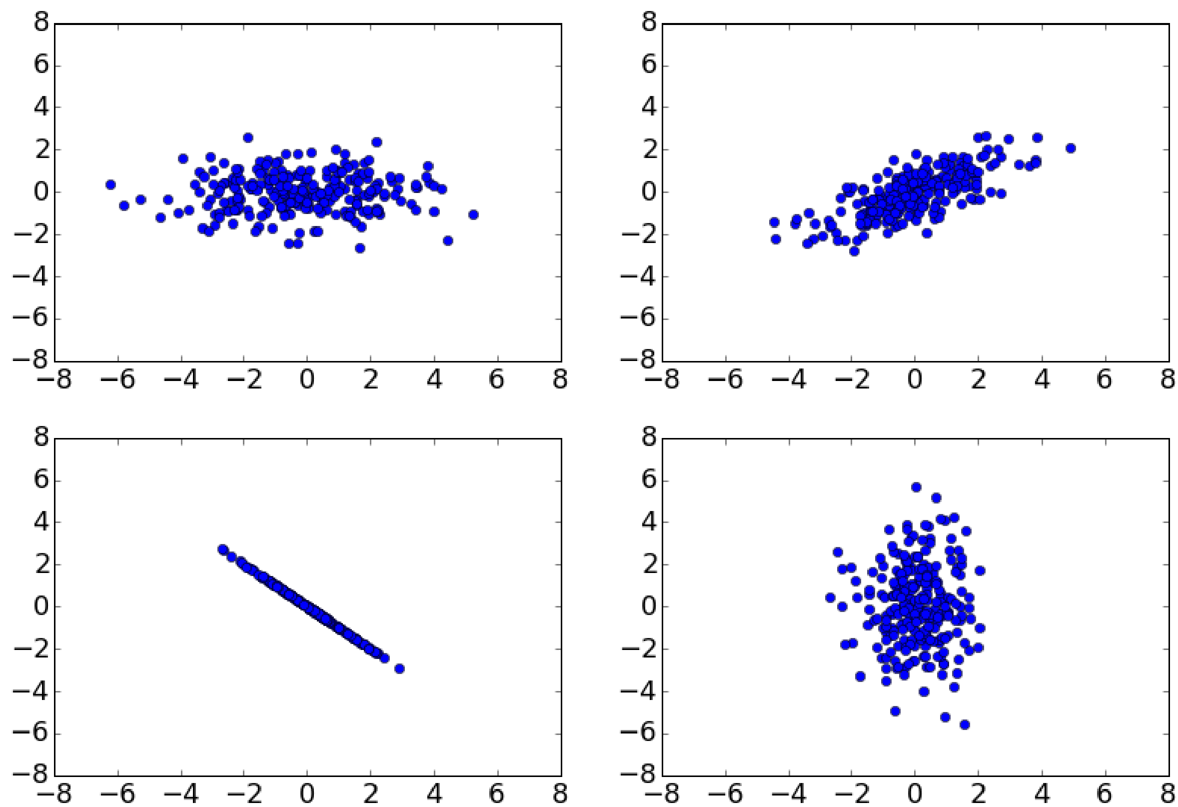
DSE 220: Machine learning

Worksheet 11 — Informative projections

1. Is the following set of vectors an orthonormal basis of $\mathbb{R}^3$? Explain why or why not.

$$\begin{pmatrix}3\\4\\0\end{pmatrix}, \begin{pmatrix}4\\-3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

2. The following four figures show different 2-dimensional data sets. In each case, make a rough sketch of an ellipsoidal contour of the covariance matrix and indicate the directions of the first and second eigenvectors (mark which is which).



3. Let $u_1, u_2 \in \mathbb{R}^d$ be two vectors with $\|u_1\| = \|u_2\| = 1$ and $u_1 \cdot u_2 = 0$. Define

$$U = \begin{pmatrix}\uparrow & \uparrow\\ u_1 & u_2\\ \downarrow & \downarrow\end{pmatrix}$$

(a) What are the dimensions of each of the following?

- U
- U^T
- UU^T
- $u_1 u_1^T$

(b) What are the differences, if any, between the following four mappings?

- $x \mapsto (u_1 \cdot x, u_2 \cdot x)$
- $x \mapsto (u_1 \cdot x)u_1 + (u_2 \cdot x)u_2$
- $x \mapsto U^T x$
- $x \mapsto UU^T x$

4. A certain random variable $X \in \mathbb{R}^3$ has mean and covariance as follows:

$$\mathbb{E}X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathrm{cov}(X) = \begin{pmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(a) Consider the direction $u = (1,1,1)/\sqrt{3}$. What are the mean and variance of $X \cdot u$?

(b) The eigenvectors of $\mathrm{cov}(X)$ can be found in the following list; which ones are they?

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

(c) Find the eigenvalues corresponding to each of the eigenvectors in part (b). Make it clear which eigenvalue belongs to which eigenvector.

(d) Suppose we used principal component analysis (PCA) to project points X into *two* dimensions. Which directions would it project onto?

(e) Continuing from part (d), what would be the resulting two-dimensional projection of the point $x = (4, 0, 2)$?

(f) Continuing from part (e), suppose that starting from the 2-d projection, we tried to reconstruct the original x. What would the three-dimensional reconstruction be, exactly?

5. M is a 2×2 real-valued symmetric matrix with eigenvalues $\lambda_1 = 2, \lambda_2 = -1$ and corresponding eigenvectors

$$u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(a) What is M?

(b) What are the eigenvalues of the matrix $M + 2I$?

(c) What are the eigenvalues of the matrix $M^2 = MM$?