DSE 210 (Probability and Statistics Using Python)
Homework 1, due Friday 1/22 9:00am

Instructions

Please follow these instructions when completing your assignment:

- Please upload your written answers to Gradescope by the due date. **Late submissions will not be graded**.
- You can write up your answers using pencil and paper or using document editing software (LaTeX, Word, etc...). If you write your answers using pencil and paper, you can scan your answers and upload the resulting file to Gradescope or take pictures of each page and upload those.
- For written answers you are not required to show work. However, showing work will enable better feedback.
- Collaboration is encouraged, but all submissions should be in your own writing and completed with your own understanding.

Worksheet 1

1. (a) $A \times A \times A = A^3$
   (b) $|A|^3 = 125$
2. (a) $|A \cup B| \leq |A| + |B| = 12$
   (b) $|A \cup B| \geq \max(|A|, |B|) = 7$
   (c) $|A \cap B| \leq \min(|A|, |B|) = 5$
   (d) $|A \cap B| \geq 0$
3. $2^{10} = 1024$
4. Assuming we don't want to repeat flavors $\binom{10}{3} = 120$
5. There are 6 choices for the first, 5 for the second, and 4 for the third, so $6 \cdot 5 \cdot 4 = 120$

Worksheet 2

1. (a) $\Omega = \{\texttt{H}, \texttt{T}\}$
   (b) $\Omega = \{\text{red, black, silver, blue}\} \times \{\text{beige, black}\}$
   (c) $\Omega = \{\text{Jan,Feb,...,Mar} \times \text{Mon,Tues,...,Sun}\}$
   (d) $\Omega = \{\texttt{H}, \texttt{T}\}^{100}$
2. (a) $E = A \cap B \cap C$
   (b) $E = A \cup B \cup C$
   (c) $E = (A \cup B) \setminus C$

5 $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 1 - \Pr(A^c) + \Pr(B) - \Pr(A \cap B) = 11/12$

6 The rolls are independent so $\sum_{n=1}^{6} \Pr(A = n \cap B = n) = \sum_{n=1}^{6} \Pr(A = n)\Pr(B = n) = 1/6$

7 Any placement of the first rook is equivalent. Given some placement of the first rook, there are 63 possible placements of the second rook, of which 14 are attacking so $\Pr(\text{attacking}) = 14/63 = 2/9$

9 Let N be the number on the roll of the die. Let $\Pr(N = 1) = x$. Then $\Pr(N = 2) = 2x, \Pr(N = 3) = 3x, \ldots$. Furthermore: $\sum_{n=1}^{6} \Pr(N = n) = \sum_{n=1}^{6} n \cdot x = 1$. Therefore, $x = 1/21$ and so $\Pr(N = 2) + \Pr(N = 4) + \Pr(N = 6) = 12/21 = 4/7$

10 There are 5! ways to arrange 5 people, and exactly 1 is sorted in increasing order of height, so: $1/(5!) = 1/120$

11 Any choice of suit for the first card is equivalent, so without loss of generality assume the first card was a heart. Given this, there are 12/51 remaining hearts in the deck, so $\Pr(\text{same suit}) = 12/51$.

13 We can think of each outcome as a binary string of length 6. There are $2^6 = 64$ such strings of which $\binom{6}{3} = 20$ have exactly 3 ones. So $\Pr(3 \text{ girls}) = 20/64 = 5/16$.

15 It's easier to bound the probability of the complement in this case. Let E_n be the event that at least one digit was a 7. Then E_n^c is the event that there are *no* 7's in a string of n digits. We want $\Pr(E_n) \geq 0.9 \Rightarrow 1 - \Pr(E_n^c) \geq 0.9 \Rightarrow \Pr(E_n^c) \leq 0.1$. $\Pr(E_n^c) = 0.9^n$ so we can solve the bound for n to obtain $n \geq 22$.

Worksheet 3

1 Let E be the event that there was at least one 6. Then E^c is the event that there are no sixes in any of the rolls. $\Pr(E) = 1 - \Pr(E^c) = 1 - \left(\frac{5}{6}\right)^3 = 0.4213$

2 The complement of the event that there are two or more heads is that there were zero or one heads. Now, $\Pr(\text{no heads}) = 1/1024$ and $\Pr(\text{one heads}) = 10/1024$. Therefore $\Pr(\text{two or more heads}) = 1 - 11/1024 = 0.9893$.

6 (a) Given that we already got one head, we need exactly one of the two remaining tosses to be heads, which occurs with probability 1/2.

(b) Now we need both remaining tosses to be heads which occurs with probability 1/4

(c) We need the final toss to be a tails which occurs with probability 1/2

(d) We're out of luck. It's impossible to get two heads.

(e) The second outcome needs to be a heads which occurs with probability 1/2

8 $\Pr(A \cap B) = \Pr(A|B)\Pr(B) = \Pr(A|B)(1 - \Pr(B^c)) = 3/8$.

9 (a) The second roll must be a $5, 6$ or 7 which occurs with probability 1/2

(b) We're out of luck - it's impossible

(c) Let A be the event that the first roll was > 3 and B be the event that the sum of rolls is > 7. We can use the rule $\Pr(B|A) = \Pr(A \cap B)/\Pr(A)$. We can just count up the possible outcomes to obtain $\Pr(A \cap B) = 12/36$. Since $\Pr(A) = 1/2$, we have $\Pr(B|A) = 24/36 = 2/3$

(d) Using the same process as in part (c), we obtain $\Pr(A \cap B) = 6/36$ and $\Pr(A) = 4/6$, so $\Pr(B|A) = 1/4$

11 Consider some arbitrary student. Let $G = 1$ if the student is from Gryffindor and define H, R, S analogously. Let $D = 1$ if the student is good at dark arts. Then by the summation rule: $\Pr(D) = \Pr(D|G)\Pr(G) + \Pr(D|H)\Pr(H) + \Pr(D|R)\Pr(R) + \Pr(D|S)\Pr(S) = 1/2$

12 (a) Let $D = 1$ if a randomly chosen car is defective and let $F_i = 1$ if a randomly chosen car was made in factory i. Then $\Pr(D) = \sum_{i=1}^{3} \Pr(D|F_i)\Pr(F_i) = 0.0345$

(b) By definition $\Pr(F_1|D) = (\Pr(D|F_1)\Pr(F_1))/\Pr(D) = 0.3623$

14 (a) Let $P = 1$ if the outcome of the test was positive and let $D_i = 1$ if the patient has disease i. Then: $\Pr(P) = \sum_{i=1}^{3} \Pr(P|D_i)\Pr(D_i) = 0.6$

(b) As in 12 part (b): $\Pr(D_1|P) = (\Pr(P|D_1)\Pr(D_1))/\Pr(P) = 4/9$. Using the same process: $\Pr(D_2|P) = 1/3$, $\Pr(D_3|P) = 2/9$

16 (a) $\Pr(B|A) = \frac{12}{51}$ but $\Pr(B) = \frac{13}{52}$ so the events are **dependent**.

(b) $\Pr(B|A) = 1/13$ and $\Pr(B) = \frac{4}{52} = 1/13$ so the events are **independent**.

(c) $\Pr(B|A) = \frac{4}{51}$ but $\Pr(B) = \frac{4}{52}$ so the events are **dependent**

(d) $\Pr(B|A) = \frac{1}{13} = \Pr(B)$ so the events are **independent**.