# Sistemas Complejos en Máquinas Paralelas

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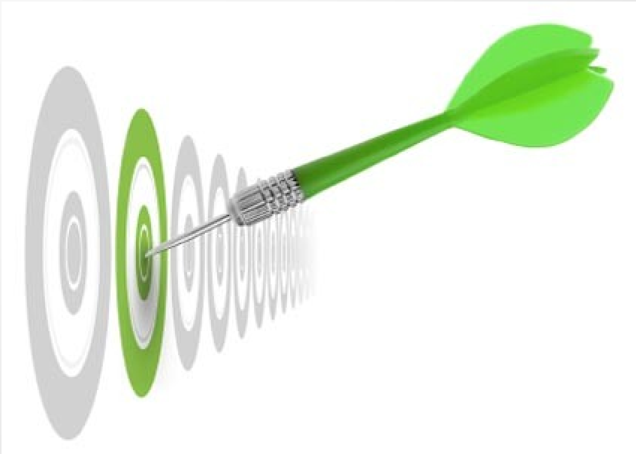
# Objetivos

- Clasificación de ecuaciones
- Métodos iterativos para Sistemas de Ecuaciones Lineales
- Problema del Valor Inicial
- Ecuación de difusión de temperatura, 2D, estado estacionario



# Clasificación Ecuaciones

- Lineales / No lineales
- Ordinarias / Parciales
- Orden
- Homogéneas / No homogéneas
- Elípticas, Parabólicas, Hiperbólicas

# Clasificación Ecuaciones

- Lineales / No lineales
- Ordinarias (1 var. indep.) / Parciales (n var. indep.)
- Orden (máxima derivada)
- Homogéneas ($b=0$) / No homogéneas ($b \neq 0$)
- Elípticas, Parabólicas, Hiperbólicas (más adelante...)

# Clasificación Ecuaciones

- Ecuación Lineal: $a_n x_n + \ldots + a_1 x_1 + a_0 x_0 = b$
  - Para 2 var.: $a_1 x_1 + a_0 x_0 = b \Rightarrow y = c_0 x + c_1 \qquad a_i, b, c_i \ \ constantes$
- Sistema de Ecuaciones Lineales

$$\begin{array}{l} a_{0,0} x_0 + \quad \ldots \quad + a_{0,n-1} x_{n-1} + a_{0,n} x_n = b_0 \\ \qquad\qquad\qquad \ldots \\ a_{n,0} x_0 + \quad \ldots \quad + a_{n,n-1} x_{n-1} + a_{n,n} x_n = b_n \end{array}$$

  - Forma matricial: $A X = B$

# Clasificación Ecuaciones

- Ecuación Diferencial Lineal
  - Ordinaria (una sola var. indep.)

$$a_n(x)\frac{d^n y}{dx^n}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)\,y=b(x)$$

$$Si\ a_n\neq 0,\ la\ ecuación\ es\ de\ orden\ n$$

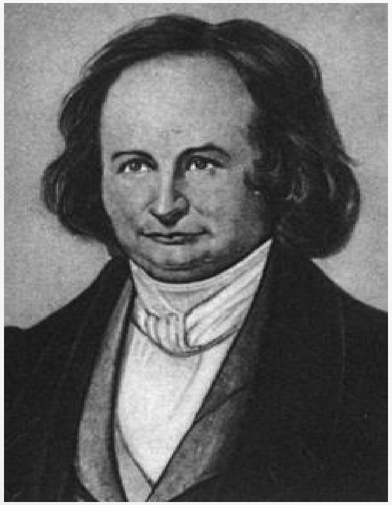
  - En Derivadas Parciales (más de una var. indep.)

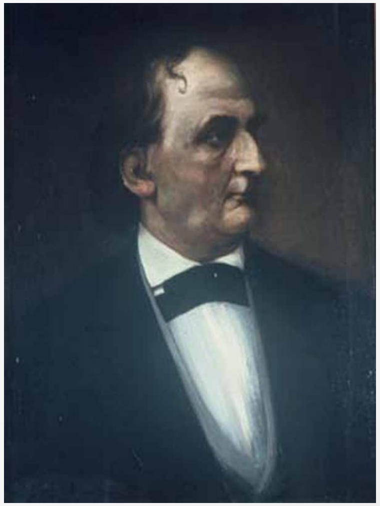
Ejemplo Orden 2, cantidad de var. Indep. 2

$$a(x,t)\frac{\partial^2 y}{\partial x^2}+b(x,t)\frac{\partial^2 y}{\partial x\,\partial t}+c(x,t)\frac{\partial^2 y}{\partial t^2}+d(x,t)\frac{\partial y}{\partial x}+e(x,t)\frac{\partial y}{\partial t}+f(x,t)\,y=b(x,t)$$

# Métodos Iterativos para Sistemas de Ecuaciones Lineales

## Jacobi y Gauss-Seidel



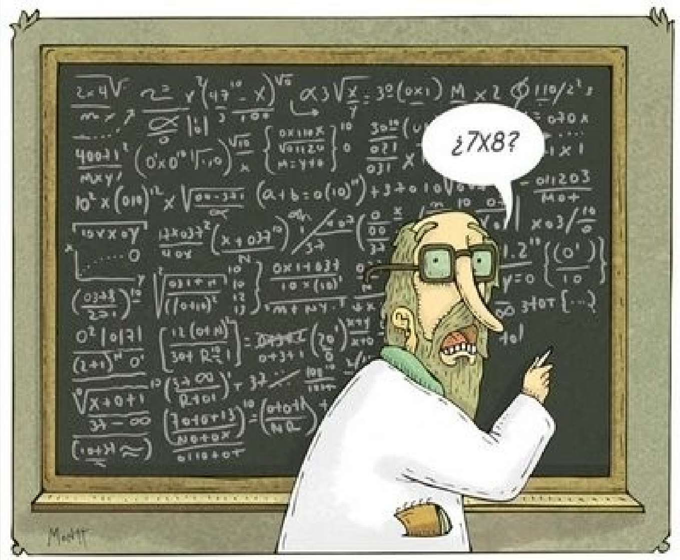


# Métodos numéricos para SEL

- Directos:
  - Consumen gran cantidad de operaciones
  - No aprovechan matrices ralas (=> mayor cantidad de operaciones)
  - Ejemplos:
    - Gauss
    - Gauss-Jordan
- Indirectos o iterativos
  - Buenos para matrices grandes
  - Aprovechan matrices ralas
  - Ejemplos:
    - **Jacobi o método de los desplazamientos simultáneos**
    - **Gauss-Seidel o método de los desplazamientos sucesivos**

# Ejemplo: Jacobi

- Dado un sistema de ecuaciones lineales: $AX=B$
  - Resolución por Jacobi: $x_i^k = \dfrac{b_i - \sum_{j, i \neq j} a_{i,j} x_j^{k-1}}{a_{i,i}}$

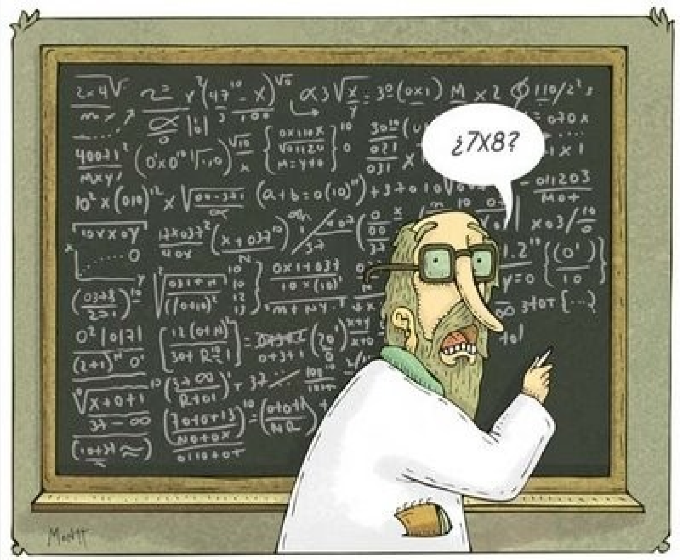



*Vamos al pizarrón...*

# Ejemplo: Gauss-Seidel

- Dado un sistema de ecuaciones lineales: $AX = B$
  - Resolución por Gauss-Seidel:

$$x_i^{k+1} = \frac{b_i}{a_{i,i}} - \sum_{j=1}^{i-1} \frac{a_{i,j}}{a_{i,i}} x_j^{k+1} - \sum_{j=i+1}^{n} \frac{a_{i,j}}{a_{i,i}} x_j^k$$




*Vamos al pizarrón...*

# Normas

- Vectoriales
  - Norma Euclidiana: $\|X\|_E = \left(\sum_i x_i^2\right)^{1/2}$
  - Norma-1: $\|X\|_1 = \sum_i |x_i|$
  - Norma del máximo, infinito o M: $\|X\|_M = \max_i |x_i|$

- Matriciales
  - Norma del máximo, infinito o M: $\|A\|_M = \max_i \sum_j |a_{i,j}|$
  - Norma de Lebesgue o L: $\|A\|_L = \max_j \sum_i |a_{i,j}|$

# Esquemas Iterativos: Gauss-Seidel y Jacobi

- Queremos resolver: **A X = B**

A=D-L-U

→ (D-L-U) X = B
→ D x - (L+U) X = B
→ D x = B + (L+U) X
→ x = $D^{-1}$ (B + (L+U) X)
→ $\mathbf{X^{(k+1)} = \underbrace{D^{-1} (L+U)}_{E} X^{(k)} + \underbrace{D^{-1}\ B}_{C}}$ **Jacobi**

→ (D-L-U) X = B
→ (D-L) X - U X = B
→ (D-L) X = U X + B
→ X = $(D-L)^{-1}$ (U X + B)
→ $\mathbf{X^{(k+1)} = \underbrace{(D-L)^{-1}\ U}_{E} X^{(k)} + \underbrace{(D-L)^{-1}\ B}_{C}}$ **Gauss-Seidel**

# Convergencia

- Una sucesión de vectores converge al vector X según una norma vectorial dada si $\lim_{k \to \infty} \|x - x_k\| = 0$
- Para cualquier método iterativo: $X_k = E\, X_{k-1} + C$
  - Si $\|E\|<1$ para alguna norma matricial inducida, la sucesión $\{X_k\}$ generada por la fórmula de iteración arriba mencionada converge a la única solución X del sistema X=EX+C (equivalente al original) cualquiera sea la aproximación inicial.
  - Para cualquier $X^0$, la sucesión $\{X_k\}$ definida por la fórmula de iteración $X_k=E\, X_{k-1}+C$ converge a la única solución X del sistema X=EX+C $\Leftrightarrow$ el radio espectral $\rho(E)<1$, donde
    - $\rho(E)=\max \{ \ |\lambda| \ / \ \lambda$ es autovalor de E }
    - Para calcular los autovalores de E: $| E - \lambda \ Id | = 0 \ \Rightarrow \ \lambda$
- Si la matriz A de un sistema A X = B es E.D.D. por filas entonces el método iterativo de Jacobi y Gauss-Seidel converge a la única solución X del sistema A X = B, cualquiera sea $X^0$.

  E.D.D. (estrictamente dominante diagonalmente) por filas $\sum_{j=1, j \neq i} |a_{i,j}| < |a_{i,i}|$

# Problema del Valor Inicial

- Problema bien planteado:
  - Ecuación gobernante. Ej. $\frac{dX}{dt}=2$
  - Condiciones **iniciales**/contorno. Ej. $X(t=0)=10$
- Integrado... $X(t)=2t+c$ ¿c?
- Aplicando la condición inicial/contorno... $X(t)=t+10$

# Problema del Valor Inicial

- Ejemplo difusión temperatura 1D, estado estacionario:
  - Ecuación gobernante. Ej. $\frac{d^2T}{dx^2}=0$
  - Condiciones iniciales/**contorno**. Ej. $T(x=0)=100$ $T(x=1)=0$
- Integrado... $T(x)=c_1x+c_2$ ¿$c_1$ y $c_2$?
- Aplicando las condiciones de inicial/**contorno**...

$$T(x)=-100x+100$$

# Planteo de un Ejercicio

- Conducción de calor en un disipador
- Modelado simplificado de sección transversal

- Ecuación de Laplace: $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$ y Condiciones de Contorno
- Discretización: $T(i,j-1)+T(i,j+1)+T(i-1,j)+T(i+1,j)-4T(i,j)=0$

  y Condiciones de Contorno Discretizadas

# Resolución: Armado del Sistema de Ecuaciones

- $T(i,j-1)+T(i,j+1)+T(i-1,j)+T(i+1,j)-4T(i,j)=0$

- $Supongamos\; i=1,2,3 \; y \; j=1,2,\ldots 7$

$$i=1, j=1 \Rightarrow \cancel{T(1,0)}+T(1,2)+\cancel{T(0,1)}+T(2,1)-4T(1,1)=0$$

$$\Rightarrow T(1,2)+T(2,1)-4T(1,1)=0$$

$$i=1, j=2 \Rightarrow T(1,1)+T(1,3)+\cancel{T(0,2)}+T(2,2)-4T(1,2)=0$$

$Etc\ldots$

# Resolución: Armado de la Representación Matricial

$$AT = B \Leftrightarrow$$

| | T(1,1) | T(1,2) | T(1,3) | T(1,4) | T(1,5) | T(1,6) | T(1,7) | T(2,1) | T(2,2) | T(2,3) | T(2,4) | T(2,5) | T(2,6) | T(2,7) | T(3,1) | T(3,2) | T(3,3) | T(3,4) | T(3,5) | T(3,6) | T(3,7) | T(i,j) | | B |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| T(1,1) | 4 | -1 | | | | | | -1 | | | | | | | | | | | | | | T(1,1) | | 0 |
| T(1,2) | -1 | 4 | -1 | | | | | | -1 | | | | | | | | | | | | | T(1,2) | | 0 |
| T(1,3) | | -1 | 4 | -1 | | | | | | -1 | | | | | | | | | | | | T(1,3) | | 0 |
| T(1,4) | | | -1 | 4 | -1 | | | | | | -1 | | | | | | | | | | | T(1,4) | | 0 |
| T(1,5) | | | | -1 | 4 | -1 | | | | | | -1 | | | | | | | | | | T(1,5) | | 0 |
| T(1,6) | | | | | -1 | 4 | -1 | | | | | | -1 | | | | | | | | | T(1,6) | | 0 |
| T(1,7) | | | | | | -1 | 4 | -1 | | | | | | -1 | | | | | | | | T(1,7) | | 0 |
| T(2,1) | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | | | | | | T(2,1) | | 0 |
| T(2,2) | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | | | | | T(2,2) | | 0 |
| T(2,3) | | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | | | | T(2,3) | = | 0 |
| T(2,4) | | | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | | | T(2,4) | | 0 |
| T(2,5) | | | | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | | T(2,5) | | 0 |
| T(2,6) | | | | | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | T(2,6) | | 0 |
| T(2,7) | | | | | | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | T(2,7) | | 0 |
| T(3,1) | | | | | | | | -1 | | | | | | -1 | 4 | -1 | | | | | | T(3,1) | | 0 |
| T(3,2) | | | | | | | | | -1 | | | | | | -1 | 4 | -1 | | | | | T(3,2) | | 0 |
| T(3,3) | | | | | | | | | | -1 | | | | | | -1 | 4 | -1 | | | | T(3,3) | | 0 |
| T(3,4) | | | | | | | | | | | -1 | | | | | | -1 | 4 | -1 | | | T(3,4) | | 0 |
| T(3,5) | | | | | | | | | | | | -1 | | | | | | -1 | 4 | -1 | | T(3,5) | | 0 |
| T(3,6) | | | | | | | | | | | | | -1 | | | | | | -1 | 4 | -1 | T(3,6) | | 0 |
| T(3,7) | | | | | | | | | | | | | | -1 | | | | | | -1 | 4 | T(3,7) | | 0 |

# Resolución: Agregado de Condiciones de Contorno

| | | | | | | |
|---|---|---|---|---|---|---|
| $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{CPU}$ |
| $T_{AMB}$ | ? | ? | ? | ? | ? | $T_{CPU}$ |
| $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{CPU}$ |

$$T(1,1)=T(1,2)=T(1,3)=T(1,4)=T(1,5)=T(1,6)=T_{AMB}$$

$$T(2,1)=T_{AMB}$$

$$T(3,1)=T(3,2)=T(3,3)=T(3,4)=T(3,5)=T(3,6)=T_{AMB}$$

$$T(1,7)=T(2,7)=T(3,7)=T_{CPU}$$

# Resolución: Agregado de Condiciones de Contorno

| | T(1,1) | T(1,2) | T(1,3) | T(1,4) | T(1,5) | T(1,6) | T(1,7) | T(2,1) | T(2,2) | T(2,3) | T(2,4) | T(2,5) | T(2,6) | T(2,7) | T(3,1) | T(3,2) | T(3,3) | T(3,4) | T(3,5) | T(3,6) | T(3,7) | T(i,j) | | B |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| T(1,1) | 1 | | | | | | | | | | | | | | | | | | | | | T(1,1) | | $T_{AMB}$ |
| T(1,2) | | 1 | | | | | | | | | | | | | | | | | | | | T(1,2) | | $T_{AMB}$ |
| T(1,3) | | | 1 | | | | | | | | | | | | | | | | | | | T(1,3) | | $T_{AMB}$ |
| T(1,4) | | | | 1 | | | | | | | | | | | | | | | | | | T(1,4) | | $T_{AMB}$ |
| T(1,5) | | | | | 1 | | | | | | | | | | | | | | | | | T(1,5) | | $T_{AMB}$ |
| T(1,6) | | | | | | 1 | | | | | | | | | | | | | | | | T(1,6) | | $T_{AMB}$ |
| T(1,7) | | | | | | | 1 | | | | | | | | | | | | | | | T(1,7) | | $T_{CPU}$ |
| T(2,1) | | | | | | | | 1 | | | | | | | | | | | | | | T(2,1) | | $T_{AMB}$ |
| T(2,2) | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | | | | | T(2,2) | | 0 |
| T(2,3) | | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | | | | T(2,3) | = | 0 |
| T(2,4) | | | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | | | T(2,4) | | 0 |
| T(2,5) | | | | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | | T(2,5) | | 0 |
| T(2,6) | | | | | | -1 | | | | | | -1 | 4 | -1 | | | | | | -1 | | T(2,6) | | 0 |
| T(2,7) | | | | | | | | | | | | | | 1 | | | | | | | | T(2,7) | | $T_{CPU}$ |
| T(3,1) | | | | | | | | | | | | | | | 1 | | | | | | | T(3,1) | | $T_{AMB}$ |
| T(3,2) | | | | | | | | | | | | | | | | 1 | | | | | | T(3,2) | | $T_{AMB}$ |
| T(3,3) | | | | | | | | | | | | | | | | | 1 | | | | | T(3,3) | | $T_{AMB}$ |
| T(3,4) | | | | | | | | | | | | | | | | | | 1 | | | | T(3,4) | | $T_{AMB}$ |
| T(3,5) | | | | | | | | | | | | | | | | | | | 1 | | | T(3,5) | | $T_{AMB}$ |
| T(3,6) | | | | | | | | | | | | | | | | | | | | 1 | | T(3,6) | | $T_{AMB}$ |
| T(3,7) | | | | | | | | | | | | | | | | | | | | | 1 | T(3,7) | | $T_{CPU}$ |

- $AT=B$, A es Estrictamente Dominante Diagonalmente

# Resolución: Pseudocódigo Jacobi

- Jacobi:

$$A\,X = B \qquad x_{i'}^{k} = \frac{B_{i'} - \sum\limits_{j',\, i' \neq j'} a_{i',j'}\, x_{j'}^{k-1}}{a_{i',i'}}$$

- Respecto del problema actual $x_{i'}^{k} = T^{k}(i, j)$

$$x_1^k = T^k(2,2)$$
$$x_2^k = T^k(2,3)$$
$$x_3^k = T^k(2,4)$$
$$x_4^k = T^k(2,5)$$
$$x_5^k = T^k(2,6)$$

$$\Rightarrow T^{k}(i,j) = \frac{T^{k-1}(i, j-1) + T^{k-1}(i, j+1) + T^{k-1}(i-1, j) + T^{k-1}(i+1, j)}{4}$$

*Nota: sólo para los elementos que no pertenecen al contorno, puesto que éstos ya son dato.*

# Resolución: Pseudocódigo Jacobi

1. *Definir e inicializar* ***variables y constantes*** *que se usarán en el problema*

   *$T=T^k$, $Tant=T^{k-1}$, error_it, tol_it, Tcpu, Tamb*

2. *Agregar* ***condiciones de borde*** *a T(i,j) y Tant(i,j)*
3. *Agregar* ***valores semilla*** *a T(i,j) y Tant(i,j)*
4. ***Calcular matriz***

   *Mientras no haya convergencia:* ***error_it>tol_it***

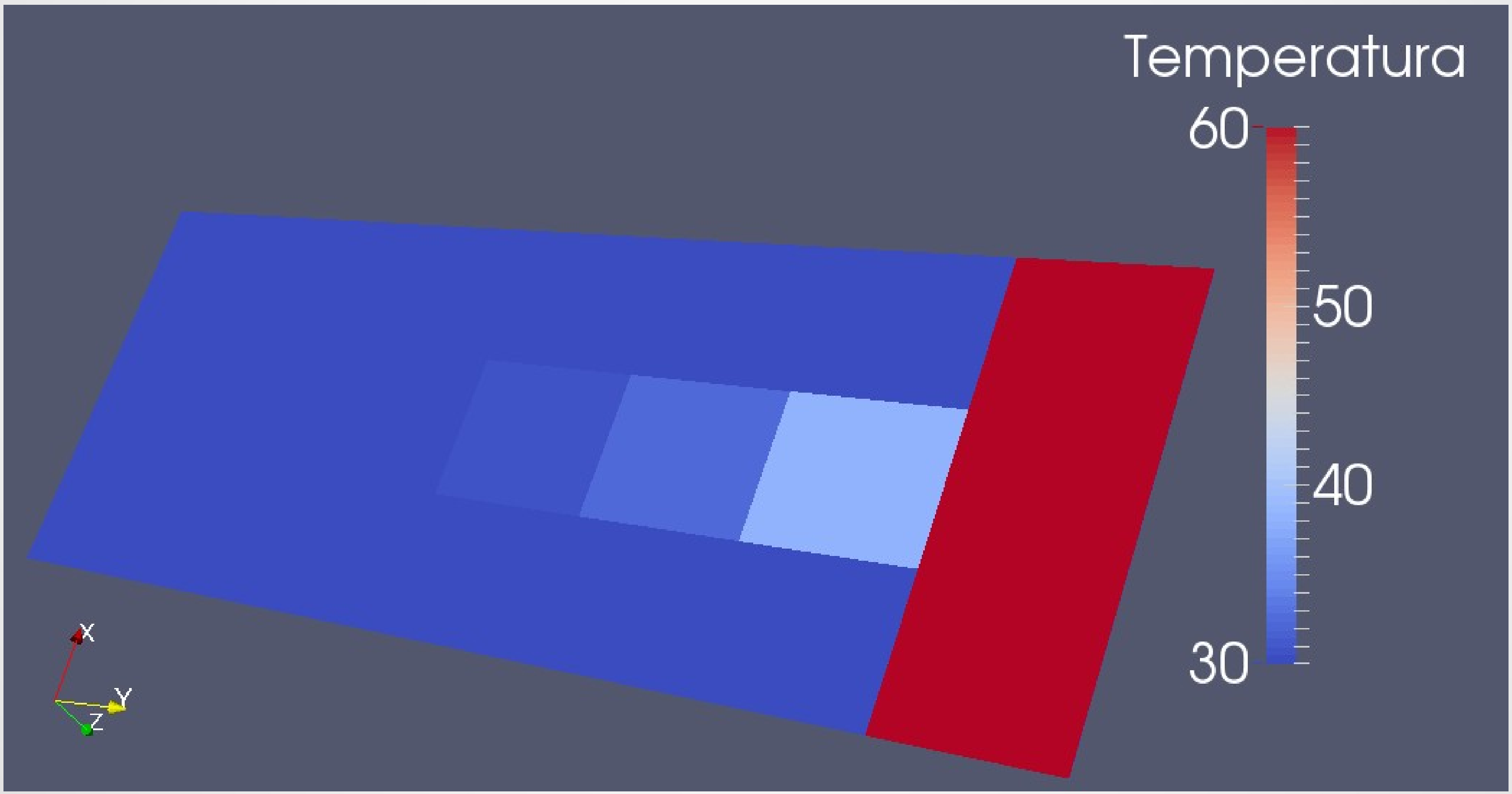
   1. *Para los T(i,j) que no pertenecen al contorno*

      $$T(i,j)=(T(i,j-1)+T(i,j+1)+T(i-1,j)+T(i+1,j))/4$$

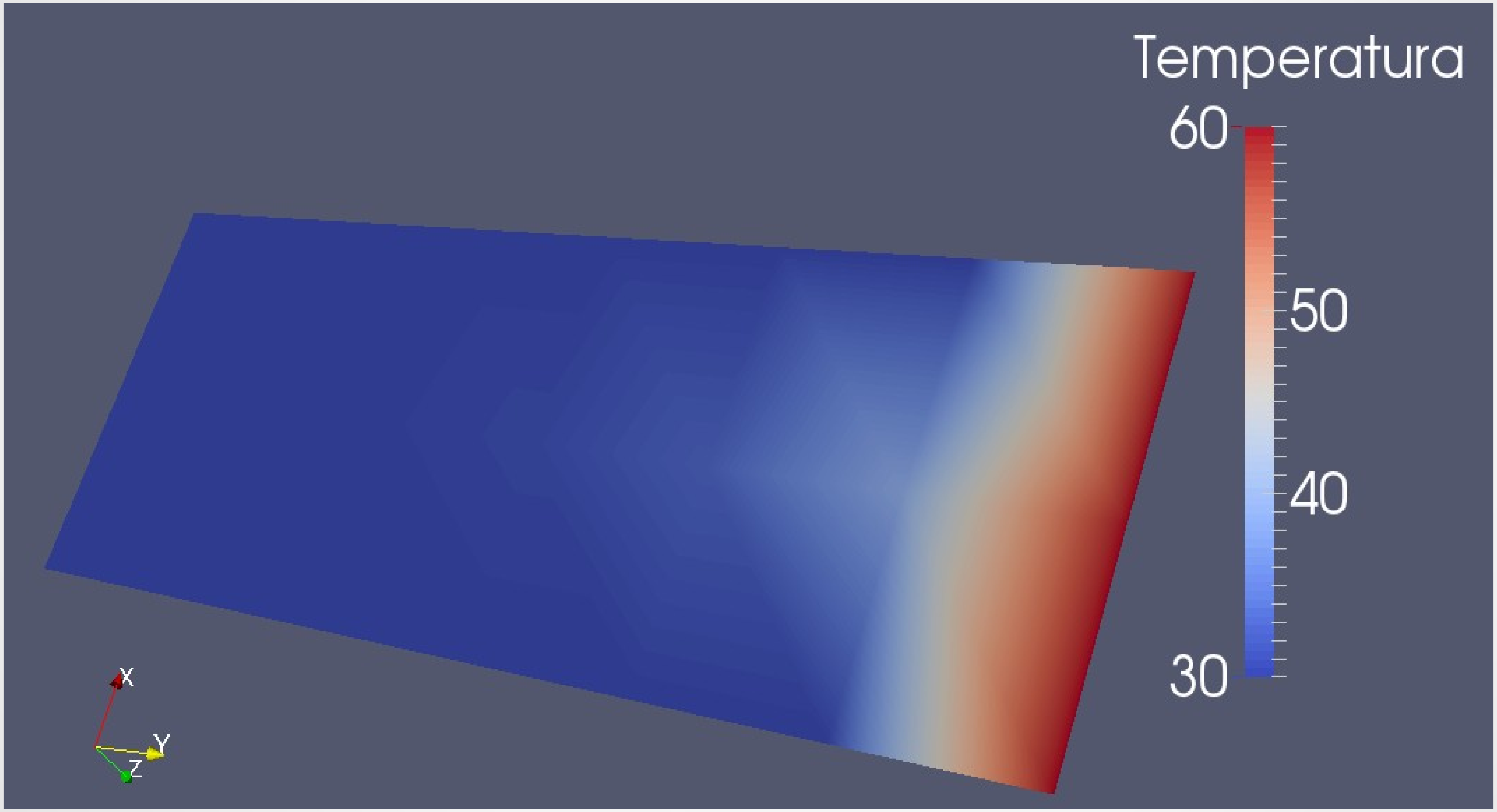
   2. *Calcular error:* ***error_it = normMax(T,Tant)***
   3. *Salvar estado actual:* ***Tant=T***

5. *Graficar y* ***salvar*** *los resultados: T*

# Resultado



# Resultado

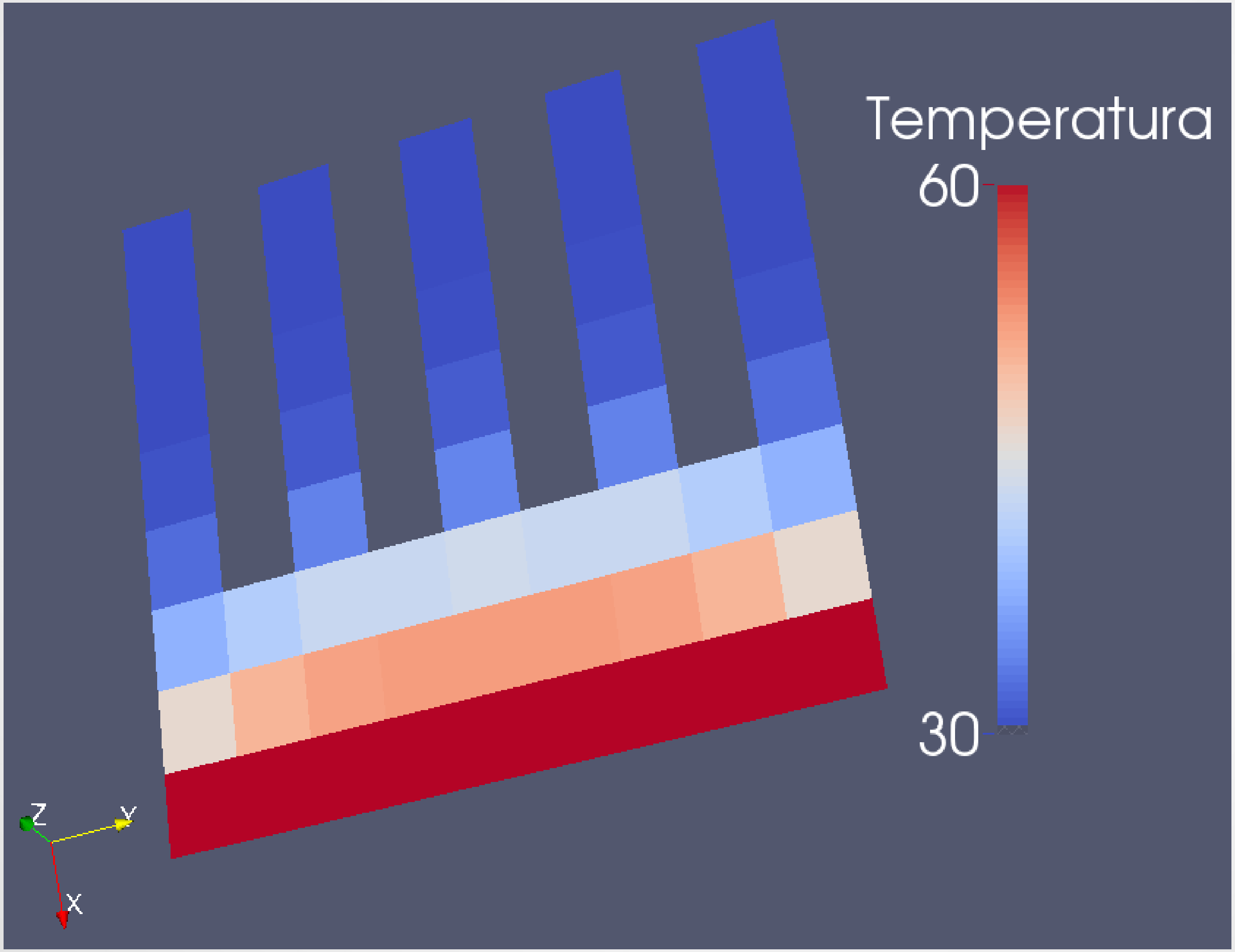



- Ojo con la interpolación!

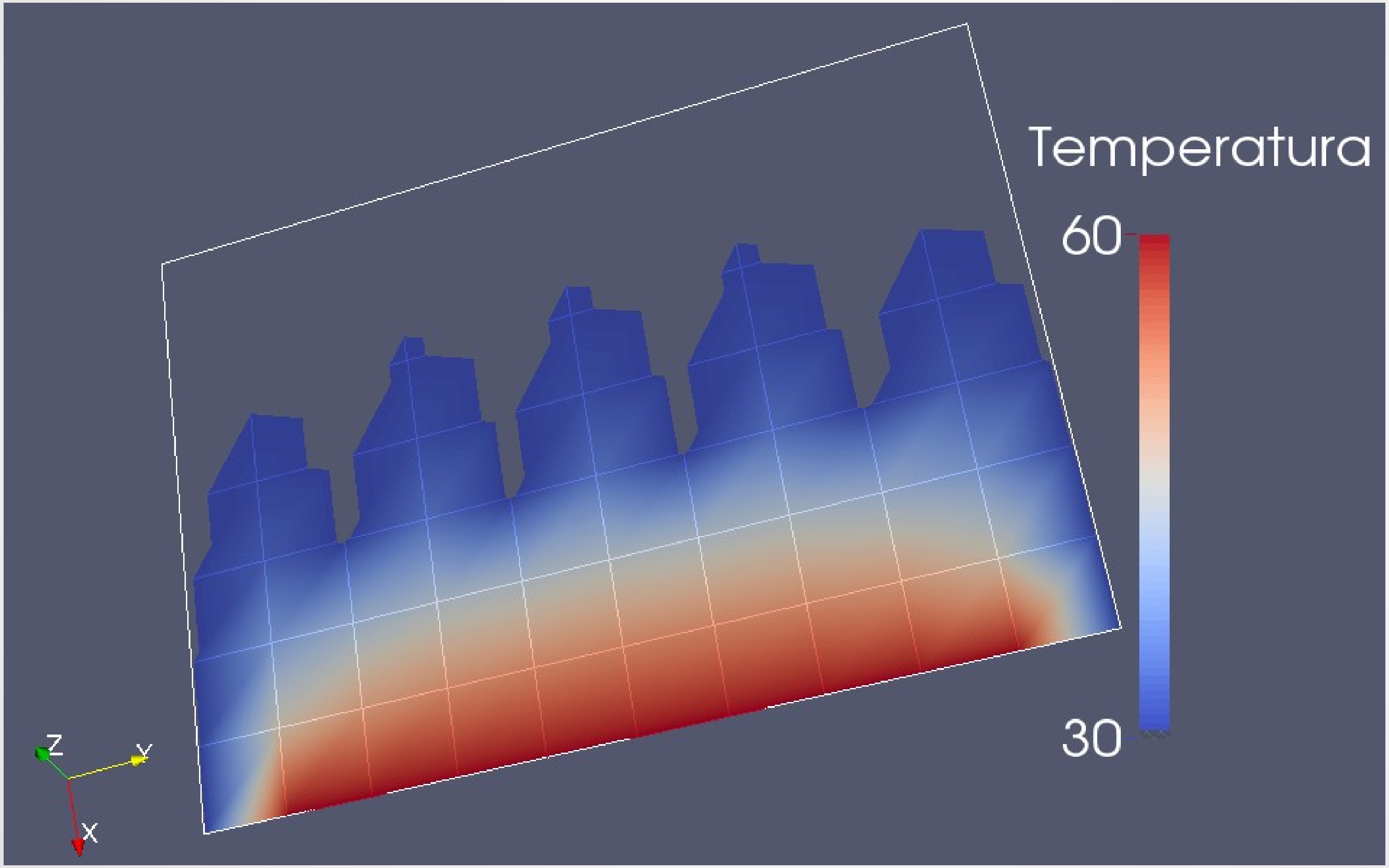
# Planteo y Resolución de Ejercicio

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ | $T_{AMB}$ |
| $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ |
| $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ |
| $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ |
| $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ |
| $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ | ? | $T_{AMB}$ |
| $T_{AMB}$ | ? | ? | ? | ? | ? | ? | ? | ? | ? | $T_{AMB}$ |
| $T_{AMB}$ | ? | ? | ? | ? | ? | ? | ? | ? | ? | $T_{AMB}$ |
| $T_{AMB}$ | $T_{CPU}$ | $T_{CPU}$ | $T_{CPU}$ | $T_{CPU}$ | $T_{CPU}$ | $T_{CPU}$ | $T_{CPU}$ | $T_{CPU}$ | $T_{CPU}$ | $T_{AMB}$ |

# Resultado



# Resultado




- Ojo con la interpolación!

# Un caso más complejo...

Thermal simulation of a power supply:

https://www.youtube.com/watch?v=XxoenSujLSY