Performance of utility-based strategies for hedging basis risk

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Abstract. The performance of optimal strategies for hedging a claim on a non-traded asset is analyzed. The claim is valued and hedged in a utility maximization framework, using exponential utility. A traded asset, correlated with that underlying the claim, is used for hedging, with the correlation ρ typically close to 1. Using a distortion method [30, 31] we derive a nonlinear expectation representation for the claim's ask price and a formula for the optimal hedging strategy. We generate a perturbation expansion for the price and hedging strategy in powers of $\epsilon^2 = 1 - \rho^2$. The terms in the price expansion are found to be proportional to the central moments of the claim payoff under a measure equivalent to the physical measure. The resulting fast computation capability is used to carry out a simulation based test of the optimal hedging program, computing the terminal hedging error over many asset price paths. These errors are compared with those from a naive strategy which uses the traded asset as a proxy for the non-traded one. The distribution of the hedging error acts as a suitable metric to analyze hedging performance. We find that the the optimal policy improves hedging performance, in that the hedging error distribution is more sharply peaked around a non-negative profit. The frequency of profits over losses is increased, and this is measured by the median of the distribution, which is always increased by the optimal strategies.

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1. Introduction

This article investigates the extent to which the use of an optimal hedging method, based on utility maximization, can improve the management of *basis risk*. By this term we mean the risk associated with the trading of a derivative security on an underlying asset that is not traded. Examples include weather derivatives, or options on baskets of stocks, where the basket is illiquid. In such a scenario, a correlated traded asset might be used for hedging purposes. (In the stock basket example, the claim on the basket might be hedged using liquid futures on a stock index, where the composition of the basket and the index are similar but not identical.)

In such a situation perfect hedging will not generally be possible, and to approach the problem systematically some optimal hedging method is sought. This can be done by embedding the problem in a utility maximization framework, in a manner that is now well established in derivative pricing. Indeed, the optimal valuation and hedging of claims on non-traded assets has been studied by other authors [5, 6, 11, 14, 24]. These papers have been concerned with solving the associated utility maximization problems, involving a portfolio of the traded asset and a random endowment of the claim payoff, from a variety of perspectives.

This paper takes the solution of the utility maximization problem as given, though we do present it briefly for completeness, and generalize the representation for prices given in [11, 24]. Our main contribution is, first, to derive a perturbation series which gives accurate analytic approximations for the price and hedging strategy of the claim. Further details and results on such perturbation expansions are provided in [23]. Second, we use the ensuing fast computation of prices and hedging strategies to conduct a simulation-based test of the efficacy of the optimal hedge relative to a naive strategy which simply uses the traded asset as a proxy for the non-traded one. We take the view that it is important to establish whether optimal risk management procedures offer a significant improvement to more *ad hoc* procedures.

We use an exponential utility function to express the investor's risk preferences, though future work will explore strategies across different preferences and risk measures, such as "expected shortfall" [8]. This risk measure has recently been analyzed in the context of hedging in a stochastic volatility model [17], though a full-blooded test over many asset path histories was not carried out. This is also a fertile topic for future research.

Our testing procedure is to simulate many paths for the traded and non-traded asset prices, and to implement a self-financing hedging strategy implied by both optimal and naive methods. We compute the terminal tracking error for each path, plot the histogram for the tracking error distribution, and compute some relevant statistics of the distribution. Recall that in the Black-Scholes (BS) [3] world the hedging error is zero with probability one, implying a Dirac δ-function distribution for the terminal hedging error.

We do indeed find that the optimal method improves hedging performance over the naive method, and the improvement is greater for lower absolute values of the correlation, and for higher values of risk aversion. The hedging error distribution has a lower standard deviation under the optimal strategy, and a higher median, indicating a higher relative occurrence of positive hedging errors.

The structure of the paper is as follows. In Sections 2–4 we set up the model, define utility-based prices, and classes of equivalent probability measures that arise in the sequel. In Section 5 we derive representations for the asking price and optimal hedging strategy for the claim, and perturbation expansions are derived in Section 6, with explicit results for a put option on the non-traded asset. Section 7 analyzes hedging performance via simulation, and Section 8 concludes.

2. The Basis Risk Model

Two asset prices $(S,Y) := (S_t, Y_t)_{0\le t\le T}$ follow log-normal diffusions:

$$dS_t = \mu S_t dt + \sigma S_t dw_t, \tag{1}$$

$$dY_t = \mu_0 Y_t dt + \sigma_0 Y_t dw_t^0, \tag{2}$$

for $0 \le t \le T$, where the Brownian motions $(w, w^0) = (w_t, w_t^0)_{0\le t\le T}$ have correlation ρ, so that $dw_t^0 dw_t = \rho dt$, with $-1 \le \rho \le 1$. The parameters $\mu, \sigma, \mu_0, \sigma_0, \rho$ are constants, and equations (1) and (2) are written in the physical measure $\mathbb{P}$. The riskless interest rate r is constant. The asset with price S is a *traded* asset but the asset with price Y is *non-traded*. A European option on asset Y has non-negative payoff $h(Y_T)$ at maturity time T, where h is a function.

Denote by $(w, w') := (w_t, w'_t)_{0\le t\le T}$ a two-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, and let the filtration $(\mathcal{F}_t)_{0\le t\le T}$ be the one generated by $(w_t, w'_t)_{0\le t\le T}$. Then w' is independent of w and we can write w_t^0 in (2) as

$$w_t^0 = \rho w_t + \epsilon w'_t, \tag{3}$$

where $\epsilon = \sqrt{1-\rho^2}$. Denote by $(\mathcal{G}_t)_{0\le t\le T}$ the filtration generated by $(w_t^0)_{0\le t\le T}$, the Brownian motion driving the non-traded asset price.

An agent with risk preferences expressed via an exponential utility function

$$U(x) = -\exp(-\gamma x), \tag{4}$$

with constant risk aversion parameter $\gamma \in (0,1)$, has the objective of maximizing expected utility of terminal wealth at time T. The investor can trade a dynamic self-financing portfolio containing Δ_t shares of the traded asset S_t at time $t \in [0,T]$, with the remainder invested in a cash account at interest rate r. In addition, the investor's account is credited at time T with n units of the derivative payoff $h(Y_T)$.

The wealth in the investor's cash and share portfolio, $(X_t)_{0\le t\le T}$, then follows the process

$$dX_t = rX_t dt + \pi_t((\mu - r)dt + \sigma dw_t), \tag{5}$$

where we have defined $\pi_t := \Delta_t S_t, 0 \leq t \leq T$, as the wealth invested in the stock. We note that there is no explicit dependence on S in (5), so that we may use (5) in place of (1) in the equations describing the dynamics of the state variables (X, Y) instead of (S, Y).

The investor's optimization problem is as follows: starting at time $t \in [0, T]$ with endowment $X_t = x$, and with initial non-traded asset price $Y_t = y$, the investor seeks a trading strategy $\pi := (\pi_t)_{0 \leq t \leq T}$ to achieve the supremum

$$F^n(t, x, y) := \sup_{\pi \in \mathcal{P}} \mathbb{E}_{t,x,y} U(X_T + nh(Y_T)). \tag{6}$$

The supremum is taken over a suitable class $\mathcal{P}$ of admissible trading strategies, defined precisely below, and $\mathbb{E}_{t,x,y}$ denotes $\mathbb{P}$-expectation conditional on $X_t = x, Y_t = y$. The superscript n on the left-hand-side of (6) will denote the number of derivative payoffs credited at time T, and the cases $n = 0$ and $n = \pm 1$ will concern us for the most part.

As is well known [6, 11], to ensure that (6) results in a meaningful optimization problem with exponential utility, we must assume that the random endowment $nh(Y_T)$ is bounded below. This covers long positions in calls and puts, short positions in puts, but excludes short call positions. The case of hedging short calls on the non-traded asset will be revisited in future papers.

A trading strategy is an adapted process $(\pi_t)_{0 \leq t \leq T}$ satisfying $\int_0^T \pi_t^2 dt < \infty$ almost surely. Denote by $\mathcal{P}_0$ the set of trading strategies. The set of admissible trading strategies is defined following [27] via the following construction:

$$\begin{aligned}
\mathcal{P}_b &= \{\pi \in \mathcal{P}_0 : X_t \geq a_\pi \in \mathbb{R} \text{ a.s. } \forall t \in [0, T]\}, \\
\mathcal{U}_b &= \{F \in L^0(\Omega, \mathcal{F}_T, \mathbb{P}) : F \leq X_T + nh(Y_T), \text{ for } \pi \in \mathcal{P}_b \text{ and } \mathbb{E}|U(F)| < \infty\} \\
\mathcal{U} &= \{U(F) : F \in \mathcal{U}_b\}^c, \\
\mathcal{P} &= \{\pi \in \mathcal{P}_0 : U(X_T + nh(Y_T) \in \mathcal{U}\},
\end{aligned} \tag{7}$$

where $\{\ldots\}^c$ denotes the closure in $L_1(\Omega, \mathcal{F}_T, \mathbb{P})$.

The intuition behind the above definitions is that one first seeks trading strategies whose gains processes are bounded below, in order to eliminate doubling strategies [10], resulting in the class $\mathcal{P}_b$. But this class is not big enough to ensure locating the optimal strategy by searching only within it. When the utility function $U(x)$ is defined for all $x \in \mathbb{R}$, it is necessary to consider strategies with wealths which are not necessarily bounded from below.

Denote the optimal trading strategy that achieves the supremum in (6) by $\pi^* = (\pi_t^*)_{0 \leq t \leq T}$. We shall use the optimization problem (6) to define various candidate time-t prices $p(t, x, y)$ for the claim, consistent with the investor's utility maximization objective, as shown in the next section.

2.1. The Case of Perfect Correlation

If $\rho = 1$, then as shown in [5], absence of arbitrage implies that, given σ, σ_0, the drifts μ, μ_0 are related by

$$\frac{\mu_0 - r}{\sigma_0} = \frac{\mu - r}{\sigma}. \tag{8}$$

In this case, perfect hedging of the claim on Y is possible by trading S, the hedging strategy at time $t \in [0, T]$ being to hold a number of shares given by

$$\frac{\sigma_0 Y_t}{\sigma S_t} \frac{\partial}{\partial s} \mathrm{BS}(Y_t, 0, \sigma_0), \tag{9}$$

where $\mathrm{BS}(s, q, \sigma)$ denotes the BS formula with underlying asset price s, dividend yield q and volatility σ.

3. Utility Based Pricing

Consider some special cases of the optimization problem (6). For $n = 0$ there is no dependence on the claim. The dynamics of the non-traded asset Y do not influence the problem at all and we recover a variant of the classical Merton problem [19, 20]. We set $F^0(t, x, y) = F(t, x)$ to signify that there is no dependence on n or y in this case. The cases $n = \pm 1$ correspond to a credit and debit of one unit of the option payoff $h(Y_T)$, so with a suitable adjustment to the initial endowment of $\mp p(t, x, y)$, represent the cases where the investor buys or sells one claim for price $p(t, x, y)$.

We can use these special cases to define various utility based prices for the claim. At time t, the utility indifference selling price (or simply the *ask* price) of the claim, $p^a(t, x, y)$, is defined by

$$F(t, x) = F^{-1}(t, x + p^a(t, x, y), y). \tag{10}$$

Similarly the utility indifference buying price (or the *bid* price) of the claim, $p^b(t, x, y)$, is defined by

$$F(t, x) = F^{1}(t, x - p^b(t, x, y), y). \tag{11}$$

The marginal price $p^m(t, x, y)$ for the claim is given by

$$p^m(t, x, y) = \frac{\mathbb{E}_{t,x,y}[U'(X_T^*) h(Y_T)]}{F_x(t, x)}, \tag{12}$$

where $U'(x)$ denotes the derivative of $U(x)$, $F_x(t, x)$ denotes the partial derivative of $F(t, x)$ with respect to x, and $(X_t^*)_{0 \le t \le T}$, denotes the optimal wealth process under the optimal trading strategy $(\pi_t^*)_{0 \le t \le T}$, which achieves the supremum in (6) for $n = 0$. The original definition of the marginal price in [4] was as the price which left the investor's maximum utility unchanged for an infinitesimal diversion of funds into the purchase or sale of a claim, and this reduces to the representation in (12) when the value function in (6) satisfies appropriate smoothness conditions, as shown in [18]. For a recent analysis of a general definition of the marginal price, which involves treating the number of claims traded as a variable of the optimization problem, see [15, 16].

4. Equivalent Measures

4.1. Measures Equivalent on $\mathcal{F}_T$

Consider how the asset price dynamics under $\mathbb{P}$ in (1) and (2) alter under a change of measure. Measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ have densities of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T, \tag{13}$$

where $(Z_t)_{0\le t\le T}$ is the $\mathbb{P}$-local martingale given by

$$Z_t = \exp\left(-\int_0^t m_u dw_u - \int_0^t g_u dw'_u - \frac{1}{2}\int_0^t m_u^2 du - \frac{1}{2}\int_0^t g_u^2 du\right), \tag{14}$$

with m_t, g_t being $\mathcal{F}_t$-adapted processes satisfying $\int_0^T m_t^2 dt < \infty$, $\int_0^T g_t^2 dt < \infty$, $\mathbb{P}$-almost surely.

Under $\mathbb{Q}$ the two-dimensional process $(\widetilde{w}, \widetilde{w}') = (\widetilde{w}_t, \widetilde{w}'_t)_{0\le t\le T}$, defined by

$$\widetilde{w}_t := w_t + \int_0^t m_u du, \tag{15}$$

$$\widetilde{w}'_t := w'_t + \int_0^t g_u du, \tag{16}$$

is two-dimensional Brownian motion.

Then under $\mathbb{Q}$ the asset price dynamics become

$$dS_t = (\mu - m_t\sigma)S_t dt + \sigma S_t d\widetilde{w}_t, \tag{17}$$

$$dY_t = (\mu_0 - \sigma_0(\rho m_t + \epsilon g_t))Y_t dt + \sigma_0 Y_t d\widetilde{w}_t^0, \tag{18}$$

where $\widetilde{w}_t^0$ is a Brownian motion defined by

$$\widetilde{w}_t^0 = \rho\widetilde{w}_t + \epsilon\widetilde{w}'_t, \tag{19}$$

so that $d\widetilde{w}_t^0 d\widetilde{w}_t = \rho dt$.

4.2. Local Martingale Measures

For $\mathbb{Q}$ to be a local martingale measure we require the process $(e^{-rt}S_t)_{0\le t\le T}$ to be a $\mathbb{Q}$-local martingale. From (17) this is true only if $\mu - m_t\sigma = r$, that is if

$$m_t = \lambda := \frac{\mu - r}{\sigma}, \tag{20}$$

while g_t can be arbitrary. Therefore the set $\mathcal{M}$ of equivalent local martingale measures is in one-to-one correspondence with the set of integrands g_t in (14).

Definition 1 (Minimal Martingale Measure) *The minimal martingale measure $\mathbb{Q}^0 \in \mathcal{M}$ corresponds to $g_t = 0, 0 \le t \le T$.*

There are many characterizations of the minimal martingale measure, and the reader is referred to the review by Schweizer [28] for further details.

4.3. Measures Equivalent on $\mathcal{G}_T$

Consider measures $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ on $\mathcal{G}_T$. Recall that $(\mathcal{G}_t)_{0\le t\le T}$ is the filtration generated by $(w^0_t)_{0\le t\le T}$, the Brownian motion driving the non-traded asset price Y. We shall have recourse to discuss such measures in the sequel. They have densities of the form

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \widetilde{Z}_T, \tag{21}$$

where $(\widetilde{Z}_t)_{0\le t\le T}$ is the $\mathbb{P}$-local martingale given by

$$\widetilde{Z}_t = \exp\left(-\int_0^t \theta_u dw^0_u - \frac{1}{2}\int_0^t \theta^2_u du\right), \tag{22}$$

and where θ_t is a $\mathcal{G}_t$-adapted process satisfying $\int_0^T \theta^2_t dt < \infty$, $\mathbb{P}$-almost surely.

Under $\widetilde{\mathbb{P}}$ the process $(\widetilde{w}^0_t)_{0\le t\le T}$, defined by

$$\widetilde{w}^0_t := w^0_t + \int_0^t \theta_u du, \tag{23}$$

is Brownian motion, and the process followed by asset price Y becomes

$$dY_t = (\mu_0 - \sigma_0\theta_t)Y_t dt + \sigma_0 Y_t d\widetilde{w}^0_t. \tag{24}$$

Comparing (24) with (18) shows that the dynamics of the non-traded asset Y are the same under $\mathbb{Q}$ and $\widetilde{\mathbb{P}}$ whenever the integrands m_t, g_t, θ_t are related by

$$\rho m_t + \epsilon g_t = \theta_t, \quad 0 \le t \le T. \tag{25}$$

5. The Asking Price of a Claim

5.1. The Hamilton-Jacobi-Bellman Equation

By the Bellman optimality principle for dynamic programming (which amounts to the fact that the utility process is a supermartingale, and a martingale at the optimum strategy), $F^n(t,x,y)$ is conjectured to satisfy the PDE

$$\max_{\pi_t} \mathcal{L}F^n(t,x,y) = 0, \tag{26}$$

where $\mathcal{L}$ is the differential operator defined by

$$\begin{aligned}\mathcal{L}\phi(t,x,y) = {} & \phi_t(t,x,y) + (rx + \pi_t(\mu - r))\phi_x(t,x,y) + \mu_0 y\phi_y(t,x,y) \\ & + \frac{1}{2}\sigma^2\pi_t^2\phi_{xx}(t,x,y) + \frac{1}{2}\sigma_0^2 y^2\phi_{yy}(t,x,y) \\ & + \rho\sigma\sigma_0\pi_t y\phi_{xy}(t,x,y).\end{aligned} \tag{27}$$

If one can find a classical solution to this equation to which Itô's lemma can be applied, then proof of optimality follows from standard verification theorems. See [7], for instance.

Formally carrying out the maximization over π_t yields the optimal strategy π^*_t as

$$\pi^*_t = -\frac{[(\mu - r)F^n_x(t,x,y) + \rho\sigma\sigma_0 y F^n_{xy}(t,x,y)]}{\sigma^2 F^n_{xx}(t,x,y)}. \tag{28}$$

Substituting this into (26) gives the HJB equation for $F^n(t,x,y)$ in the form

$$F_t^n(t,x,y) + rxF_x^n(t,x,y) + \mu_0 y F_y^n(t,x,y) + \frac{1}{2}\sigma_0^2 y^2 F_{yy}^n(t,x,y) - \frac{1}{2F_{xx}^n(t,x,y)}\left[\lambda F_x^n(t,x,y) + \rho\sigma_0 y F_{xy}^n(t,x,y)\right]^2 = 0, \tag{29}$$

with terminal boundary condition $F^n(T,x,y) = -e^{-\gamma(x+nh(y))}$, and λ defined in (20).

Under exponential utility, it is possible to factor out the initial cash endowment x because the index of risk aversion, $-U''(x)/U'(x) = \gamma$, is constant. To be more precise about this commonly made argument, note that the solution to the stochastic differential equation (5) gives the terminal wealth X_T (given $X_t = x$) as

$$X_T = \beta(t,T)x + G(t,T), \tag{30}$$

where we have defined the accumulation factor

$$\beta(t,T) := e^{r(T-t)}, \quad 0 \le t \le T, \tag{31}$$

and the gains from trading process

$$G^\pi(t,T) \equiv G(t,T) := (\mu - r)e^{rT}\int_t^T e^{-ru}\pi_u du + \sigma e^{rT}\int_t^T e^{-ru}\pi_u dw_u. \tag{32}$$

Consequently, with $U(x)$ given by (4), we have

$$U(X_T + nh(Y_T)) = e^{-\gamma\beta(t,T)x}\left(-e^{-\gamma(G(t,T)+nh(Y_T))}\right). \tag{33}$$

The constant term involving the initial capital x then factors out of the value function $F^n(t,x,y)$, so that

$$F^n(t,x,y) = e^{-\gamma\beta(t,T)x}F^n(t,0,y) =: e^{-\gamma\beta(t,T)x}W^n(t,y). \tag{34}$$

We have thus reduced the dimensionality of the problem, expressing it in terms of the function $W^n(t,y) := F^n(t,0,y)$.

Using (34) we rewrite the HJB equation (29) for $F^n(t,x,y)$ in terms of $W^n(t,y)$. All terms involving the initial capital x disappear, and we are left with the following non-linear equation for $W^n(t,y)$:

$$W_t^n(t,y) + (\mu_0 - \sigma_0\rho\lambda)\, yW_y^n(t,y) + \frac{1}{2}\sigma_0^2 y^2 W_{yy}^n(t,y) - \frac{1}{2}(\sigma_0\rho y)^2\frac{(W_y^n(t,y))^2}{W^n(t,y)} - \frac{1}{2}\lambda^2 W^n(t,y) = 0, \tag{35}$$

with terminal boundary condition $W^n(T,y) = -e^{-\gamma nh(y)}$.

5.2. Distortion

At first sight it appears difficult to find a simple representation for the solution to the PDE (35). However, a simple power transformation can help. To this end, write

$$W^n(t,y) = (f^n(t,y))^\delta, \tag{36}$$

for some arbitrary parameter δ and a function $f^n(t,y)$. This technique is called *distortion* by Zariphopoulou [30, 31] and is also employed in [11, 12]. There are links to the dual approach to solving the optimization problem, involving the Legendre transform of the value function. These links are discussed further in [22].

Rewriting the PDE (35) as a PDE for $f^n(t,y)$ results in

$$f_t^n(t,y) + (\mu_0 - \sigma_0\rho\lambda)\, y f_y^n(t,y) + \frac{1}{2}\sigma_0^2 y^2 f_{yy}^n(t,y) + \frac{1}{2}\sigma_0^2 y^2[(\delta-1) - \delta\rho^2]\frac{(f_y^n(t,y))^2}{f^n(t,y)} - \frac{1}{2}\frac{\lambda^2}{\delta} f^n(t,y) = 0, \tag{37}$$

with terminal boundary condition $f^n(T,y) = -e^{-\gamma n h(y)/\delta}$. This non-linear equation is readily reduced to a linear one by an appropriate choice of δ, namely

$$\delta = \frac{1}{1-\rho^2}. \tag{38}$$

With this choice of δ, (37) becomes

$$f_t^n(t,y) + \mathcal{A} f^n(t,y) - \alpha f^n(t,y) = 0, \tag{39}$$

with terminal condition $f^n(T,y) = -\exp(-\gamma(1-\rho)^2 n h(y))$. The parameter α in (39) is given by

$$\alpha = \frac{1}{2}\lambda^2(1-\rho^2) = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (1-\rho)^2, \tag{40}$$

and $\mathcal{A}$ is a differential operator given by

$$\mathcal{A}\phi(y) = (\mu_0 - \sigma_0\rho\lambda)\, y\phi_y(y) + \frac{1}{2}\sigma_0^2 y^2 \phi_{yy}(y). \tag{41}$$

In other words $\mathcal{A}$ is the generator of the one-dimensional diffusion

$$dY_t = (\mu_0 - \sigma_0\rho\lambda)\, Y_t dt + \sigma_0 Y_t d\widetilde{w}_t^0, \tag{42}$$

where $\widetilde{w}^0$ is a Brownian motion.

The dynamics in (42) are the same as those of asset Y in (2) with an adjusted drift, and therefore under some measure $\mathbb{P}^0$, equivalent to $\mathbb{P}$ on some σ-algebra $\mathcal{B}$, large enough to contain the information from observing Y over $[0,T]$.

There are two possible choices for $\mathcal{B}$: either $\mathcal{B} = \mathcal{F}_T$, or $\mathcal{B} = \mathcal{G}_T$, since both $\mathcal{F}_T$ and $\mathcal{G}_T$ contain the information from observing the non-traded asset price Y.

If we choose $\mathcal{B} = \mathcal{F}_T$, then comparing (42) with (18) we see that $\mathbb{P}^0 = \mathbb{Q}$, corresponding to integrands $(m_t, g_t)_{0\le t\le T}$ in (14) that satisfy

$$\rho m_t + \epsilon g_t = \rho\lambda, \quad 0 \le t \le T. \tag{43}$$

If we choose $\mathcal{B} = \mathcal{G}_T$, then comparing (42) with (24) we see that $\mathbb{P}^0 = \widetilde{\mathbb{P}}$, corresponding to an integrand $(\theta_t)_{0\le t\le T}$ in (22) satisfying

$$\theta_t = \rho\lambda, \quad 0 \le t \le T. \tag{44}$$

Definition 2 *Define by $\mathcal{N}$ the class of probability measures equivalent to the physical measure $\mathbb{P}$ on $\mathcal{F}_T$ and which correspond to Girsanov densities with integrands $(m_t, g_t)_{0\le t\le T}$ satisfying (43). In other words*

$$\mathcal{N} := \{\mathbb{P}^0 \sim \mathbb{P} \text{ on } \mathcal{F}_T : \rho m_t + \epsilon g_t = \rho\lambda, \;\; 0 \le t \le T.\} \tag{45}$$

Since $\mathcal{G}_T \subset \mathcal{F}_T$, the class $\mathcal{N}$ includes the measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ which satisfies (44). In this case $(m_t, g_t) = (\rho^2\lambda, \rho\epsilon\lambda), 0 \leq t \leq T$, as shown in Lemma 1 below.

Returning to the solution of (39), the Feynman-Kac Theorem implies that $f^n(t, y)$ has the probabilistic representation

$$f^n(t,y) = \mathbb{E}^0_{t,y}\left(-e^{-\alpha(T-t)-\gamma(1-\rho)^2 nh(Y_T)}\right), \tag{46}$$

where $\mathbb{E}^0_{t,y}$ denotes expectation under $\mathbb{P}^0 \in \mathcal{N}$, conditional on $Y_t = y$.

Then, using (36) and (38) we get the following representation for $W^n(t, y)$:

$$W^n(t,y) = \left[\mathbb{E}^0_{t,y}\left(-e^{-\alpha(T-t)-\gamma(1-\rho)^2 nh(Y_T)}\right)\right]^{(1-\rho^2)^{-1}}. \tag{47}$$

The value function for the original optimization problem is then obtained from (34) as

$$F^n(t,x,y) = e^{-\gamma\beta(t,T)x}\left[\mathbb{E}^0_{t,y}\left(-e^{-\alpha(T-t)-\gamma(1-\rho)^2 nh(Y_T)}\right)\right]^{(1-\rho^2)^{-1}}. \tag{48}$$

Finally, using the above result along with (10) we obtain the following representation for the ask price of the claim.

Theorem 1 *The utility indifference asking price at time $t \leq T$ of a European claim with payoff $h(Y_T)$ is given by*

$$p^a(t,y) = \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\log\left[\mathbb{E}^0_{t,y}\left(e^{\gamma(1-\rho^2)h(Y_T)}\right)\right], \tag{49}$$

where $\mathbb{E}^0_{t,y}$ denotes expectation conditional on $Y_t = y$ under any probability measure $\mathbb{P}^0 \in \mathcal{N}$.

We observe that $p^a(t, y)$ is independent of the agent's initial cash endowment x, as is always the case under exponential preferences.

Remark 1 The expectation in (49) is under *any* equivalent probability measure in the class $\mathcal{N}$. The distribution of the non-traded asset price is the same under any measure in $\mathcal{N}$, so the price in (49) is indeed uniquely fixed.

Henderson [11] and Musiela & Zariphopoulou [24] obtain similar (but not identical) representations to (49) for the ask price, the results differing in the probability measure appearing in the non-linear representation (49). In [11] the measure used was the minimal martingale measure $\mathbb{Q}^0$, corresponding to $(m_t, g_t) = (\lambda, 0)$. This is the only martingale measure in $\mathcal{N}$, that is, $\mathcal{N} \cap \mathcal{M} = \mathbb{Q}^0$, and measures outside $\mathcal{M}$ were not considered in [11].

In [24] the chosen measure corresponded to $(m_t, g_t) = (\rho^2\lambda, \rho\epsilon\lambda)$. This choice was arrived at by considering measures $\widetilde{\mathbb{P}}$, equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ (as opposed to $\mathcal{F}_T$), and under which the drift of asset Y would be that given in (42). This leads to $(m_t, g_t) = (\rho^2\lambda, \rho\epsilon\lambda)$, as shown in Lemma 1 below. The measures appearing in [11] and [24] are both in the class $\mathcal{N}$, so the results in [11] and [24] are special cases of Theorem 1.

Lemma 1 *The unique measure $\widetilde{\mathbb{P}} \sim \mathbb{P}$ on $\mathcal{G}_T$ which lies in $\mathcal{N}$ corresponds to $(m_t, g_t) = (\rho^2\lambda, \rho\epsilon\lambda), 0 \leq t \leq T$.*

Proof First note that any measure in $\mathcal{N}$, and therefore equivalent to $\mathbb{P}$ on $\mathcal{F}_T$, will necessarily be equivalent to $\mathbb{P}$ on $\mathcal{G}_T \subset \mathcal{F}_T$. Second, if $\widetilde{\mathbb{P}} \sim \mathbb{P}$ on $\mathcal{F}_T$ corresponds to $(m_t, g_t) = (\rho^2\lambda, \rho\epsilon\lambda), 0 \le t \le T$, then (43) is satisfied, so that $\widetilde{\mathbb{P}}$ is indeed in $\mathcal{N}$. Then the integrand $(\theta_t)_{0\le t\le T}$ in (22) must satisfy (44). Using the definition (3) of w^0, along with (21) and (22), we then have, with $\theta_t = \rho\lambda, 0 \le t \le T$:

$$\begin{aligned}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} &= \exp\left(-\int_0^T \theta_t dw_t^0 - \frac{1}{2}\int_0^T \theta_t^2 dt\right) \\ &= \exp\left(-\int_0^T \rho\lambda(\rho dw_t + \epsilon dw_t') - \frac{1}{2}\int_0^T (\rho\lambda)^2 dt\right) \\ &= \exp\left(-\int_0^T \rho^2\lambda dw_t - \int_0^T \rho\epsilon\lambda dw_t' - \frac{1}{2}\int_0^T \left[(\rho^2\lambda)^2 + (\rho\epsilon\lambda)^2\right] dt\right),\end{aligned}$$

where we have applied the identity $(\rho\lambda)^2 = [(\rho^2\lambda)^2 + (\rho\epsilon\lambda)^2]$. Comparing with (13) and (14) completes the proof.

■

5.3. A PDE for the Reservation Ask Price

We can derive a PDE satisfied by the ask price. From (49) we have

$$p^a(t,y) = \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\log\psi(t,y), \tag{50}$$

where by the Feynmac-Kac formula, $\psi(t,y)$ solves

$$\psi_t(t,y) + \mathcal{A}\psi(t,y) = 0, \quad \psi(T,y) = e^{\gamma(1-\rho^2)h(y)}, \tag{51}$$

and $\mathcal{A}$ is the operator defined in (41). It readily follows that $p^a(t,y)$ solves

$$p_t^a(t,y) + \mathcal{A}p^a(t,y) - rp^a(t,y) + \frac{1}{2}\gamma\sigma_0^2(1-\rho^2)y^2\beta(t,T)(p_y^a(t,y))^2 = 0, \tag{52}$$

with terminal condition $p^a(T,y) = h(y)$.

Remark 2 For $\rho = 1$ the above PDE reduces to the BS PDE with volatility σ_0, and the asking price becomes the BS price with this volatility.

The nonlinear nature of (52) illustrates the usefulness of the distortion method and the expectation representation (49), which would certainly not be obvious from (52). Note that the left-hand-side of (52) contains terms reminiscent of a BS-type equation, with the last term being a non-linear perturbation, which can be regarded as small for values of ρ close to 1. One can envisage trying to solve the PDE via classical perturbation analysis, familiar in physics [2]. A natural perturbation parameter would be $\epsilon = 1 - \rho^2$. We shall not solve the PDE in this way, but instead derive a perturbation expansion directly from the expectation representation (49).

5.4. Optimal Hedging Strategy

The optimal trading strategy in the presence of the random endowment $nh(Y_t)$ at the terminal time is given by (28). For $n = 0$, and using (48), this gives the optimal trading strategy in the absence of the claim as

$$\pi_t^* = e^{-r(T-t)} \left(\frac{\mu - r}{\sigma^2 \gamma} \right), \tag{53}$$

which is the well-known solution to the Merton optimal investment problem with exponential utility.

For the case of the writer of a claim, we must take $n = -1$ in (28). Now, for general n, differentiating (48) yields

$$F_x^n(t,x,y) \; = \; -\gamma\beta(t,T)F^n(t,x,y), \tag{54}$$

$$F_{xx}^n(t,x,y) = \gamma^2\beta^2(t,T)F^n(t,x,y), \tag{55}$$

$$F_{xy}^n(t,x,y) = \; -\gamma\beta(t,T)F_y^n(t,x,y). \tag{56}$$

The derivatives of the value function with respect to x are proportional to the value function itself. To get a similar result for $F_{xy}^n(t,x,y) = -\gamma\beta F_y^n(t,x,y)$ in the case $n = -1$, proceed as follows. Differentiate (10) with respect to y, and recall that the ask price is independent of the initial capital x (i.e. $p^a(t,x,y) = p^a(t,y)$), to give

$$F_y^{-1}(t,x,y) = -F_x^{-1}(t,x,y)p_y^a(t,y). \tag{57}$$

Using this in (56), along with (54), 55) and (28) gives the optimal trading strategy of the writer as

$$\pi_t^* = e^{-r(T-t)} \left(\frac{\mu - r}{\sigma^2 \gamma} \right) + \frac{\rho\sigma_0 y}{\sigma} p_y^a(t,y). \tag{58}$$

The strategy in (58) is very intuitive. The first term represents the optimal investment strategy in the absence of a claim, and the second term is the adjustment to this strategy caused by the introduction of the claim, that is, the *hedging strategy* for the claim. This definition of a hedging strategy for a claim associated with a utility-based pricing scheme has been used in models with transaction costs [21], and shown to be a natural one. We therefore have the following result.

Theorem 2 *The hedging strategy for the sale of the claim at the asking price $p^a(t,y)$ at time $t \in [0,T]$ is to hold Δ_u^a shares of the traded asset S at time $u > t$, given by*

$$\Delta_u^a = \frac{\rho\sigma_0 Y_u}{\sigma S_u} \frac{\partial p^a}{\partial y}(u, Y_u), \quad t \le u \le T. \tag{59}$$

It is easy to see that this reduces to the strategy in (9) when $\rho = 1$.

6. Perturbation Expansions

Having presented the derivation of the representation (52) for the ask price of the claim, we proceed to derive a power series expansion for it, and also for its derivative with respect to y, which has application in hedging, as given by Theorem 2. Further perturbative expansions of the type described below, and for other utility functions, are derived in [23].

Let a random variable X have variance Σ^2 and write $\mu_k = \mathbb{E}(X^k), k \in \mathbb{N}$. Define the skewness $\mathrm{skw}(X)$ and kurtosis $\mathrm{kur}(X)$ of X by

$$\mathrm{skw}(X) := \frac{\mathbb{E}\left[(X-\mu_1)^3\right]}{\Sigma^3}, \tag{60}$$

$$\mathrm{kur}(X) := \frac{\mathbb{E}\left[(X-\mu_1)^4\right]}{\Sigma^4} - 3. \tag{61}$$

Observe that with the above definitions we have the identities

$$\Sigma^3 \mathrm{skw}(X) = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 \tag{62}$$

$$\Sigma^4 \mathrm{kur}(X) = \mu_4 - 3\mu_2^2 + 12\mu_1^2\mu_2 - 4\mu_1\mu_3 - 6\mu_1^4. \tag{63}$$

We then have the following expansion for the asking price $p^a(t,y)$ of the claim on the non-traded asset with payoff $h(Y_T)$.

Theorem 3 *The function $p^a(t,y)$ representing the asking price of the claim with payoff $h(Y_T)$ at time $T \geq t$ has the perturbative representation*

$$p^a(t,y) = \frac{1}{\beta(t,T)}\left[\mathbb{E}^0_{t,y}h(Y_T) + \frac{1}{2}\gamma\epsilon^2 \mathrm{var}^0_{t,y}h(Y_T) + \frac{1}{3!}(\gamma\epsilon^2)^2\Sigma^3\mathrm{skw}^0_{t,y}h(Y_T) \right.$$
$$\left. + \ \frac{1}{4!}(\gamma\epsilon^2)^3\Sigma^4\mathrm{kur}^0_{t,y}h(Y_T) + O(\epsilon^8)\right], \tag{64}$$

where $O(\epsilon^8)$ denotes terms proportional to ϵ^8 and to higher powers of ϵ. The expansion is valid for model parameters satisfying $\mathbb{E}^0_{t,y}\exp(\gamma\epsilon^2 h(Y_T)) \leq 2$.

In the above Theorem, $\mathrm{var}^0_{t,y}$ denotes the variance operator conditional on $Y_t = y$, under any measure $\mathbb{P}^0 \in \mathcal{N}$, with a similar convention for $\mathrm{skw}^0_{t,y}$ and $\mathrm{kur}^0_{t,y}$.

Proof Expanding the exponential in (49) using Taylor's Theorem gives

$$p^a(t,y) = \frac{1}{\beta(t,T)\gamma\epsilon^2}\log\left(1 + \gamma\epsilon^2\mathbb{E}^0_{t,y}h(Y_T) + \frac{1}{2}\gamma^2\epsilon^4\mathbb{E}^0_{t,y}h^2(Y_T)\right.$$
$$\left. + \ \frac{1}{3!}\gamma^3\epsilon^6\mathbb{E}^0_{t,y}h^3(Y_T) + \frac{1}{4!}\gamma^4\epsilon^8\mathbb{E}^0_{t,y}h^4(Y_T) + O(\epsilon^{10})\right). \tag{65}$$

The power series expansion of $f(x) = \log(1+x)$ is valid for $-1 < x \leq x$. The terms inside the logarithm in (65) are non-negative, and when summed over all powers of ϵ^2 they give the exponential in (49). This implies that the logarithm in (65) can be expanded as a Taylor series provided $\mathbb{E}^0_{t,y}\exp(\gamma\epsilon^2 h(Y_T)) \leq 2$. This proves the last assertion in the theorem.

Expanding (65), initially keeping all terms up to order ϵ^{10}, then simplifying, gives

$$p^a(t,y) = \frac{1}{\beta(t,T)}\left[M_1 + \frac{1}{2}\gamma\epsilon^2\left(M_2 - M_1^2\right) + \frac{1}{3!}\gamma^2\epsilon^4\left(M_3 - 3M_1M_2 + 2M_1^3\right)\right.$$
$$\left. + \ \frac{1}{4!}\gamma^3\epsilon^6\left(M_4 - 3M_2^2 + 12M_1^2M_2 - 4M_1M_3 - 6M_1^4\right) + O(\epsilon^8)\right], \tag{66}$$

where, for brevity, we have introduced the notation

$$M_k := \mathbb{E}^0_{t,y} h^k(Y_T), \quad k \in \mathbb{N}. \tag{67}$$

Then, in view of the identities (62) and (63), the proof is complete.

■

6.1. Explicit Results for a Put Option

Suppose $h(y) = (K-y)^+$ for a positive constant K. Then it is a straightforward, though lengthy, process to establish explicit results for $p^a(t,y)$ and $p^a_y(t,y)$. We use the fact that under $\mathbb{P}^0 \in \mathcal{N}$, and conditional on $Y_t = y$, $\log Y_T$ is normally distributed with mean m and variance s^2, given by

$$m = \log y + \left(r - q - \sigma_0^2/2\right)(T-t), \tag{68}$$

$$s^2 = \sigma_0^2(T-t), \tag{69}$$

where we have defined the "dividend yield" q by

$$q = r - (\mu_0 - \sigma_0\rho\lambda). \tag{70}$$

We make extensive use of the (easily verifiable) integrals

$$\begin{aligned} \mathbb{E}^0_{t,y}\left[Y_T^k I_{Y_T \le K}\right] &= e^{k\left(m+ks^2/2\right)} N\left(-d_1 - (k-1)s\right) \\ &= y^k e^{k(r-q+(k-1)\sigma_0^2/2)(T-t)} N(-d_1 - (k-1)s), \\ &\quad (k \in \{0,1,2,3,4\}). \end{aligned} \tag{71}$$

In (71), I_A denotes the indicator function of event A, $N(.)$ denotes the standard cumulative normal distribution function and we have defined the variable d_1 by

$$d_1 = \frac{\log(y/K) + (r - q + \sigma_0^2/2)(T-t)}{\sigma_0\sqrt{T-t}}. \tag{72}$$

This is the familiar argument of $N(\cdot)$ which appears in the BS formula.

As an illustration, the zeroth order term in the expansion for $p^a(t,y)$ is $p^{a,0}(t,y)$ given by

$$p^{a,0}(t,y) = e^{-r(T-t)}\mathbb{E}^0_{t,y} h(Y_T) = e^{-r(T-t)}\mathbb{E}^0_{t,y}\left[(K - Y_T) I_{Y_T \le K}\right]. \tag{73}$$

Using (71) this becomes

$$\begin{aligned} p^{a,0}(t,y) &= Ke^{-r(T-t)} N(-d_1 + \sigma_0\sqrt{T-t}) - ye^{-q(T-t)} N(-d_1) \\ &= \mathrm{BS}^p(y, K, q, \sigma_0, T-t), \end{aligned} \tag{74}$$

where $\mathrm{BS}^p(y,K,q,\sigma_0,T-t)$ denotes the Black-Scholes put option formula with underlying asset price y, strike K, dividend yield q, volatility σ_0 and time to expiration $T-t$.

In a similar manner we establish all other necessary results. The essential formulae are summarized below.

$$\mathbb{E}^0_{t,y} h(Y_T) = M_1 = KN(-d_1+s) - ye^{(r-q)(T-t)}N(-d_1), \tag{75}$$

$$\begin{aligned}\mathbb{E}^0_{t,y} h^2(Y_T) = M_2 &= K^2N(-d_1+s) - 2Kye^{(r-q)(T-t)}N(-d_1) \\ &+ y^2e^{(2(r-q)+\sigma_0^2)(T-t)}N(-d_1-s),\end{aligned} \tag{76}$$

$$\begin{aligned}\mathbb{E}^0_{t,y} h^3(Y_T) = M_3 &= K^3N(-d_1+s) - 3K^2ye^{(r-q)(T-t)}N(-d_1) \\ &+ 3Ky^2e^{(2(r-q)+\sigma_0^2)(T-t)}N(-d_1-s) \\ &- y^3e^{3(r-q+\sigma_0^2)(T-t)}N(-d_1-2s),\end{aligned} \tag{77}$$

$$\begin{aligned}\mathbb{E}^0_{t,y} h^4(Y_T) = M_4 &= K^4N(-d_1+s) - 4K^3ye^{(r-q)(T-t)}N(-d_1) \\ &+ 6K^2y^2e^{(2(r-q)+\sigma_0^2)(T-t)}N(-d_1-s) \\ &- 4Ky^3e^{3(r-q+\sigma_0^2)(T-t)}N(-d_1-2s) \\ &+ y^4e^{2(2(r-q)+3\sigma_0^2)(T-t)}N(-d_1-3s).\end{aligned} \tag{78}$$

These results can then be substituted into (64) or (66) for numerical computation of the asking price.

6.1.1. Put Option Delta Differentiating (64) with respect to y gives the following expansion for $p^a_y(t,y)$:

Corollary 1 *The derivative of the asking price $p^a(t,y)$ with respect to y has the perturbative expansion*

$$\begin{aligned}\frac{\partial p^a}{\partial y}(t,y) = \frac{1}{\beta(t,T)}\Big[&\partial M_1 + \frac{1}{2}\gamma\epsilon^2\left(\partial M_2 - 2M_1\partial M_1\right) \\ &+ \frac{1}{3!}\gamma^2\epsilon^4\left(\partial M_3 - 3M_2\partial M_1 - 3M_1\partial M_2 + 6M_1^2\partial M_1\right) \\ &+ \frac{1}{4!}\gamma^3\epsilon^6\Big(\partial M_4 - 6M_2\partial M_2 + 12M_1^2\partial M_2 + 24M_1M_2\partial M_1 \\ &- 4M_1\partial M_3 - 4M_3\partial M_1 - 24M_1^3\partial M_1\Big) + O(\epsilon^8)\Big],\end{aligned} \tag{79}$$

where we have used the notation

$$\partial M_k \equiv \frac{\partial M_k}{\partial y} = \frac{\partial \mathbb{E}^0_{t,y} h^k(Y_T)}{\partial y}. \tag{80}$$

The partial derivatives needed to apply the above corollary are obtained by differentiating (75)–(78). This yields the following formulae:

$$\partial M_1 = -e^{(r-q)(T-t)}N(-d_1), \tag{81}$$

$$\partial M_2 = -2e^{(r-q)(T-t)}\left[KN(-d_1) - ye^{(r-q+\sigma_0^2)(T-t)}N(-d_1-s)\right], \tag{82}$$

$$\begin{aligned}\partial M_3 = &-3e^{(r-q)(T-t)}\Big[K^2N(-d_1) - 2Kye^{(r-q+\sigma_0^2)(T-t)}N(-d_1-s) \\ &+ y^2e^{(2(r-q)+3\sigma_0^2)(T-t)}N(-d_1-2s)\Big]\end{aligned} \tag{83}$$

$$\begin{aligned}\partial M_4 = &-4e^{(r-q)(T-t)}\Big[K^3N(-d_1) - 3K^2ye^{(r-q+\sigma_0^2)(T-t)}N(-d_1-s) \\ &+ 3Ky^2e^{(2(r-q)+3\sigma_0^2)(T-t)}N(-d_1-2s) \\ &- y^3e^{3(r-q+2\sigma_0^2)(T-t)}N(-d_1-3s)\Big].\end{aligned} \tag{84}$$

The above recipe is sufficient to give fast computation of the asking price of the put option on the non-traded asset and the associated hedging strategy.

6.2. Numerical Results

Using the expectation representation (49) it is a simple matter to produce numerical values for the ask price of the claim, and for its derivative with respect to y, by simulation. This was done for 2 million samples, and the numerical values compared with those from the perturbation expansions in the last section. The goal is to establish the accuracy (or otherwise) of the expansions across a range of values of the correlation ρ. The simulations were also used to check that the model parameters we used did indeed satisfy the restrictions of Theorem 2, needed for the perturbation expansions to be valid. All results reported below were for valid model parameters. It was found that risk aversion values γ below about 0.05 guaranteed validity, regardless of other parameter choices. Typical risk aversion parameters for market participants are around 10^{-6} [13], so this is a very mild restriction.

A detailed account of the accuracy of the perturbation expansions is given in [23]. We limit ourselves here to the results shown in Table 1 for $p^a(t,y)$ and $p^a_y(t,y)$ at time zero, for $\gamma = 0.001$ and various values of ρ. The results produced by the perturbation expansion at order ϵ^2 and beyond are remarkably in line with those from simulation. Accurate results are obtained across all values of correlation when the risk aversion parameter is below about 0.05, with the accuracy increasing with increasing $|\rho|$ and decreasing γ [23].

The significance of these results is that we now have a very fast route to computing option prices and hedging strategies. This allows for practical implementation, and for an efficient testing program of the hedging performance of optimal strategies versus the "naive" strategies which simply use the traded asset as a proxy for the non-traded one. Such a testing procedure is carried out below.

7. Hedging Performance of Optimal Strategies

To analyze hedging performance, we suppose that a put option on asset Y is sold at time zero for price $p^a(0,Y_0)$, defining the initial endowment in our hedging portfolio, and hedged using strategy $(\Delta^a_t)_{0\leq t\leq T}$ given in Theorem 2. Denote the wealth in the hedging portfolio by $(X^a_t)_{0\leq t\leq T}$, given by (5) with $\pi_t = \Delta^a_t S_t$. The evolution of this wealth in discrete time will be used in the numerical simulations below.

We simulate a path for both asset prices $(S,Y) := (S_t,Y_t)_{0\leq t\leq T}$ with given correlation ρ, and choose a number of times that the hedge is rebalanced in the option lifetime. The formulae established in the previous section are used to compute the hedge portfolio "delta" at each rehedging time. Then for each asset price path simulated we compute the terminal tracking error

$$\mathcal{E}_T := X^a_T - (K - Y_T)^+. \tag{85}$$

Table 1. Put ask prices $p^a(0, Y_0)$ and "deltas" $p^a_y(0, Y_0)$ from the perturbative expansion and from simulation. The parameters are those in Table 2. The exception to this is the case $\rho = 1$, in which case no-arbitrage considerations fix $\mu_0 = \mu - \sigma_0 \lambda = 0.11$, and the option value is the BS value with volatility σ_0 and dividend yield 0. Figures in parentheses are standard deviations of the observations that were averaged for the simulation results.

PUT OPTION ASKING PRICES, $\gamma = 0.001$, 2×10^6 simulations

ρ	$o(\epsilon^0)$	$o(\epsilon^2)$	$o(\epsilon^4)$	$o(\epsilon^6)$	Simulation
-0.95	5.3914	5.4016	5.4016	5.4016	5.4001 (0.0111)
-0.75	5.6320	5.6566	5.6567	5.6567	5.6564 (0.0023)
-0.50	6.0493	6.0944	6.0946	6.0946	6.0970 (0.0246)
-0.25	6.4870	6.5471	6.5474	6.5474	6.5465 (0.0131)
0	6.9451	7.0133	7.0138	7.0138	7.0113 (0.0034)
0.25	7.4238	7.4917	7.4922	7.4922	7.4913 (0.0020)
0.50	7.9231	7.9806	7.9809	7.9809	7.9791 (0.0128)
0.75	8.4428	8.4783	8.4784	8.4784	8.4806 (0.0241)
0.95	8.8733	8.8815	8.8815	8.8815	8.8790 (0.0136)
1	9.3542	9.3542	9.3542	9.3542	9.3514 (0.0180)

PUT OPTION DELTAS

ρ	$o(\epsilon^0)$	$o(\epsilon^2)$	$o(\epsilon^4)$	$o(\epsilon^6)$	Simulation
-0.95	-0.2634	-0.2639	-0.2639	-0.2639	-0.2632
-0.75	-0.2715	-0.2726	-0.2726	-0.2726	-0.2723
-0.50	-0.2850	-0.2870	-0.2870	-0.2870	-0.2866
-0.25	-0.2986	-0.3011	-0.3012	-0.3012	-0.3006
0	-0.3123	-0.3151	-0.3151	-0.3151	-0.3145
0.25	-0.3260	-0.3287	-0.3287	-0.3287	-0.3280
0.50	-0.3397	-0.3418	-0.3419	-0.3419	-0.3411
0.75	-0.3533	-0.3546	-0.3546	-0.3546	-0.3540
0.95	-0.3641	-0.3644	-0.3644	-0.3644	-0.3644
1	-0.3757	-0.3757	-0.3757	-0.3757	-0.3752

The above calculation is repeated over a large number M (say, 10,000) of asset price paths.

Finally, we repeat the entire calculation over the same simulated paths, but use a "naive" approach which assumes we sell the option for $\mathrm{BS}^p(Y_0, K, 0, \sigma_0, T)$ and hedge using the strategy given in (9).

7.1. Results

The results reported below used the parameters shown in Table 2 as a base case, and the options were re-hedged 200 times during their life.

Figures 1 and 2 illustrate the nature of the simulations. The upper graphs show the traded (solid line) and non-traded (broken line) asset prices along a path, while

Table 2. Model parameters.

S_0	Y_0	K	r	μ	σ	μ_0	σ_0	T
100	100	100	5%	10%	25%	12%	30%	1year

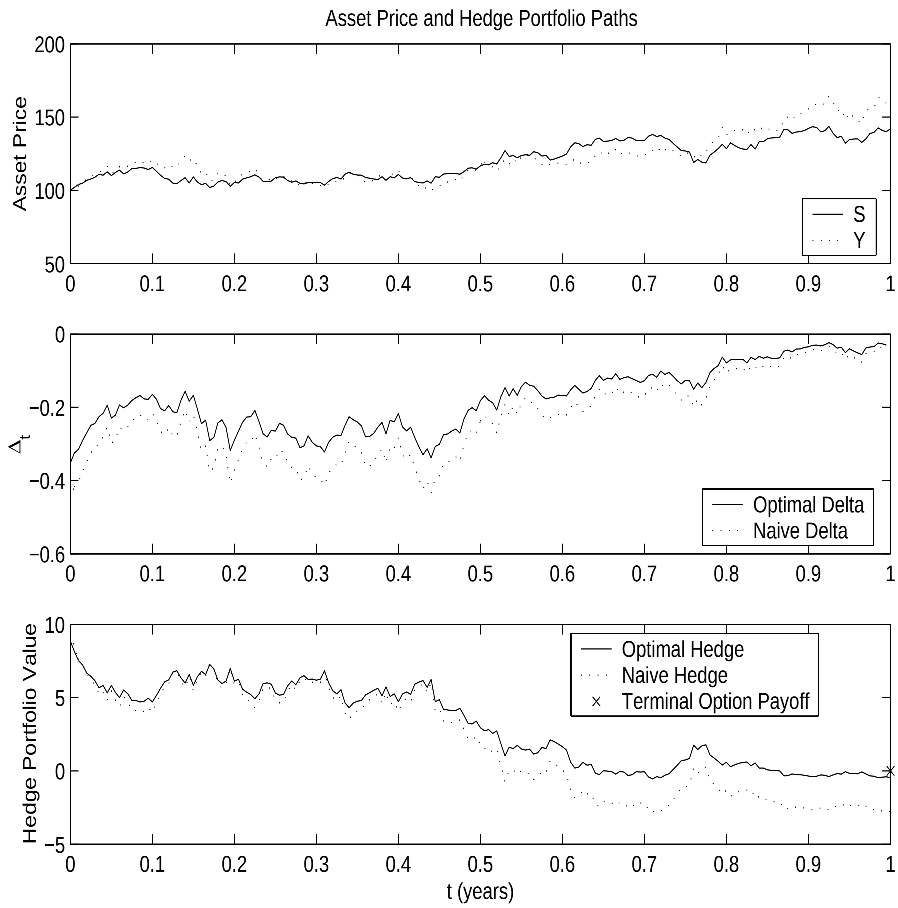

Figure 1. Asset prices (upper graph), hedge ratios (middle graph) and hedge portfolio wealths (lower graph) along a simulated path. The solid line in the lower two graphs corresponds to the optimal hedge, while the broken line corresponds to the naive hedge. The parameters are as in Table 2, and $\rho = 0.8$, $\gamma = 0.01$.

the middle and lower graphs show the hedge ratios and hedge portfolio values along the paths for the optimal (solid line) and naive (broken line) strategies. The terminal option payoff is also marked with a cross ($\times$).

Figure 3 shows histograms illustrating the distribution of the terminal hedging error produced by the optimal (upper graph) and naive (lower graph) hedging strategies. The results, over 10,000 simulations, are for $\rho = 0.65$ and $\gamma = 0.001$. Both graphs are plotted on the same scales for ease of comparison. It is immediately apparent that the optimal hedging procedure produces a more sharply peaked distribution, with a higher proportion of errors around and just above zero, compared with the naive hedging

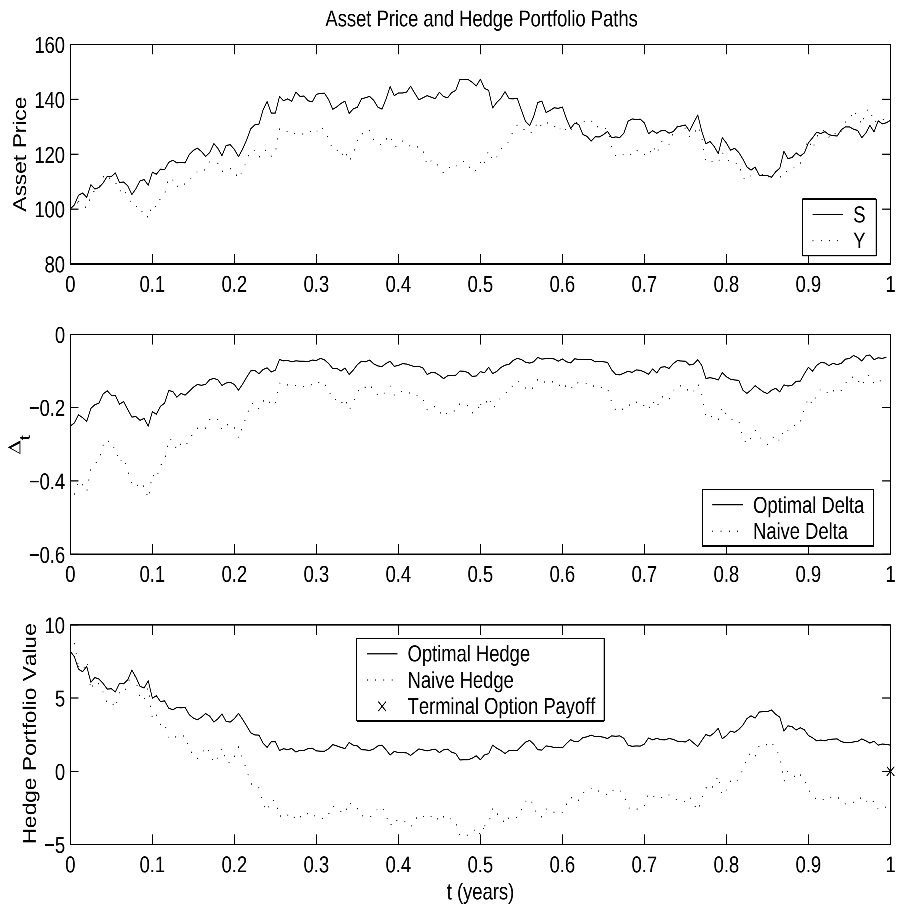

Figure 2. Asset prices (upper graph), hedge ratios (middle graph) and hedge portfolio wealths (lower graph) along a simulated path. The solid line in the lower two graphs corresponds to the optimal hedge, while the broken line corresponds to the naive hedge. The parameters are as in Table 2, and $\rho = 0.6$, $\gamma = 0.001$.

strategy. The shapes of the histograms show how the optimal method will tolerate small negative errors, but not large losses.

To put some concrete numbers on these visual observations, we give summary statistics for the distributions in Table 3. The standard deviation of the naive hedging error distribution is about 7% higher than that of the optimal hedging policy. The really significant statistic, however, is the *median* of the distributions. The median hedging error from the optimal policy is 78% higher than that from the naive hedging policy. In other words, the optimal policy results in positive hedging errors far more frequently than the naive policy. This is precisely what one would require of a good hedging policy. The mean of the distribution is fairly meaningless in this context, as the figures in the Table show. Note also how the range of the hedging error is larger with the naive hedging policy. In other words, sometimes one will be lucky and make a large profit, while at other times one will incur a large loss. Systematic improvements are therefore made by the optimal procedure.

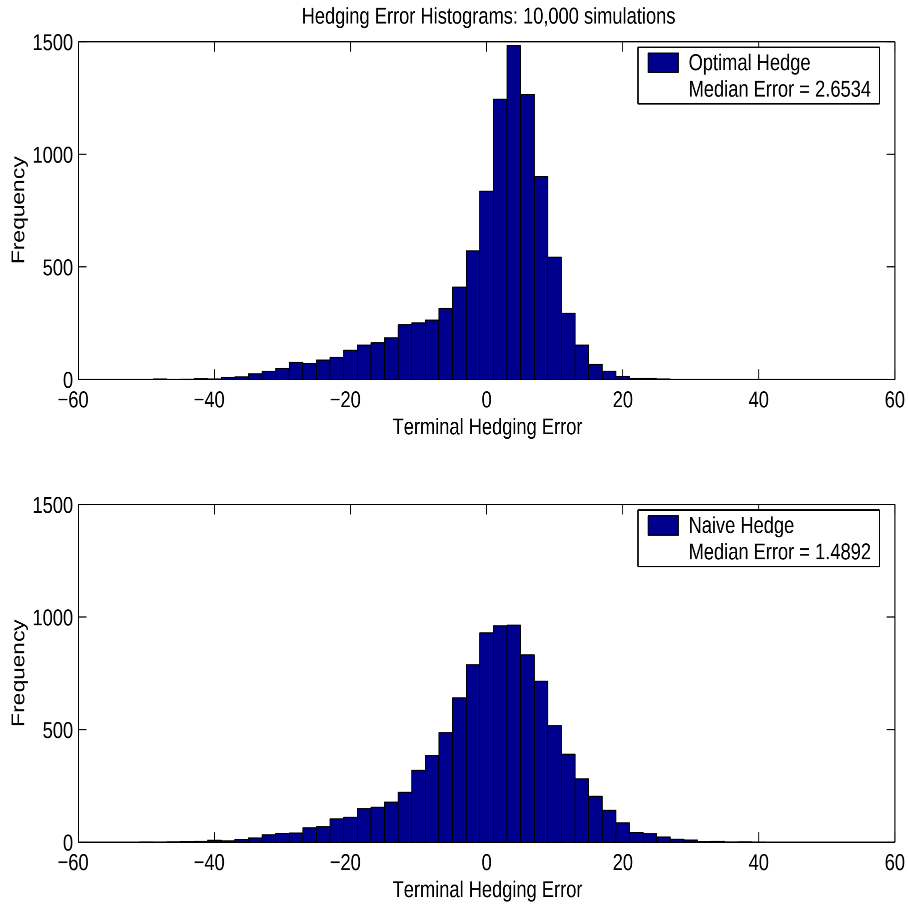

Figure 3. Histograms of terminal hedging error over 10,000 sample paths for the optimal hedging strategy (upper graph) and the naive strategy (lower graph). The parameters are as in Table 2, and $\rho = 0.65$, $\gamma = 0.001$.

Table 3. Hedging error statistics for the histograms in Figure 3.

	Max	Min	Mean	SD	Median
Optimal Hedge	25.65	-48.09	0.1145	9.6342	2.6534
Naive Hedge	37.22	049.68	0.4303	10.3618	1.4892

Figure 4 shows similar histograms for a higher value of the correlation, namely $\rho = 0.85$. The pattern is similar, as the summary statistics in Table 4 show. This time, the median hedging error for the optimal strategy is about 45% higher than that for the naive strategy, and the standard deviation is about 1% higher for the naive strategy. In other words, the optimal strategy is still an improvement over the naive policy, even for a higher correlation.

Figures 5 and 6 show hedging error distributions for $\rho = 0.65$ and $\rho = 0.85$, but now with a larger risk aversion parameter, $\gamma = 0.01$. Summary statistics for these distributions are given in Tables 5 and 6 respectively. The results are similar to those

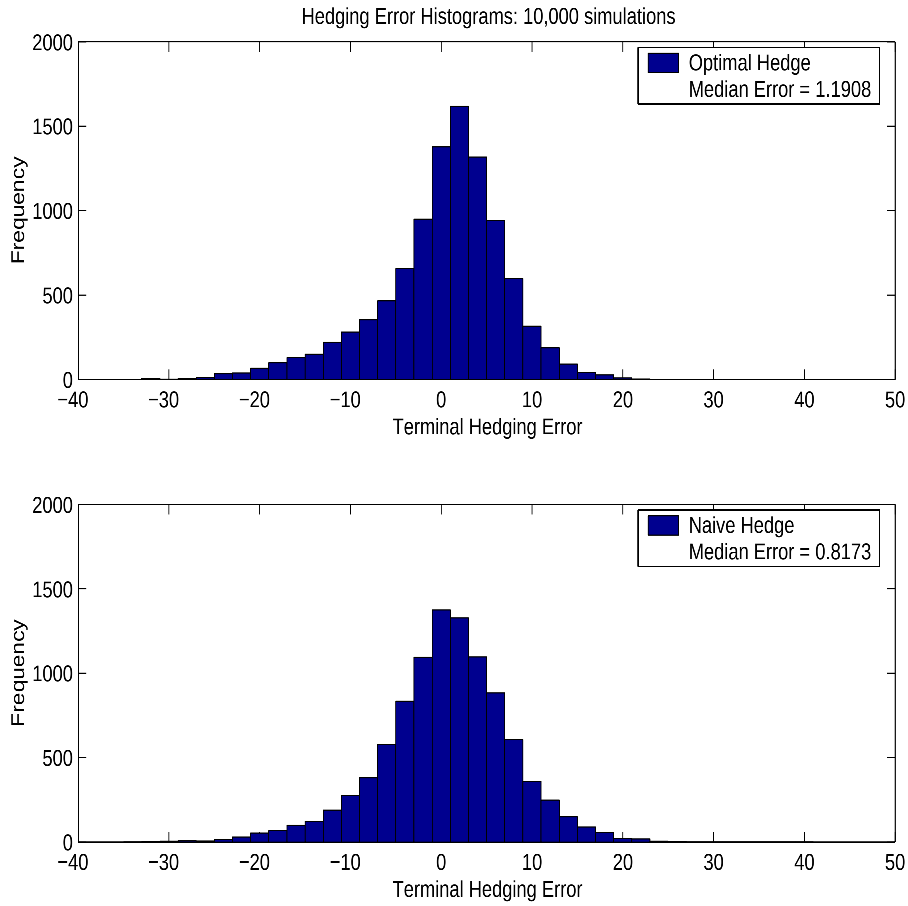

Figure 4. Histograms of terminal hedging error over 10,000 sample paths for the optimal hedging strategy (upper graph) and the naive strategy (lower graph). The parameters are as in Table 2, and $\rho = 0.85$, $\gamma = 0.001$.

Table 4. Hedging error statistics for the histograms in Figure 4.

	Max	Min	Mean	SD	Median
Optimal Hedge	22.24	-32,78	0.1816	6.9951	1.1908
Naive Hedge	26.49	-32.27	0.5098	7.0880	0.8173

reported earlier. For $\rho = 0.65$, the median hedging error for the optimal strategy is about twice (100% higher) that for the naive strategy, and the standard deviation is about 7% higher for the naive strategy. For $\rho = 0.85$, the median hedging error for the optimal strategy is about 75% higher that for the naive strategy, and the standard deviation is about 1% higher for the naive strategy. In other words, the improvements are similar, and in terms of the median, perhaps even greater for the case of a higher risk aversion. This is intuitively correct, of course, as “optimality” should be of greater benefit when one is more sensitive to risk. Similar results, not reported here, hold for other model parameters.

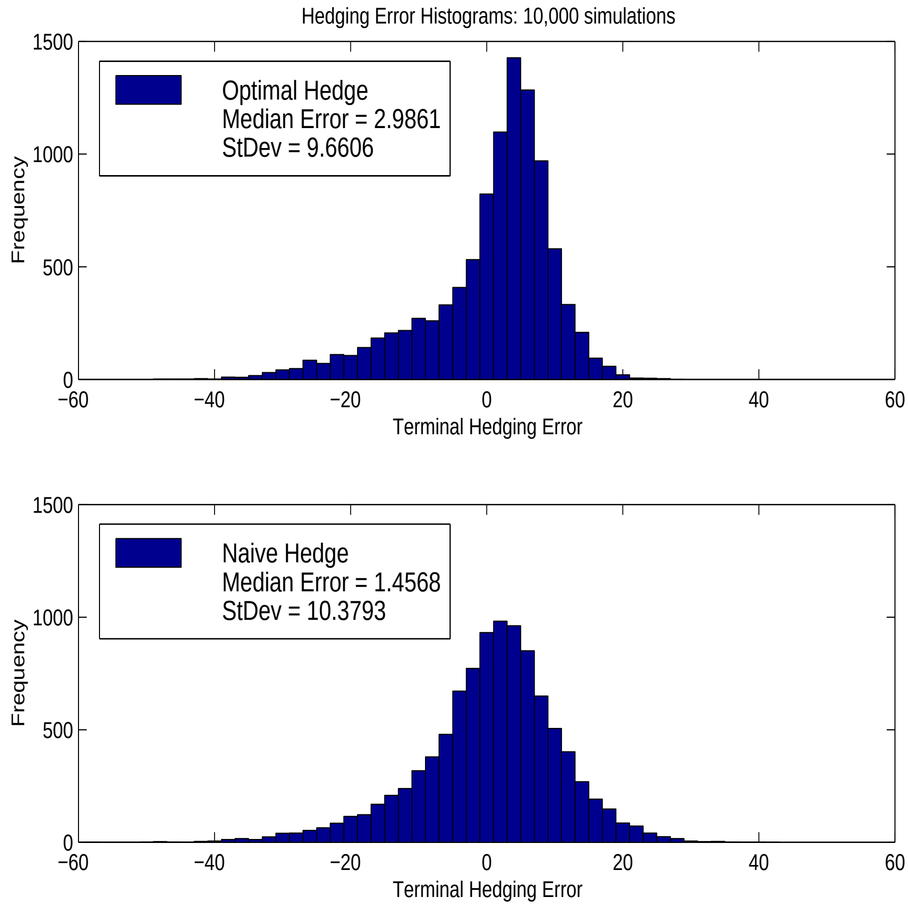

Figure 5. Histograms of terminal hedging error over 10,000 sample paths for the optimal hedging strategy (upper graph) and the naive strategy (lower graph). The parameters are as in Table 2, and $\rho = 0.65$, $\gamma = 0.01$.

Table 5. Hedging error statistics for the histograms in Figure 5.

	Max	Min	Mean	SD	Median
Optimal Hedge	28.28	-47.46	0.5155	9.6606	2.9861
Naive Hedge	40.13	-57.04	0.4808	10.3793	1.4568

Table 6. Hedging error statistics for the histograms in Figure 6.

	Max	Min	Mean	SD	Median
Optimal Hedge	24.70	-34.17	0.3879	6.9340	1.2318
Naive Hedge	28.53	-35.94	0.5183	7.0033	0.7019

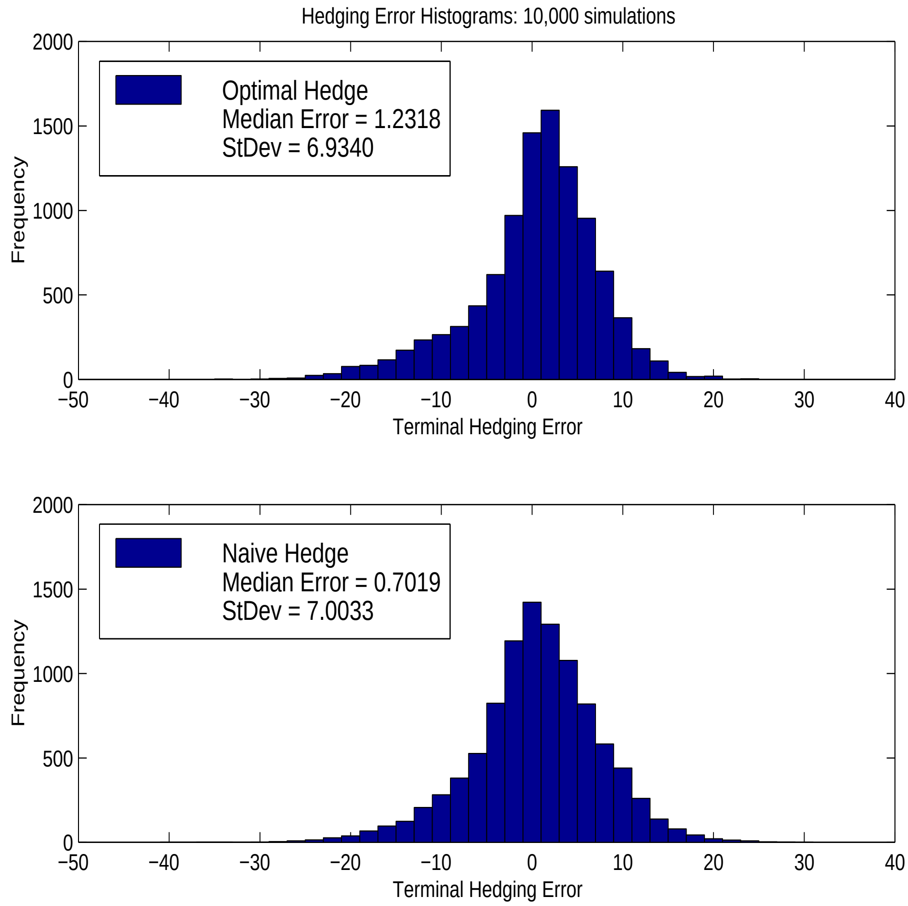

Figure 6. Histograms of terminal hedging error over 10,000 sample paths for the optimal hedging strategy (upper graph) and the naive strategy (lower graph). The parameters are as in Table 2, and $\rho = 0.85$, $\gamma = 0.01$.

8. Conclusions

Using a non-linear expectation representation for the asking price of a claim on a non-traded asset we have derived analytic perturbation expansions for the price and hedging strategy of the claim. These formulae were used to show how optimal risk management, arising from the embedding of the pricing problem in a utility maximization framework, gives marked improvement in hedging performance over naive policies which use a traded asset as a proxy for the non-traded one. This improvement was measured by computing the distribution of terminal hedging error, and noting the increased frequency of profits over losses, as measured by the median hedging error.

The tests initiated here could be carried out using different risk measures and utility functions, as it would be interesting to see what sort of hedging strategies offer the greatest improvement. The issue of formalizing appropriate metrics to measure risk management performance enters the fray here, and there are presumably links with the coherent measures of risk in [1].

In general, the computation of hedging error distributions is a task that has not received much attention, despite being a natural way to assess the merits of a risk management program. Most studies have simply taken a "snapshot" of the hedging error over a limited number of scenarios [17]. The application of the methods advocated here to other incomplete markets scenarios, such as stochastic volatility models, is certainly feasible and desirable.

It would also be interesting to add features such as transaction costs to the model analyzed in this paper. If one could develop suitable analytic formulae for prices and hedging strategies, along the lines of [29], then it becomes feasible to determine which market imperfection, (basis risk or transaction costs) is the most severe, in terms of the hedging errors that must be tolerated.

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