$$x = \begin{pmatrix} 3 \\ 5 \\ -3 \\ 2 \end{pmatrix} \quad 4 \times 1$$

Probabilistic reasoning using Bayes' rule

DSE 210

Pearl's alarm scenario

You wake up in the middle of the night to the shrill sound of your burglar alarm. What is the chance that a burglary has been attempted?

Pearl's alarm scenario

You wake up in the middle of the night to the shrill sound of your burglar alarm. What is the chance that a burglary has been attempted?

The facts:

- There is a 95% chance that an attempted burglary will trigger the alarm.

$$\Pr(\text{alarm}|\text{burglary}) = 0.95$$

- There is a 1% chance of a false alarm.

$$\Pr(\text{alarm}|\text{no burglary}) = 0.01$$

- Based on local crime statistics, there is a 1-in-10,000 chance that a given house will be burglarized on a given night.

$$\Pr(\text{burglary}) = 10^{-4}$$

We need to compute $\Pr(\text{burglary}|\text{alarm})$.

Apply $\Pr(A \cap B) = \Pr(A)\Pr(B|A)$ twice to get

$\Pr(B|A) = \dfrac{\Pr(A,B)}{\Pr(A)}$

$$\rightarrow \Pr(\text{burglary}|\text{alarm}) = \frac{\Pr(\text{burglary, alarm})}{\Pr(\text{alarm})} = \frac{\Pr(\text{alarm}|\text{burglary})\Pr(\text{burglary})}{\Pr(\text{alarm})} \leftarrow ?$$

- $\Pr(\text{alarm}|\text{burglary}) = 0.95$
- $\Pr(\text{alarm}|\text{no burglary}) = 0.01$
- $\Pr(\text{burglary}) = 10^{-4}$

$$\Pr(\text{alarm}) = \underbrace{\Pr(\text{alarm, burglary})}_{\Pr(\text{burglary})\Pr(\text{alarm}|\text{burglary})} + \underbrace{\Pr(\text{alarm, no burglary})}_{\Pr(\text{no burglary})\Pr(\text{alarm}|\text{no burglary})}$$

$$= 10^{-4} \cdot 0.95 + (1 - 10^{-4})\, 0.01$$

$$\Pr(\text{burglary}|\text{alarm}) = \frac{0.95 \times 10^{-4}}{10^{-4} \times 0.95 + (1-10^{-4})(0.01)} \approx 0.01$$

The alarm increases in our belief in burglary about a hundredfold.

Bayes' rule for reasoning about evidence

Two events A, B

- We are interested in A
- We can observe B

If we find out B occurred, how does it alter the probability of A?

Bayes' rule: $\Pr(A|B) = \Pr(A) \times \dfrac{\Pr(B|A)}{\Pr(B)}$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)\Pr(B|A)}{\Pr(B)}$$

Example: Ten coins

You have ten coins. Nine are fair, but one is a bad coin that always comes up tails.

- You close your eyes and pick a coin at random.
- You toss it four times, and it comes up tails every time.

What is the probability you picked the bad coin?

$$\Pr(\text{bad coin} \mid \text{four tails}) = \underbrace{\Pr(\text{bad coin})}_{1/10} \times \frac{\overbrace{\Pr(\text{four tails} \mid \text{bad coin})}^{1.0}}{\Pr(\text{four tails})} \leftarrow ?$$

$$\begin{aligned}
\Pr(\text{four tails}) &= \Pr(\text{four tails, bad coin}) + \Pr(\text{four tails, not bad coin}) \\
&= \Pr(\text{bad coin}) \Pr(\text{four tails} \mid \text{bad coin}) + \\
&\quad \Pr(\text{not bad coin}) \Pr(\text{four tails} \mid \text{not bad coin}) \\
&= \frac{1}{10} \cdot 1 + \frac{9}{10} \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{10} + \frac{9}{10} \cdot \frac{1}{16}
\end{aligned}$$

- Ten coins: nine are fair, one is a bad coin that always comes up tails.
- You pick a coin at random, toss it four times, and it's tails every time.

$$\Pr(\text{bad coin} \mid \text{four tails}) = \Pr(\text{bad coin}) \times \frac{\Pr(\text{four tails} \mid \text{bad coin})}{\Pr(\text{four tails})}$$

$$= \frac{1}{10} \times \frac{1}{\frac{1}{10} + \frac{9}{10} \times \frac{1}{16}}$$

$$= \frac{1}{1 + 9/16} = \frac{16}{25} = 0.64 \quad [\text{ie. } 64\%]$$

Chance of a bad coin went from 10% in the absence of information to 64% with the additional information.

The three prisoners paradox

Three prisoners – A, B, C – are in a jail one night and one of them (they don't know whom) will be declared guilty and executed in the morning. Racked by worry, prisoner A calls the prison guard and begs to be told whether he is the unlucky one. The guard is not allowed to tell him – but he can say only that B will be declared innocent. Now A thinks to himself, "previously my chance of being executed was 1/3, and now, because of an innocuous inquiry, it seems to have gone up to 1/2. How can this be?"

Analyze using these events:

- G_A = the event that A will be declared guilty
- I_B = the event that the guard, when prompted, will declare B innocent

What is $\Pr(G_A|I_B)$?

$$\Pr(G_A|I_B) = \Pr(G_A) \cdot \frac{\Pr(I_B|G_A)}{\Pr(I_B)} \quad \text{(Bayes' rule)}$$

$$= \frac{1}{3} \cdot \frac{1/2}{\underbrace{\Pr(G_A)}_{1/3}\underbrace{\Pr(I_B|G_A)}_{1/2} + \underbrace{\Pr(\text{not } G_A)}_{2/3}\underbrace{\Pr(I_B|\text{not } G_A)}_{1/2}} = \boxed{\frac{1}{3}}$$