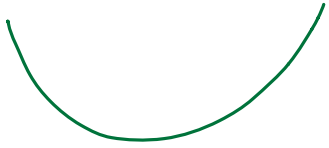


Linear classification

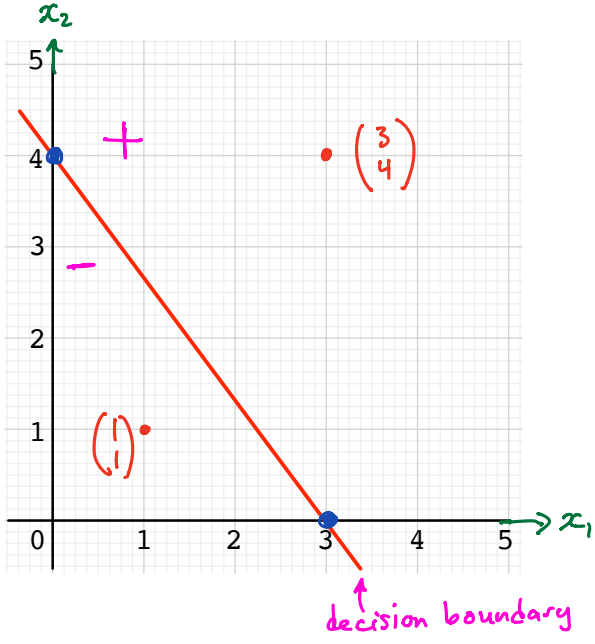
DSE 220

Outline

1. Linear decision boundary for binary classification
2. The Perceptron algorithm
3. Maximizing the margin
4. The soft-margin SVM

Linear decision boundary for classification: example

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



<linear function> = 0

Slope = $\frac{-4}{3}$

$x_2 = -\frac{4}{3} x_1 + 4$

$\Leftrightarrow \quad 4x_1 + 3x_2 - 12 = 0$

Classify $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Plug into $4x_1 + 3x_2 - 12$:

$4 \cdot 3 + 3 \cdot 4 - 12 = 12 > 0$ } predict +1

For $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$4 \cdot 1 + 3 \cdot 1 - 12 = -5 < 0$ } -1

- What is the formula for this boundary?
- What label would we predict for a new point x?

Prediction:

SIGN $(4x_1 + 3x_2 - 12)$

Linear classifiers

Binary classification problem: data $x \in \mathbb{R}^d$ and labels $y \in \{-1, +1\}$

- Linear classifier:
  - Parameters: $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$
  - Decision boundary $w \cdot x + b = 0$
  - On point x, predict label $\text{sign}(w \cdot x + b)$
- If the true label on point x is y:
  - Classifier correct if $y(w \cdot x + b) > 0$

Say true label for x is y.

Our prediction is $\text{sign}(w \cdot x + b)$

Correct $\iff$ $y = \text{sign}(w \cdot x + b)$

$\iff$ EITHER $y = +1$ and $w \cdot x + b > 0$

OR $y = -1$ and $w \cdot x + b < 0$

$\iff$ $y(w \cdot x + b) > 0$

Outline

1. Linear decision boundary for binary classification
2. The Perceptron algorithm
   - A loss function for classification
   - A stochastic gradient descent approach
   - The Perceptron
   - Convergence
3. Maximizing the margin
4. The soft-margin SVM

A loss function for classification

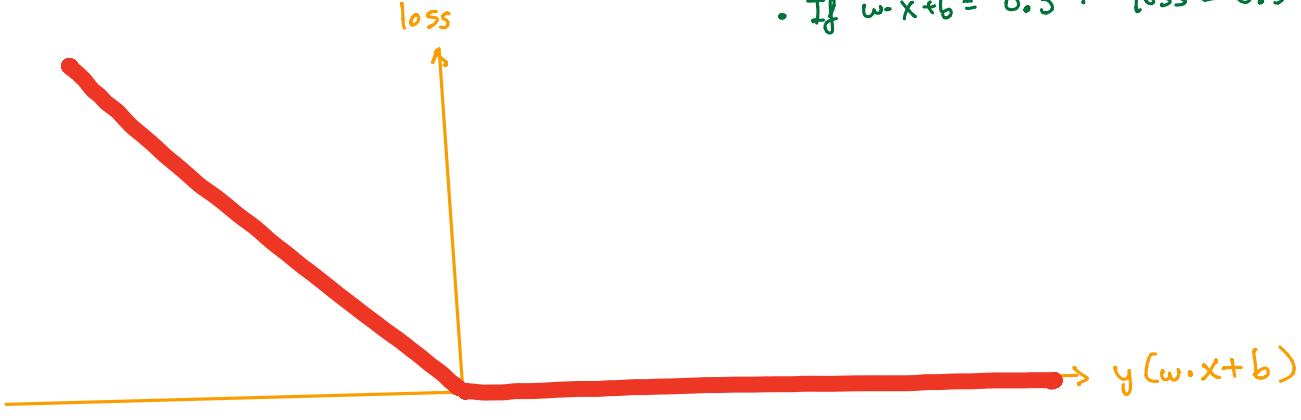
What is the **loss** of our linear classifier (given by w, b) on a point (x, y)?

One idea for a loss function:

- If $y(w \cdot x + b) > 0$: correct, no loss
- If $y(w \cdot x + b) < 0$: loss $= -y(w \cdot x + b)$

E.g. $y = -1$

- If $w \cdot x + b = -3$: correct prediction, no loss
- If $w \cdot x + b = 3$: loss = 3
- If $w \cdot x + b = 0.3$: loss = 0.3



A simple learning algorithm

Fit a linear classifier w, b to the training set using **stochastic gradient descent**.

- Update w, b based on just one data point (x, y) at a time
- If $y(w \cdot x + b) > 0$: zero loss, no update (correct on (x,y))
- If $y(w \cdot x + b) \leq 0$: loss is $-y(w \cdot x + b)$ (wrong on (x,y))

Wrong on (x,y):

Loss $\ell = -y(w \cdot x + b)$

$\frac{d\ell}{dw} = -yx$

$\frac{d\ell}{db} = -y$

learning rate

SGD update

$w \leftarrow w + \eta\, yx$

$b \leftarrow b + \eta\, y$

just use $\eta = 1$

A simple learning algorithm

Fit a linear classifier w, b to the training set using **stochastic gradient descent**.

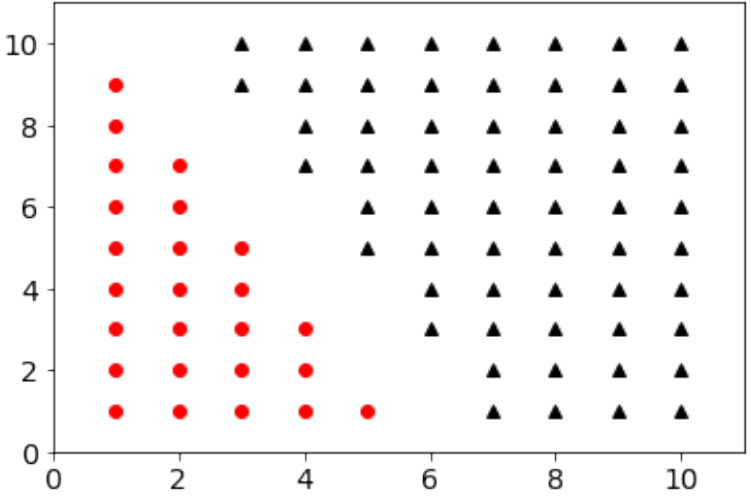
- Update w, b based on just one data point (x, y) at a time
- If $y(w \cdot x + b) > 0$: zero loss, no update
- If $y(w \cdot x + b) \leq 0$: loss is $-y(w \cdot x + b)$

The Perceptron algorithm

- Initialize $w = 0$ and $b = 0$
- Keep cycling through the training data (x, y):
  - If $y(w \cdot x + b) \leq 0$ (i.e. point misclassified):
    - $w = w + yx$
    - $b = b + y$

The Perceptron in action

85 data points, linearly separable.



The Perceptron in action

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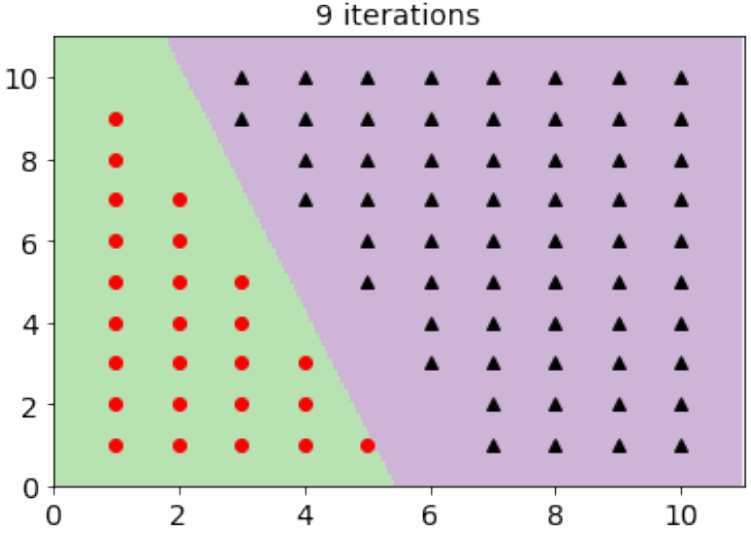

The Perceptron in action

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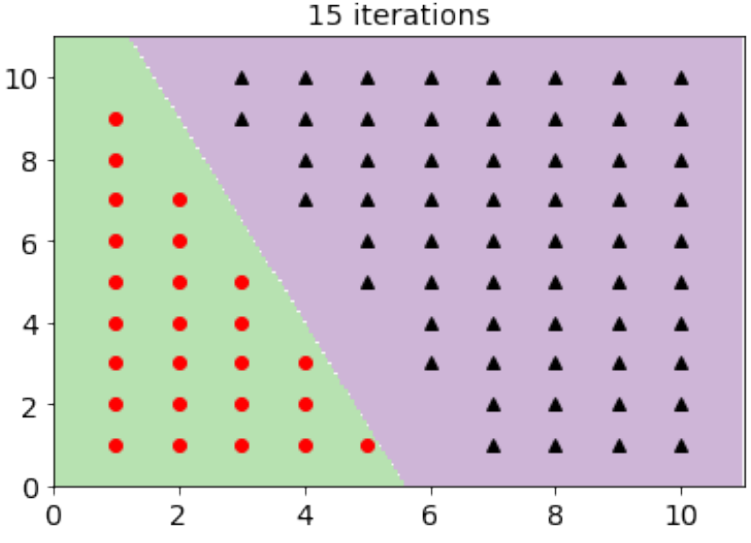

The Perceptron in action

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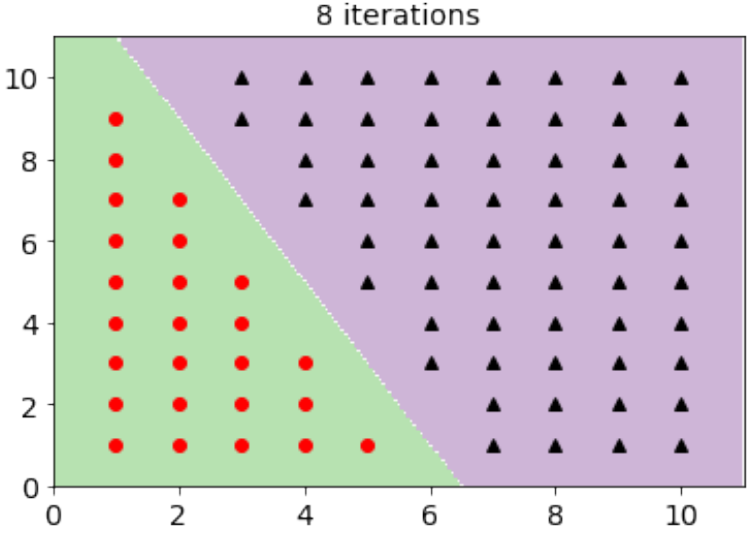

The Perceptron in action

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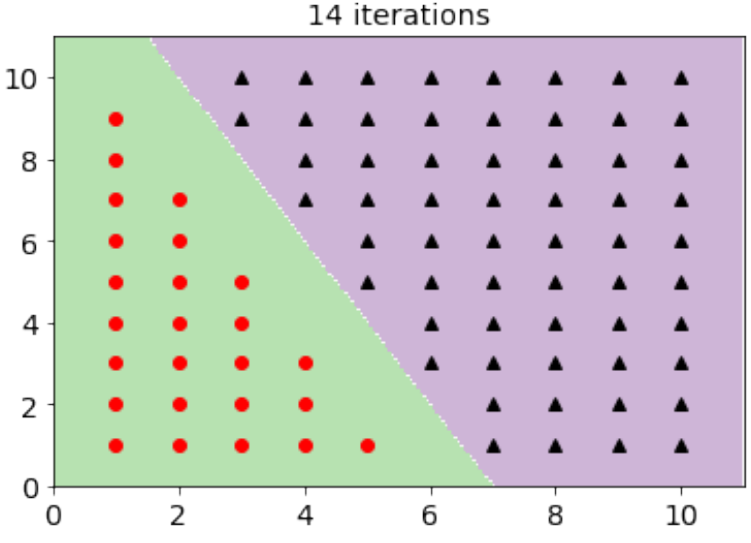

The Perceptron in action

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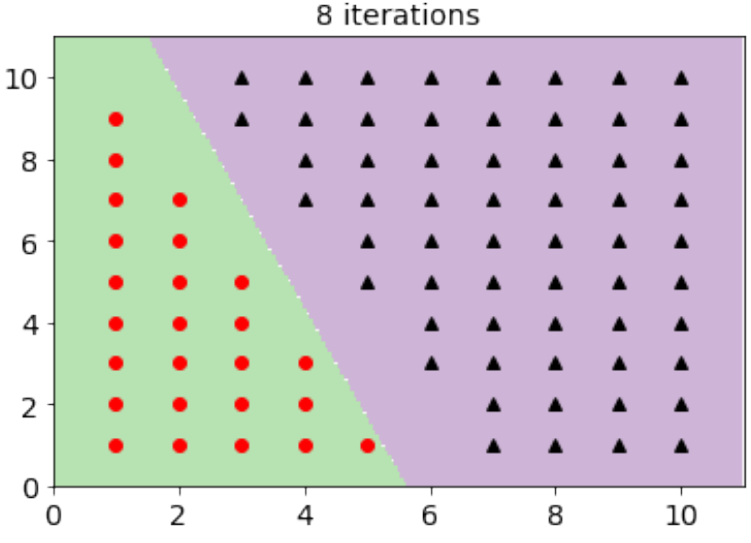


Perceptron: convergence

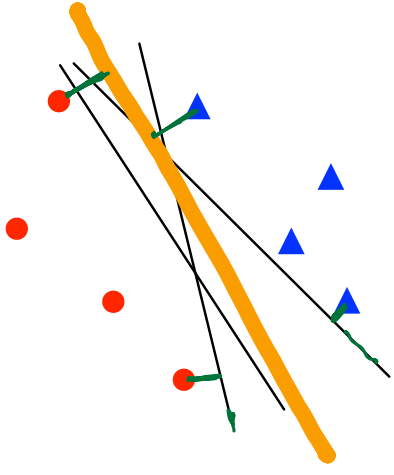
If the training data is linearly separable:

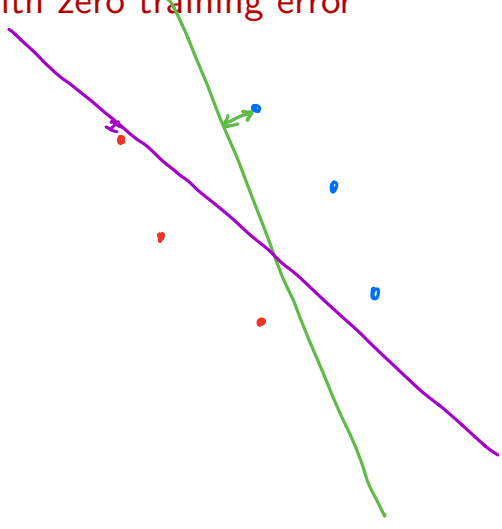
- The Perceptron algorithm will find a linear classifier with zero training error
- It will converge within a finite number of steps.

Perceptron: convergence

If the training data is linearly separable:

- The Perceptron algorithm will find a linear classifier with zero training error
- It will converge within a finite number of steps.





But is there a better, more systematic choice of separator?

Outline

Worksheet 9 # 1, 2, 3

Lab 2 # 1

1. Linear decision boundary for binary classification
2. The Perceptron algorithm
3. Maximizing the margin
   - The margin of a linear classifier
   - Maximizing the margin
   - A convex optimization problem
   - Support vectors
4. The soft-margin SVM

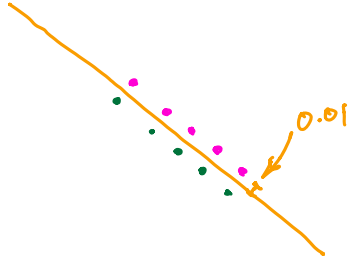
Support vector machines

The learning problem

Given: training data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$.

Find: $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that $y^{(i)}(w \cdot x^{(i)} + b) > 0$ for all i.

The learning problem



Given: training data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$.

Find: $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that $y^{(i)}(w \cdot x^{(i)} + b) > 0$ for all i.

By scaling w, b, can equivalently ask for

multiply w, b by a suitable factor

$$y^{(i)}(w \cdot x^{(i)} + b) \geq 1 \quad \text{for all } i$$

If $y^{(i)}(w \cdot x^{(i)} + b) > 0$ for all data points i,
then multiplying w, b by a scaling factor will give
$y^{(i)}(w \cdot x^{(i)} + b) \geq 1$ for all i.

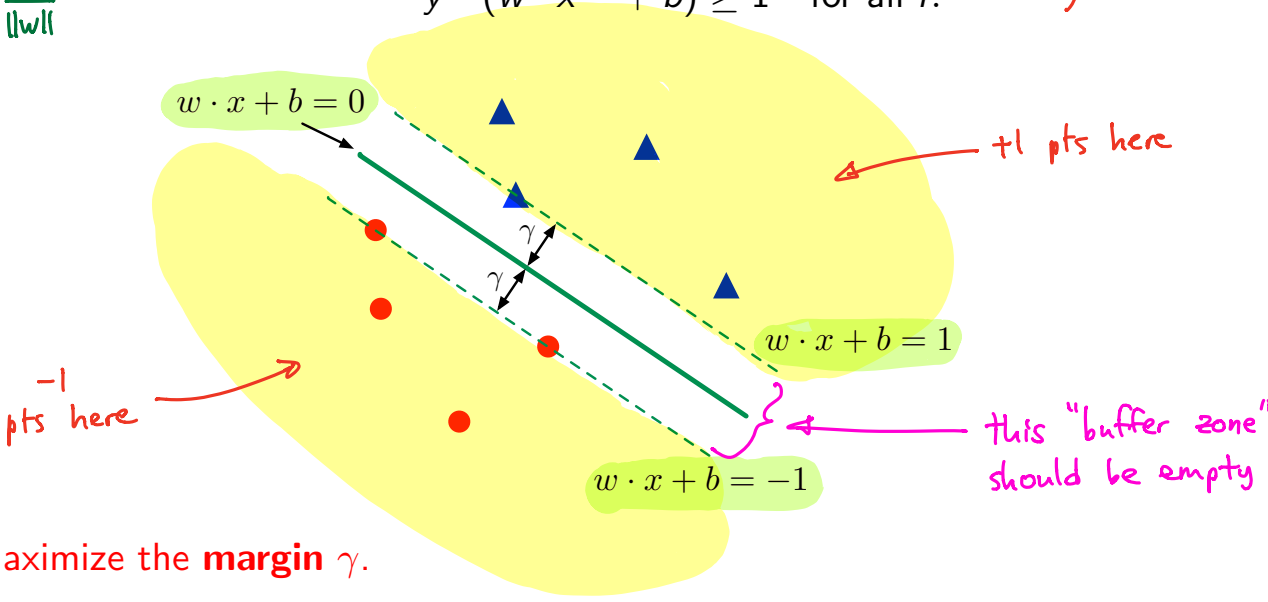
Maximizing the margin

Given: training data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$.
Find: $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

$$y^{(i)}(w \cdot x^{(i)} + b) \geq 1 \quad \text{for all } i.$$

} data is correctly classified

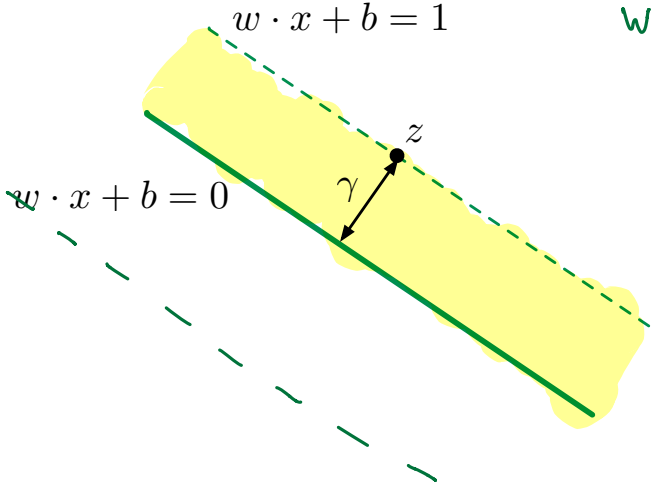
$\gamma = \frac{1}{\|w\|}$



Maximize the **margin** γ.

A formula for the margin

Close-up of a point z on the positive boundary.



A quick calculation shows that $\gamma = 1/\|w\|$.

In short: to maximize the margin, minimize $\|w\|$.

$$\|w\| = \sqrt{w_1^2 + w_2^2 + \cdots + w_d^2}$$

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix}$$

Goal: maximize margin γ

$\equiv$ minimize $\|w\|$

$\equiv$ minimize $\|w\|^2$

Maximum-margin linear classifier

- Given $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$

$$\begin{aligned} &\min_{w \in \mathbb{R}^d, b \in \mathbb{R}} \quad \|w\|^2 \\ &\text{s.t.:} \quad y^{(i)}(w \cdot x^{(i)} + b) \geq 1 \quad \text{for all } i = 1, 2, \ldots, n \end{aligned}$$

Maximum-margin linear classifier

convex function $w_1^2 + w_2^2 + \ldots + w_d^2$

- Given $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$

this is the optimization problem we want to solve

$$\min_{w \in \mathbb{R}^d, b \in \mathbb{R}} \|w\|^2$$

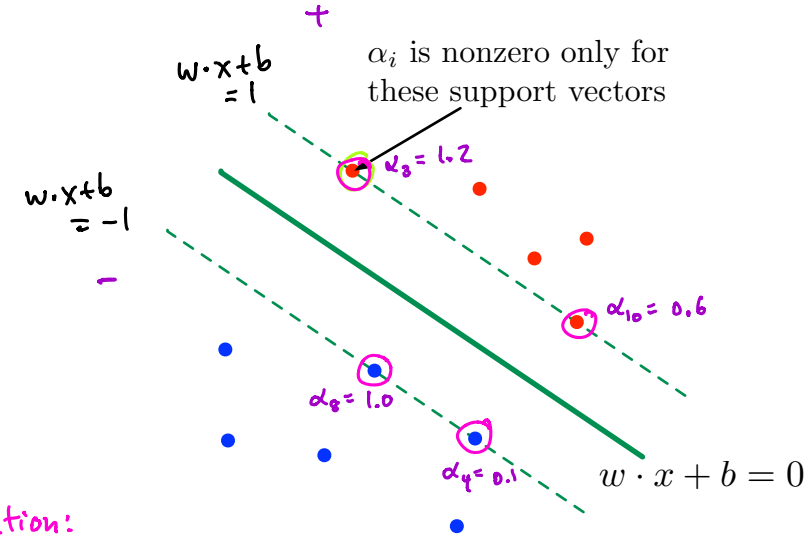
$$\text{s.t.: } y^{(i)}(w \cdot x^{(i)} + b) \geq 1 \quad \text{for all } i = 1, 2, \ldots, n$$

- This is a **convex optimization problem**:
  - Convex objective function
  - Linear constraints

- This means that:
  - the optimal solution can be found efficiently
  - **duality** gives us information about the solution

Support vectors

Support vectors: training points that lie exactly on the margin, i.e. $y^{(i)}(w \cdot x^{(i)} + b) = 1$.



$W = 1.2\, x^{(3)} + 0.6\, x^{(10)} - 1.0\, x^{(8)} - 0.1\, x^{(4)}$

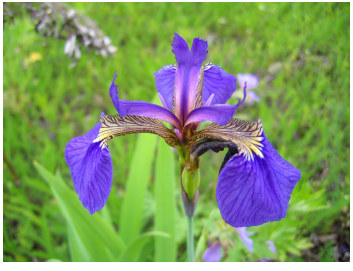
Form of the solution:

$w = \sum_{i=1}^{n} \alpha_i y^{(i)} x^{(i)}$ is a function of just the support vectors.

← these coefficients are all ≥ 0 and they > 0 only on support vectors.

Small example: Iris data set

Fisher's **iris** data



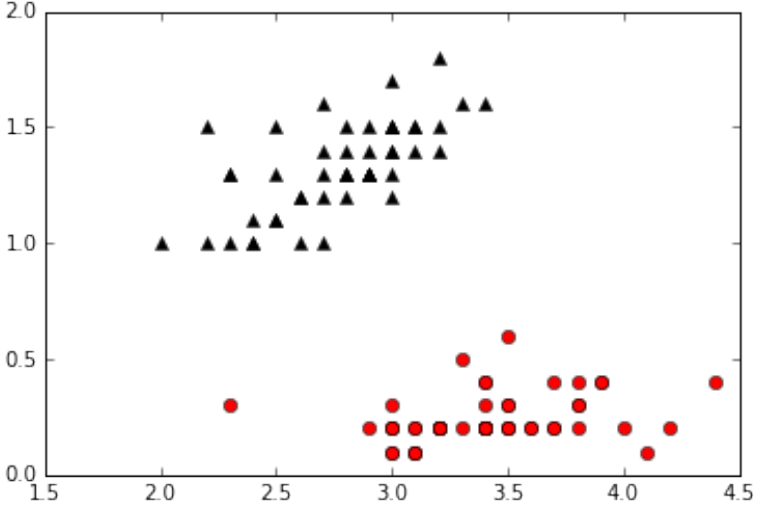
150 data points from three classes:

- iris setosa
- iris versicolor
- iris virginica

Four measurements: petal width/length, sepal width/length

Small example: Iris data set

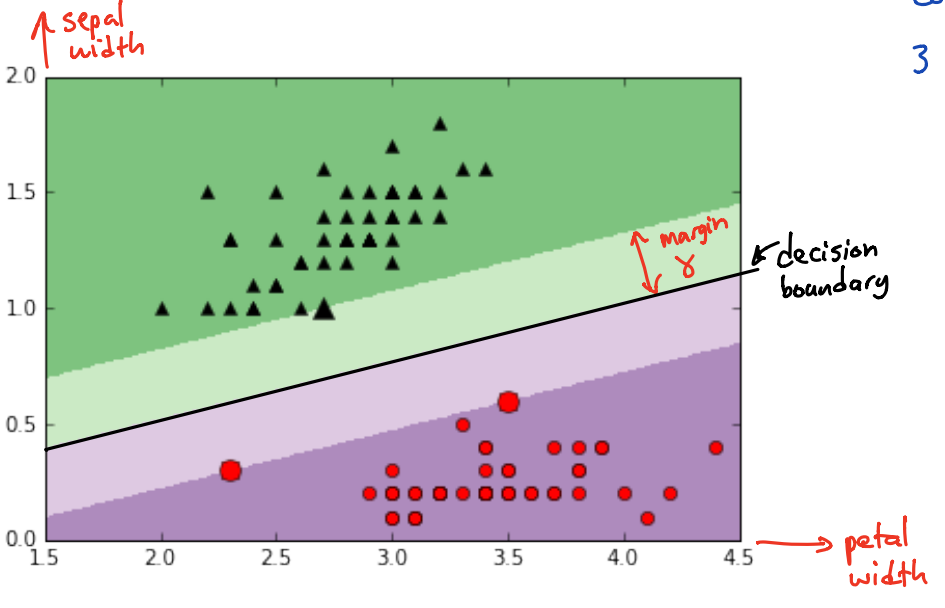
Two features: sepal width, petal width.
Two classes: setosa (red circles), versicolor (black triangles)



Small example: Iris data set

Two features: sepal width, petal width.

Two classes: setosa (red circles), versicolor (black triangles)



3 support vectors

⇒ w is a linear combination of those 3 data points

Outline

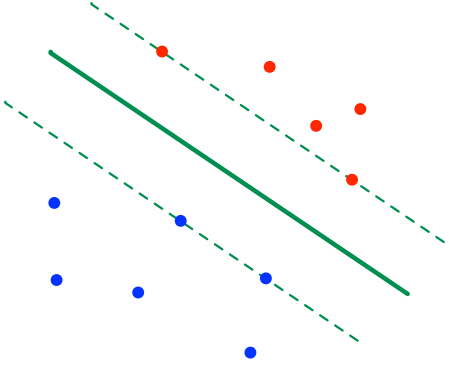
1. Linear decision boundary for binary classification
2. The Perceptron algorithm
3. Maximizing the margin
4. The soft-margin SVM ← the version of the SVM that is used in practice
   - Data that isn't linearly separable
   - Adding slack variables for each point
   - Revised convex optimization problem
   - Setting the slack parameter

Recall: maximum-margin linear classifier

Given: $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$.
Find: the linear separator w that perfectly classifies the data and has maximum margin.

$$\min_{w \in \mathbb{R}^d, b \in \mathbb{R}} \|w\|^2$$

s.t.: $y^{(i)}(w \cdot x^{(i)} + b) \geq 1 \quad \text{for all } i = 1, 2, \ldots, n$



Solution $w = \sum_{i=1}^{n} \alpha_i y^{(i)} x^{(i)}$ is a function of just the support vectors.

Recall: maximum-margin linear classifier

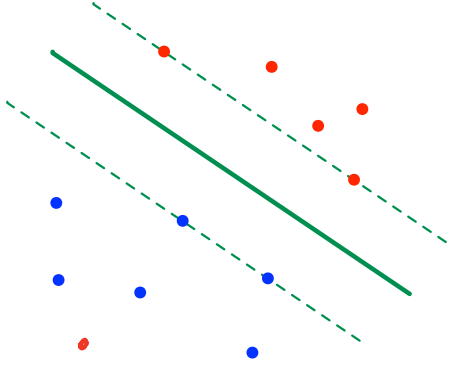
Given: $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \{-1, +1\}$.

Find: the linear separator w that perfectly classifies the data and has maximum margin.

$$\min_{w \in \mathbb{R}^d, b \in \mathbb{R}} \|w\|^2$$

$$\text{s.t.: } y^{(i)}(w \cdot x^{(i)} + b) \geq 1 \quad \text{for all } i = 1, 2, \ldots, n$$

no longer possible to satisfy all these constraints; we must give up on some of them...



Solution $w = \sum_{i=1}^{n} \alpha_i y^{(i)} x^{(i)}$ is a function of just the support vectors.

What if data is not separable?

The non-separable case

maximize margin as before

minimize total amount of slack used

$$\min_{w \in \mathbb{R}^d, b \in \mathbb{R}, \xi \in \mathbb{R}^n} \|w\|^2 + C \sum_{i=1}^{n} \xi_i$$

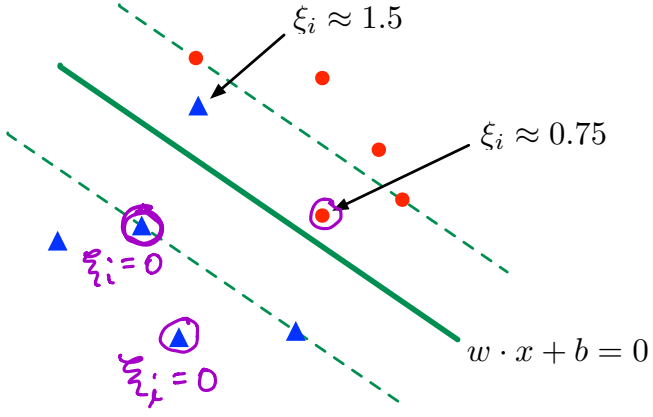
s.t.: $y^{(i)}(w \cdot x^{(i)} + b) \geq 1 - \xi_i$ for all $i = 1, 2, \ldots, n$

$\xi \geq 0$

Allow each data point $x^{(i)}$ some **slack** ξ_i.

the extent to which the constraint on $(x^{(i)}, y^{(i)})$ is violated

$\xi \approx 3$

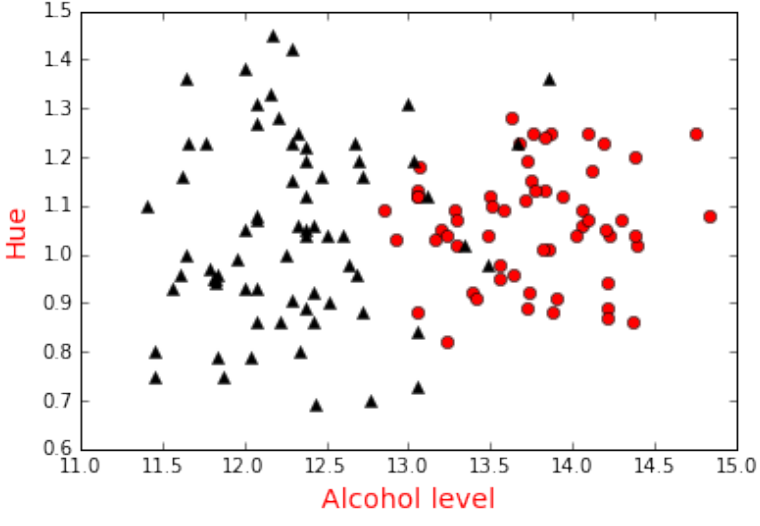


$x_i : \xi$

total slack used is $\sum_{i=1}^{n} \xi_i \approx 2.25$

Wine data set

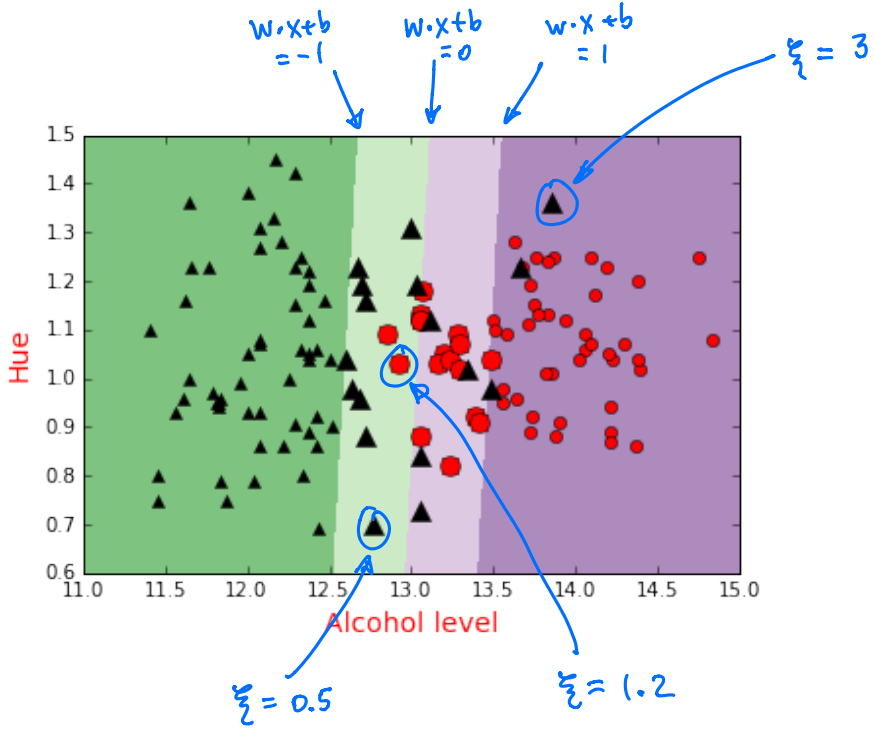
Here $C = 1.0$



Wine data set

Here $C = 1.0$

Support vectors = all points right on the margin AND all points on which we have used slack



The tradeoff between margin and slack

What effect does this have?

$$\min_{w \in \mathbb{R}^d, b \in \mathbb{R}, \xi \in \mathbb{R}^n} \quad \|w\|^2 + C \sum_{i=1}^{n} \xi_i$$

$$\text{s.t.:} \quad y^{(i)}(w \cdot x^{(i)} + b) \geq 1 - \xi_i \quad \text{for all } i = 1, 2, \ldots, n$$

$$\xi \geq 0$$

$C = 0$

Slack is FREE

Misclassify as many pts as we like

We'll end up with $w = 0$ (infinite margin)

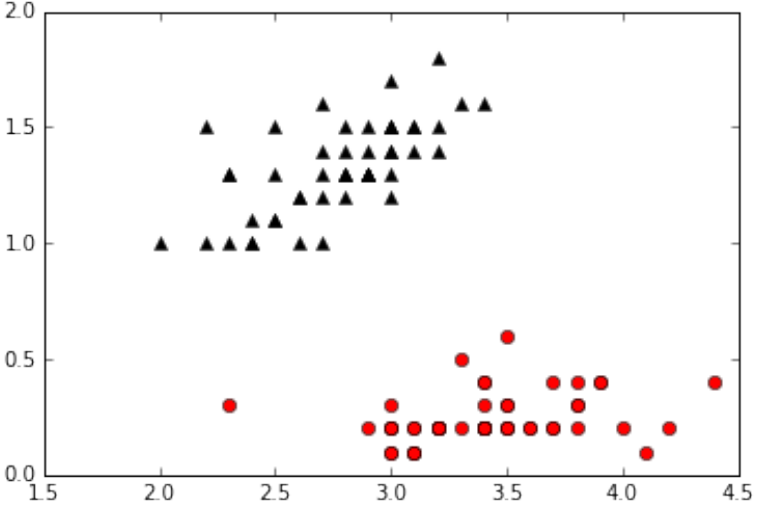
$C \to \infty$

Slack is EXPENSIVE

Like the hard-margin SVM that assumed separability

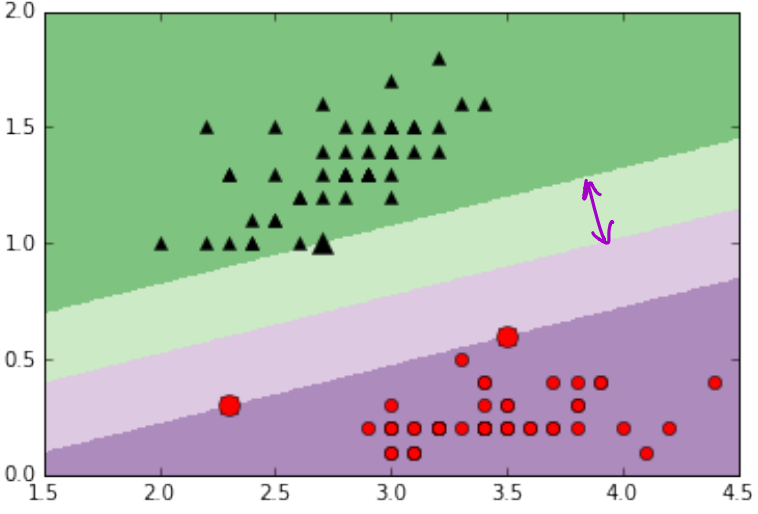
Back to Iris

$C = 10$



Back to Iris

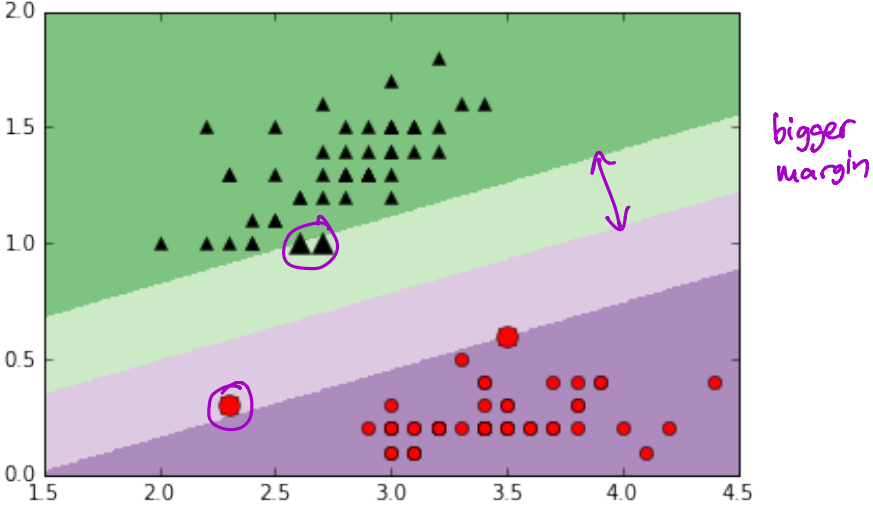
$C = 10$



no slack used

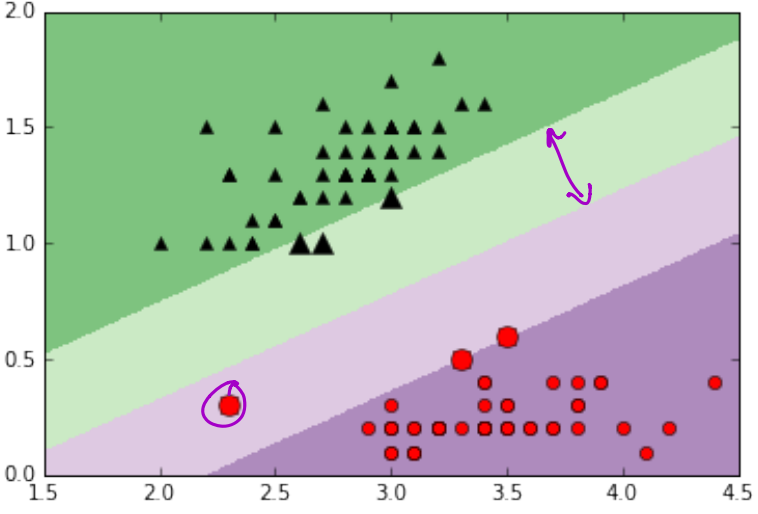
Back to Iris

$C = 3$



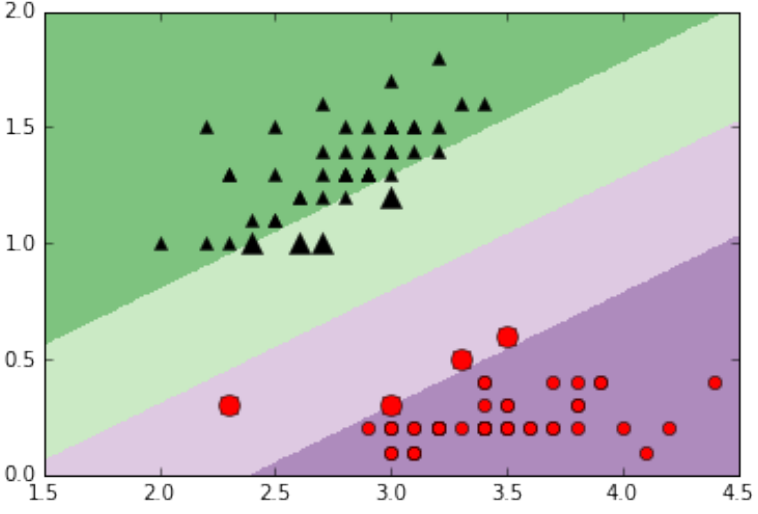
Back to Iris

$C = 2$



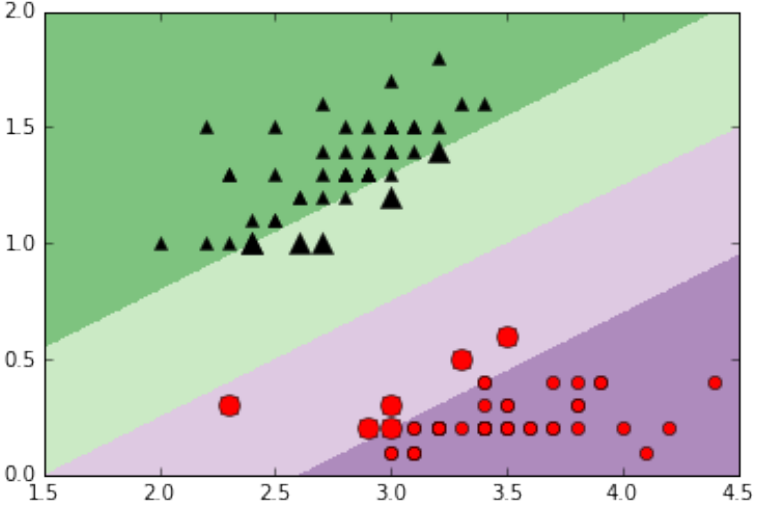
Back to Iris

$C = 1$



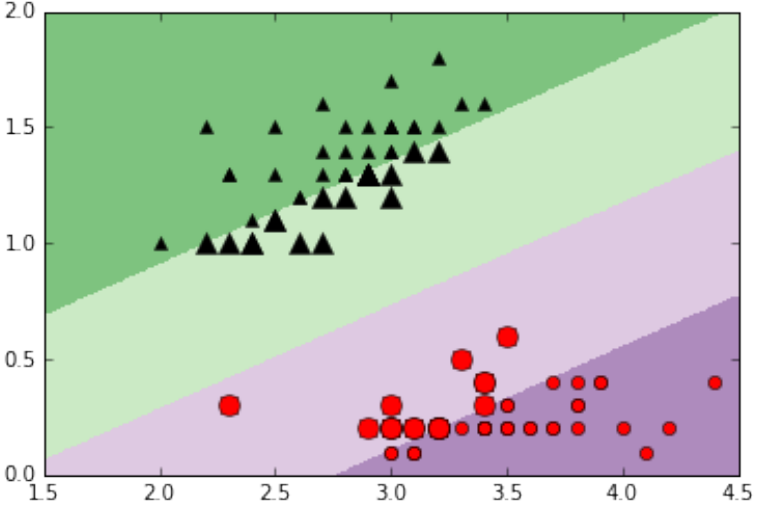
Back to Iris

$C = 0.5$



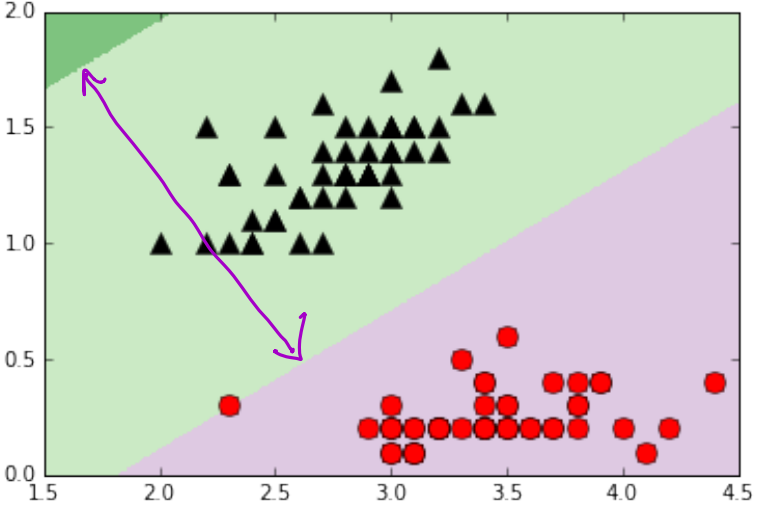
Back to Iris

$C = 0.1$



Back to Iris

$C = 0.01$



Sentiment data

Sentences from reviews on Amazon, Yelp, IMDB, each labeled as positive or negative.

- `Needless to say, I wasted my money.`
- `He was very impressed when going from the original battery to the extended battery.`
- `I have to jiggle the plug to get it to line up right to get decent volume.`
- `Will order from them again!` +1

Dimension $d = 4500$

Num pts $n = 2500$

Data details:

- Bag-of-words representation using a vocabulary of size 4500
- 2500 training sentences, 500 test sentences

What C to use?

As C grows, the charge for violations grows
$\Rightarrow$ fewer violations in training set
$\Rightarrow$ smaller training error

C	training error (%)	test error (%)	# support vectors
0.01	23.72	28.4	2294
0.1	7.88	18.4	1766
1	1.12	16.8	1306
10	0.16	19.4	1105
100	0.08	19.4	1035
1000	0.08	19.4	950

How to choose C?

Cross-validation.

Why is this decreasing as C grows?

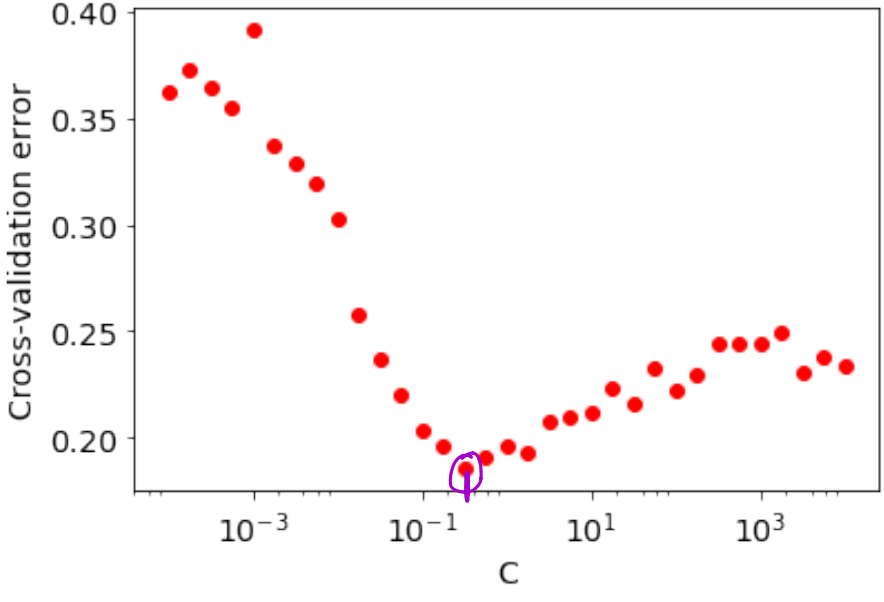
Fewer violations (every violation is a support vector)

Cross-validation

Training set size = 2500

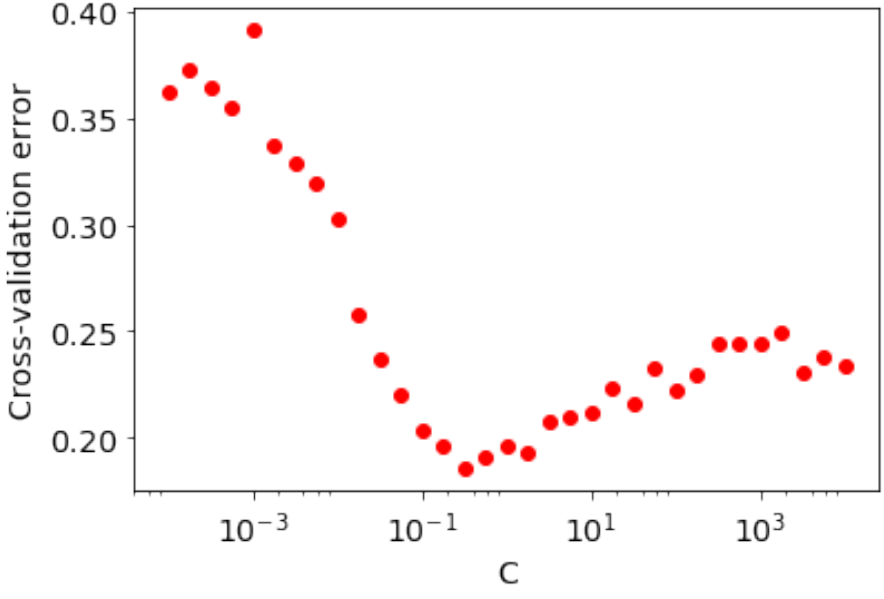
Results of 5-fold cross-validation:

5 trials;
each validation set has size 500



Cross-validation

Results of 5-fold cross-validation:



Chose $C = 0.32$. Test error: 15.6%

(recall LR error was > 21%)

Worksheet 9: all

Lab 2: #1-4

Worksheet 9 probs