Transition matrices of real, periodic, quadratic polynomials

Balázs Strenner, Agniva Roy, Jacob Shulkin, Logan White

Georgia Institute of Technology (GIT), GIT, University of Michigan, University of Chicago

Important Definitions

Given polynomial $f:\mathbb{C}\to\mathbb{C}$, we have critical set

$$C_f = \{p\in\mathbb{C} : f'(p)=0\}$$

and postcritical set

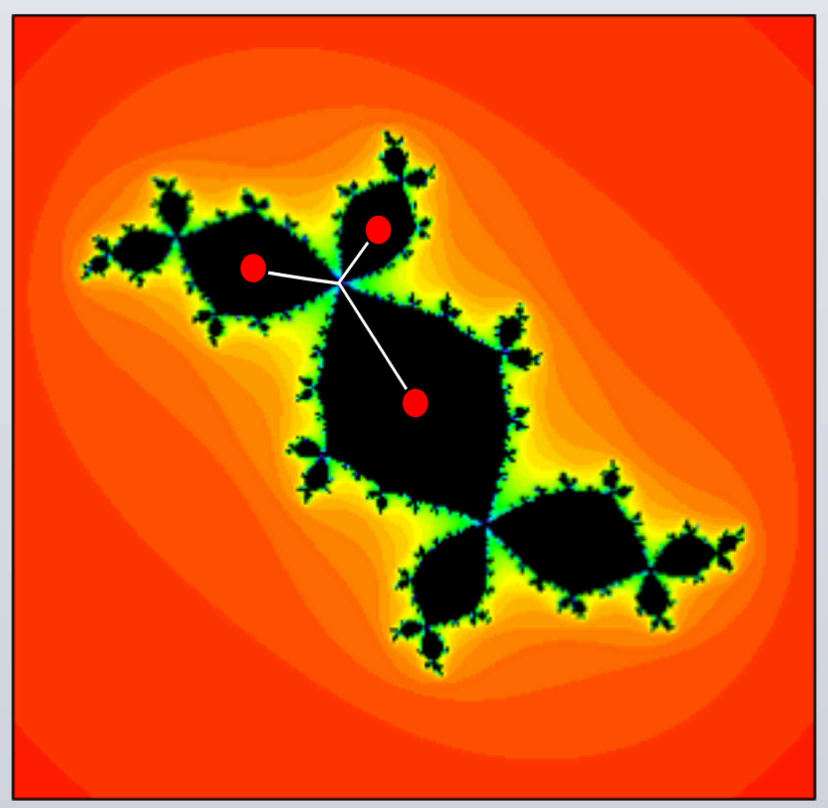
$$P_f = \cup_{n>0}\ f^n(C_f)$$

The Julia set is the complex points with bounded image under iteration of f.

Hubbard tree is the hull of P_f in the Julia set.

Ex: Douady Rabbit

Black: Julia set
Red dots: P_f
White: Hubbard tree



Fact: Hubbard tree maps to itself under f.

Obtain a "transition matrix" from a polynomial by looking at the action of f on the edges of the Hubbard tree.

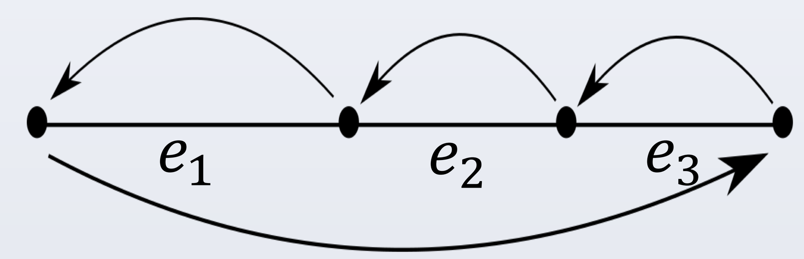
Objectives

We aim to examine the transition matrices of polynomials of the form $f(z)=z^2+c, c\in\mathbb{C}$, where f has a single periodic critical point.

In particular, we are interested in the numbers that arise as eigenvalues of these matrices.

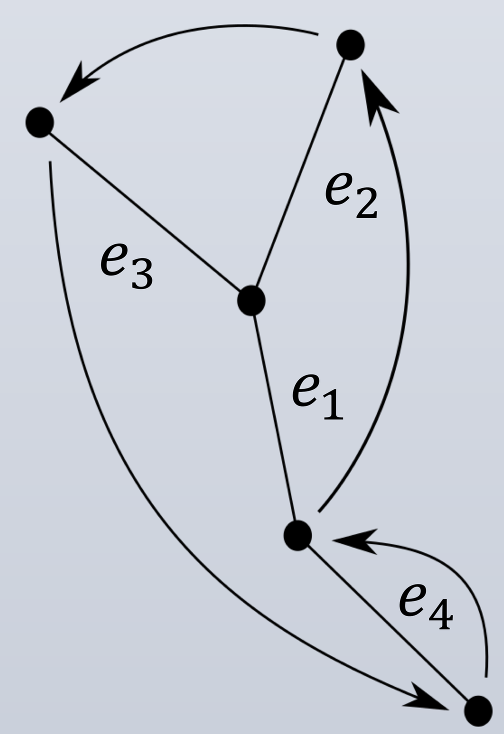
Examples/Computation

Ex. 1: Real period 4 polynomial/transition matrix



$$\begin{array}{cc} \begin{matrix} e_1 & e_2 & e_3 \end{matrix} & \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix} \end{array}$$

Ex. 2: Non-real period 4 with four edges/transition matrix



$$\begin{array}{cc} \begin{matrix} e_1 & e_2 & e_3 & e_4 \end{matrix} & \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{matrix} \end{array}$$

Method

Thurston gives criteria classifying which periodic sequences of post-critical points arise from polynomials $f(z)=z^2+c, c\in\mathbb{R}$.

We used these criteria to write a program that generates a list containing all possible periodic sequences of a given length and computes the associated transition matrices.

Theorems

Each of the following results holds when $c\in\mathbb{R}$:

1. Characteristic polynomial of transition matrix M for polynomial of period $n+1$ is

$$ch_M(T)\equiv T^n+T^{n-1}+\ \dots+1\ (mod\ 2)$$

Corollaries:

i) Some edge vector v will give a basis $\{v, Mv, M^2v, \dots, M^{n-1}\}$. Equivalently, $ch_M(T)$ is always the minimal polynomial of M.

ii) M is invertible

2. There exists a unique edge of the Hubbard tree that maps over itself

Corollary: Trace of M is 1

Conjecture(s)

1. All coefficients of $ch_M(T)$ are ±1

Further Questions

1. In "Entropy in Dimension One", Thurston classifies all numbers that arise as largest eigenvalues of transition matrices for real polynomials of any degree. Which numbers can arise in the case of real, periodic, quadratic polynomials?

2. What can be said about transition matrices when $c\in\mathbb{C}\setminus\mathbb{R}$? What about when 0 is non-periodic?