DSE210_HW1

Worksheet 1

1.

(a)

$$A = \{1, 2, 3, 4, 5\}$$

$$A \times A \times A = A^3$$

(b)

$$5 \times 5 \times 5 = 5^3 = 125$$

2.

$$|A| = 5 \ and \ |B| = 7$$

(a) Largest size $A \cup B$ = 5 + 7 = 12

(b) Smallest size $A \cup B = 7$

(c) Largest size $A \cap B = 5$

(d) Smallest size $A \cap B = 0$

3.

2^{10}

4.

$\binom{10}{3}$

$= \frac{10\times9\times8}{3\times2\times1} = 120$

5.

$\binom{6}{3} \times 3! = \frac{6\times5\times4}{3\times2\times1} \times 3 \times 2 \times 1 = 120$

Worksheet 2

1.

(a)

$$\Omega = \{H, T\}$$

(b)

$$\Omega = \{(red, beige), (black, beige), (silver, beige), (blue, beige), (red, black), (black, black), (silver, black), (blue, black)\}$$

Or

$$\Omega = \{(exterior, interior) : exterior \in (red, black, silver, blue), interior \in (beige, black)\}$$

(c)

$$\Omega = \{(M, D) : M \in \{Jan,\ Feb, \ldots, Dec\}, D \in \{Mon,\ Tue, \ldots, Sun\}\}$$

Or

$$\Omega = \{1, 2, 3, \ldots, 12\} \times \{1, 2, 3, \ldots, 7\}$$

(d)

$$\Omega = \{H, T\}^{100}$$

2.

(a) $A \cap B \cap C$

(b) $A \cup B \cup C$

(c) $(A \cap B) \setminus C$

5.

$$P_r(A \cup B) = P_r(A) + P_r(B) - P_r(A \cap B)$$
$$= 1 - P_r(A^c) + P_r(B) - P_r(A \cap B)$$
$$= 1 - \tfrac{1}{3} + \tfrac{1}{2} - \tfrac{1}{4}$$
$$= \tfrac{11}{12}$$

6.

$$\tfrac{1}{6} \times \tfrac{1}{6} \times 6 = \tfrac{1}{6}$$

7.

$|\Omega| = \binom{64}{2}$

$|A| = 64 \times (7+7)/2$

$\frac{|A|}{|\Omega|} = \frac{2}{9}$

9.

$P_r(1) + P_r(2) + P_r(3) + P_r(4) + P_r(5) + P_r(6) = 1$

$21P_r(1) = 1$

$P_r(1) = \frac{1}{21}$

$P_r(2) \cup P_r(4) \cup P_r(6)$

$= 12P_r(1)$

$= \frac{12}{21}$

$= \frac{4}{7}$

10.

$\Omega = \{order\ of\ heights\ for\ 5\ people\}$

$|\Omega| = 5!$

$P_r(\omega) = 1/5! = \frac{1}{120}$

$I = \{increasing\ order\ of\ height\}$

$P_r(I) = 1/5! = \frac{1}{120}$

11.

$\Omega = \{13\ cards\ are\ dealt\ from\ a\ deck\ of\ cards.\}$

$|\Omega| = \binom{52}{2}$

$P_r(\omega) = 1/\binom{52}{2}$

$S = \{first\ and\ second\ cards\ are\ of\ the\ same\ suit\}$

$|S| = 4 \times \binom{13}{2}$

$P_r(S) = \frac{|S|}{|\Omega|} = \binom{13}{2} \times 4/\binom{52}{2}$

$= \frac{12}{51}$

13.

$\Omega = \{6 \textit{ children's gender.}\}$

$|\Omega| = 2^6$

$P_r(\omega) = 1/2^6$

$S = \{3 \textit{ girls and } 3 \textit{ boys}\}$

$|S| = \binom{6}{3}$

$P_r(S) = \frac{|S|}{|\Omega|}$

$= \binom{6}{3}/2^6$

$= \frac{5}{16}$

15.

$\Omega = \{n \textit{ length sequence of random decimal digits.}\}$

$|\Omega| = 10^n$

$P_r(\omega) = \frac{1}{10^n}$

$$S = \{7 \ appears \ in \ the \ n \ length \ sequence\}$$
$$S^c = \{7 \ not \ appears \ in \ the \ n \ length \ sequence\}$$

$$P_r(S) = 1 - P_r(S^c)$$
$$= 1 - \left(\tfrac{9}{10}\right)^n \geq 0.9$$

$$\left(\tfrac{9}{10}\right)^n \leq 0.1$$
$$nlog(0.9) \leq log(0.1)$$
$$n \geq \tfrac{log(0.1)}{log(0.9)}$$
$$n \geq 22$$

Worksheet 3

1.

$$\Omega = \{1, 2, 3, 4, 5, 6\}^3$$
$$|\Omega| = 6^3$$

$$A = \{at\ least\ one\ of\ the\ roll\ is\ 6\}$$
$$|A| = 5^2 \times \binom{3}{1} + 5 \times \binom{3}{2} + 1$$

$$P_r(A) = \frac{|A|}{|\Omega|}$$
$$= (5^2 \times \binom{3}{1} + 5 \times \binom{3}{2} + 1)/6^3$$
$$= \frac{91}{216}$$

2.

$$\Omega = \{H, T\}^{10}$$
$$|\Omega| = 2^{10}$$

$A = \{at\ least\ two\ heads\}$
$A^c = \{no\ heads,\ one\ heads\}$
$|A^c| = 1 + 10 = 11$

$P_r(A) = \frac{|A|}{|\Omega|}$
$= (2^{10} - 11)/2^{10}$
$= \frac{1013}{1024}$

6.

$\Omega = \{H, T\}^3$
$|\Omega| = 2^3$

$B = \{there\ are\ exactly\ two\ heads\}$

(a)
$A = \{the\ first\ outcome\ is\ a\ head\}$
$|A| = 2^2$
$P_r(A) = \frac{2^2}{2^3} = \frac{1}{2}$

$|B \cap A| = 2$

$P_r(B \cap A) = \frac{2}{2^3} = \frac{1}{4}$

$P_r(B|A) = P_r(B \cap A)/P_r(A)$

$= \frac{\frac{1}{4}}{\frac{1}{2}}$

$= \frac{1}{2}$

(b)

$A = \{the\ first\ outcome\ is\ a\ tail\}$

$|A| = 2^2$

$P_r(A) = \frac{2^2}{2^3} = \frac{1}{2}$

$|B \cap A| = 1$

$P_r(B \cap A) = \frac{1}{2^3} = \frac{1}{8}$

$P_r(B|A) = P_r(B \cap A)/P_r(A)$

$= \frac{\frac{1}{8}}{\frac{1}{2}}$

$= \frac{1}{4}$

(c)

$A = \{the\ first\ two\ outcomes\ are\ both\ heads\}$

$|A| = 2$

$P_r(A) = \frac{2}{2^3} = \frac{1}{4}$

$|B \cap A| = 1$

$P_r(B \cap A) = \frac{1}{2^3} = \frac{1}{8}$

$P_r(B|A) = P_r(B \cap A)/P_r(A)$

$= \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$

(d)

$A = \{the\ first\ two\ outcomes\ are\ both\ tails\}$

$|A| = 2$

$P_r(A) = \frac{2}{2^3} = \frac{1}{4}$

$|B \cap A| = 0$

$P_r(B \cap A) = \frac{0}{2^3} = 0$

$P_r(B|A) = P_r(B \cap A)/P_r(A)$

$= \frac{0}{\frac{1}{4}} = 0$

(e)

$A = \{the\, first\ outcome\ is\ a\ head\ and$
$\qquad the\ third\ outcome\ is\ a\ tail\}$

$|A| = 2$

$P_r(A) = \frac{2}{2^3} = \frac{1}{4}$

$|B \cap A| = 1$

$P_r(B \cap A) = \frac{1}{2^3} = \frac{1}{8}$

$P_r(B|A) = P_r(B \cap A)/P_r(A)$

$= \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$

8.

$P_r(B^c) = \frac{1}{4}$

$P_r(A|B) = P_r(A \cap B)/P_r(B) = \frac{1}{2}$

$P_r(A \cap B) = P_r(B)P_r(A|B)$

$= (1 - P_r(B^c))P_r(A|B)$

$= (1 - \frac{1}{4}) \times \frac{1}{2}$

$= \frac{3}{8}$

9.

$$B = \{the\ sum\ of\ the\ two\ rolls\ is\ >\ 7\}$$

$$\Omega = \{1, 2, 3, 4, 5, 6\}^2$$
$$|\Omega| = 6^2$$

(a)

$$A = \{the\, first\ row\ is\ a\ 4\}$$
$$|A| = 6$$
$$P_r(A) = \tfrac{6}{6^2} = \tfrac{1}{6}$$
$$B \cap A = 3$$
$$P_r(B \cap A) = \tfrac{3}{6^2} = \tfrac{1}{12}$$

$$P_r(B|A) = P_r(B \cap A)/P_r(A)$$
$$= \frac{\frac{1}{12}}{\frac{1}{6}}$$
$$= \tfrac{1}{2}$$

(b)

$A = \{the\,first\,row\,is\,a\ 1\}$

$|A| = 6$

$P_r(A) = \frac{6}{6^2} = \frac{1}{6}$

$B \cap A = 0$

$P_r(B \cap A) = \frac{0}{6^2} = 0$

$P_r(B|A) = P_r(B \cap A)/P_r(A)$

$= \frac{0}{\frac{1}{6}}$

$= 0$

(c)

$A = \{the\,first\,row\,is\ > 3\}$

$|A| = 3 \times 6 = 18$

$P_r(A) = \frac{18}{6^2} = \frac{1}{2}$

$B \cap A = 3 + 4 + 5 = 12$

$P_r(B \cap A) = \frac{12}{6^2} = \frac{1}{3}$

$P_r(B|A) = P_r(B \cap A)/P_r(A)$

$= \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{1}{6}$ $2/3$

(d)

$A = \{the\ first\ row\ is\ < 5\}$

$|A| = 4 \times 6 = 24$

$P_r(A) = \frac{24}{6^2} = \frac{2}{3}$

$B \cap A = 0 + 1 + 2 + 3 = 6$

$P_r(B \cap A) = \frac{6}{6^2} = \frac{1}{6}$

$P_r(B|A) = P_r(B \cap A)/P_r(A)$

$= \frac{\frac{1}{6}}{\frac{2}{3}}$

$= \frac{1}{4}$

11.

$A_1 = Gryffindor$

$A_2 = Hufflepuff$

$A_3 = Ravenclaw$

$A_4 = Slytherin$

$A_5 = Dark\ Arts$

$$P_r(A_1) = \frac{1}{3}$$
$$P_r(A_2) = \frac{1}{4}$$
$$P_r(A_3) = \frac{1}{6}$$
$$P_r(A_4) = \frac{1}{4}$$

$$P_r(A_5|A_1) = \frac{1}{2}$$
$$P_r(A_5|A_2) = \frac{1}{3}$$
$$P_r(A_5|A_3) = \frac{1}{2}$$
$$P_r(A_5|A_4) = \frac{2}{3}$$

$$\begin{aligned} P_r(A_5) &= P_r(A_5|A_1)P_r(A_1) + P_r(A_5|A_2)P_r(A_2) \\ &\quad + P_r(A_5|A_3)P_r(A_3) + P_r(A_5|A_4)P_r(A_4) \\ &= \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{6} + \frac{2}{3} \times \frac{1}{4} \\ &= \frac{1}{2} \end{aligned}$$

12.

$$P_r(F_1) = 25\%$$
$$P_r(F_2) = 35\%$$
$$P_r(F_3) = 40\%$$

$$P_r(D|F_1) = 5\% \ (D : Defective)$$
$$P_r(D|F_2) = 4\% \ (D : Defective)$$
$$P_r(D|F_3) = 2\% \ (D : Defective)$$

(a)

$$P_r(D) = P_r(D|F_1)P_r(F_1) + P_r(D|F_2)P_r(F_2)$$
$$+ P_r(D|F_3)P_r(F_3)$$
$$= 5\% \times 25\% + 4\% \times 35\% + 2\% \times 40\%$$
$$= 3.45\%$$

(b)

$$P_r(F_1|D) = P_r(F_1 \cap D)/P_r(D)$$
$$= 25\% \times 5\%/3.45\%$$
$$\approx 36.23\%$$

14.

$P_r(d_1) = \frac{1}{3}$

$P_r(d_2) = \frac{1}{3}$

$P_r(d_3) = \frac{1}{3}$

$p : positive$

$P_r(p|d_1) = 0.8$

$P_r(p|d_2) = 0.6$

$P_r(p|d_3) = 0.4$

(a)

$P_r(p) = P_r(p|d_1)P_r(d_1) + P_r(p|d_2)P_r(d_2) +$
$P_r(p|d_3)P_r(d_3)$
$= 0.8 \times \frac{1}{3} + 0.6 \times \frac{1}{3} + 0.4 \times \frac{1}{3}$
$= 0.6$

(b)

$$
\begin{aligned}
P_r(d_1|p) &= P_r(d_1 \cap p)/P_r(p) \\
&= P_r(p|d_1)P_r(d_1)/P_r(p) \\
&= 0.8 \times \tfrac{1}{3}/0.6 \\
&= \tfrac{4}{9}
\end{aligned}
$$

$$
\begin{aligned}
P_r(d_2|p) &= P_r(d_2 \cap p)/P_r(p) \\
&= P_r(p|d_2)P_r(d_2)/P_r(p) \\
&= 0.6 \times \tfrac{1}{3}/0.6 \\
&= \tfrac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
P_r(d_3|p) &= P_r(d_3 \cap p)/P_r(p) \\
&= P_r(p|d_3)P_r(d_3)/P_r(p) \\
&= 0.4 \times \tfrac{1}{3}/0.6 \\
&= \tfrac{2}{9}
\end{aligned}
$$

16.

(*i*) *Dependent*

(*ii*) *Independent*

(*iii*) *Dependent*

(*iv*) *Independent*