Boosting

DSE 220

Worksheet 10

Class 1 : $\omega_1 = (1,1)$, $b_1 = 0$ $\Rightarrow$ $f_1(x) = x_1 + x_2$

Class 2 : $\omega_2 = (1,0)$, $b_2 = 1$ $\Rightarrow$ $f_2(x) = x_1 + 1$

Class 3 : $\omega_3 = (0,1)$, $b_3 = -1$ $\Rightarrow$ $f_3(x) = x_2 - 1$

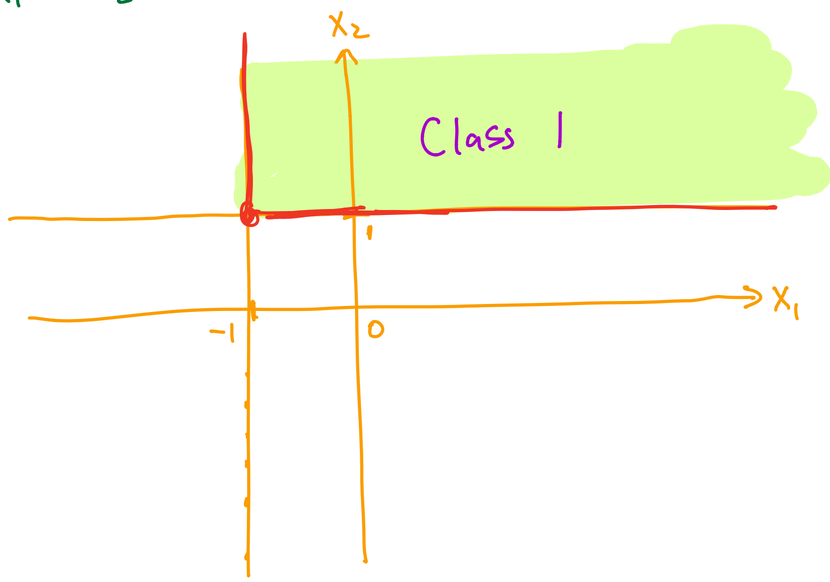
E.g. To classify point $(1,2)$

$$\left.\begin{aligned} f_1(1,2) &= 3 \\ f_2(1,2) &= 2 \\ f_3(1,2) &= 1 \end{aligned}\right\} \quad \therefore \text{ Predict class } \underline{\underline{1}}$$

$\therefore$ Predict class 1 when $f_1(x) \geq f_2(x)$ and $f_1(x) \geq f_3(x)$:

$$x_1 + x_2 \geq x_1 + 1 \iff x_2 \geq 1$$

$$x_1 + x_2 \geq x_2 - 1 \iff x_1 \geq -1$$



Choosing a classifier

So many choices:

- Nearest neighbor
- Different generative models
- Linear predictors with different loss functions
- Different kernels
- Neural nets
- etc.

Can one **combine** them?
And get a classifier that is better than any of them individually?

Voting or weighted majority ? (eg. can fit weights using logistic regression)

Combining simple classifiers

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   So why not keep the individual components simple?

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3. The full (combined) models may get enormous.
   Is this bad for generalization? (overfitting)

Weak learners

It is often easy to come up with a **weak classifier**, one that is marginally better than random guessing:

$$\Pr(h(X) \neq Y) \leq \frac{1}{2} - \epsilon$$

e.g. 45% error

A learning algorithm that can consistently generate such classifiers is called a **weak learner**. ← black-box learning alg

Is it possible to systematically boost the quality of a weak learner?

The blueprint for boosting

Weak learner takes a weighted data set

Given: data set $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$. ← $y \in \{-1, +1\}$

- Initially give all points equal weight.
- Repeat for $t = 1, 2, \ldots$:
  - Feed weighted data set to the weak learner, get back a weak classifier h_t
  - Reweight data to put more emphasis on points that h_t gets wrong
- Combine all these h_t's linearly

weak learner → h_1 coeff. α_1 ; increase weight on points h_1 got wrong

cannot be the same as h_1; weighting is constructed so that h_1 has 50% error on the new weights ← h_2 coeff. α_2 ; increase weight on points h_2 got wrong

h_3 coeff. α_3

$\vdots$

h_T coeff. α_T

Prediction on a new point x is: $\text{sign}\left(\alpha_1 h_1(x) + \alpha_2 h_2(x) + \cdots + \alpha_T h_T(x)\right)$

AdaBoost

Data set $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$, labels $y^{(i)} \in \{-1, +1\}$.

1. Initialize $D_1(i) = 1/n$ for all $i = 1, 2, \ldots, n$ ← $D_t(1), \ldots, D_t(n)$: weights on the n data points at time t; sum to 1

2. For $t = 1, 2, \ldots, T$:
   - Give D_t to weak learner (black box), get back some $h_t : \mathcal{X} \to [-1, 1]$ ← allow it to be more general than $\{-1,+1\}$ (measure of confidence, e.g.)
   - Compute h_t's margin of correctness:

$$r_t = \sum_{i=1}^{n} D_t(i) y^{(i)} h_t(x^{(i)}) \in [-1, 1]$$

← how accurate is h_t on D_t? +1 : perfect, 0 : random

$$\alpha_t = \frac{1}{2} \ln \frac{1 + r_t}{1 - r_t}$$

← coefficient of h_t in the final classifier

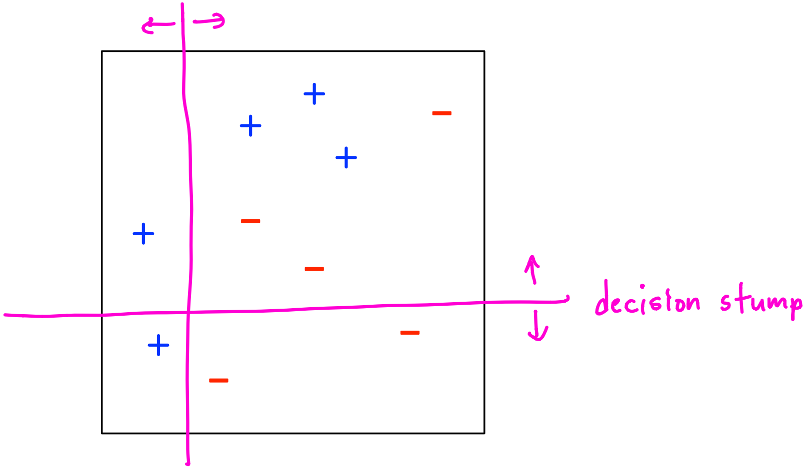
   - Update weights: $D_{t+1}(i) \propto D_t(i) \exp\left(-\alpha_t y^{(i)} h_t(x^{(i)})\right)$ ← put larger weight on the points that h_t got wrong

     ↑ "proportional to" (ie. normalize to sum to 1)

3. Final classifier: $H(x) = \text{sign}\left(\sum_{t=1}^{T} \alpha_t h_t(x)\right)$

Example (Freund-Schapire)

Training set:

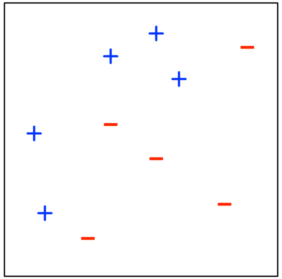


Use "decision stumps" (single-feature thresholds) as weak classifiers

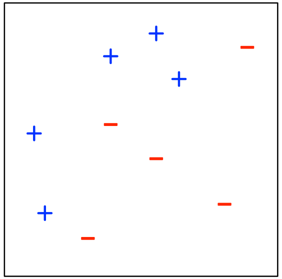
n data pts in $\mathbb{R}^d$: how many distinct decision stumps are there? $(n-1)\,d$

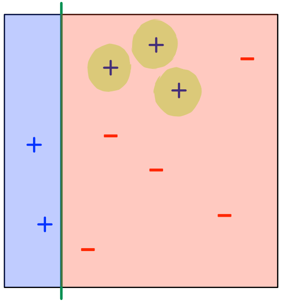
$\therefore$ Easy to choose best stump.

D_1



D_1



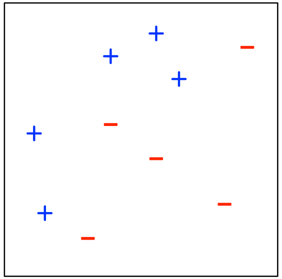


h_1

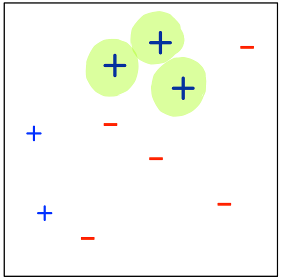
$r_1 = 0.40$, $\alpha_1 = 0.42$

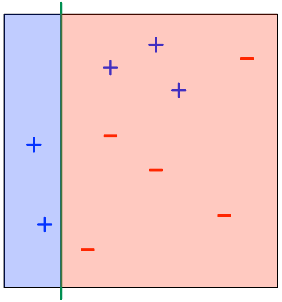
70% accurate

D_1



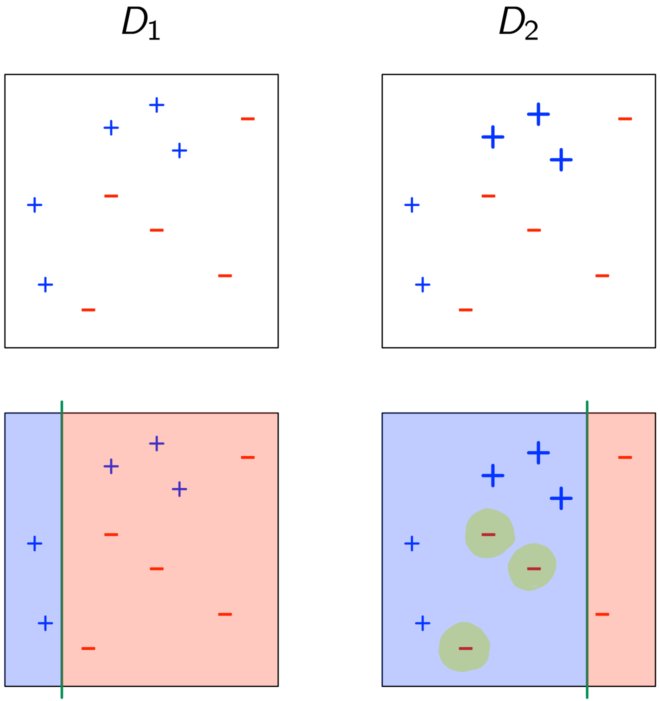
D_2





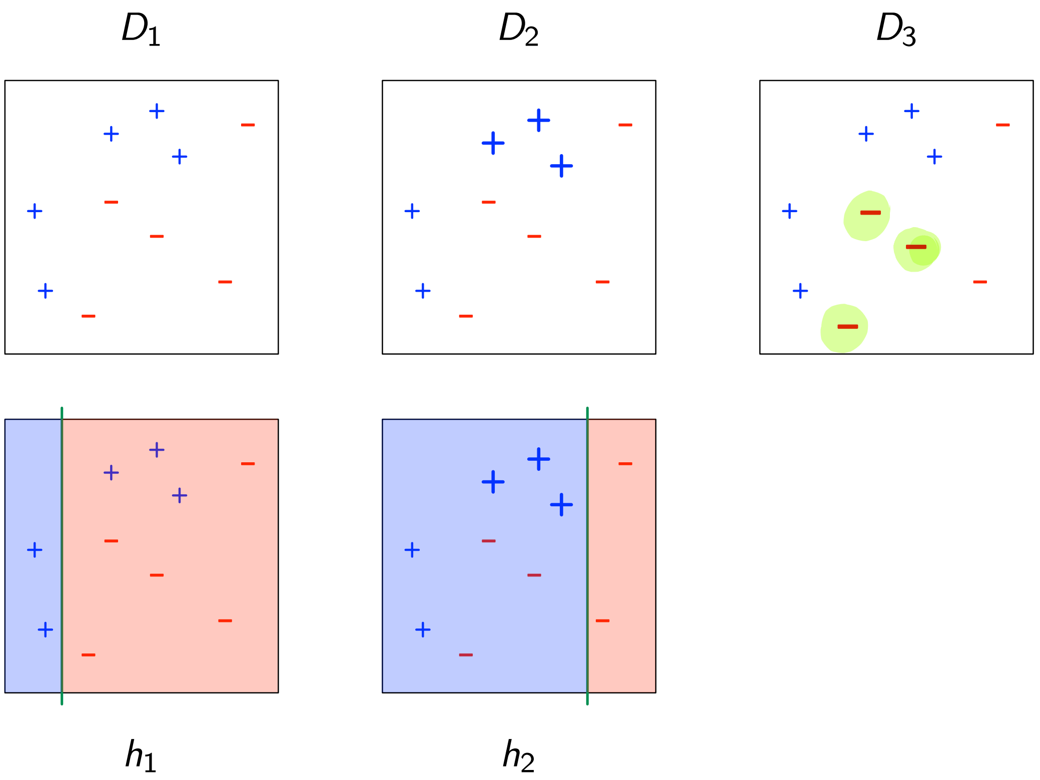
h_1

$r_1 = 0.40, \alpha_1 = 0.42$

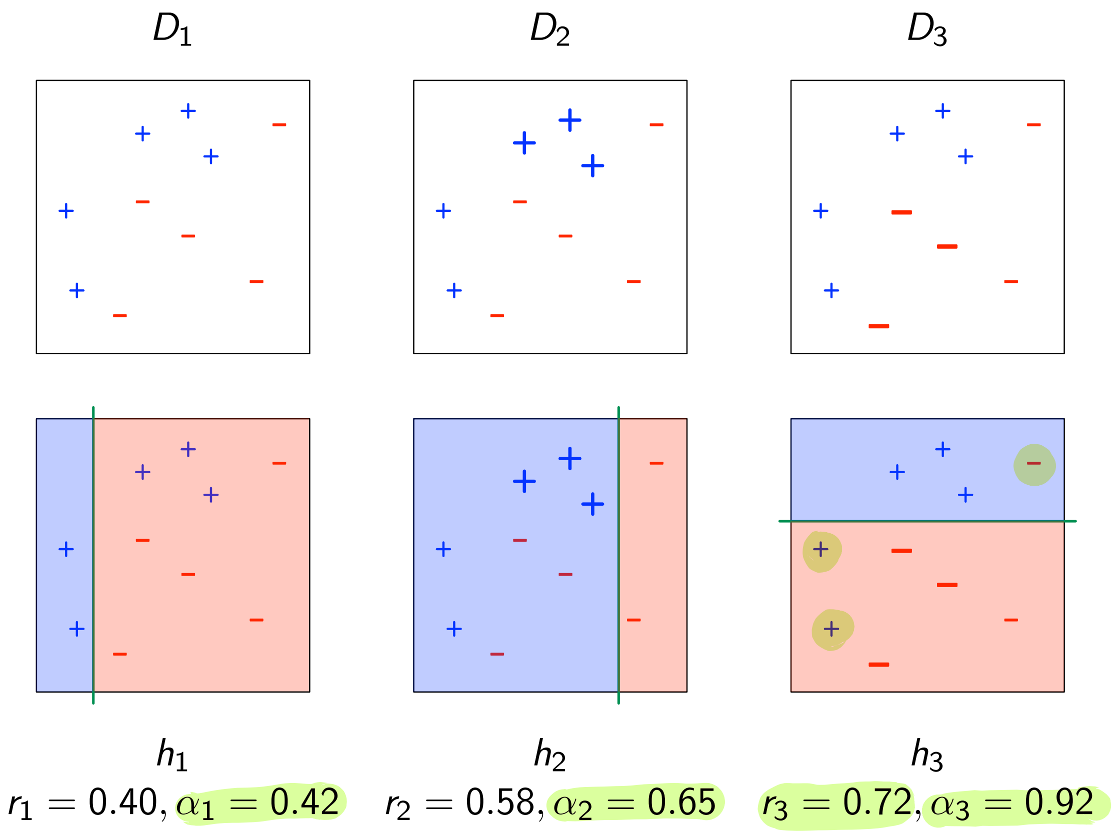


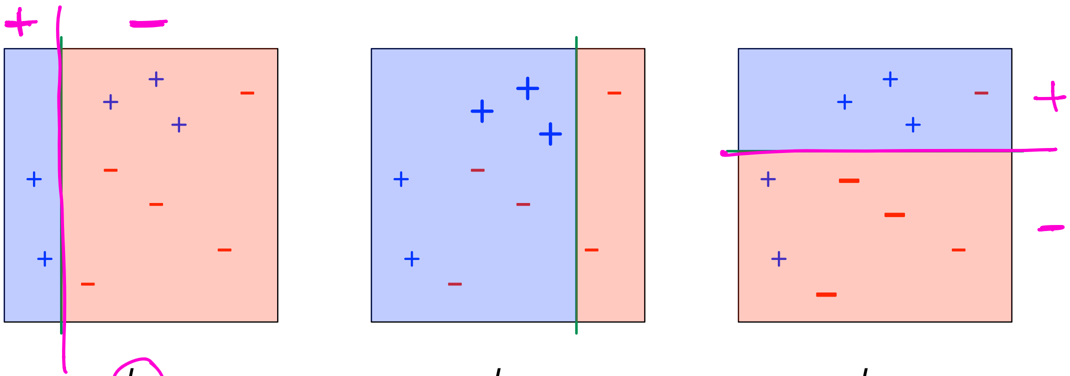
h_1 h_2

$r_1 = 0.40, \alpha_1 = 0.42$ $r_2 = 0.58, \alpha_2 = 0.65$



$r_1 = 0.40, \alpha_1 = 0.42$ $\quad$ $r_2 = 0.58, \alpha_2 = 0.65$

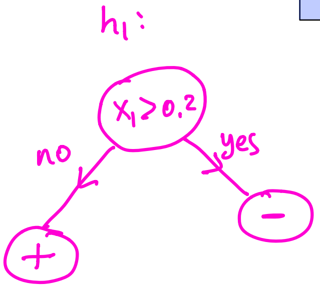

D_1
D_2
D_3
h_1
h_2
h_3
$r_1 = 0.40, \alpha_1 = 0.42$
$r_2 = 0.58, \alpha_2 = 0.65$
$r_3 = 0.72, \alpha_3 = 0.92$



h_1 $\alpha_1 = 0.42$

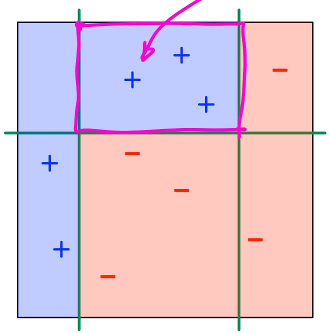
h_2 $\alpha_2 = 0.65$

h_3 $\alpha_3 = 0.92$



Final classifier:

$$\mathrm{sign}\left(0.42h_1(x) + 0.65h_2(x) + 0.92h_3(x)\right)$$



h_1: − 0.42
h_2: + 0.65
h_3: + 0.92

∴ +

The surprising power of weak learning

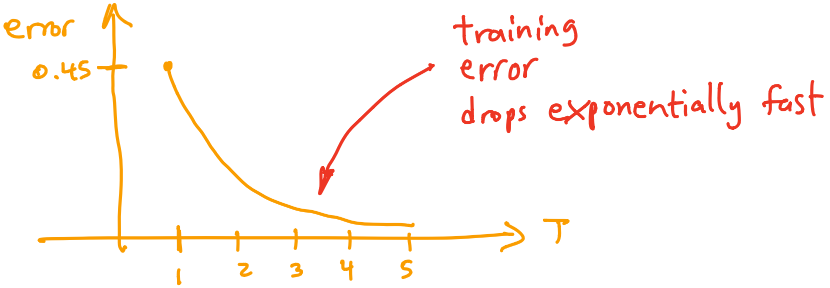
black box

Suppose that on each round t, the weak learner returns a rule h_t whose error on the time-t weighted data distribution is $\leq 1/2 - \gamma$. e.g. 0.05 (at most 45% error)

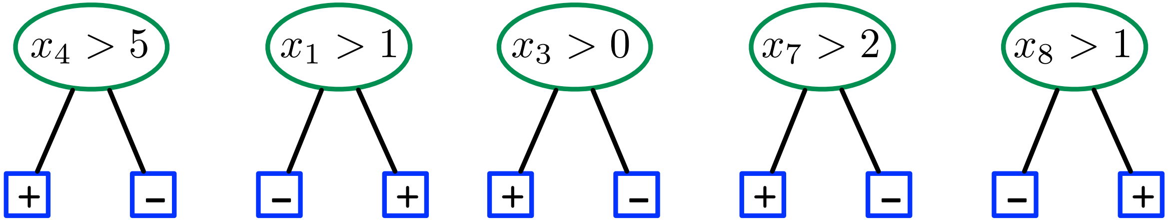
Then, after T rounds, the training error of the combined rule

$$H(x) = \text{sign}\left(\sum_{t=1}^{T} \alpha_t h_t(x)\right)$$

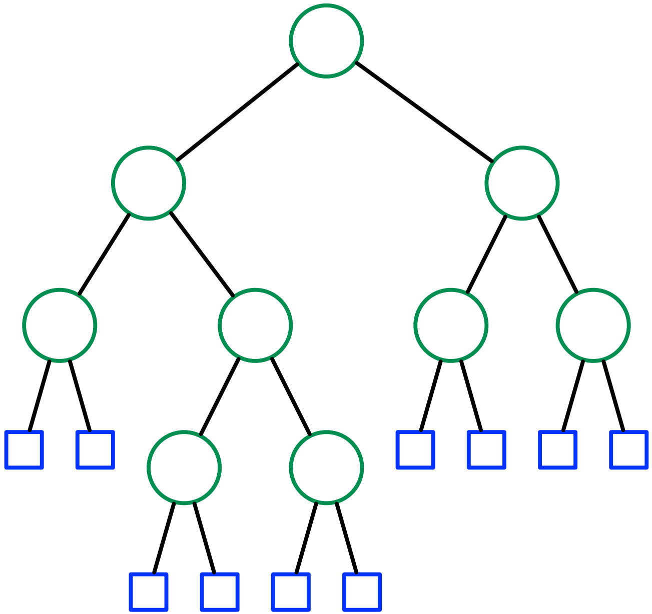
is at most $e^{-\gamma^2 T/2}$.



Boosting decision stumps



Boosting decision trees



Boosting decision trees