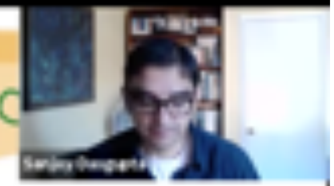
Sets and counting

DSE 210

Tuples (sequences)

$|C^k| = |C$



In a tuple (sequence), the order of elements matters:

$$(H, T) \neq (T, H).$$

Let $C = \{H, T\}$. ← set of two possible outcomes when you flip a coin

- All sequences of two elements from C:

$$\{(H, H), (H, T), (T, H), (T, T)\} = C \times C = C^2$$

- All sequences of three elements of C:

$$\{(H, H, H), (H, H, T), (H, T, H), \ldots\} = C \times C \times C = C^3$$

- All sequences of k elements from C: denoted $C^k = C \times C \times \cdots \times C$.

How many sequences of length k are there?

$|C|^k$

$\underset{1^{st}}{\_\_}, \underset{2^{nd}}{\_\_}, \underset{3^{rd}}{\_\_}, \underset{4^{th}}{\_\_}, \cdots, \underset{k^{th}}{\_\_}$

$(c, o, m, p, u, t, e, r, s, c)$ $= 26^{10}$ (s, k) (c, t)

$26 \cdot 26 \cdot 26$

Consider sequences drawn from set $A = \{a, b, c, \ldots, z\}$. $\leftarrow$

- How many sequences are there of length 2? 26^2

  $(\_, \_)$

- How many sequences of length 10? 26^{10}

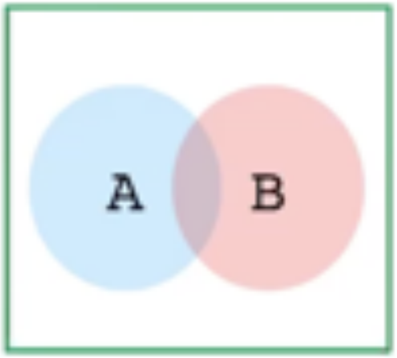
- How many sequences of length n? 26^n

An alien language has an alphabet of size 10. Every sequence of ≤ 5 of these characters is a valid word. How many words are there in this language?

#words of length 1 + #words length 2 + .. + #words length 5

$= 10 + 10^2 + 10^3 + 10^4 + 10^5 = 111{,}110$

Union and intersection



Venn diagram

$A \cup B = \{\text{any element in } A \text{ or in } B \text{ or in both}\}$
$A \cap B = \{\text{any element in } A \text{ and in } B\}$

$M = \{2, 3, 5, 7, 11\}$ and $N = \{1, 3, 5, 7, 9\}$

- What is $M \cup N$? $\{1, 2, 3, 5, 7, 9, 11\}$
- What is $M \cap N$? $\{3, 5, 7\}$

$S = \{\text{all even integers}\}$ and $T = \{\text{all odd integers}\}$

- What is $S \cup T$? $\{\text{all integers}\}$
- What is $S \cap T$? ϕ empty set $\{\}$

Permutations

How many ways to order the three letters A, B, C?

$$ABC, ACB, BAC, BCA, CAB, CBA$$

3 choices for the first, 2 choices for the second, 1 choice for the third
$3 \times 2 \times 1 = 6$. Call this 3!

General rule: The number of ways to order n distinct items is:

$$n! = n(n-1)(n-2)\cdots 1.$$

- How many ways to order A, B, C, D, E? $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
- How many ways to place 6 men in a line-up? $6! = 720$
- How many possible outcomes of shuffling a deck of cards? $52!$

Combinations

Worksheet #1
→ all problems

An ice-cream parlor has flavors {chocolate, vanilla, strawberry, pecan}. You are allowed to pick two of them. How many options do you have?

$$CV, CS, CP, VS, VP, SP$$

In general, the number of ways to pick k items out of n is:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

For instance, $\binom{4}{2} = \frac{4 \cdot 3}{2!} = 6.$ $\qquad \frac{4!}{2!\,2!} = \frac{4\cdot 3\cdot 2\cdot 1}{(2\cdot 1)(2\cdot 1)} = 6$

- How many ways to pick three ice-cream flavors? $\binom{4}{3} = \frac{4!}{3!\,1!} = 4$
- Pick any 4 of your favorite 100 songs. How many ways to do this?

$$\binom{100}{4} = \frac{100!}{4!\,96!} = \frac{100\cdot 99\cdot 98\cdot 97}{4\cdot 3\cdot 2\cdot 1} = ?$$

Probability spaces

DSE 210

Discrete probability spaces

How should we interpret a statement like the following?

The chance of getting a flush in a five-card poker hand is about 0.20%.
(Flush = five of the same suit.)

One possible outcome:
$\{3\heartsuit, 5\clubsuit, J\diamondsuit, A\heartsuit, 2\clubsuit\}$

The underlying **probability space** has two components:

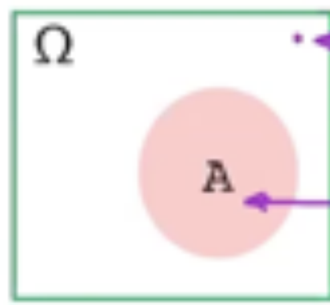
1. The **sample space** (the space of outcomes).
   In the example, $\Omega = \{\text{all possible five-card hands}\}$.
2. The **probabilities of outcomes**.
   In the example, all hands are equally likely: probability $1/|\Omega|$.

$|\Omega| = \binom{52}{5}$

Note: $\sum_{\omega \in \Omega} \Pr(\omega) = 1$.

Event of interest: the set of outcomes
$A = \{\omega : \omega \text{ is a flush}\} \subset \Omega$.

$$\Pr(A) = \sum_{\omega \in A} \Pr(\omega) = \frac{|A|}{|\Omega|}$$



some five-card hand

all hands that are flushes

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Examples

prob. space

event of interest

Roll a die. What is the chance of getting a number > 3?

- What is the sample space Ω?

  $$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- What are probabilities of outcomes?

  $$Pr(\omega) = 1/6$$

this fully defines the probability space

- What is the event of interest?

  $$A = \{4, 5, 6\}$$

always a subset of Ω

- What is the probability of the event of interest?

  $$Pr(A) = \sum_{\omega \in A} Pr(\omega) = Pr(4) + Pr(5) + Pr(6) = \frac{1}{2}$$

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Roll three dice. What is the chance that their sum is 3?

$$\Omega = \{(a,b,c) : \quad a,b,c \in \{1,2,3,4,5,6\}\}$$

$$= \{1,2,..,6\} \times \{1,2,..,6\} \times \{1,2,..,6\}$$

$$= \{1,2,3,4,5,6\}^3$$

$$|\Omega| = 6^3 = 216$$

$Pr(\omega) = \frac{1}{216}$ for each $\omega \in \Omega$

Event of interest $\quad A = \{(1,1,1)\}$

$$Pr(A) = \frac{1}{216}$$

Roll n dice. What is the chance their sum is $6n$?

Sample space $\Omega = \{1,2,3,4,5,6\}^n$

$$|\Omega| = 6^n$$

$Pr(\omega) = \frac{1}{6^n}$ for each $\omega \in \Omega$

Event of interest $A = \{(6,6,\ldots,6)\}$, so $|A| = 1$

$$Pr(A) = \frac{1}{6^n}$$

outcome = (first sock color, second sock color)

Socks in a drawer. A drawer has three blue socks and three red socks. You put your h
and pull out two socks at random. What is the probability that they match?

(Think of grabbing one sock first, then another.)

$$\Omega = \{ (B,B), (B,R), (R,B), (R,R) \} = \{B,R\}^2$$

$$Pr((B,B)) = \frac{3}{6} \cdot \frac{2}{5} = \frac{1}{5}$$

$$Pr((B,R)) = \frac{3}{6} \cdot \frac{3}{5} = \frac{3}{10}$$

$$Pr((R,B)) = \frac{3}{6} \cdot \frac{3}{5} = \frac{3}{10}$$

$$Pr((R,R)) = \frac{3}{6} \cdot \frac{2}{5} = \frac{1}{5}$$

these add up to 1

$$\sum_{\omega \in \Omega} Pr(\omega) = 1$$

We have fully defined the probability space

Event of interest $A = \{(B,B), (R,R)\}$

$$Pr(A) = Pr((B,B)) + Pr((R,R)) = \frac{2}{5}$$

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Socks in a drawer, cont'd. This time the drawer has three blue socks and two red soc[ks] put your hand in and pull out two socks at random. What is the probability that they match?

$$\Omega = \{B, R\}^2$$

$$\Pr((B,B)) = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$$

$$\Pr((B,R)) = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$$

$$\Pr((R,B)) = \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{10}$$

$$\Pr((R,R)) = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}$$

they add up to 1

Event of interest $A = \{(B,B), (R,R)\}$

$$\Pr(A) = \frac{2}{5}$$

probability space

event of interest

Shuffle a pack of cards. What is the probability that all the reds come before all the blacks?

$\Omega = \{\text{all orderings of 52 cards}\}$

$|\Omega| = 52!$

$Pr(\omega) = \frac{1}{52!}$ for each $\omega \in \Omega$

$A = \{\text{all orderings in which all the reds precede all the blacks}\}$

$|A| = 26! \cdot 26!$

$\therefore Pr(A) = \dfrac{26!\,26!}{52!}$

Toss a fair coin 10 times.

$\Omega = \{H, T\}^{10}$ = all sequences of 10 coin tosses

$|\Omega| = 2^{10} = 1024$

$Pr(\omega) = \frac{1}{1024}$ for each $\omega \in \Omega$

Probability space

- What is the chance none are heads?

$Pr((T, T, \ldots, T)) = \frac{1}{1024}$

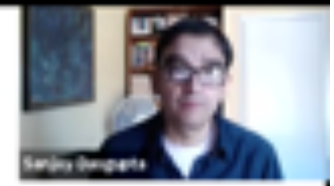
- What is the probability of exactly one head? (Each such sequence can be specified by the location of the one H, and there are 10 choices for this.)

$A = \{(H,T,T,\ldots,T), (T,H,T,\ldots,T), \ldots, (T,T,\ldots,T,H)\}$

there are 10 places that one H could be

$|A| = 10$, so $Pr(A) = \frac{10}{1024}$

Toss a fair coin 10 times. What is the chance of exactly two heads?



$A = \{$sequences of 10 coin tosses with exactly two heads$\}$

$$(\_, \underline{H}, \_, \_, \_, \_, \_, \underline{H}, \_, \_)$$

$|A| = \binom{10}{2} = 45 \leftarrow$ there are ten slots
\# of ways to pick two of these slots (in which to put H)

$$\therefore \Pr(A) = \frac{45}{1024}$$

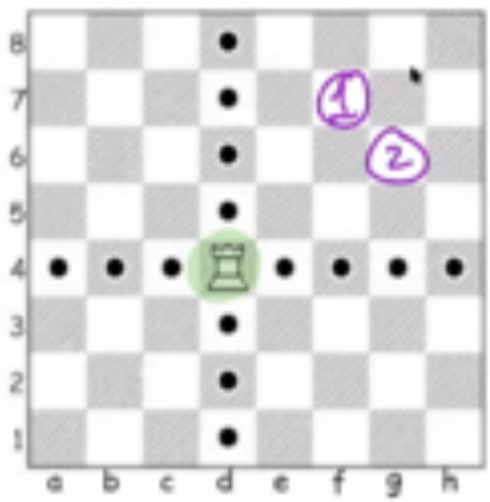
What is the probability of exactly k heads?

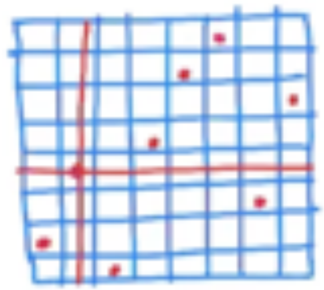
$A = \{$sequences of 10 coin tosses with exactly k heads$\}$

$$|A| = \binom{10}{k}$$

$$\Pr(A) = \frac{\binom{10}{k}}{1024}$$

Rooks on a chessboard.





What is the maximum number of rooks you can place so that no rook is attacking any other?

8

How many ways are there to place 8 rooks on the board, attacking or not? $\boxed{\binom{64}{8}}$

Choose 8 squares out of 64 $\longrightarrow$

How many non-attacking placements of 8 rooks are there? $8!$

$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Randomly place 8 rooks on the board. What is the probability that it is a non-attacking placement?

$$8! / \binom{64}{8} = \frac{8!\,8!}{64 \cdot 63 \cdot 62 \cdot 61 \cdots 57}$$

Five-card poker. You are dealt 5 cards from a deck of 52.

- What is the probability space?

$\Omega = \{\text{all five-card hands}\}$ $\quad |\Omega| = \binom{52}{5}$

$Pr(\omega) = \frac{1}{\binom{52}{5}}$ for any given five-card hand ω

- What is the probability of getting a flush (five of the same suit)?

$F = \{\text{five cards of the same suit}\}$

$|F| = 4 \times \binom{13}{5}$

$Pr(F) = \frac{4 \times \binom{13}{5}}{\binom{52}{5}}$

- What is the probability of a straight flush (flush, and in sequence)?

$S = \{\text{same suit and they form a sequence}\}$ $\quad \{6\heartsuit, 4\heartsuit, 3\heartsuit, 7\heartsuit, 5\heartsuit\}$

$|S| = \underbrace{4}_{\text{choose suit}} \times 10$ $\quad \therefore Pr(S) = \frac{4 \times 10}{\binom{52}{5}}$

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Birthday paradox. Interesting fact: in a group of 23 random people, chances are some of them have a common birthday!

- A room contains k people.
  Number the people $1, 2, \ldots, k$.
  Number the days of the year $1, 2, \ldots, 365$.

  $\omega = (\omega_1, \omega_2, \ldots, \omega_k) \in \{1, 2, \ldots, 365\}$

  ω_1: birthday of person 1; ω_k: birthday of person k

  Let $\omega = (\omega_1, \ldots, \omega_k)$, where $\omega_i \in \{1, 2, \ldots, 365\}$ is the birthday of person i.
  Thus $\Omega = \{1, 2, \ldots, 365\}^k$.
- What is $|\Omega|$?

  365^k

- Event of interest: $A = \{(\omega_1, \ldots, \omega_k) : \text{all } \omega_i \text{ different}\}$. What is $|A|$?

  $365 \cdot 364 \cdot 363 \cdot \cdots (365 - k + 1)$ ← # outcomes in which all birthdays are different
- What is $\Pr(A)$?

$$\Pr(A) = \frac{365 \cdot 364 \cdots \cdot (365 - k + 1)}{365^k} = 1 \cdot \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\left(1 - \frac{3}{365}\right) \cdots \left(1 - \frac{k-1}{365}\right)$$

When $k \geq 23$, this is $< 1/2$.

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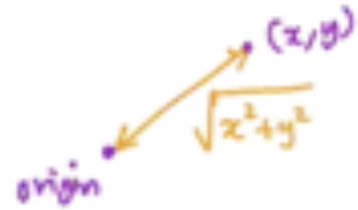
Dartboard. A dartboard has radius 1 and its central bullseye has radius 0.1. You throw and it lands at a random location on the board.



$$\Pr(\text{lands in } A) = \frac{\text{area of } A}{\text{area of board}}$$

- What is the sample space?

$\Omega = \{(x,y) : x^2+y^2 \le 1\}$ ← all points in the unit circle

- What is the probability of hitting the bullseye?

$$\Pr(\text{falls in bullseye}) = \frac{\text{area of bullseye}}{\text{area of dartboard}} = \frac{\pi(0.1)^2}{\pi(1)^2} = \frac{1}{100}$$

- What is the probability of hitting the exact center? 0

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Uniform distribution on [0, 1].

- What is Ω?

$\Omega = [0,1]$

$Pr(\text{outcome is } \frac{1}{2}) = 0$

$\int$

- What is the probability of interval [0.2, 0.8]?

0.6

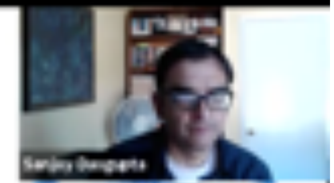
- What density function $p(x)$ can we use?

$$p(x) = \begin{cases} 1 & \text{for } x \in [0,1] \\ 0 & \text{outside } [0,1] \end{cases}$$

$$Pr(\text{fall in } [a,b]) = \underbrace{\int_a^b p(x)\,dx}_{\text{integral of } p(x) \text{ between } a \text{ and } b} = \int_a^b 1\,dx = x\Big]_a^b = b-a$$

$0 \le a \le b \le 1$

Uniform distribution on $[-1, 1]$.



- What is the probability of interval $[0, 0.4]$?

0.2, because $[0, 0.4]$ is 0.2 of the entire interval $[-1, 1]$

- What density function can we use?

$$p(x) = \frac{1}{2}$$

Same density everywhere, so say it is c.

$$\int_{-1}^{1} p(x)\, dx = \int_{-1}^{1} c\, dx = [cx]_{-1}^{1} = 2c$$

Uniform distribution on $[a, b]$. What density function can we use?

density $p(x) = \frac{1}{b-a}$

Uniform on $[a, b]$

- Pr (land at a specific pt) $= 0$
- Pr (land in an interval) $= \frac{\text{length of interval}}{b-a}$

Multiple events, conditioning, and independence

DSE 210

People's probability judgements

Experiment by Kahneman-Tversky. Subjects were told:

> *Linda is 31, single, outspoken, and very bright. She majored in philosophy in college. As a student, she was deeply concerned with racial discrimination and other social issues, and participated in anti-nuclear demonstrations.*

They were then asked to rank three possibilities:

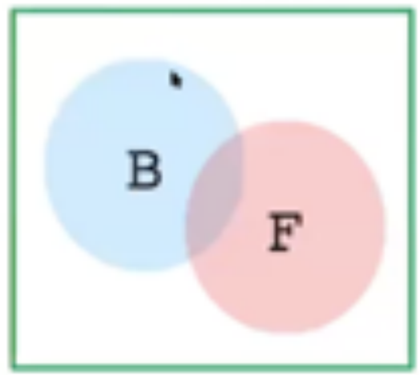
(a) Linda is active in the feminist movement.

(b) Linda is a bank teller.

(c) Linda is a bank teller and is active in the feminist movement.

Over 85% respondents chose (a) $>$ (c) $>$ (b).

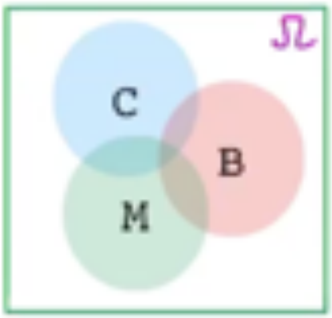
But: $\Pr(\text{bank teller}, \text{feminist}) \leq \Pr(\text{bank teller})$.



In a city, 60% of people have a car, 20% of people have a bike, and 10% of people ha[...] motorcycle. Anyone without at least one of these walks to work. What is the minimu[...] fraction of people who walk to work?

Let $\Omega = \{\text{people in the town}\}$.
Let $C = \{\text{has car}\}$, $B = \{\text{has bike}\}$, $M = \{\text{has motorcyle}\}$, $W = \{\text{walks}\}$.

General picture:



$\Pr(W) \geq 1 - \Pr(C \cup B \cup M)$
What is the least $\Pr(W)$ could be?

$$\Pr(C \cup B \cup M) \leq \Pr(C) + \Pr(B) + \Pr(M)$$

$$= 0.6 + 0.2 + 0.1 = 0.9$$

$$\therefore \Pr(W) \geq 0.1$$

Complements and unions

Let Ω be a sample space.

The union bound: For any events $E_1, \ldots, E_k \subset \Omega$:

$$\Pr(E_1 \cup \cdots \cup E_k) \leq \Pr(E_1) + \cdots + \Pr(E_k).$$

When is this inequality exact? ← when $E_1, E_2, \ldots, E_k$ are DISJOINT (zero intersection)

The **complement** of an event:

For any event $E \subset \Omega$, let E^c be the event that E does **not** occur. That is, $E^c = \Omega \setminus E$.

How is $\Pr(E^c)$ related to $\Pr(E)$? $\Pr(E^c) = 1 - \Pr(E)$

Either E happens, or it doesn't happen.

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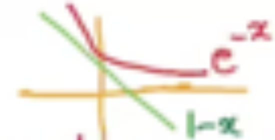
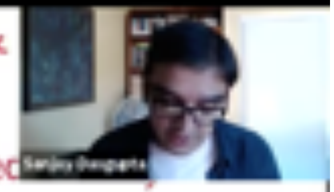
Coupon-collector problem

$\boxed{1-x \le e^{-x}}$





Question: Each cereal box has one of k action figures. How many boxes do you need so that you are likely to get all k figures?

Answer: To get all k figures with probability $\geq 1/2$, buy $O(k \log k)$ boxes.

Say we buy n boxes. Let A_i be the event that the ith action figure is *not* obtained.

- Give an exact expression for $\Pr(A_i)$.

$$\Pr(A_i) = \Pr(\text{not in } 1^{st} \text{ box}) \cdot \Pr(\text{not in } 2^{nd} \text{ box}) \cdot \dots \cdot \Pr(\text{not in } n^{th} \text{ box})$$
$$= \left(1-\frac{1}{k}\right) \cdot \left(1-\frac{1}{k}\right) \cdot \dots \cdot \left(1-\frac{1}{k}\right) = \left(1-\frac{1}{k}\right)^n$$

- Give an upper bound on $\Pr(A_1 \cup A_2 \cup \cdots \cup A_k)$.

$$\Pr(\text{miss at least one figure}) = \Pr(A_1 \cup A_2 \cup \cdots \cup A_k)$$
$$\leq \Pr(A_1) + \Pr(A_2) + \Pr(A_3) + \dots + \Pr(A_k) = k\left(1-\frac{1}{k}\right)^n$$
$$\leq k e^{-n/k}$$

- What value of n will make this bound $\leq 1/2$?

If $n = k \ln 2k$ then $k e^{-n/k} = k e^{-\ln 2k} = \frac{1}{2}$

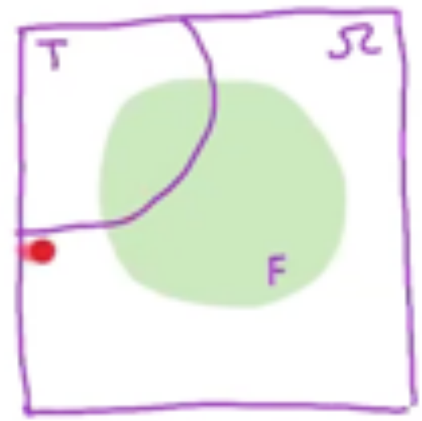
$\therefore n \geq k \ln 2k \Rightarrow \Pr(\text{fail to get all } k \text{ action figures}) \leq \frac{1}{2}$.

Conditioning

You meet a stranger at a bar. What is the chance that he/she could become a lifelong friend?

Just use the average for your town. 0.3, say

You notice that he/she is wearing a "Star Wars" t-shirt.



Sample space Ω = {all people in your town}. Two events of interest:

- F = {could become lifelong friend}
- T = {wears Star Wars t-shirts}

What we want is:

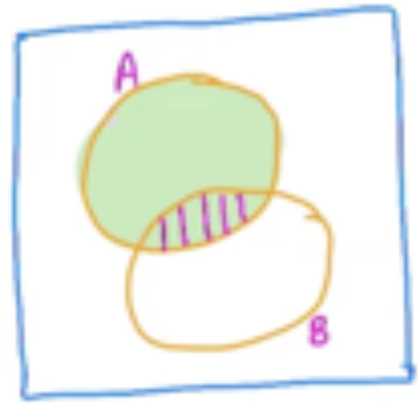
$\Pr(F|T)$ = fraction of Star Wars t-shirt wearers who are potential lifelong friends

"given"

Can you express this in terms of $\Pr(F \cap T)$, $\Pr(F)$, $\Pr(T)$?

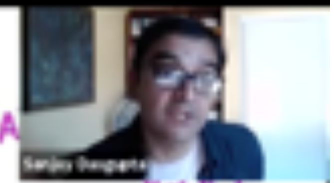
$$\Pr(F|T) = \frac{\Pr(F \cap T)}{\Pr(T)}$$

Conditional probability



Pr(B|A

Pr(A)



For two events A, B, **conditional probability**

$$\Pr(B|A) = \text{probability that } B \text{ occurs, given that } A \text{ occurs}$$

↑ "given that"

Conditioning formula: $\boxed{\Pr(A \cap B) = \Pr(A)\Pr(B|A)}$

Virus test.

The following data is obtained on a virus test:

- $\Omega = \{\text{people who use the test}\}$
- $V = \{\text{people using the test who have the virus}\}$
- $P = \{\text{people for whom the test comes out positive}\}$

It turns out that $P \subset V$ and $\Pr(V) = 0.4$ and $\Pr(P) = 0.3$.

- Suppose the test comes out positive. What is the chance of having the virus?

$$\Pr(V \mid P) = \frac{\Pr(V \cap P)}{\Pr(P)} = \frac{0.3}{0.3} = 1$$

- Suppose the test comes out negative. What is the chance of having the virus?

$$\Pr(V \mid P^c) = \frac{\Pr(V \cap P^c)}{\Pr(P^c)} = \frac{0.4 - 0.3}{1 - 0.3} = \frac{0.1}{0.7} = \frac{1}{7}$$

Rolls of a die.

(first, second) e.g. (4,2)

You roll a die twice. What is the probability that the sum is ≥ 10:

- If the first roll is 6?

$$Pr(\text{sum} \geq 10 \mid \text{first} = 6) = Pr(\text{second} \geq 4) = \frac{1}{2}$$

- If the first roll is ≥ 3?

(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)

$$Pr(\text{sum} \geq 10 \mid \text{first} \geq 3) = \frac{Pr(\text{first} \geq 3 \ \underline{\text{AND}} \ \text{sum} \geq 10)}{Pr(\text{first} \geq 3)} = \frac{6/36}{2/3} = \frac{1}{4}$$

- If the first roll is < 6?

(5,5), (5,6), (4,6)

$$Pr(\text{sum} \geq 10 \mid \text{first} < 6) = \frac{Pr(\text{first} < 6 \ \underline{\text{AND}} \ \text{sum} \geq 10)}{Pr(\text{first} < 6)} = \frac{3/36}{5/6} = \frac{1}{10}$$

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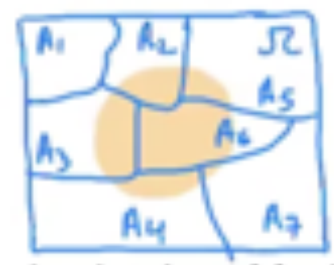
Summation rule

Suppose events $A_1, \ldots, A_k$ are disjoint and $A_1 \cup \cdots \cup A_k = \Omega$: that is, one of these events must occur. Then for any other event E,

$$\begin{aligned}\Pr(E) &= \Pr(E \cap A_1) + \Pr(E \cap A_2) + \cdots + \Pr(E \cap A_k) \\ &= \Pr(E|A_1)\Pr(A_1) + \Pr(E|A_2)\Pr(A_2) + \cdots + \Pr(E|A_k)\Pr(A_k)\end{aligned}$$

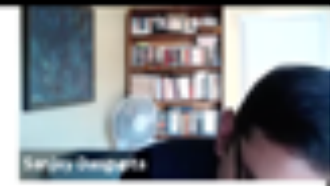


Example: What fraction of North Americans play ice hockey?

$$\begin{aligned}&\Pr(\text{play ice hockey}) \\ &= \Pr(\text{play ice hockey, Canada}) + \Pr(\text{play ice hockey, USA}) + \Pr(\text{play ice hockey, Mexico}) \\ &= \Pr(\text{play ice hockey}|\text{Canada})\ \Pr(\text{Canada}) \\ &\quad + \Pr(\text{play ice hockey}|\text{USA})\ \Pr(\text{USA}) \\ &\quad + \Pr(\text{play ice hockey}|\text{Mexico})\ \Pr(\text{Mexico})\end{aligned}$$

Sex bias in graduate admissions

Simpson's



In 1969, there were 12673 applicants for graduate study at Berkeley.
44% of the male applicants were accepted, and 35% of the female applicants.

Define:

- $\Omega = \{\text{all applicants}\}$
- $M = \{\text{male applicants}\}$
- $F = \{\text{female applicants}\} = M^c$
- $A = \{\text{accepted applicants}\}$

What do the percentages 44% and 35% correspond to?

- $\Pr(A|M) = 0.44$ and $\Pr(A|F) = 0.35$.

Engineering / Science

Accept everyone

Men applying to these

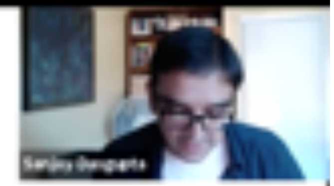
Humanities

Reject everyone

Women applying to these

The administration found, however, that in every department, the accept rate for female applicants was at least as high as the accept rate for male applicants. How could this be?

Independence



Two events A, B are **independent** if the probability of B occurring is the same whether or not A occurs.

Example: toss two coins.
$A = \{\text{first coin is heads}\}$
$B = \{\text{second coin is heads}\}$

Formally, we say A, B are independent if $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

The independence of A and B implies:

- $\Pr(A|B) = \Pr(A)$
- $\Pr(B|A) = \Pr(B)$
- $\Pr(A|B^c) = \Pr(A)$

Examples: independent or not?

1. You have two children.
   $A = \{\text{first child is a boy}\}$, $B = \{\text{second child is a girl}\}$

   indpt

2. You throw two dice.
   $A = \{\text{first is a six}\}$, $B = \{\text{sum} > 10\}$

   dependent

   $Pr(B|A) = Pr(\text{second} = 5 \text{ or } 6) = 1/3$

   $Pr(B) = Pr(\{(5,6), (6,5), (6,6)\}) = 3/36 = 1/12$

   not the same

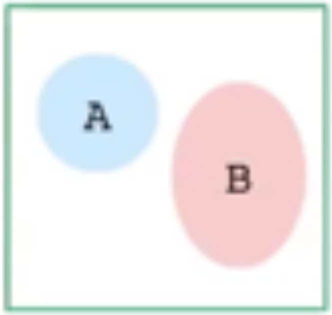
3. You get dealt two cards at random from a deck of 52.
   $A = \{\text{first is a heart}\}$, $B = \{\text{second is a club}\}$

   dependent

   $\underbrace{Pr(B|A)}_{13/51} > \underbrace{Pr(B)}_{\frac{1}{4}}$

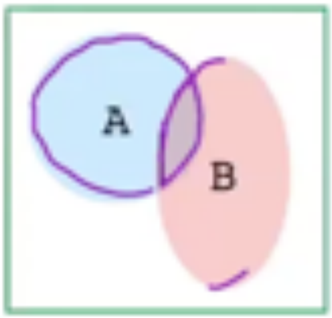
4. You are dealt two cards.
   $A = \{\text{first is a heart}\}$, $B = \{\text{second is a 10}\}$

   indpt

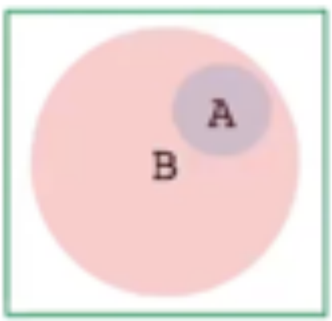


Dependent : $Pr(A|B) = 0$

$Pr(A) > 0$



May be indpt, not enough information to decide



Dependent $Pr(A|B) = 1$

$Pr(A) < 1$