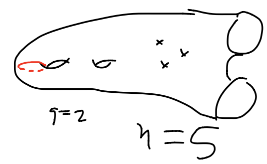
Class 1 (1/7)

constant curvature -1

X – hyperbolic surface with negative Euler char

g – genus of X

n – number of boundaries/punctures of X



$s_X(L) = \#$ of simple closed geodesics on X with length $\leq L$

no self-intersections

locally length-minimizing curve

parametrized by S^1

$$a(L) \sim b(L) \iff \lim_{L\to\infty} \frac{a(L)}{b(L)} = 1$$

<u>Thm</u> (Mirzakhani, 2008): $s_X(L) \sim c(X) \cdot L^{6g-6+2n}$ for some $c(X) > 0$.

<u>Detour</u>: Gaussian Curvature, Theorem Egregium (bending a sheet of paper, squeezing an orange), Negative curvature, Hyperbolic crochets, corals, lettuce

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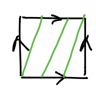
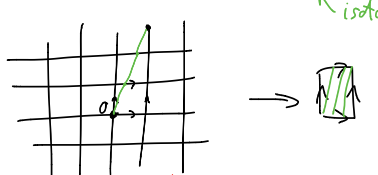
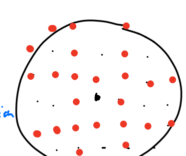
<u>Q</u>: What is the asymptotics of $s_X(L)$ when X is the square torus?

isotopy classes of SCG





origin



$s_X(L) = \big(\#$ of primitive lattice points of norm $\leq L\big) - 1$

orientation

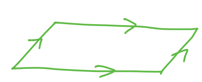
2

<u>Fact 1</u>: # of lattice points of norm $\leq L \sim$ Area(disk of radius L) $= \pi L^2$

<u>Fact 2</u>: "Probability of two integers being relatively prime" $= \frac{6}{\pi^2}$

$$\Rightarrow s_X(L) \sim \frac{6}{\pi^2} \cdot \frac{\pi L^2}{2} = \frac{3L^2}{\pi}$$

<u>Q</u>: What if X is any flat torus?

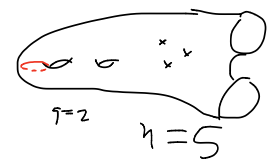


$$s_X(L) \sim \frac{6}{\pi^2} \cdot \frac{\pi L^2}{2\,\mathrm{Area}(X)} = \frac{3L^2}{\pi \cdot \mathrm{Area}(X)}$$

Class 2 (1/9)

constant curvature -1, complete, finite-area, geodesic boundary

X – hyperbolic surface



g – genus of X

n – number of boundaries/punctures of X

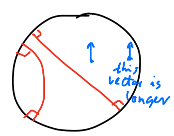
$s_X(L)$ = # of simple closed geodesics on X with length $\leq L$

Thm (Mirzakhani, 2008): $s_X(L) \sim c(X)\cdot L^{6g-6+2n}$ for some $c(X)$.

Examples of hyperbolic surfaces

(1) Hyperbolic plane

- Poincaré model



Set: $\mathbb{D}=\{z : |z|<1\}$

Metric: $4\cdot\frac{(dx)^2+(dy)^2}{(1-x^2-y^2)^2}$

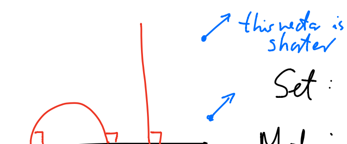
Geodesics: arcs of circles orthogonal to boundary

Isometries: Möbius transformations fixing $\mathbb{D}$

$$f(z) = a\cdot\frac{z+b}{\bar{b}z+1}$$

$|a|=1$

- Upper half plane model



Set: $\mathbb{H}=\{z : \mathrm{Im}(z)>0\}$

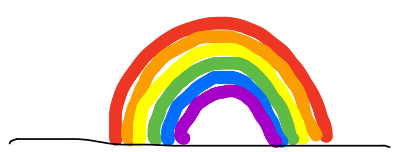
Metric: $\frac{(dx)^2+(dy)^2}{y^2}$

Geodesics: same

Isometries: Möbius transformations fixing $\mathbb{H}$

$$f(z)=\frac{az+b}{cz+d}$$

$\begin{bmatrix}a&b\\c&d\end{bmatrix}/\pm1 \in PSL(2,\mathbb{R})$ $\quad ad-bc=1$, $a,b,c,d\in\mathbb{R}$



geodesic rainbow

Isometry group $\cong PSL(2,\mathbb{R})$

$\underline{Q}$: Is $\mathbb{H}$ homeomorphic to some $S_{g,n}$?
$\underline{A}$: Yes, it is a once punctured sphere, so $g=0$, $n=1$.
$\underline{Q}$: Does Mirzakhani's theorem hold?
$\underline{A}$: No, there are no simple closed geodesics.

<u>Missing hypothesis 1</u>: X has to have finite area.

$\underline{Q}$: How can we construct finite-area examples.
$\underline{A}$: Just like $\mathbb{R}^2/\mathbb{Z}^2$, take a quotient of $\mathbb{H}$ by a group of isometries.

Classification of Isometries of $\mathbb{H}$ by fixed points

Natural Compactification of $\mathbb{H}$: $\overline{\mathbb{H}} = \mathbb{H} \cup \underbrace{\mathbb{R} \cup \{\infty\}}_{\partial \mathbb{H}}$

Every isometry f extends to $\overline{\mathbb{H}}$ continuously.

Brouwer fixed point theorem $\Rightarrow f$ has a fixed point $x \in \overline{\mathbb{H}}$

<u>Case I</u>: There is a fixed point in $\mathbb{H}$. Elliptic isometry. (rotation about a point)

That is the only fixed point. No fixed points on $\partial\mathbb{H}$

Conjugates of $f(z) = az$ $(|a|=1)$ in disk model.

<u>Case II</u>: No fixed points in $\mathbb{H}$, one fixed point in $\partial\mathbb{H}$. Parabolic isometry.

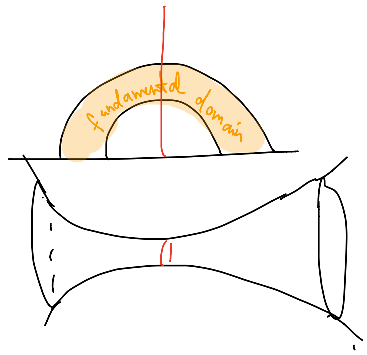
Conjugates of $f(z) = z + c$ in half-plane model

<u>Case III</u>: No fixed points in $\mathbb{H}$, two fixed points in $\partial\mathbb{H}$. Hyperbolic isometry

Conjugates of $f(z) = \lambda z$ $(\lambda > 0)$ in half-plane model.

Translation along an axis.

② $\mathbb{H}/\langle g \rangle$ where g is hyperbolic

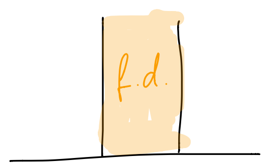


- 1 simple closed geodesic
- ∞ area

$X = \mathbb{H}/G$

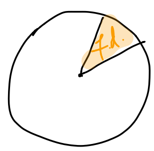
X	G
closed geodesic	conj. class of a hyperbolic element of G
primitive closed geodesic	— " — primitive — " —
simple closed geodesic	—— " —— so that the axes of elements in the conjugacy class are disjoint
cusp	conj. class of a parabolic element of G

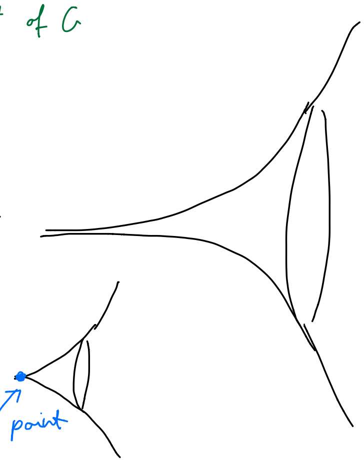
③ $\mathbb{H}/\langle g \rangle$ where g is parabolic



- no simple closed geodesics
- ∞ area

④ $\mathbb{H}/\langle g \rangle$ where g is elliptic finite order



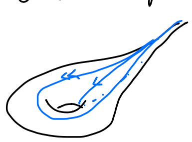


topologically, the quotient is a manifold. But the metric has a singularity.

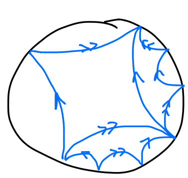
Orbifold: quotient of $\mathbb{H}$ by a discrete (not necessarily fixedpoint-free) group of isometries.

Note: We have to quotient with a discrete group, otherwise the quotient is not a manifold (not Hausdorff).

⑤ Torus with one cusp

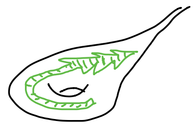


$\mathbb{H}/G$



G = symmetry group of this picture

finite volume!

⑥ 

Torus with one cusp minus closed Christmas tree

- Topologically $S_{g,n}$? Yes, $S_{1,2}$.
- hyperbolic? Yes
- finite volume? Yes
- Mirzakhani's thm holds? No, finitely many simple closed geodesics.

Missing hypothesis II : X has to be complete

⑦ 

Torus with one cusp minus open Christmas tree

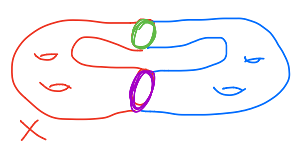
- finite volume ✓
- complete ✓

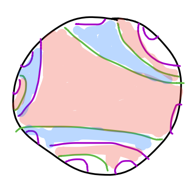
Missing hypothesis III : Boundary components have to be geodesics.

Thm (Uniformization) Let X be a complete hyperbolic surface without boundary. Then X is the quotient of $\mathbb{H}$ by a discrete, fixed point-free group.

Q: What if we have geodesic boundary?

A: Double the surface





X is a quotient of one of these (convex) shaded regions.

Revisit
- Farey triangulation picture
- dictionary

Prime ______ theorems

Class 3 (1/14)

(1) $\pi(N) = \#$ of pos. primes less than N

Thm (Prime Number Theorem) $\pi(N) \sim \dfrac{N}{\log N}$

Hadamard, de la Vallée Poussin 1896

(2) M – compact n-manifold of negative curvature

$PC_M(L)$ – primitive closed geodesics in M of length $\leq L$

Thm (Prime Geodesic Theorem, Margulis' thesis, 1970)

Huber '59: M hyperbolic surface
+Desart, Selberg

$$PC_M(L) \sim \frac{e^{hL}}{hL}, \text{ where } h = \text{top. entropy of geodesic flow on } M.$$

Notice: $PC_M(\log L) \sim \dfrac{L^h}{h \cdot \log L}$

But what the heck is h?

Thm (Manning, 1979): $h = \lim\limits_{L \to \infty} \dfrac{1}{L} \log \text{Area}(\text{disk of radius } L \text{ in universal cover}) = \sqrt{-K} = 1$ for hyperbolic surfaces

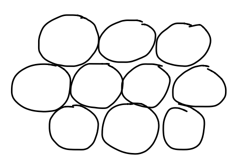
Fact: constant curvature $K < 0 \Rightarrow \text{Area}(\text{disk of radius } L) \sim c \cdot e^{L\sqrt{-K}}$

$C_X(\log L) = \dfrac{L}{\log L}$ for a hyperbolic surface X.

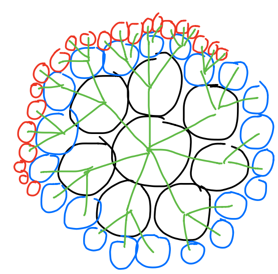
In class: show hyperbolic games instead.

Why?

Euclidean circle packing



Hyperbolic circle packing



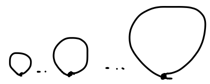
this picture describes the universal cover of certain hyperbolic surfaces. Within radius L, we have $\sim c \cdot e^L$ circles, each corresponding to a closed geodesic. $\frac{1}{cL}$ of them corresponds to primitive geodesics

Each circle has 2-3 outer circles associated with it.

(3) Mirzakhani's Theorem = Prime Simple Geodesic Theorem

Remarks:
- the non-prime versions of these theorems are trivial lattice counting
- the prime, non simple versions are primitive lattice point counting problems
- the prime simple version (Mirzakhani) is much more complex, since the condition of simpleness doesn't nicely translate to the lattice.

Can you hear the shape of ______ theorems

① Metric circles:  W_r = circle of length r

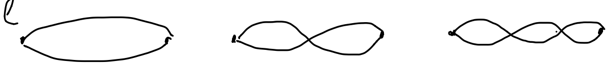
length spectrum
$$Sp(W_r) = \{\text{lengths of closed geodesics (with multiplicity)}\} = (r, 2r, 3r, \dots)$$

asymptotic length spectrum
$$C_{W_r}(L) = \#\text{ of closed geodesics of length} \le L$$

$$C_{W_r}(L) \sim \frac{L}{r}$$

Note: both $sp(W_r)$ and C_{W_r} determine r. So you can hear the "shape" of a circle.

Now consider a resonating string and the frequencies it generates.

Length: ℓ  ... Laplace spectrum

Freq: $\frac{1}{\ell}$ Freq: $\frac{2}{\ell}$ Freq: $\frac{3}{\ell}$

Close relationship between length spectrum and Laplace spectrum.

② Hyperbolic manifolds — here the spectrum includes lengths of primitive geodesics

Lots of examples that are isospectral but not isometric.

- Vignéras '80: Arithmetic construction
- Sunada '85: General construction using finite covers for nonpositively curved manifolds.

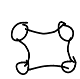
So the length spectrum of a hyperbolic surface X does not determine X.

Q: What does the asymptotic behavior of $\#C_X(L) \sim \frac{e^L}{L}$ determine? Nothing. Not even the genus of the surface.

Q: What does the asymptotic behavior of $S_X(L) \sim c(X) \cdot L^{6g-6+2n}$ determine?

If $n=0$, it determines the genus and gives information about the metric as well.

Open problem: Are there nonisometric hyperbolic surfaces with the same simple length spectrum?

McShane-Parlier (2008): two nonisometric  with same interior simple length spectrum.
They have different bdy lengths.

Maungchang (2013): Many isospectral, non-isometric hyperbolic surfaces arising from Sunada's construction are not simple isospectral.

Mapping class groups

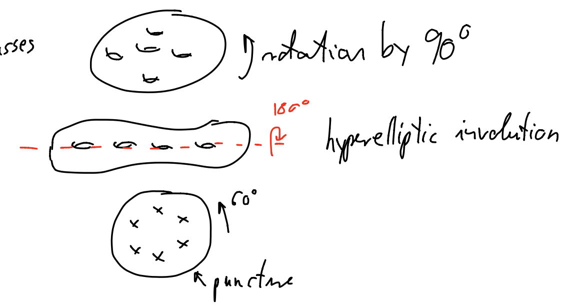
Class 4 (1/16)

$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S)/\text{isotopy}$

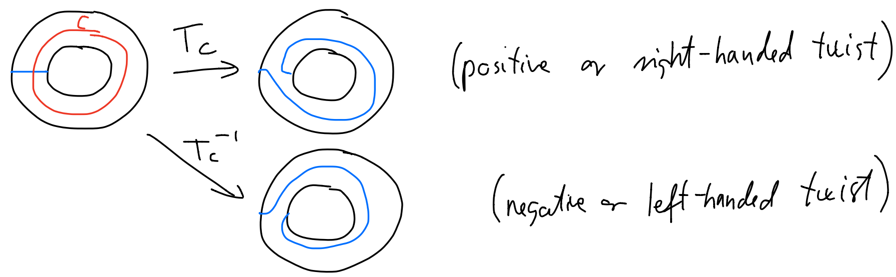
- boundary fixed pointwise
- punctures may be permuted

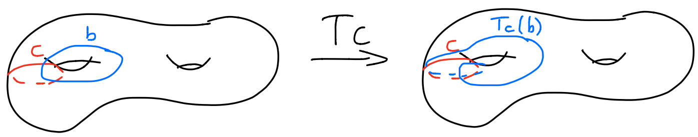
Examples:

(1) Finite order mapping classes



(2) Dehn twists



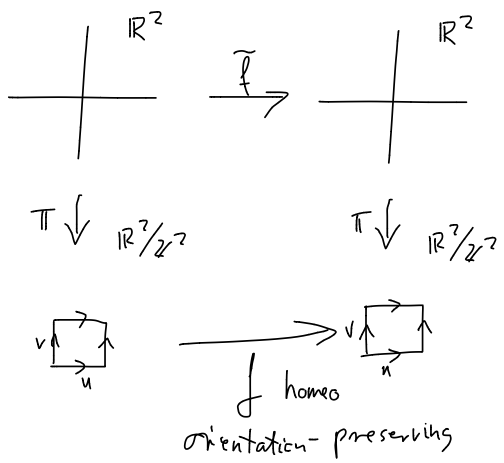


(3) Multi-twists: products of powers of Dehn twists about disjoint curves

But there are much more... If only we could classify them just like hyperbolic isometries....

Thurston ('70s): No problem!

Mapping class group of the torus (Nielsen)



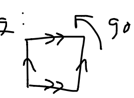
After isotopy:
- $\tilde{f}(0) = 0$, so $\tilde{f}(\mathbb{Z}^2) = \mathbb{Z}^2$
- $\pi^{-1}(u), \pi^{-1}(v)$ are straight line segments
- $\tilde{f}$ is linear

Area is preserved, so $\det(\tilde{f}) = 1$.
$\tilde{f}(\mathbb{Z}^2) = \mathbb{Z}^2$, so $\tilde{f} \in SL(2,\mathbb{Z})$.

$$\mathrm{Mod}(\circledcirc) \cong SL(2,\mathbb{Z})$$

We can classify $M \in SL(2,\mathbb{Z})$ by $tr(M)$

E.g.:



Case I: $|tr(M)| \leq 1$ Char. poly $=$ x^2+1 order 4
x^2+x+1 order 3
x^2-x+1 order 6

cyclotomic polynomials

E.g.

Case II: $|tr(M)| = 2$ Char. poly $=$ $x^2+2x+1 = (x+1)^2$
$x^2-2x+1 = (x-1)^2$

If geom. multiplicity $=$ alg. multiplicity: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(2) (2)

identity

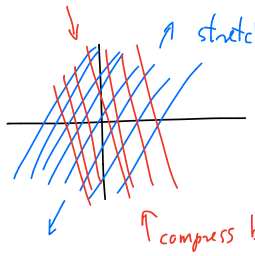
If geom. multiplicity $=1$ Jordan normal form: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & k \\ 0 & -1 \end{bmatrix}$

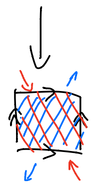
k-th power of Dehn twist

same, times 180° rotation

Case III : $|\mathrm{tr}(M)| > 2$ Char. poly : $x^2 - k\cdot x + 1$ $(|k| \geq 3)$

Eigenvalues : $\lambda, \frac{1}{\lambda}$ where $\lambda \in \mathbb{R}$, $|\lambda| > 1$.





$|\lambda|$ – stretch factor (dilatation)

Note : Mod() = Mod()

Moduli space and Teichmüller space

Riemann surface : surface with complex structure
$S_{g,n}$, no boundary

better definition : in addition, the complex structure extends to the puncture

Uniformization theorem (Poincaré, Koebe) There are three simply connected Riemann surfaces

$$\mathbb{C}\ ,\ \mathbb{C}\cup\{\infty\}\ ,\ \mathbb{H}=\{z : \mathrm{Im}(z)>0\}$$

$\mathrm{Aut}(U)$ — Conformal automorphisms of U = Möbius transformations mapping U to itself

$\mathrm{Isom}^+(U)$ — Or. preserving Isometry group of constant curvature metric on U compatible with complex structure

U	$\mathrm{Aut}(U)$	const curvature metric	$\mathrm{Isom}(U)$		
$\mathbb{C}\cup\{\infty\}$	$\left\{\frac{az+b}{cz+d} : ad-bc=1,\ a,b,c,d\in\mathbb{C}\right\}$	spherical (>0)	$SO(3)$		
$\mathbb{C}$	$\mathrm{Aff}(\mathbb{R}^2) \supseteq \{az+b : a,b\in\mathbb{C},\ a\neq 0\}$	flat ($=0$)	same with $\lvert a\rvert=1$		
$\mathbb{H}$	$\left\{\frac{az+b}{cz+d} : ad-bc=1,\ a,b,c,d\in\mathbb{R}\right\}$	hyperbolic (<0)	same		

Class 5 (1/23)

$T(S)=\{$marked Riemann surfaces homeomorphic to $S\}/$marked biholomorphism

+ condition on punctures

Uniformization

$=\{$faithful (injective) representations $\rho : \pi_1(S)\to \mathrm{Aut}(U)$ with discrete image such that $U/\mathrm{Im}(\rho)\overset{\text{homeo}}{\cong} S\}/$conjugation by $\mathrm{Aut}(U)$

$=\{$marked constant curvature surfaces homeomorphic to $S\}/$marked isometry and scaling

Marked Riemann surface : homeo $S\to R$ ← Riemann surface

$i_1 : S\to R_1$ and $i_2 : S\to R_2$ are equivalent if there is a biholomorphic map $h : R_1\to R_2$ such that

$$\begin{tikzcd} & R_1 \arrow[d, "h"] \\ S \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & R_2 \end{tikzcd}$$

commutes up to isotopy.

Eg :

are not equivalent

$\mathrm{Mod}(S) \curvearrowright T(S)$
$\quad f \in \mathrm{Mod}(S)$, $i \in T(S)$

$f\cdot i = i\circ f$

$M(S) = \{\text{Riemann surfaces homeomorphic to } S\} / \text{biholomorphism}$

Uniformization

$= \{\text{faithful (injective) representations } \rho : \pi_1(S) \to \mathrm{Aut}(U) \text{ with discrete image such that } U/\mathrm{Im}(\rho) \overset{\text{homeo}}{\cong} S\} / \text{conjugation by } \mathrm{Aut}(U)$

$\mathrm{Mod}(S) \backslash$

$= \{\text{constant curvature surfaces homeomorphic to } S\} / \text{isometry and scaling}$

$$\boxed{M(S) = \mathcal{T}(S) / \mathrm{Mod}(S)}$$

Examples

① $S = $ sphere $\quad \mathcal{T}(S) = $ one point $= \{\hat{\mathbb{C}}\}$

$\pi_1(S) = \{1\}$ $\quad = \{\pi_1(S) \xrightarrow{1 \mapsto 1} \mathrm{Aut}(\hat{\mathbb{C}})\}$

$\mathrm{Mod}(S) = \{1\}$ $\quad = \{\text{sphere with curvature } +1\}$

$M(S) = \mathcal{T}(S)$

Class 6 (1/28)

② $S = $ open disk $\quad \mathcal{T}(S) = $ two points $= \{\mathbb{C}, \mathbb{H}\}$

$\pi_1(S) = \{1\}$ $\quad = \{\pi_1(S) \xrightarrow{1 \mapsto 1} \mathrm{Aut}(\mathbb{C}), \pi_1(S) \xrightarrow{1 \mapsto 1} \mathrm{Aut}(\mathbb{H})\}$

$\mathrm{Mod}(S) = \{1\}$ $\quad = \{\mathbb{C} \text{ with flat metric}, \mathbb{H} \text{ with hyp. metric}\}$

$M(S) - \mathcal{T}(S)$

③ $S = $ open annulus

$\pi_1(S) = \mathbb{Z}$

- $U = \hat{\mathbb{C}}$ The universal cover of S is $\mathbb{R}^2$, not S^2, so this won't work
- $U = \mathbb{C}$ $f(z) = az + b$ has fixed point if $a \neq 1$.

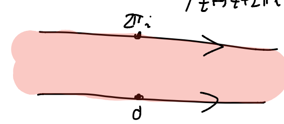
So we need to map into $\{z \mapsto z + b : b \in \mathbb{C}\}$

We can send the generator to any of these for $b \neq 0$.

Q: Which ones are conjugate by $\mathrm{Aut}(\mathbb{C})$?

A: $\frac{1}{c}[(cz+d) + b - d] = z + \frac{b}{c}$. All conjugate.

<u>Note</u>: $g(z) = e^z = e^{\mathrm{Re}(z) + i\,\mathrm{Im}(z)}$ is a conformal map between $\mathbb{C}/_{z \mapsto z+2\pi i}$ and $\mathbb{C} - \{0\}$



- $U = \mathbb{H}$ The image of $\pi_1(S)$ is either a parabolic or hyperbolic subgroup.
  - parabolic: all parabolic isometries are conjugate, so we get one point.

    <u>Note</u>: $e^{2\pi i z}$ is conformal $\mathbb{H}/_{z \mapsto z+1} \to \{z : 0 < |z| < 1\}$

  - hyperbolic: the generator of $\pi_1(S)$ can map to $z \mapsto \lambda z$ for any $0 < \lambda < 1$.

    these are not conjugate

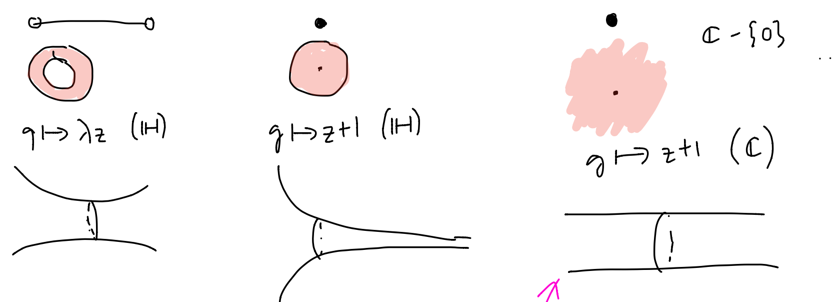
    <u>Note</u>: the annuli $\mathbb{H}/\langle z \mapsto \lambda z \rangle$ $(0<\lambda<1)$ are biholomorphic to the annuli $\{z : r < |z| < \frac{1}{r}\}$ $(0 < r < 1)$

$$\mathcal{T}(S) = \{\text{annuli } r_1 < |z| < r_2 \text{ where } 0 \le r_1 < r_2 \le \infty\}/\text{scaling}$$

Riemann surface

Representation $(\pi_1(S) = \langle g \rangle)$

Const. curvature surface

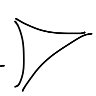


with the better definition, only this one survives

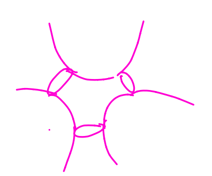
$\mathrm{Mod}(S) = \{1\}$

$M(S) = \mathcal{T}(S)$

④ $S = $ (sphere with three marked points) $\quad T(S) = M(S) = \{1\}$ Double of



The better definition is needed to rule out such Riemann surfaces:



Class 7 (1/30)

(5) $S =$ 

$\pi_1(S) = \mathbb{Z}^2$
$\quad \| $
$\langle u, v \rangle$

- $U = \hat{\mathbb{C}}$ doesn't work $\mathrm{Aut}(\hat{\mathbb{C}}) = \{1\}$, no injective $\pi_1(S) \to \mathrm{Aut}(\hat{\mathbb{C}})$
- $U = \mathbb{C}$ As for the annulus the image is in $\{z \mapsto z+b\}$

  After conjugation: $\rho(u) = z \mapsto z+1$

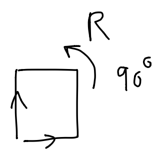
  $\rho(v) = z \mapsto z + c \quad (\mathrm{Im}(c) > 0)$

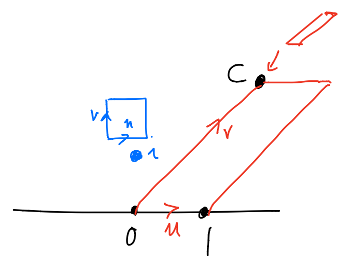
  All of these are different.

- $U = \mathbb{H}$ <u>Exercise</u>: there is no discrete faithful representation $\mathbb{Z}^2 \to \mathrm{Aut}(\mathbb{H}) \cong PSL(2,\mathbb{R})$

So $T(S) \cong \mathbb{H}$.

$M(S) = T(S) / \mathrm{Mod}(S)$

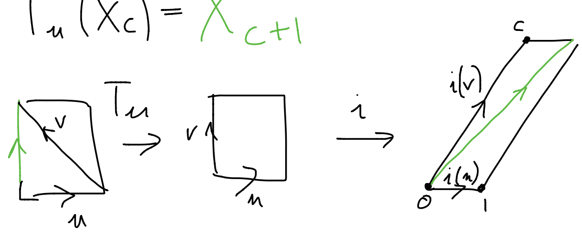




$\mathrm{Mod}(S) = \langle T_u, T_v \rangle = \langle T_u, R \rangle$

Let $(i : S \to X_c) \in T(S)$ for $c \in \mathbb{H}$.

$T_u(X_c) = X_{c+1}$



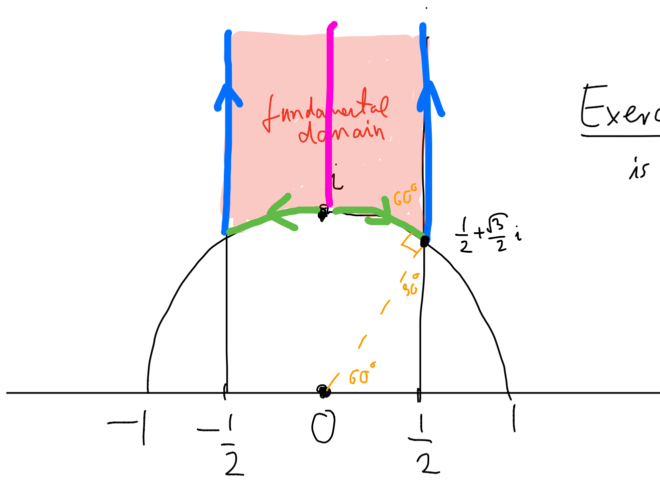
<u>Exercise</u>: $T_v(X_c) = X_{\frac{c}{1-c}}$

$R(X_c) = X_{-\frac{1}{c}}$

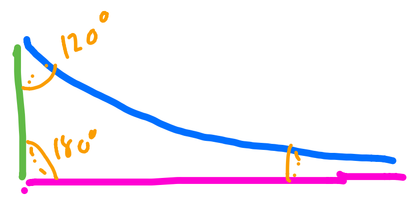
- $c \mapsto c+1$ parabolic with fixed point at ∞
- $c \mapsto -\frac{1}{c}$ elliptic with fixed point i

Order 2: $-\frac{1}{-\frac{1}{c}} = c$

$c = -\frac{1}{c}$
$c^2 = -1$
$c = \pm i$

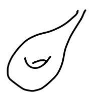


Exercise: Show that this region is indeed a fundamental domain.



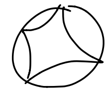
Orbifold

Ex: Show that the area of this hyperbolic orbifold is $\pi/3$.
(One way to do this: Integrate the hyp. metric over the fundamental domain.)

⑥ $S =$ 

Exercise: Describe natural bijections $\mathcal{T}(\,\cdot\,) \cong \mathcal{T}(\,\cdot\,)$ and $M(\,\cdot\,) \cong M(\,\cdot\,)$

Hint: Use Riemann surfaces. The isomorphism is not at all obvious from the constant curvature viewpoint, since [torus] has flat metric, but [surface] has hyperbolic

From the hyperbolic point of view, $\mathcal{T}(\,\cdot\,) = \{$configurations  $\}/\mathrm{Isom}(\mathbb{H})$

equivalence classes $\Longleftrightarrow$ cross ratios of the 4 bdy points
$\Uparrow$
$\mathbb{H}$

$M(\,\cdot\,) =$ quotient out by some equivalence on cross-ratios

⑦ $S = S_{g,n}$ $\quad \chi(S) = 2 - 2g - n < 0$

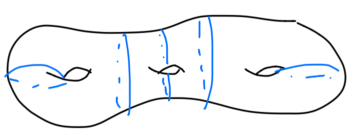
Thm (Fricke-Klein, 1897) $\mathcal{T}(S) \overset{\text{real analytic}}{\cong} \mathbb{R}^{6g-6+2n}$ (there is also natural complex structure)

Cor: $\mathcal{M}(S)$ is a $6g-6+2n$-dimensional orbifold.

n \ g	0	1	2
0	•	$\mathbb{H}$	dim=6
1	•	$\mathbb{H}$	
2	•	dim=4	
3	•		
4	$\mathbb{H}$ (dim=4)		

Fenchel-Nielsen coordinates

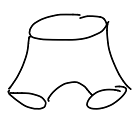
Q: How to parametrize $\mathcal{T}(S)$?



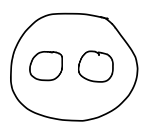
pants decomposition : Maximal collection of disjoint, pairwise non-isotopic essential simple closed curves

↑ not null-homotopic or homotopic to bdy or puncture

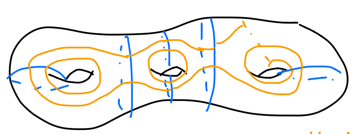
Complementary regions :

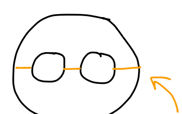


pairs of pants



Marked pants decomposition





collection of curves so that each pant looks like this

Fenchel-Nielsen coordinates : $\mathcal{T}(S) \to \mathbb{R}^{3g-3+n}_{>0} \times \mathbb{R}^{3g-3+n}$ $\quad$ $3g-3+n = \#$ of pants curves

$$X \mapsto \left(\ell_i(X), t_i(X)\right)_{i=1}^{3g-3+n}$$

length parameter = length of pants curve i in X

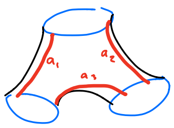
twist parameter = TBD

Class 8 (2/4)

<u>Prop</u>: A hyperbolic pair of pants is determined by the lengths of the three boundary curves.
I.e. "$\mathrm{Teich}(\text{pants}) \cong \mathbb{R}^3_{>0}$" ← Although later on, we will fix the boundary lengths for Teichmüller spaces of surfaces with bdry.

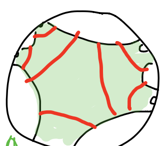
With that philosophy: $\mathrm{Teich}(\text{pants}, \ell_1, \ell_2, \ell_3) = \{\bullet\}$

<u>Note</u>: This implies that the length parameters determine the shapes of the pairs of pants of the decomposition.

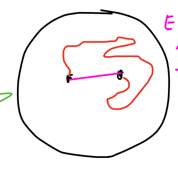
<u>Proof</u>: 

<u>Claim 1</u>: For each boundary pair, there is a unique simple geodesic arc between them, perpendicular to the boundaries. Moreover, the the three arcs are disjoint.

<u>Sketch of proof</u>: (Existence) Choose disjoint simple arcs like this, and lift to universal cover

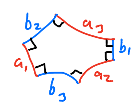
Universal cover: 

For any geodesic pair, there is a unique shortest arc between them. These arcs are disjoint and $\perp$ to the boundaries.

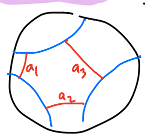


Every arc is homotopic to the straight arc.

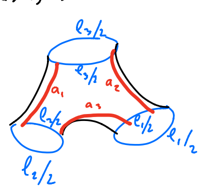
(Uniqueness) Up to isotopy, there is a unique choice for a_1, a_2, a_3
So this picture is canonical up to isotopy. ∎

Our pair of pants falls apart into two rectangular hexagons: 

<u>Claim 2</u>: The lengths a_1, a_2, a_3 determine b_1, b_2, b_3.
<u>Proof</u>: Exercise. <u>Hint</u>: Place the hexagon in $\mathbb{H}$ and extend the blue sides to bi-infinite geodesics.



<u>Corollary</u>: The two hexagons are isometric and the arcs cut each boundary component in half.



In other words, every hyperbolic pair of pants is a "double" of a rectangular hexagon.

Hyp. pair of pants $\longleftrightarrow$ rectangular hexagons $\longleftrightarrow$ positive triples $(\ell_1/2, \ell_2/2, \ell_3/2)$ ∎

<u>Exercise</u>: Explain how one can define the twist parameters. (Possible reference: Farb-Margalit)

Compactifying Teichmüller space

$\mathrm{Mod}(S)$ acts on $\mathcal{T}(S) \cong \mathbb{R}^{6g-6+2n}$

__Thurston__: Let's give $\mathcal{T}(S)$ a boundary to get a compactification $\overline{\mathcal{T}(S)} \cong \{x \in \mathbb{R}^{6g-6+2n} : \|x\| \le 1\}$ so that $\mathrm{Mod}(S)$ acts on $\overline{\mathcal{T}(S)}$ continuously. Then use the Brouwer fixed point theorem to classify elements of $\mathrm{Mod}(S)$.

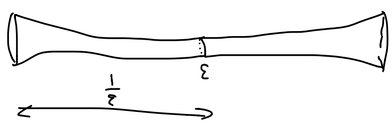
isotopy classes of simple closed curves

__Idea__: Every $X \in \mathcal{T}(S)$ induces a function $\ell_X : \mathcal{S}(S) \to \mathbb{R}_+$

Points on $\partial\mathcal{T}(S)$ will be "limit functions" $\mathcal{S}(S) \to \mathbb{R}_+$

__Q__: What is the limit of ℓ_X as we shrink the length of a (pants) curve α to 0?

__Collar lemma__: A simple closed geodesic of length ε has an $\approx \frac{1}{\varepsilon}$ wide embedded collar around it.



__A__: $\ell_{X_\varepsilon}(\gamma) = \begin{cases} \text{bounded}, & \text{if } \gamma \text{ is disjoint from } \alpha \\ \approx \frac{2}{\varepsilon} + \text{bounded} & \text{if } \gamma \text{ intersects } \alpha \text{ once} \\ \approx \frac{2k}{\varepsilon} + \text{bounded} & \text{if } \quad - \text{ " } - \quad k \text{ times} \end{cases}$

So $\lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \ell_{X_\varepsilon}(\gamma) \to i(\gamma, \alpha) =$ intersection number of γ and α

So $i_\alpha : \mathcal{S}(S) \to \mathbb{R}_{\ge 0}$ should be a function in $\partial\mathcal{T}(S)$
$\qquad \gamma \mapsto i(\gamma, \alpha)$

I.e, simple closed curves naturally sit in $\partial\mathcal{T}(S)$!

__Q__: How can we think of this geometrically?

A:

Think of a "metric" that is concentrated at α. Outside α, the length of the γ is 0, every time it crosses α, it gains length 1.

__More formally__.

$\mathbb{R}^{\mathcal{S}}$ = functions $\mathcal{S} \to \mathbb{R}$

$\mathcal{P}(\mathbb{R}^{\mathcal{S}}) = \mathbb{R}^{\mathcal{S}} - \{0\} / x \sim \lambda x$

Natural embeddings $\mathcal{T}(S) \to \mathcal{P}(\mathbb{R}^{\mathcal{S}})$
$\qquad X \mapsto [\ell_X]$

$$\mathcal{S} \to \mathbb{P}(\mathbb{R}^{\mathcal{S}})$$
$$\alpha \mapsto [i_\alpha]$$

Ex: Check that the map $\mathcal{S} \to \mathbb{P}(\mathbb{R}^{\mathcal{S}})$
- is well-defined (i.e., $i_\alpha \neq 0$)
- is injective (i.e., if $\alpha \neq \beta$, then $i_\alpha \neq i_\beta$)

Class 9 (2/6)

By definition, $\overline{\mathcal{T}(S)}$ is the closure of the image of $\mathcal{T}(S)$ in $\mathbb{P}(\mathbb{R}^{\mathcal{S}})$.
A countable (dense) subset of $\partial\mathcal{T}(S)$ is realized as "geometrically" as i_α.

Q: Can we think of all points of $\partial\mathcal{T}(S)$ geometrically?

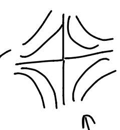
A: (Thurston) Yes!

Singular measured foliations

Foliation: locally 

E.g. 

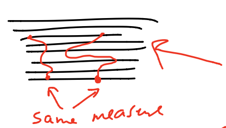
Every point is covered by exactly one leaf.

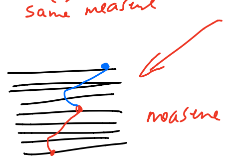
Singular foliation: locally  or  or  and so on

at punctures, we allow one-pronged singularities, as well.

Measured foliation: A foliation with a transverse measure

function from transverse arcs to $\mathbb{R}_{\geq 0}$ that is
- invariant under isotopy along leaves
- additive





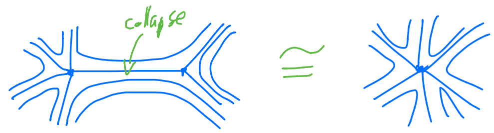
measure of concatenation = sum of measures

Alternatively: a collection of charts to $\mathbb{R}^2$ so that all transition maps are of the form $h(x,y) = (f(x,y), c \pm y)$

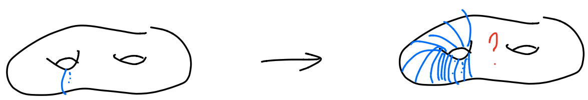
If the charts can be chosen so that there is always a $+$ sign, then the foliation is transversely orientable. (Oriented arcs have signed measure.)

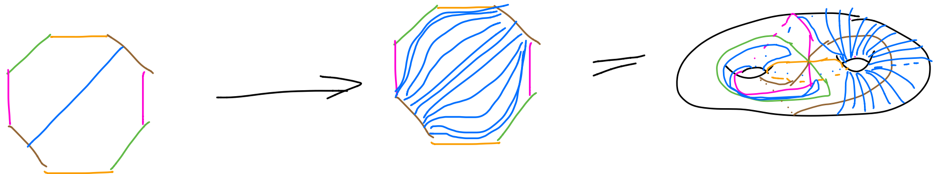
Ex: Give a definition of <u>singular</u> measured foliations using charts and transition maps.

$\mathcal{MF}(S)$ = space of (singular) measured foliations on S, up to isotopy and <u>Whitehead equivalence</u>



There is a natural inclusion $\mathcal{S} \subset \mathcal{MF}(S)$





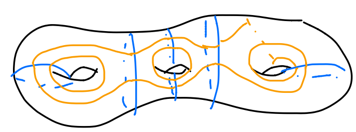
If $\mathcal{F} \in \mathcal{MF}(S)$, then $\displaystyle\int_{[\gamma]} \mathcal{F} = \inf_{\beta \in [\gamma]} \int_\beta \mathcal{F}$ i.e, infimum measure in isotopy class

$[\gamma] \in \mathcal{S}(S)$

Natural map $\mathcal{MF}(S) \longrightarrow \mathbb{R}^{\mathcal{S}}$

$\mathcal{F} \longmapsto \int_\cdot \mathcal{F}$

Dehn-Thurston coordinates



Once again, fix a marked pants decomposition.

For $\mathcal{F} \in \mathcal{MF}(S)$, there are $3g-3+n$ length parameters (measures of pants curves), and $3g-3+n$ twist parameters

$$\mathcal{MF}(S) \cong \mathbb{R}^{6g-6+2n}$$

$$\mathcal{PMF}(S) = \mathcal{MF}(S) - \{0\} /_{\mathcal{F} \sim \lambda\mathcal{F}} \cong S^{6g-7+2n}$$

projective measured foliation space

Natural map $\mathcal{PMF}(S) \to \mathbb{P}(\mathbb{R}^{\mathcal{S}})$.

<u>Thm</u> (Thurston): The image of $\mathcal{PMF}(S)$ in $\mathbb{P}(\mathbb{R}^{\mathcal{S}})$ is the boundary of the image of $\mathcal{T}(S)$.

Class 10 (2/11)

Measured laminations

Pull tight bi-infinite leaves of a singular foliation in a hyperbolic surface X

$\Longrightarrow$ geodesic lamination : • closed subset of X
• disjoint union of simple geodesics

Examples : See internet.

Measure defined similarly to foliations.

$\mathcal{ML}(S) \cong \mathcal{MF}(S)$
$\mathcal{PML}(S) \cong \mathcal{PMF}(S)$

can be used interchangeably

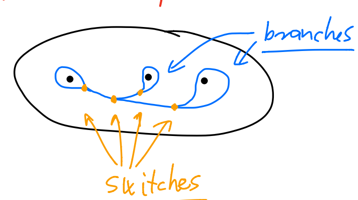
Charts on $\mathcal{ML}(S)$: train tracks

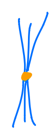
Why would we want local charts on $\mathcal{ML}(S)$? We already have global coordinates (Dehn-Thurston)

Disadvantages :
- only local coordinates

Advantages :
- properties of the transition maps tell us what geometric structure the space has

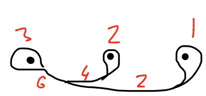
Train tracks :



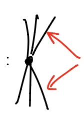


At each switch, the incident branches are tangent to some direction

Measures on train tracks

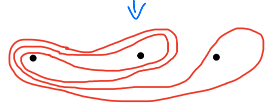


Switch condition :

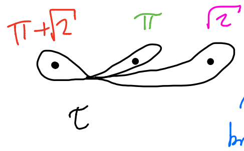


total measure on the two sides are equal

this encodes a (multi)curve

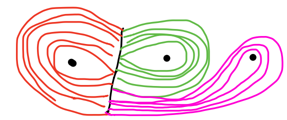


We don't need to restrict to integral measures.

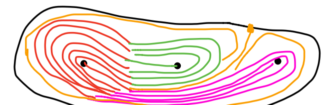


replace each branch with a foliated rectangle of given width

glue



collapse complementary regions



measured foliation

$$U(\tau)=\{\mathcal{F}\in\mathcal{MF}(S) : \mathcal{F} \text{ is carried on } \tau\} \subseteq \mathcal{MF}(S)$$

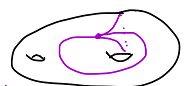
carried: can be represented by a measure

Fact: If τ is complete and recurrent, then $U(\tau)$ is a full-dimensional "cone" in $\mathcal{MF}(S)$.

complete: complementary regions are ▽ or ◯.

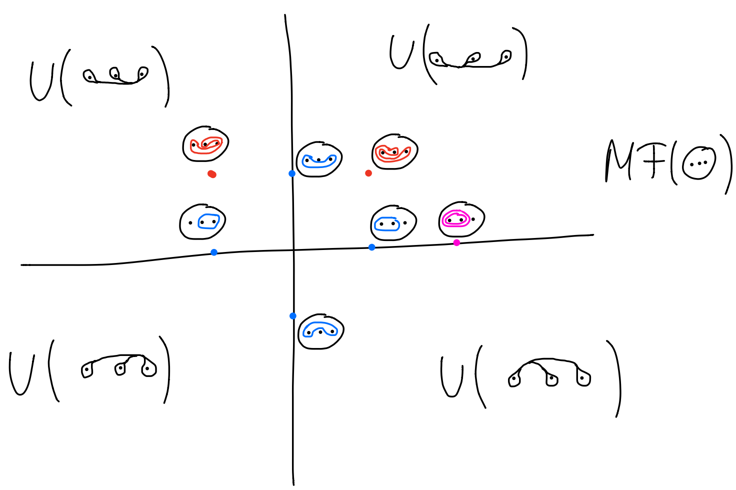
I.e., we cannot add more branches.

recurrent: τ admits a strictly positive measure



Not recurrent.

Example: $S=$ (⋯) $\quad g=0,\ n=4 \quad \dim(\mathcal{MF})=6g-6+2n=2$



$\underline{Q}$: The $U(\tau)$ are just subsets of $\mathcal{MF}(S)$. To set charts, we need to define maps $U(\tau) \to \mathbb{R}^d$, where $d = 6g-6+2n$. But how?

$\underline{A}$: There is the natural map $m: U(\tau) \to \mathbb{R}^{\#\text{ of branches}}$

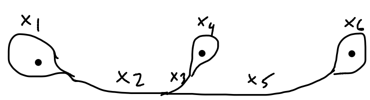
coordinates are measures of branches

But usually # of branches $> d$

E.g. if τ is , then

$$\#\text{ of branches} = 6$$
$$d = 2$$

$\underline{\text{Note}}$: The image $m(U(\tau))$ is contained in a subspace cut out by switch conditions



$$2x_1 = x_2$$
$$x_2 = x_3 + x_5$$
$$x_3 = 2x_4$$
$$x_5 = 2x_6$$

4 independent conditions $\Rightarrow$ they cut out a 2-dimensional subspace of $\mathbb{R}^6$!

Let $i: \mathbb{R}^d \to m(U(\tau)) \subset \mathbb{R}^{\#\text{ of branches}}$ be a linear isomorphism so that $i(\mathbb{Z}^d) = m(\underline{\mathcal{S}'})$

integral multicurves = finite disjoint union of simple closed curves

$$U(\tau) \xrightarrow{m} \mathbb{R}^{\#\text{ of branches}} \xleftarrow{i} \mathbb{R}^d$$

Then $i^{-1} \circ m : U(\tau) \to \mathbb{R}^d$ is a chart on $\mathcal{MF}(S)$.

$\underline{Q}$: What can we say about the transition maps?

$\underline{\text{Note}}$: The charts depend on two choices: τ and i.

① What if we change i?

$$\begin{tikzcd} & & \mathbb{R}^d \arrow[dl, "i_1"'] \\ U(\tau) \arrow[r] & \mathbb{R}^{\#\text{ of branches}} & \\ & & \mathbb{R}^d \arrow[ul, "i_2"] \arrow[uu, "i_1^{-1} \circ i_2"'] \end{tikzcd}$$

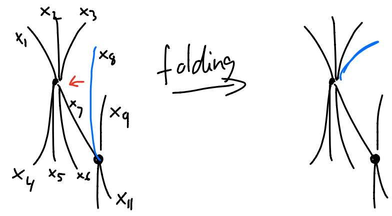
$i_1^{-1} \circ i_2 \in GL(d, \mathbb{Z})$

- linear
- $\mathbb{Z}^d \to \mathbb{Z}^d$

Class 11 (2/13)

② What if we change τ?

There are various ways to change τ, we'll just check one elementary operation.



τ_1 $\qquad\qquad\qquad$ τ_2

Q: What is the relationship between $U(\tau_1)$ and $U(\tau_2)$?

A: $U(\tau_1) \subseteq U(\tau_2)$. τ_1 is carried on τ_2, so any foliation carried on τ_1 is also carried on τ_2.

$$\begin{tikzcd}
U(\tau_1) \arrow[r] \arrow[d, phantom, "\bigcap"] & \mathbb{R}^{\#\text{ of branches}} \arrow[d, "{(x_1,\dots,x_n) \mapsto (x_1,\dots,x_6,\, x_7+x_8,\, x_8,\dots,x_{11})}"] & \mathbb{R}^d \arrow[l, "i_1"'] \\
U(\tau_2) \arrow[r] & \mathbb{R}^{\#\text{ of branches}} & \mathbb{R}^d \arrow[l, "i_2"']
\end{tikzcd} \qquad \in GL(d,\mathbb{Z})$$

So the $\mathbb{R}^d \to \mathbb{R}^d$ map is also in $GL(d,\mathbb{Z})$.

So: $ML(S)$ has a natural $GL(d,\mathbb{Z})$-structure!

Recall: $GL(d,\mathbb{R})$-structure $=$ piecewise linear structure

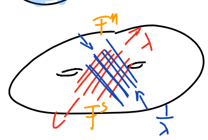
So $GL(d,\mathbb{Z})$-structure $= \underbrace{\text{piecewise integral linear}}_{PIL}$ structure

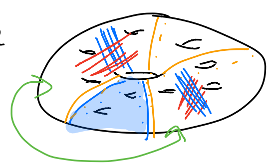
Note: • In a PIL structure, there is a well-defined set of integral lattice points.
• The PIL structure defines a natural Lebesgue measure where the covolume of the integral lattice is 1 under any chart. (Elements of $GL(d,\mathbb{Z})$ have determinant ± 1.)

μ_{Th} — the Thurston measure (natural Lebesgue measure) on $ML(S)$.

Classification of mapping classes

Nielsen-Thurston classification: $f \in \mathrm{Mod}(S)$ is either

- finite order 
- pseudo-Anosov 

- reducible 

Sketch of Thurston's proof: $\mathrm{Mod}(S)$ acts on $\overline{T(S)} = T(S) \cup PMF(S)$ continuously

$\overline{T(S)} \cong$ closed ball

Let $f \in \mathrm{Mod}(S)$.

Brower Fixed Point Theorem $\Rightarrow f$ has a fixed point in $\overline{T(S)}$

1. $\exists$ fixed point in $T(S)$ $\Rightarrow f$-invariant hyperbolic metric $\Rightarrow$ finite order

   Ex: Let X be a closed hyperbolic surface. Show that any isometry $X \to X$ has finite order.

2. $\exists$ fixed point in $PMF(S)$ with associated eigenvalue $1 \Rightarrow$ reducing multicurve

   (action on $MF(S)$ is piecewise linear) well-defined eigenvalue for fixed points in PMF

3. Otherwise two fixed points in $PMF(S)$, one with eigenvalue $\lambda > 1$, the other with $\frac{1}{\lambda}$. $\Rightarrow$ pseudo-Anosov, fixed points in PMF are the invariant foliations

Main Thm (Margalit-S.-Yurttas): Quadratic-time algorithm to find NT-type, order, reducing curves, pA stretch factors, foliations

Idea: Use piecewise linear action of $\mathrm{Mod}(S)$ on MF

Class 12 (2/18)

A more precise statement of Mirzakhani's theorem

<u>Recall</u>: $\ell_\gamma : \mathcal{T}(S) \to \mathbb{R}_+$ length of simple closed curve γ. $(\gamma \in \mathcal{S})$

But the definition also makes sense for any $\gamma \in \mathcal{S}'$ (integral multicurve).

Recall that $\mathcal{S}' \subseteq \mathcal{MF}$ is the integral lattice.

So we have $\ell : \mathcal{S}' \times \mathcal{T}(S) \to \mathbb{R}_+$.

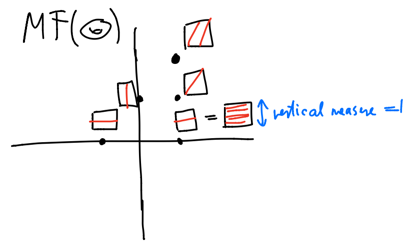
<u>Note</u>: Homogeneous in first coordinate.

There is unique continuous and homogeneous extension

$$\ell : \mathcal{MF} \times \mathcal{T}(S) \to \mathbb{R}_+$$

So it makes sense to talk about the length of a measured foliation on a hyperbolic surface.

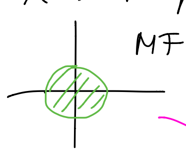
<u>Def</u>: For each $X \in \mathcal{T}(S)$, let B_X denote the unit ball in $\mathcal{MF}$ with respect to the metric X.

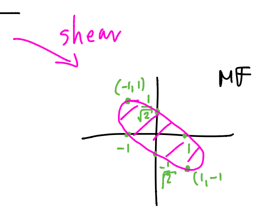


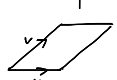
<u>Example</u> (torus) Recall: $\mathcal{T}(S) \cong \mathbb{H}$

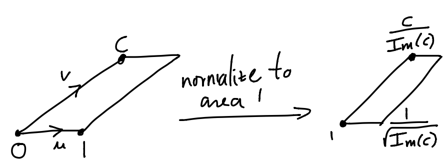
$\mathcal{MF} \cong \mathbb{R}^2/_{x \sim -x} \cong \mathbb{R}^2$

<u>Q</u>: What is B_X if X is the square torus  ? $= X_i$





<u>Q</u>: What is $B_{X_{1+i}}$? 



<u>Q</u>: What is B_{X_c} for a general $c \in \mathbb{H}$?

$$B_{X_c} = \left\{ (x,y) \in \mathbb{R}^2 : \left| x \frac{1}{\sqrt{\mathrm{Im}(c)}} + y \frac{c}{\sqrt{\mathrm{Im}(c)}} \right| \le 1 \right\}$$

$$\left\{ (x,y) : \left\| \begin{bmatrix} 1 & \mathrm{Re}(c) \\ 0 & \mathrm{Im}(c) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 \le \mathrm{Im}(c) \right\}$$

$$\left\{ \underline{v} \in \mathbb{R}^2 : \begin{bmatrix} 1 & Re(c) \\ 0 & Im(c) \end{bmatrix} \underline{v} \in B(0, \sqrt{Im(c)}) \right\}$$

$$\parallel$$

$$\begin{bmatrix} 1 & 0 \\ -Re(c) & Im(c)^{-1} \end{bmatrix} \cdot B(0, \sqrt{Im(c)})$$

ellipse of area π

Def: $B(X) = \mu_{Th}(B_X)$

$\Rightarrow$ function $B : \mathcal{T}(S) \to \mathbb{R}_+$, in fact B is $Mod(S)$-invariant, so $B : \mathcal{M}(S) \to \mathbb{R}_+$

Example (torus): $B : \mathcal{M}(S) \to \mathbb{R}_+$ is the constant $\pi/2$ function

Thm (Mirzakhani): If $\chi(S) < 0$, then $B : \mathcal{M}(S) \to \mathbb{R}_+$ is an integrable proper function.

integrable: with respect to Weil-Petersson metric on $\mathcal{M}(S)$, see later

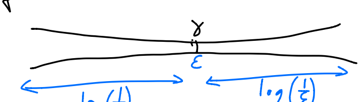
proper: preimages of compact sets are compact

Intuitively: $\lim_{X \to \partial \mathcal{M}(S)} B(X) = \infty$

(B goes to infinity at the boundaries)

Q: Intuitively, why should B be proper?

A: X is "close" to the boundary of $\mathcal{M}(S)$ if there is a very short curve γ. But then there is a large collar.



So the unit ball B_X of measured laminations should either be in the complement of γ (but this is not a full-dimensional subset) or go through the collar.

Let t_i and ℓ_i be the twisting and length parameters of a curve/foliation in Dehn-Thurston coordinates. Then the unit ball condition is roughly $\varepsilon |t_i| + \log(\frac{1}{\varepsilon}) \cdot \ell_i \leq 1$, and the area of these (t_i, ℓ_i) pairs go to ∞.

Def: A rational multicurve is $\gamma = \sum_{i=1}^{k} a_i \gamma_i \in \mathcal{MF}$, where the γ_i are disjoint essential simple closed curves and $a_i \in \mathbb{Q}$.

$$s_X(L, \gamma) = \#\{\alpha \in Mod(S) \cdot \gamma \mid \ell_\alpha(X) \leq L\}$$

Thm (Mirzakhani) Let $X \cong S_{g,n}$ be a complete, finite-volume hyperbolic surface with geodesic boundary. For any rational multicurve γ, we have

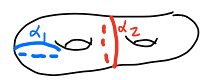
$$s_X(L,\gamma) \sim \frac{c(\gamma)\cdot B(X)}{\int_{M(S_{g,n})} B(X)\cdot dX} \cdot L^{6g-6+2n}$$

for some $c(\gamma) \in \mathbb{Q}_{>0}$

Corollary: $\lim_{L\to\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)} \in \mathbb{Q}_{>0}$, independent of the metric

Example (Mirzakhani): For $i=1,2$, let α_i be a curve on S_2 that cuts the surface into i components. Then

$$\frac{s_X(L,\alpha_1)}{s_X(L,\alpha_2)} \to 6$$



That is, there are 6 times as many nonseparating geodesics as separating ones.

Class 13 (2/20)

Lattice counting via convergence of measures

B_X = unit ball in $\mathcal{MF}$ with respect to hyp. metric

$$s_X(L,\gamma) = \#\{\alpha \in \mathrm{Mod}(S)\cdot\gamma \mid \ell_\alpha(X) \le L\}$$
$$= \#\{\alpha \in \mathrm{Mod}(S)\cdot\gamma \cap L\cdot B_X\}$$
$$= \#\left(\frac{\mathrm{Mod}(S)\cdot\gamma}{L} \cap B_X\right)$$

$\gamma \in \mathcal{ML}$
rational multicurve

Define the following measures on $\mathcal{MF}$

δ_α : Dirac measure concentrated at $\alpha \in \mathcal{MF}$.

$$\delta_\alpha(U) = \begin{cases} 1 & \text{if } \alpha \in U \\ 0 & \text{if } \alpha \notin U \end{cases}$$

$$\mu_\gamma = \sum_{f\in \mathrm{Mod}(S)} \delta_{f(\gamma)}$$

Q: Using $\mu_{1,\gamma}$, how can we write $s_X(1,\gamma)$?

A: $\mu_\gamma(B_X)$

Q: How can we write $s_X(L,\gamma)$?

A: $\mu_\gamma(L\cdot B_X)$

So $\lim_{L\to\infty} s_X(L,\gamma)$ is the same as studying $\lim_{L\to\infty} \mu_\gamma(L\cdot B_X)$

fixed measure, varying sets ☹

Would be better to have a fixed set and varying measures.

$$\mu_{L,\gamma} = \frac{1}{L^{6g-6+2n}} \sum_{f\in \mathrm{Mod}(S)} \delta_{\frac{f(\gamma)}{L}}$$

Mirzakhani's theorem is equivalent to

$$\lim_{L\to\infty} \mu_{L,\gamma}(B_X) = \frac{c(\gamma)\cdot B(X)}{\int_{\mathcal{M}(S)} B(X)\,dX} \qquad B(X) = \mu_{Th}(B_X)$$

More generally:

$$\lim_{L\to\infty} \mu_{L,\gamma}(U) = \frac{c(\gamma)}{\int_{\mathcal{M}(S)} B(X)\,dx} \cdot \mu_{Th}(U) \quad \text{for all } U \subseteq \mathcal{MF}$$

+ some hypothesis?

That is:

$$\mu_{L,\gamma} \xrightarrow{w^*} \frac{c(\gamma)}{\int_{\mathcal{M}(S)} B(X)\,dX}\,\mu_{Th}$$

Weak * convergence: $\mu_n \xrightarrow{w*} \mu$

$\Updownarrow$

$$\lim_{n\to\infty} \int f\,d\mu_n = \int f\,d\mu \quad \text{for all compactly supported continuous } f.$$

E.g.: $\delta_{\frac{1}{n}} \xrightarrow{w*} \delta_0$

(weak * convergence = pointwise convergence of operators)

So Mirzakhani's theorem translates to the fact that the measures $\mu_{L,\gamma}$ converge to a particular scalar multiple of the natural Lebesgue measure on $\mathcal{MF}$.

Proof of Mirzakhani's prime simple geodesic theorem

Note: WLOG we can assume that γ is an integral multicurve.

Steps of the proof

1. Any subsequence of $\mu_{L,\gamma}$ contains a weakly convergent subsequence.
2. The limit of any such subsequence is a multiple of μ_{Th}.
3. For all subsequences, it is the same multiple: $\frac{c(\gamma)}{\int_{\mathcal{M}(S)} B(X)\,dX}\,\mu_{Th}$.

Proof of step 1: The space of compactly supported continuous functions has a countable dense subset (e.g. "rational" piecewise linear functions).

So weak-* convergence $\Leftrightarrow$ convergence of integrals on these functions

Enough to show that for all $T>0$ there is $C(T,X,\gamma)$, independent of L s.t.

$$\int_{T\cdot B_X} \mu_{L,\gamma} \leq C(T,X,\gamma) < \infty$$

Then by the diagonal method, we can pick a convergence subsequence.

Prop: $\exists C \;\; \forall L \quad \#\{\alpha \in \mathrm{Mod}(S)\cdot\gamma : \ell_X(\gamma) \leq T\cdot L\} \leq C(T,X,\gamma)\cdot L^{6g-6+2n}$

Proof idea: Define Combinatorial length of γ using Dehn-Thurston coordinates

Step I (trivial): estimate holds for the combinatorial length instead of ℓ_X.

There are $6g-6+2n$ DT coordinates.

Step II: compare combinatorial and hyperbolic length

(not at all trivial, but not very hard either. Hyperbolic geometry)

Class 14 (2/25)

Proof of step 2: Suppose some convergent subsequence has limit ν.

1) $\mu_{Th}(V)=0 \Rightarrow \nu(V)=0$. (I.e., ν isn't concentrated to small sets like the Dirac measures.)

Let $U \subset \mathcal{MF}$ be a convex open subset of a train track chart.

counting all integral points not only γ-orbit

# of lattice points $\approx$ volume

$$\nu(U) \underset{\text{weak-* conv.}}{\leq} \liminf_{i\to\infty} \underbrace{\mu_{L_i,\gamma}(U)}_{= \frac{\#\{\text{integral points in the orbit of } \gamma \text{ in } L_i\cdot U\}}{L_i^{6g-6+2n}}} \leq \lim_{i\to\infty} \frac{\#\{\text{integral points in } L_iU\}}{L_i^{6g-6+2n}} = \lim_{i\to\infty} \frac{\mu_{Th}(L_iU)}{L_i^{6g-6+2n}}$$

$$= \lim_{i\to\infty} \frac{L_i^{6g-6+2n}\,\mu_{Th}(U)}{L_i^{6g-6+2n}} = \mu_{Th}(U)$$

Then approximate an arbitrary V with sets like U.

2) Ergodicity of $\mathrm{Mod}(S) \curvearrowright \mathcal{MF}$ implies $\nu = k\cdot\mu_{Th}$

Theorem of Masur ('82)

Intuitively: ν is $\mathrm{Mod}(S)$ invariant, and $\mathrm{Mod}(S)$ "mixes up" $\mathcal{MF}$ well, so ν has to be "just as uniformly distributed" as the Lebesgue measure.

Note: We are using Part I) here. The measures $\mu_{L_i,\gamma}$ are also $\mathrm{Mod}(S)$-invariant, but not of the form $k \cdot \mu_{Th}$.

Proof of step 3: For any convergent subsequence indexed by J,

$$\mu_{L_i,\gamma} \to k_J \cdot \mu_{Th}$$

where $k_J = k$ is independent of J.

We have

$$\begin{array}{ccc} \mu_{L_i,\gamma}(B_X) & \to & k_J \, \mu_{Th}(B_X) \\ \| & & \| \\ \dfrac{s_X(L_i,\gamma)}{L_i^{6g-6+2n}} & \longrightarrow & k_J \cdot B(X) \end{array}$$

Now what? How can we get rid of the dependence on the subsequence? The problem is that both s_X and $B(X)$ are complicated expressions, subtly depending on the geometry of X, so we can't expect to obtain nice formulas for them.

MIRZAKHANI: INTEGRATE BOTH SIDES OVER MODULI SPACE !!!

Before, even the WP-volume of moduli space was unknown. That is, nobody knew how to integrate the constant one function.

In addition, $s_X(L_i,\gamma)$ is a quite complex function defined using hyperbolic geometry.

Thm (Mirz.) For any multicurve γ, $\int_{M(S)} s_X(L,\gamma)\, dX$,

is a polynomial of degree $6g-6+2n$ in L.

Def: $c(\gamma)$ – leading coeff. If γ is rational, then $c(\gamma) \in \mathbb{Q}$

Continuing the proof:

$$\frac{s_X(L_i,\gamma)}{L_i^{6g-6+2n}} \longrightarrow k_J \cdot B(X)$$

Lebesgue dominated convergence theorem

$$\frac{\int_{M(S)} s_X(L_i,\gamma)\, dX}{L_i^{6g-6+2n}} \longrightarrow k_J \int_{M(S)} B(X)\, dX$$

Thm

$$\longrightarrow c(\gamma)$$

rational

So $k_J = \dfrac{c(\gamma)}{\int_{M(S)} B(X)\, dX}$ is indeed independent of J.

$\pi^{6g-6+2n}$ · rational

Volume of moduli spaces aka integrating the constant function

McMullen ('98): $\mathrm{vol}_{\mathrm{Teich}}(M_{g,n}) < \infty$

Exercise previously: $\mathrm{vol}_{\mathrm{Teich}}(M_{1,0}/M_{1,1}/M_{0,4}) = \frac{\pi}{3}$

The exact value of $\mathrm{vol}_{\mathrm{Teich}}(M_{g,n})$ seems to be unknown in other cases.

Wolpert ('83): $\mathrm{vol}_{WP}(M_{1,0}/M_{1,1}/M_{0,4}) = \frac{\pi^2}{6}$

Nakanishi–Näätänen (2001): $\mathrm{vol}_{WP}(M_{1,1}(L)) = \frac{\pi^2}{6} + \frac{L^2}{24}$

Zograf ('98): recursive formula for $\mathrm{vol}_{WP}(M_{0,n})$.

Mirzakhani (2007): Recursive formula for all $\mathrm{vol}_{WP}(M_{g,n}(L_1,\ldots,L_n))$

Mirzakhani's computation of $\mathrm{vol}_{WP}(M_{1,1})$

We need to integrate the constant 1 function on $M_{1,1}$.

Brilliant idea: geometrically inspired partition of unity

McShane's identity ('91): Let X be any hyperbolic once-punctured torus.

Then
$$\sum_{\gamma} \frac{1}{1+e^{\ell(\gamma)}} = \frac{1}{2}$$

where the sum is over simple closed geodesics γ on X, and $\ell(\gamma)$ is the length of γ.

<u>Rewrite</u> (1st try): $\sum_{\gamma} \frac{2}{1+e^{\ell_\gamma(X)}} = 1$ ← constant 1 function on $M(S)$

we want to think about these terms as functions on $M(S)$, but unfortunately ℓ_γ only works on X, since γ is a geodesic on X.

What if we define $\ell_\gamma(X)$ as the length of the geodesic on X in the homotopy class of γ?

But then, what would γ iterate over?
Isotopy classes of simple closed curves on .... what?
A topological torus?
... The problem is that the surfaces X are not marked in any particular way, so no particular curve on X would be specified.

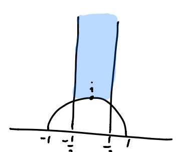
Bottom line: ℓ_γ is not a well-defined function on $M_{1,1}$.

Class 15 (2/27)

<u>Rewrite</u> (2nd try): $M^*_{1,1} = \{(X,\gamma) : X \in \overset{M(S_{1,1})}{\overset{\parallel}{M_{1,1}}},\ \gamma \text{ is a simple closed geodesic on } X\}$

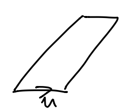
covering of $M(S)$

$$\pi : M^*_{1,1} \to M_{1,1}$$
$$(X,\gamma) \mapsto X$$

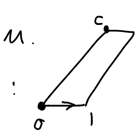
<u>Q</u>: We know $M_{1,1}$ can be identified with 

What about $M^*_{1,1}$?

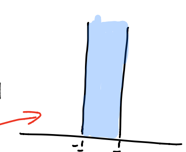
Think about $M^*_{1,0} \to M_{1,0}$ instead.

$M^*_{1,0}$ is the set of flat tori marked with one curve. 

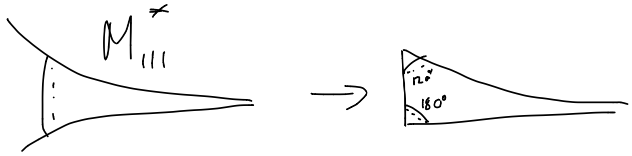
Two such marked tori are equivalent if there is a scaling + isometry between them fixing u.

After scaling and rotation, we can drawing the tori in $\mathbb{H}$ like this: 

c is well-defined up to addition of integers. So $M^*_{1,0} \cong \mathbb{H}/_{z\mapsto z+1}$

much simpler fd than $M_{1,0}$ 

$M_{1,1} \cong M_{1,0}$ and $M_{1,1}^{*} \cong M_{1,0}^{*}$, so the covering $M_{1,1}^{*} \to M_{1,1}$ can be visualized like this.



Now we have a well-defined function

$$\ell : M_{1,1}^{*} \to \mathbb{R}_{+}$$

$$\ell(X,\gamma) = \ell_X(\gamma)$$

So

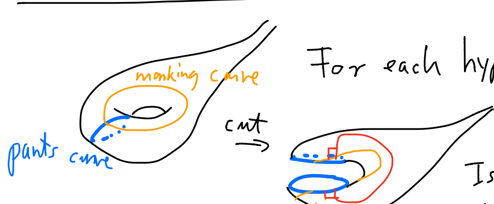
$$1 = \sum_{\gamma} \frac{2}{1+e^{\ell_\gamma(X)}} = \sum_{\pi(Y)=X} \frac{2}{1+e^{\ell(Y)}} \quad \text{holds for any } X \in M_{1,1}$$

So

$$\mathrm{vol}(M_{1,1}) = \int_{M_{1,1}} 1\, dX = \int_{M_{1,1}} \sum_{\pi(Y)=X} \frac{2}{1+e^{\ell(Y)}} dX = \int_{M_{1,1}^{*}} \frac{2}{1+e^{\ell(Y)}} dY$$

McShane's identity crucially uses the hyperbolic metric, so at this point, we really have to think about $M_{1,1}^{*}$ as space of hyp. surfaces, not flat surfaces.

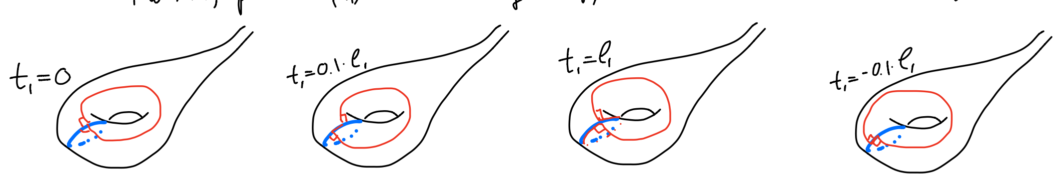
<u>Fenchel-Nielsen coordinates</u>



For each hyp. metric X,

Isotope the arc obtained from the marked curve so that it is perpendicular to the boundary.

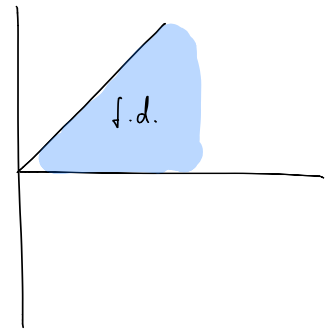
Twisting parameter (t_1) = "amount of isotopy" needed along the boundary.



Remark: there are different ways to normalize the twist parameter.
A full twist is 2π in Farb-Margalit, but we'll see that in some sense the length of the twisting curve is more natural.

$T_{1,1} \cong \mathbb{R}_+ \times \mathbb{R}$

$X \to (\ell_1(X), t_1(X))$



We have seen that $M_{1,1}^* = T_{1,1} \Big/ {\scriptstyle \text{Dehn twist about marked curve}} \cong \mathbb{R}_+ \times \mathbb{R} \big/ (\ell,t) \sim (\ell, t+\ell)$

Class 16 (3/4)

Thm (Wolpert '82) The Weil-Petersson symplectic form in Fenchel-Nielsen coordinates is $\sum_{i=1}^{3g-3+n} d\ell_i \wedge dt_i$.

(This result is so nice only if the twist parameter is ℓ_i for a full twist, and not normalized to 2π.)

Therefore the WP metric is just the standard Euclidean metric in Fenchel-Nielsen coords!!!

Cor (Wolpert '75): The WP metric is incomplete.

Completion of $M_{g,n}$ with WP metric = Deligne-Mumford compactification

So $\mathrm{vol}_{WP}(M_{1,1}^*) = \int_{M_{1,1}^*} \frac{2}{1+e^{\ell(Y)}} \, dY_{WP} = \int_0^\infty \int_0^\ell \frac{2}{1+e^\ell} \, dt \, d\ell = 2 \int_0^\infty \frac{\ell}{1+e^\ell} \, d\ell = 2 \cdot \frac{\pi^2}{12} = \frac{\pi^2}{6}$

So what happened that made the integration possible?

$\int_{M_{1,1}} 1 \, dX$ was reduced to $\int_{M_{1,1}^*} \cdots \, dY$. Using the method above, can we integrate any function on $M_{1,1}^*$?

We have to be able to express the function in Fenchel-Nielsen coords.

In particular any function $f(\ell(Y))$ is fine.

$\uparrow$

call these geometric functions

Integrating geometric functions on moduli space (higher genus surfaces)

$\gamma = \sum_{i=1}^{k} c_i \gamma_i$ multicurve

$\ell_\gamma(X) = \sum_{i=1}^{k} c_i \ell_{\gamma_i}(X)$

$f : \mathbb{R}_+ \to \mathbb{R}_+$

Geometric function: $f_\gamma : M_{g,n} \to \mathbb{R}_+$

$$f_\gamma(X) = \sum_{\alpha \in \mathrm{Mod}\cdot\gamma} f(\ell_\alpha(X))$$

Examples

① $f = \chi_{[0,L]}$. $\Rightarrow f_\gamma(X) = \#\{\alpha \in \mathrm{Mod}\cdot\gamma : \ell_\alpha(X) \le L\} = s_X(L,\gamma)$

Therefore $s_\cdot(L,\gamma)$ are geometric functions.

② Γ = set of different types of integral multicurves on the surface S (i.e, Mod(S)-orbits of integral multicurves)

Then $B(X) = \mu_{Th}(B_X) = \mu_{Th}\left(\{F \in \mathcal{MF} : \mu_{Th}(F) \le 1\}\right)$

$$= \lim_{L\to\infty} \frac{\#\{\text{integral lattice points in } L \cdot B_X\}}{L^{6g-6+2n}} = \lim_{L\to\infty} \frac{\sum_{\gamma_i \in \Gamma} s_X(L,\gamma_i)}{L^{6g-6+2n}}$$

Therefore $B(\cdot)$ can be approximated by geometric functions.

Passing to a cover of $M_{g,n}$ — Class 17 (3/6)

Simple case: γ = simple closed curve

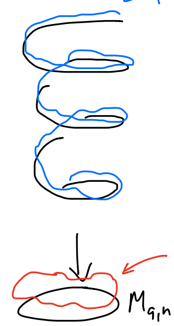
$$M^*_{g,n} = \{(X,\gamma) : X \in M_{g,n},\ \gamma \text{ simple closed geodesic on } X\}$$

Natural projection: $\pi : M^*_{g,n} \to M_{g,n}$
(in fact, covering)
$$\pi(X,\gamma) = X$$

Length function: $\ell : M^*_{g,n} \to \mathbb{R}_+$
$$\ell(X,\gamma) = \ell_\gamma(X)$$

We can rewrite $f_\gamma(X) = \sum_{\alpha \in \mathrm{Mod}\cdot\gamma} f(\ell_\alpha(X))$

$$= \sum_{\pi(Y)=X} f(\ell(Y))$$



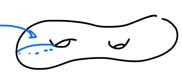
So
$$\int_{M_{g,n}} f_\gamma(X)\,dX = \int_{M_{g,n}} \sum_{\pi(Y)=X} f(\ell(Y))\,dX = \int_{M^*_{g,n}} f(\ell(Y))\,\underbrace{dY}_{\text{lift of } dX}.$$

Problem: Even though $T_{g,n}$ is nice in Fenchel-Nielsen coordinates, $M^*_{g,n} = T_{g,n}/\mathrm{Stab}(\gamma)$ is now complicated, since $\mathrm{Stab}(\gamma)$ is huge.

Key observation: The level sets of ℓ are lower genus moduli spaces!!!

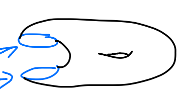
$\{Y : \ell(Y) = t\}$ = pairs (X,γ) where the length of γ on X is the prescribed value t.

Surfaces where this curve has length t



$\Updownarrow$

Cut-up surfaces where these two boundary components have length t



$\Rightarrow$ surfaces with boundary naturally arise.

$S_{g,n}$ – genus g surface with n boundary components

$T_{g,n}(L_1,\dots,L_n)$ = Teichmüller space of hyperbolic structures on $S_{g,n}$ with bdy lengths L_i

$M_{g,n}(L_1,\dots,L_n) = T_{g,n}(L_1,\dots,L_n)/\mathrm{Mod}_{g,n}$ ← boundary components fixed

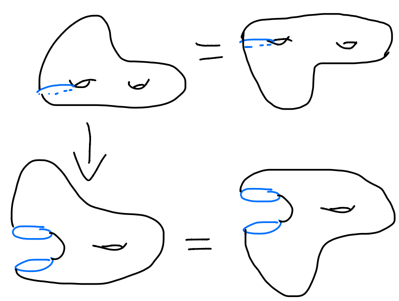
If $L_i = 0$, the i-th boundary is a cusp.

So $T_{g,n}(0,\dots,0) = T_{g,n}$ $\qquad \cong M_{1,2}(t,t)\big/$ involution reversing the orientation of γ

$\quad M_{g,n}(0,\dots,0) = M_{g,n}$

<u>Example</u>: On S_2, we have a circle bundle

$$\begin{array}{c} \{Y \in M^{\gamma}_{2,0} : \ell(Y) = t\} = \ell^{-1}(t) \\ \downarrow \\ M^{\text{unlabeled}}_{1,2}(t,t) \end{array}$$



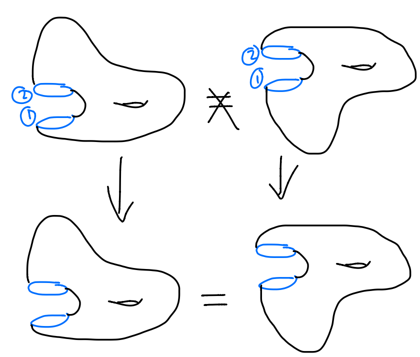
(bdy components not labeled)

There is a fiberwise S^1-action on the top space by partial twisting.

Mirzakhani: this is a Hamiltonian action, so

$$\mathrm{vol}(\ell^{-1}(t)) = t \cdot \mathrm{vol}(M^{\text{unlabeled}}_{1,2}(t,t))$$

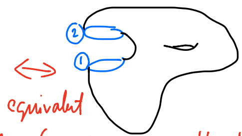
Double cover $\qquad$ $$\begin{array}{c} M_{1,2}(t,t) \\ \Big\downarrow \text{forget labelling} \\ M^{\text{unlabeled}}_{1,2}(t,t) \end{array}$$



$$\mathrm{vol}(M_{1,2}(t,t)) = 2\,\mathrm{vol}(M^{\text{unlabeled}}_{1,2}(t,t))$$

So

$$vol\left(\ell^{-1}(t)\right) = \frac{vol(M_{1,2}(t,t))\cdot t}{2}$$



Not equivalent in $M_{1,2}(L,L)$, since the bdy is numbered

$$\text{So } \int_{M_{2,0}} f_\gamma(X)\,dX = \int_{M^*_{2,0}} f(\ell(Y))\,dY = \int_0^\infty f(t)\cdot vol\left(\{Y\in M^*_{2,0} : \ell(Y)=t\}\right)dt = \int_0^\infty f(t)\frac{vol\left(M_{1,2}(t,t)\right)\cdot t}{2}\,dt$$

<u>substitute</u>: $\ell(Y)=t$

Therefore we can integrate geometric functions of $M_{g,n}$ if we can compute volumes of moduli spaces of bordered Riemann surfaces!!

How do we compute volumes?, Integrate the constant 1 function. But how? It would be nice to have a higher dimensional McShane's identity that would turn the constant 1 function a geometric function...

Mirzakhani: no problem! $\Rightarrow$ Mirzakhani–McShane identity for higher genus surfaces.

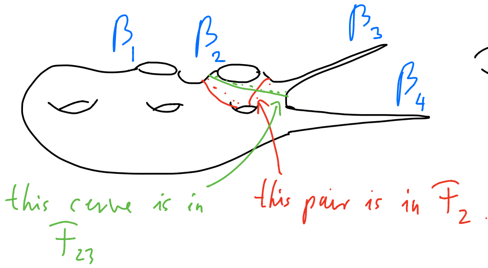
$\Rightarrow$ one can recursively compute volumes of moduli spaces by writing the constant 1 function as a geometric function and using the level set trick.

Class 18 (3/11)

McShane's identity (torus '91): Let M be a hyperbolic once-punctured torus. Then

$$\sum_{\gamma} \frac{1}{1+e^{\ell(\gamma)}} = \frac{1}{2}$$

where the sum is taken over all simple closed geodesics on M.



$\mathcal{F}_i$ = set of unordered pairs of isotopy classes of nonperipheral simple closed curves $\{\gamma_1, \gamma_2\}$ bounding a pair of pants with β_i

$\mathcal{F}_{ij}$ = set of isotopy classes of simple closed curves γ bounding a pair of pants with β_i, β_j.

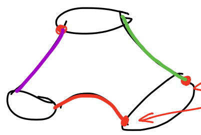
McShane's identity (higher genus '98): Let X be a complete hyperbolic surface with punctures $p_1, \ldots, p_n$ and no boundary. Then

$$\sum_{\{\gamma_1,\gamma_2\} \in \mathcal{F}_1} \frac{1}{1+e^{\frac{\ell_{\gamma_1}(X)+\ell_{\gamma_2}(X)}{2}}} + \underbrace{\sum_{i=2}^{n} \sum_{\gamma \in \mathcal{F}_{1,i}} \frac{1}{1+e^{\frac{\ell_\gamma(X)}{2}}}}_{\text{special case of the expression in the first sum when one bdy of the pair of pants has length 0.}} = \frac{1}{2}$$

(Actually, McShane proved this in the case $n=1$, but Mirzakhani generously credited this more general version to McShane, as well.)

The Mirzakhani - McShane identity

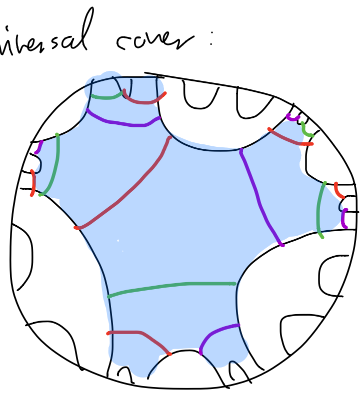
Recall: there are unique simple geodesic arc $\perp$ to each pair of boundaries of a hyp. pair of pants with geodesic boundary.



Recall: these are antipodal points.

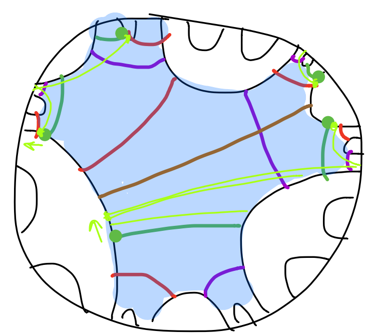
σ: involution interchanging the two hexagons

Universal cover:

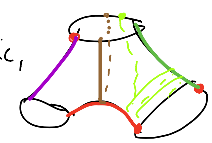


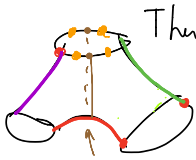
Vary the point the green segments emanate from perpendicular.

until the segment becomes infinite-length



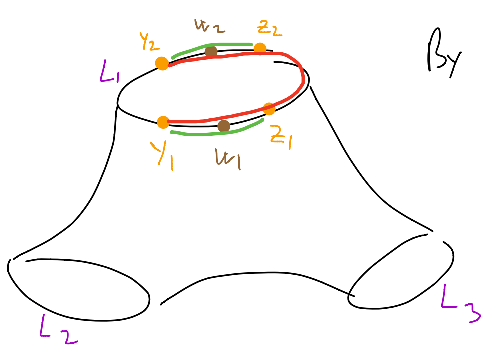
We obtain a simple geodesic, $\perp$ to a boundary that spirals toward another boundary component.





There are 4 such geodesics from each boundary component, since we can vary both points in both directions.

There is also a geodesic arc connecting the ∂ with itself perpendicularly.



By symmetry, we have

$\sigma(y_1) = y_2$

$\sigma(w_1) = w_2$

$\sigma(z_1) = z_2$

$$\mathcal{R}(L_1, L_2, L_3) = \ell(y_1\, y_2)$$

containing w_i

$$\mathcal{D}(L_1, L_2, L_3) = \ell(y_1\, z_1) + \ell(y_2\, z_2)$$

containing w_i

Mirzakhani-McShane identity for bordered surfaces (Mirzakhani '07)

For any $X \in \mathcal{T}_{g,n}(L_1, \dots, L_n)$ with $3g-3+n>0$, we have

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \mathcal{D}\left(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X)\right) + \sum_{i=2}^{n} \sum_{\gamma\in\mathcal{F}_{1,i}} \mathcal{R}\left(L_1, L_i, \ell_\gamma(X)\right) = L_1$$

Ex: Show that this implies the orignal McShane identities. (see Mirzakhani's paper "Simple geodesics and ...")

Birman-Series ('85): The union of complete simple geodesics in a closed hyp. surface has Hausdorff dimension 1.

Ex: Summarize the proof using Birman-Series: "Geodesics with bounded ..."

Class 19 (3/13)

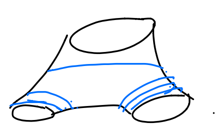
Complete simple geodesics on hyperbolic surfaces

① Simple closed geodesics

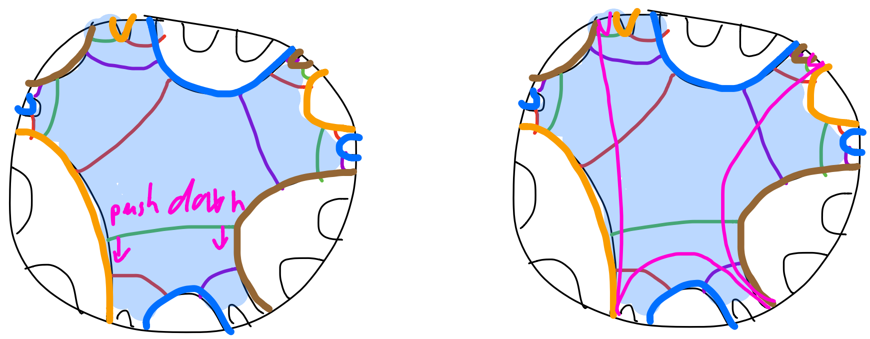
② Spiraling to simple closed geodesics

Q: How do we know these exist for any pair of disjoint simple closed curves?

A: Take a third curve that bounds a pair of pants with the two curves. Need to construct

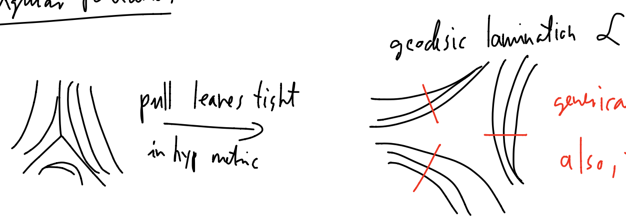


Universal cover:



Def: A geodesic lamination is a compact subset of a hyperbolic surface that is a disjoint union of complete simple geodesics.

Singular foliation

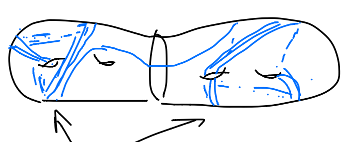


generically, cross-sections are Cantor sets

also, the closure of each leaf is $\mathcal{L}$

③ Closure of the simple geodesic has Cantor set cross-sections (the closure can fill the whole surface or a subsurface)

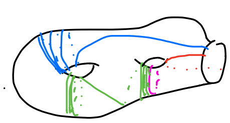
④ The limit sets of the simple geodesics in the two directions (Ω_+, Ω_-) are different.



also, either of the two limit sets can be a simple closed curve.

$E(X)$ = union of simple complete geodesics in X meeting ∂X perpendicularly

$\cap$
X

E.g. 

$E_i = E \cap \beta_i$

Lemma 1: The sets E_i have measure 0.

Proof: Double X. Doubling E is a union of simple geodesics in X.

$$\dim_{Haus}(E) = 1 \Rightarrow \dim_{Haus}(E_i) = 0 \Rightarrow \mu(E_i) = 0.$$

Lemma 2: Each E_i is homeomorphic to a Cantor set union finitely many isolated points.

Let γ_x be the complete geodesic emanating $\perp$ from $x \in E_i$.

Lemma 3: For any $x \in E_i$, exactly one of the following holds:

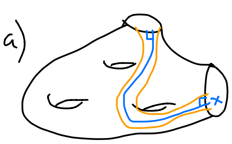
(a) If the other end of γ_x approaches a boundary component, then x is an isolated component of x.

(b) If γ_x spirals to a simple closed curve inside X, then x is a boundary point of E_i.

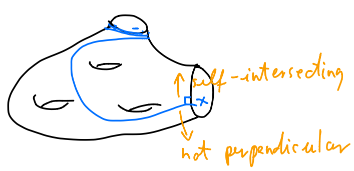
(c) If γ_x spirals to a lamination that is not a simple closed curve, then x is neither a boundary nor an isolated point.

Ex: Summarize the proof of Lemma 2 and 3 using Mirzakhani's paper "Simple geodesics..."

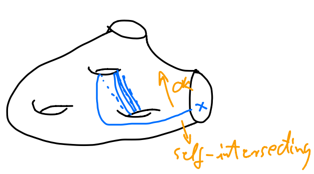
Intuitively: a)



Either way we move x, the new perpendicularly emanating geodesic is not perpendicular to the other ∂.



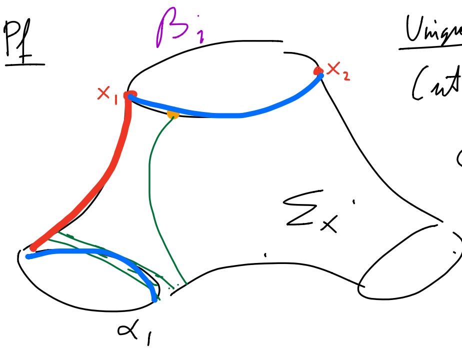
b)



Class 20 (3/25)

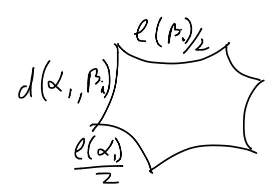
Prop: Let $x \in E_i$ such that γ_x spirals into a simple closed geodesic α_1. Then there is a unique embedded pair of pants Σ_x on X such that $\gamma_x \subseteq \Sigma_x$.

Pf



__Uniqueness__

Cut Σ_x into the two hexagons. The lengths of 3 consecutive sides are known:



These 3 length determine the shape of the hexagon, hence the shape of Σ_x.

__Existence__: Take the abstract hexagon determined by the 3 lengths, and observe that there is no obstructions for embedding it into X in the two ways we want. These two embedded hexagons form the desired Σ_x.

$I_i = \{$isolated points of $E_i\}$

$$\underbrace{I_i \cup (\beta_i - E_i)}_{\text{circle} - \text{Cantor set}} = \bigcup_{h \in H} (a_h, b_h)$$

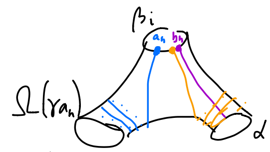
__Note__: $a_h, b_h \in \partial E_i$ and they are not isolated, so γ_{a_h} and γ_{b_h} spiral to simple closed curves in the interior.

__Lemma__: There is a natural map from H to embedded pairs of pants containing β_i.

This map is 2-to-1 for pairs of pants with one peripheral boundary (only β_i) and 1-to-1 for pairs of pants with two peripheral boundaries (β_i and another)

__Pf__: Let $h \in H$. $\underbrace{\Omega(\gamma_{a_h})}_{\text{limit set of } \gamma_{a_h}}$ is a simple closed curve. Let Σ_h be the unique pair of pants containing γ_{a_h} such that $\partial \Sigma_h = \{\beta_i, \Omega(\gamma_{a_h}), \alpha\}$. We will show that $\gamma_{b_h} \subseteq \Sigma_h$.

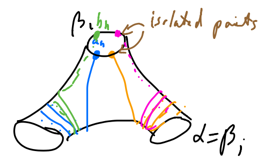
Case I: α is non-peripheral. Then $\gamma_{b_n} \subseteq \Sigma_n$, otherwise there would be a point $c \in (a_n, b_n)$ such that γ_c spirals to α. But then (since α is non-peripheral), by Lemma 3, c is a non-isolated point of E_i, a contradiction.



So $\gamma_{b_n} \subseteq \Sigma_n$ and $\alpha = \Omega(\gamma_{b_n})$

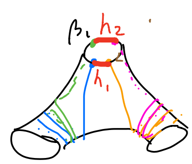
Class 21 (3/27)

Case II: α is peripheral. In that case, we have two isolated points of E_i corresponding to α, and b_n must be the point such that γ_{b_n} also spirals around $\Omega(\gamma_{a_n})$, but in the other direction.

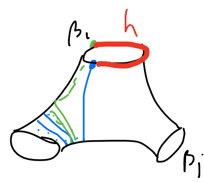


Therefore the associated pair of pants Σ_x is well-defined (using γ_{a_n} gives us the same Σ_x as γ_{b_n})

Conversely, let Σ be an embedded pair of pants so that $\partial\Sigma = \{\beta_i, \alpha_1, \alpha_2\}$.
If both α_1 and α_2 are non-peripheral, then we have one pair of spiraling geodesics starting in one hexagon, and another pair in the other hexagon. Each pair corresponds to some $h \in H$.



If one of α_1 and α_2 is peripheral, then the pair spiraling to the non-peripheral boundary yields an $h \in H$.

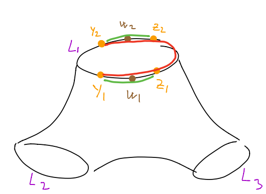


Proof of the Mirzakhani–McShane identity

Recall: $\mathcal{R}(L_1, L_2, L_3) = \ell(y_1 y_2)$

$\mathcal{D}(L_1, L_2, L_3) = \ell(y_1 z_1) + \ell(y_2 z_2)$

containing w_i



We need to show

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \underbrace{\mathcal{D}\left(L_1, \ell_{\gamma_1}(X), \ell_{\gamma_2}(X)\right)}_{\substack{\text{sum of the lengths of the 2 intervals } (a_n,b_n) \\ \text{corresponding to the pair of} \\ \text{pants with bdy } \beta_1\, \gamma_1 \gamma_2}} + \sum_{i=2}^{n} \sum_{\gamma\in\mathcal{F}_{1i}} \underbrace{\mathcal{R}\left(L_1, L_i, \ell_\gamma(X)\right)}_{\substack{\text{length of interval } (a_n,b_n) \\ \text{corresponding to the pair of} \\ \text{pants with bdy } \beta_1\, \beta_i\, \gamma.}} = L_1$$

We count every interval (a_n, b_n) exactly once and the complementary Cantor set has measure 0, therefore the sum is L_1, the length of the bdy β_1.

Summary of proof of simple closed geodesic counting theorem

- translate the problem to convergence of measures
- show that every convergent subsequence converges to some constant times μ_{Th}
- integrate on moduli space to show it is the same constant

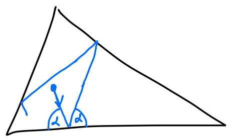
integrating geometric functions $\rightarrow$ volumes of moduli spaces of simpler surfaces

$\uparrow$ $\quad\quad\quad\quad$ $\downarrow$

Mirzakhani-McShane identity describes the constant 1 function as a geometric function $\leftarrow$ integrating the constant 1 function

Triangular billiards

Class 23 (4/3)

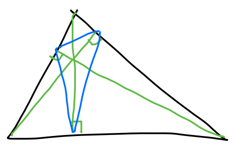


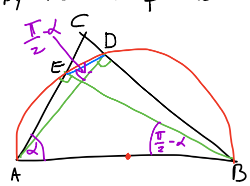
Periodic trajectory: ball eventually returns to the initial position, going in the same direction

Q: Given a triangle T, is there a periodic billiard trajectory?

Thm (Fagnano, 1775): If T is acute, then yes.

Proof: The triangle formed by the bases of the altitudes is a periodic trajectory.



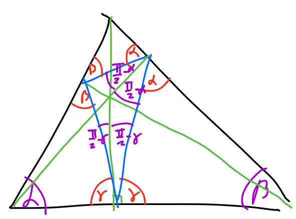


ABDE lie on the same circle.

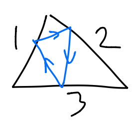
$ABE\measuredangle = ADE\measuredangle$, since both angles belong to the same arc (AE) arc of the circle.

$\underset{\frac{\pi}{2}-\alpha}{\underset{\parallel}{ABE\measuredangle}} + \underset{\alpha}{\underset{\parallel}{EAB\measuredangle}} = \frac{\pi}{2}$, so $ADE\measuredangle = \frac{\pi}{2} - \alpha$.

By symmetry, we have the following angles



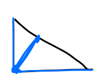
Note: We encode this trajectory as (1 2 3).



Class 24 (4/8)

Q: Does this work for right-angled triangles?

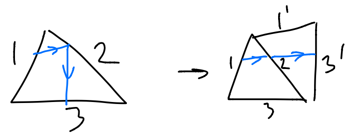
A: No, because the altitude triangle is degenerate.

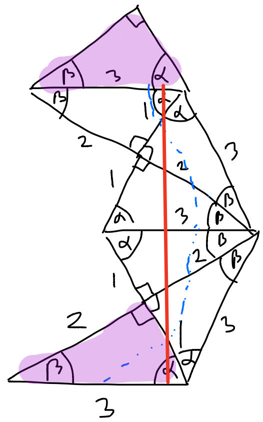


Prop. In every right-angled triangle there is a periodic path with encoding (3 1 2 3 2 1).

Pf: How to find this path? Idea: the unfolding trick:

instead of reflecting the trajectory and leave the triangle fixed, reflect the triangle and leave the trajectory fixed.

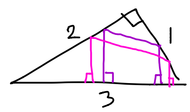




We get a translated copy of the original!
So just pull the blue path tight and we get our closed billiard path!

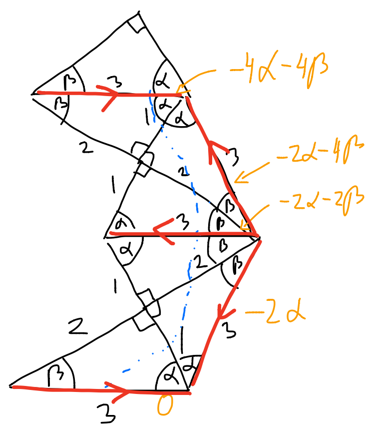
- Check that there is a straight path inside the triangles.
- there are multiple such paths.

Alternatively:



Q: Great! Does this work for obtuse triangles?

A: No, the first and last triangles won't be translates. Let's track the angle side 3 encloses with the positive real axis.

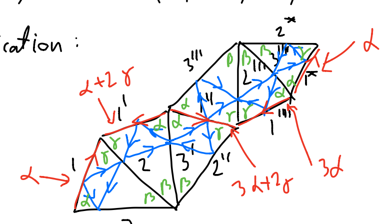


$4\alpha+4\beta$ is a multiple of 2π precisely when $\alpha+\beta=\frac{\pi}{2}$, i.e. when the triangle is a right angle.

Therefore the comb. path (312321) is not stable. A path is stable if it works for an open set of triangles.

Example: The path (123) is stable (by Fagnano)

Alternative justification:



We get a parallel side no matter what the angles are.

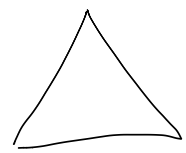
However, stable does not mean that it works for all triangles. Even though the segments are always parallel, there is no straight segment connecting them for obtuse △s.

Q: Ok, so what about obtuse triangles?

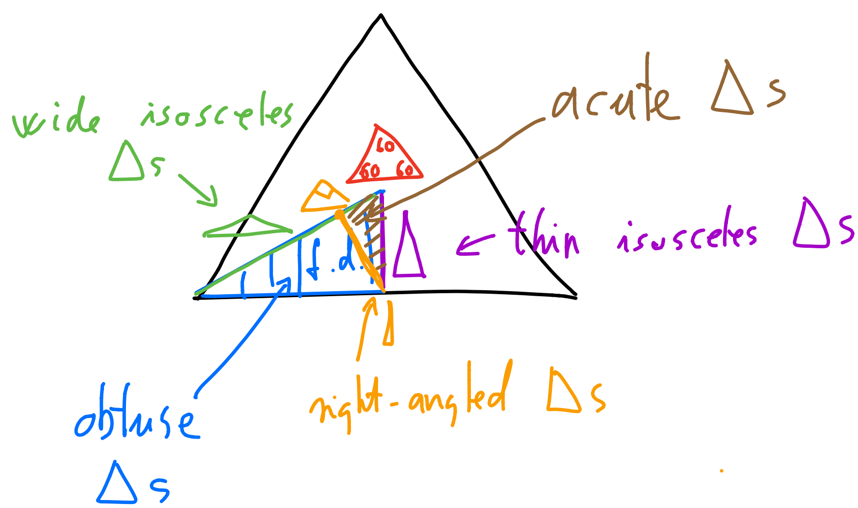
Thm (Schwartz, 2006) Let S_ε denote the set of Euclidean triangles whose two small angles are within ε radians of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, respectively.

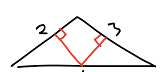
(1) For any $\varepsilon>0$ there exists a triangle in S_ε that has no periodic billiard path of comb. length $\leq \frac{1}{\varepsilon}$

(2) Every triangle in $S_{1/400}$ has a periodic billiard path.

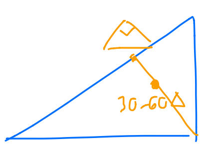
Moduli space of triangles $\cong \{\{\alpha,\beta,\gamma\} : \alpha,\beta,\gamma>0, \alpha+\beta+\gamma=\pi\} \cong$  / rotation by 120°, reflection symmetries.

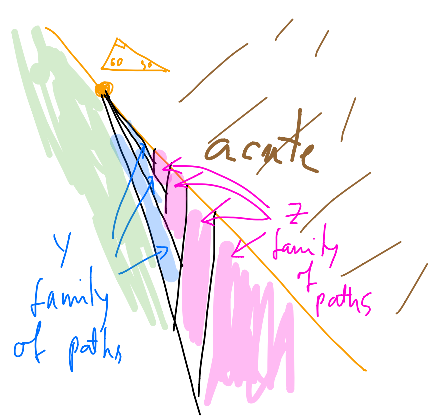
unordered



Fact: $\exists$ path of type (2131) if and only if the Δ is isosceles. 

Schwartz's trajectories



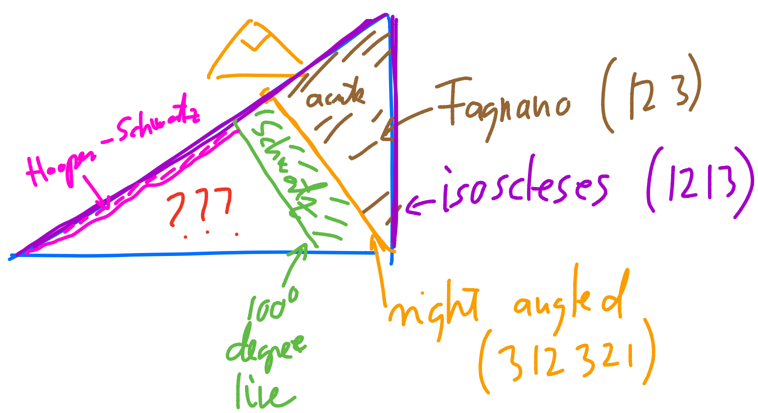


Thm (Hooper, 07') All periodic billiard paths in right triangles are unstable.

(hence it is difficult to cover a neighborhood of the right-angled triangle locus)

Thm (Hooper-Schwartz, 09') Any sufficiently small perturbation of an isosceles triangle has a periodic billiard path. (82 pages)

Thm (Schwartz, '09): Every triangle with angles $\leq 100^\circ$ have a periodic billiard path.



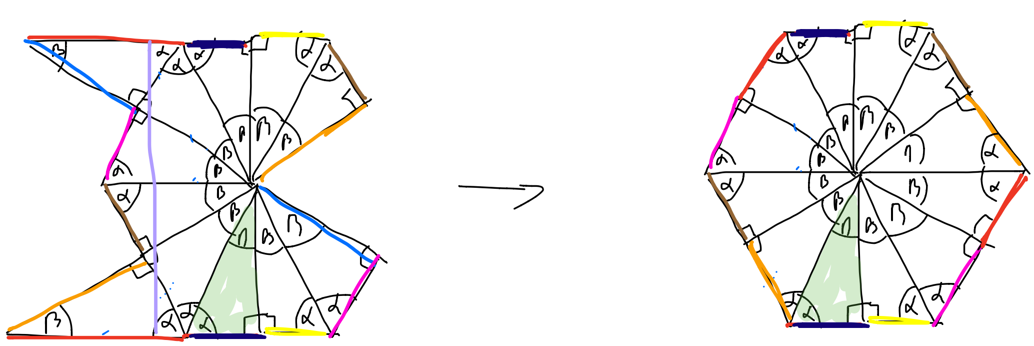
Rational billiards

Def: A polygon is rational if all its angles are rational multiples of π.

Thm (Masur '86) Every rational polygon has (many) periodic billiard trajectories.

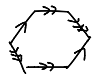
Thm (Boshernitzan, Galperin, Krüger, Troubetzkoy '98) For rational polygons, periodic points of the billiard flow are dense in the phase space of the billiard flow.

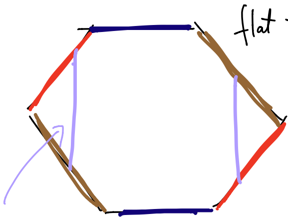
$\alpha = 60^\circ$
$\beta = 30$



Periodic billiard trajectories in 

$\updownarrow$

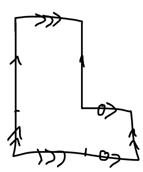
closed geodesics in the flat torus 



Note: we get a closed geodesics of all rational slopes, so the periodic trajectories are indeed abundant.

Q: What if we have a general rational polygon instead of the 90-60-30 $\triangle$?
A: We can do the same unfolding procedure $\Rightarrow$ finitely many flat polygons so that pairs of edges are glued together by translations. That's called a translation surface.

Example:

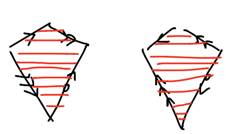


Def 2: A translation surface is a Riemann surface endowed with a holomorphic 1-form.

On each polygon we have the standard complex structure and the hol. 1-form dz. Transition maps are translations that preserve both.
Conversely, given a Riemann surface with a holomorphic 1-form $f(z)dz$, $|f(z)||dz|$ induces a Euclidean metric outside the zeros of $f(z)$. At a zero of $f(z)$ of multiplicity m, the metric has cone angle $2\pi(m+1)$. Using straight segments connecting these singularities, one can chop up the surface to polygons.

There is also the notion of half-translation surfaces where the sides of the polygons are glued together either by a translation or a $180°$ rotation. These correspond to surfaces with holomorphic quadratic differentials $f(z)\,dz^2$. (The form dz^2 is invariant with respect to the rotation $z \mapsto -z$ as well).

Ex:

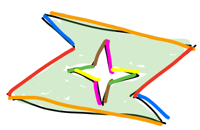


For both translation and half-translation surfaces, there is a well-defined horizontal and vertical direction $\Rightarrow$ horizontal and vertical singular measured foliations. For translation surfaces, these foliations are transversely orientable.

In fact, there is a foliation in every direction.

Thm (Masur '86): Let q be any holomorphic quadratic differential on a compact Riemann surface of genus at least 2. There exists a dense set of θ for which $e^{i\theta} q$ has a closed vertical trajectory.

Note: This is more general than the rational billiard thm, since an unfolding of a rational polygon is always a translation surface, and not even all translation surfaces arise that way.

Ex:  In this translation surface, there is a dense set of directions with a closed orbit.

Idea for detecting closed vertical orbits

Given a half-translation surface, we can stretch it horizontally be e^t and compress vertically by e^{-t}. (This map is linear, so parallel edges remain parallel.) We get a 1-parameter family of half-tr. surfaces $\rightarrow$ Teichmüller flow.

Project to moduli space $\Rightarrow$ geodesic in Teichmüller metric.

1) If there is a closed vertical orbit, then this flow shrinks that curve to "length 0", so the Teichmüller geodesic goes to the boundary of moduli space.

2) Many other cases, but for example if horizontal and vertical foliations are the invariant foliations of a pseudo-Anosov mapping class, then the Teich geodesic is repeating over a closed loop in moduli space.

Rough idea of Masur's proof: take a small interval of angles, and start the Teichmüller geodesic in all those directions. $\Rightarrow$ immersion of an infinite sector  in moduli space. Then show that it accumulates at the boundary.