Constructor University Bremen

CO-520-B

Signal and Systems Lab

Fall Semester 2023

Lab Experiment 1 – RLC-Circuits - Transient Response

Experiment conducted by: Abhigyan Deep Barnwal, Sumit Giri

Place of execution: Room 54, Research I, Jacobs University, 28759 Bremen

Date of execution: 23th October 2023

Author of the Report: Abhigyan Deep Barnwal

IRC Mailbox Number: NB-124

Introduction

Objective

In this lab, we studied the transient response of second-order systems, namely the RLC circuit with a resistor, an inductor, and a capacitor in series. In the experiment, we observed the under, over, and critically damped responses in the RLC circuits.

Theory

Second Order Systems and Differential Equations

A second order circuit is a circuit that contains exactly two energy-storing parts (a combination of inductors and capacitors). These circuits can be modeled by a second-order differential equation, which in their general form appear as the following:

$$a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = x(t)$$

Where y(t) is the response of the system to an input in the form of x(t), and a_2, a_1 and a_0 are the system parameters. However, while applying this to electrical engineering, it is more convenient to write the above equation in the form:

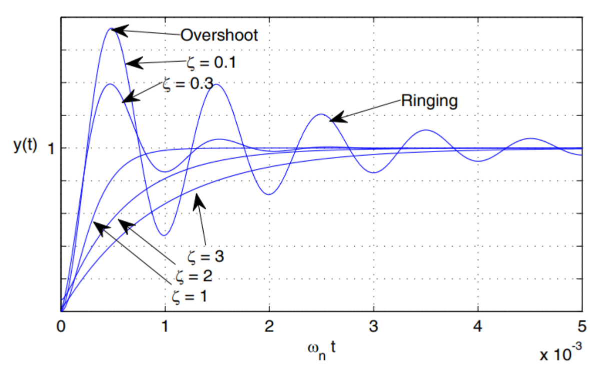
$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t)$$

Such that the system parameters change as follows:

$$\omega_n = \sqrt{\frac{a_0}{a_2}} \qquad \zeta = \frac{a_1}{2\sqrt{a_0 a_2}} \qquad K = \frac{1}{a_0}$$

Where ω_n is the natural frequency, ζ is the damping ratio, and K is the system's gain. According to the values of ζ, we can classify the transient response into three cases, which affect the output signal as shown in the figure to the right:

1. Under-damped case : $0 < \zeta < 1$
2. Critically damped case : $\zeta = 1$
3. Over-damped case : $\zeta > 1$



Underdamped case solution: $y(t) = \exp(-\zeta\omega_n t)(C_1\cos(\omega_d t) + C_2\sin(\omega_d t))$ where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Critically damped case solution: $y(t) = C_1\exp(-\zeta\omega_n t) + C_2 t\exp(-\zeta\omega_n t)$

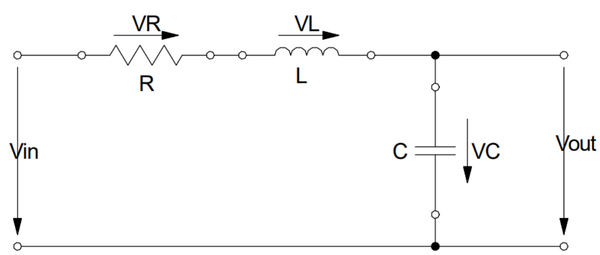
Overdamped case solution: $y(t) = C_1\exp((-\zeta + \sqrt{\zeta^2-1})\omega_n t) + C_2\exp((-\zeta - \sqrt{\zeta^2-1})\omega_n t)$

The complete solution y(t) of a second-order differential equation is given by first solving it homogeneously with x(t)=0, and then adding it to the forced solution given by finding the solution to the non-homogeneous equation with x(t)≠0, which gives us the steady-state response of the system.

The above information is useful for RLC circuits, as they are a commonly used type of second-order system in electrical engineering.

RLC-Circuits

RLC circuits are a specific type of second-order circuits, and their general solution can be calculated by using simple circuit analysis and substituting it into the general solution for all second-order circuits.



$$V_{in} = V_R + V_L + V_{out}$$

$$i = i_C = C\frac{dV_{out}}{dt} \qquad V_R = iR = RC\frac{dV_{out}}{dt}$$

$$V_L = L\frac{di}{dt} = L\frac{d}{dt}(C\frac{dV_{out}}{dt}) = LC\frac{d^2V_{out}}{dt^2}$$

Putting all of this together, we get:

$$LC\frac{d^2V_{out}}{dt^2} + RC\frac{dV_{out}}{dt} + V_{out} = V_{in}$$

This allows us to equate the system parameters of the differential equation:

$$\omega_n = \frac{1}{\sqrt{LC}} \qquad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}} \qquad K = 1$$

Steady-state Value

The steady-state value is the magnitude of the voltage or current after the system has reached stability.

Ringing

Ringing is the oscillation phenomenon that occurs if the system is under-damped,

The Complete Response

The necessary steps to determine the complete response of a second-order system based on an RLC network with DC sources are:

- For transient response
    1. Using Ohm's law, KVL, and/or KCL, obtain a second-order differential nonhomogeneous equation
    2. Solve the homogeneous equation corresponding to the obtained second-order differential non-homogeneous equation
    3. Obtain the complete solution by adding the forced solution to the homogeneous solution. The complete solution still contains unknown coefficients C_1 and C_2
    4. Use the initial conditions to determine the value of C_1 and C_2
- For steady-state response
    1. Replace all capacitances with open circuits
    2. Replace all inductances with short circuits

Experimental Set-up and Results

Experimental Setup and Procedure

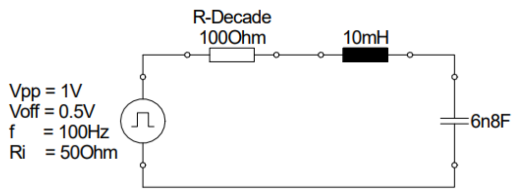
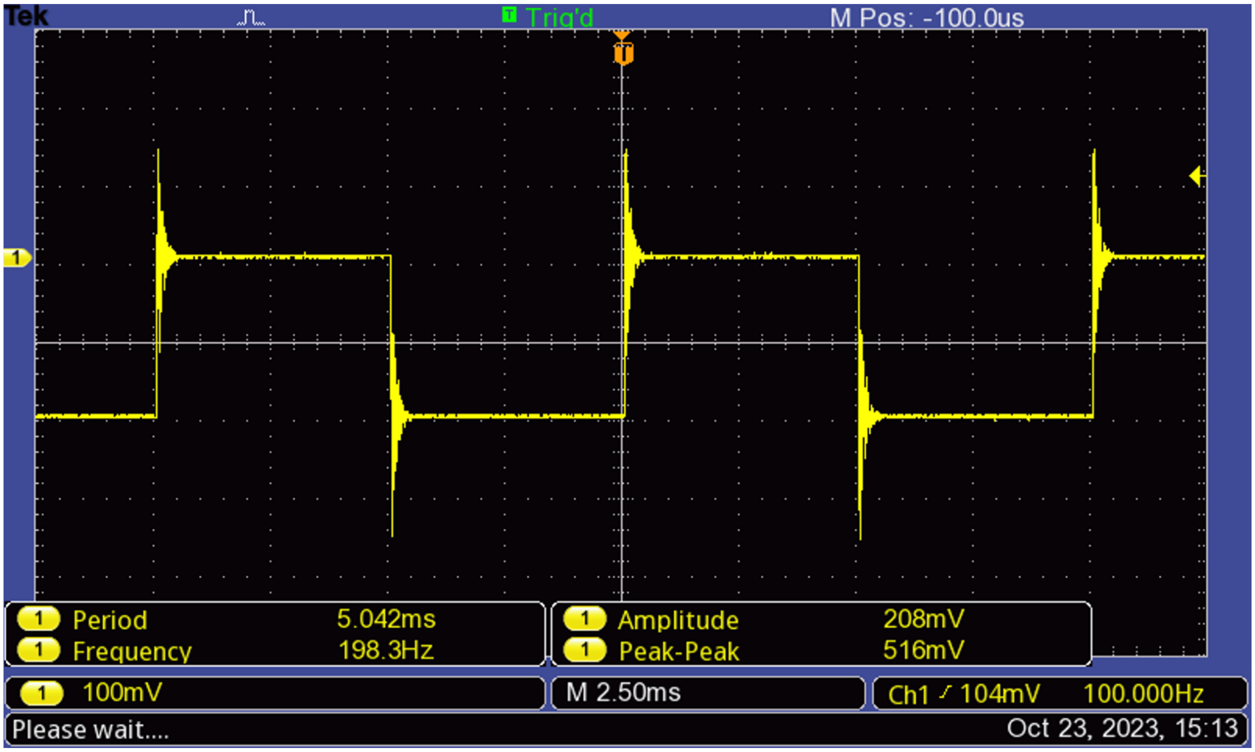


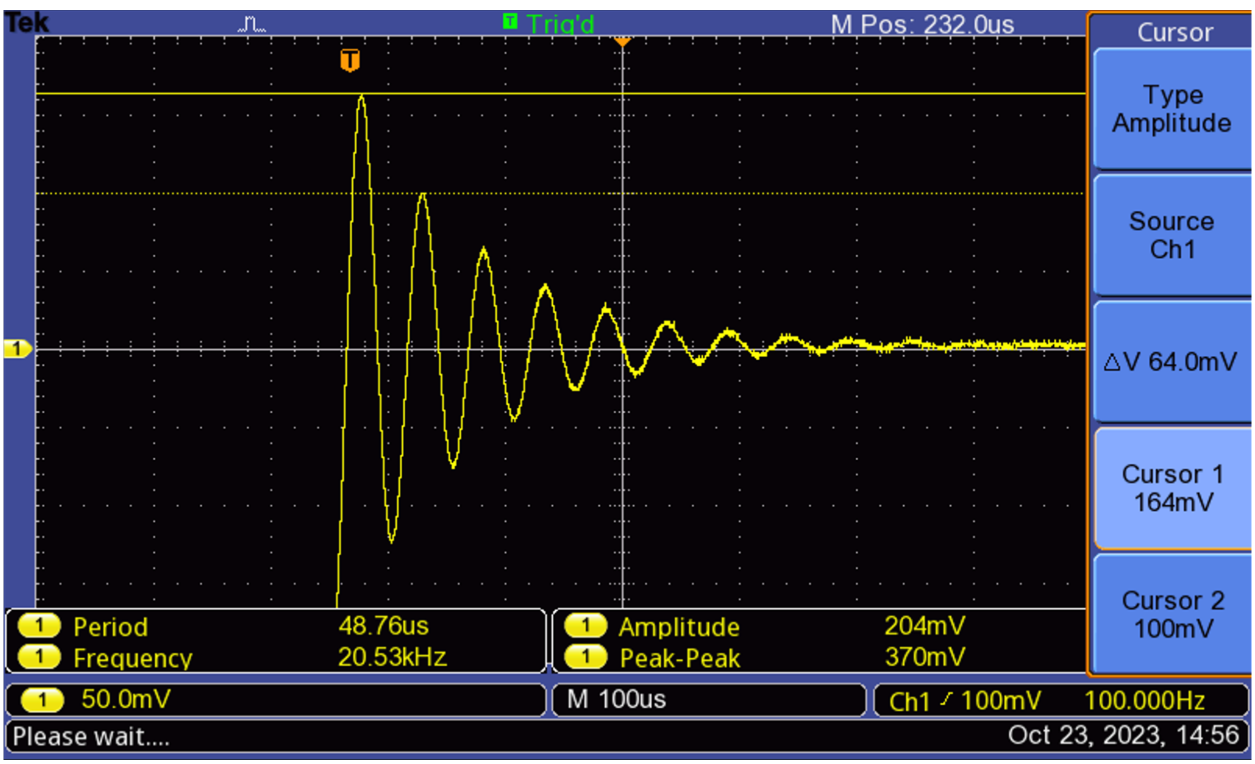
Figure 1: Experimental Setup

1. Set the function generator to produce a 100 Hz square wave with an amplitude of 0.5 V and an offset of 0.5 V.
2. Connect the oscilloscope in parallel to the capacitor
3. Measure the underdamped frequency - Take a hardcopy of one signal period and one focusing on the ringing phenomenon.
4. Calculate the damped radian frequency ω_d. In your calculation, consider the internal resistance of the function generator to be 50Ω. Compare the calculated value with the measured value in step 3. If they are consistent, proceed with the next steps.
5. Calculate the resistance so that the circuit is critically damped. Set that resistance to R Decade accounting for the 50Ω internal resistance of the function generator. Display the signal and take a hard copy.
6. Check if the practical signal is critically damped. If not, then vary the R-decade value such that it is and take a hardcopy of the final result. Remember to record the final R decade value.
7. Set the R-decade to 30kΩ, so that the circuit is overdamped. Take a hardcopy of the signal.

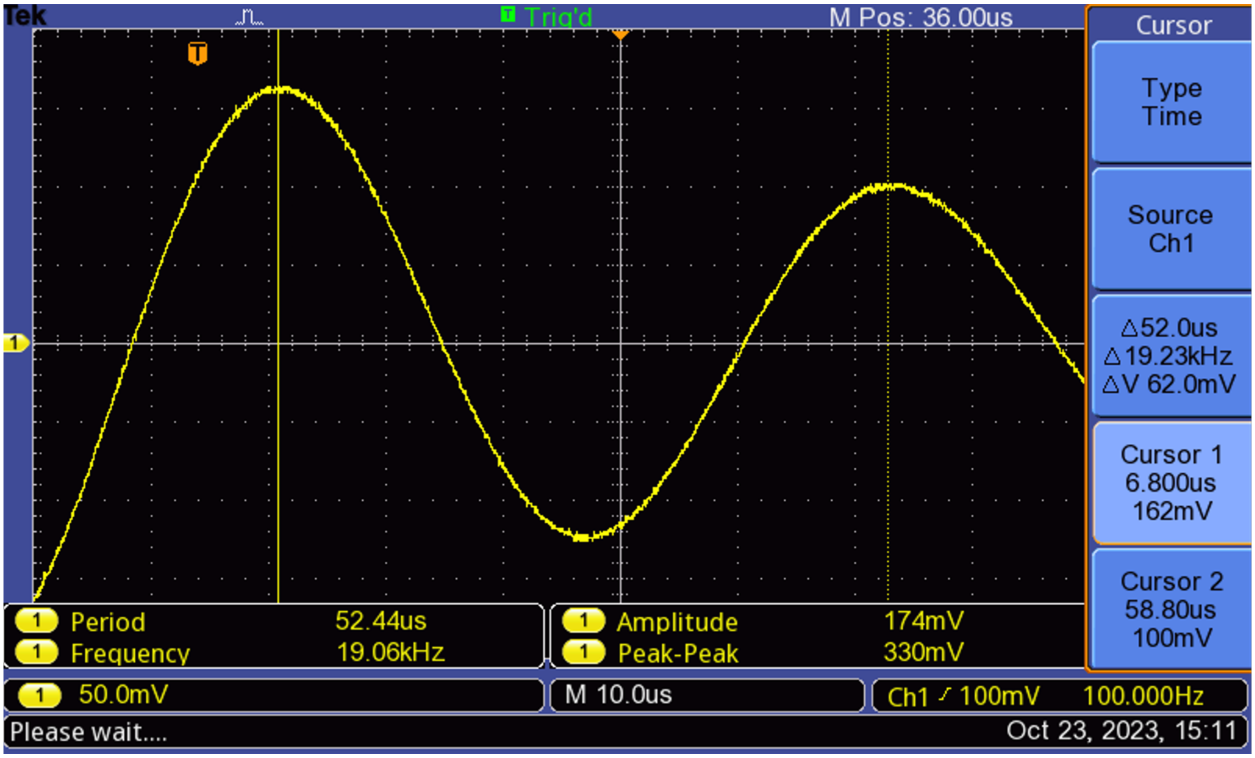
Results

Underdamped Frequency - Ringing Phenomenon





Underdamped Frequency - Zoomed in



<u>Calculating theoretical underdamped frequency</u>

$\omega_n = \frac{1}{\sqrt{LC}}$ $\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$ $\omega = 2\pi f$

$\omega_d = \frac{1}{\sqrt{10\cdot10^{-3}\cdot6.8\cdot10^{-9}}}\sqrt{1-(\frac{150}{2}\sqrt{\frac{6.8\cdot10^{-9}}{10\cdot10^{-3}}})^2}$ $f = \frac{\omega}{2\pi} = 19.28Hz$

The experimental value of $19.23Hz$ is very close to the theoretical value of $19.28Hz$. The minuscule error might possibly be from the unaccounted resistances in the inductor and capacitor.

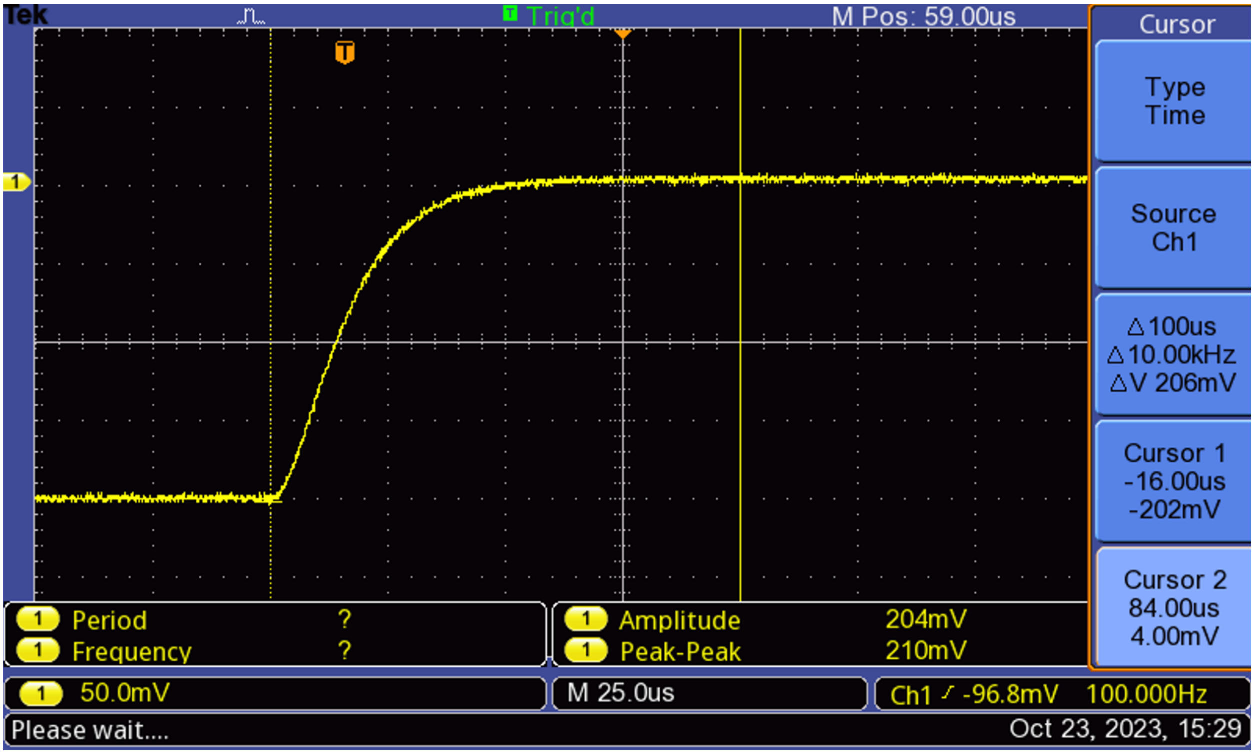
<u>Critically Damped Signal - Theoretical Resistance</u>

To calculate the theoretical resistance for a critically damped signal, we had to find the resistance such that the damping ratio is 1.

$$1 = \frac{R}{2}\cdot\sqrt{\frac{C}{L}} = \frac{R+50}{2}\cdot\sqrt{\frac{6.8\cdot10^{-9}}{10\cdot10^{-3}}}$$

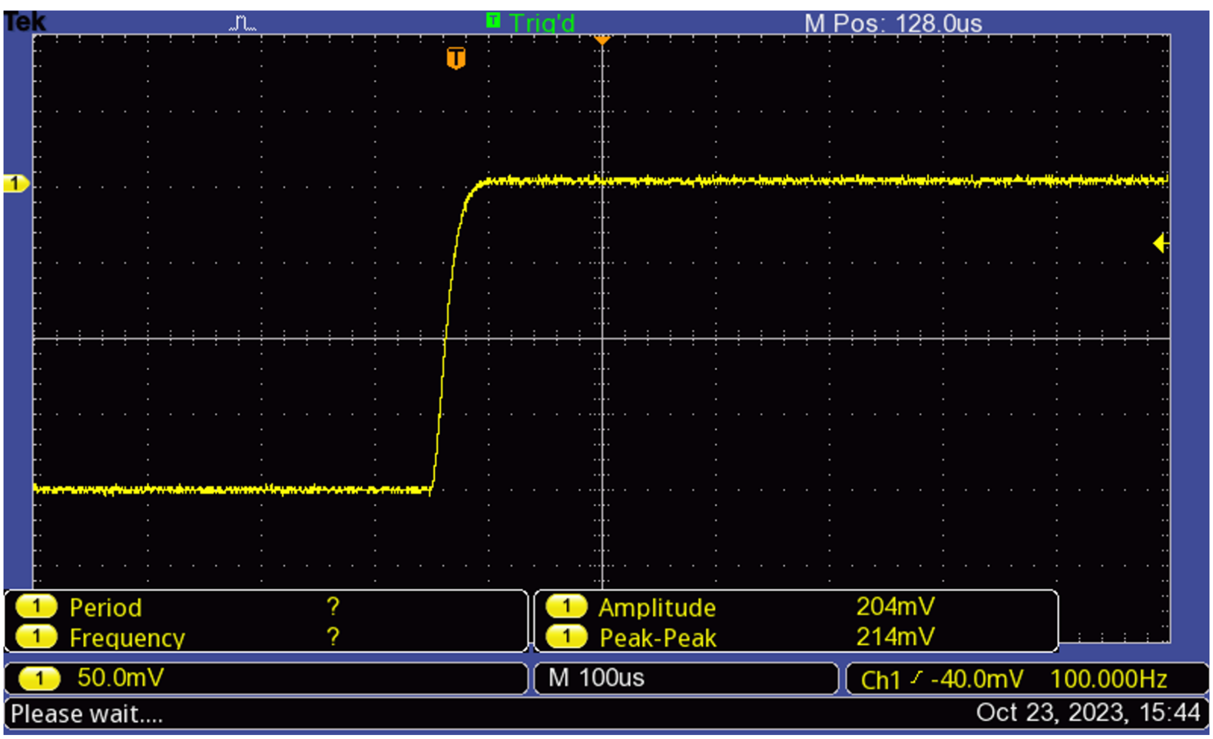
$R = 2375\Omega$

The signal obtained using the calculated resistance is shown in the figure below.

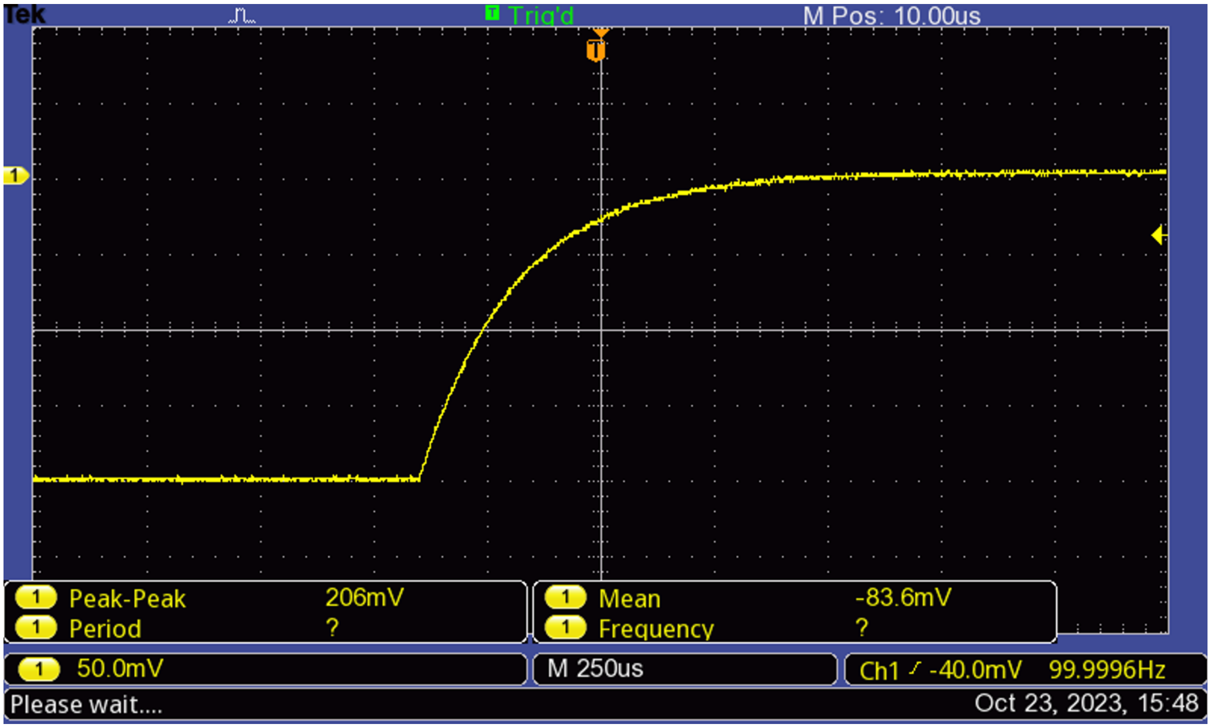


Critically Damped Signal - Actual Resistance

The figure below shows the signal after changing R Decade such that the signal appears to be critically damped on the oscilloscope to the naked eye. $R = 2100\Omega$



Overdamped Signal with 30kΩ



Evaluation

Part 1 - Use the circuit from the experiment and obtain the differential equation for the voltage $V_c(t)$ across the capacitor when R = 100 Ω, identify the damping nature of the circuit and determine the values for the coefficients C_1 and C_2.

As mentioned in the introduction of this report, the differential equation for an RLC circuit is given by:

$$LC\frac{d^2V_{out}}{dt^2} + RC\frac{dV_{out}}{dt} + V_{out} = V_{in}$$

As such, the specific differential equation for the voltage across can be calculated by plugging in the given values for our circuit.

$$(10 \cdot 10^{-3} \cdot 6.8 \cdot 10^{-9} \cdot v_c''(t)) + (100 \cdot 6.8 \cdot 10^{-9} \cdot v_c'(t)) + v_c(t) = v_s(t)$$

$$(6.8 \cdot 10^{-11} \cdot v_c''(t)) + (6.8 \cdot 10^{-7} \cdot v_c'(t)) + v_c(t) = v_s(t)$$

The damping nature of the circuit can be determined by finding the damping ratio, again by using a formula discussed in the theory part:

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\zeta = \frac{100}{2} \cdot \sqrt{\frac{6.8 \cdot 10^{-9}}{10 \cdot 10^{-3}}} = 50 \cdot \sqrt{6.8 \cdot 10^{-7}} = 0.04$$

$$0 < 0.04 < 1$$

Hence we can say the transient response of the circuit should be underdamped.

Now, we determine the values of the coefficients C_1 and C_2:

$$v_c(t) = e^{-\zeta\omega_n t}(C_1 cos(\omega_d t) + C_2 sin(\omega_d t)) + v_{ss}$$

$$\omega_n = \frac{1}{\sqrt{LC}} \quad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}} \quad \omega_d = \omega_n\sqrt{1-\zeta^2} \quad v_{ss} = 1V$$

$$v_c(t) = e^{-5000t}(C_1 cos(121165t) + C_2 sin(121165t)) + 1$$

We know that the voltage drop across a capacitor cannot change instantaneously, so $V_c(0)=0$.

$$v_c(0) = e^{0}(C_1 cos(0) + C_2 sin(0) + 1$$

$$C_1 = \; -1$$

Differentiating once, we get:

$$v_c'(t) = e^{-\zeta\omega_n t}(-\omega_d C_1 sin(\omega_d t) + \omega_d C_2 cos(\omega_d t)) - \zeta\omega_n e^{-\zeta\omega_n t}(C_1 cos(\omega_d t) + C_2 sin(\omega_d t))$$

We know that the current through an inductor cannot change instantaneously, so $i_L(0)=0$

$$i_L(0) = i_C(0) = 0 = C\frac{dV}{dt}$$

$$\Rightarrow v_c'(0) = 0$$

$$v_c'(0) = e^0(-\omega_d C_1 sin(0) + \omega_d C_2 cos(0)) - \zeta\omega_n e^0(C_1 cos(0) + C_2 sin(0))$$

$$C_2 = \frac{-\zeta\omega_n}{\omega_d} = -0.041$$

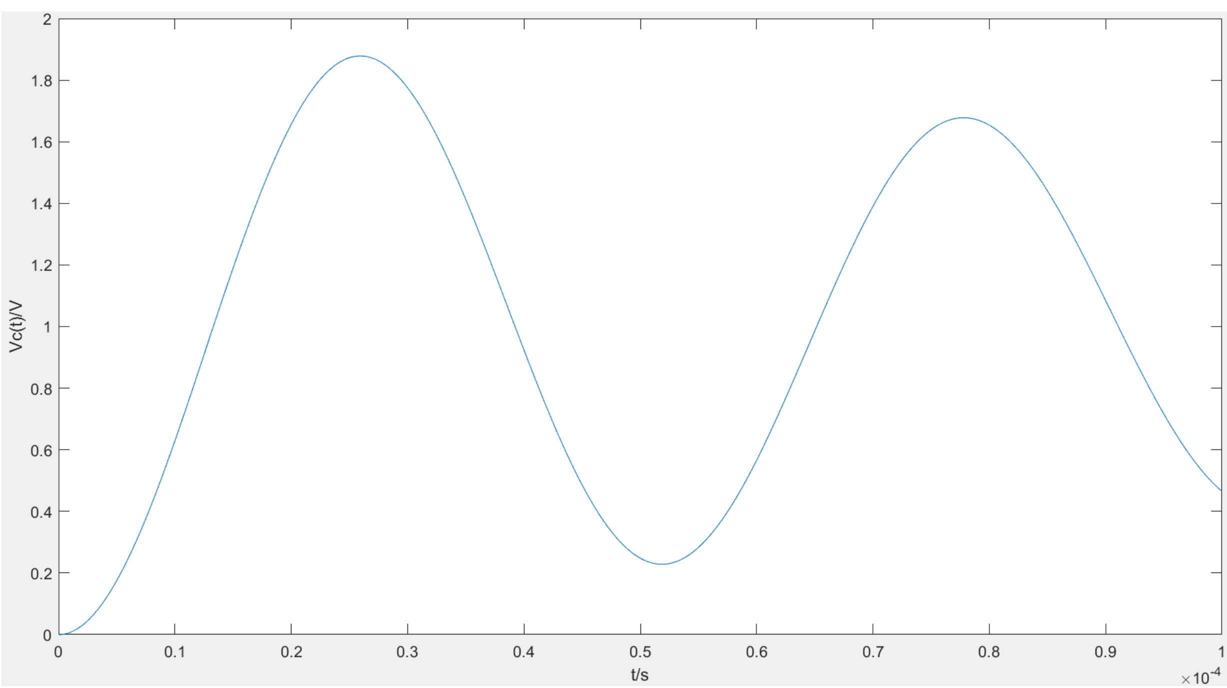
Hence, the final equation for $v_c(t)$ is:

$$v_c(t) = 1 - e^{-5000t}(cos(121165t) + 0.041sin(121165t))$$

Part 2 - Plot the voltage $v_c(t)$ using Matlab

The code on the right was entered into matlab to plot the graph below:

```
t=[0:0.0000001:0.0001]
v=1-exp(t.*(-5000)).*(cos(t.*121165)+0.041*sin(t.*121165))
plot(t,v)
xlim([0 0.0001])
ylim([0 2])
xlabel('t/s')
ylabel('Vc(t)/V')
```



Part 3 - Calculate the resistor value to obtain a critically damped case and obtain the corresponding equation describing the voltage $v_c(t)$ including the values for C_1 and C_2. Plot the voltage $v_c(t)$ using Matlab

The theoretical resistance for a critically damped case was calculated during the experiment and determined to be 2375Ω.

$$v_c(t) = C_1 e^{-\zeta\omega_n t} + C_2 t e^{-\zeta\omega_n t} + v_{ss}$$

We know that the damping ratio is 1, and that $v_{ss} = 1V$

$$v_c(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t} + 1$$

We can use the same initial values as with the calculations for the underdamped part:

$v_c(0) = 0$ and $v_c'(0) = 0$

$$v_c(0) = C_1 e^{0} + C_2 (0) e^{0} + 1$$

$$C_1 = -1$$

$$v_c'(t) = -\omega_n C_1 e^{-\omega_n t} + C_2 (e^{-\omega_n t} - \omega_n t e^{-\omega_n t})$$

$$v_c'(0) = -\omega_n C_1 e^{0} + C_2 (e^{0} - \omega_n (0) e^{0})$$

$$0 = \omega_n + C_2$$

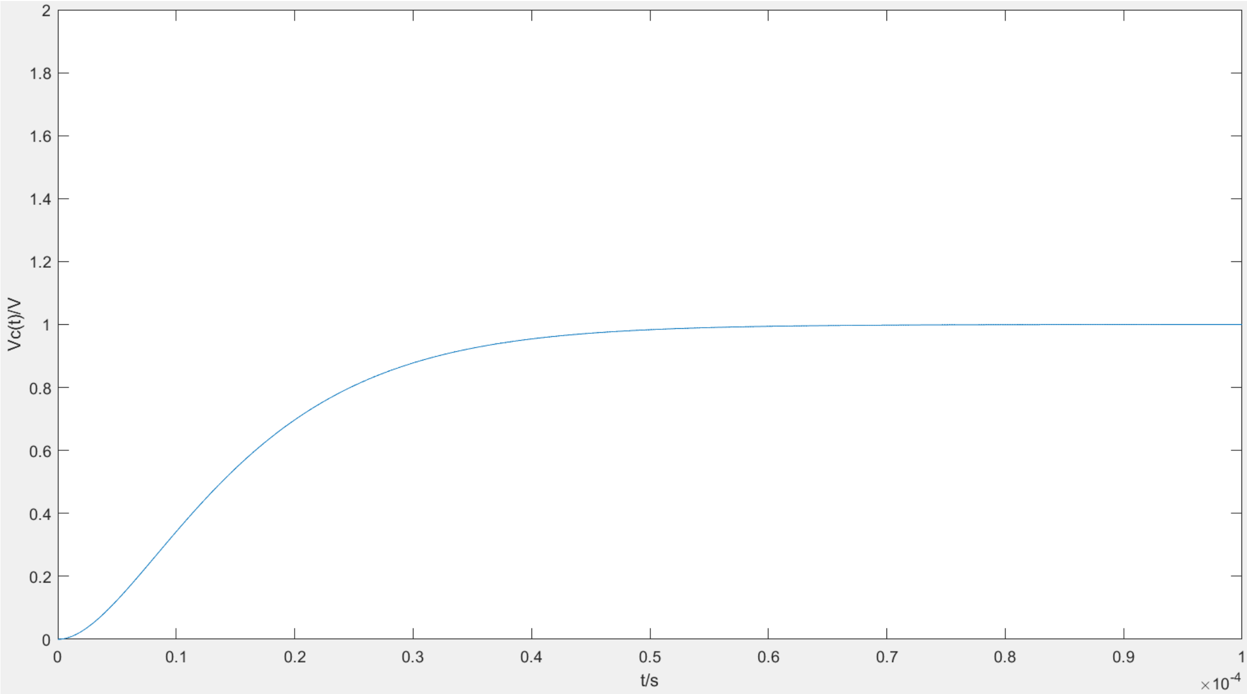
$$C_2 = -\omega_n = -121268$$

Hence, the final equation for $v_c(t)$ is:

$$v_c(t) = 1 - (1 + 121268t)e^{-121268t}$$

The code on the right was entered into Matlab to plot the graph below:

```
t=[0:0.0000001:0.0001]
v=1-(1.+t.*121268).*exp(t.*(-121268))
plot(t,v)
xlim([0 0.0001])
ylim([0 2])
xlabel('t/s')
ylabel('Vc(t)/V')
```



Part 4 - Compare the experimental results obtained in the lab with the calculations. Provide a detailed explanation if the experimental results deviate. Discuss the origin of the deviation

The theoretical and experimental values are presented below:

$$R_{theoretical} = 2375\Omega$$

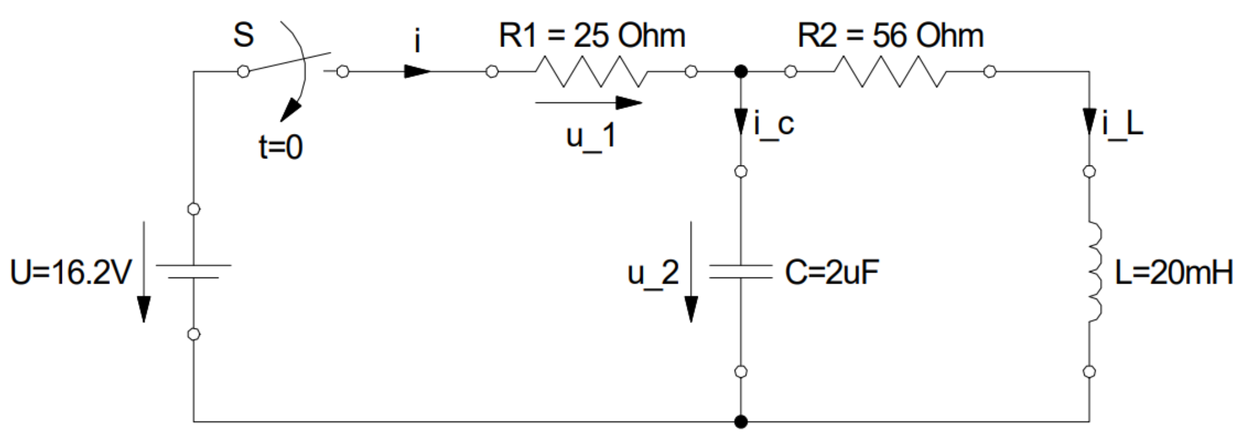
$$R_{experimental} = 2100\Omega$$

The values have a difference of 275Ω, which is relatively high (~12%). This could have had multiple reasons, such as:

- The actual values of the equipment used (capacitor and inductor mainly) may not have been completely accurate
- R-decade switches also have errors. For example, turning on the ‘4’ and ‘1’ switches in a range has a slightly noticeable difference as compared to turning on just the ‘2’ and ‘3’ switches in the same range, despite them representing the same resistance value
- Internal resistances of wires and other equipment were not accounted for

All of these reasons may have caused small errors, which pile on top of each other to give us an experimental value with a difference from the theoretical value in the end.

Part 5 - Solve the following problem. The switch in the circuit below is closed at t = 0.



a) Obtain the differential equation for the current $i_L(t)$.

To obtain the differential equation of $i_L(t)$, we can use Ohm's law, KCL, KVL, capacitance, and inductance formulae to get the following:

According to KCL:

$$i = i_C + i_L$$

$$\frac{di}{dt} = \frac{di_C}{dt} + \frac{di_L}{dt}$$

Loop 1:

$$16.2 = v_{R1} + v_C$$

$$0 = \frac{dv_{R1}}{dt} + \frac{dv_C}{dt}$$

$$0 = R_1\frac{di}{dt} + \frac{i_C}{C}$$

Loop 2:

$$v_c = v_{R2} + v_L$$

$$\frac{dv_c}{dt} = \frac{dv_{R2}}{dt} + \frac{dv_L}{dt}$$

$$\frac{i_C}{C} = R_2\frac{di_L}{dt} + L\frac{d^2i_L}{dt^2}$$

$$\frac{di_c}{dt} = CR_2\frac{d^2i_L}{dt^2} + CL\frac{d^3i_L}{dt^3}$$

So combining the equations we get:

$$0 = R_1(CR_2\frac{d^2i_L}{dt^2} + CL\frac{d^3i_L}{dt^3} + \frac{di_L}{dt}) + R_2\frac{di_L}{dt} + L\frac{d^2i_L}{dt^2}$$

$$(R_1CL)\frac{d^3i_L}{dt^3} + (R_1R_2C + L)\frac{d^2i_L}{dt^2} + (R_1 + R_2)\frac{di_L}{dt} = 0$$

$$(R_1CL)\frac{d^2i_L}{dt^2} + (R_1R_2C + L)\frac{di_L}{dt} + (R_1 + R_2)i_L = 0$$

b) Identify the damping nature of the circuit and determine the values for the coefficients C1 and C2. Show the formula for the complete response!

$$\zeta = \frac{a_1}{2\sqrt{a_0a_2}} = \frac{(R_1R_2C+L)}{2\sqrt{(R_1+R_2)(R_1CL)}} \sim 1.3 > 1$$

Therefore, the circuit is overdamped.

The steady-state response also needs to be accounted for. In the steady state, $t = \infty$, the capacitor behaves like an open circuit and the inductor behaves like a short circuit.

As such, i_L can be calculated as:

$$i_L(\infty) = i = \frac{V}{R_1+R_2} = \frac{16.2}{25+56} = 0.2A$$

Now, the full response is given by:

$$i_L(t) = 0.2 + C_1e^{((-\zeta+\sqrt{\zeta^2-1})\omega_n t)} + C_2e^{((-\zeta-\sqrt{\zeta^2-1})\omega_n t)}$$

$$\omega_n = \sqrt{\frac{a_0}{a_2}} = \sqrt{\frac{(R_1+R_2)}{(R_1CL)}} = 9000rads^{-1}$$

To solve for C_1 and C_2 we can first look at the initial conditions,

$$i_L(0) = 0 = 0.2 + C_1e^{((-\zeta+\sqrt{\zeta^2-1})\omega_n 0)} + C_2e^{((-\zeta-\sqrt{\zeta^2-1})\omega_n 0)} = 0.2 + C_1 + C_2$$

$$C_1 + C_2 = (-0.2)$$

Now, since the current through an inductor can't change instantly:

$$\frac{di_L(0)}{dt} = 0$$

$$\frac{di_L(0)}{dt} = ((-\zeta + \sqrt{\zeta^2 - 1})\omega_n)C_1 e^{((-\zeta+\sqrt{\zeta^2-1})\omega_n 0)} + ((-\zeta - \sqrt{\zeta^2 - 1})\omega_n)C_2 e^{((-\zeta-\sqrt{\zeta^2-1})\omega_n 0)}$$

$$\frac{di_L(0)}{dt} = 0 = ((-\zeta + \sqrt{\zeta^2 - 1})\omega_n)C_1 + ((-\zeta - \sqrt{\zeta^2 - 1})\omega_n)C_2$$

$$C_1 + C_2 = (-0.2)$$

Solving the two equations gives us the values for C_1 and C_2 as:

$$C_1 =- 263mA$$

$$C_2 = 63mA$$

So, the complete response is:

$$i_L(t) = 0.2 - 0.263e^{(-0.4403t)} + 0.063e^{(-18397t)}$$

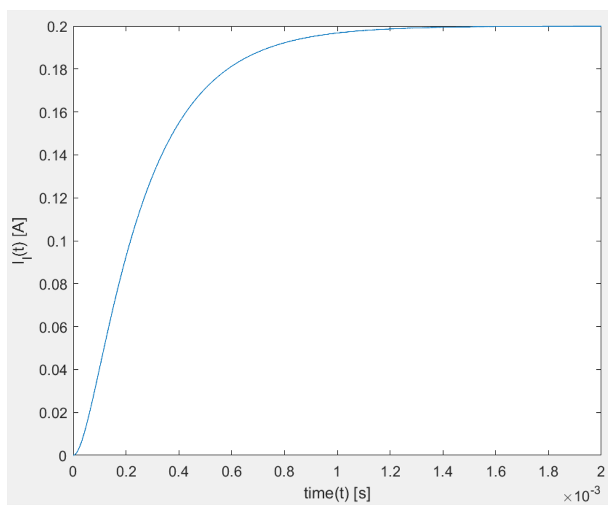
c) Plot the current $i_L(t)$ using Matlab.

```
r1=25;
r2=56;
c=2*10^-6;
l=20*10^-3;
a0=r1+r2;
a1=r1*r2*c+l;
a2=r1*c*l;
zeta=a1/(2*sqrt(a0*a2));
omegan=sqrt(a0/a2);

c1=(-263*10^-3);
c2=(63*10^-3);
expc1=(-zeta+sqrt(zeta^2-1));
expc2=(-zeta-sqrt(zeta^2-1));
t=0:0.0000002:0.002;

il=0.2+c1*exp((expc1)*omegan*t)+c2*exp((expc2)*omegan*t);

plot(t,il);
xlabel("time(t) [s]");
ylabel("I_l(t) [A]");
```



Conclusion

The primary goal of our experiment was to delve into the transient response characteristics of second-order RLC circuits, specifically exploring how changes in resistance affected the behavior of the system. Initially, we set up an underdamped RLC system according to provided instructions. Utilizing an oscilloscope, we observed the circuit's output across the capacitor, enabling us to analyze the ringing phenomenon and determine the damped frequency. Our calculations were validated through a comparative analysis of our results with the oscilloscope data. Moving forward, we investigated the impact of resistance changes on the system. We aimed to achieve a critically damped RLC circuit by calculating the theoretical value of resistance. However, during experimentation, we discovered that the experimentally determined critical damping resistance was slightly lower than our calculated value. This discrepancy was attributed to deviations in circuit component characteristics from the expected values. Subsequently, we explored the characteristics of an overdamped system by setting the R-Decade to 30 $k\Omega$, concluding this phase of our experimentation. In the evaluation section, we revisited the theoretical aspects of our circuit, solving for voltage across the capacitor under various conditions. Additionally, we examined a different circuit structure with distinct initial conditions, solving for current through the resistor as a function of time. MATLAB played a crucial role in visualizing the circuit characteristics, providing a practical means to compare our theoretical predictions with experimental data obtained from the oscilloscope.

Our experiment underscored the importance of transient analysis in RLC circuits, emphasizing how changes in resistor values influenced the transition from underdamped to critically damped to overdamped responses. Throughout the experiment, we encountered real-world challenges, such as discrepancies in device precision and internal resistance, prompting a deeper consideration of the practical implications of theoretical calculations. In summary, our comprehensive approach allowed us to gain a nuanced understanding of RLC circuit dynamics and their correlation with theoretical predictions, all of which were visually represented through MATLAB plots.

Appendix

Lab Experiment 2 - RLC-Circuits - Frequency Response - PRELAB

Given is a series RLC resonant circuit with R = 390Ω, C = 270 nF and L = 10 mH. Name the filter characteristic measured over the different components, component combinations.

Resistor: Band Pass Filter

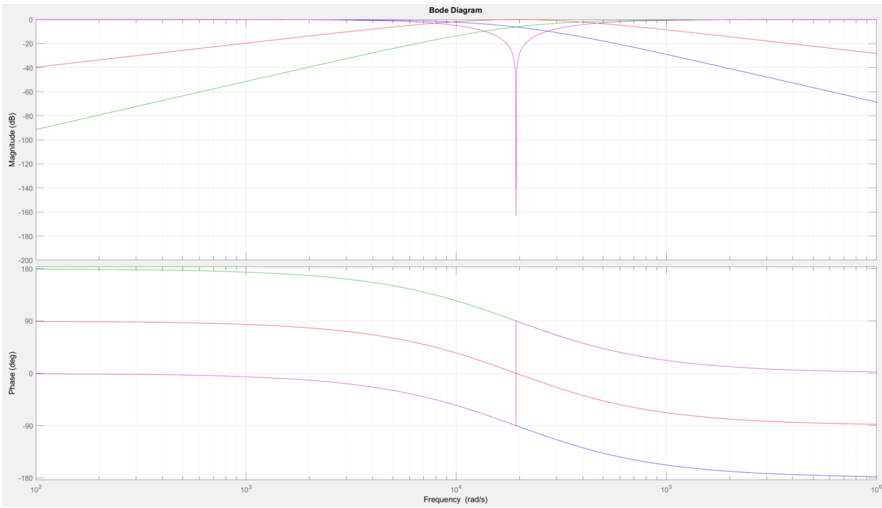
Capacitor: Low Pass Filter

Inductor: High Pass Filter

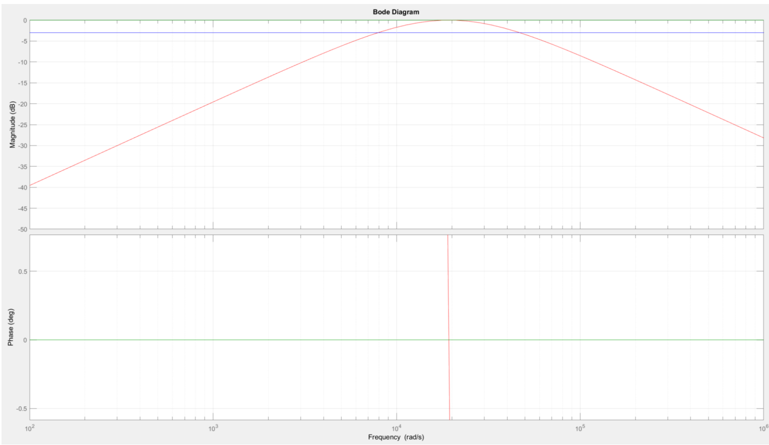
Inductor and Capacitor: Band Stop/Notch Filter

Show the Bode magnitude plot across the resistor, the capacitor, the inductor and across the capacitor and the inductor together. Use a 5 V amplitude and vary the frequency starting at 100 Hz up to 100 KHz. Develop a Matlab script to plot the four characteristic in one graph. Attach the script to the prelab!

```
%% Initialization
R = 390;
C = 270*10^-9;
L = 10*10^-3;
V = 5;
s = tf('s');
impC = 1/(s*C);
impL = s*L;
impR = R;
impT = impC + impL + impR;
%% Bode plots
%Capacitor
HC = impC/impT;
bode(HC, 'b');
grid on;
hold on;
%Resistor
HR = impR/impT;
bode(HR, 'r');
grid on;
hold on;
%Inductor
HL = impL/impT;
bode(HL , 'g');
grid on;
hold on;
%Capacitor and Inductor
HLC = (impC + impL)/impT;
bode(HLC, 'm');
grid on;
hold on;
```



Taking the magnitude across the resistance represents a band-pass filter. Calculate the bandwidth and the Q factor of the circuit. Extract the bandwidth from the Matlab plot and compare.



On zooming in on the points of intersection, we see that the curves intersect at the following radian frequencies:

$\omega 1 = 7924 rad/s$

$\omega 2 = 46743 rad/s$

From the information above, we can calculate the -3dB frequencies and bandwidth:

$f1 = \frac{7924}{2\pi} = 1261.14377 Hz$

$f2 = \frac{46743}{2\pi} = 7439.3975 Hz$

$B = |f1 - f2| = 6178.23573 Hz$

The cutoff frequency was also extracted from the plot using the same method:

$\omega 0 = 19250 rad/s$

$f0 = \frac{19250}{2\pi} = 3063.73265 Hz$

$Q = \frac{f0}{B} = 0.495891187$

References

Pagel, Uwe. *CH-520-B Systems and Signals Lab Manual*. 2023.