A COMPUTATIONAL FRAMEWORK FOR THE MEASURED LAMINATION SPACE VIA TRAIN TRACKS

Abstract. The space of measured laminations (ML) on a surface has a piecewise linear structure which is preserved by the action of the mapping class group on ML. Explicit formulas of this piecewise linear action in various coordinate systems have been worked out in the past.

In this paper, we give an algorithm that allows one to derive such formulas automatically in a variety of coordinate systems (triangle coordinates, Dehn–Thurston coordinates and even on nonorientable surfaces). The algorithm computes more than just the change of coordinates: it also computes a geometric representation of a linear map between two charts as maps between train tracks.

1. Introduction

The change of coordinates under elementary moves (flips) of an ideal triangulation are very simple [Mos88, p. 30]. In his thesis, Penner [Pen82, Pen84] derived formulas for the change of Dehn–Thurston coordinates under elementary moves of a pants decomposition. Hall and Yurttaş gave formulas for the change of Dynnikov coordinates for the Artin generators of the braid group [HYs09]. Hamidi-Tehrani and Chen [HTC96] worked out formulas for coordinates coming from ideal polygon decompositions of surface.

This work is

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In this paper, we consider undirected graphs. However, whenever we refer to an edge, we almost always mean the edge with one of the two possible orientation. For an oriented edge e, we denote by $\bar{e}$ the edge with the opposite orientation. Talking about oriented edges allows us to talk about their starting and ending vertices, denoted by $\mathrm{START}(e)$ and $\mathrm{END}(e)$, respectively. For any vertex, the set of outgoing (resp. incoming) edges $\mathcal{OUT}(v)$ (resp. $\mathcal{IN}(v)$) is the set of oriented edges whose starting (resp. ending) vertex is v.

2.1. **Templates.** A *template* is an unoriented graph (V, E) with some extra structure. For each vertex, the incident edges are classified into one of three gates: the positive gate, the negative gate and the neutral gate. Formally, for each $v \in V$, the set $\mathcal{OUT}(v)$ is decomposed into a disjoint union

$$\mathcal{OUT}(v) = \mathcal{OUT}^+(v) \cup \mathcal{OUT}^-(v) \cup \mathcal{OUT}^0(v).$$

There is an analogous decomposition of $\mathcal{IN}(v)$ as well, determined by the rule that if $e \in \mathcal{OUT}^\varepsilon(v)$ and only if $\bar{e} \in \mathcal{IN}^\varepsilon(v)$ for $\varepsilon = \pm, 0$. If $e \in \mathcal{OUT}^\varepsilon(v)$ (or $e \in \mathcal{IN}^\varepsilon(v)$), then we say that the starting (or ending) gate of e is ε and we write $\mathrm{INGATE}(e) = \varepsilon$ (or $\mathrm{OUTGATE}(e) = \varepsilon$). We require that an edge that starts at the neutral gate of vertex ends at the neutral gate of the same vertex. Formally, if $e \in \mathcal{OUT}^0(v)$, then $e \in \mathcal{IN}^0(v)$. These edges are called *neutral edges*.

Two kinds of templates will be important for us.

2.1.1. *Ideal polygon decompositions.* The first one arises from an ideal polygon decomposition of a punctured surface. By this, we mean a set A of disjoint essential arcs connecting punctures that divide the surface into polygons $P_1, \dots, P_n$. The corresponding template has one vertex on each arc $a \in A$. For each polygon, each pair of vertices on the boundary of the polygon are connected by an edge inside the polygon. The neighborhood of each every vertex is divided into two regions by the corresponding arcs. We (arbitrarily) declare one side as positive and the other as negative. Edges emanating into the positive and negative sides are assigned into the positive and negative gates, respectively. The neutral gates in this case are all empty.

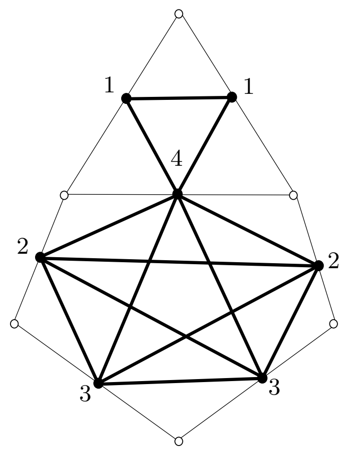


FIGURE 1. The template of a polygon decomposition. Vertex 1 has two outgoing edges at both the positive and the negative gate. Vertex 4 has two and four outgoing edges at the positive and negative gates (in either order), respectively.

2.1.2. *Marked pants decompositions.* The other kind of templates arise from (marked) pants decompositions of surfaces. For simplicity, we restrict ourselves to compact surfaces (possibly with boundary) here. We will mention the minor modifications required for punctured surfaces later.

A *pants decomposition* $\mathcal{P}$ of a compact surface S with negative Euler characteristic is a collection C of disjoint simple closed curves on the surface that separate the surface into pairs of pants. The boundary components of S are also included in C. There are two types of pants curves: *inner pants curves*, that are contained in the interior of S and *boundary pants curve* that are boundary components of S.

When referring to a pair of pants P of a pants decomposition $\mathcal{P}$, we will write $P \in \mathcal{P}$.

A *marking* of a pair of pants P consists of

- a cyclic ordering of the three components of ∂P. When we refer to the three components by $\partial_1 P$, $\partial_2 P$ and $\partial_3 P$, we mean that $\partial_{i+1} P$ comes after $\partial_i P$ in the cyclic ordering (modulo 3).
- a choice of a *marking point* $p(c)$ on each component of ∂P, and

- for each boundary curve $\partial_i P$, an arc, called a *marking arc*, connecting $p(\partial_i P)$ and $p(\partial_{i+1} P)$ inside P. The marking arcs are required to be disjoint. It is also required that the arc from $\partial_i P$ to $\partial_{i+1} P$ is on the right of the arc from $\partial_i P$ to $\partial_{i-1} P$ when looking from the intersection of the two arcs towards the interior of P (Figure 2).

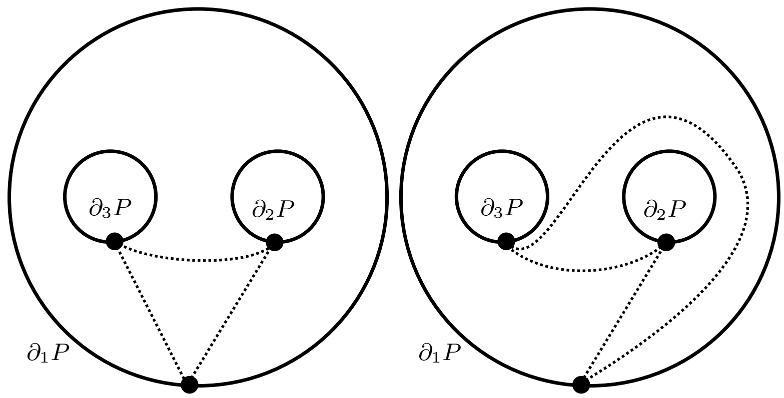


FIGURE 2. On the left: a marking of a pair of pants. On the right: a bad marking, because the marking arc from $\partial_1 P$ to $\partial_2 P$ is not in the right of the arc from $\partial_1 P$ to $\partial_3 P$.

A *marked pants decomposition* is a pants decomposition with a marking on each pair of pants. The marking points of adjacent pants are required to coincide.

Associated to every marked pants decomposition is a template defined as follows. The vertices are the marking points on the inner pants curves. There are three types of edges:

- the marking arcs connecting marking points on inner pants curves.
- for each P and $\partial_i P$ which is an inner pants curve, there is an edge connecting the $p(\partial_i P)$ to itself in P. This edge is homotopic to the loop obtained by going from $p(\partial_i P)$ to $p(\partial_{i+1} P)$ along a marking arc, going around $\partial_{i+1} P$ and the coming back along the marking arc.
- the inner pants curves.

For every vertex, the corresponding inner pants curve enters and exits at the neutral gate. We choose one side of each inner pants curve to be positive, the other to be negative. Edges emanating into the positive and negative sides belong to the positive and negative gates, respectively.

2.2. **Illegal paths and replacement rules.** We say that $\gamma = e_1 \cdots e_n$ is an edge path if $\mathrm{END}(e_i) = \mathrm{START}(e_{i+1})$ for $1 \leq i \leq n-1$. Roughly speaking, paths are legal if they "do not turn back". More specifically, there are three types of illegal paths.

(1) (*backtracking*) a path of length two which is of the form $e\bar{e}$.
(2) (*short illegal path*) a path of length two which is not a backtracking and is of the form $e_1 e_2$ so that $\mathrm{OUTGATE}(e_1) = \mathrm{INGATE}(e_2)$ and this gate is not neutral.

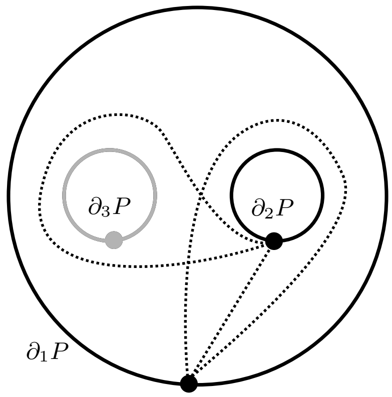


Figure 3. The template in a pair of pants, two of whose boundary components is an inner pants curve and the third is a boundary pants curve.

(3) (*long illegal path*) a path of length three which is of the form $e_1e_2e_3$ so that $\mathrm{OUTGATE}(e_1) = \mathrm{INGATE}(e_3)$ is non-neutral gate and e_2 is a neutral edge.

Each illegal path can be replaced by a legal path.

A backtracking is always replaced by the "empty path". The replacement paths for short and long illegal paths depend on the situation. We will describe these replacement rules on a case by case basis.

2.2.1. *Ideal polygon decompositions.* Since there are no neutral edges in this case, there are only two types of illegal paths: backtrackings and short illegal paths. Short illegal paths are replaced by the edge obtained by pulling tight. (Figure 4)

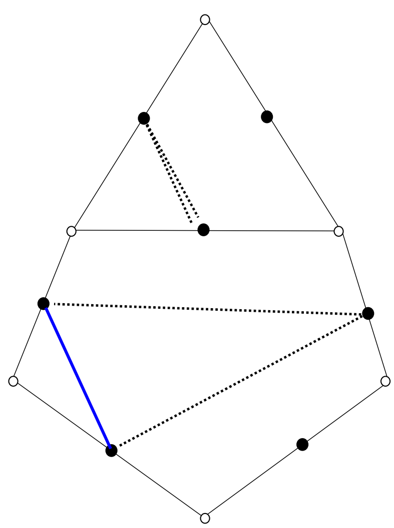
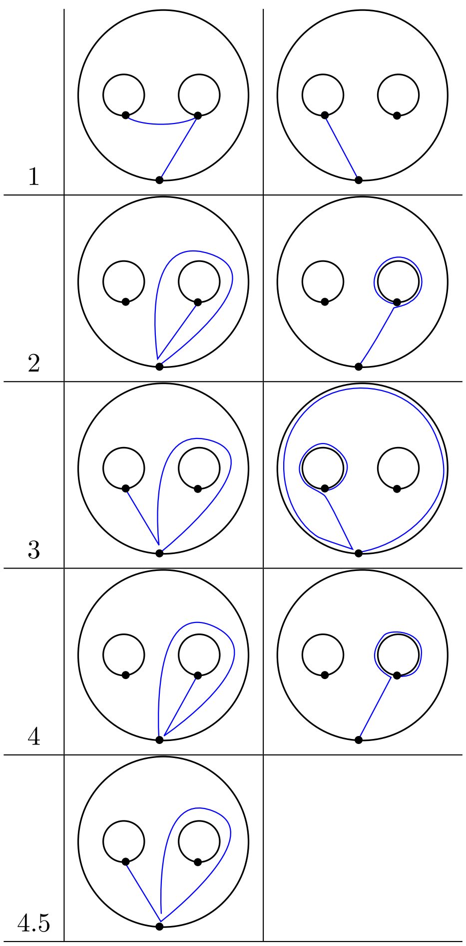


Figure 4. A backtracking and a short illegal path. The path replacing the short illegal path is shown in blue.

2.2.2. *Marked pants decompositions.* We start by classifying short illegal paths e_1e_2 (up to reversing the path and cyclic symmetries of the containing pair of pants). We will call an edge e *self-connecting* if $\mathrm{INGATE}(e) = \mathrm{OUTGATE}(e)$. If neither e_1 nor e_2 is self-connecting, then there only one possible picture, case 1 on **??**. If exactly one of e_1 and e_2 is self-connecting, then there are two possibilities for the other edge and two possibilities for the direction of the self-connecting edges. See the cases 2, 3, 4 and 4.5 on **??**. We ignore the replacement rule for case 4.5, since an closed embedded path containing this illegal path also have to contain either a backtracking or a type 2 or type 3 illegal path.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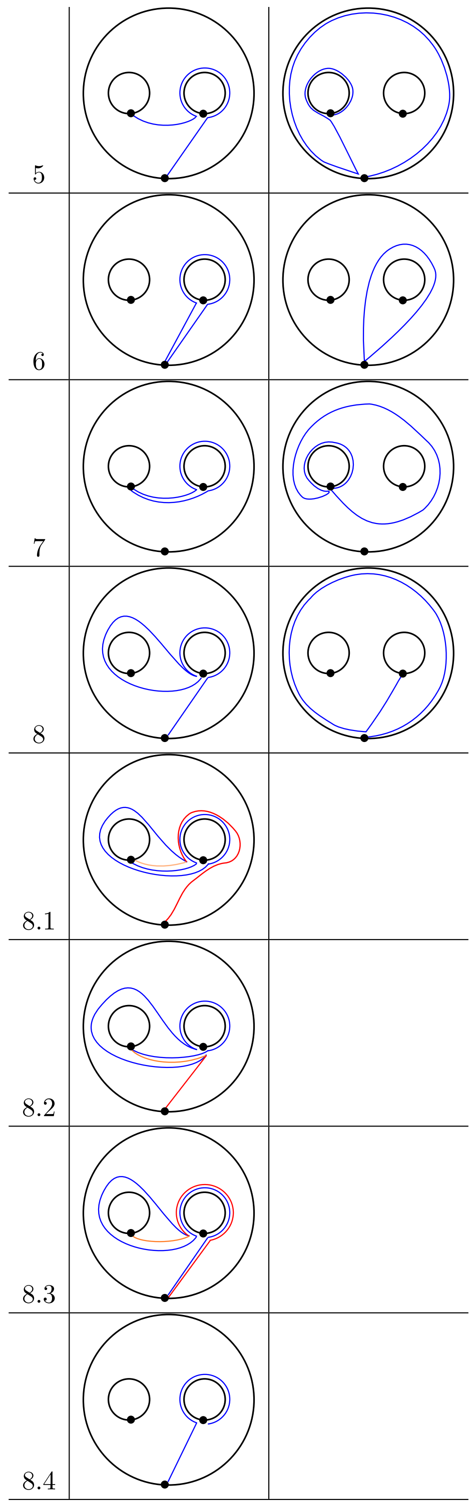
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Now we turn towards long illegal paths. Without loss of generality, we can assume that the neutral edge of the path is $\partial_2 P$ on Figure 3. If none of the other two edges are self-connecting, then there are three possibilities:

(1) they are the same, connecting to $\partial_1 P$ (case 6)
(2) they are the same, connecting to $\partial_3 P$ (case 7)
(3) one of them connects to $\partial_1 P$, the other connects to $\partial_3 P$ (case 5).

Now suppose that e_1 is self-connecting, but e_3 is not. One possible illegal path is shown in case 8.

Now assume in addition that e_3 connects $\partial_2 P$ and $\partial_3 P$. For one of the two possible orientations of e_2, there is only one way to orient e_1 to get a curve without self-intersections (case 8.1). There are three ways to continue the path $\bar{e}_3\bar{e}_2\bar{e}_1$ at $\bar{e}_1$: back along e_2 which produces a backtracking, to $\partial_3 P$ (shown in orange) which produces a subpath as in case 4, or around $\partial_2 P$ and to $\partial_1 P$ (shown in red) which produces a subpath as in case 8. In this last case, it might also happen that this red path keeps going around $\partial_2 P$ and self-connecting multiple times, cycling outwards, before eventually going to $\partial_1 P$. In this case, there is a subpath as in case 8 as well.

Still assuming that that e_3 connects $\partial_2 P$ and $\partial_3 P$, but considering the other orientation of e_2, we arrive to case 8.2. There are three possibilities again: $\bar{e}_1$ is either continued by a backtracking, or it connects to $\partial_3 P$ (orange) in which case there is subpath of type 2, or it connects to $\partial_1 P$ (red) in which case there is a subpath of type 3. Again, if there is some cycling around before connecting to $\partial_1 P$, we still have a subpath of type 3.

Now consider the case when e_3 connects $\partial_2 P$ and $\partial_1 P$. If e_1 is oriented differently from case 8, then we get to case 8.3. Assuming there is no backtracking, there are two ways to continue the path. In one case (orange) we get a subpath of type 4, in the other case (red) we get a subpath of type 8.

Finally, note that the case when e_3 still connects $\partial_2 P$ and $\partial_1 P$, but e_2 is oriented in the other direction is impossible, since there is no way to draw e_1 without self-intersections (case 8.4).

The very last case we did not explain is why cannot both e_1 and e_3 be self-connecting. It is not hard to see this also implies the existence of an illegal path of type 8. Thus we have shown:

Proposition 1. *Any closed path which has an illegal subpath has an illegal subpath of type 1–8.*

2.3. **Tightening closed paths.**

Proposition 2. *Let γ be a closed path in either a polygon decomposition or a pants decomposition template. Then there is a finite sequence of replacement operations that changes the closed path into one without any illegal turns.*

The length of this sequence is bounded by the total length of the path for polygon decompositions, and by the three times the total length for pants decompositions.

Proof. For polygon decompositions, every replacement decreases the total length, so the statement follows.

For pants decompositions, first observe that is that removing a backtracking reduces the total length TL of the path and does not increase the number NNE of non-neutral edges.

Second, notice that any replacement for short or long illegal paths decreases NNE. (However, it might increase TL by one, see case 3.)

Overall, we observe that the quantity $2 * NNE + TL$ decreases for every replacement. Since $NNE \leq TL$, it follows that the process terminates in at most three times the total length steps. □

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