Utility-Based Hedging of Stochastic Income

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Abstract

In this dissertation, we study and examine utility-based hedging of the optimal portfolio choice problem in stochastic income. By assuming that the investor has a preference governed by negative exponential utility, we a derive a closed-form solution for the indifference price through the pricing methodology based on utility maximization criteria. We perform asymptotic analysis on this closed-form solution to develop the analytic approximation for the indifference price and the optimal hedging strategy as a power series expansion involving the risk aversion and the correlation between the income and a traded asset. This gives a fast computation route to assess these quantities and perform our analysis. We implemented the model to perform simulations for the optimal hedging policy and produce the distributions of the hedging error at terminal time over many sample paths histories. In turn, we analyze the performance of the utility-based hedging strategy together with the strategy which arises from employing the traded asset as a substitute for the stochastic income.

ACKNOWLEDGEMENTS

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INTRODUCTION

In 1973, Black and Scholes [2] and Merton [29] developed methods for pricing options. They demonstrated that it was possible to replicate an option by trading in the riskless and underlying asset. Their work however, is crucially based on the assumption that the market is *complete*. In other words, the assets underlying the derivatives to be hedged are traded in the market and so every claim written on any underlying asset can be can be perfectly replicated by constructing a hedging portfolio through dynamic trading in the stock, thus removing all risk and uncertainty. Therefore, assuming no arbitrage in the market, the price of the claim is equivalent to the cost associated with constructing the corresponding hedging portfolio[1]. The existing financial asset markets are often *incomplete*. Market frictions (such as transactions costs, nontraded assets, and portfolio constraints) prevalent in the economy drive the incompleteness of the market. This implies that there is a range of prices that are consistent with no-arbitrage, with each corresponding to a different martingale measure leaving no longer a unique price. Thus, it is not possible to remove all the risk inherent in the claim solely through replication.

However, the situation is not as bleak as it seems; even when the investor is placed in an incomplete market, the investor is allowed to maximize the expected utility of wealth and may be able to reduce the risk due to the uncertain payoff through dynamic trading. One approach to the pricing problem from the utility maximization framework is to provide a compensation to the writer/receiver of the claim so that he is indifferent in terms of expected utility. This corresponds precisely to the notion of *utility indifference pricing* (also known as *certainty equivalence*) due to Hodges and Neuberger [22]. Utility-indifference pricing provides the machinery for deriving the price of a contingent claim in an incomplete market. It is an extension of the static certainty equivalence conception from economics to the dynamic setting. The appealing feature of utility-indifference pricing method is that it provides economic justification and encapsulates the risk preferences of investors into the pricing mechanism through a (concave) utility function. It assesses the optimal behavior of the investor where she faces the dilemma of buying/selling the claim at the current instant and receiving the payoff later, and not buying the claim at that instant.

Henderson [17] studied the problem of pricing and hedging claims on nontraded assets whereby the investor's preference can be expressed through either the power or exponential utility. Musiela and Zariphopoulou [35] also investigated a similar problem by solely considering exponential utility and performed a similar analysis to Henderson [17], but in a slightly more general setting where the drift and volatility parameters of the nontraded asset are functions of the nontraded asset. By appealing to the dynamic programming principle and solving the associated Hamilton-Jacobi-

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[1]Note, that the 2nd Fundamental Theorem of Asset Pricing [9] shows that the pricing measure in a complete market is unique and hence the price of any claim is unique regardless of the investor's risk preferences.

Bellman (HJB) equation, they were able to derive a nonlinear partial differential equation (PDE) describing the value function for both utility functions. For the case under exponential utility, they [17] employed the Cole-Hopf (also referred to as *distortion power*) transformation, introduced by Zariphopoulou [42], to reduce the nonlinear PDE to a linear one. From this, they obtained an explicit solution for the value function which is represented as a nonlinear expectation under the minimal martingale measure of Föllmer and Schweizer [13], and consequently, the indifference price and optimal hedging strategy. A survey of indifference pricing and its characterization can be found in Henderson and Hobson [20].

The performance strategy of utility-based heding methods was analyzed by Monoyios [30] where he conducted numerical computations to assess its effectiveness by making comparison with that of Black-Scholes. He studied the problem concerning pricing and hedging of European put option on a nontraded asset. By considering the exponential utility function, a nonlinear expectation representation for the indifference price was acquired and based on this expression, he was able to derive analytic approximations for the indifference price in the form of a power series expansion and also for the optimal hedging strategy. These approximations enable a fast and efficient testing program to perform the analysis for the hedge of a put option on simulated paths and a basket of nine stocks based on real data of the FTSE100 futures dated from 1990 to 2003. The results obtained indicated that utility-based hedging outperforms the Black-Scholes hedging.

An alternative application of utility indifference pricing from valuing claims on nontraded assets, can be used to value *labor income*[2] which are streams of income payments received by the investor over time. From an economics perspective, an individual's labor income can be interpreted as a dividend/yield on the individual's implicit holding of human capital. Human capital[3] is a nontradable asset, since individuals cannot sell claims against future labor income[4]. Thus, incompleteness of the market is generated by labor income. Moreover, the future labor earnings are typically uncertain for most individuals possibly due to the nature of the profession or the prevailing economic conditions. Thus, human capital becomes a risky rather than a safe nontradable asset. These risk factors affect decision problems of allocation of financial assets and the amount required to be saved. There is a large body of literature concerned with the role of labor income risk on asset allocation decisions (see [4]).

Consider the the scenario where an investor seeks to receive a loan from a bank. It is crucial banks that they evaluate the investor's background and credit history. From a risk management point of view, their modeling should, *ceteris paribus*, take into account the exposure of the investor's to income risk (for instance, individuals with sound jobs are less sensitive to the general economic conditions, while individuals with uncertain income are more sensitive to them). Thus, banks issue loans to individuals on the basis that they have assurance that those individuals maintain a safe investments so that they do not withstand the possibility of incurring losses. We ask how should an individual optimally invest (or consume) so as to remove the uncertainty from their wealth?

The problem involving stochastic income has been studied extensively in literature. The seminal work of Merton [27] studies the effect optimal investment decisions under subject to uncertain labor

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[2]Under our context, we refer to labor income as stochastic income without introducing confusion to the reader.

[3]Human capital is the net present value of an individual's future labour income.

[4]The contract committing an individual to work constitutes to a form of slavery which is illegal in most countries. An individual is entitled with the legal right to stop working. For this reason, the claim on an individual's income is bound to be of trivial value from the buyer of a claim perspective.

income which follows a stochastic process that is perfectly correlated to the traded asset. It turns out that when the income can be reproduced by trading (so the market is complete), the problem with the initial wealth adjusted by the risk-adjusted present value[5] of the cumulative income. This result does not generally hold in an incomplete market, which is the motivation for this study. Bodie et al. [**3**], extended the model by incorporating labor flexibility that allows the investor to select allocation for her work in each period. El Karoui and Jeanblanc-Piqué [**12**] derived properties of the optimal asset allocation assuming that the income is spanned by the market assets but imposed by liquidity constraints in that individuals are not allowed to borrow against future labor income. Duffie et al. [**11**] and Munk [**33**] both studied the optimal consumption/investment problem over the infinite time horizon with stochastic income that is imperfectly correlated with the risky asset. Munk [**33**] performed numerical computations to obtain the value of the nontraded income and optimal portfolio by using a Markov chain approximation. Henderson [**18**] found explicit solutions under exponential utility and studies the effect of the stochastic income on optimal investment decision.

In this dissertation, we further examine the hedging of a stochastic income stream in an incomplete market using the model of Henderson [**18**] as a motivation for our study. We establish and provide a framework which will enable us to derive representations for the value function and indifference price in terms of the minimal martingale measure through the use of techniques from dynamic programming. Our contribution is to derive explicit asymptotic formulae for the indifference price and hedging strategy in the manner similar to Monoyios [**30**]. The problem that we face however, is more involved as we are dealing with a claim that pays out continuously rather than one which pays at a single fixed date. We also compute these quantities via Monte-Carlo simulations to assess the accuracy of the asymptotic expansion. Finally, we adopt the approach of Monoyios [**30**] to develop an approximation or numerical solution for the hedge, and compute/simulate the residual risk. For a set of simulations, we determine the distribution of the residual risk at terminal time. This would allow us to make inferences about the quality of the optimal hedging policy by using the Black-Scholes hedging strategy as a proxy for our analysis.

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[5]More precisely, the present value is equivalent to the expectation under the unique equivalent martingale measure.

CHAPTER 1

INVESTOR'S PORTFOLIO CHOICE PROBLEM

1.1 MARKET MODEL

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space that supports correlated standard Brownian motions $W = (W_t)_{t \geq 0}$ and $W^Y = (W_t^Y)_{t \geq 0}$ whose correlation ρ between them satisfies $|\rho| \leq 1$. The filtrations $(\mathscr{F}_t)_{t \geq 0}$ and $(\mathscr{F}_t^Y)_{t \geq 0}$ both satisfy the usual conditions of completeness and right-continuity and $\mathscr{F}$ is the σ-algebra generated by the pair (W, W^Y) and $\mathscr{F}^Y$ be the σ-algebra generated by W^Y.

Consider a financial market consisting of a stochastic process S representing the asset (stock) price, and a bond $B_t = B_0 e^{rt}$ at time t with constant interest rate r. The price dynamics are assumed to evolve according to

$$dB_t = rB_t dt, \quad B_0 = 1,$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad S_0 = s > 0,$$

where μ and $\sigma > 0$ are constants and W_t is a standard Brownian motion.

We further consider that the investor is endowed with a continuous stochastic income over time which is modeled as a correlated diffusion.

Assumption 1. The income rate at time t is $I(t, Y_t)$ where $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ is nonnegative and continuous, and Y_t that follows a lognormal distribution

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_t^Y, \qquad Y_0 = y > 0.$$

where μ_Y and σ_Y are constants and W_t^Y is a standard Brownian motion.

Since W^Y is correlated with W, we can write $W^Y := \rho W + \sqrt{1 - \rho^2} W^\perp$ where $W^\perp$ is a standard Brownian motion independent of W with $|\rho| \leq 1$. It is important to note that for $|\rho| < 1$, the filtration $\mathscr{F}$ is not generated by a single Brownian motion W which means the market is *incomplete* because the nontraded income cannot be perfectly hedged through the stock S.

Remark 1.1. The dynamics of the income is correlated with that of the stock encapsulates that the financial market has an influence on the labor income of an individual. Note that the dynamics of the stochastic income and the definition of W^Y capture the labor income risk profiles. Further, if $\sigma_Y = 0$, the labor income of an individual behaves like a riskless asset; if $\rho = 1$, the labor income behaves as a stock.

Assumption 2. Assume that the income rate is $nI(t, y) \geq 0$ (or bounded from below) where $n \in \{-1, 1\}$ and is continuous. For $n = 1$, the investor is said to be receiving the income stream and otherwise for $n = -1$.

Note that it is reasonable to suppose that one is 'selling' the income. For example, a pension fund pays out (possibly not stochastic) income–as it does for certain insurance products. However, one should be aware that there is a technical issue involved as one may not be able to price an outflow of an income stream due to the potential unboundedness of the payoff received at final time T (see Remark 2.2 below for amplification of the fact the income stream in our model can be thought as a claim which pays at terminal time T).

Remark 1.2. We could have very well assumed that the drift and volatility coefficients of the process Y have a general Markovian structure $\mu_Y(t, Y_t)$ and $\sigma_Y(t, Y_t)$ respectively. It should be noted that in [18], and [41], several other forms such as the normally distributed model and the lognormal model with mean-reversion were studied. However, we should emphasize that the lognormality model of the income enables us to derive expressions, which will follow below, that could not have been done otherwise.

If the number of traded assets spanning financial market model is equal to the number of Brownian motions driving the model, the market will be complete. Otherwise, the market will be incomplete if there are more Brownian motions than traded assets (and consequently there are some risks that cannot be hedged). Our model is an example where we have one traded asset and two Brownian motions if $|\rho| \neq 1$. Hence, in this case, the market is incomplete. It can be shown (see for example [7]) that if $\rho = 1$, then $W = W^Y$ and the no arbitrage condition require that that the market price of risk (Sharpe ratio) of both the traded and nontraded income, defined by λ and λ_Y respectively, are related by

$$\lambda := \frac{\mu - r}{\sigma} = \frac{\mu_Y - r}{\sigma_Y} =: \lambda_Y.$$

Thus, when the correlation between the traded and nontraded income becomes perfect, the market becomes complete and we can perfectly replicate the claim on a nontraded income by using the Black-Scholes type hedging strategy.

Consider the class $\mathscr{M}$ of equivalent local martingale measures $\mathbb{Q} \sim \mathbb{P}$ defined on $\mathscr{F}$ whose density process is given by

$$Z_t^{\mathbb{Q}} \equiv \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \mathscr{E}(-\lambda \cdot W - \psi \cdot W^\perp)_t,$$

where the ψ is an $\mathscr{F}_t$-adapted process satisfying $\int_0^T \psi_u du < \infty$. Further, we insist that the process ψ satisfies the Novikov condition. This ensures that $Z^{\mathbb{Q}}$ is a martingale and consequently, the measures $\mathbb{Q} \in \mathscr{M}$ are equivalent to $\mathbb{P}$. Further, the class $\mathscr{M}$ has a one-to one correspondence with the set of integrands ψ.

By the Cameron-Martin-Girsanov theorem, we see that S and Y has the following dynamics under $\mathbb{Q} \in \mathscr{M}$

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}},$$
$$dY_t = [\mu_Y - \sigma_Y(\rho\lambda + \sqrt{1-\rho^2}\psi_t)]Y_t dt + \sigma_Y Y_t dW_t^{Y,\mathbb{Q}},$$

where $W^{\mathbb{Q}} := W + \lambda t$ and, $W^{\perp,\mathbb{Q}} + \int_0^{\cdot} \psi_u du$ are independent Brownian motions under $\mathbb{Q}$ with $W^{Y,\mathbb{Q}} = \rho W^{\mathbb{Q}} + \sqrt{1-\rho^2}W^{\perp,\mathbb{Q}}$. Consequently, $W^{Y,\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$ satisfying

$dW_t^{\mathbb{Q}} dW_t^{Y,\mathbb{Q}} = \rho dt$. Note that for $|\rho| \neq 1$, each choice of ψ gives rise to different measures $\mathbb{Q} \in \mathcal{M}$, which implies that the market is incomplete. If we let $\psi = 0$, we recover the *minimal martingale measure* $\overline{\mathbb{Q}}$ of Föllmer and Schweizer [13], that is, the density process with respect to $\mathbb{P}$ given by[1]

$$Z_t^{\overline{\mathbb{Q}}} \equiv \left.\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \mathscr{E}(-\lambda \cdot W)_t.$$

Under $\overline{\mathbb{Q}}$ the traded asset price becomes a local martingale, but the drift of any Brownian motion orthogonal to W is left unchanged. This corresponds to leaving unhedgeable risk unpriced; we simply ignore the risk of the nontraded income. When $\psi = 0$, we see that the traded asset and nontraded income dynamics under $\overline{\mathbb{Q}}$ evolve according to

$$dS_t = rS_t dt + \sigma S_t dW_t^{\overline{\mathbb{Q}}},$$

$$dY_t = (\mu_Y - \sigma_Y \rho \lambda) Y_t dt + \sigma_Y Y_t dW_t^{Y,\overline{\mathbb{Q}}},$$

where $W^{\overline{\mathbb{Q}}}$ and $W^{Y,\overline{\mathbb{Q}}}$ are Brownian motions under $\overline{\mathbb{Q}}$ satisfying $dW_t^{\overline{\mathbb{Q}}} dW_t^{Y,\overline{\mathbb{Q}}} = \rho dt$.

1.2   The Utility Maximization Problem

Let $X_t = \pi_t + \pi_t^0$ be the investor's wealth (which also represents the investor's budget constraint) at time $t \geq 0$, where π_t denotes the wealth invested in the stock and π_t^0 denotes wealth from an inflow of stochastic income along with investment in the riskless asset. The processes π and π^0 are both $\mathscr{F}_t$-measurable. Since π_t is the amount of wealth in the stock at time t, we write $\pi_t := H_t S_t$ where $H \equiv (H_t)_{t \geq 0}$ denotes the holdings of the stock S. Let $x > 0$ be the *initial wealth endowment*. If the portfolio containing S is self-financing, then the dynamics of the wealth X can be expressed as

$$dX_t = rX_t dt + \pi_t \sigma (\lambda dt + dW_t) + nI(t, Y_t) dt.$$

We use the notation $X_t \equiv X_t^{\pi,I}$ to emphasize the dependence of the stochastic income and control in the wealth. By a standard calculation, the wealth process is then given by

$$X_t^{\pi,I} = e^{rt} \left(x + \int_0^t e^{-ru} \pi_u \sigma (\lambda du + dW_u) + \int_0^t e^{-ru} nI(u, Y_u) du \right), \qquad t \geq 0. \qquad (1.1)$$

Note that from the boundedness assumptions on the parameters of our model, it is sufficient to assume that $\int_0^t \pi_u^2 du < \infty$ almost surely for all $t \geq 0$ so that the stochastic integral in equation (1.1) is well-defined.

From now on, we will work with a finite time horizon $T \in [0, \infty)$. We assume that the investor's utility function at time T is given by $U(X_T^{\pi,I})$, for some deterministic function U. The investor's goal then is to maximize $\mathbb{E}[U(X_T^{\pi,I})]$ over a set of *admissible strategies* $\mathscr{A}$ and characterize the optimal strategy.

Assumption 3. Throughout our analysis, we assume that the individual preferences exhibit Constant Absolute Risk Aversion (CARA) which are modeled through an exponential utility function

$$U(x) = -\frac{1}{\gamma} e^{-\gamma x}, \quad x \in \mathbb{R}.$$

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[1]We use the notation $\mathscr{E}(\cdot)$ to denote Dooléans-Dade exponential.

with the constant risk aversion parameter satisfying $\gamma \in (0, 1)$, and it remains the same, even if the investor is being subject to receiving an income flow.

The choice of this utility function enables one to reduce the dimension of the state space since the initial wealth x can be factored out. Consequently, the portfolio problems do not depend on the investor's wealth level X. More precisely, a strategy π maximizes the expected utility $\mathbb{E}[U(X_T^{\pi,I})]$ if and only if it maximizes the expected utility $\mathbb{E}[U(\tilde{X}_T^{\pi,I})]$ for a different wealth level $\tilde{X}$. Further, the (negative) exponential utility is the only utility function that possess this invariant under any translation of wealth property (modulo a positive affine transformations) (see Section 2.3 in [**14**]). This independence of the current wealth level is appealing because it promotes mathematical tractability.

Remark 1.3. Note that the set $\mathscr{A}$ of admissible strategies depend on the choice of the utility function V. The least requirement is the assumption that the set of strategies is such that the expected utility is well-defined $\mathbb{E}[|U(X_T^{\pi,I})|] < \infty$. However, due to the presence of pathological strategies (such as the *doubling strategy* introduced by Harrison and Kreps [**15**]), it is necessary for us to impose stricter integrability conditions to our set so that it is economically justified.

We could restrict ourselves to the class of admissible strategies to those such that wealth processes are bounded below (see Harrison and Pliska [**16**]). In other words, we could insist that the investor cannot go arbitrarily far into debt. However, it should be noted that exponential utility admits negative wealth levels. Schachermayer [**37**] highlighted that this restricted class of admissible strategies is not large enough to ensure inclusion of the optimal trading strategy. Also, this constraint will somewhat influence the indifference prices which may generate prices that are significantly high and therefore not suitable for practice. Indeed, the class needs to be enlarged to include some strategies with wealths that are not necessarily bounded from below. To this end, we make the following definitions

$$\mathscr{A}_0 := \left\{ \pi_t \in \mathscr{F}_t : \int_0^T \pi_t^2 \leq \infty \text{ almost surely} \right\},$$

$$\mathscr{A}_b := \{ \pi \in \mathscr{A}_0 : X_t \geq k_\pi \text{ almost surely for all } t \in [0, T], k_\pi \in \mathbb{R} \},$$

$$\mathscr{U} := Cl\{ U(X_T^{\pi,I}) : \pi \in \mathscr{A}_b \}$$

where $Cl\{\cdot\}$ denotes the closure[2] of a set in the space $L^1(\Omega, \mathscr{F}_T, \mathbb{P})$. We consider the enlarged class of $\mathscr{A}_b$ and specify the set of admissible strategies to be

$$\mathscr{A} := \{ \pi \in \mathscr{A}_b : U(X_T^{\pi,I}) \in \mathscr{U} \}.$$

From above, we first provide a bound for the wealth from below, but as already mentioned, the restricted class may not large enough. To enlarge the class, we admit utilities which may be approximated by the utility which characterizes investors with debt limits. Further discussion on this subtlety issue can be found in Schachermayer [**37**] and Davis [**7**].

Remark 1.4. It has been shown by Delbaen et al. [**8**] and noted by Becherer [**1**] that the same dual (and hence primal) solutions are obtained if we considered a variety of definitions of admissible strategy. Consider the set of martingale measures $\mathbb{Q}$ with finite relative entropy defined by

$$\mathscr{M}_f := \{ \mathbb{Q} \in \mathscr{M} : \mathscr{H}(\mathbb{Q}|\mathbb{P}) < \infty \},$$

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[2]A subset $\{U(X_T^{\pi,I}) : \pi \in \mathscr{A}_b\}$ of $\mathbb{R}$ is closed if whenever a sequence $\{x_n\}$ with $x_n \in \{U(X_T^{\pi,I}) : \pi \in \mathscr{A}_b\}$ converges to some x, we must have $x \in \{U(X_T^{\pi,I}) : \pi \in \mathscr{A}_b\}$.

where $\mathscr{H}(\mathbb{Q}|\mathbb{P})$ denotes the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$. We take the set of admissible strategies to be

$$\mathscr{A} := \left\{ H \in L(S) : \int_0^t H_u dS_u \text{ is a } \mathbb{Q}\text{-martingale for all } \mathbb{Q} \in \mathscr{M}_f \right\},$$

where $L(S)$ denotes the set of adapted S-integrable processes. Thus this alternative set specification insists that we trade in such a way that the wealth process is a $\mathbb{Q}$-martingale.

Since (X, Y) are jointly Markov[3], the objective function for the investor is given by

$$V(t, x, y) = \sup_{\pi \in \mathscr{A}} \mathbb{E}[U(X_T^{\pi, I}) | X_t = x, Y_t = y], \tag{1.2}$$

which is the utility attained by the investor if the optimal portfolio policy is followed.

We tackle the optimal portfolio choice problem (1.2) through the Dynamic Programming Principle. To this end, we invoke the Davis-Varaiya Martingale Principle of Optimal Control [**5**], which tells us that we should have that the value function V is a supermartingale for any controls $\pi \in \mathscr{A}$, and there exist optimal admissible $\pi^* \in \mathscr{A}$ (to be found below) such that V is a true martingale.

Throughout this dissertation, we use a subscript notation to denote partial derivatives, e.g $V_x \equiv \frac{\partial V}{\partial x}$, etc. Assuming that V possesses sufficient regularity, we apply Itô's formula to V to get

$$\begin{aligned} d(V(t, X_t, Y_t)) = &\left(V_t(t, X_t, Y_t) + \mathscr{L}^{X,Y} V(t, X_t, Y_t) \right) dt \\ &+ \sigma \pi_t V_x(t, X_t, Y_t) dW_t + \sigma_Y Y_t V_y(t, X_t, Y_t) dW_t^Y, \end{aligned} \tag{1.3}$$

where $\mathscr{L}^{X,Y}$ is the *infinitesimal generator*[4] of the controlled diffusion process $(X_t, Y_t)_{t \geq 0}$ (under $\mathbb{P}$):

$$\mathscr{L}^{X,Y} V = (rx + \sigma \lambda \pi + nI) V_x + \frac{1}{2} \sigma^2 \pi^2 V_{xx} + \mu_Y y V_y + \frac{1}{2} \sigma_Y^2 y^2 V_{yy} + \rho \sigma \sigma_Y y \pi V_{xy}.$$

For V to be a supermartingale for any controls, we require the drift term in the Itó expansion (1.3) to be nonpositive. Further, for V and a martingale under the optimal control, we require the supremum of the drift term to be zero. Thus, the derived utility function V of wealth and stochastic income, satisfies the (nonlinear) *Hamilton-Jacobi-Bellman* (HJB) equation:

$$V_t(t, x, y) + \sup_{\pi \in \mathscr{A}} \mathscr{L}^{X,Y} V(t, x, y) = 0, \quad V(T, x, y) = -\frac{1}{\gamma} e^{-\gamma x}. \tag{1.4}$$

By performing the maximization over π_t yields the optimal feedback Markov control process (abusing notation) $\pi_t^* \equiv \pi^*(t, X_t^*, Y_t)$ for $t \in [0, T]$, where the function $\pi : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is given by

$$\boxed{\pi^*(t, x, y) = -\frac{\lambda}{\sigma} \frac{V_x(t, x, y)}{V_{xx}(t, x, y)} - \frac{\rho \sigma_Y y}{\sigma} \frac{V_{xy}(t, x, y)}{V_{xx}(t, x, y)}.} \tag{1.5}$$

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[3]The solutions to stochastic differential equations (SDE) are known to have the (strong) Markov property.

[4]More formally, an infinitesimal generator of a multi-dimensional Markov process X, denoted by $\mathscr{L}^X$, is defined by

$$\mathscr{L}^X \varphi(x) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}[\varphi(X_{t+\varepsilon}) | X_t = x, \pi_t = \pi] - \varphi(x)}{\varepsilon} = \int \frac{\varphi(y) - \varphi(x)}{\varepsilon} \mathbb{P}(t, t + \varepsilon, x, \pi; dy),$$

where φ is a test function. Throughout this dissertation, we suppose that φ is twice differentiable with continuous second derivative and compactly supported, and so

$$\mathscr{L}^X \varphi(x) = \frac{1}{2} \sum_{i,j} \sigma_{ij}(t, \pi, x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_j \mu_j(t, \pi, x) \frac{\partial \varphi}{\partial x_i},$$

where $(\sigma_{ij}(t, \pi, x))_{i,j}$ is nonnegative definite.

with X_t^* solving

$$dX_t^* = rX_t^* dt + \pi_t^* \sigma(\lambda dt + dW_t) + nI(t, Y_t)dt.$$

From equation (1.5), we see that the optimal control can be decomposed into two different strategies given by

$$\pi_t^0 := -\frac{\lambda}{\sigma} \frac{V_x(t, X_t, Y_t)}{V_{xx}(t, X_t, Y_t)} \qquad \text{and} \qquad \pi_t^H := -\frac{\rho \sigma_Y Y_t}{\sigma} \frac{V_{xy}(t, X_t, Y_t)}{V_{xx}(t, X_t, Y_t)}. \tag{1.6}$$

where π^0 denotes the Merton strategy, and π^H is essential a correction term. The correction term π^H corresponds precisely to the utility-based hedging strategy (which we will formally define below). Note that the form of π^0 is identical to that of the Merton strategy provided in [**27**] in the absence of the stochastic income. In general, π^0 need not be identical to the Merton strategy but if we consider the negative exponential utility function for our problem, then the strategy holds identically.

Remark 1.5. Notice that when ρ or σ_Y are zero, optimal hedging strategy π^H vanishes and the existence of nontraded income does not affect stock holdings. This makes sense as we are unable to hedge against the stochastic income by trading in the stock when they are both uncorrelated. Also, nontraded income becomes deterministic when $\sigma_Y = 0$ and so we need not perform any hedging as there is no risk needed to be removed. Even in these cases, however, the existence of the income still has an effect on the solution through the value function V.

CHAPTER 2

UTILITY INDIFFERENCE PRICING WITH CARA PREFERENCES

2.1 EXPLICIT SOLUTIONS TO UTILITY MAXIMIZATION PROBLEM

We operate under the framework of the classical Merton portfolio optimization problem, appropriately modified to accommodate stochastic income. We want to maximize the expected utility of terminal wealth and so the objective is

$$V(t, x, y) := \sup_{\pi \in \mathscr{A}} \mathbb{E}[U(X_T^{\pi, I}) | X_t = x, Y_t = y].$$

Under the general setting where the investor is either subject to receiving income or possibly debiting income, we can establish the following definition due to Hodges and Neuberger [**22**].

Definition 2.1. (Utility Indifference Price). The utility-based indifference (bid) price of the claim paying the stochastic income at $t \in [0, T]$ given $(X_t, Y_t) = (x, y)$ is $p(t, x, y)$, which is defined implicitly by

$$V(t, x - np(t, x, y), y) = V^0(t, x).$$

where V^0 is defined to be the value function in the absence of the stochastic income.

The definition above is also known as the *certainty equivalent value*. For $n = 1$, the investor will attain the same utility as when the investor does not receive the income. In other words, the indifference price is the least amount of initial wealth that the investor would be willing to pay in exchange for the stream of stochastic income, (see Munk [**33**]). If the income is not received, then there is no dependence on the nontraded asset Y and this is precisely the classical Merton problem. We can also arrive at a symmetric definition for the indifference *ask* price when $n = -1$.

Remark 2.1. In general, the indifference price may depend on t, x, and y as provided in the Definition 3.1. However, if we consider the (negative) exponential utility, we lose the dependence on x and thus, reduce the dimensionality of our problem.

With the utility indifference price of a claim, we can compare the optimal trading strategies associated with the case when we have the inflow of uncertain income stream to when we do not. This gives rise to the following definition:

Definition 2.2. (Optimal hedging strategy). The optimal hedging strategy π^H for the stochastic income is defined by

$$\pi^H := \pi^* - \pi^0,$$

where π^* is the optimal portfolio with the stochastic income and π^0 corresponds to the Merton strategy.

The strategy π^H corresponds to the inter-temporal hedging demand from Merton [28]. From above, we see that it is the difference between the optimal strategy π^* which we obtained from performing the optimization and the optimal strategy when the investor does not receive the income. This difference is the extra trading that the investor has to perform to hedge against a long position in the stochastic income.

We now present the main result of this section.

Proposition 2.1. *The value function V and the indifference (bid) price for the optimal portfolio choice problem (1.2) with exponential utility are both given by*

$$V(t,x,y) = V^0(t,x)F(t,y)^{\frac{1}{1-\rho^2}}, \tag{2.1}$$

$$V^0(t,x) = -\frac{1}{\gamma}\exp\left\{-\gamma x e^{r(T-t)} - \frac{1}{2}\lambda^2(T-t)\right\}, \tag{2.2}$$

$$F(t,y) = \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\exp\left\{-\gamma(1-\rho^2)\int_t^T e^{r(T-u)}nI(u,Y_u)du\right\}\Bigg|Y_t = y\right], \tag{2.3}$$

$$p(t,y) = -\frac{e^{-r(T-t)}}{n\gamma(1-\rho^2)}\log F(t,y), \quad p(T,y) = 0, \tag{2.4}$$

for $y \in \mathbb{R}$ and $t \in [0,T)$, under the measure[1] $\tilde{\mathbb{Q}}$ where

$$Z_t^{\tilde{\mathbb{Q}}} \equiv \frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\bigg|_{\mathscr{F}_t} = \mathscr{E}(-\rho\lambda \cdot W^Y)_t,$$

is a unique state price density and $\tilde{\mathbb{Q}} \sim \mathbb{P}$.

Proof. By substituting the optimal control function π^* into the HJB equation (1.4), we have

$$V_t + rxV_x + nI(t,y)V_x + \mu_Y yV_y + \frac{1}{2}\sigma_Y^2 y^2 V_{yy} - \frac{1}{2}\frac{(\lambda V_x + \rho\sigma_Y yV_{xy})^2}{V_{xx}} = 0,$$
$$V(T,x,y) = -\frac{1}{\gamma}e^{-\gamma x}. \tag{2.5}$$

One seeks solution to the HJB equation in (1.4). The scaling properties of exponential utility suggests that we suppose that the candidate solution V for our problem is of the form

$$V(t,x,y) = -\frac{1}{\gamma}\exp\left\{-\gamma x e^{r(T-t)}\right\}f(t,y). \tag{2.6}$$

We suppose that the candidate solution V satisfies the (2.5) and we see that f satisfies the semi-linear partial differential equation (PDE)

$$f_t + \mathscr{L}^{Y,\tilde{\mathbb{Q}}}f - \left(\frac{1}{2}\lambda^2 + \gamma nI(t,y)e^{r(T-t)}\right)f - \frac{1}{2}\rho^2\sigma_Y^2 y^2\frac{f_y^2}{f} = 0, \quad f(T,y) = 1,$$

---

[1]Note, by Cameron-Martin-Girsanov theorem, $W^{Y,\tilde{\mathbb{Q}}} := W^Y + \rho\lambda t$ is a standard Brownian motion under the measure $\tilde{\mathbb{Q}}$. The probability measure $\tilde{\mathbb{Q}}$ is known as the *indifference measure* given in [35] which is the projection of the minimal martingale onto the σ-algebra $\mathscr{F}^Y$.

where $\mathscr{L}^{Y,\tilde{\mathbb{Q}}}$ is the generator of Y under $\tilde{\mathbb{Q}}$ satisfying

$$\mathscr{L}^{Y,\tilde{\mathbb{Q}}}f(t,y) = (\mu_Y y - \rho\lambda\sigma_Y y)f_y + \frac{1}{2}\sigma_Y^2 y^2 f_{yy}.$$

We now use the distortion transformation $f = g^\delta$ (introduced by Zariphopoulou [42]) for some constant δ and find that g satisfies PDE

$$g_t + \mathscr{L}^{Y,\tilde{\mathbb{Q}}}g - \frac{1}{\delta}\left(\frac{1}{2}\lambda^2 + \gamma n I(t,y)e^{r(T-t)}\right)g + \frac{1}{2}(\delta - 1 - \delta\rho^2)\sigma_Y^2 y^2 \frac{g_y^2}{g} = 0, \quad g(T,y) = 1,$$

By making a clever choice $\delta = 1/(1-\rho^2)$, we obtain a linear PDE for g:

$$g_t + (\mu_Y y - \rho\lambda\sigma_Y y)g_y + \frac{1}{2}\sigma_Y^2 y^2 g_{yy} - (1-\rho^2)\left(\frac{1}{2}\lambda^2 + \gamma n I(t,y)e^{r(T-t)}\right)g = 0.$$

Note that the structure for coefficients of the process Y allows the linear PDE for g to admit a unique, smooth and bounded solution (see Theorem 2.9.10 of [25]). From Feynman-Kac theorem, we arrive at the following the stochastic representation

$$g(t,y) = \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\exp\left\{-(1-\rho^2)\left(\frac{1}{2}\lambda^2(T-t) + \gamma\int_t^T e^{r(T-u)}nI(u,Y_u)du\right)\right\}\Bigg| Y_t = y\right].$$

under the probability measure $\tilde{\mathbb{Q}}$, such that Y satisfies

$$dY_t = (\mu_Y - \rho\lambda\sigma_Y)Y_t dt + \sigma_Y Y_t dW_t^{Y,\tilde{\mathbb{Q}}},$$

where $W^{Y,\tilde{\mathbb{Q}}}$ is a standard Brownian motion under $\tilde{\mathbb{Q}}$. Using $V(t,x,y) = U(xe^{r(T-t)})g(t,y)^\delta$ with $\delta = 1/(1-\rho^2)$ then gives (2.1). The indifference price formula then follows from its definition. ∎

Remark 2.2. From the viscosity solution argument standpoint, we have to show the value function V is the unique solution of the HJB equation. Duffie and Zariphopoulou [10] provided an analysis of the value function of an optimal investment model with stochastic income and general utility function. They mentioned that uniqueness of V is valid in the class of functions that are concave and is strictly increasing in x, and are uniformly bounded in the variable y. In the special case for our model, this is justified from our specification of V which has exponential growth in the wealth argument and g being a bounded function. Since candidate solution V is smooth and so it is a viscosity solution of (2.5). Moreover, the assumption of the market parameters ensures us that V belongs to the class of viscosity solutions in which uniqueness holds. Thus, uniqueness is indeed satisfied.

Remark 2.3. It is interesting to observe that the stochastic income can be thought as a future value payment or claim that pays the future value $\int_t^T e^{r(T-u)}nI(u,Y_u)du$ at terminal time T; we are effectively taking the sum of the income stream which pays at a rate of $nI(t,Y_t)\Delta t$ from time t to terminal time T. From (2.3), the value function is the one we would expect if the investor were receiving the future value of cumulative income as a single payment at time T. Also, the terminal condition $p(T,y) = 0$ makes sense as one will not have the time to receive the income then.

We can obtain an expression for the optimal hedging strategy by comparing the optimal portfolio choice problem with stochastic income to one without the stochastic income income but with an adjusted initial endowment and this is shown in the following proposition.

Proposition 2.2. *The optimal hedging strategy for the stochastic income is to hold H_t shares at $t \in [0, T]$, given by*

$$\pi_t^H = -\frac{n\rho\sigma_Y Y_t}{\sigma} p_y(t, Y_t).$$ (2.7)

Moreover, the sign[2] of the optimal hedging strategy π^H depends on the sign of $\rho n I_y$.

Proof. **Part 1**. From Proposition 2.1, we can use equation (2.1) and (2.3) to get a relation for V and V^0 given by

$$V(t, x, y) = V^0(t, x)e^{-n\gamma p(t,y)e^{r(T-t)}}.$$ (2.8)

Substitute the expression of equation (2.8) into (1.5), we get

$$\pi^*(t, x, y) = -\frac{V_x^0(t,x)}{\sigma V_{xx}^0(t,x)}(\lambda - n\rho\sigma_Y \gamma y e^{r(T-t)} p_y(t, y)).$$

However, from equation (2.2), we can deduce that $V_x^0(t,x)/V_{xx}^0(t,x) = -e^{-r(T-t)}/\gamma$. Consequently, we can remove the dependence on x from π^* to obtain

$$\pi^*(t, y) = \frac{\lambda}{\sigma\gamma e^{r(T-t)}} - \frac{n\rho\sigma_Y y}{\sigma} p_y(t, y).$$ (2.9)

By using Definition 3.2, the optimal hedging strategy π^H for a long position in the stochastic income can be easily obtained.

Part 2. Note first that all terms in (2.7), were assumed positive, apart from ρ and p_y. Since p and F are related by (2.4), we have

$$p_y(t, y) = -\frac{e^{-r(T-t)}}{n\gamma(1 - \rho^2)} \frac{F_y(t, y)}{F(t, y)}.$$ (2.10)

From this, we investigate the monotonicity of F in the y argument because it will give us an indication of the sign of the derivative F_y. From (2.3), we know that $F > 0$. Since the process Y is assumed to follow a geometric Brownian motion (this assumption can be relaxed as in Henderson [**18**]), it has a strong solution which is unique in law and has the Strong Markov Property. Consider two independent realizations of Y started at $Y_t^{(1)}$, $Y_t^{(2)}$ with $Y_t^{(2)} > Y_t^{(1)}$. Let τ be the first time where the two processes $Y^{(1)}$ and $Y^{(2)}$ coincide, that is, $\tau := \inf_{u \geq t}\{u : Y_u^{(2)} = Y_u^{(1)}\}$ and

$$\tilde{Y}_u := \begin{cases} Y_u^{(1)} & \text{for } u < \tau \\ Y_u^{(2)} & \text{for } u \geq \tau \end{cases}.$$

Note that $Y_u^{(2)} \geq \tilde{Y}_u$ by construction. By the Strong Markov Property, the process Y begins afresh after time τ, which implies that $\tilde{Y}_u \stackrel{d}{=} Y_u^{(1)}$ and so $Y_u^{(2)} \geq Y_u^{(1)}$. If the derivative $nI_y > 0$, we have

$$F(t, Y_t^{(1)}) \geq F(t, Y_t^{(2)}).$$

On the other hand, we get a reversed inequality if $nI_y < 0$. Thus, if $nI_y > 0$, we have $F_y < 0$, therefore the sign of π^H depends on the sign of $\rho n I_y$. ∎

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[2]The sign of ρ and a_y is important in analyzing the optimal investment behavior of an individual.

Remark 2.4. We can make the following observations:

1. The optimal hedging strategy becomes greater in magnitude if the initial income state variable, volatility of income greater, volatility of stock smaller, and the investment horizon T is further away.

2. For $n = 1$ and $I(t, y) = y$, we observe[3] that as the time to maturity collapses, the optimal hedging strategy, in terms of time T money defined by $\hat{\pi}_t^H := \pi_t^H e^{r(T-t)}$, decreases in magnitude over time to zero. Thus, we see that the investor reduces his position of wealth in the stock (we will see below how this result also holds through simulation) over the time horizon of the investment. This is consistent with practice since investment professionals advise investors to reduce their holdings of assets in stocks over time and as they approach retirement (to reduce risk in the portfolio). Further, Henderson [18] also noted that it is also consistent with the findings in [4] and with the background risk literature.

3. If correlation is positive, then $\hat{\pi}^H$ is concave in time, and convex otherwise.

We can use (2.4), (2.7) and (2.9) to implement utility-based hedging. By considering (2.4) and the PDE satisfied by the function F in (2.3) (via Feynman-Kac theorem), we can deduce that p satisfies the following nonlinear PDE:

$$p_t + \left(\mu_Y y - \rho \sigma_Y y \lambda\right) p_y + \frac{1}{2}\sigma_Y^2 y^2 p_{yy} - rp - \frac{1}{2}\sigma_Y^2 y^2 n\gamma(1-\rho^2)e^{r(T-t)}p_y^2 + nI(t, y) = 0, \qquad (2.11)$$

with terminal condition $p(T, y) = 0$. We note that if we allow the correlation $\rho \to 1$, we recover a similar Black-Scholes type PDE (slightly modified from the standard one) for an asset which pays continuous income in a complete market:

$$p_t + ryp_y + \frac{1}{2}\sigma_Y^2 y^2 p_{yy} - rp + nI(t, y) = 0. \qquad (2.12)$$

On the other hand, if we allow the risk-aversion $\gamma \to 0$, we get a linear PDE instead; invoke the Feynmac-Kac theorem to this PDE to obtain a stochastic representation which corresponds to the *marginal price*[4] given by

$$\hat{p}(t, y) = \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\int_t^T e^{-r(u-t)} nI(u, Y_u)du \,\bigg|\, Y_t = y\right]. \qquad (2.13)$$

Note that the expression above differs from the Black-Scholes price in the sense that we are computing the expectation under the indifference measure $\tilde{\mathbb{Q}}$.

2.2 PERFECT CORRELATION CASE

If the correlation is perfect, $\rho = 1$, the income becomes a traded asset. Since the market now is complete, under the no-arbitrage condition, the drift of Y has to be the riskless interest rate r

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[3]This can be seen by differentiating (see Appendix A for derivation) the indifference price p with respect to y using equation (2.4) and noticing that $p(T, y) = 0$ which corresponds to the income ceasing at time T (it would be different though if there was some additional payoff at time T).

[4]The concept of marginal price was introduced by Davis in [7]. It can be interpreted as the price that an investor would trade an infinitely small position of the claim which can be used to approximate the price for a small number of claims.

under $\tilde{\mathbb{Q}}$. Therefore, both the Brownian motions W and W^Y are identical. In a complete market, there could only be one equivalent martingale measure. Furthermore, as $|\rho| \to 1$, we see that the state price density $Z_t^{\tilde{\mathbb{Q}}} \to Z_t^{\mathcal{Q}}$ under the so-called risk-neutral measure $\mathcal{Q}$ related by

$$\frac{d\mathcal{Q}}{d\mathbb{P}} = Z_t^{\mathcal{Q}} := \mathscr{E}(-\lambda \cdot W)_t.$$

In a complete market, the value of future uncertain cash flows can be obtained by discounting the expected value of the cash flow under the unique risk-neutral measure. Intuitively, the indifference price should coincide with the no arbitrage (Black-Scholes) price, as there is no risk that cannot be hedged. Introducing the abbreviation, the indifference price is given by

$$\mathrm{BS}(t, y) \equiv p(t, y) = \mathbb{E}^{\mathcal{Q}} \left[\int_t^T e^{-r(u-t)} nI(u, Y_u) du \bigg| Y_t = y \right],$$

Note that we can verify this expression by finding a stochastic representation for the PDE in (2.12) via Feynman-Kac theorem.

Remark 2.5. By specializing to the case where $I(t, y) \equiv y$, we can easily obtain a closed-form solution for the indifference price:

$$\boxed{\mathrm{BS}(t, Y_t) = \frac{nY_t}{\mu_Y - \sigma_Y \lambda - r} \left(e^{(\mu - \sigma_Y \lambda - r)(T-t)} - 1 \right),}$$

provided $\mu_Y - \sigma_Y \lambda \neq r$. We can also easily recover the strategy associated with the given in (2.7) if we set $\rho = 1$:

$$\boxed{\pi_t^N := -\frac{n\sigma_Y Y_t}{\sigma} \frac{\partial \mathrm{BS}}{\partial y}(t, Y_t).} \tag{2.14}$$

In this case, perfect hedging of the claim on Y is possible by trading S, and our strategy at time $t \in [0, T]$ is to hold a proportion π^N of wealth in the stock. These results for the perfect correlation case are important as it allow us to use then as a proxy to make comparison with those under imperfect correlation.

NUMERICAL IMPLEMENTATION

3.1 THE RESIDUAL RISK PROCESS

We now implement the model numerically to examine the hedging performance of the model in the style initiated by Monoyios [**30**]. His model however, investigates the effect of hedging based on a *short position of a European put option*. We attempt to adapt his methodology to conduct our analysis.

Consider an investor receiving a claim (stochastic income) at time 0 at a price of $p(0, Y_0)$ per claim and hedges this position over $[0, T]$ using the strategy π^H. We suppose that our initial wealth endowment, x in our hedging portfolio is $-np(0, Y_0)$[1]. We define the *residual risk process* $(R_t)_{t \geq 0}$ as

$$dR_t = dX_t^H + ndp(t, Y_t),$$

where $X^H \equiv (X_t^H)_{t \geq 0}$ denotes the wealth of the hedging portfolio in S, satisfying

$$dX_t^H = rX_t^H dt + \pi_t^H[(\mu - r)dt + \sigma dW_t] + nI(t, Y_t)dt, \quad X_0^H = -np(0, Y_0).$$

By applying Itó's formula to the indifference price p and using the PDE of (2.11), we obtain[2]

$$dR_t = rR_t dt + \frac{1}{2}n^2\gamma(1 - \rho^2)e^{r(T-t)}\sigma_Y^2 Y_t^2 p_y(t, Y_t)^2 dt + n\sqrt{1 - \rho^2}\sigma_Y Y_t p_y(t, Y_t)dW_t^\perp, \quad (3.1)$$

with $R_0 = 0$. Observe that the term W is not present in equation (3.1). This is a consequence of hedging optimally in the tradable asset S as it eliminated the risks inherent in that particular component. However, since we have the component $W^\perp$ in the expression (3.1), it is interpreted as the component of unhedgeable risks; even by trading solely in the tradable asset, it is not possible to eliminate all the risks.

Remark 3.1. By construction, the value of residual risk at terminal time should equal to the terminal wealth value, that is $R_T = X_T$. If the market is complete with $\rho = 1$, the residual risk process becomes deterministic. But since $R_0 = 0$, the terminal hedging error is zero in such market which is expected.

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[1] For $n = 1$, the claim is essentially bought at time 0 for a price $p(0, Y_0)$. Whatever initial wealth we start with, we assumed that we have paid some money for the claim and with exponential utility, wealth factors out of the problem. The minus sign is necessary as the investor is hedging against a long position in the stochastic income. For instance, a university graduate who took on a loan to finance education will have negative wealth just upon graduation. The future earnings serve as labor income for the individual which is subject to a certain degree of uncertainty. A similar interpretation can be made for $n = -1$.

[2] See Appendix A for derivation.

Monoyios [30] also considered the residual risk process which arises from using the so-called "naive" strategy, which he defined to be the Black-Scholes hedging strategy which perfectly replicates the stochastic income in the complete market setting . This strategy can be seen to be utility-based optimal hedging strategy when $\rho = 1$. To this end, we substitute the optimal hedging strategy π^H for $\rho = 1$ into wealth process to get

$$dX_t^N = rX_t^N dt + \pi_t^N[(\mu - r)dt + \sigma dW_t] + nI(t, Y_t)dt, \quad X_0^N = -n\text{BS}(0, Y_0). \tag{3.2}$$

The hedging error process associated with this naive strategy by R^N then we can use (2.7) and (2.11) at the limit $\rho \to 1$ to get

$$\boxed{dR_t^N = rR_t^N dt + n\sigma_Y Y_t \frac{\partial \text{BS}}{\partial y}(t, Y_t)[(\rho - 1)dW_t + \sqrt{1 - \rho^2}dW_t^\perp],} \tag{3.3}$$

with $R_0^N = 0$. One can clearly see that this residual risk process associated with the naive strategy in (3.3) consists of an extra random term W (this should be contrasted with (3.1)). Thus, this term contributes to the extra risk that one faces when using the naive strategy as oppose to the optimal hedging strategy.

Remark 3.2. If we hedge using the optimal strategy, we are assuming that we begin with an initial wealth of $-np(0, Y_0)$ as oppose to $-n\text{BS}(0, Y_0)$. For the case where $n = -1$, one might argue that for someone who charges with a Black-Scholes price will not start off with as much wealth as one who charges at the indifference price, so one is clearly likely to do better in the latter. However, this is precisely how the utility indifference pricing mechanism is designed which tell us what wealth one should begin with (or price to charge) and how we should optimally invest. We will also see below that for $n = 1$, the initial wealth X_0 is less than X_0^N. Moreover, we could always check what would happen if we started at some wealth by a trivial subtraction.

3.2   PERTURBATION EXPANSION

We would like to simulate the residual risk process R but in order to do so reasonable efficiently[3], we have to first approximate the terms p and p_y (since there is no known closed-form expression for the indifference price p).

Assumption 4. In the incomplete case under the lognormal model, we have explicit expressions for the investor's value function (2.3) which hold for the case of general function I for the income rate. Bodie et al. [3] studied the form where at time t, we have $I(t, Y_t) = L_t Y_t$ where L is the supply of labor and Y is the wage per unit labor. We suppose for simplicity that $L = 1$ and $n = 1$, specializing to the function $nI(t, Y_t) \equiv Y_t$. The expression for F in (2.3) now reduces to

$$F(t, y) = \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\exp\left\{-n\gamma(1 - \rho^2)Y_t \int_t^T e^{r(T-u)}e^{(\mu_Y - \lambda\rho\sigma_Y - \frac{1}{2}\sigma_Y^2)(u-t) + \sigma_Y(W_u^{\tilde{\mathbb{Q}}} - W_t^{\tilde{\mathbb{Q}}})}du\right\}\bigg| Y_t = y\right].$$

This explicit representation allows us to perform Monte Carlo simulations for estimating the expectation under the indifference measure $\tilde{\mathbb{Q}}$.

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[3]One could for instance, perform a finite difference scheme for (2.11) to numerically determine the quantities of interest but we will not investigate this method in this dissertation.

In the following theorem, we generate perturbation expansion for the indifference price. Throughout this dissertation, we use the short-hand notation $\mathbb{E}_{t,y}$ to denote the expectation condition on $Y_t = y$.

Theorem 3.1. *The indifference price $p(t, y)$ has the perturbative representation*

$$p(t, y) = \frac{1}{\beta_t} \left(\mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}} \left[\int_t^T \beta_u Y_u du \right] + \frac{1}{2} \varepsilon \text{Var}_{t,y}^{\tilde{\mathbb{Q}}} \left[\int_t^T \beta_u Y_u du \right] + \mathcal{O}(\varepsilon^2) \right), \tag{3.4}$$

where $\beta_u := e^{r(T-u)}$ is defined to be the accumulation factor, $\varepsilon := -\gamma(1-\rho^2)$, and $\mathcal{O}(\varepsilon)$ denotes terms proportional to ε. The expansion is valid provided that the model parameters satisfy the following condition:

$$\boxed{0 < F(t, y) < 2.} \tag{3.5}$$

Proof. By expanding the exponential in (2.3) for ε using Taylor's theorem, we have

$$p(t, y) = \frac{1}{\varepsilon \beta_t} \log \left(1 + \varepsilon \mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}} \left[\int_t^T \beta_u Y_u du \right] + \frac{1}{2} \varepsilon^2 \mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}} \left[\left(\int_t^T \beta_u Y_u du \right)^2 \right] + \mathcal{O}(\varepsilon^3) \right). \tag{3.6}$$

Note that the power series expansion of $\log(1 + x)$ holds for $|x| < 1$. This implies that the logarithm in (3.6) can be expanded as a Taylor series provided the condition (3.5) is satisfied. This proves the last assertion. Moreover, by writing out the power expansion for (3.6), it is a straightforward procedure to then verify the first assertion. ∎

Remark 3.3. Observe from equation (3.4) that as $|\rho|$ tends to 1, we are only left with the first term in the expansion. Moreover, we have seen that the state price density $Z^{\tilde{\mathbb{Q}}}$ converges to that of $Z^{\mathcal{Q}}$ under the risk-neutral measure $\mathcal{Q}$. Thus, when the correlation becomes perfect, the indifference price essentially becomes the price under the complete market setting.

It turns out that we can actually find closed-form formulas for at least the first two terms of the perturbation expansion which are given below in the following proposition.

Proposition 3.1. *The first two moments for the future value of cumulative income under the indifference measure $\tilde{\mathbb{Q}}$ is given by*

$$\mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}} \left[\int_t^T \beta_u Y_u du \right] = \frac{y e^{r(T-t)}}{\nu - r} [e^{(\nu-r)(T-t)} - 1], \tag{3.7}$$

$$\mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}} \left[\left(\int_t^T \beta_u Y_u du \right)^2 \right] = 2y^2 \left(-\frac{e^{(2\nu+\sigma_Y^2)(T-t)} - e^{2r(T-t)}}{(\nu - r)(2\nu + \sigma_Y^2 - 2r)} + \frac{e^{(2\nu+\sigma_Y^2)(T-t)} - e^{(\nu+r)(T-t)}}{(\nu - r)(\nu + \sigma_Y^2 - r)} \right). \tag{3.8}$$

where $\nu := \mu_Y - \lambda \rho \sigma_Y$ and $|\rho| \leq 1$. These expressions hold only when the denominators are non-zero.

Proof. Under the measure $\tilde{\mathbb{Q}}$, the process Y_u is given by

$$Y_u = Y_t e^{(\mu_Y - \lambda \rho \sigma_Y - \frac{1}{2}\sigma_Y^2)(u-t) + \sigma_Y(W_u^{\tilde{\mathbb{Q}}} - W_t^{\tilde{\mathbb{Q}}})}.$$

With this expression, we can compute the first order term inside the expansion (3.4):

$$\mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}\left[\int_t^T \beta_u Y_u du\right] = y \int_t^T e^{r(T-u)} e^{(\nu-\frac{1}{2}\sigma_Y^2)(u-t)} \mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}[e^{\sigma_Y(W_u^{\tilde{\mathbb{Q}}}-W_t^{\tilde{\mathbb{Q}}})}] du$$

$$= y \int_t^T e^{r(T-u)} e^{(\nu-\frac{1}{2}\sigma_Y^2)(u-t)+\frac{1}{2}\sigma_Y^2(u-t)} du$$

$$= y e^{rT-\nu t} \int_t^T e^{(\nu-r)u} du$$

$$= \frac{y e^{r(T-t)}}{\nu-r}[e^{(\nu-r)(T-t)} - 1].$$

We further compute the second order term inside the expansion:

$$\mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}\left[\left(\int_t^T \beta_u Y_u du\right)^2\right] = \mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}\left[\int_t^T \int_t^T \beta_u \beta_s Y_u Y_s du ds\right].$$

By Tonelli's theorem, we now face the problem of computing $\mathbb{E}^{\tilde{\mathbb{Q}}}[Y_u Y_s | Y_t = y]$. The way to solve this problem is to exploit in independence property of the Brownian motion. To this end, assume $u > s$ and let $\nu := \mu_Y - \lambda\rho\sigma_Y$. We have

$$\mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}[Y_u Y_s] = Y_t^2 e^{(\nu-\frac{1}{2}\sigma_Y^2)(s+u-2t)} \mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}[e^{\sigma_Y(W_u^{\tilde{\mathbb{Q}}}-W_t^{\tilde{\mathbb{Q}}})+\sigma_Y(W_s^{\tilde{\mathbb{Q}}}-W_t^{\tilde{\mathbb{Q}}})}]$$

$$= Y_t^2 e^{(\nu-\frac{1}{2}\sigma_Y^2)(s+u-2t)} \mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}[e^{\sigma_Y(W_u^{\tilde{\mathbb{Q}}}-W_s^{\tilde{\mathbb{Q}}})+2\sigma_Y(W_s^{\tilde{\mathbb{Q}}}-W_t^{\tilde{\mathbb{Q}}})}]$$

$$= Y_t^2 e^{(\nu-\frac{1}{2}\sigma_Y^2)(s+u-2t)} e^{\frac{1}{2}\sigma_Y^2(u-s)+2\sigma_Y^2(s-t)}$$

$$= Y_t^2 e^{(\nu-\frac{1}{2}\sigma_Y^2)(s+u-2t)+\frac{1}{2}\sigma_Y^2(u-s)+2\sigma_Y^2(s-t)}$$

$$= Y_t^2 e^{\nu(s+u-2t)+\sigma_Y^2(s-t)}.$$

Going back to our original problem, observe that the line $u = s$ in the us-plane splits the rectangle $[0, t] \times [0, t]$ into two triangles with the lower and upper triangle both having explicit expressions for the expectation. Thus, we have

$$\mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}\left[\left(\int_t^T \beta_u Y_u du\right)^2\right] = \int_t^T \int_t^T \beta_u \beta_s \mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}[Y_u Y_s] du ds$$

$$= \int_t^T \int_s^T \beta_u \beta_s \underbrace{\mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}[Y_u Y_s]}_{u>s} du ds + \int_t^T \int_t^s \beta_u \beta_s \underbrace{\mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}[Y_u Y_s]}_{u<s} du ds$$

$$= y^2 \int_t^T \int_s^T \beta_u \beta_s e^{\nu u+\nu(s-2t)+\sigma_Y^2(s-t)} du ds$$

$$+ y^2 \int_t^T \int_u^T \beta_u \beta_s e^{\nu s+\nu(u-2t)+\sigma_Y^2(u-t)} ds du$$

$$= 2y^2 \int_t^T \int_u^T \beta_u \beta_s e^{\nu s+\nu(u-2t)+\sigma_Y^2(u-t)} ds du \qquad \text{(by symmetry)}$$

$$= 2y^2 \left(-\frac{e^{(2\nu+\sigma_Y^2)(T-t)} - e^{2r(T-t)}}{(\nu-r)(2\nu+\sigma_Y^2-2r)} + \frac{e^{(2\nu+\sigma_Y^2)(T-t)} - e^{(\nu+r)(T-t)}}{(\nu-r)(\nu+\sigma_Y^2-r)}\right).$$

∎

With these results at hand, we can substitute them back into (3.4) which enable us to perform our numerical calculations that involve the indifference price p and its derivative p_y.

3.3 Accuracy of Perturbation Expansion

The objective is to investigate the distribution of the terminal hedging error R_T and R_T^N by using the SDE of the residual risk process R to simulate over many price histories. We perform an analysis for different values of γ and ρ, to check if the perturbation expansion we have for the indifference price p and its derivative p_y was reasonable to what we got if we used Monte-Carlo simulation. By considering the stochastic representation given in equation (2.3), we can generate numerical values for the indifference (bid) price p of the income stream and its derivative p_y through simulation. We performed 40000 simulations, and compared the numerical values against those that arise from the perturbation expansion to get an idea for the accuracy of our perturbation expansions. Furthermore, the simulations are also used to determine whether our model parameters do indeed fulfill the condition provided in (3.5).

Table 3.1: Model parameters

S_0	Y_0	r	μ	σ	μ_Y	σ_Y	T (years)
100	100	0.05	0.10	0.25	0.12	0.3	1

Table 3.2 shows the accuracy of our perturbation expansions for p and p_y at time $t = 0$ based on the parameters which are summarized in Table 3.1 with risk aversion to be $\gamma = 0.001$, and various of values for ρ. The results produced by the perturbation expansion are close with that from simulations which establishes the accuracy. Further tests were also conducted but not reported here, show that accurate results are obtained across all values of correlation when the risk aversion parameter is below about 0.05. In addition, the accuracy improves when $|\rho|$ increases and γ decreases.

With these results, we are able to generate indifference prices and hedging strategies to a reasonable degree of accuracy. The closed form expressions obtained from the asymptotic expansion up to the first order serve as a machinery for our further evaluation of the hedging performance of optimal strategies against naive strategies.

Remark 3.4. At this point, the author would like to point out that there is a small error in [18]. In pp. 20, equation (44) is not quite accurate in general due to the first term of the equation which only holds in the limit when the correlation when $\rho = 1$. More importantly, it was mentioned in the paper that the indifference value in given equation (44) is lower than the value of the indifference price when the correlation is perfect for the lognormal model. This is in contrast with the numerical results produced in Table 3.2 which shows that indifference prices arising from imperfect correlation is greater than that of the perfect correlation. It is worth mentioning that Hobson [21] showed that for a utility function satisfying certain properties, the utility indifference (bid) price of a nonnegative contingent claim is bounded above by the marginal price under the measure $\overline{\mathbb{Q}}$ (for our case, this is related with (2.13)) which is indeed reflected in the results from Table 3.2 (up to a first order approximation).

3.4 Hedging Performance

In this section, we illustrate the optimal strategy and the effect of the utility-based hedging. We suppose that stochastic income Y is received at time zero in exchange for price charged at $p(0, Y_0)$

Table 3.2: Income bid prices and deltas from the perturbation expansion and from simulations. Input parameters used are listed in Table 3.1. Note that under the special case where $\rho = 1$, we set $\nu = \mu_Y - \sigma_Y \lambda = 0.06$. Figures in parentheses indicate the standard deviation from the simulations.

ρ	$\mathcal{O}(\varepsilon^0)$	$\mathcal{O}(\varepsilon^1)$	Simulation	ρ	$\mathcal{O}(\varepsilon^0)$	$\mathcal{O}(\varepsilon^1)$	Simulation
-0.95	106.6276	106.6091	106.6174 (0.02672)	-0.95	1.066276	1.065907	1.066418
-0.75	105.9769	105.8963	105.9260 (0.00956)	-0.75	1.059769	1.058137	1.057825
-0.5	105.1709	105.0337	105.0176 (0.02545)	-0.5	1.051709	1.048965	1.049105
-0.25	104.3730	104.2047	104.2439 (0.02132)	-0.25	1.043730	1.040364	1.040447
0	103.5831	103.4070	103.3581 (0.03519)	0	1.035831	1.032308	1.033305
0.25	102.8011	102.6391	102.6621 (0.02313)	0.25	1.028011	1.024770	1.024116
0.50	102.0269	101.8997	101.9567 (0.02436)	0.50	1.020269	1.017725	1.016785
0.75	101.2605	101.1877	101.1937 (0.01883)	0.75	1.012605	1.011148	1.010774
0.95	100.6528	100.6328	100.5932 (0.01635)	0.95	1.006528	1.006209	1.005810
1	100.5017	100.5017	100.4960 (0.01583)	1	1.005020	1.005019	1.004695

Income bidding prices, $p(0, Y_0)$ Income deltas, $p_y(0, Y_0)$

which defines the initial endowment for the hedging portfolio, and the hedged using strategy H. Note that all results reported below were carried out using valid model parameters. The parameters were methodically checked if they had satisfy the constraint (3.5) and produced an accurate approximation as well as whether our closed-form expressions for the perturbation expansion can accept those parameters.

Figure 3.1 provides examples of the simulated paths of both the stock price (traded asset) along with the uncertain income stream and plots the evolutionary paths for both the hedge ratios H, and hedge portfolio values X for the optimal and naive strategies using (3.1) and (3.3), all against time t based on the simulations. This provide us a qualitative, intuitive view of the overall performance of the hedging performance associated with two different strategies. Notice that when the correlation is positive, the hedge ratio is negative and otherwise for negative correlation which is consistent with proposition 2.2. Also, the asymptotic behavior of the hedge ratio process can be seen to tend to zero when approaching maturity which also coincides with the observation in Remark 2.4.

In Figure 3.1(a), we notice that the optimal delta in the traded asset is smaller than the naive delta. This is justified[4] from our market parameters with the drift of the traded asset μ being greater than the interest rate r. The hedge portfolio processes are seen to have a steady upward trend which one would expect to see since the wealth process is injected with the (stochastic) income over the investment horizon. A similar observation can be made for Figure 3.1(b).

While such plots are extremely useful, the graphical description can be made quantitative by computing the statistics of the mean, median, standard deviation of the hedging error distribution. These statistics indicate several important aspects that arises from both strategies and help us make more accurate inferences. To assess the quality of the hedging strategy, the median would tell us how frequent the a policy results in positive hedging error simply because there are cases where one

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[4]Indeed, the utility-based hedging mechanism acknowledges that taking a large short position in the stock would likely lead to a loss.

(a) Simulations with $\rho = 0.2$, $\gamma = 0.01$ (b) Simulations with $\rho = -0.2$, $\gamma = 0.01$
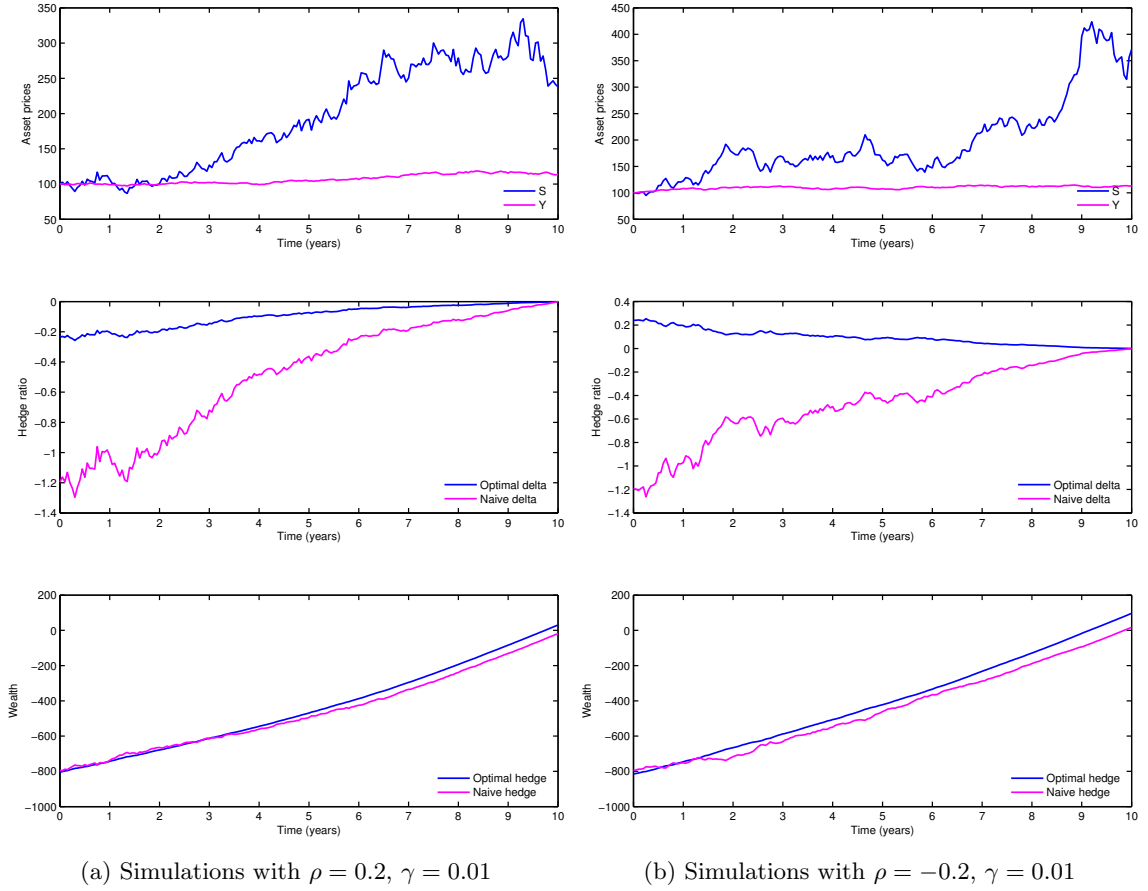
Figure 3.1: Simulations of asset prices (upper graph), hedge ratios (middle graph) and hedge portfolio wealths (lower graph). Input parameters: $S_0 = 100$, $Y_0 = 100$, $r = 0.05$, $\mu = 0.10$, $\mu_Y = 0.01$, $\sigma = 0.2$, $\sigma_Y = 0.03$, and $T = 10$.

might not be able to achieve a terminal wealth of zero (that is, $X_T = 0$), and in those cases we would like our hedge portfolio value to be positive which indicates that we have made a profit from that strategy. We also comment on the range of the hedging errors which is the difference between maximum and minimum of the terminal values of the residual risks from the samples[5].

Figure 3.2(a) illustrates the distribution of the terminal hedging error through histograms produced by the optimal and naive hedging strategies whose results, were conducted for 10 000 simulations, with correlation $\rho = 0.2$ and risk aversion $\gamma = 0.01$. We observe that the naive hedge restricted our terminal hedging error to a minimum of about -44.3 and a maximum of 42.6. However, the hedging strategy is not adequate since it only manages to achieve a median of about 0.19. Moreover, the terminal hedging errors have negative mean and a relatively high standard deviation. By making a comparison with the distributions of the terminal hedging error arising from the optimal hedging strategy in Figure 3.2(a), we notice an improvement in hedge performance. The median of the hedging strategy has raised to about 0.35. Furthermore, the standard deviation of

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[5]Note that this statistic is only useful for comparison when the distributions do not statistically differ significantly and when the mean of the distributions are closely centered at 0. In that case, a large hedging error range would indicate that there would be times when one may make large profits but also large loses.

(a) Hedging Error with $\rho = 0.2$, $\gamma = 0.01$ (b) Hedging Error with $\rho = 0.8$, $\gamma = 0.01$
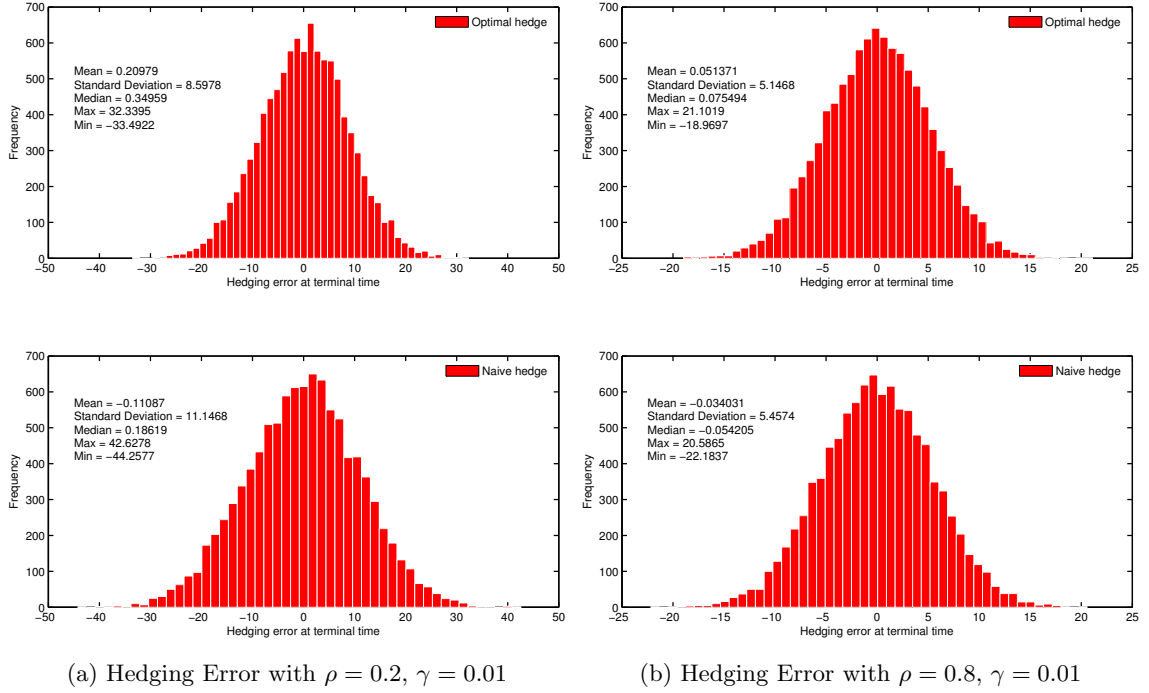
Figure 3.2: Distributions of terminal hedging error for the optimal hedging strategy (upper graph) and the naive strategy (lower graph). Input parameters: $\gamma = 0.01$, $S_0 = 100$, $Y_0 = 100$, $r = 0.05$, $\mu = 0.10$, $\mu_Y = 0.01$, $\sigma = 0.2$, $\sigma_Y = 0.03$, and $T = 10$.

the optimal hedging strategy is 22.9% lower than that of the naive hedge. We also notice that the range of terminal hedging errors has also been reduced for the optimal hedge to a maximum value of 32.3 and minimum values at about -33.5. Finally, this statistics were reported for a low value of correlation which would indicate that the optimal hedge is worthwhile.

Figure 3.2(b) illustrates histograms of the terminal hedging error but for a higher value of the correlation, namely $\rho = 0.8$. We notice that by increasing the correlation from 0.2 to 0.8, the across all statistics for the both strategies becomes closely related statistically. A reason that can be attributed to this observation is that with higher correlation, the nontraded asset behaves more like a traded asset and so their performances are likely to give almost similar results. The median for the optimal hedging strategy is still seen to be higher when compared with the naive strategy but the medians are now closely centered about 0. Also, the standard deviation and minimum value are only slightly higher for the naive strategy. Thus, even if we increase the correlation between the traded and nontraded asset, optimal strategy still serves as an improvement over the naive strategy.

Figures 3.3(a) and 3.3(b) also illustrate the distributions of the terminal hedging error with similar correlations $\rho = 0.2$ and 0.8, but now with a lower risk aversion parameter $\gamma = 0.001$. We still continue to see an improvement of the optimal hedging strategy over the naive strategy. For $\rho = 0.2$ the median for the optimal strategy is about twice higher than that for the naive strategy, and the standard deviation is about 30% higher for the naive strategy. We also notice a reduced in the range of terminal hedging error compared with the naive one. For $\rho = 0.8$, the median hedging error for the optimal strategy is higher than that for the naive strategy, and the standard deviation

(a) Hedging Error with $\rho = 0.2$, $\gamma = 0.001$ (b) Hedging Error with $\rho = 0.8$, $\gamma = 0.001$
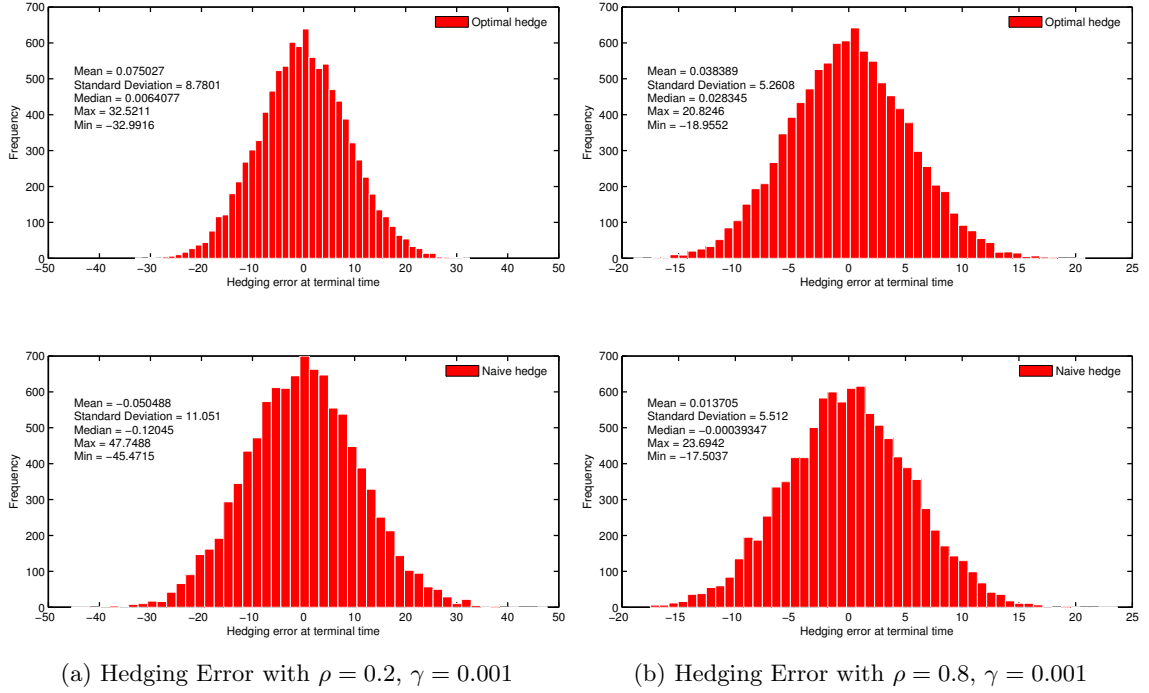
Figure 3.3: Distributions of terminal hedging error for the optimal hedging strategy (upper graph) and the naive strategy (lower graph). Input parameters: $\gamma = 0.001$, $S_0 = 100$, $Y_0 = 100$, $r = 0.05$, $\mu = 0.10$, $\mu_Y = 0.01$, $\sigma = 0.2$, $\sigma_Y = 0.03$, and $T = 10$.

is about 5% higher for the naive strategy whilst the range of terminal hedging error between the two strategies do not differ significantly.

Remark 3.5. For low values of the risk aversion, we can clearly observe that although the optimal hedging strategy performed better, the improvements of the strategy are not statistically significant. This result does not come as a surprise. Indeed, utility functions are designed to build in insurance and so the effect of the utility-based methods have a greater impact on insurance as one increases the risk aversion. In other words, as the risk aversion increases, the investor becomes more conservative and the optimal policies should provide better benefits. On the other hand, if we take the risk aversion to tend to zero, we are effectively going to get the price under the minimal martingale measure[6]. So as we take the risk aversion to be lower, the effect from exponential utility diminishes which corresponds to leaving the unhedgeable risk untouched.

These observations and results indicate that by applying utility-based hedging instead of naive Black-Scholes hedging, the residual risk has been reduced. Through a simple example, we have demonstrated that utility-based hedging methods can be used to hedge claims on the nontradable income stream by using a traded-asset to improve the hedging strategy. If an investor is allowed to take long/short positions, she can hedge her labor income risk by using the available financial assets.

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[6]In fact, it is known that in the limit of zero risk aversion, the exponential indifference pricing corresponds to a *quadratic hedging* criterion and residual risk evolves under the *local risk minimization* (see for example, [**38**]).

CONSUMPTION IN INCOMPLETE MARKET

4.1 MOTIVATION

The formula for indifference price shows that one has to charge/pay for some prices as if one would received the future value of income at terminal time T. The reason is because the objective only measures the utility at time T. From this perspective, the income is equivalent to a terminal payoff problem as we have demonstrated previously. To see differences in the valuation arising due to income, we would need to use some *intertemporal* utility objective (for instance, consumption or possibly wealth). The motivation for investigating this problem is that the results that we have obtained previously might now be different (particularly for the hedging strategy) if one achieves utility from consumption at *intermediate times*.

4.2 PROBLEM FORMULATION

In the classical Merton portfolio selection problem [**27**], an investor invests in the financial market and consumes part of her wealth at each instant up to a horizon T, which can be interpreted as the planned retirement time of the investor. The investor seeks to maximize her expected utility of terminal wealth and integrated utility of consumption which entails searching for an optimal investment strategy π and a consumption rate C such that the maximal utility is attained. The problem can be casts as

$$\mathbb{E}\left[\int_0^T U_1(s, C_s)ds + U_2(X_T^{\pi, C})\right],$$

where $U_i, i = 1, 2$ are utility functions, with U_1 a time-dependent utility. Merton solved the optimal investment/consumption problem under uncertainty by assuming that labor income follows a stochastic process which is *perfectly correlated* to the traded asset. If the investor is allowed to take short positions in the traded asset, then she can hedge the risk associated with her labor income with the available financial assets.

We continue to assume that the investor's preference is modeled through CARA over both consumption and terminal wealth. This time the investor not only chooses to invest in the financial market while receiving the income (which is correlated with the traded asset) but also consumes at a nonnegative rate C_u on a finite time horizon. Similar calculations carried forward from Chapter 1 yield the dynamics of the wealth process:

$$dX_t = rX_t dt + \pi_t \sigma(\lambda dt + dW_t) - C_t dt + I(t, Y_t)dt.$$

Her objective at time t is then given by

$$\mathcal{V}(t,x,y) := \sup_{\pi,C \in \tilde{\mathscr{A}}} \mathbb{E}\left[\int_t^T U_1(u,C_u)du + U_2(X_T^{\pi,C,I})\middle| X_t = x, Y_t = y\right] \tag{4.1}$$

where $\tilde{\mathscr{A}}$ denotes the set of admissible consumption-strategy pairs:

$$\tilde{\mathscr{A}} = \{(\pi,C) : C \in \mathscr{F}_t, C \geq 0, \text{and } \pi \in \mathscr{A}\}.$$

The HJB equation for our problem is given by

$$\mathcal{V}_t(t,x,y) + \sup_{\pi,C \in \tilde{\mathscr{A}}}\left\{\mathscr{L}_{\pi,C}^{X,Y}\mathcal{V}(t,x,y) + U_1(t,C)\right\} = 0, \quad \mathcal{V}(T,x,y) = -\frac{1}{\gamma}e^{-\gamma x}. \tag{4.2}$$

where $\mathscr{L}_{\pi,C}^{X,Y}$ is defined by

$$\mathscr{L}_{\pi,C}^{X,Y}\mathcal{V} := (rx + \sigma\lambda\pi + I - C)\mathcal{V}_x + \frac{1}{2}\sigma^2\pi^2\mathcal{V}_{xx} + \mu_Y y\mathcal{V}_y + \frac{1}{2}\sigma_Y^2 y^2\mathcal{V}_{yy} + \rho\sigma\sigma_Y y\pi\mathcal{V}_{xy}.$$

If we can solve for the value function in the HJB equation above, we can find the utility indifference bid price for the stochastic income.

By performing the maximization over π and C in (4.2), yield the optimal consumption and investment controls

$$C^*(t,x,y) = -\frac{1}{\gamma}\log V_x(t,x,y),$$

$$\pi^*(t,x,y) = -\frac{\lambda}{\sigma}\frac{V_x(t,x,y)}{V_{xx}(t,x,y)} - \frac{\rho\sigma_Y y}{\sigma}\frac{V_{xy}(t,x,y)}{V_{xx}(t,x,y)}.$$

By substituting the optimal control pair (π^*, C^*) in to the HJB equation (4.2) yields

$$V_t + (rx + I(t,y))V_x + \mu_Y yV_y + \frac{1}{2}\sigma_Y^2 y^2 V_{yy} - \frac{1}{2}\frac{(\lambda V_x + \rho\sigma_Y yV_{xy})^2}{V_{xx}} + \frac{1}{\gamma}V_x\log V_x - \frac{1}{\gamma}V_x = 0.$$

At this point, we have to solve the above nonlinear PDE through for example, finite difference method.

In the incomplete market situation, the investment/consumption problem is generally not possible to solve explicitly. The formidable task lies in the fact that it is now difficult to linearize the reduced function f which appears as part of the candidate solution for $\mathcal{V}$ (see for instance (2.6)). Only under certain assumptions imposed to the model would allow the function f to be obtained in closed form. Stoikov and Zariphopoulou [40] studied a similar investment/consumption problem where the drift and volatility of the nontraded asset are driven by the nontraded asset themselves and assuming that the investor has Constant Relative Risk Aversion (CRRA) preference. In doing so, they noted that *"If the market is incomplete, the reduced equation can still be linearized, but only if there is no intermediate consumption and, at the same time, low dimensionality"*. Thus, the inclusion of intermediate consumption further complicates the model making it difficult to derive analytic expressions and as Henderson [18] mentioned, we have to resort to numerical methods to solve the optimal portfolio choice problems that include both incompleteness and consumption.

CHAPTER 5

CONCLUSION

In this dissertation, we have studied the stochastic income model of Henderson [**18**]. With an explicit representation for the indifference price of an uncertain income stream through a nonlinear expectation, we have succeeded in developing a reasonable approximation for the indifference price (and its 'delta') which was close to that generated by Monte Carlo simulation. The formulae for the perturbation expansion were used to demonstrate the effectiveness of utility-based hedging method. By adopting the methodology of Monoyios [**30**], we performed simulations and managed to show the improvements in the utility-based hedge performance compared to the strategy which corresponds to using the traded asset as an alternative for the non-traded income stream. The improvements were assessed through the statistics implied by the distribution of the terminal hedging error which was emphasized by a decrease in standard deviation and an increase in median of the hedging error. However, it should be noted that our approximation is only of the first-order of accuracy. Thus, further efforts to compute the *kurtosis* of the perturbation expansion would be desirable as it would enhanced the accuracy for our numerical results.

Having established a methodology to the pricing problem involving stochastic income, we should check the performance and quality of this type of pricing mechanism against empirical data which would enable us to establish the model risk. Also, we note that one crucial assumption inherent in the analysis we have conducted above, is the assumed dynamics of the assets, particularly for the income, which follows a log-normal distribution. For instance, if we considered a log-normal with mean reversion model for the stochastic income, we might not observe the same behavior that the holdings in stock decreases in absolute value over the investment horizon. Different specifications of the model could lead to different investment strategies which is worthwhile to examine from the utility maximization framework.

As we have pointed out previously, a natural extension of our work would be to include the feature of consumption into the investment problem with income. The introduction of consumption promotes mathematical difficulties, which can solved numerically through finite differences. Work could be done to adopt techniques from Forsyth and Labahn [**26**] where they provided a methodology for solving the single factor optimal control problems where they used the combination of both numerical methods of policy iteration and Crank-Nicolson. It is worth mentioning though that in a recent paper, Rogers and Zaczkowski [**36**] has also provided a method to numerically solve optimal investment problems in incomplete markets by using Monte Carlo simulations coupled with duality methods which could also be applied to our problem.

Further work could involve by using the recently developed *forward utility* of Zariphopoulou [**34**] (which is also known as a *horizon-unbiased utility*, developed independently by Henderson and

Hobson [**19**]) to accomplish the indifference valuation. The shortcomings of the classical utility is that the investor's risk preferences at intermediate times and the optimal investment decisions depend on an a priori chosen investment horizon. Indeed, the classical utility does not capture and relate the risk preferences of an investor to the (random) market environments and her investment decision. The forward utility is an extension from the classical utility in that it attempts to address this issue of the dependence on the horizon time T for problem's solution. It will be interesting to make some comparative analysis of both of these utilities.

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APPENDIX TO CHAPTER 2 & 3

A.1 Derivations for Remark 2.3

We specialize to $n = 1$ and $I(t, y) = y$. Using the expression in (2.10), the optimal hedging strategy in terms of time T money is given by

$$\hat{\pi}^H(t, y) \equiv e^{r(T-t)} \pi^H(t, y) = \frac{\rho \sigma_Y y}{\sigma \gamma (1 - \rho^2)} \frac{F_y(t, y)}{F(t, y)}.$$

Differentiating with respect to t yields

$$\frac{\partial \hat{\pi}^H}{\partial t} = \frac{\rho \sigma_Y y}{\sigma \gamma (1 - \rho^2)} \left(F_y \frac{\partial}{\partial t} \frac{1}{F} + \frac{1}{F} \frac{\partial}{\partial t} F_y \right).$$

Note that the partial derivative of F with respect to y is given by

$$F_y(t, y) = \mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}} \left[-(1 - \rho^2) \gamma \int_t^T e^{r(T-u)} \frac{Y_u}{y} du \cdot \exp \left\{ -\gamma (1 - \rho^2) \int_t^T e^{r(T-u)} Y_u du \right\} \right].$$

Based on this expression, we can easily deduce the observation that the optimal hedging strategy decreases in magnitude over time to zero at time T. Computing the partial derivatives of F and F_y with respect to t, yields

$$\frac{\partial}{\partial t} \frac{1}{F} = -\gamma (1 - \rho^2) e^{r(T-t)} y \frac{1}{F},$$

$$\frac{\partial}{\partial t} F_y = \gamma (1 - \rho^2) e^{r(T-t)} F + \gamma (1 - \rho^2) e^{r(T-t)} y F_y.$$

Hence, we have

$$\frac{\partial \hat{\pi}^H}{\partial t} = \frac{\rho \sigma_Y y}{\sigma} e^{r(T-t)},$$

whose sign only depends on the sign of ρ. We further differentiate with respect to t to get

$$\frac{\partial^2 \hat{\pi}^H}{\partial t^2} = -\frac{r \rho \sigma_Y y}{\sigma} e^{r(T-t)}.$$

which is negative if $\rho > 0$ and positive if $\rho < 0$. Thus, this establishes concavity and convexity.

A.2 Derivation of the Indifference Price PDE

From equation (2.3), we invoke Feynmac-Kac theorem to derive a PDE for the stochastic representation F:

$$F_t(t, Y_t) + \mathscr{L}^{\tilde{\mathbb{Q}}, Y} F(t, Y_t) - n\gamma (1 - \rho^2) e^{r(T-t)} I(t, Y_t) F(t, Y_t) = 0.$$

Rewrite equation (2.4) in terms of the indifference price, p, to get

$$F(t, y) = \exp\{-n\gamma(1 - \rho^2)e^{r(T-t)}p(t, y)\}$$

By computing the derivatives of F, we can arrive at the (nonlinear) PDE from the indifference price:

$$p_t + (\mu_Y y - \rho\sigma_Y \lambda y)\, p_y + \frac{1}{2}\sigma_Y^2 y^2 p_{yy} - rp - \frac{1}{2}\sigma_Y^2 y^2 n\gamma(1 - \rho^2)e^{r(T-t)}p_y^2 + nI = 0, \qquad \text{(A.1)}$$

with terminal condition $p(T, y) = 0$. The PDE above can be written more compactly as follows

$$p_t - \rho\sigma_Y \lambda y p_y + \mathscr{L}^Y p - rp - \frac{1}{2}\sigma_Y^2 y^2 n\gamma(1 - \rho^2)e^{r(T-t)}p_y^2 + nI = 0. \qquad \text{(A.2)}$$

where $\mathscr{L}^Y$ is the infinitesimal generator of the diffusion process $(Y_t)_{t \geq 0}$ under the measure $\mathbb{P}$:

$$\mathscr{L}^Y p(t, y) = \mu_Y y p_y(t, y) + \frac{1}{2}\sigma_Y^2 y^2 p_{yy}(t, y).$$

A.3 Derivation of the Residual Risk Process

Recall that under the measure $\mathbb{P}$, the process Y follows

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t(\rho dW + \sqrt{1 - \rho^2}dW^\perp).$$

By substituting the optimal hedging strategy π^H in (2.7) into the value of the hedging portfolio in S, we have

$$\begin{aligned} dX_t^H &= rX_t^H dt + \pi_t^H \sigma(\lambda dt + dW_t) + nI(t, Y_t)dt \\ &= rX_t^H dt - n\rho\sigma_Y Y_t p_y(t, Y_t)(\lambda dt + dW_t) + nI(t, Y_t). \end{aligned}$$

Apply Itô's formula to the indifference price process $p(t, Y_t)$ to get

$$\begin{aligned} dp(t, Y_t) &= (p_t(t, Y_t) + \mu_Y(t, Y_t)p_y(t, Y_t) + \frac{1}{2}\sigma_Y^2 Y_t^2 p_{yy}(t, Y_t)dt + \sigma_Y Y_t p_y(t, Y_t)dW_t^Y \\ &= (p_t(t, Y_t) + \mathscr{L}^Y p(t, Y_t))dt + \sigma_Y(t, Y_t)p_y(t, Y_t)(\rho dW_t + \sqrt{1 - \rho^2}dW_t^\perp). \end{aligned}$$

Putting this together, we have the residual risk associated with the optimal hedging strategy:

$$\begin{aligned} dR_t &= dX_t^H + ndp(t, Y_t) \\ &= rX_t^H dt - n\rho\sigma_Y Y_t p_y(t, Y_t)\,(\lambda dt + dW_t) - n(p_t(t, Y_t) + \mathscr{L}^Y p(t, Y_t))dt \\ &\quad + n\sigma_Y Y_t p_y(t, Y_t)(\rho dW_t + \sqrt{1 - \rho^2}dW_t^\perp) + nI(t, Y_t)dt \\ &= rR_t dt + \frac{1}{2}\sigma_Y^2 n^2 \gamma(1 - \rho^2)e^{r(T-t)}Y_t^2 p_y(t, Y_t)^2 dt + n\sigma_Y \sqrt{1 - \rho^2}Y_t p_y(t, Y_t)dW_t^\perp - n(n-1)I, \end{aligned}$$

with $R_0 = 0$ by definition. For the case where $n = 1$, we have

$$dR_t = rR_t dt + \frac{1}{2}\sigma_Y^2 n^2 \gamma(1 - \rho^2)e^{r(T-t)}Y_t^2 p_y(t, Y_t)^2 dt + n\sigma_Y \sqrt{1 - \rho^2}Y_t p_y(t, Y_t)dW_t^\perp.$$

A.4   Derivation of the Naive Residual Risk Process

Let $\rho \to 1$ in the indifference price PDE (A.2) to get

$$p_t - \sigma_Y \lambda y p_y + \mathscr{L}^Y p - rp + nI = 0. \tag{A.3}$$

We substitute the optimal hedging strategy π^H for $\rho = 1$ into wealth process to obtain:

$$dX_t^N = rX_t^N dt - n\sigma_Y Y_t p_y(t, Y_t)(\lambda dt + dW_t) + nI(t, Y_t)dt. \tag{A.4}$$

With equation (A.3) and (A.4), we can derive the residual process associated with the naive strategy

$$
\begin{aligned}
dR_t^N &= dX_t^N + ndp(t, Y_t) \\
&= rX_t^N dt - n\sigma_Y Y_t p_y(t, Y_t)(\lambda dt + dW_t) + nI(t, Y_t)dt \\
&\quad + n(p_t(t, Y_t) + \mathscr{L}^Y p(t, Y_t))dt + n\sigma_Y Y_t p_y(t, Y_t)(\rho dW_t + \sqrt{1 - \rho^2} dW_t^\perp) \\
&= rR_t^N dt + n\sigma_Y Y_t p_y(t, Y_t)[(\rho - 1)dW_t + \sqrt{1 - \rho^2} dW_t^\perp] - n(n-1)I.
\end{aligned}
$$

with $R_0^N = 0$ by definition. For the case where $n = 1$, we have

$$dR_t^N = rR_t^N dt + n\sigma_Y Y_t p_y(t, Y_t)[(\rho - 1)dW_t + \sqrt{1 - \rho^2} dW_t^\perp].$$

A.5   Asymptotic Expansion Derivation

Let $\beta_u := e^{r(T-u)}$. We generate an analytic expansion for the function F by expanding (2.3) in powers of $\varepsilon := -\gamma(1 - \rho^2)$, to obtain

$$
\begin{aligned}
F(t, y) &= \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\exp\left\{\varepsilon \int_t^T \beta_u Y_u du\right\} \middle| Y_t = y\right] \\
&= \mathbb{E}^{\tilde{\mathbb{Q}}}\left[1 + \varepsilon \int_t^T \beta_u Y_u du + \frac{1}{2}\varepsilon^2 \left(\int_t^T \beta_u Y_u du\right)^2 \middle| Y_t = y\right] + \mathcal{O}(\varepsilon^3) \\
&= 1 + \varepsilon \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\int_t^T \beta_u Y_u du \middle| Y_t = y\right] + \frac{1}{2}\varepsilon^2 \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\left(\int_t^T \beta_u Y_u du\right)^2 \middle| Y_t = y\right] + \mathcal{O}(\varepsilon^3).
\end{aligned}
$$

This implies

$$
\begin{aligned}
p(t, y) &= \frac{1}{\epsilon \beta_t} \log\left\{1 + \varepsilon \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\int_t^T \beta_u Y_u du \middle| Y_t = y\right] + \frac{1}{2}\varepsilon^2 \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\left(\int_t^T \beta_u Y_u du\right)^2 \middle| Y_t = y\right] + \mathcal{O}(\varepsilon^3)\right\} \\
&= \frac{1}{\beta_t}\left(\mathbb{E}_{t,y}^{\tilde{\mathbb{Q}}}\left[\int_t^T \beta_u Y_u du\right] + \frac{1}{2}\varepsilon \mathrm{Var}_{t,y}^{\tilde{\mathbb{Q}}}\left[\int_t^T \beta_u Y_u du\right] + \mathcal{O}(\varepsilon^2)\right).
\end{aligned}
$$

MATLAB Code

B.1   Accuracy of Perturbation Expansion

```matlab
1  %% Simulated indifference price
2  clear all
3  format long
4
5  % Parameters
6  y0 = 100;
7  r = 0.05;
8  mu = 0.1;
9  sigma = 0.25;
10 rho = 0.75; % Vary rho for different values
11 lambda = (mu-r)/sigma; % market price of risk (for stock)
12 muY = 0.12; % drift of income
13 sigmaY = 0.3; % volatility of income
14
15 nu = muY-rho*sigmaY*lambda; % drift under MMM
16 gamma = 0.001; % risk aversion
17 epsilon = -gamma*(1-rho^2);
18
19 N = 200; % no of time steps
20 M = 10000; % no of paths
21 T = 1; % maturity
22 dt = T/N; % time step
23
24 price = zeros(15,1);
25
26 for k=1:15
27     % Implement Euler Scheme
28     y = zeros(N,M);
29     y(1,:) = y0; % set initial value
30     w = randn(N,M); % generate M random variables of w
31     temp = zeros(N,M);
32     temp(1,:) = exp(r*T*(1-0))*y(1,:)*dt;
33
34     for j=1:N-1
35         y(j+1,:) = max(y(j,:),0) + nu*max(y(j,:),0)*dt + sigmaY*max(y(j,:),0)*sqrt(dt).*w(j,:);
36         temp(j+1,:) = exp(r*T*(1-j/(N-1)))*y(j+1,:)*dt;
37     end
38
39     integral = zeros(1,M);
40
41     for i=1:M
42         integral(i) = sum(temp(:,i));
43     end
44
45     mean = sum(exp(epsilon*integral))/M; % for accuracy of parameter/ condition check
46     check    = mean > 0                                      ; %Condition check
```

```matlab
47      if        check == 0
48             disp('Warning: Condition Violated!!')
49      end
50
51      check2      = mean < 2                                              ; %Condition check
52      if        check2 == 0
53             disp('Warning: Condition Violated!!')
54      end
55
56      % Investigate MC-error
57      square = sum(exp(epsilon*integral).^2)/M;
58      stddev = sqrt(square-mean^2)/sqrt(M-1);
59
60      price(k) = 1/(epsilon*exp(r*T*(1-0)))*log(mean);
61      %mean;
62      %price;
63  end
64
65  price
66  pricemean = sum(price)/15;
67  squareprice = sum(price.^2)/15;
68  stddev2 = sqrt(squareprice - pricemean^2)/sqrt(15-1)*sqrt(15);
69  pricemean;
70
71  % Indifference Price when correlation is perfect (rho=1)
72  syms x;
73
74  delta = mu-sigmaY*lambda - r;
75  price2 = y0*(exp(delta*(T-0))-1)/delta;
76
77  price2/exp(r*(T-0));
78
79  % 1st-order approximation
80  expectation = y0*exp(r*(T-0))*(exp((nu-r)*(T-0))-1)/(nu-r);
81
82  price3 = expectation/exp(r*(T-0))
83
84  % 2nd-order approximation
85  term1 = (exp((2*nu+sigmaY^2)*(T-0)) - exp(2*r*(T-0)))/((nu - r)*(2*nu + sigmaY^2 - 2*r));
86  term2 = (exp((2*nu+sigmaY^2)*(T-0))- exp((nu + r)*(T-0)))/((nu - r)*(nu + sigmaY^2 - r));
87
88  variance = 2*y0^2*(-term1 + term2) - expectation*expectation;
89  price4 = (expectation + 0.5*epsilon*variance)/exp(r*(T-0))
90
91  %% Simulated derivative of indifference price
92
93  N = 1000; % no of time steps
94  M = 10000; % no of paths
95  %T = 1; % maturity
96  dt = T/N; % time step
97
98  % Implement Euler Scheme
99  y = zeros(N,M);
100 ydelta = zeros(N,M);
101 y(1,:) = 100; % set initial value
102 ydelta(1,:) = 100 + 0.001; % % set initial value; change in y
103 w = randn(N,M); % generate M random variables of w
104 w2 = randn(N,M);
105 temp = zeros(N,M);
106 temp2 = zeros(N,M);
107 temp(1,:) = exp(r*(1-0))*y(1,:)*dt;
108 temp2(1,:) = exp(r*(1-0))*ydelta(1,:)*dt;
109
110 for j=1:N-1
111     y(j+1,:) = max(y(j,:),0) + nu*max(y(j,:),0)*dt + sigmaY*max(y(j,:),0)*sqrt(dt).*w(j,:);
```

```matlab
112        ydelta(j+1,:) = max(ydelta(j,:),0) + nu*max(ydelta(j,:),0)*dt + sigmaY*max(ydelta(j,:),0)*sqrt(dt)
               .*w(j,:);
113        temp(j+1,:) = exp(r*T*(1-j/(N-1)))*y(j+1,:)*dt;
114        temp2(j+1,:) = exp(r*T*(1-j/(N-1)))*ydelta(j+1,:)*dt;
115 end
116
117 integral = zeros(1,M);
118 integral2 = zeros(1,M);
119 for j=1:M
120        integral(j) = sum(temp(:,j));
121        integral2(j) = sum(temp2(:,j));
122 end
123
124 mean = sum(exp(epsilon*integral))/M; % for accuracy of parameter
125 mean2 = sum(exp(epsilon*integral2))/M; % for accuracy of parameter
126 price = 1/(epsilon*exp(r*T*(1-0)))*log(mean);
127 price2 = 1/(epsilon*exp(r*T*(1-0)))*log(mean2);
128
129 mean;
130 deriv = (price2-price)/0.001
131
132 % 1st-order approximation (for derivative)
133 expectation_deriv = exp(r*T*(1-0))*(exp((nu-r)*T*(1-0))-1)/(nu-r);
134
135 price_deriv3 = expectation_deriv/exp(r*T*(1-0))
136
137
138 % 2nd-order approximation (for derivative)
139 term1 = (exp((2*nu+sigmaY^2)*T*(1-0)) - exp(2*r*T*(1-0)))/((nu - r)*(2*nu + sigmaY^2 - 2*r));
140 term2 = (exp((2*nu+sigmaY^2)*T*(1-0)) - exp((nu + r)*T*(1-0)))/((nu - r)*(nu + sigmaY^2 - r));
141
142 variance_deriv = 4*y0*(-term1 + term2) - 2*expectation_deriv*expectation;
143 price_deriv4 = (expectation_deriv + 0.5*epsilon*variance_deriv) /exp(r*T*(1-0))
```

Accuracy.m

B.2   SIMULATED PATHS

```matlab
1 %% Simulations for asset prices
2
3 % Model parameters
4 rho = 0.2;
5
6 N = 200; % number of time steps
7 T = 10; % terminal time
8 mu = 0.1; % drift of traded asset
9 muY = 0.01; % drift of non-traded asset
10 sigma = 0.2; % volatility of traded asset
11 sigmaY = 0.03; % volatility of non-traded asset
12 r = 0.05; % interest rate
13 S0 = 100; % initial traded asset price
14 Y0 = 100; % initial non-traded asset price
15
16 t = (0:1:N)'/N;                 % t is the column vector [0 1/N 2/N ... 1]
17 W1 = [0; cumsum(randn(N,1))]/sqrt(N); % S is running sum of N(0,1/N) variables
18 W2 = [0; cumsum(randn(N,1))]/sqrt(N); % S is running sum of N(0,1/N) variables
19 t = t*T;
20 W1 = W1*sqrt(T); % create brownian motion
21 W2 = W2*sqrt(T); % create orthogonal brownian motion
22
23 S = S0*exp((mu-(sigma^2)/2)*t + sigma*W1);
24 Y = Y0*exp((muY-(sigmaY^2)/2)*t + sigmaY*(rho*W1 + sqrt(1-rho^2)*W2));
```

```matlab
25
26
27  % Generate Hedge Ratio processes
28
29  % Further parameter
30  lambda = (mu-r)/sigma; % market price of risk (for stock)
31  nu = muY - rho*sigmaY*lambda; % drift under MMM
32  nuNaive = muY - sigmaY*lambda; % drift under MMM (rho=1)
33  gamma = 0.01; % risk aversion
34  epsilon = -gamma*(1-rho^2);
35
36  % 0th-order approximation
37  expectation = Y.*exp(r*(T-t)).*(exp((nu-r)*(T-t))-1)/(nu-r);
38  price3 = expectation./exp(r*(T-t)); % indifference price approximation (up to 0th order)
39  expectationNaive = Y.*exp(r*(T-t)).*(exp((nuNaive-r)*(T-t))-1)/(nuNaive-r);
40  price3Naive = expectationNaive./exp(r*(T-t));% BS price, NAIVE
41
42  % 1st-order approximation
43  term1 = (exp((2*nu+sigmaY^2)*(T-t)) - exp(2*r*(T-t)))/((nu-r)*(2*nu + sigmaY^2 - 2*r));
44  term2 = (exp((2*nu+sigmaY^2)*(T-t)) - exp((nu+r)*(T-t)))/((nu-r)*(nu + sigmaY^2 - r));
45  variance = 2*Y.^2.*(-term1 + term2) - expectation.*expectation;
46  price4 = (expectation + 0.5*epsilon*variance)./exp(r*(T-t)); % indifference price approximation (up to
        1st order)
47
48  % 0th-order approximation (for derivative)
49  expectation_deriv = exp(r*(T-t)).*(exp((nu-r)*(T-t))-1)/(nu-r);
50  expectation_derivNaive = exp(r*(T-t)).*(exp((nuNaive-r)*(T-t))-1)/(nuNaive-r);
51  price_deriv3 = expectation_deriv./exp(r*(T-t)); % delta of the claim (up to 0th order)
52  price_deriv3Naive = expectation_derivNaive./exp(r*(T-t)); % BS-delta of the claim (up to 0th order,
        NAIVE)
53
54  % 1st-order approximation (for derivative)
55  variance_deriv = 4*Y.*(-term1 + term2) - 2*expectation_deriv.*expectation; % variance
56  price_deriv4 = (expectation_deriv + 0.5*epsilon*variance_deriv)./exp(r*(T-t)); % delta of the claim (up
         to 1st order)
57
58  % delta of stock
59  delta = -rho*sigmaY*Y.*price_deriv4./(sigma*S);
60  deltaNaive = -sigmaY*Y.*price_deriv3Naive./(sigma*S);
61
62
63  % Generates outcomes of solutions of the SDE dx(t)=rx(t)dt-delta(t)*S(t)*[(mu-r)dt-sigma*dW(t)]+Y(t)dt
64
65  dt=T/N; % time step
66  t=(0:dt:T); % t is the vector [0 1dt 2dt 3dt ... Ndt]
67
68  % place to store locations
69  X = zeros(size(t));
70  XN = zeros(size(t));
71
72  X(1) = -price4(1); % initial hedge portfolio value
73  XN(1) = -price3Naive(1); % initial hedge portfolio value (NAIVE)
74
75  for i=1:N          % take N steps
76      X(i+1) = X(i) + r*X(i)*dt + delta(i)*S(i)*((mu-r)*dt + sigma*(W1(i+1) - W1(i))) + Y(i)*dt;
77      XN(i+1) = XN(i) + r*XN(i)*dt + deltaNaive(i)*S(i)*((mu-r)*dt + sigma*(W1(i+1) - W1(i))) + Y(i)*dt;
78  end;
79
80  F = exp(epsilon*exp(r*(T-t)).*price4')'  ;
81
82  %% PLOTS
83
84  subplot(3,1,1);
85  plot(t,S,'linewidth',1.5);
86  hold on
```

```
 87  subplot(3,1,1);
 88  plot(t,Y,'linewidth',1.5,'Color','m');
 89  xlabel('Time (years)')
 90  ylabel('Asset prices')
 91  hold off
 92
 93  h = legend('show','S','Y','location','SouthEast');
 94  legend boxoff
 95  set(gcf, 'PaperPositionMode', 'auto');
 96  set(gca,'linewidth',1.1)
 97  set(gcf, 'Units', 'centimeters');
 98  afFigurePosition = [10 10 20 30.5]; % [pos_x pos_y width_x width_y]
 99  set(gcf, 'Position', afFigurePosition); % [left bottom width height]
100
101
102  subplot(3,1,2);
103  plot(t,delta,'linewidth',1.5);
104  hold on
105  subplot(3,1,2);
106  plot(t,deltaNaive,'linewidth',1.5,'Color','m');
107  %plot(t,exp(r*t),':');
108  xlabel('Time (years)')
109  ylabel('Hedge ratio')
110  hold off
111
112  h = legend('show','Optimal delta','Naive delta','location','SouthEast');
113  legend boxoff
114  set(gcf, 'PaperPositionMode', 'auto');
115  set(gca,'linewidth',1.1)
116  set(gcf, 'Units', 'centimeters');
117  afFigurePosition = [10 10 20 30.5]; % [pos_x pos_y width_x width_y]
118  set(gcf, 'Position', afFigurePosition); % [left bottom width height]
119
120  subplot(3,1,3);
121  plot(t,X,'linewidth',1.5);
122  hold on
123  subplot(3,1,3);
124  plot(t,XN,'linewidth',1.5,'Color','m');
125  xlabel('Time (years)')
126  ylabel('Wealth')
127  hold off
128
129  h = legend('show','Optimal hedge','Naive hedge','location','SouthEast');
130  legend boxoff
131  set(gcf, 'PaperPositionMode', 'auto');
132  set(gca,'linewidth',1.1)
133  set(gcf, 'Units', 'centimeters');
134  afFigurePosition = [10 10 20 30.5]; % [pos_x pos_y width_x width_y]
135  set(gcf, 'Position', afFigurePosition); % [left bottom width height]
136
137
138  print -depsc2 figure2
```

Simulated_Processes.m

B.3   Hedging Error Distributions

```
1  %% Generate the Residual Risk Process
2  clear
3  format short
4  rho = 0.2;
5
```

```matlab
6  N = 200; % number of time steps
7  T = 10; % terminal time
8  mu = 0.1; % drift of traded asset
9  muY = 0.01; % drift of non-traded asset
10 sigma = 0.2; % volatility of traded asset
11 sigmaY = 0.03; % volatility of non-traded asset
12 r = 0.05; % interest rate
13
14
15 lambda = (mu-r)/sigma; % market price of risk (for stock)
16 nu = muY - rho*sigmaY*lambda; % drift under MMM
17 nuNaive = muY - sigmaY*lambda; % drift under MMM (rho=1)
18 gamma = 0.01; % risk aversion
19 epsilon = -gamma*(1-rho^2);
20
21 dt = T/N;
22
23 M = 40000; % number of sample paths
24
25 W1 = randn(N,M);
26 W2 = randn(N,M);
27 S = zeros(N+1,M);
28 Y = zeros(N+1,M);
29
30 S(1,:) = 100; % set initial traded asset price
31 Y(1,:) = 100; % set initial non-traded asset price
32
33 expectation = zeros(N+1,M);
34 price3 = zeros(N+1,M);
35 expectationNaive = zeros(N+1,M);
36 price3Naive = zeros(N+1,M);
37 term1= zeros(N+1,M);
38 term2= zeros(N+1,M);
39 variance = zeros(N+1,M);
40 price4 = zeros(N+1,M);
41 expectation_deriv = zeros(N+1,M);
42 expectation_derivNaive = zeros(N+1,M);
43 price_deriv3 = zeros(N+1,M);
44 price_deriv3Naive = zeros(N+1,M);
45 variance_deriv = zeros(N+1,M);
46 price_deriv4 = zeros(N+1,M);
47
48 for j=1:N
49     S(j+1,:) = mu*S(j,:)*dt + sigma*S(j,:)*sqrt(dt).*W1(j,:);
50     Y(j+1,:) = muY*Y(j,:)*dt + sigmaY*Y(j,:)*sqrt(dt).*(rho*W1(j,:) + sqrt(1-rho^2)*W2(j,:));
51 end
52
53 for k=1:N+1
54     % 0th-order approximation
55     expectation(k,:) = Y(k,:)*exp(r*T*(1-(k-1)/N))*(exp((nu-r)*T*(1-(k-1)/N))-1)/(nu-r);
56     price3(k,:) = expectation(k,:)/exp(r*T*(1-(k-1)/N)); % indifference price approximation (up to 0th
           order)
57     expectationNaive(k,:) = Y(k,:)*exp(r*T*(1-(k-1)/N))*(exp((nuNaive-r)*T*(1-(k-1)/N))-1)/(nuNaive-r);
58     price3Naive(k,:) = expectationNaive(k,:)/exp(r*T*(1-(k-1)/N));% BS price, NAIVE
59
60     % 1st-order approximation
61     term1(k,:) = (exp((2*nu+sigmaY^2)*T*(1-(k-1)/N)) - exp(2*r*T*(1-(k-1)/N)))/((nu-r)*(2*nu + sigmaY^2
           - 2*r));
62     term2(k,:) = (exp((2*nu+sigmaY^2)*T*(1-(k-1)/N)) - exp((nu+r)*T*(1-(k-1)/N)))/((nu-r)*(nu + sigmaY
           ^2 - r));
63     variance(k,:) = 2*Y(k,:).^2.*(-term1(k,:) + term2(k,:)) - expectation(k,:).*expectation(k,:);
64     price4(k,:) = (expectation(k,:) + 0.5*epsilon*variance(k,:))/exp(r*T*(1-(k-1)/N)); % indifference
           price approximation (up to 1st order)
65
66     % 0th-order approximation (for derivative)
```

```matlab
67        expectation_deriv(k,:) = exp(r*T*(1-(k-1)/N))*(exp((nu-r)*T*(1-(k-1)/N))-1)/(nu-r);
68        expectation_derivNaive(k,:) = exp(r*T*(1-(k-1)/N))*(exp((nuNaive-r)*T*(1-(k-1)/N))-1)/(nuNaive-r);
69        price_deriv3(k,:) = expectation_deriv(k,:)/exp(r*T*(1-(k-1)/N)); % delta of the claim (up to 0th
              order)
70        price_deriv3Naive(k,:) = expectation_derivNaive(k,:)/exp(r*T*(1-(k-1)/N)); % BS-delta of the claim
              (up to 0th order, NAIVE)
71
72        % 1st-order approximation (for derivative)
73        variance_deriv(k,:) = 4*Y(k,:).*(-term1(k,:) + term2(k,:)) - 2*expectation_deriv(k,:).*expectation(
              k,:); % variance
74        price_deriv4(k,:) = (expectation_deriv(k,:) + 0.5*epsilon*variance_deriv(k,:))/exp(r*T*(1-(k-1)/N))
              ; % delta of the claim (up to 1st order)
75 end
76
77 % place to store locations
78 R = zeros(N+1,M);
79 RN = zeros(N+1,M);
80 term3 = zeros(N,M);
81 term4 = zeros(N,M);
82 term5 = zeros(N,M);
83
84 R(1,:) = 0; % initial hedge portfolio value
85 RN(1,:) = 0; % initial hedge portfolio value (NAIVE)
86
87 for i=1:N            % take N steps
88     term3(i,:) = -0.5*epsilon*exp(r*T*(1-k/N))*sigmaY^2*Y(i,:).^2.*price_deriv4(i,:).^2*dt;
89     term4(i,:) = sigmaY*sqrt(1-rho^2)*Y(i,:).*price_deriv4(i,:)*sqrt(dt).*W2(i,:);
90     term5(i,:) = sigmaY*Y(i,:).*price_deriv3Naive(i,:).*((rho-1)*sqrt(dt).*W1(i,:) + sqrt(1-rho^2)*sqrt
              (dt)*W2(i,:));
91     R(i+1,:) = R(i,:) + r*R(i,:)*dt + term3(i,:) + term4(i,:);
92     RN(i+1,:)= RN(i,:) + r*RN(i,:)*dt + term5(i,:);
93 end;
94
95 counter1 = 0;
96 counter2 = 0;
97 for m=1:M
98    if (R(N+1,m)>=0)
99         counter1 = counter1 + 1;
100    end
101
102    if (RN(N+1,m)>=0)
103        counter2 = counter2 + 1;
104    end
105 end
106
107 meanR = mean(R(N+1,:));
108 stddevR = std(R(N+1,:));
109 medianR = median(R(N+1,:));
110 maxR = max(R(N+1,:));
111 minR = min(R(N+1,:));
112 rangeR = maxR - minR;
113 %probR = counter1/M;
114
115 meanRN = mean(RN(N+1,:));
116 stddevRN = std(RN(N+1,:));
117 medianRN = median(RN(N+1,:));
118 maxRN = max(RN(N+1,:));
119 minRN = min(RN(N+1,:));
120 rangeRN = maxRN - minRN;
121 %probRN = counter2/M;
122
123 Analysis = [meanR meanRN;stddevR stddevRN;medianR medianRN;maxR maxRN;minR minRN; rangeR rangeRN];
124 csvwrite('analysis.csv',Analysis); % saves the error results
125
126 %%
```

```
127
128 subplot(2,1,1)
129 hist(R(N+1,:)',50);
130 axis([-50 50 0 700])
131 xlabel('Hedging error at terminal time')
132 ylabel('Frequency')
133 h = findobj(gca,'Type','patch');
134 set(h,'FaceColor',[1 0 0],'EdgeColor','w');
135 h = legend('show','Optimal hedge');
136 legend boxoff
137 set(gca,'linewidth',1.1)
138 str = {['Mean = ' num2str(meanR)] ['Standard Deviation = ' num2str(stddevR)] ...
139     ['Median = ' num2str(medianR)] ['Max = ' num2str(maxR)] ['Min = ' num2str(minR)]};
140
141 annotation('textbox',[.16 .75 .1 .1],'EdgeColor','none','String',str);
142
143 subplot(2,1,2)
144 hist(RN(N+1,:)',50);
145 %axis([-2500 2000 0 700])
146 xlabel('Hedging error at terminal time')
147 ylabel('Frequency')
148 h = findobj(gca,'Type','patch');
149 set(h,'FaceColor',[1 0 0],'EdgeColor','w');
150 h = legend('show','Naive hedge');
151 legend boxoff
152 set(gcf, 'PaperPositionMode', 'auto');
153 set(gca,'linewidth',1.1)
154 str2 = {['Mean = ' num2str(meanRN)] ['Standard Deviation = ' num2str(stddevRN)] ...
155     ['Median = ' num2str(medianRN)] ['Max = ' num2str(maxRN)] ['Min = ' num2str(minRN)]};
156 annotation('textbox',[.16 .3 .1 .1],'EdgeColor','none','String',str2);
157 set(gcf, 'Units', 'centimeters');
158 afFigurePosition = [10 200 20 30.5]; % [pos_x pos_y width_x width_y]
159 set(gcf, 'Position', afFigurePosition); % [left bottom width height]
160
161
162 print -depsc2 figure5
```

Residual_Risk.m