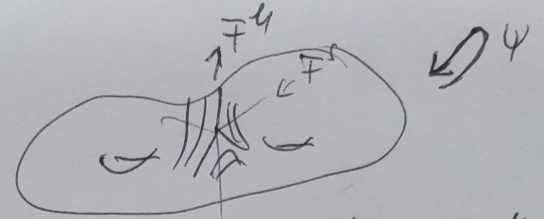
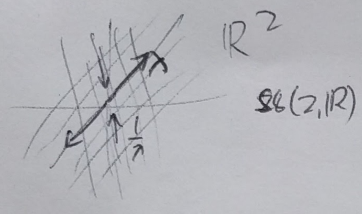
# Number theoretic aspects of surface homeomorphisms

**Nielsen–Thurston classification**: Every $[\psi] \in \mathrm{Mod}(S)$ is either
- finite order
- reducible
- pseudo-Anosov



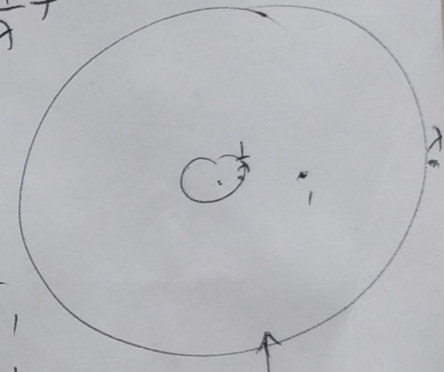

$\boxed{\lambda}$ stretch factor




$$\psi(\mathcal{F}^u) = \lambda \mathcal{F}^u$$
$$\psi(\mathcal{F}^s) = \frac{1}{\lambda} \mathcal{F}^s$$

**Fact**: $\lambda$ is a bi-Perron algebraic unit

min. poly. $x^n \pm \dots \pm 1$




all Galois conjugates lie here.

## $\boxed{\text{I}}$ Fried's problem

$\boxed{\text{NT}}$ $\boxed{\text{TOP}}$

**Conj**: Every bi-Perron alg. unit is a pA stretch factor.

"Examples of Theorem 7 show that this example is sharp".

## $\boxed{\text{II}}$ Algebraic degrees

$\boxed{\text{NT}}$ $\boxed{\text{TOP}}$

Thurston (70's): $2 \le \deg(\lambda) \le 6g-6 = \dim(\mathrm{Teich}(S_g))$

Long ('84): $\deg(\lambda) \le 3g-3$ if $\deg(\lambda)$ is odd

Arnoux–Yoccoz ('91): $g$ ✓

Shin ('14): $2g$ ✓

**Thm** (S.) The possible degrees of $\lambda$ on $S_g$ are

$$[2, 6g-6]_{\text{even}} \cup [3, 3g-3]_{\text{odd}}.$$

## III Degrees and covers

Thm (Franks – Rykken '99) $\mathcal{F}^u, \mathcal{F}^s$ orientable.

$\deg(\lambda) = 2 \iff \psi$ is a lift of an Anosov map of the torus by a branched covering.

$\boxed{NT}$ $\boxed{TOP}$

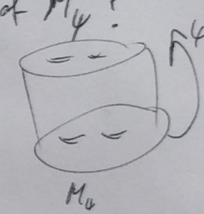
Conj (Farb) $\forall\, d \geq 2\ \exists\, h(d)$ $\deg(\lambda(\psi)) = d \implies \psi$ is a lift of pA on a genus $\leq h(d)$ surface

($h(2) = 1$ if $\mathcal{F}^u, \mathcal{F}^s$ or.)

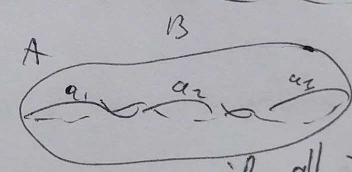
## IV Surface bundles

**Q**: How are the NT properties of $\lambda$ reflected in the top/geom of $M_\psi$?

$\boxed{NT}$ $\boxed{TOP}$




## V Penner's construction ('88)




A, B filling multicurves

Any product of $T_{a_i}$ and $T_{b_j}^{-1}$ is pA if all twists are used.

E.g. $T_{a_3} T_{a_1}^{4} T_{b_1}^{-5} T_{a_1} T_{b_2}^{-200} T_{a_2}^{2}$ is pA.

Penner's conjecture ('88) Every pA has a power arising like this.

Thm (Shin – S.) If $\lambda$ has a Galois conjugate on the unit circle, $\boxed{NT}$ then it does not arise from Penner's construction. $\boxed{TOP}$

Thm (S.) $g \geq 2$, $\exists\, A, B$ Galois conjugates of Penner stretch factors are dense in $\mathbb{C}$.

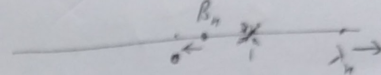
# Proof of degree theorem

## Asymptotic irreducibility criterion

**Lemma** (S.) Let $p_n \in \mathbb{Z}[x]$, degree $r$. Suppose $p_n(\lambda_n) = 0$
$p_n = x^r \pm \dots \pm 1$

$\lambda_n \to \infty$, $p_n(1) \neq 0$, $\dfrac{p_n(x)}{x - \lambda_n} \to x(x-1)^{r-2}$

Then $p_n(x)$ is irreducible if $n$ is large.




**Proof**: $\lambda_n, \beta_n$ eventually in same factor.

If reducible for $\infty$ many $n$, then $(x-1)^k$ eventually a factor.

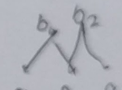
---

## Computing Penner stretch factors

$T_{a_i}$, $T_{b_i}^{-1}$ $\Rightarrow$ $Q_i$ — $5\times5$ transition matrices, ~~functions of~~ depending only on $\Omega$ — intersection matrix of $A, B$

product of $T_{a_i}, T_{b_i}^{-1}$ $\to$ product of $Q_i$ $\to$ spectral radius $= \lambda$

## Recipe for degree $r$ pA

$G(\Omega) =$


intersection graph

**Thm** (S.) 1) Pick $A, B$ so that $\mathrm{rank}(\Omega) = r$
2) Pick a contractible closed path $\gamma = (i_1 i_2 \dots i_k i_1)$ in $G(\Omega)$

Then $\deg(\lambda(Q_{i_k}^n \dots Q_{i_1}^n)) = r$ if $n$ is large enough.

---

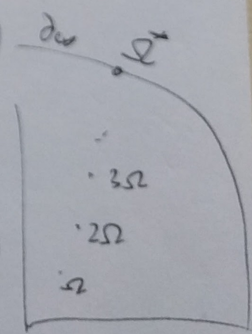
**Sketch of proof**: $Q_{i_k}^n(\Omega) \dots Q_{i_1}^n(\Omega) = Q_{i_k}(n\Omega) \dots Q_{i_1}(n\Omega)$ dir $\Omega^*$

$= (Q_{i_k} \dots Q_{i_1})(n\Omega)$

$\wr$

$(Q_{i_1 \leftarrow i_k} \dots Q_{i_2 \leftarrow i_1})(\Omega^*)$

projections



$5\times5$ intersection matrices

$\gamma$ contractible $\Rightarrow$ projections cancel $\Rightarrow$ composition is a projection

$\dfrac{p_n(x)}{x - \lambda_n} \longrightarrow x(x-1)^{r-2}$