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MATH: 7251
(Final Exam)

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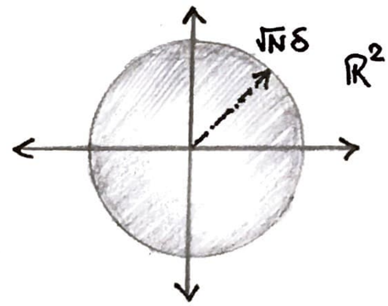
Problem 1.

Here; $(X_1, X_2, \dots, X_N)$ is a random vector with independent components. and each component X_K is absolutely continuous with density bounded by 1.

<u>Prove:</u> $\mathbb{P}\{\|X\| \le \sqrt{N}\delta\} \le (C\delta)^N$ for some constant $C>0$ & $\forall \delta>0$.



Let; X denote the random vector.

Consider a ball $B(0,\sqrt{N}\delta)$ centered at origin in $\mathbb{R}^N$ with radius $\sqrt{N}\delta$ as shown in the figure alongside.

We want to find the probability that X lies inside the ball $B(0,\sqrt{N}\delta)$.

The joint PDF of X can be written as the product of its marginals since each components are independent.

i.e. $f(x_1, x_2, \dots, x_N) = f(x_1)\, f(x_2) \cdots f(x_N)$

From the question; $f(x_K)$ is bounded by 1. ie. $0 \le f(x_K) \le 1 \quad \forall x_K$

and $1 \le K \le N$

Therefore;

$$0 \le f(x_1, x_2, \dots x_N) \le 1$$

By definition:
$$\mathbb{P}\{\|X\| \le N^{1/2}\delta\} = \int_{B(0,\sqrt{N}\delta)} f(x_1, x_2, \dots, x_N)\, dx_1\, dx_2 \dots dx_N$$
$$\le \int_{B(0,\sqrt{N}\delta)} 1\, dx_1\, dx_2 \dots dx_N$$
$$= \text{Vol. of } B(0,\sqrt{N}\delta)$$
$$= \frac{\pi^{N/2}}{\Gamma(N/2+1)} (\sqrt{N}\delta)^N = \frac{\pi^{N/2} N^{N/2}}{\Gamma(\frac{N}{2}+1)} \delta^N$$

$$\therefore \mathbb{P}\{\|X\| \le \sqrt{N}\delta\} \le (C\delta)^N \quad //.$$

(proved)*

Problem 2.

- By definition of Lipschitz function we have; $|f(x)-f(y)| \le L|x-y|$

Here; $Z = n^{-1}\sum_{j=1}^{n} f(\lambda_j(x))$; $\mathbb{E}Z = n^{-1}\sum_{j=1}^{n} f(\lambda_j(\mathbb{E}x))$

Now;

$$|Z-\mathbb{E}Z| = \left| n^{-1}\sum_{j=1}^{n}\left(f(\lambda_j(x)) - f(\lambda_j(\mathbb{E}x))\right)\right| \quad \text{— (i)}$$

Let; $a_j = |\lambda_j(x) - \lambda_j(\mathbb{E}x)|$ and $b_j = 1 \;\; \forall\, j = 1,\dots,n$

By Cauchy-Schwarz inequality:

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \le \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right)$$

In our case: $\sum_{j=1}^{n} a_j b_j = \sum_{j=1}^{n} |\lambda_j(x) - \lambda_j(\mathbb{E}x)|$ and $\sum_{j=1}^{n} b_j^2 = n$

Using the provided hint:

$$\sum_{j=1}^{n}(\lambda_j(x) - \lambda_j(\mathbb{E}x))^2 \le \|X - \mathbb{E}X\|_{HS}^2$$

Rewrite Cauchy-Schwarz as;

$$\left(\sum_{j=1}^{n} |\lambda_j(x) - \lambda_j(\mathbb{E}X)|\right)^2 \le n.\sum_{j=1}^{n}(\lambda_j(x) - \lambda_j(\mathbb{E}X))^2$$

i.e. $$\left(\sum_{j=1}^{n} |\lambda_j(x) - \lambda_j(\mathbb{E}x)|\right)^2 \le n.\|X - \mathbb{E}X\|_{HS}^2$$

Taking square root on both sides;

$$\sum_{j=1}^{n} |\lambda_j(x) - \lambda_j(\mathbb{E}x)| \le \sqrt{n}.\|X - \mathbb{E}X\|_{HS} \quad \text{— (ii)}$$

From -(i) and (ii) using Lipschitz property:

$$|Z - \mathbb{E}Z| \le n^{-1}\sqrt{n}\, L\, \|X - \mathbb{E}X\|_{HS}$$

or; $$|Z - \mathbb{E}Z| \le \frac{L}{\sqrt{n}}\|X - \mathbb{E}X\|_{HS}. \le \frac{L}{\sqrt{n}}\sqrt{\sum_{j=1}^{n}\lambda_j(x)^2}$$

And by concentration Inequalities for Lipschitz functions of random matrices we get; $\|Z - \mathbb{E}Z\|_{\psi_2} \lesssim \frac{L}{\sqrt{n}}$ //.

(proved).*

Problem 3.

Here; $X_1, X_2, \dots, X_N$ are independent R.V; where X_j is Bernoulli with parameter $P_j \in [0,1]$. Also; $\xi(x_1, \dots x_n)$ satisfies bounded difference condition (BDC).

To show:- $\quad \mathrm{Var}(\xi(X_1, X_2, \dots, X_N)) \leq \sum_{j=1}^{n} c_j^2 P_j(1-P_j)$.

→. Since; $\xi(x_1, \dots, x_n)$ satisfies B.D.C. we have;

$$c_j \geq |\xi(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n) - \xi(x_1, \dots x_{j-1}, x_j', x_{j+1}, \dots, x_n)|$$

Squaring both sides we get;

$$(\xi(x_1, \dots x_{j-1}, x_j, x_{j+1}, \dots, x_n) - \xi(x_1, \dots x_{j-1}, x_j', x_{j+1}, \dots x_n))^2 \leq c_j^2 \quad \text{—(i)}$$

Here; X_j and X_j' are independent Bernoulli R.V.

So; $\mathbb{P}(X_j \neq X_j') = P_j(1-P_j) + (1-P_j)P_j = 2P_j(1-P_j)$

when $X_j = X_j'$; L.H.S. of —(i) is 0.

Now; Taking expectation on both sides of —(i).

$$\mathbb{E}(\xi(x_1, \dots, x_j, \dots x_n) - \xi(x_1, \dots x_j', \dots x_n))^2 \leq 2c_j^2 P_j(1-P_j) \quad \text{—(ii)}$$

By Efron-Stein Inequality we also have;

$$\begin{aligned}
\mathrm{Var}(\xi(x_1, \dots, x_n)) &\leq \frac{1}{2} \sum_{j=1}^{n} \mathbb{E}(\xi(\dots x_j \dots) - \xi(\dots x_j' \dots))^2 \\
&\leq \frac{1}{2} \sum_{j=1}^{n} 2c_j^2 P_j(1-P_j) \quad (\text{from —(ii)})^* \\
&= \sum_{j=1}^{n} c_j^2 P_j(1-P_j)
\end{aligned}$$

$\therefore \mathrm{Var}(\xi(X_1, \dots X_n)) \leq \sum_{j=1}^{n} c_j^2 P_j(1-P_j)$ //.

(proved).*

Problem 4.

Here; $T = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$. and $\|\cdot\|$ is the Euclidean Norm.

A stochastic process $X(t), t \in T$ satisfies

$$|X(t) - X(s)| \leq L\|t-s\| ; \quad t, s \in T$$

with r.v. $L \geq 0$ and $\mathbb{E} L^2 < +\infty$.

$X_1, \dots, X_n$ are iid. copies of $X(t)$ and $\xi_1, \dots, \xi_n$ are iid $N(0,1)$. r.v. which are independent of $X_1, \dots, X_n$.

To show:

$$\mathbb{E} \sup_{t \in T} n^{-1} \sum_{j=1}^{n} \xi_j X_j(t) \leq \sqrt{\mathbb{E} L^2 \frac{d}{n}}$$

>> Solution;

Let; $\quad Y(t) := n^{-1} \sum_{j=1}^{n} \xi_j X_j(t) \quad \forall\, t \in T.$

The covariance of this Gaussian process is given by;

$$\begin{aligned}
\mathbb{E}\,(Y(t) - Y(s))^2 &= n^{-2}\, \mathbb{E} \Big(\sum_{j=1}^{n} \xi_j \,(X_j(t) - X_j(s))\Big)^2 \\
&= n^{-2}\, \mathbb{E} \Big[\sum_{j=1}^{n} \xi_j\,(X_j(t) - X_j(s))\Big]^2 \\
&= n^{-2} \sum_{j=1}^{n} \mathbb{E}(\xi_j)^2 \cdot \mathbb{E}\,[X_j(t) - X_j(s)]^2 \\
&= n^{-2} \sum_{j=1}^{n} 1 \cdot \mathbb{E}\,[X_j(t) - X_j(s)]^2 \qquad \because \operatorname{Var}(\xi_j) = 1. \\
&\leq n^{-2} \sum_{j=1}^{n} L^2 \|t-s\|^2 = n^{-1} L^2 \|t-s\|^2
\end{aligned}$$

i.e. $\mathbb{E}\,[Y(t) - Y(s)]^2 \leq n^{-1} L^2 \|t-s\|^2$

Now;

Define another process $Z(t) := \sqrt{\frac{d}{n}}\, L\|t\| * N(0,1)$.

and $\mathbb{E}\,[Z(t) - Z(s)]^2 = \left(\frac{d}{n}\right) \cdot L^2 \|t-s\|^2$

Since ; $\mathbb{E}\,[Y(t) - Y(s)]^2 \leq \mathbb{E}\,[Z(t) - Z(s)]^2 \;; \forall\, t, s \in T$

we can now apply Sudakov-Fernique Inequality:

$$\text{i.e. } \mathbb{E} \sup_{t\in T} Y(t) \le \mathbb{E} \sup_{t\in T} Z(t)$$

Here;

$$\mathbb{E} \sup_{t\in T} Z(t) = \mathbb{E} \sup_{t\in T} \sqrt{\frac{d}{n}}\, L \|t\| \mathcal{N}(0,1) = \sup_{t\in T} \sqrt{\frac{d}{n}}\, \mathbb{E} L \|t\| \mathcal{N}(0,1)$$

$$= \sqrt{\frac{d}{n}} \sqrt{\mathbb{E}L^2} \cdot 1 \cdot 1. \quad \text{since ; } t\in T \text{ and } \|t\| \le 1$$

$$= \sqrt{\mathbb{E}L^2 \frac{d}{n}}$$

From Sudakov-Fernique ; it follows that;

$$\mathbb{E} \sup_{t\in T} Y(t) = \mathbb{E} \sup_{t\in T} n^{-1} \sum_{j=1}^{n} \xi_j X_j(t) \le \sqrt{\mathbb{E}L^2 \frac{d}{n}} \quad //$$

(proved)*

Problem 5.

Let; (T,d) be a metric space and $X(t); t\in T$ is a sub-gaussian process w.r.t. d.

To prove:
$$\Big\| \sup_{t,s\in T} |X(t) - X(s)| \Big\|_{\psi_2} \lesssim \int_0^{D} H_d^{1/2}(T;\varepsilon)\, d\varepsilon.$$

where D is the diameter of T and $H_d(T;\varepsilon)$ is its ε-entropy.

>>> To prove this; we'll use similar argument used in proving Dudley's entropy bound in class.

- Let; $\varepsilon_k = 2^{-k} D \;\; \forall\; k \ge 0$. Define $T_k \subset T$ such that $\text{card}(T_k) = N_d(T;\varepsilon_k)$

  T_k is an ε-net for T.

  For all $t\in T$, denote $\pi_k t \in \underset{s\in T_k}{\arg\min}\, d(t,s)$ and $d(t,s) \le \varepsilon_k$

Now; we can write the difference between the process at two points 't' and 's' as telescopic sum. i.e.

$$(X(t) - X(s)) = \sum_{k\ge 0} \left[X(\pi_{k+1}t) - X(\pi_k t) - \left(X(\pi_{k+1}s) - X(\pi_k s) \right) \right]$$

Using triangle Inequality we get the following.

$$|X(t)-X(s)| \leq \sum_{K\geq 0}\left[|X(\pi_{K+1}t)-X(\pi_K t)| + |X(\pi_{K+1}s)-X(\pi_{K+1}s)|\right]$$

Applying triangle inequality for ψ-2 norm and taking supremum over 't' and 's' in T we get;

$$\left\| \sup_{t,s\in T} |X(t)-X(s)| \right\|_{\psi_2} \leq \sum_{K\geq 0}\left[\left\| \sup_{t\in T} |X(\pi_{K+1}t)-X(\pi_K t)| \right\|_{\psi_2} + \left\| \sup_{s\in T} |X(\pi_{K+1}s) \cdots \cdots - X(\pi_K s)| \right\|_{\psi_2} \right]$$

Using Dudley's entropy bound for each term in the sum;

$$\left\| \sup_{t\in T} |X(\pi_{K+1}t)-X(\pi_K t)| \right\|_{\psi_2} \lesssim \int_{\varepsilon_{K+2}}^{\varepsilon_{K+1}} H_d^{1/2}(T;\varepsilon)\,d\varepsilon$$

Therefore;

$$\left\| \sup_{t,s\in T} |X(t)-X(s)| \right\|_{\psi_2} \lesssim \sum_{K\geq 0} 2\int_{\varepsilon_{K+2}}^{\varepsilon_{K+1}} H_d^{1/2}(T;\varepsilon)\,d\varepsilon$$

Combining the sum and integral we get;

$$\left\| \sup_{t,s\in T} |X(t)-X(s)| \right\|_{\psi_2} \lesssim \int_0^D H_d^{1/2}(T;\varepsilon)\,d\varepsilon \quad //$$

(Proved)*.

Factor of 2 is incorporated in '$\lesssim$' symbol.