DSE 220: Machine learning

Worksheet 11 — Solutions

1. No. Although they are orthogonal to each other, the first two aren't unit length.
2. *Principal axes of two-dimensional data.*
   (a) First points roughly east, second north
   (b) First points roughly northeast, second northwest
   (c) First is roughly southwest, second northwest
   (d) First is roughly north, second east.
3. *Projections*
   (a) $u_1, u_2 \in \mathbb{R}^d$ and $U = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix}$. So we the following dimensions.
      - U: $d \times 2$
      - U^T: $2 \times d$
      - UU^T: $d \times d$
      - $u_1 u_1^T$: $d \times d$

   (b) We can collect the mappings into two groups:
      - $x \mapsto (u_1 \cdot x, u_2 \cdot x)$ and $x \mapsto U^T x$ are the same projection onto the u_1, u_2 directions
      - $x \mapsto (u_1 \cdot x)u_1 + (u_2 \cdot x)u_2$ and $x \mapsto UU^T x$ are the same reconstruction from the above projection
4. *Eigenvectors*
   (a) We can work out

$$\mathbb{E}[X \cdot u] = \mathbb{E}[X] \cdot u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\mathrm{var}(X \cdot u) = u^T \mathrm{cov}(X) u = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}(5 - 3 - 3 + 5 + 4) = \frac{8}{3}$$

   (b) + (c) By multiplying out each of the vectors with $\mathrm{cov}(X)$, we can find which ones are the eigenvectors. We see that $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ satisfies

$$\begin{pmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus this is an eigenvector with eigenvalue 4. We see $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$ satisfies

$$\begin{pmatrix}5 & -3 & 0\\-3 & 5 & 0\\0 & 0 & 4\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}2\\2\\0\end{pmatrix} = \frac{2}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

Thus this is an eigenvector with eigenvalue 2. We see $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$ satisfies

$$\begin{pmatrix}5 & -3 & 0\\-3 & 5 & 0\\0 & 0 & 4\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}8\\-8\\0\end{pmatrix} = \frac{8}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

Thus this is an eigenvector with eigenvalue 8.

(d) PCA would choose the two eigenvectors with the largest eigenvalues: $u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $u_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

(e) The two dimensional projection of $x = (4, 0, 2)$ would be

$$\begin{pmatrix}x \cdot u_1\\x \cdot u_2\end{pmatrix} = \begin{pmatrix}\frac{4}{\sqrt{2}}\\2\end{pmatrix} = \begin{pmatrix}2\sqrt{2}\\2\end{pmatrix}$$

(f) The three-dimensional reconstruction would be

$$(u_1 \cdot x)u_1 + (u_2 \cdot x)u_2 = 2\sqrt{2}u_1 + 2u_2 = \frac{2\sqrt{2}}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\-2\\2\end{pmatrix}$$

5. *Spectral decomposition*

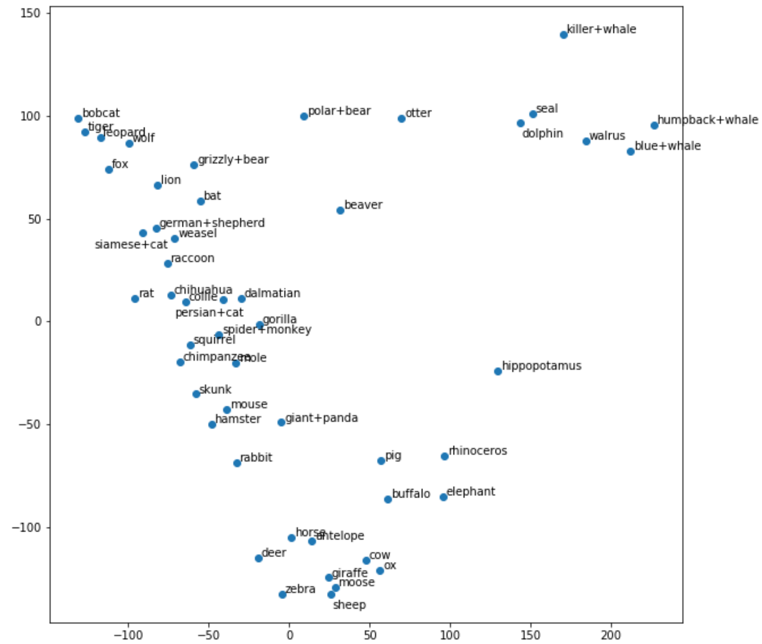
(a) We can construct M from its spectral decomposition:

$$M = \begin{pmatrix}| & |\\u_1 & u_2\\| & |\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}\begin{pmatrix}— & u_1 & —\\— & u_2 & —\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}7 & 6\\6 & -2\end{pmatrix}$$

(b) If $Mu = \lambda u$ then $(M + 2I)u = (\lambda + 2)u$. Thus the eigenvalues of $M + 2I$ are $\lambda_1 + 2 = 4$ and $\lambda_2 + 2 = 1$.

(c) If $Mu = \lambda u$ then $M^2u = M(Mu) = M(\lambda u) = \lambda(Mu) = \lambda^2 u$. Thus the eigenvalues of M^2 are $\lambda_1^2 = 4$ and $\lambda_2^2 = 1$.

6. *PCA on animals.* We can use the PCA package from scikit-learn to fit and transform the data to two dimension. Below is the result of this procedure. The embedding seems to be fairly reasonable. Many land predators are mapped closely together in the upper left-hand corner, while some sea mammals seem to also be mapped to the upper right-hand region.


killer+whale
polar+bear
otter
seal
dolphin
humpback+whale
walrus
blue+whale
bobcat
tiger
leopard
wolf
fox
grizzly+bear
lion
bat
beaver
german+shepherd
weasel
siamese+cat
raccoon
rat
chihuahua
collie
dalmatian
persian+cat
gorilla
spider+monkey
squirrel
chimpanzee
mole
hippopotamus
skunk
mouse
hamster
giant+panda
rabbit
pig
rhinoceros
buffalo
elephant
horse
antelope
deer
cow
ox
giraffe
moose
zebra
sheep