Cyclical Monotonicity and Kantorovich Problem

📅Date: 01-11-2023
🕐Time: 14:21

Bipin Koirala

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Continuous Problem in Higher Dimension

Similar to Monge's Formulation in 1-D, in $\mathbb{R}^N$ we have;

Goal: Find $T(x)$ for

$$\min \frac{1}{2} \int_{\mathbb{R}} (x - T(x))^2 \, f(x) dx \quad \text{s.t} \quad \int_{T^{-1}(A)} f(x) dx = \int_A g(y) \, dy \ \forall \ A \subset \mathbb{R}^N$$

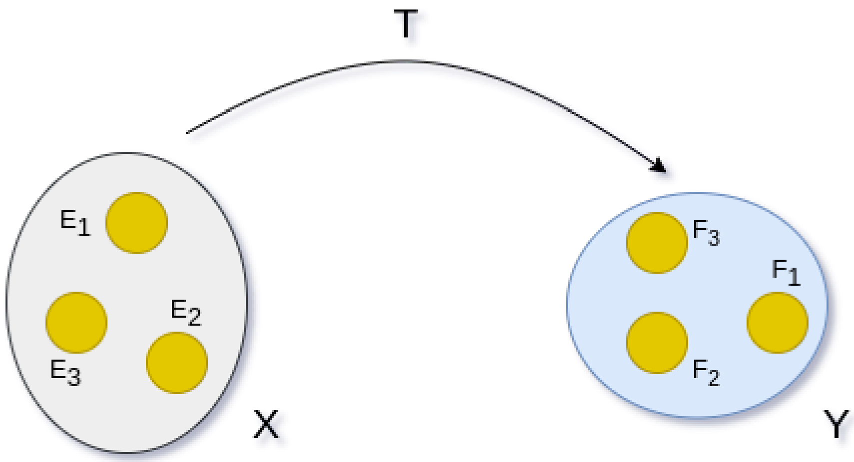
with the change of variables; $y = T(x)$:

$$\int_{T^{-1}(A)} f(x) dx = \int_{T^{-1}(A)} g(T(x)) \left|\nabla T(x)\right| dx$$

Here; $|\nabla T(x)|$ is the determinant of the #Jacobian .

$$\therefore f(x) = g(T(x)) \left|\nabla T(x)\right|$$

At this stage, assume we have an optimal map T already. Choose, some $x_1, x_2, \ldots, x_N \in X$ and let $y_i = T(x_i)$. Let, E_i be a ball centered at x_i s.t. $\int_{E_i} f(x) \, dx = \epsilon$ and let $F_i = T(E_i)$. Below is a visualization of this scenario.



Let's create a new map, $\widetilde{T}(x)$ that is measure-preserving and we want the following:

$$\widetilde{T}(x_i) = y_{i+1}$$
$$\widetilde{T}(E_i) = F_{i+1}$$
$$\widetilde{T}(x) = T(x) \text{ if } x \notin \cup_{i=1}^{N} E_i$$

Note: the above transformation is cyclic i.e. $E_1 \to F_2$ and $E_3 \to F_1$.

Now, if T is optimal, we have:

$$\frac{1}{2}\int_{\mathbb{R}} (x - T(x))^2 f(x)dx \le \frac{1}{2}\int_{\mathbb{R}} (x - \widetilde{T}(x))^2 f(x)dx$$
$$\Rightarrow \frac{1}{\epsilon}\sum_{i=1}^{N}\int_{E_i} x(\widetilde{T}(x) - T(x))\, f(x)\, dx \le 0$$

Above expression is similar to 1-D case.
As $\epsilon \to 0$: $x \to x_i, \widetilde{T}(x_i) \to y_{i+1}, T(x) \to y_i$ and integrating $f(x)$ over E_i gives ϵ, so

$$\sum_{i=1}^{N} x_i(y_{i+1} - y_i) \le 0$$

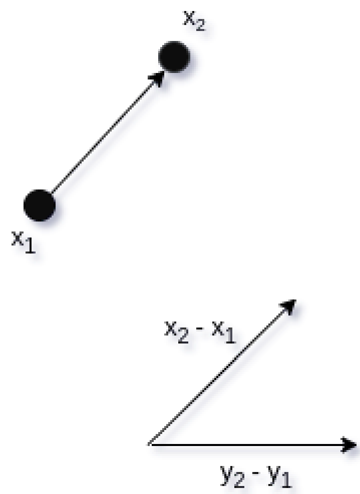
This condition is called #Cyclical-Monotonicity.

Example:

When $N = 2$, from #Cyclical-Monotonicity we have:

$$x_1(y_2 - y_1) + x_2(y_1 - y_2) \le 0$$
$$(x_2 - x_1) \cdot (y_2 - y_1) \ge 0$$

In other words, upon transformation the vectors when super-imposed, make an acute angle.



This condition holds true for all N and restricts our ability to 'twist' mass. This is a stronger condition than being irrotational.

Gradient of Convex Function

> 💡 **Theorem (Rockafellar):**
> A cyclically monotone map can be expressed as the gradient of a convex function.

From the optimality condition; we can write $T(x) = \nabla u(x)$ where u is a convex function. Now, by the conservation of mass we have already established that $f(x) = g(T(x))\left|\nabla T(x)\right|$
Now,

$$\begin{aligned}\left|\nabla T(x)\right| &= \frac{f(x)}{g(T(x))} \\ \left|\nabla^2 u(x)\right| &= \frac{f(x)}{g(T(x))}\end{aligned}$$

Above equation is called the #Monge-Ampere equation and $|\nabla^2 u(x)|$ is the determinant of a Hessian.

Kantorovich Problem

We have already seen the objective function in #Kantorovich formulation.

$$\inf \int_{X\times Y} c(x,y)\, d\pi(x,y) \middle| \pi \in \Pi(\mu,\nu)$$

Here, $\Pi(\mu,\nu)$ is the set of measures whose marginals on X and Y are μ and ν respectively.

#Kantorovich formulation is feasible under the following conditions:

- $\Pi(\mu,\nu)$ is feasible if there is a mass balance

- As long as the cost function is bounded below and X, Y are bounded then there exists an infimum that is finite.

Is there a minimizer?

We need a compactness argument.

> 💡 **Theorem (Weierstrass):**
> If $f : U \to \mathbb{R}$ is continuous and U is compact then f attains a minimum on U.

So we need to show that $c(x,y)\ d\pi(x,y)$ is continuous and $\Pi(\mu,\nu)$ is compact in order for the #Kantorovich formula to contain infimum.

> 💡 **Theorem**
> Suppose $X, Y \subseteq \mathbb{R}^N$ are compact and that $c(x,y)$ is continuous. Under this assumptions #Kantorovich problem contains minimum.

Proof:
Before we talk about compactness we need a notion of convergence. We identify $U = \Pi(\mu,\nu)$ and without loss of generality these are probability measures. Furthermore, a sequence of measures converges $\gamma_n \to \gamma$ if

$$\int_{X\times Y} g(x,y) d\gamma_n(x,y) \to \int_{X\times Y} g(x,y) d\gamma \quad \forall\, g \in C^0(X\times Y)$$

> Is the set $\Pi(\mu,\nu)$ compact?
> Choose any sequence $\pi_n \in \Pi(\mu,\nu)$. We need to extract a convergent sub-sequence.
> Since, π_n are probability measures, we can extract a sub-sequence.
> $\pi_{n_k} \to \pi$, which is also a probability measure.

🔔 Need to check: Does $\pi \in \Pi(\mu,\nu)$?
`Check marginals` Choose any $g \in C(X)$ and

$$\begin{aligned}\int_{X\times Y} g(x)\ d\pi(x,y) &= \lim_{k\to\infty} \int_{X\times Y} g(x)\ d\pi_{n_k}(x,y) \quad\ \because\ \pi_{n_k} \to \pi \\ &= \int_X g(x)\ d\mu(x) \quad\ \because\ \pi_{n_k} \in \Pi(\mu,\nu),\ \text{so is the marginal over } X \text{ is } \mu\end{aligned}$$

With the same argument, the marginal of π over Y is ν.

$$\therefore\ \ \pi \in \Pi(\mu,\nu) \qquad \text{i.e. } \Pi \text{ is compact}$$

> Is the value of f continuous?
> Let, $f(\pi) = \int_{X\times Y} c(x,y)\ d\pi(x,y) \quad \forall \pi \in U$
> Choose any $\pi_n \in \Pi(\mu,\nu)$ s.t. $\pi_n \to \pi \in \Pi(\mu,\nu)$. >

🔔 Need to show:

$$f(\pi_n) \to f(\pi)$$

Here;

$$\begin{aligned} f(\pi_n) &= \int_{X\times Y} c(x,y)\, d\pi_n(x,y) \\ &= \int_{X\times Y} c(x,y)\, d\pi(x,y) \quad \because \pi_n \to \pi \;\;\&\;\; c \text{ is continuous} \\ &= f(\pi) \end{aligned}$$

$\therefore\ f$ is continuous.

Therefore, from Weierstrass theorem we can say that the #Kantorovich problem has a minimizer.

📝 #Kantorovich problem seeks to minimize a real-valued, continuous function over a compact set. It admits a minimizer.
