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1 Connection Between Entropy Regularized Optimal Transport and Covariance Steering for Motion Planning

Entropy regularized optimal transport problem can be formulated as

$$\inf_{\pi \in \prod(\mu,\nu)} \int_{X\times Y} c(x,y)\ \pi(x,y)\ dx\ dy + \varepsilon \int_{X\times Y} \pi(x,y)\ \log \pi(x,y)\ dx\ dy \tag{1}$$

In robotics, the motion planning problem can be layed out as follows;

$$\begin{aligned}
&\min_{x,u} \mathbb{E} \int_0^1 \big[\frac{1}{2}||u(t)||^2_{Q_c^{-1}} + ||h(x(t))||^2_{\Sigma_{obs}^{-1}}\big]\ dt \\
&\dot{x}(t) = A(t)x(t) + F(t)\big[u(t) + Tw(t)\big] + b(t) \\
&x(t_0) \sim \mathcal{N}(\mu_0, \Sigma_0); \quad x(t_1) \sim \mathcal{N}(\mu_1, \Sigma_1)
\end{aligned} \tag{2}$$

Need to show: Entropy regularized optimal transport is related to the constrained stochastic optimal control problem above.

We can see that the system dynamics and the control input are affected by stochastic process. Due to this structure of the cost and dynamics in (2), we can re-formulate this motion planning problem as an optimization problem over probability measures. Let, $\mathcal{P}^u$ be the distribution induced by the stochastic process $\dot{x}(t)$ over the space $\Omega := C([0,1], \mathbb{R}^n)$. Furthermore, $\mathcal{P}^0$ is the distribution induced by the same process when $u(t) = 0$. Then Girsanov theorem states that,

$$\frac{d\mathcal{P}^u}{d\mathcal{P}^0} = \exp\Big(\int_0^1 \frac{1}{2\varepsilon}||u_t||^2\ dt + \frac{1}{\sqrt{\varepsilon}} u_t^T dw_t\Big) \tag{3}$$

It follows from Grisanov theorem that

$$\mathrm{KL}(\mathcal{P}^u||\mathcal{P}^0) := \int \log \frac{d\mathcal{P}^u}{d\mathcal{P}^0}\ d\mathcal{P}^u = \mathbb{E}\ \Big[\int_0^1 \frac{1}{2\varepsilon}||u_t||^2\ dt\Big] \tag{4}$$

Now the problem can be reformulated as:

$$\begin{aligned}
&\min_{\mathcal{P}^u} \int \Big[\log \frac{d\mathcal{P}^u}{d\mathcal{P}^0} + \frac{1}{\varepsilon}\|h(x(t))\|^2_{\sigma_{obs}^{-1}}\Big] \\
&\qquad (X_0)_\# \mathcal{P}^u = \rho_0, \quad (X_1)_\# \mathcal{P}^u = \rho_1
\end{aligned} \tag{5}$$

where $(X_0)_\#\mathcal{P}^u$ is the distribution of X_0 when the process X_t is associated with the distribution $\mathcal{P}^u$. Similar for $(X_1)_\#\mathcal{P}^u$

Let $\prod(\rho_0, \rho_1)$ be the set of all distributions over the path space Ω such that the constraint in equation (5) is satisfied. Then (5) can be written as;

$$\min_{\mathcal{P}^u \in \prod(\rho_0,\rho_1)} F(\mathcal{P}^u) + G(\mathcal{P}^u) \tag{6}$$

where

$$F(\mathcal{P}^u) = \int \left[\frac{1}{\varepsilon}\|h(x(t))\|^2_{\sigma_{obs}^{-1}} - \log d\mathcal{P}^0\right] d\mathcal{P}^u$$
$$G(\mathcal{P}^u) = \int d\mathcal{P}^u \log d\mathcal{P}^u$$

This is an infinite dimensional optimization. And for our purpose we use an algorithm to approximately solve this problem. Specifically, we look into an approximate solution $\mathcal{P}^u$ that is induced by Gaussian Markov process.

Therefore, we restrict our search space to $\hat{\prod}(\rho_0, \rho_1)$, a space of measures in the original space Ω that are induced by Gaussian Markov process. This new space will have the marginals $\mathcal{N}(\mu_0, \Sigma_0)$ and $\mathcal{N}(\mu_1, \Sigma_1)$.

Under the assumption that $\mathcal{P}$ has a small variance, $F(\mathcal{P})$ can be approximated by;

$$\int \left[\frac{1}{\varepsilon}\|\hat{h}(x(t))\|^2_{\sigma_{obs}^{-1}} - \log d\hat{\mathcal{P}}^0\right] d\mathcal{P}$$

where $\|\hat{h}(x(t))\|^2_{\sigma_{obs}^{-1}}$ is the second order Taylor-series approximation of $\|h(x(t))\|^2_{\sigma_{obs}^{-1}}$ and $\hat{\mathcal{P}}^0$ is a Gaussian Markov approximation of $\mathcal{P}^0$. Both of these approximations are along the mean of $\mathcal{P}$, say z_t.

Thus we have;

$$\min_{\mathcal{P}^u \in \hat{\prod}(\rho_0,\rho_1)} \int \left[\frac{1}{\varepsilon}\|\hat{h}(x(t))\|^2_{\sigma_{obs}^{-1}} - \log d\hat{\mathcal{P}}^0\right] d\mathcal{P}^u + \int d\mathcal{P}^u \log d\mathcal{P}^u \tag{7}$$

Equation (7) can also be written as follows;

$$\min_{\mathcal{P}^u \in \hat{\prod}(\rho_0,\rho_1)} \int \|\hat{h}(x(t))\|^2_{\sigma_{obs}^{-1}} d\mathcal{P}^u + \varepsilon \int d\mathcal{P}^u (\log d\mathcal{P}^u - \log d\hat{\mathcal{P}}^0) \tag{8}$$

The expression above resembles the entropy-regularized optimal transport problem:

$$\inf_{\pi \in \prod(\mu,\nu)} \int_{X\times Y} c(x,y)\ \pi(x,y)\ dx\ dy + \varepsilon \int_{X\times Y} \pi(x,y)\ \log \pi(x,y)\ dx\ dy$$