Sampling

DSE 210

Outline

1. Laws of large numbers
2. Basic sampling designs
3. Confidence intervals

Review: Expected value

The expected value of a random variable X is

$$\mathbb{E}(X) = \sum_x x \Pr(X = x).$$

Linearity properties:

- $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$ for any random variable X and any constants a, b.
- $\mathbb{E}(X_1 + \cdots + X_k) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_k)$ for any random variables $X_1, X_2, \ldots, X_k$.

Example: Toss n coins of bias p, and let X be the number of heads. What is $\mathbb{E}(X)$?

Review: Variance

Variance of an r.v. X is $\text{var}(X) = \mathbb{E}(X - \mu)^2 = \mathbb{E}(X^2) - \mu^2$, where $\mu = \mathbb{E}(X)$.

Useful variance rules:

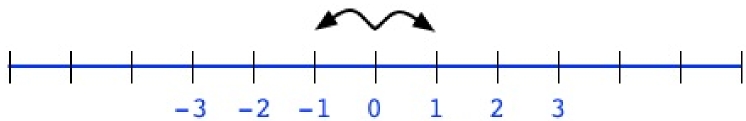
- $\text{var}(X_1 + \cdots + X_k) = \text{var}(X_1) + \cdots + \text{var}(X_k)$ if X_i's independent.
- $\text{var}(aX + b) = a^2\text{var}(X)$.

The standard deviation of X is $\sqrt{\text{var}(X)}$. It is (an approximation to) the average amount by which X differs from its mean.

Example: Toss a coin of bias p. Let $X \in \{0, 1\}$ be the outcome. What are the variance and standard deviation of X? For what p is the variance highest?

Variance of a sum

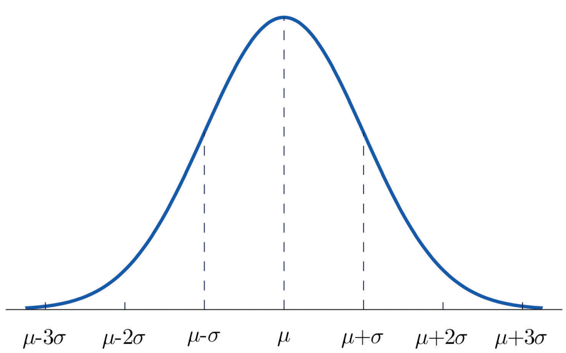
Symmetric random walk. A drunken man sets out from a bar. At each time step, he either moves one step to the right or one step to the left, with equal probabilities. Roughly where is he after n steps?



Let $X_i \in \{-1, 1\}$ be his ith step. His position after n steps is $X = X_1 + \cdots + X_n$.

- What are $\mathbb{E}(X_i)$ and $\text{var}(X_i)$?
- What are $\mathbb{E}(X)$ and $\text{var}(X)$?
- What is $\text{std}(X)$?
- What is the distribution over his possible positions?

The normal distribution



The normal (or *Gaussian*) $N(\mu, \sigma^2)$ has mean μ, variance σ^2, and density function

$$p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

- 68.3% of the distribution lies in the range $\mu \pm \sigma$
- 95.4% lies within $\mu \pm 2\sigma$
- 99.7% lies within $\mu \pm 3\sigma$

The central limit theorem

Suppose $X_1, \ldots, X_n$ are independent, each with mean μ and variance σ^2.

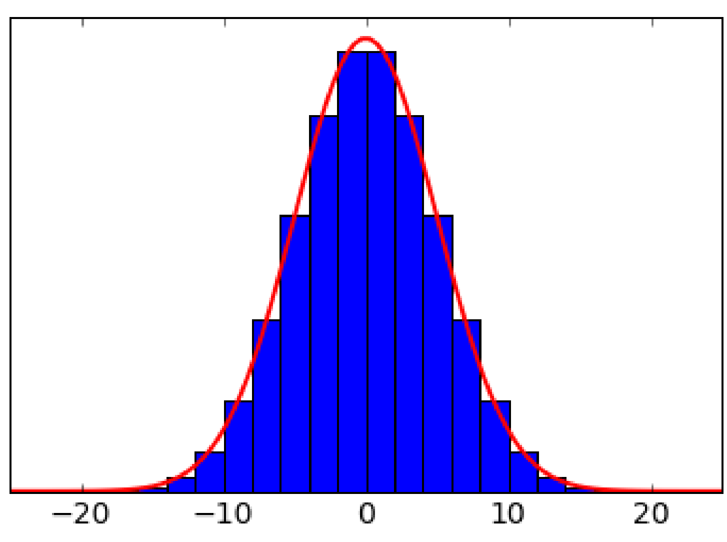
Let $S_n = X_1 + \cdots + X_n$. What are the mean and variance of S_n?

Central limit theorem, very roughly: For reasonably large n, the distribution of $S_n = X_1 + \cdots + X_n$ looks like $N(n\mu, n\sigma^2)$, the Gaussian with mean $n\mu$ and variance $n\sigma^2$.

Question: What does this imply about the distribution of the **average** $(X_1 + \cdots + X_n)/n$?

Symmetric random walk, again

Each X_i is either 1 or -1, with probability $1/2$. Thus $X_1 + \cdots + X_n$ is distributed like $N(0, n)$.



25 steps

Tosses of a biased coin

A coin of bias (heads probability) p is tossed n times.

- What is the distribution of the observed **number** of heads, roughly?
  Answer: $N(np, np(1-p))$
  Mean np, standard deviation on the order of $\sqrt{n}$.
- What is the distribution of the observed **fraction** of heads, roughly?
  Answer: $N(p, p(1-p)/n)$.
  Mean p, standard deviation on the order of $1/\sqrt{n}$.

Example: A town has 30,000 registered voters, of whom 12,000 are Democrats. A random sample of 1,000 voters is chosen. How many of them would we expect to be Democrats, roughly?

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Sampling design

In the 1948 Presidential election, the polls all predicted Thomas Dewey as the winner, with at least a five-point margin. But the outcome was quite different.

Selection bias

The Republican bias in the Gallup Poll, 1936-1948.

Year	Gallup's prediction of Republican vote	Actual Republican vote
1936	44	38
1940	48	45
1944	48	46
1948	50	45

The safest way to sample is **at random**.

Multistage cluster sampling

Sometimes random sampling is inconvenient, and careful multistage procedures are used.

For instance:

1. Stage 1
   - Divide the US into four geographical regions: Northeast, South, Midwest, West.
   - Within each region, group together all population centers of similar sizes. E.g. All towns in the northeast with 50-250 thousand people.
   - Pick a random sample of these towns.
2. Stage 2
   - Divide each town into wards, and each ward into precincts.
   - Select some wards at random from the towns chosen earlier.
   - Select some precincts at random from among these wards.
   - Then select households at random from these precincts.
   - Then select members of the selected households at random, within the designated age ranges.

Sample size versus population size

A certain town in Illinois has the same balance of Democrats and Republicans as the nation at large. We want to determine these fractions using a random sample of 1000 people. Would it be better to choose the 1000 people from the town in Illinois, or from the entire country?

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Example: Estimating a fraction

A university has 25,000 registered students. In a survey, 400 students were chosen at random, and it turned out that 317 of them were living at home. Estimate the fraction of students living at home.

Let p be the actual fraction of students living at home.

1. What is the observed fraction $\widehat{p}$?

2. Give error bars on this estimate.

Is there a problem here?

Since we don't know the true standard deviation $\sqrt{p(1-p)}$ of each sample, use the observed standard deviation $\sqrt{\widehat{p}(1-\widehat{p})}$.

- Estimate the standard deviation of $\widehat{p}$.

- The normal approximation gives confidence intervals:
  - 68.3% interval: 0.79 ± 0.02
  - 95.5% interval: 0.79 ± 0.04
  - 99.7% interval: 0.79 ± 0.06

  What is a 95% confidence interval for p?

- What does a "95% confidence interval" really mean?

Estimating an average

In a certain town, a random sample is taken of 400 people age 25 and over. The average years of schooling of this sample is 11.6 years, with a standard deviation of 4.1. Find a 95% confidence interval for the average educational level of people 25 and over in this town.

What is the distribution of the observed average?

- Let the true mean educational level be μ, with stddev σ.
- We draw n samples from this distribution, and take the average $\widehat{\mu}$.