DSE 220: Machine learning

Worksheet 8 — Convexity

1. For each of the following functions of one variable, say whether it is convex, concave, both, or neither.

   (a) $f(x) = x^2$

   (b) $f(x) = -x^2$

   (c) $f(x) = x^2 - 2x + 1$

   (d) $f(x) = x$

   (e) $f(x) = x^3$

   (f) $f(x) = x^4$

   (g) $f(x) = \ln x$

2. Show that the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not positive semidefinite.

3. Show that the matrix $M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive semidefinite.

4. For a fixed set of vectors $v_1, \ldots, v_n \in \mathbb{R}^d$, let M be the $n \times n$ matrix of all pairwise dot products: that is, $M_{ij} = v_i \cdot v_j$. Show that M is positive semidefinite.

5. For some fixed vector $u \in \mathbb{R}^d$, define
$$F(x) = \|x - u\|^2.$$
   Is $F(x)$ a convex function of x? Justify your answer.

6. For some fixed vector $u \in \mathbb{R}^d$, define the function $F : \mathbb{R}^d \to \mathbb{R}$ by $F(x) = e^{u \cdot x}$.

   (a) What is the Hessian $H(x)$?

   (b) If F a convex function of x? Justify your answer.

7. Let $p = (p_1, p_2, \ldots, p_m)$ be a probability distribution over m possible outcomes. The *entropy* of p is a measure of how much randomness there is in the outcome. It is defined as
$$F(p) = -\sum_{i=1}^{m} p_i \ln p_i,$$
   where ln denotes natural logarithm. Is this a convex function, or a concave function, or neither? Justify your answer.