}_i &= d x_i & (1 \leq i \leq n).
 \end{aligned} \tag{4.22}$$

**Rectangular version**

**Corollary 4.4.**

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{(dx_i)_{\gamma_i}} \\
& \times \frac{(a, b)_{|\gamma|}}{(e, abcq^{1-|M|}/de)_{|\gamma|}} = \frac{(e/b, de/ac)_{|M|}}{(e, de/abc)_{|M|}} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(xq^\delta)}{\Delta(x)} \\
& \times \frac{(b, d/c)_{|\delta|}}{(de/ac, q^{1-|M|}b/e)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_i / x_j)_{\delta_i}}{(qx_i / x_j)_{\delta_i}} \prod_{1 \leq i \leq n} \frac{((d/a)x_i)_{\delta_i}}{(dx_i)_{\delta_i}}.
\end{aligned} \tag{4.23}$$

**The 2nd one**

**Proposition 4.5.**

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{(dx_i)_{\gamma_i}} \\
& \times \frac{(q^{-N}, a)_{|\gamma|}}{(e, aBcq^{1-N}/de)_{|\gamma|}} = \frac{(de/ac)_N}{(de/aBc)_N} \prod_{1 \leq i \leq n} \frac{((d/b_i)x_i)_N}{(dx_i)_N} \\
& \times \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(zq^\delta)}{\Delta(z)} \frac{(q^{-N}, e/a)_{|\delta|}}{(de/ac, e)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(b_j z_i / z_j)_{\delta_i}}{(qz_i / z_j)_{\delta_i}} \prod_{1 \leq i \leq n} \frac{((e/c)z_i)_{\delta_i}}{((q^{1-N}B/d)z_i)_{\delta_i}},
\end{aligned} \tag{4.24}$$

where  $z_i = b_i/Bx_i$  for  $1 \leq i \leq n$ .

*Proof.* We use (3.15) as **(A)** and (4.8) as **(B)**. In this case, the change of variables is given by

$$\begin{aligned}
\tilde{a} &= a, & \tilde{c} &= d/c, & \tilde{e} &= de/Bc, \\
\tilde{u}_i &= \frac{d}{b_i} x_i, & \tilde{v}_i &= dx_i & (1 \leq i \leq n).
\end{aligned} \tag{4.25}$$

For the rest, one can easily verify in a similar way as the proof of Proposition 4.4.  $\square$

Note that (4.24) can also be obtained by combining (3.27) as **(A)** and (4.3) as **(B)**. In this case, the change of variables is given by

$$\begin{aligned}
\tilde{a} &= e/a, & \tilde{c} &= f/a, & \tilde{e} &= ef/aB, \\
\tilde{u}_i &= \frac{ef}{ab_i c} z_i, & \tilde{v}_i &= \frac{ef}{ac} z_i & (1 \leq i \leq n).
\end{aligned} \tag{4.26}$$

**Rectangular version**

**Corollary 4.5.**

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{(dx_i)_{\gamma_i}} \\
& \times \frac{(a, b)_{|\gamma|}}{(e, abcq^{1-|M|}/de)_{|\gamma|}} = \frac{(de/ac)_{|M|}}{(de/abc)_{|M|}} \prod_{1 \leq i \leq n} \frac{((d/b)x_i)_{m_i}}{(dx_i)_{m_i}} \\
& \times \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(zq^\delta)}{\Delta(z)} \frac{(e/a, b)_{|\delta|}}{(de/ac, e)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} z_i / z_j)_{\delta_i}}{(qz_i / z_j)_{\delta_i}} \prod_{1 \leq i \leq n} \frac{((e/c)z_i)_{\delta_i}}{((bq^{1-|M|}/d)z_i)_{\delta_i}}.
\end{aligned} \tag{4.27}$$

where  $z_i = q^{-m_i+|M|}x_i^{-1}$  for  $1 \leq i \leq n$ .

*Remark 4.5.* In the case when  $n = 1$  and  $x_1 = 1$ , all of  $A_n$  terminating balanced  ${}_4\phi_3$  transformation formulas (4.20), (4.23), (4.24) and (4.27) reduce to the Sears transformation (3.16). Especially, (4.20) has already appeared in our previous work [12] and (4.27) is originally due to Milne and Lilly (Theorem 6.5 in [23]). Note also that (4.20) can be proved by duplicating (4.24)

*Remark 4.6.* Two  $A_n$  Watson transformations (4.2) and (4.7) transpose to one another by  $A_n$  Sears transformation formula (4.24). Similarly, two  $A_n$  Watson transformations (4.6) and (4.10) transpose to one another by  $A_n$  Sears transformation formula (4.27).

#### 4.4 Another type of $A_n$ Sears transformation

Here, we give a yet another type of  $A_n$  Sears transformation formula whose form of the series in both sides are the same as that in the right hand side of  $A_n$  Watson transformation formulas in Section 4.2. We use the  $m = 1$  case (3.8) of the inversion of the duality transformation (3.7) and a certain special case (4.13) of the duality transformation formula (3.1). Our road map is as follows:

$$\begin{array}{ccc}
 {}_4\phi_3 \text{ series in } A_n & \xrightarrow{A_n \text{ Sears trans.}} & {}_4\phi_3 \text{ series in } A_n \\
 \downarrow (3.8) & & \uparrow (4.13) \\
 {}_{2n+6}W_{2n+5} \text{ series} & \xrightarrow{\text{transposition}} & {}_{2n+6}W_{2n+5} \text{ series}
 \end{array}$$

**Proposition 4.6.**

$$\begin{aligned}
 & \sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(b x_i, c x_i)_{\gamma_i}}{(e x_i, (A b c q^{1-N} / d e) x_i)_{\gamma_i}} \\
 & \times \frac{(q^{-N})_{|\gamma|}}{(d)_{|\gamma|}} = \prod_{1 \leq i \leq n} \frac{((d e / b c) z_i, (e / a_i) x_i)_N}{((d e / a_i b c) z_i, e x_i)_N} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(z q^\delta)}{\Delta(z)} \\
 & \times \frac{(q^{-N})_{|\delta|}}{(d)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(a_j z_i / z_j)_{\delta_i}}{(q z_i / z_j)_{\delta_i}} \prod_{1 \leq i \leq n} \frac{((d / b) z_i, (d / c) z_i)_{\delta_i}}{((d e / b c) z_i, (A q^{1-N} / e) z_i)_{\delta_i}},
 \end{aligned} \tag{4.28}$$

where  $z_i = a_i / A x_i$  for  $i = 1, 2, \dots, n$ .

*Proof.* We use (3.8) and (4.13). Note that both of the series in the right hand side of (3.8) and in the left hand side of (4.13) are very-well-poised-balanced  ${}_{2n+6}W_{2n+5}$  series. In this case, we consider the following change of variables:

$$\begin{aligned}
 \tilde{a} &= \frac{d e^2 q^{-1}}{A b c}, & \tilde{b} &= e / b, & \tilde{d} &= e / c, \\
 \tilde{u}_i &= (d e q^{-1} / A b c) x_i^{-1}, & \tilde{v}_i &= (e / a_i) x_i & (i &= 1, \dots, n),
 \end{aligned} \tag{4.29}$$

which is a transposition of the variables in  ${}_{2n+6}W_{2n+5}$  series in (3.8). Since the series in the right hand side of (3.8) is invariant under this change of variables. This invariance implies the desired result (4.28) by applying (4.13) according to the change (4.29).  $\square$

#### Rectangular version

**Corollary 4.6.**

$$\sum_{\gamma \in \mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(b x_i, c x_i)_{\gamma_i}}{(e x_i, (a b c q^{1-|M|} / d e) x_i)_{\gamma_i}} \tag{4.30}$$



$$\begin{aligned} \times \frac{(a)_{|\gamma|}}{(d)_{|\gamma|}} &= \prod_{1 \leq i \leq n} \frac{((de/bc)z_i, (e/a)x_i)_{m_i}}{((de/abc)z_i, ex_i)_{m_i}} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \frac{\Delta(zq^\delta)}{\Delta(z)} \\ &\times \frac{(a)_{|\delta|}}{(d)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} z_i / z_j)_{\delta_i}}{(qz_i / z_j)_{\delta_i}} \prod_{1 \leq i \leq n} \frac{((d/b)z_i, (d/c)z_i)_{\delta_i}}{((de/bc)z_i, (aq^{1-|M|}/e)z_i)_{\delta_i}}, \end{aligned}$$

where  $z_i = q^{m_i - |M|} x_i^{-1}$  for  $i = 1, 2, \dots, n$ .

*Remark 4.7.* (4.30) has originally appeared as Theorem 6.8 in Milne-Lilly [23]. Though they referred as  $C_r$  Sears transformation formula there, the sums in both side of (4.30) are  $A_n {}_4\phi_3$  series. For their point of view, it may be precise to refer it as  $A_n$  *Sears transformation formula arising from  $C_n$  Bailey transform*.

In the case when  $n = 1$  and  $x_1 = 1$ , (4.28) and (4.30) reduce to the Sears transformation (3.16). We also note that (4.28) can be obtained from  $A_n$  Bailey transformation (4.11) in the following way: First we replace  $d \rightarrow aq/d$  and  $f \rightarrow aq/f$  in (4.11). Then let the parameter  $a$  tends to infinity in the resulting equation. Finally rearrange the parameters appropriately.

## 4.5 Nonterminating ${}_8W_7$ transformations

Here we show a  $A_n$  nonterminating  ${}_8W_7$  transformation formula. Our tool to produce it is  $m = 1$  case (3.35) of of the nonterminating  ${}_8W_7$  transformation formula (3.34) in Section 3.3. One can see the way to prove the identity as

$$\begin{array}{ccc} {}_8W_7 \text{ series in } A_n & \xrightarrow{A_n \text{ nonterminating } {}_8W_7 \text{ trans.}} & {}_8W_7 \text{ series in } A_n \\ \downarrow (3.35) & & \uparrow (3.35) \\ {}_{2n+6}W_{2n+5} \text{ series} & \xrightarrow{\text{transposition}} & {}_{2n+6}W_{2n+5} \text{ series} \end{array}$$

**Proposition 4.7.**

$$\begin{aligned} &\sum_{\gamma \in \mathbb{N}^n} x_1^{-\gamma_1} \dots x_n^{-\gamma_n} \left( \frac{a^2 q^2}{bcdEf} \right)^{|\gamma|} q^{e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - aq^{|\gamma| + \gamma_i} x_i}{1 - ax_i} \\ &\times \prod_{1 \leq i, j \leq n} \frac{(e_j x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(bx_i, cx_i, dx_i)_{\gamma_i}}{((aq/f)x_i)_{\gamma_i}} \\ &\times \frac{(f)_{|\gamma|}}{(aq/b, aq/c, aq/d)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(ax_i)_{|\gamma|}}{((aq/e_i)x_i)_{|\gamma|}} \\ &= \prod_{1 \leq i \leq n} \frac{(aqx_i, (aq/e_i f)x_i, (\lambda q/e_i)z_i, (\lambda q/f)z_i)_\infty}{((aq/e_i)x_i, (aq/f)x_i, (\lambda q/e_i f)z_i, \lambda qz_i)_\infty} \\ &\times \sum_{\delta \in \mathbb{N}^n} z_1^{-\delta_1} \dots z_n^{-\delta_n} \left( \frac{aq}{Ef} \right)^{|\delta|} q^{e_2(\delta)} \frac{\Delta(zq^\delta)}{\Delta(z)} \prod_{1 \leq i \leq n} \frac{1 - \lambda q^{|\delta| + \delta_i} z_i}{1 - \lambda z_i} \\ &\times \prod_{1 \leq i, j \leq n} \frac{(e_j z_i / z_j)_{\delta_i}}{(qz_i / z_j)_{\delta_i}} \prod_{1 \leq i \leq n} \frac{((aq/cd)z_i, (aq/bd)z_i, (aq/bc)z_i)_{\delta_i}}{((\lambda q/f)z_i)_{\delta_i}} \\ &\times \frac{(f)_{|\delta|}}{(aq/b, aq/c, aq/d)_{|\delta|}} \prod_{1 \leq i \leq n} \frac{(\lambda z_i)_{|\delta|}}{((\lambda q/e_i)z_i)_{|\delta|}} \end{aligned} \quad (4.31)$$

where  $\lambda = a^2 q / bcd$  and  $z_i = \frac{e_i}{E} x_i^{-1}$ .

*Proof.* We iterate (3.35)

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{N}^n} x_i^{-\gamma_i} \cdots x_n^{-\gamma_n} \left( \frac{\mu f}{a} \right)^{|\gamma|} q^{e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - aq^{|\gamma| + \gamma_i} x_i}{1 - ax_i} \\
& \quad \times \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \prod_{1 \leq i \leq n} \frac{(cx_i, dx_i, ex_i)_{\gamma_i}}{((aq/f)x_i)_{\gamma_i}} \\
& \quad \times \frac{(f)_{|\gamma|}}{(aq/c, aq/d, aq/e)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(ax_i)_{|\gamma|}}{((aq/b_i)x_i)_{|\gamma|}} \\
& = \frac{(\mu cf/a, \mu df/a, \mu ef/a, f)_\infty}{(aq/c, aq/d, aq/e, \mu q)_\infty} \prod_{1 \leq i \leq n} \frac{(aqx_i, (\mu b_i f/a)x_i^{-1})_\infty}{((aq/b_i)x_i, (\mu f/a)x_i^{-1})_\infty} \\
& \quad \times {}_{2n+4}W_{2n+3} \left[ \mu; \{(aq/b_i f)x_i\}_n aq/cf, aq/df, aq/ef, \{(\mu f/a)x_i^{-1}\}_n; q; f \right], \\
& \quad (\mu = a^3 q^2 / Bcdef^2),
\end{aligned}$$

twice. On the way to obtain (4.31), we interchange  $(aq/b_i f)x_i$  and  $(\mu f/a)x_i^{-1}$  for all  $i = 1, \dots, n$  simultaneously in the  ${}_{2n+6}W_{2n+5}$  series.  $\square$

*Remark 4.8.* In the case when  $n = 1$  and  $x_1 = 1$ , (4.31) reduces to the following nonterminating  ${}_8W_7$  transformation:

$$\begin{aligned}
& {}_8W_7 \left[ a; b, c, d, e, f; q; \frac{a^2 q^2}{bcdef} \right] \\
& = \frac{(aq, aq/ef, \lambda q/e, \lambda q/f)_\infty}{(aq/e, aq/f, \lambda q, \lambda q/ef)_\infty} {}_8W_7 \left[ \lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f; q; \frac{aq}{ef} \right],
\end{aligned} \tag{4.32}$$

where  $\lambda = a^2 q / bcd$ . (4.31) can also be obtained by taking the limit  $N \rightarrow \infty$  in  $A_n$  Bailey transformation formula (4.11).

## 4.6 Terminating ${}_8W_7$ transformations

Here we present  $A_n$  nonterminating  ${}_8W_7$  transformation formulas. We give a proof by using  $m = 1$  case (3.40) of the nonterminating  ${}_8W_7$  transformation formula (3.39) in Section 3.3. The proof is in the same manner as in that of (4.31). One can see the way to give the identity as

$$\begin{array}{ccc}
{}_8W_7 \text{ series in } A_n & \xrightarrow{A_n \text{ terminating } {}_8W_7 \text{ trans.}} & {}_8W_7 \text{ series in } A_n \\
\downarrow (3.40) & & \uparrow (3.40) \\
{}_{2n+6}W_{2n+5} \text{ series} & \xrightarrow{\text{transposition}} & {}_{2n+6}W_{2n+5} \text{ series}
\end{array}$$

**Proposition 4.8.**

$$\begin{aligned}
& \sum_{\gamma \in \mathbb{N}^n} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \left( \frac{a^2 q^{N+2}}{Bc f g} \right)^{|\gamma|} q^{-e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - aq^{|\gamma| + \gamma_i} x_i}{1 - ax_i} \\
& \quad \times \prod_{1 \leq i \leq n} \frac{(ax_i)_{|\gamma|}}{((aq/b_i)x_i)_{|\gamma|}} \prod_{1 \leq i, j \leq n} \frac{(b_j x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \\
& \quad \times \frac{(f, g, q^{-N})_{|\gamma|}}{(aq/c)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{(aq^{N+1}x_i, (aq/f)x_i, (aq/g)x_i)_{\gamma_i}}
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
&= \prod_{1 \leq i \leq n} \frac{(aqx_i, (aq/b_i f)x_i, (aq/b_i g)x_i, (aq/f g)x_i)_N}{((aq/b_i)x_i, (aq/f)x_i, (aq/g)x_i, (aq/b_i f g)x_i)_N} \\
&\quad \times \sum_{\delta \in \mathbb{N}^n} z_1^{\delta_1} \cdots z_n^{\delta_n} \left(\frac{q}{c}\right)^{|\delta|} q^{-e_2(\delta)} \frac{\Delta(zq^\delta)}{\Delta(z)} \prod_{1 \leq i \leq n} \frac{1 - (q^{-N-1} B f g/a) q^{|\delta| + \delta_i} z_i}{1 - (q^{-N-1} B f g/a) z_i} \\
&\quad \times \prod_{1 \leq i \leq n} \frac{((q^{-N} B f g/a) z_i)_{|\delta|}}{((q^{-N} f g/a) z_i)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(b_j z_i / z_j)_{\delta_i}}{(q z_i / z_j)_{\delta_i}} \\
&\quad \times \frac{(f, g, q^{-N})_{|\delta|}}{(aq/c)_{|\delta|}} \prod_{1 \leq i \leq n} \frac{((q^{-N-1} B c f g/a^2) z_i)_{\delta_i}}{((B f g/a) z_i, (q^{-N} B g/a) z_i, (q^{-N} B f/a) z_i)_{\delta_i}}.
\end{aligned}$$

where  $z_i = \frac{e_i}{E} x_i^{-1}$ .

### Rectangular version

**Corollary 4.7.**

$$\begin{aligned}
&\sum_{\gamma \in \mathbb{N}^n} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \left(\frac{a^2 q^{|M|+2}}{bcfg}\right)^{|\gamma|} q^{-e_2(\gamma)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \leq i \leq n} \frac{1 - aq^{|\gamma| + \gamma_i} x_i}{1 - ax_i} \quad (4.34) \\
&\quad \times \prod_{1 \leq i \leq n} \frac{(ax_i)_{|\gamma|}}{(aq^{|M|+1} x_i)_{|\gamma|}} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} x_i / x_j)_{\gamma_i}}{(qx_i / x_j)_{\gamma_i}} \\
&\quad \times \frac{(b, f, g)_{|\gamma|}}{(aq/c)_{|\gamma|}} \prod_{1 \leq i \leq n} \frac{(cx_i)_{\gamma_i}}{((aq/b)x_i, (aq/f)x_i, (aq/g)x_i)_{\gamma_i}} \\
&= \prod_{1 \leq i \leq n} \frac{(aqx_i, (aq/bf)x_i, (aq/bg)x_i, (aq/f g)x_i)_{m_i}}{((aq/b)x_i, (aq/f)x_i, (aq/g)x_i, (aq/bf g)x_i)_{m_i}} \\
&\quad \times \sum_{\delta \in \mathbb{N}^n} z_1^{\delta_1} \cdots z_n^{\delta_n} \left(\frac{q}{c}\right)^{|\delta|} q^{-e_2(\delta)} \frac{\Delta(zq^\delta)}{\Delta(z)} \prod_{1 \leq i \leq n} \frac{1 - (q^{-|M|-1} b f g/a) q^{|\delta| + \delta_i} z_i}{1 - (q^{-|M|-1} b f g/a) z_i} \\
&\quad \times \prod_{1 \leq i \leq n} \frac{((q^{-|M|} b f g/a) z_i)_{|\delta|}}{((q^{-|M|} f g/a) z_i)_{|\delta|}} \prod_{1 \leq i, j \leq n} \frac{(q^{-m_j} z_i / z_j)_{\delta_i}}{(q z_i / z_j)_{\delta_i}} \\
&\quad \times \frac{(b, f, g)_{|\delta|}}{(aq/c)_{|\delta|}} \prod_{1 \leq i \leq n} \frac{((q^{-|M|-1} b c f g/a^2) z_i)_{\delta_i}}{((q^{-|M|} f g/a) z_i, (q^{-|M|} b g/a) z_i, (q^{-|M|} b f/a) z_i)_{\delta_i}}.
\end{aligned}$$

where  $z_i = q^{m_i - |M|} x_i^{-1}$ .

*Remark 4.9.* In the case when  $n = 1$  and  $x_1 = 1$ , (4.33) and (4.34) reduces to the following terminating  ${}_8W_7$  transformation:

$$\begin{aligned}
{}_8W_7 \left[ a; b, c, f, g, q^{-N}; q; \frac{a^2 q^{2+N}}{bcfg} \right] &= \frac{(aq, aq/bf, aq/bg, aq/f g)_N}{(aq/b, aq/f, aq/g, aq/bf g)_N} \quad (4.35) \\
{}_8W_7 \left[ q^{-N-1} b f g/a; b, q^{-N-1} b c f g/a^2, f, g, q^{-N}; q; \frac{q}{c} \right].
\end{aligned}$$

(4.33) can also be obtained by taking the limit  $d \rightarrow \infty$  in (4.11).

*Remark 4.10.*  $A_n$  Watson transformation (4.12) can be obtained by combining (4.33) and (4.16) and by combining (4.16) and  $A_n$  Sears transformations formula (4.28).

$$\begin{array}{ccc}
{}_8W_7 \text{ series in } A_n & \xrightarrow{(4.33)} & {}_8W_7 \text{ series in } A_n \\
\downarrow (4.16) & \searrow (4.12) & \downarrow (4.16) \\
{}_4\phi_3 \text{ series in } A_n & \xrightarrow{(4.28)} & {}_4\phi_3 \text{ series in } A_n
\end{array}$$

As we have seen in this section, our discussion here may implies not only that our class of multiple hypergeometric transformations in the previous section are broader class than Milne's class of transformation in  $A_n$  but also that our class contains more precise informations. In our terminology, one may see that Milne's hypergeometric transformations have extra hidden symmetries in the one dimensional (generalized) hypergeometric series.

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