DSE 220: Machine learning

Worksheet 12 — Singular value decomposition

1. *Singular values versus eigenvalues.* Recall from class that any $p \times q$ matrix M (with $p \leq q$, say) can be written in the form:

$$M = \underbrace{\begin{pmatrix} \uparrow & & \uparrow \\ u_1 & \cdots & u_p \\ \downarrow & & \downarrow \end{pmatrix}}_{p \times p \text{ matrix } U} \underbrace{\begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_p \end{pmatrix}}_{p \times p \text{ matrix } \Lambda} \underbrace{\begin{pmatrix} \longleftarrow & v_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & v_p & \longrightarrow \end{pmatrix}}_{p \times q \text{ matrix } V^T}$$

where $u_1, \ldots, u_p$ are orthonormal vectors in $\mathbb{R}^p$, $v_1, \ldots, v_p$ are orthonormal vectors in $\mathbb{R}^q$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$ are known as *singular values.* In this problem, we will try to understand these quantities by relating them to eigenvalues and eigenvectors of suitably defined matrices.

   (a) What is Mv_i (for $1 \leq i \leq p$)? Express the answer as simply as possible, in terms of the singular values and vectors of M.

   (b) What is $M^T u_i$?

   (c) What is $M^T M v_i$? And what is $MM^T u_i$?

   (d) Notice that MM^T is a symmetric $p \times p$ matrix and therefore has p real eigenvalues. What are its eigenvalues and eigenvectors?

   (e) How do the eigenvalues and eigenvectors of $M^T M$ relate to those of MM^T?

   (f) Suppose M has rank k. How would this be reflected in the singular values σ_i?

2. A particular 4×5 matrix M has the following singular value decomposition:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

   Find the best rank-2 approximation to M.

3. *Rank-1 matrices.*

   (a) Find the best rank-1 approximation to the matrix:

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

   If you use the SVD method in Python to solve this problem, you should use the setting `full_matrices` = 0 to get the kind of decomposition we've been discussing (sometimes called the "thin SVD").

   (b) In general, a rank-1 matrix of dimension $p \times q$ is a matrix that can be written in form uv^T, where $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$. Do you think this decomposition is unique, or is it in general possible to find a different pair of vectors $a \in \mathbb{R}^P$ and $b \in \mathbb{R}^q$ such that $uv^T = ab^T$?

(c) Let M be some $p \times q$ matrix whose singular value decomposition is as specified in problem 5. Notice that M can equally be written in the form

$$M = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_p u_p v_p^T,$$

that is, as a sum of rank-1 matrices. For $k < p$, let $\widehat{M}$ be the best rank-k approximation to M. Express $\widehat{M}$ as a sum of rank-1 matrices.