Number theoretic aspects of mapping class groups

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Georgia Tech

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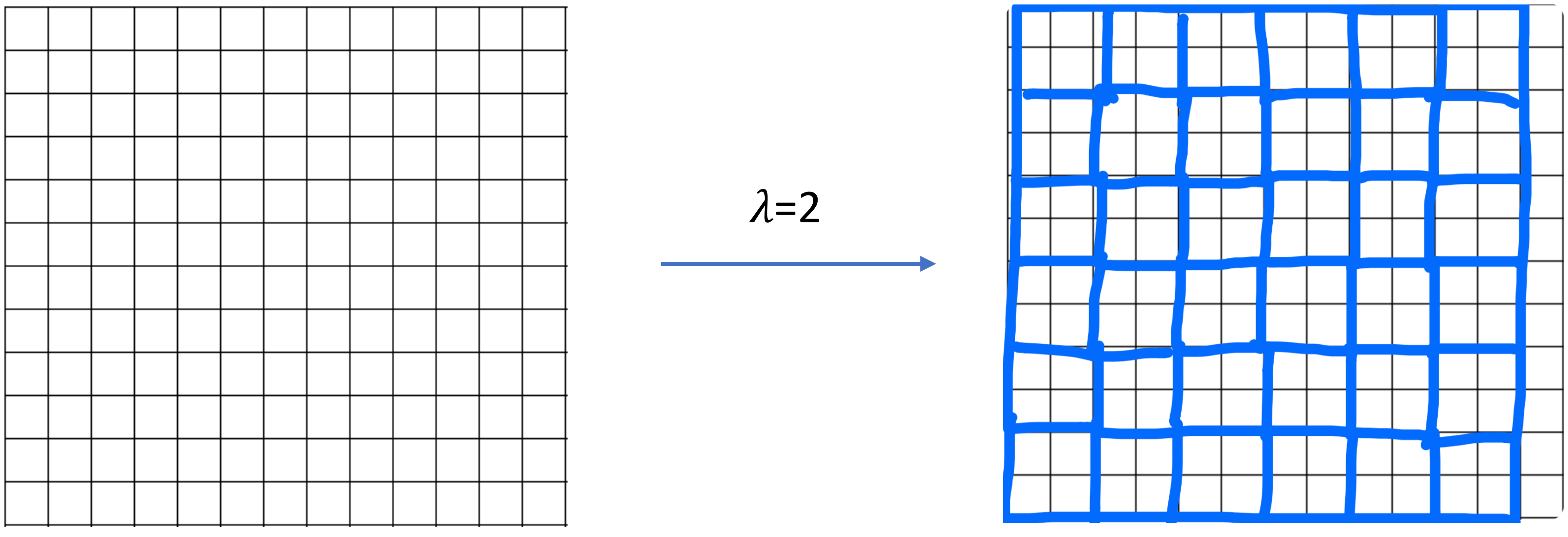
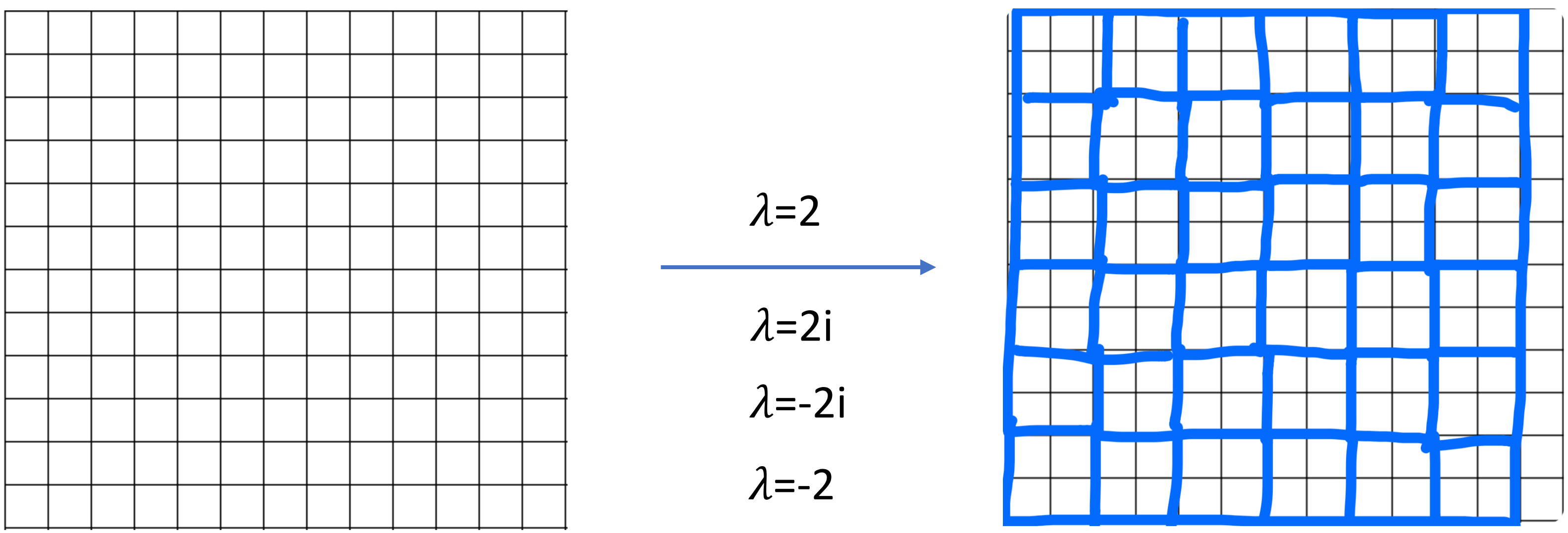
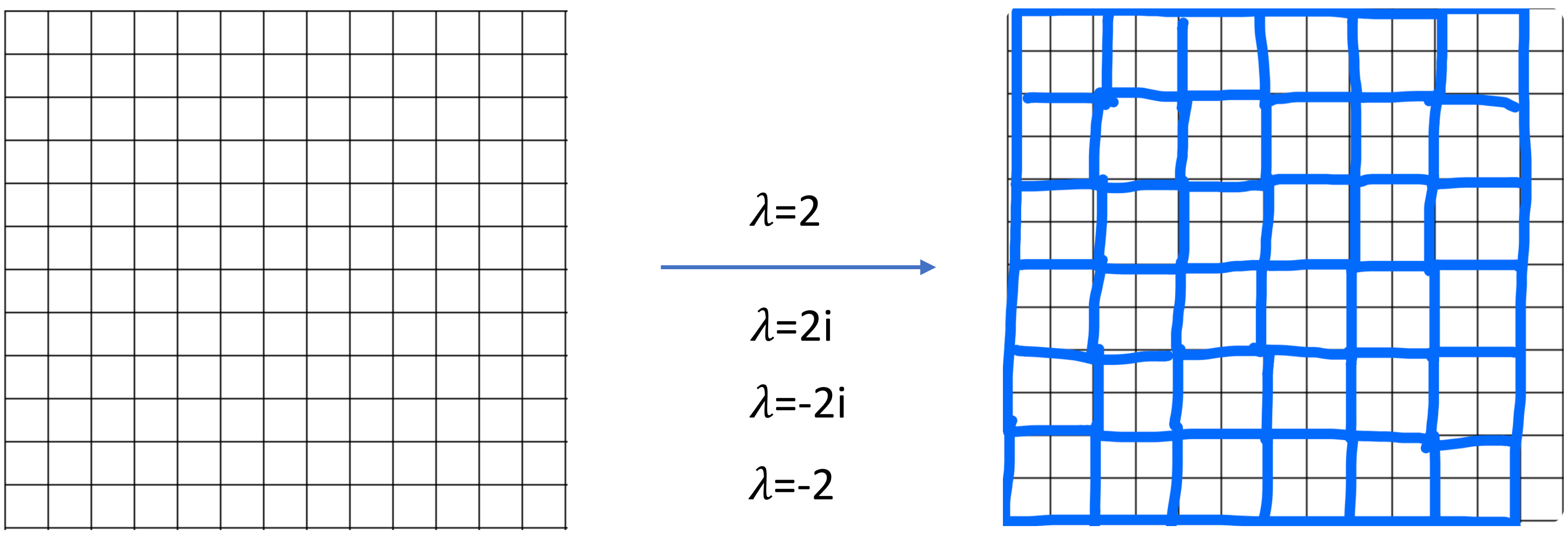
Auburn University
Auburn, Alabama, USA

1. Self-similar tilings

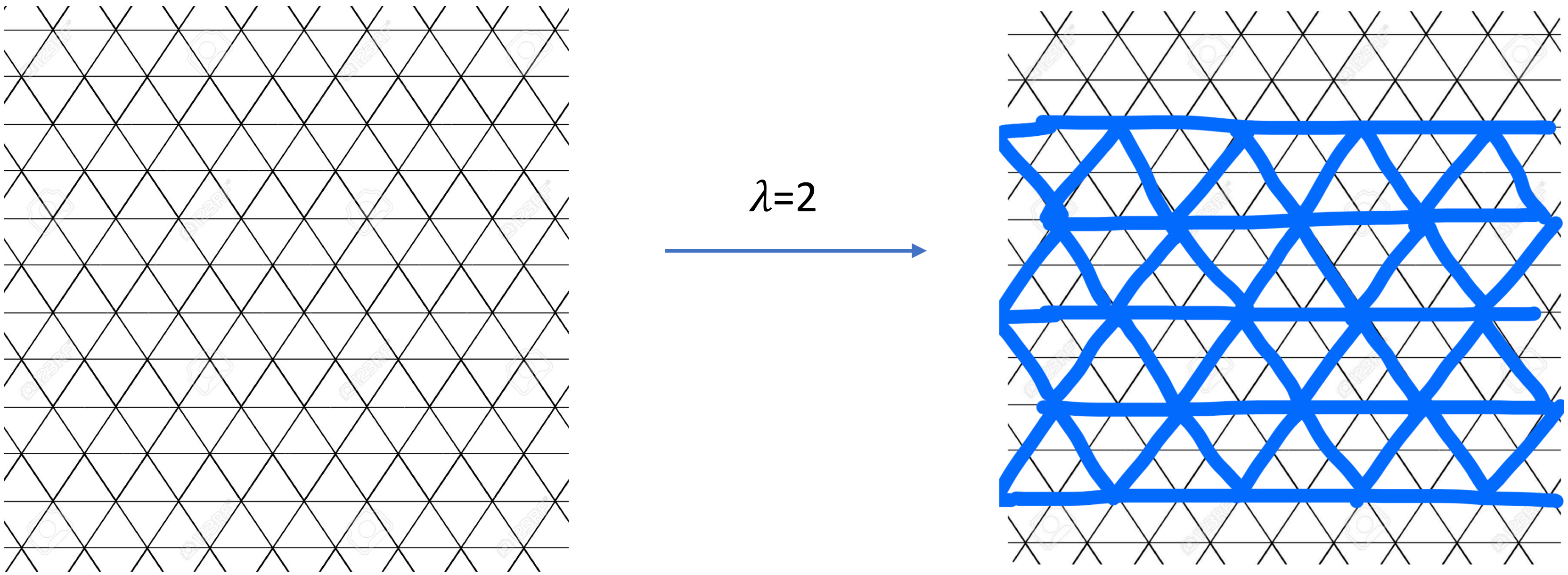

λ=2

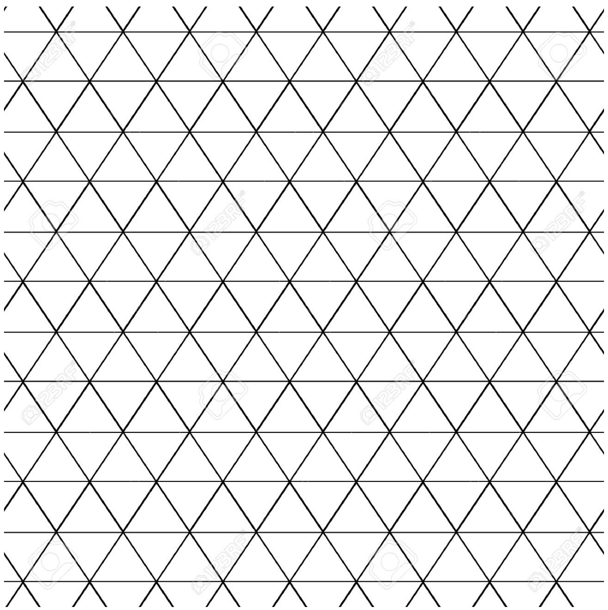


λ=2
λ=2i
λ=-2i
λ=-2



λ=3, -3, 3i, -3i, 4, -4, 4i, -4i, etc.


λ=2



$\lambda = -2$

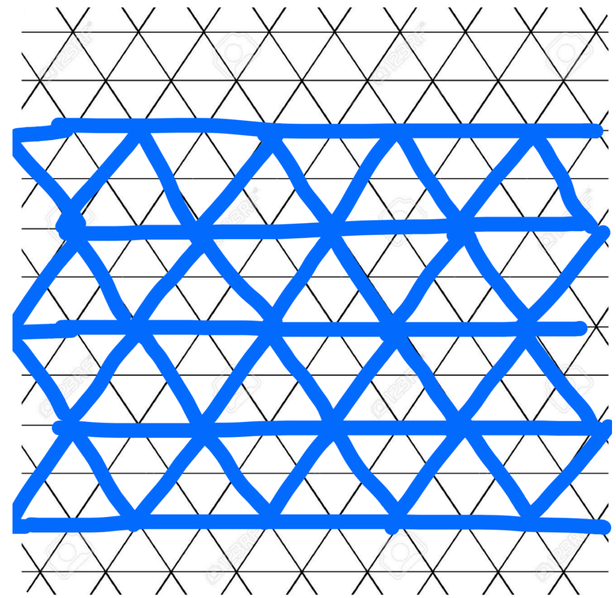
$\lambda = 2$

$\lambda = 1 + \sqrt{3}i$

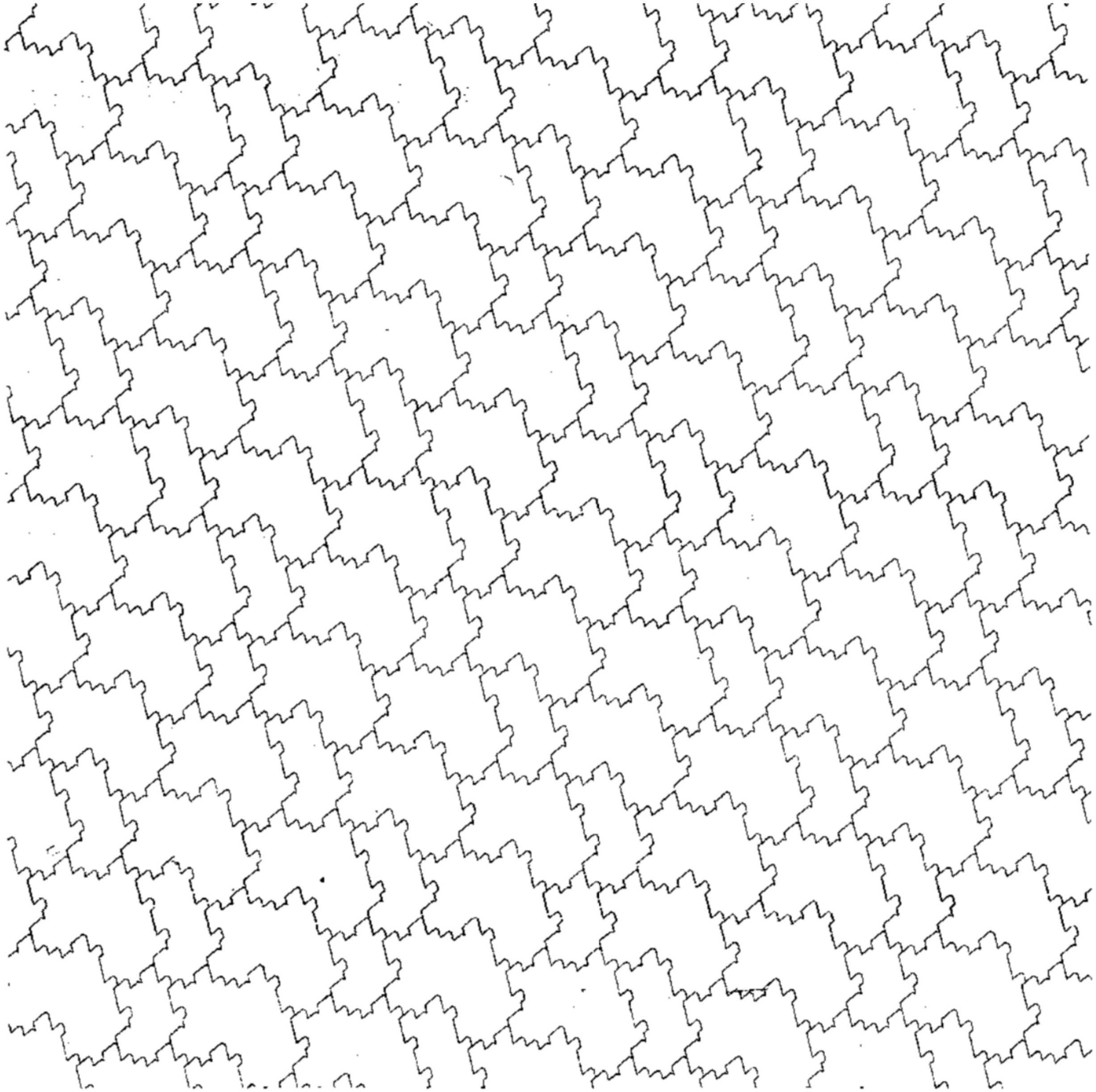
$\lambda = 1 - \sqrt{3}i$

$\lambda = -1 + \sqrt{3}i$

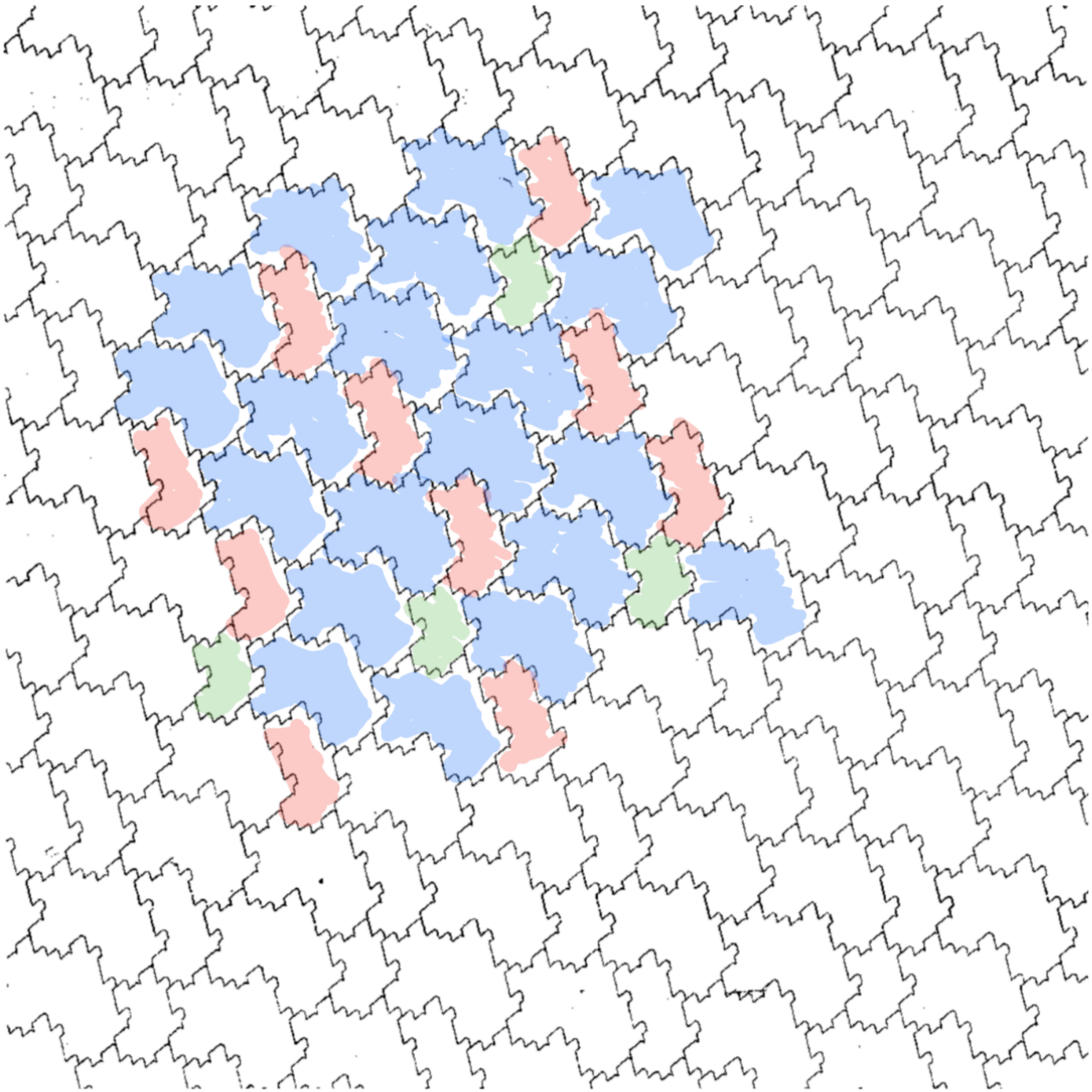
$\lambda = -1 - \sqrt{3}i$



$$\lambda = 3, -3, 3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), 3\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right), \text{ etc.}$$



by Richard Kenyon



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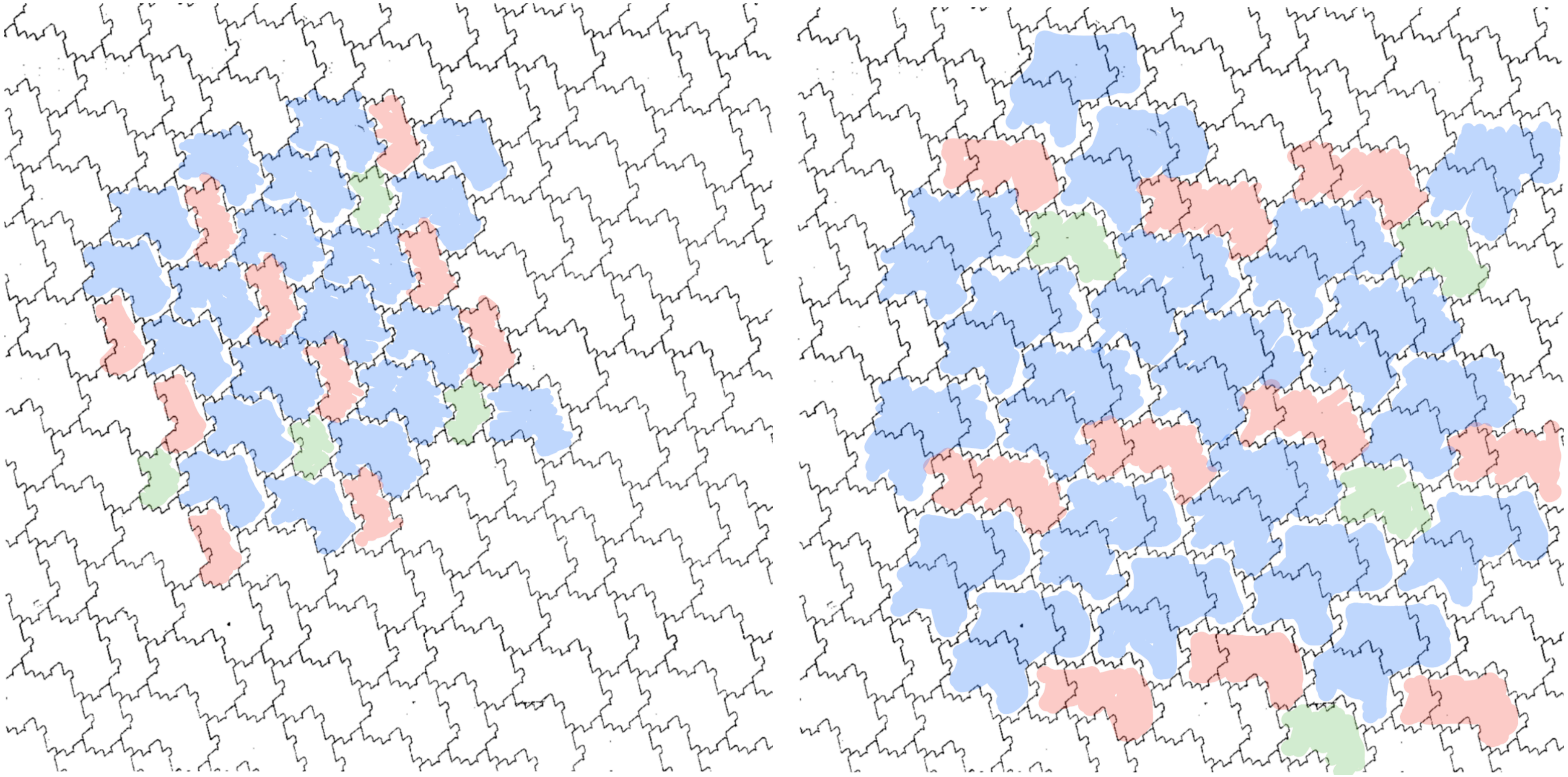
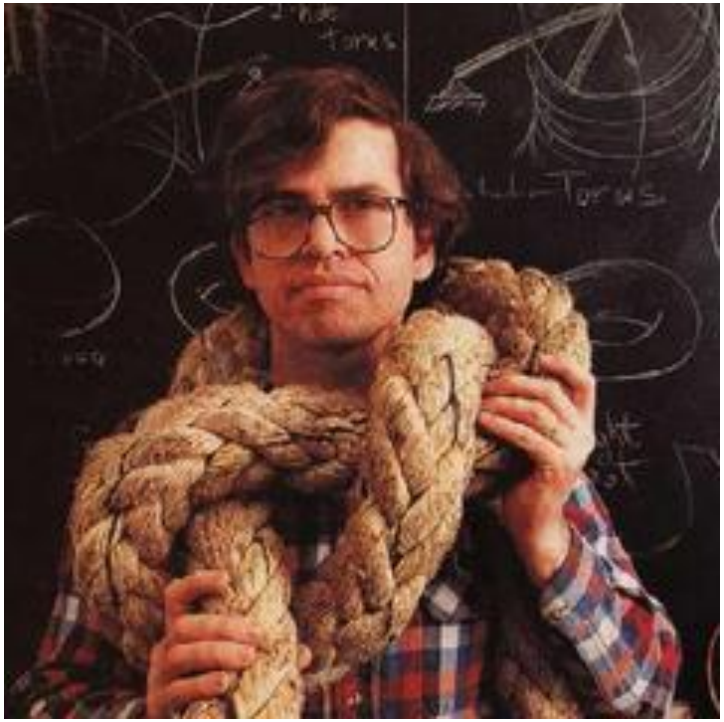


λ = .696+1.436i

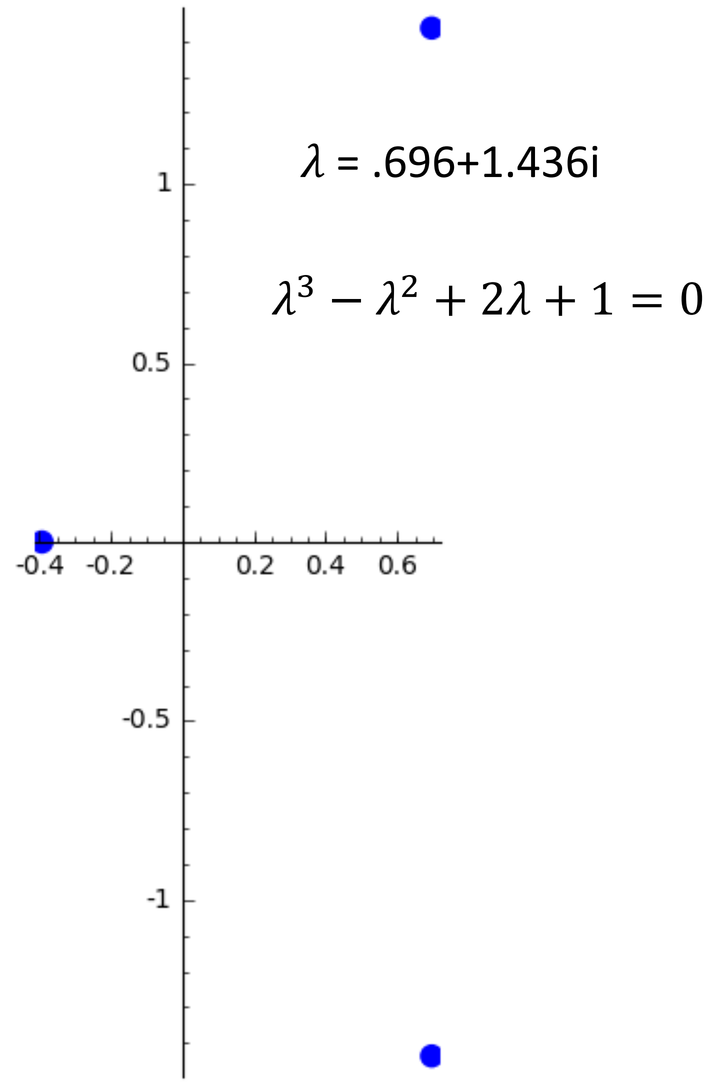
$$\lambda^3 - \lambda^2 + 2\lambda + 1 = 0$$

Theorem (Thurston, ‘80s): The expansion of a self-similar tiling is a complex Perron number.



Theorem (Thurston, '80s): The expansion of a self-similar tiling is a complex Perron number.

Complex Perron number: an algebraic integer which is strictly larger in modulus than its Galois conjugates (the other roots of its minimal polynomial), except for $\bar{\lambda}$.



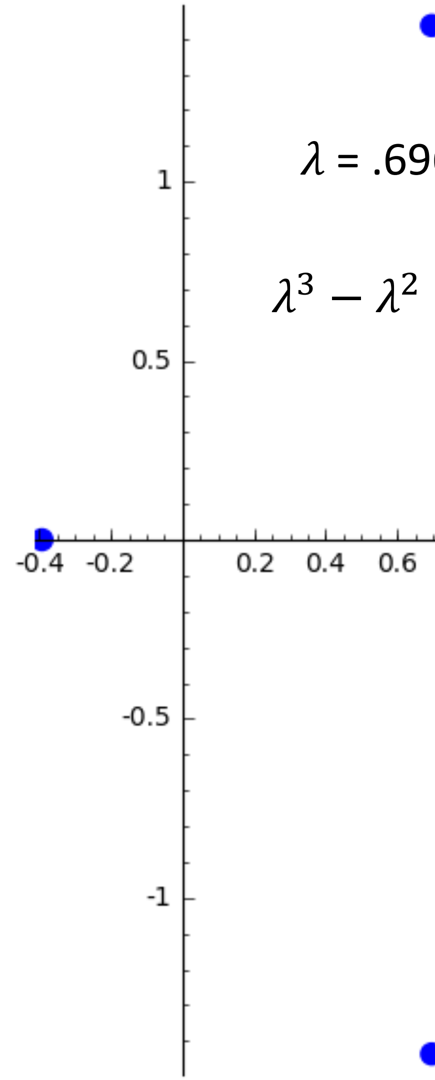
Theorem (Thurston, '80s): The expansion of a self-similar tiling is a complex Perron number.

Complex Perron number: an algebraic integer which is strictly larger in modulus than its Galois conjugates (the other roots of its minimal polynomial), except for $\bar{\lambda}$.

Algebraic integer: root of a monic polynomial with integer coefficients.

$\lambda = .696+1.436i$

$$\lambda^3 - \lambda^2 + 2\lambda + 1 = 0$$



Theorem (Kenyon '96): For each complex Perron number λ there is a self-similar tiling with expansion λ.

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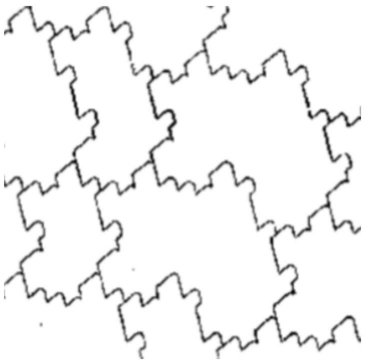

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1. Try triangulating.
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3. But if it "roughly" works, then one can refine the boundaries of the tiles iteratively to fractals.

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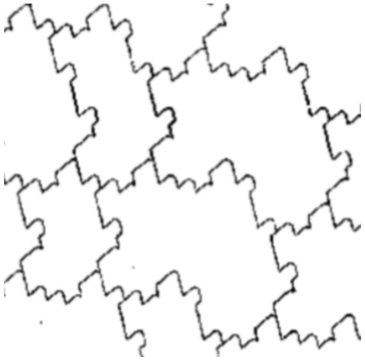
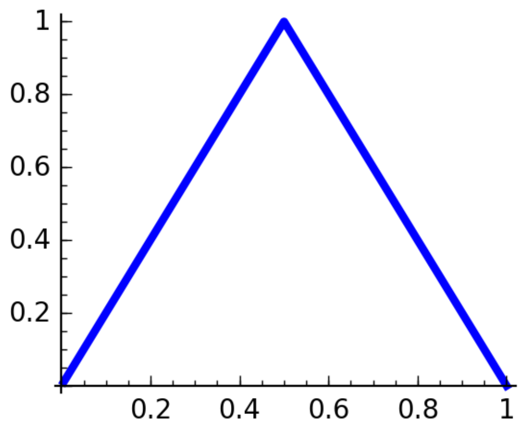


Idea of proof:

1. Try triangulating.
2. It won't work.
3. But if it "roughly" works, then one can refine the boundaries of the tiles iteratively to fractals.

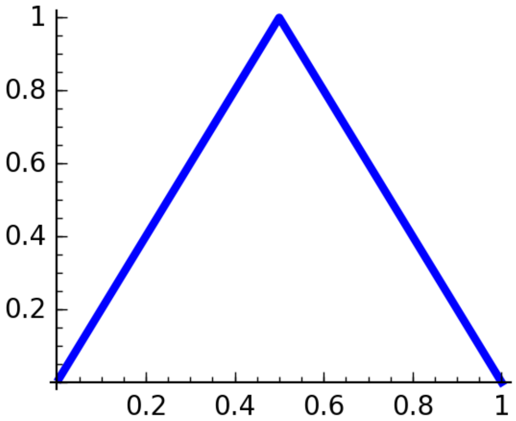
For step 1, triangulate the lattice embedding of $\mathbb{Z}[\lambda]$ into $\mathbb{R}^d$ and project it down to the subspace spanned by λ and $\bar{\lambda}$.

2. Entropies of self-maps of the interval



1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1

$c(f)$: number of turning points of f

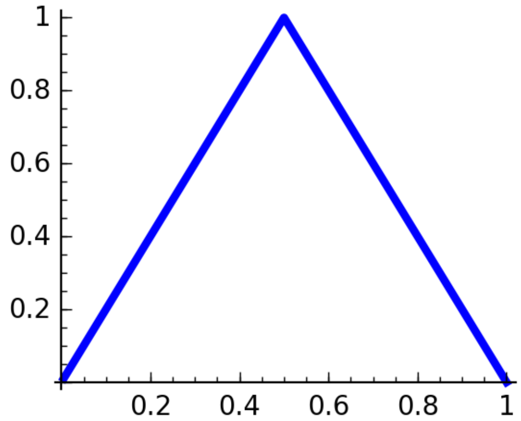


$$c(f) = 1$$

$c(f)$: number of turning points of f

$h(f)$: entropy of f

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log c(f^n)$$ (Misiurewicz-Szlenk)



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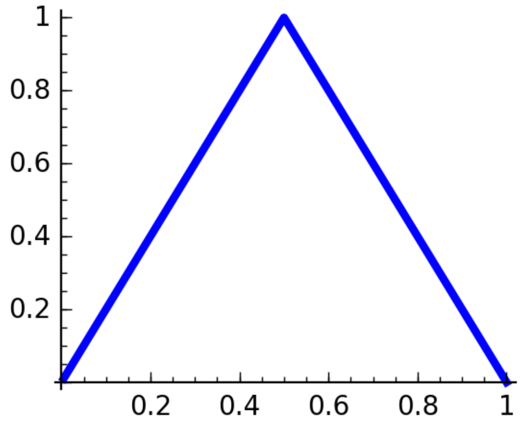

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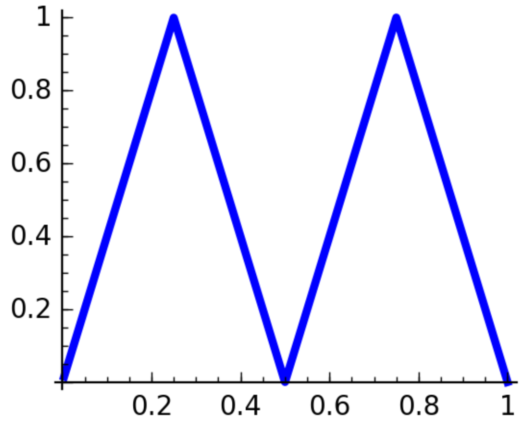

$c(f^2) = 3$

$c(f)$: number of turning points of f

$h(f)$: entropy of f

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log c(f^n)$$

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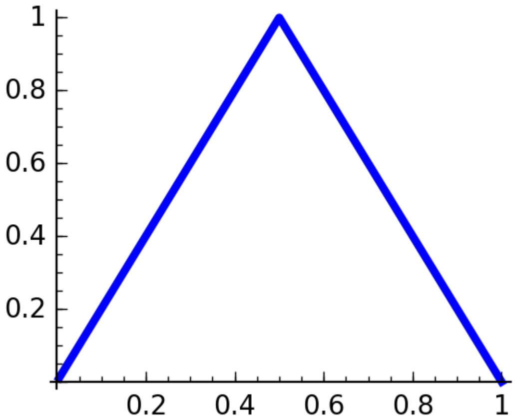
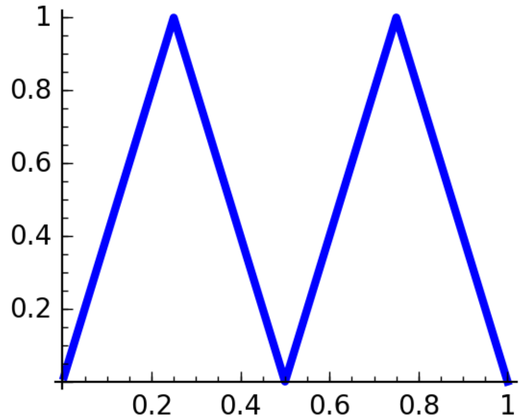
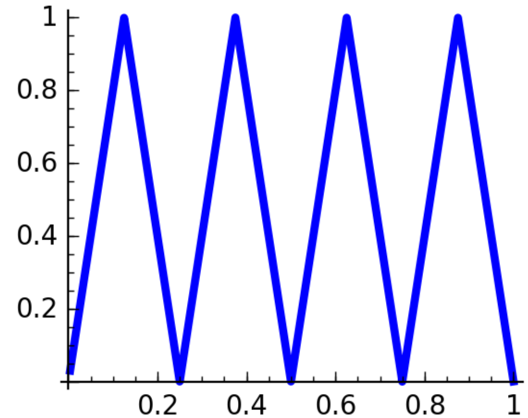


$c(f) = 1$



$c(f^2) = 3$

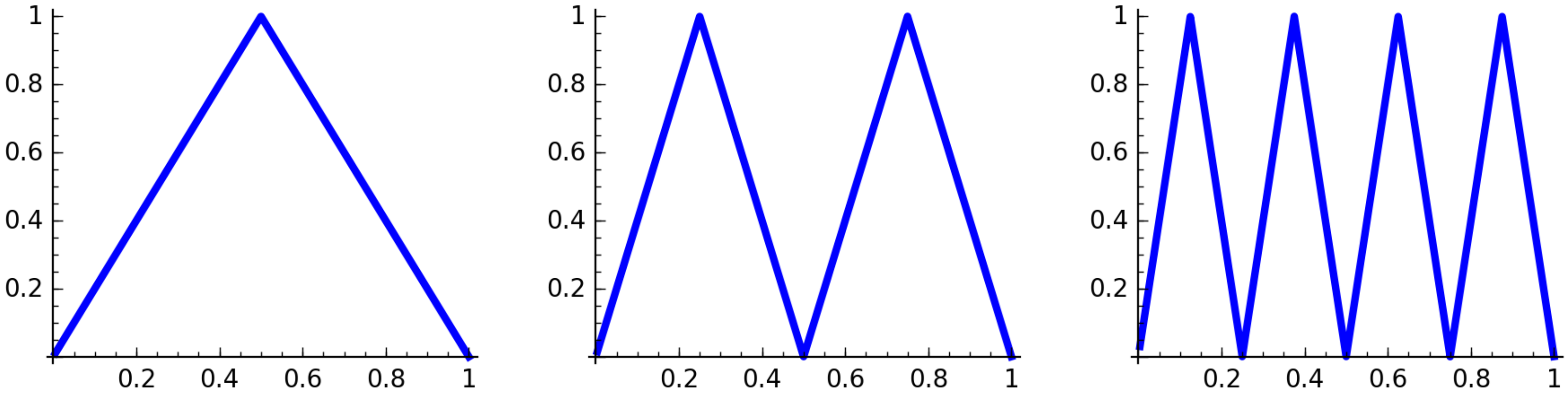


$c(f^3) = 7$

$c(f)$: number of turning points of f

$h(f)$: entropy of f

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log c(f^n)$$ (Misiurewicz-Szlenk)



$c(f) = 1$ $c(f^2) = 3$ $c(f^3) = 7$

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(2^n - 1) = \log 2$$

$$h(f) = \log 3\,?$$

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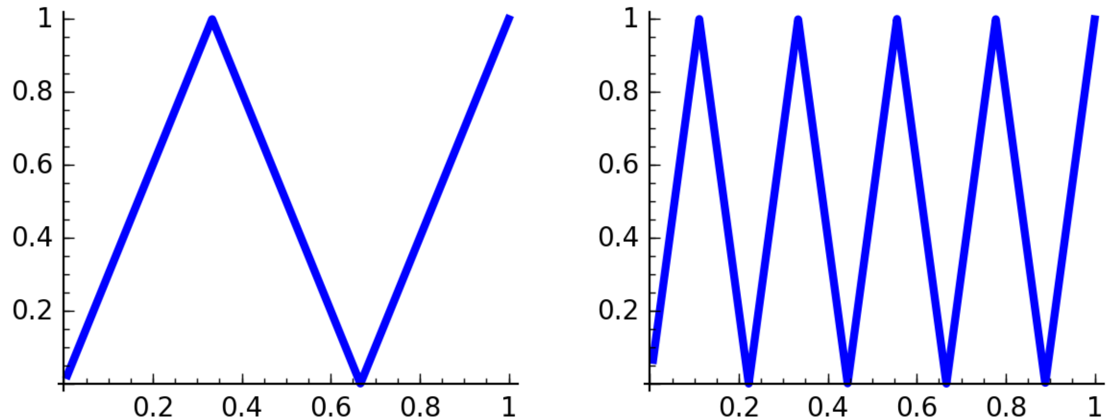

f $\qquad$ f^2

What values can $\exp(h)$ take?

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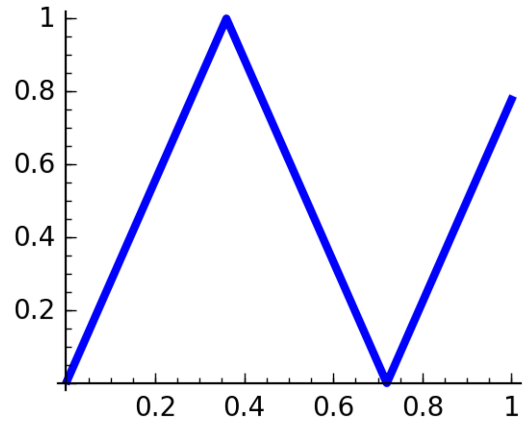
$K \geq 1$



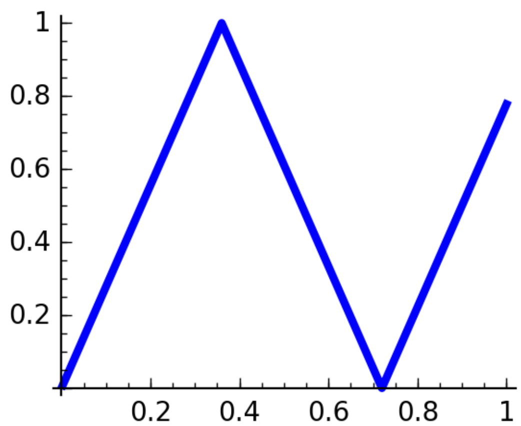
$\exp(h) = K$

$\text{slope} = \pm K$

What values can $\exp(h)$ take?

Any real number at least 1.

$K \geq 1$



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Restrict to postcritically finite case!

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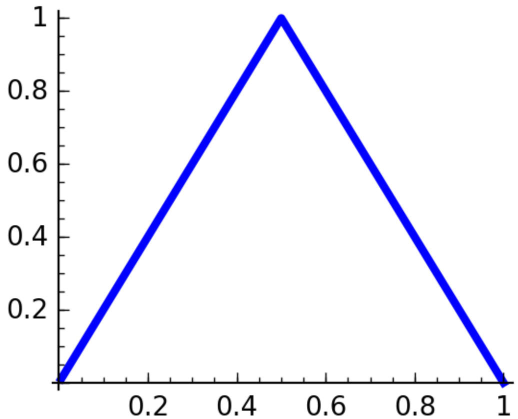
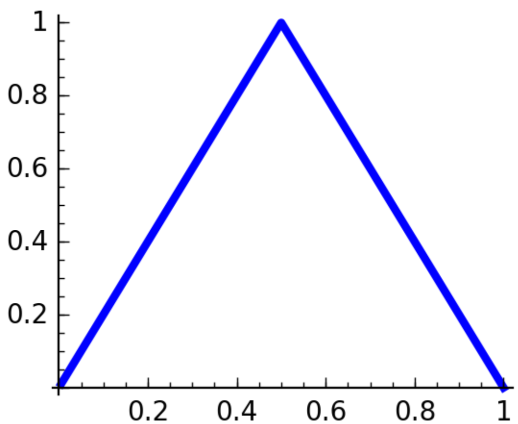
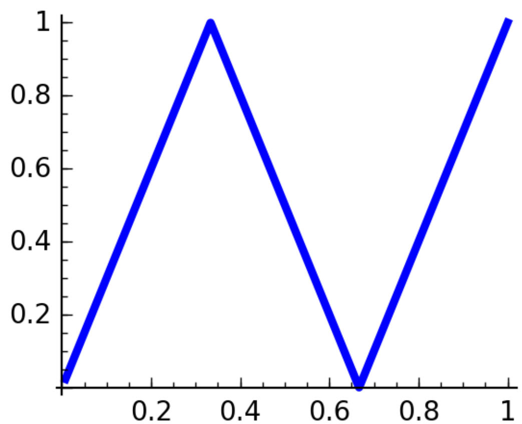


$$\frac{1}{2} \to 1 \to 0 \to 0$$

Restrict to postcritically finite case!



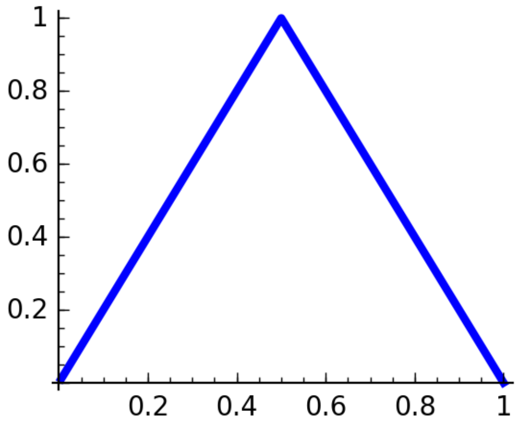
$$\frac{1}{2} \to 1 \to 0 \to 0$$



$$\frac{1}{3} \to 1 \to 1$$

$$\frac{2}{3} \to 0 \to 0$$

Restrict to postcritically finite case!



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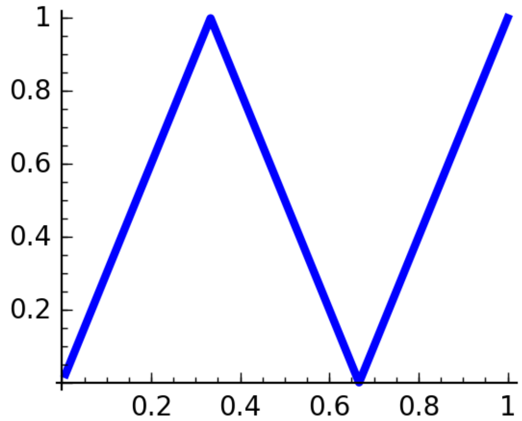

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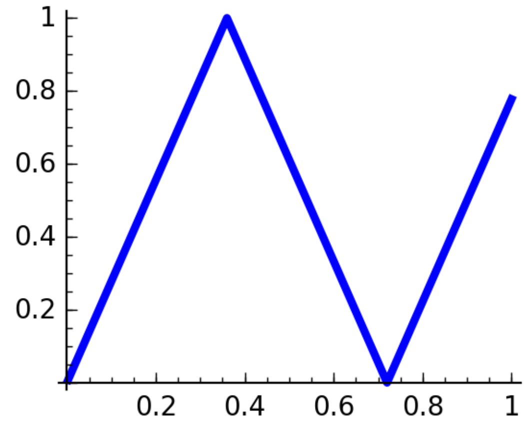
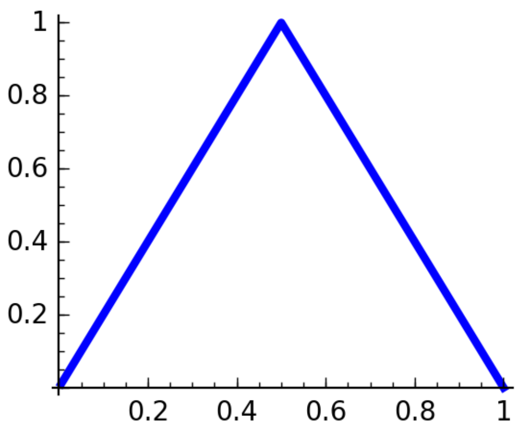
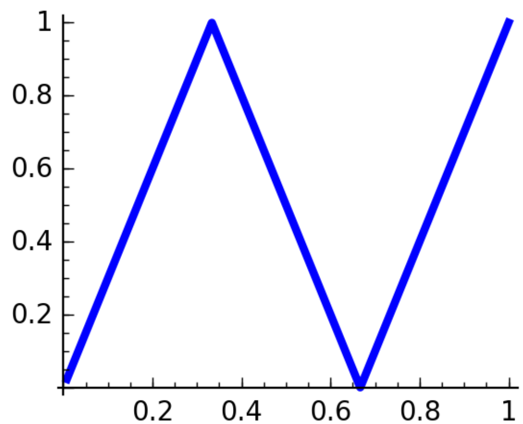


Thurston: Postcritically finite if slope is Pisot.

Restrict to postcritically finite case!



$$\frac{1}{2} \to 1 \to 0 \to 0$$



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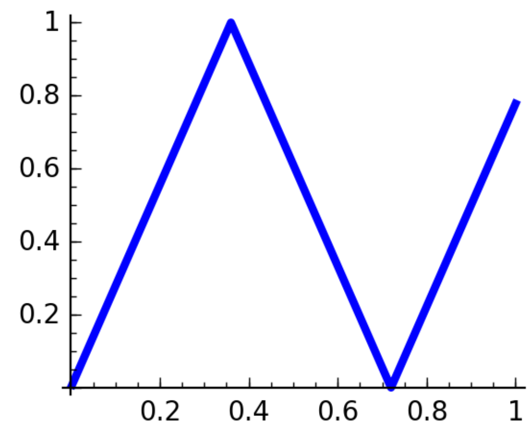

Thurston: Postcritically finite if slope is Pisot.

Pisot number: real algebraic integer $\lambda \geq 1$ whose Galois conjugates are in the open unit disk.

What values can $\exp(h)$ take if h is postcritically finite?

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PISOT ✅

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PISOT ✅

Anything else?

Thurston ('14): A positive real number h is the topological entropy of a postcritically finite self-map of the unit interval if and only if $\exp(h)$ is a weak Perron number.

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Thurston ('14): A positive real number h is the topological entropy of a postcritically finite self-map of the unit interval if and only if $\exp(h)$ is a weak Perron number.

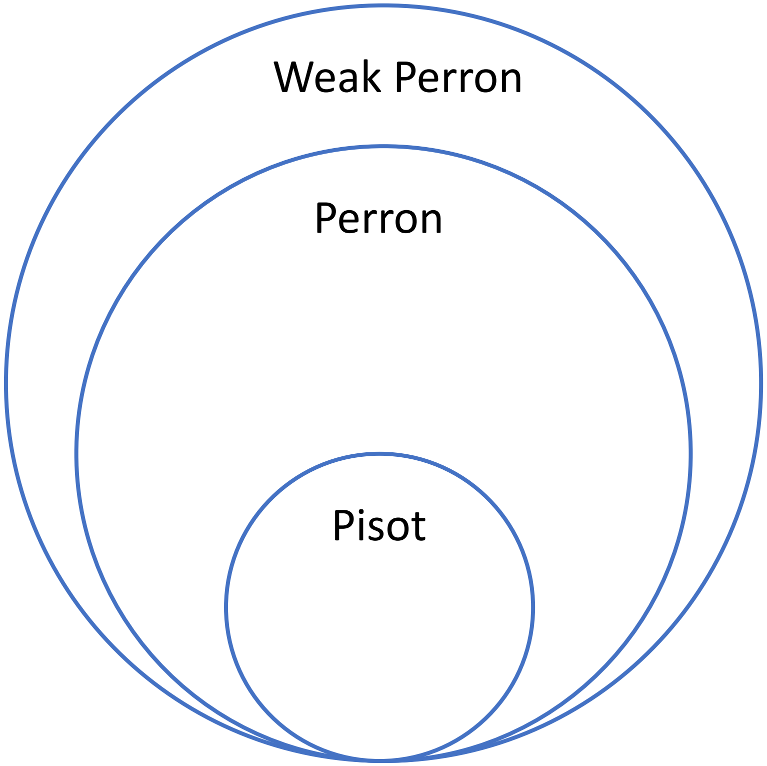
Perron number: an real algebraic integer $\lambda \geq 1$ which is strictly larger in modulus than its Galois conjugates.

Weak Perron number: an real algebraic integer $\lambda \geq 1$ which is at least as large in modulus as its Galois conjugates.

Thurston ('14): A positive real number h is the topological entropy of a postcritically finite self-map of the unit interval if and only if $\exp(h)$ is a weak Perron number.

Perron number: an real algebraic integer $\lambda \geq 1$ which is strictly larger in modulus than its Galois conjugates.

Weak Perron number: an real algebraic integer $\lambda \geq 1$ which is at least as large in modulus as its Galois conjugates.



3. Entropies of free group automorphisms

$$F_2 = \langle a, b \rangle$$

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$f: F_2 \to F_2$

$$a \mapsto b$$

$$\mathrm{b} \mapsto aB$$

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$$a \to b \to aB \to bbA \to aBaBB \to bbAbbAbA \to \ldots$$

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Entropy of f:

$$h(f) = \lim_{n \to \infty} \frac{\log(\mathrm{length}\left(f^n(x)\right))}{n}$$

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1 1 2 3 5 8

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1 1 2 3 5 8

Entropy of f:

$$h(f) = \lim_{n\to\infty} \frac{\log(\mathrm{length}(f^n(x)))}{n}$$

$$= \log(\tfrac{1+\sqrt{5}}{2}) = \log(\textit{golden ratio})$$

$$F_2 = \langle a, b\rangle$$
$$f: F_2 \to F_2$$

$$a \mapsto b$$
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$$a \to b \to aB \to bbA \to aBaBB \to bbAbbAbA \to \ldots$$

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Entropy of f:

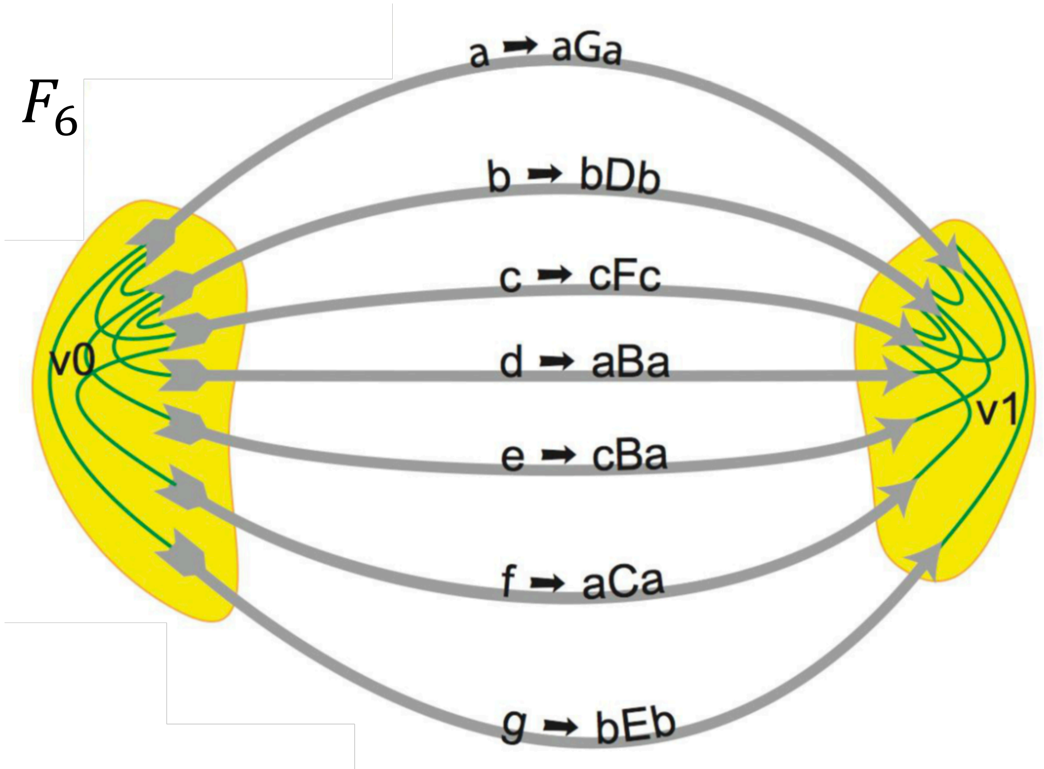
$$h(f) = \lim_{n\to\infty} \frac{\log(\mathrm{length}(f^n(x)))}{n}$$
$$= \log(\tfrac{1+\sqrt{5}}{2}) = \log(golden\ ratio)$$

What values can $\exp(h)$ take?

Free group automorphisms $\longleftrightarrow$ special graph maps
(train track maps)

Free group automorphisms ⟷ special graph maps (train track maps)

$f: F_6 \to F_6$



due to Thurston

Free group automorphisms ⟷ special graph maps (train track maps)

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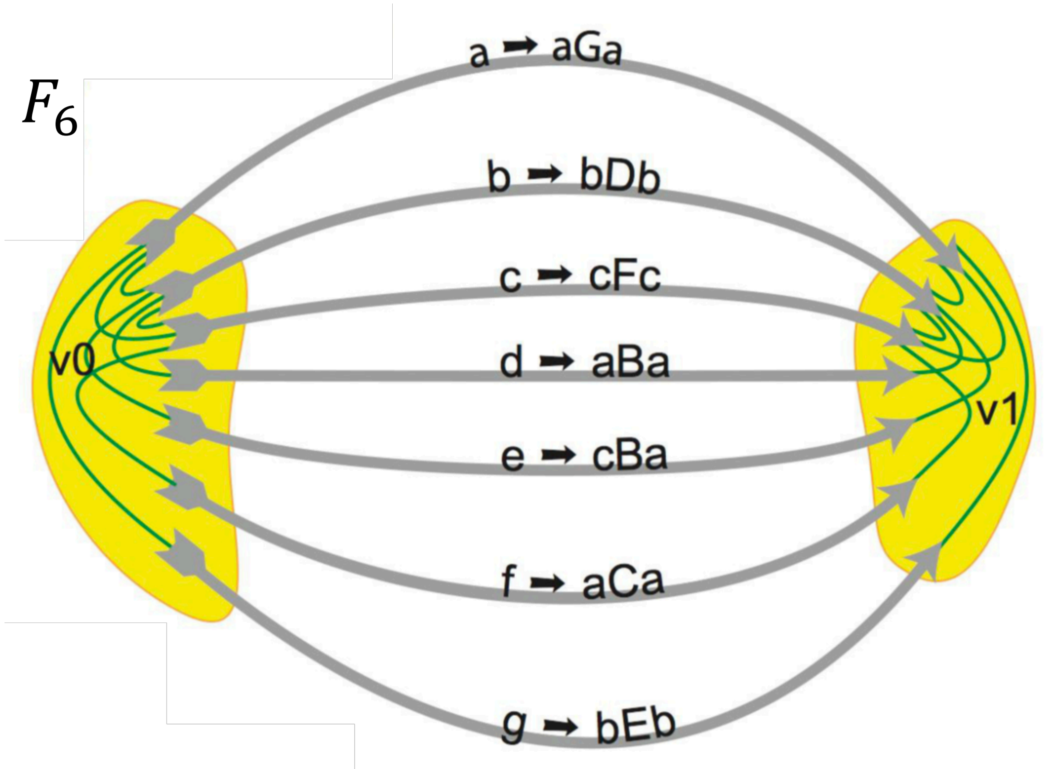

due to Thurston

$\text{length}(loop) = \#$ of "turning points" of the loop

$\rightsquigarrow$ analogous to entropy of interval maps

Free group automorphisms $\longleftrightarrow$ special graph maps (train track maps)

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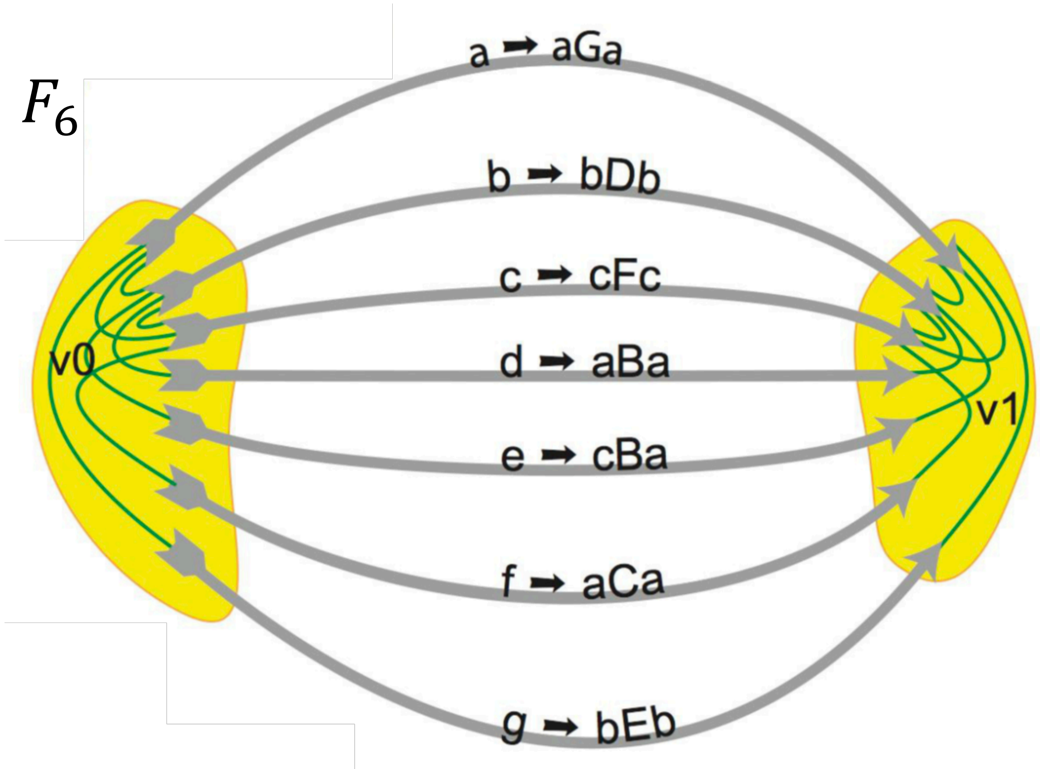


due to Thurston

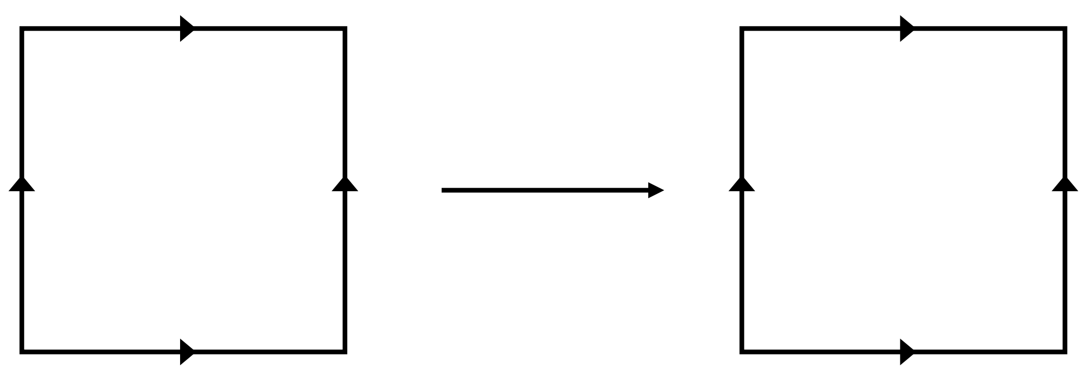
$h(f) = \log(3)$

length($loop$) $=$ # of "turning points" of the loop

$\rightsquigarrow$ analogous to entropy of interval maps

Thurston ('14): A positive real number h is the topological entropy for an ergodic train track representative of an outer automorphism of a free group if and only if its expansion constant $\exp(h)$ is a weak Perron number.

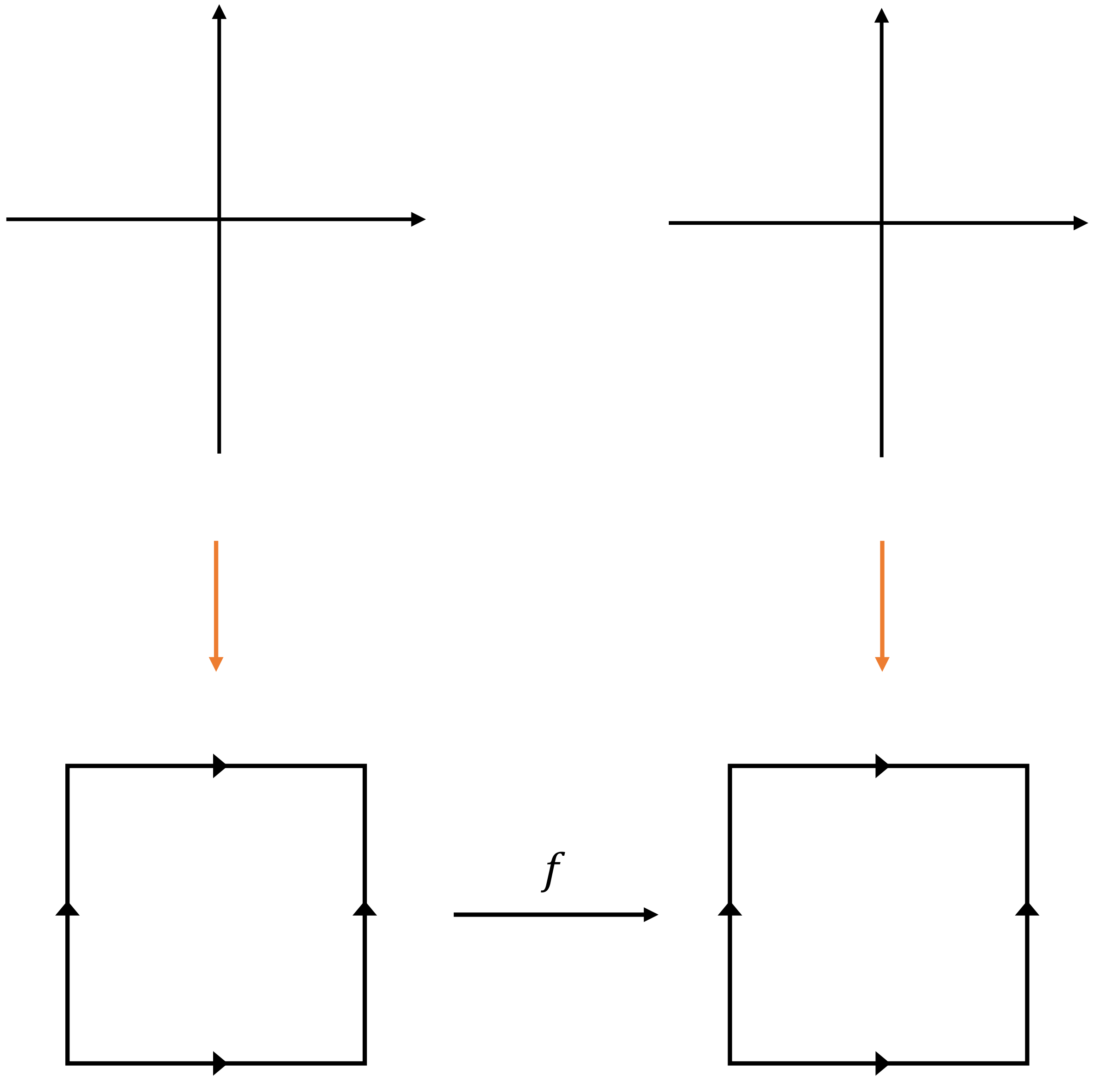
4. Stretch factors of torus homeomorphisms

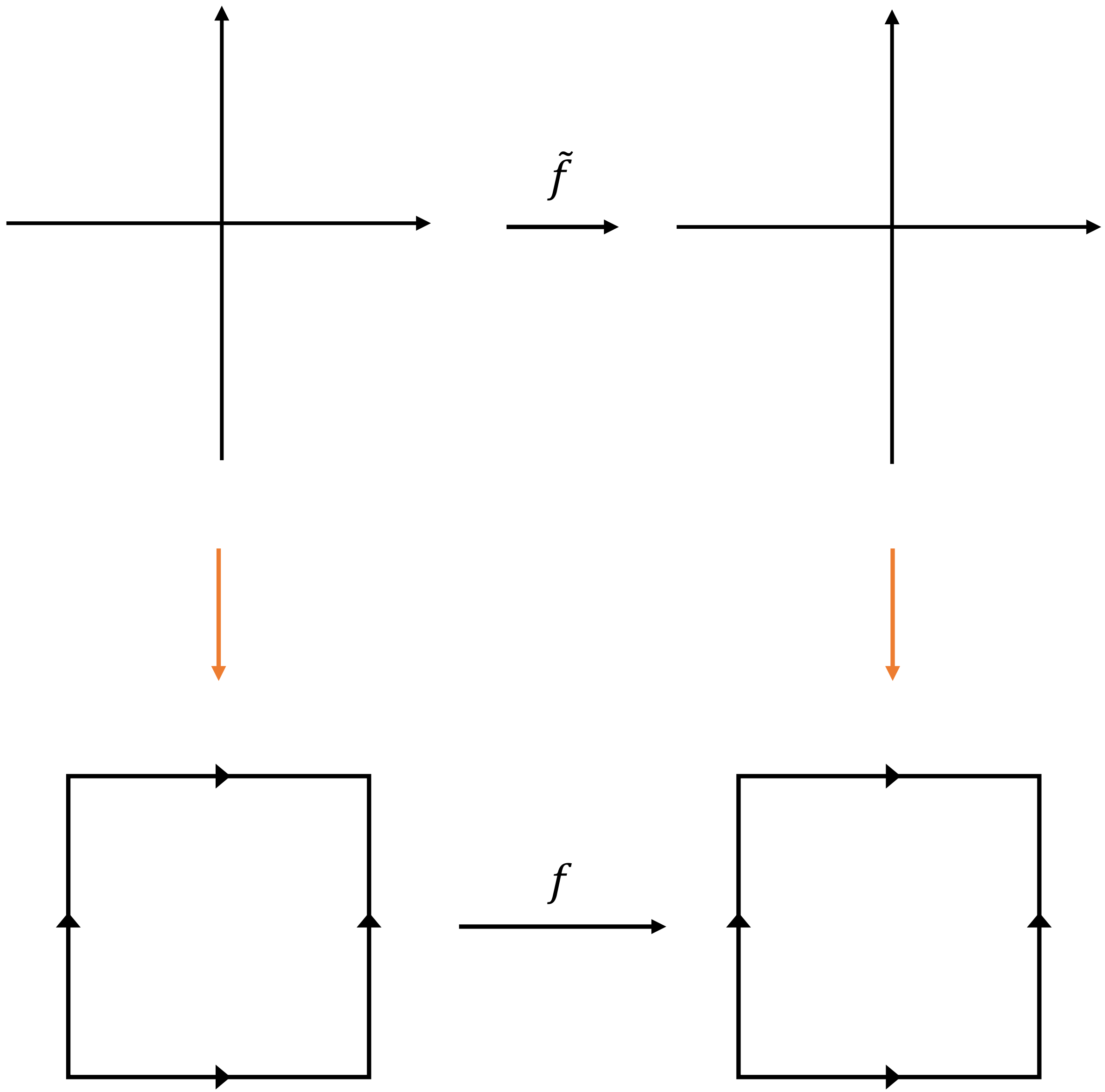


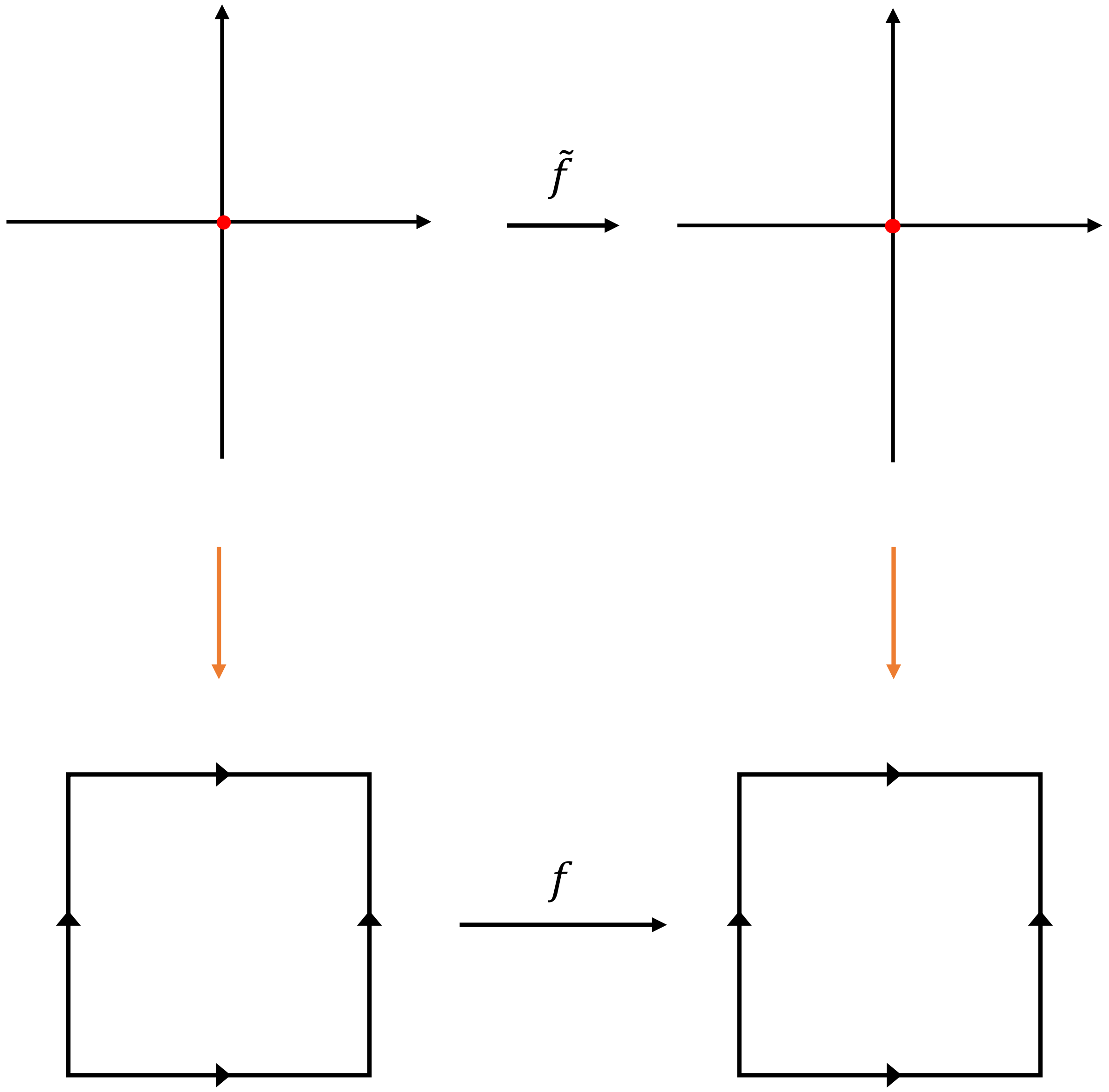
torus

torus

Stretch factor?

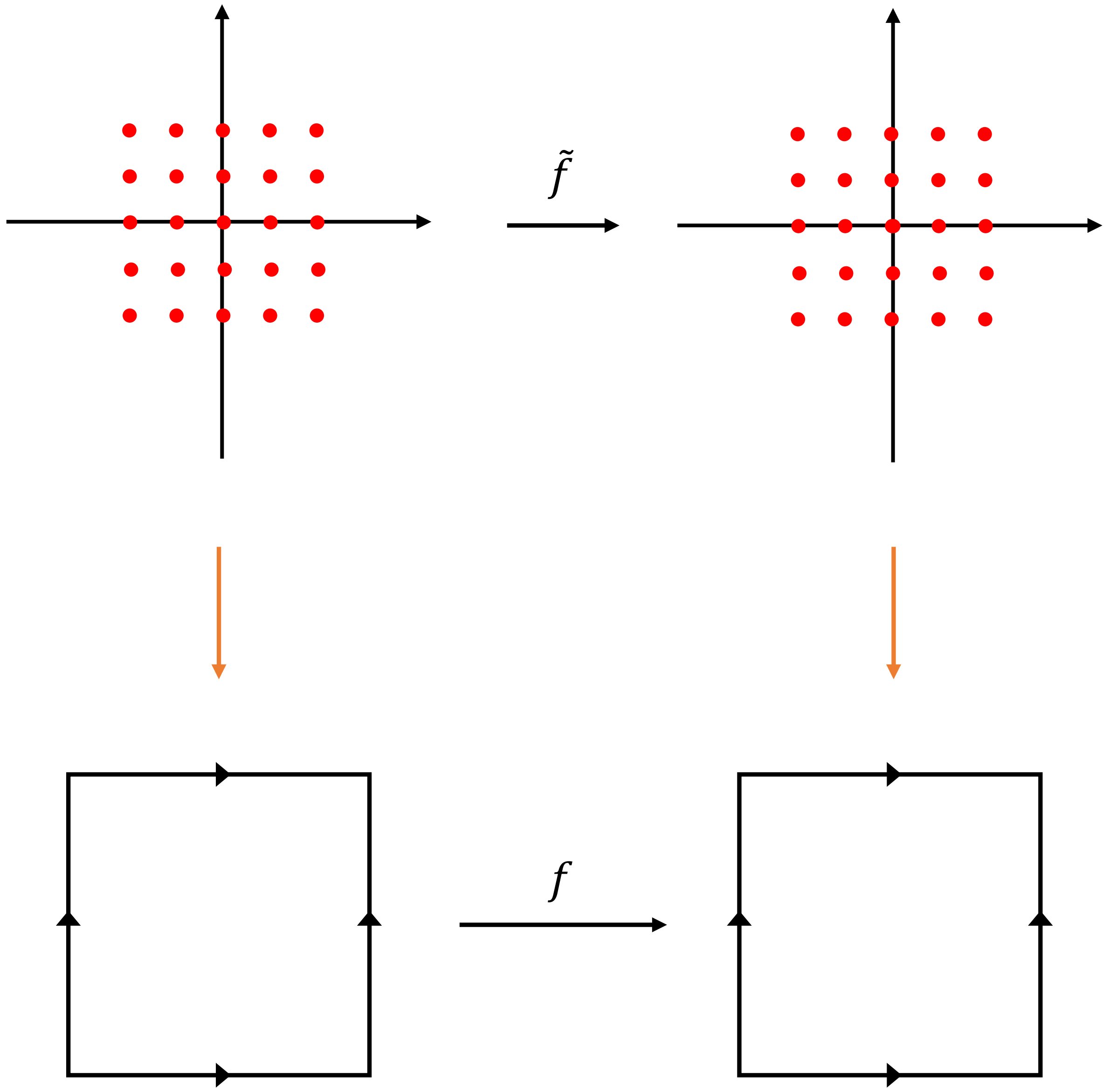

f


$\tilde{f}$
f



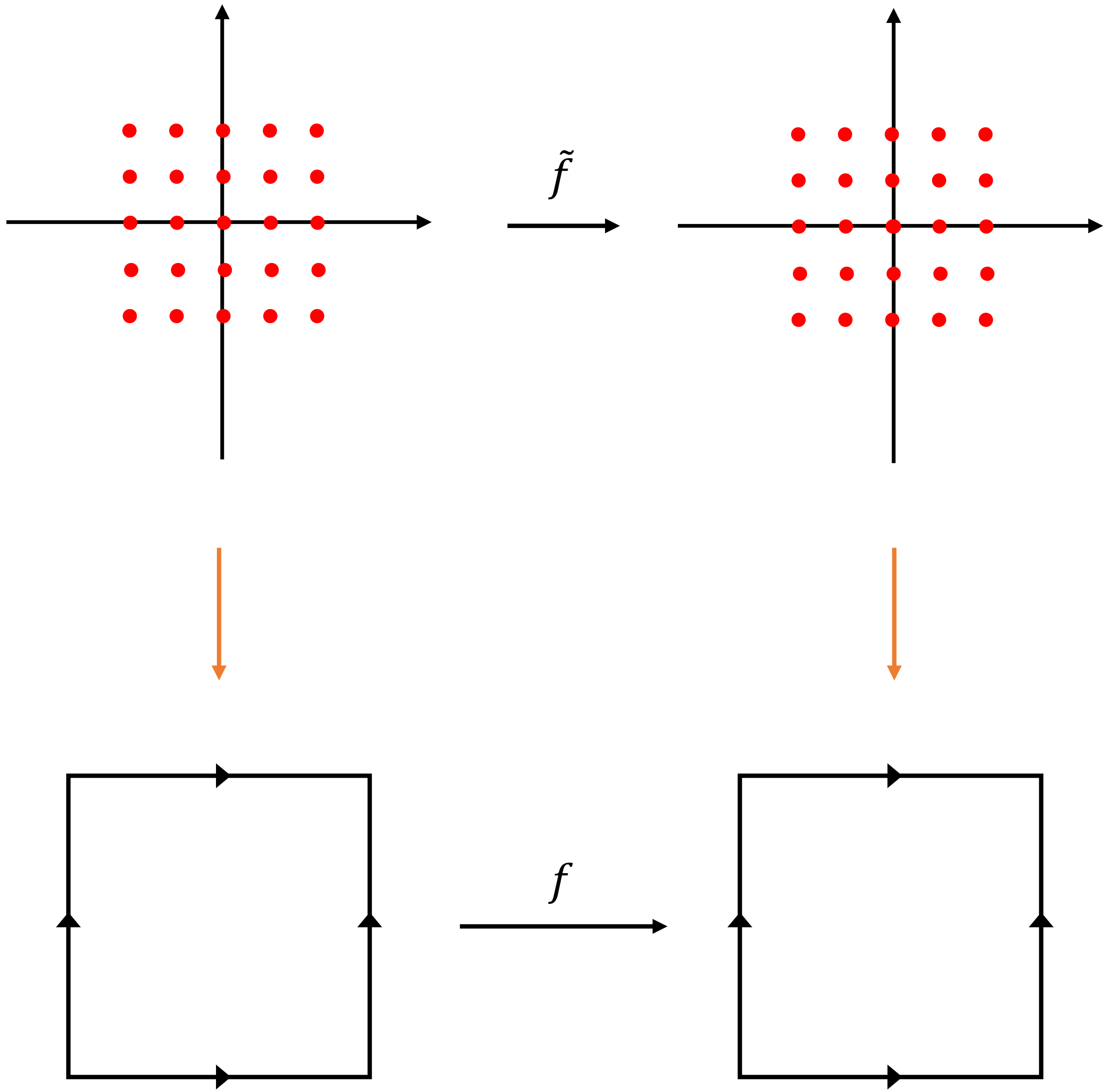
After isotopy of f:

- $\tilde{f}(0) = 0$



After isotopy of f:

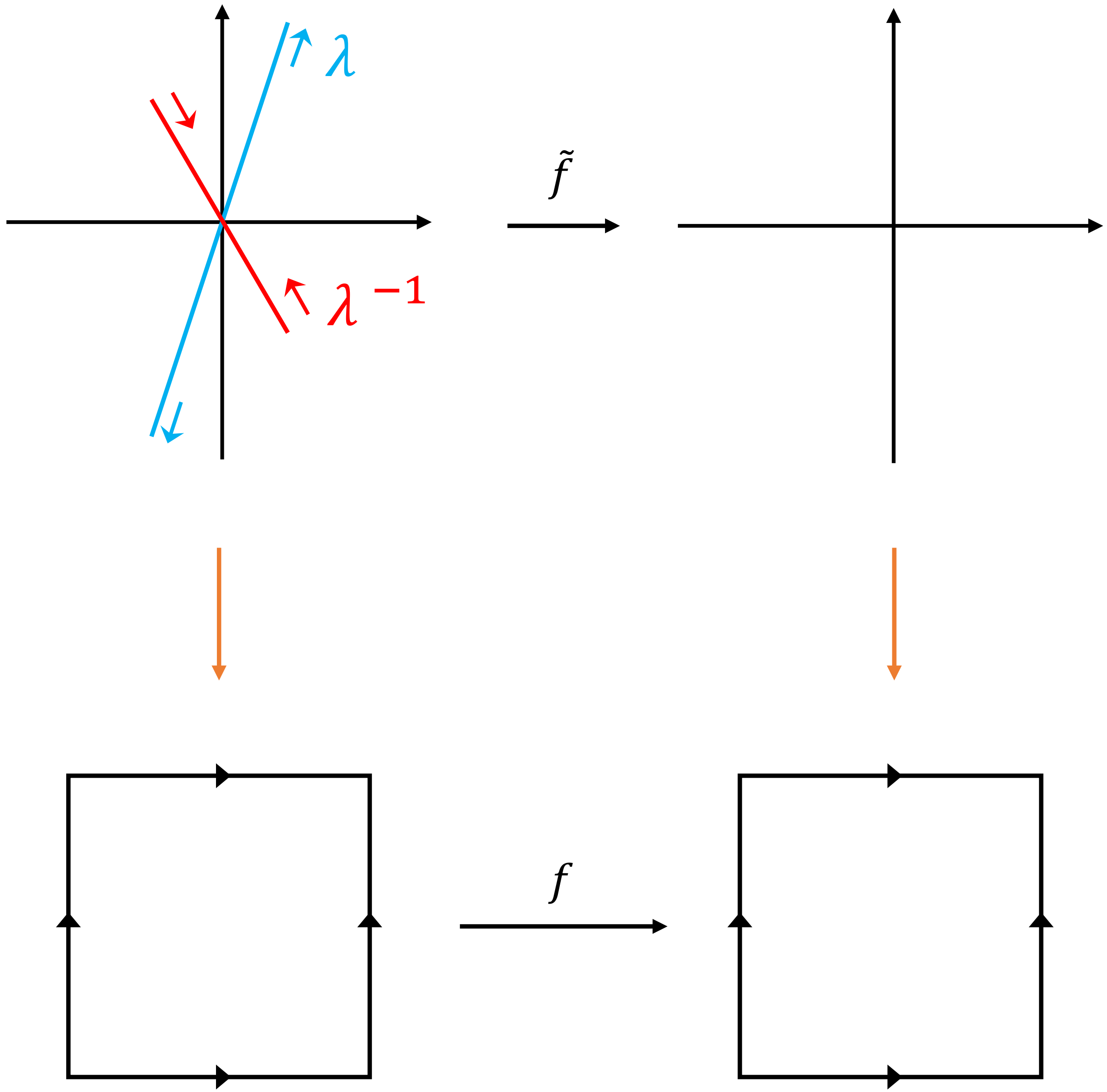
- $\tilde{f}(0) = 0$
- $\tilde{f}(\mathbb{Z}^2) = \mathbb{Z}^2$



After isotopy of f:

- $\tilde{f}(0) = 0$
- $\tilde{f}(\mathbb{Z}^2) = \mathbb{Z}^2$
- $\tilde{f}$ is linear, so

$$\tilde{f} \in SL(2, \mathbb{Z})$$



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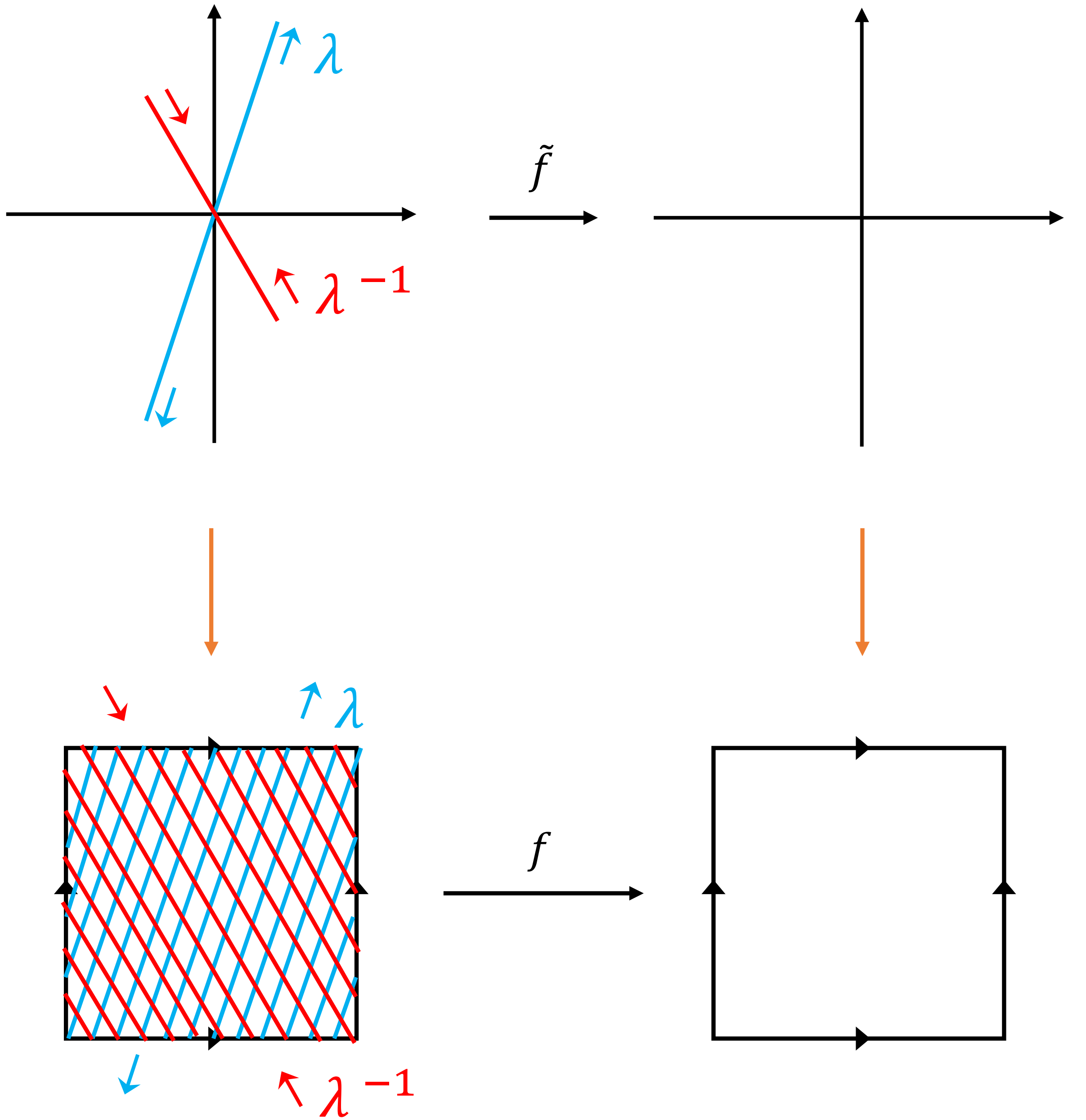

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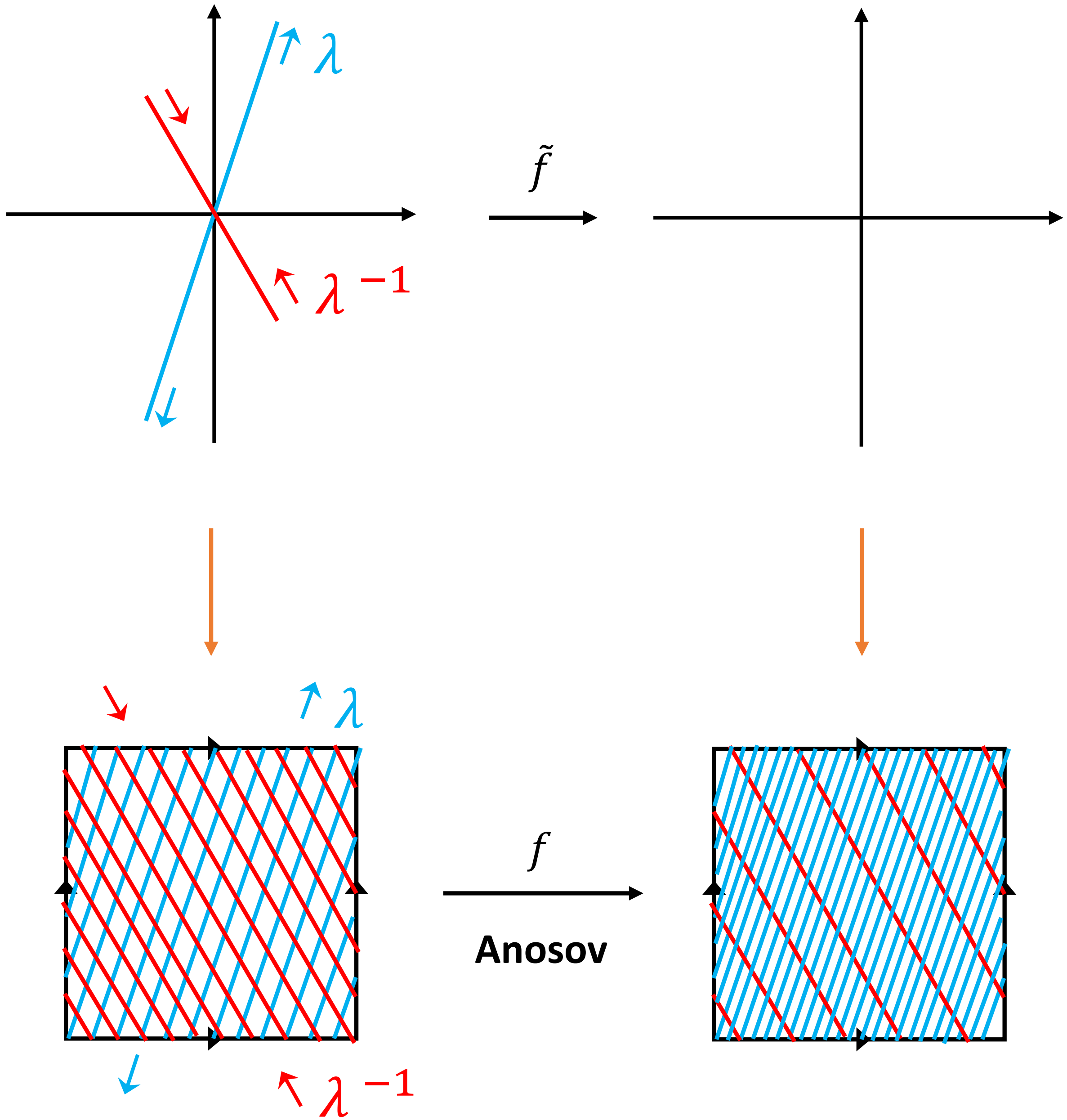

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In particular, the algebraic degree of λ can only be 2.

5. Higher genus surfaces

Mapping class group: Mod(S)=Homeo$^+$(S)/isotopy

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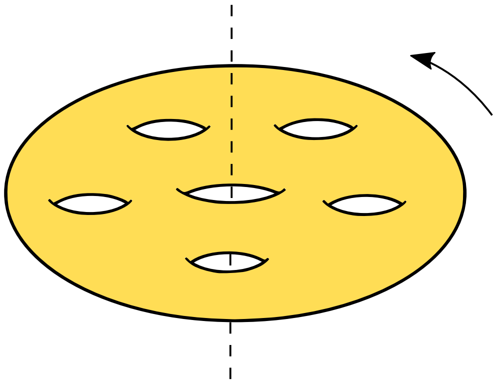
Nielsen-Thurston Classification



Finite order

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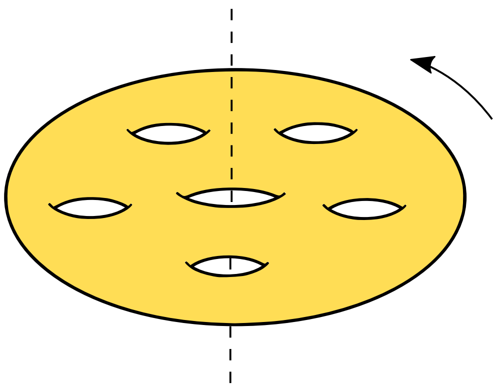

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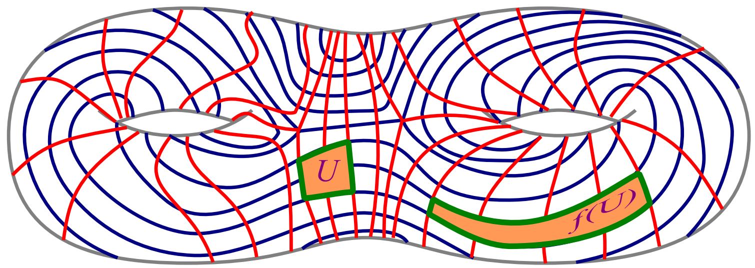

Pseudo-Anosov

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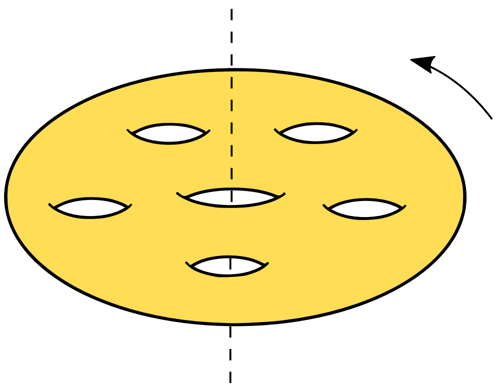

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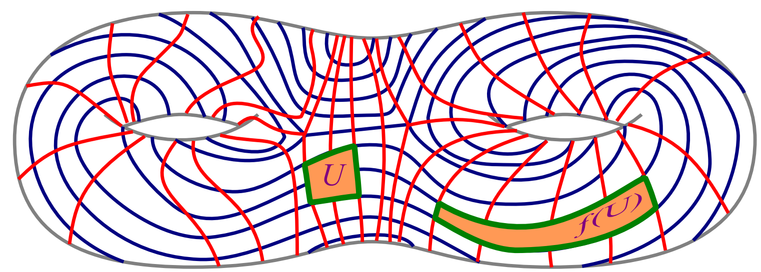

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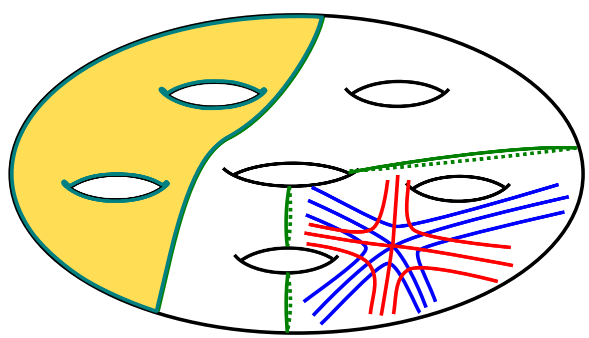
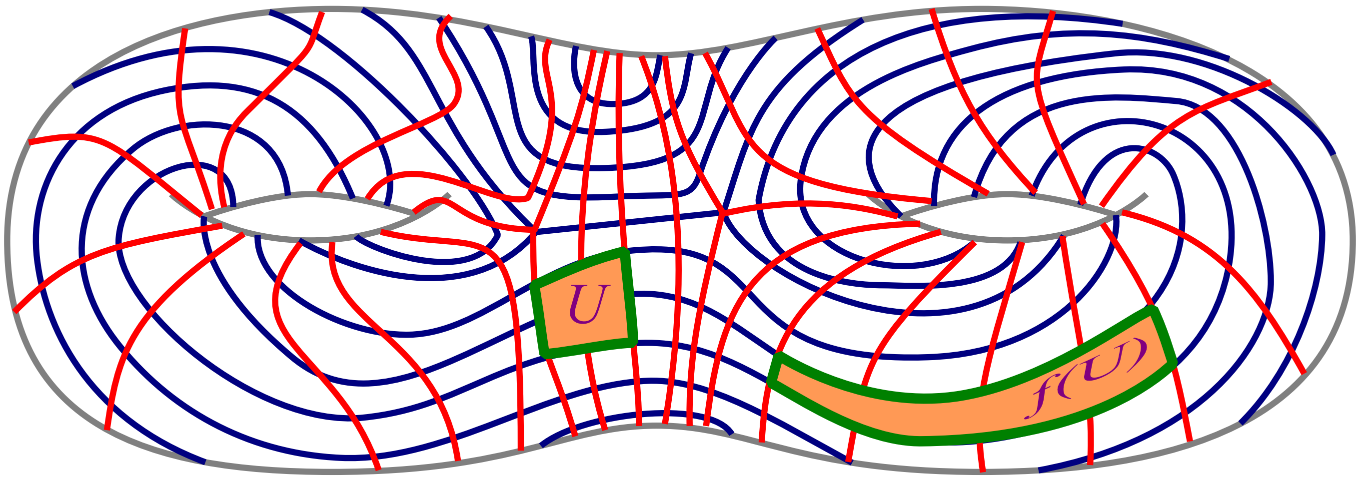
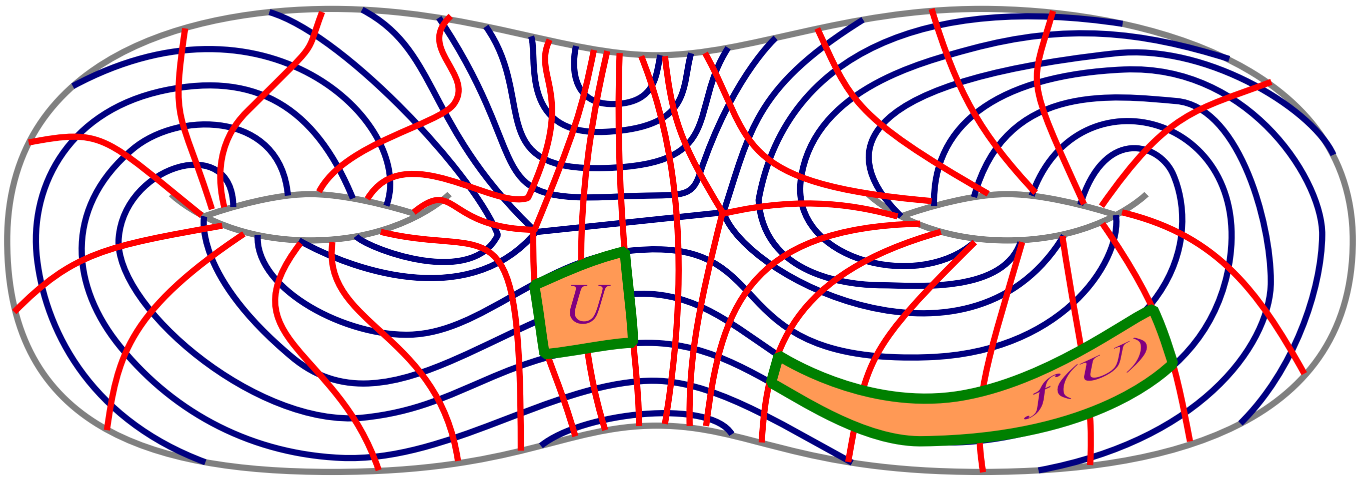


Reducible



Pseudo-Anosov

What numbers can be pseudo-Anosov stretch factors?



Pseudo-Anosov

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Algebraic unit: Unit in the ring of algebraic integers.

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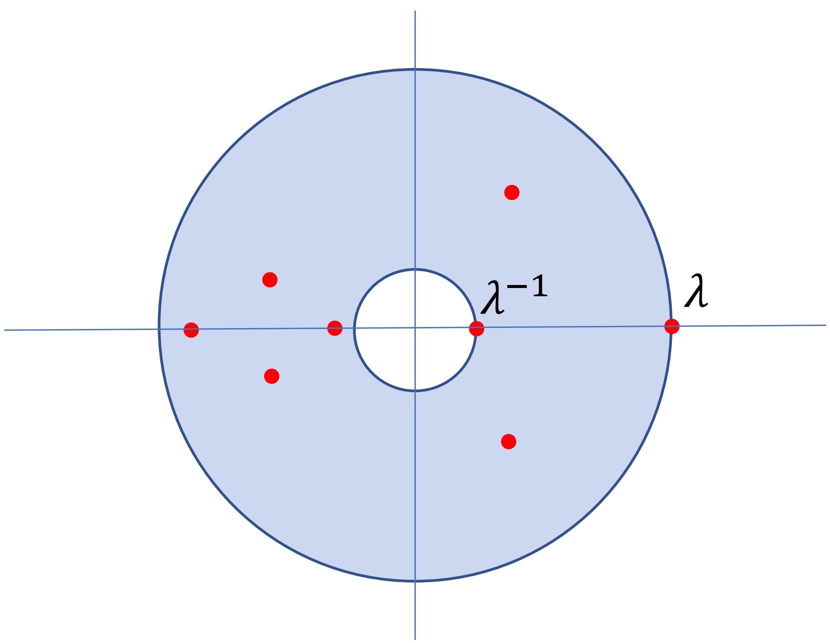
Bi-Perron: The Galois conjugates of λ (except λ^{-1}) lie in the annulus $\lambda^{-1} < |z| < \lambda$.



Fried’s problem

Does every bi-Perron unit arise as a pseudo-Anosov stretch factor?

Fibrations of 3-manifolds

Train track automata

Penner's construction

Thurston's construction

Kra's construction

Rauzy-Veech induction

Branched coverings

Casson-Bleiler's homological criterion

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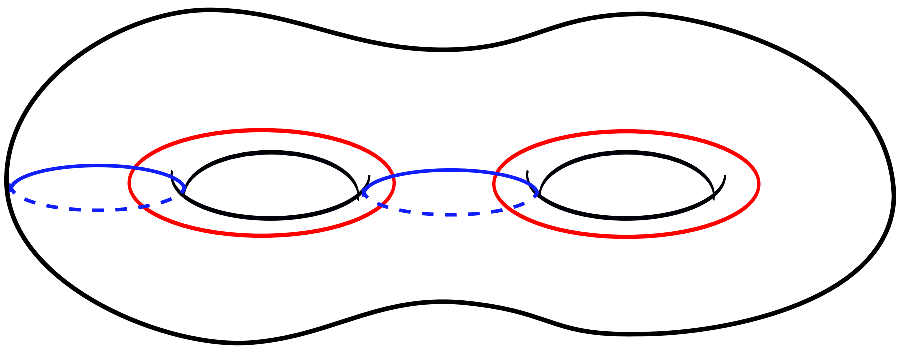
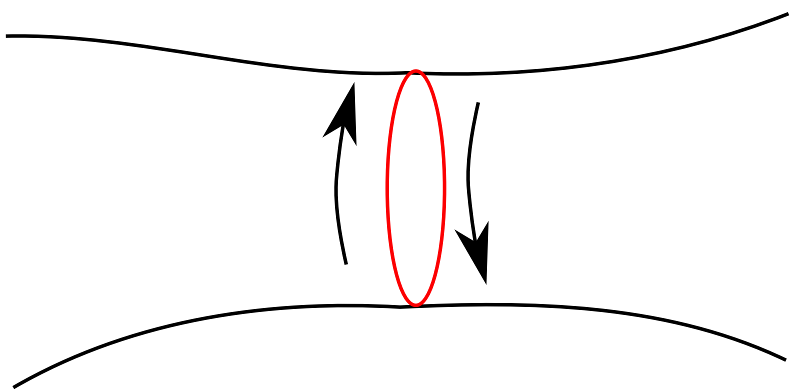
Thurston's and Penner's construction



Choose a set of filling curves.



Construct products of Dehn twists.

Thurston’s construction

Not everything arises from Thurston's construction

Hubert-Lanneau ('06): If λ is the stretch factor of a pseudo-Anosov mapping class arising from Thurston's construction, then the trace field $\mathbb{Q}(\lambda+\lambda^{-1})$ is totally real.

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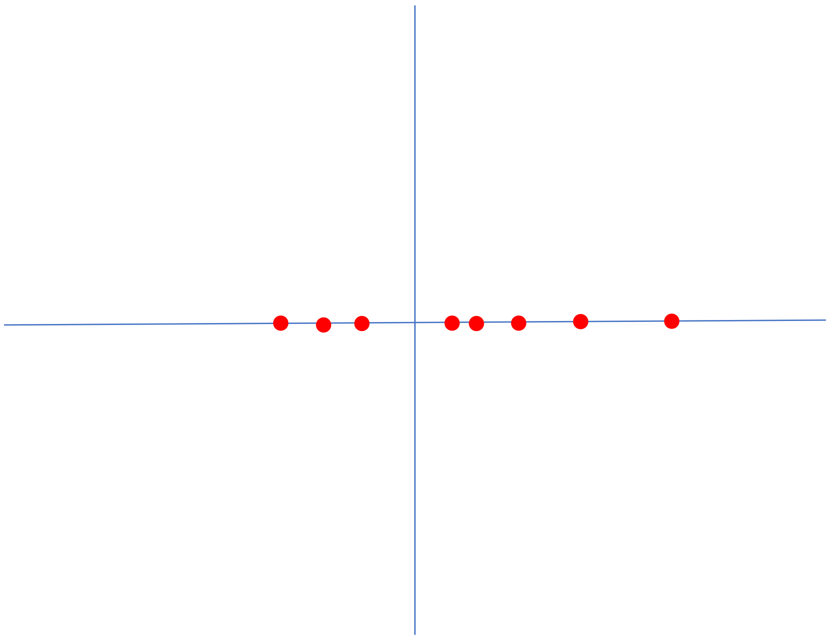
Totally real: Every Galois conjugate is real.



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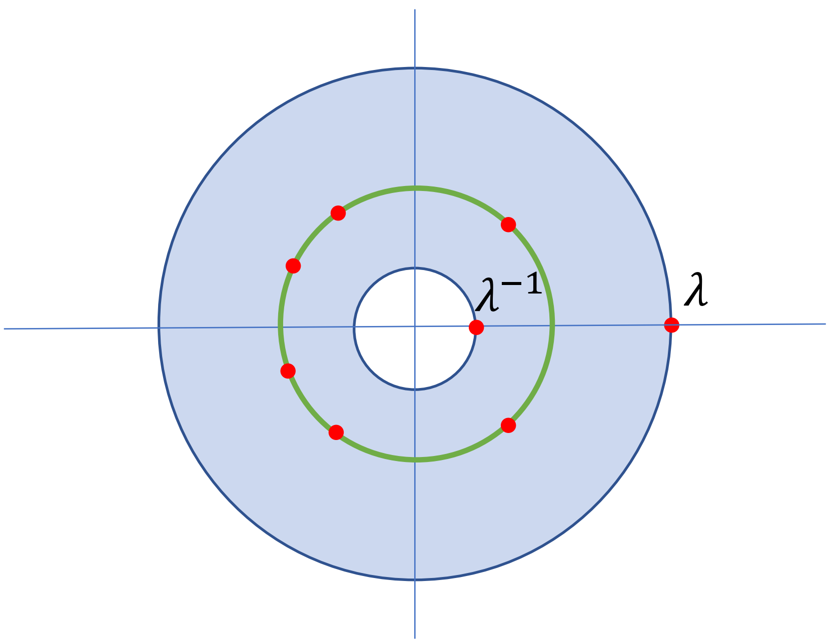

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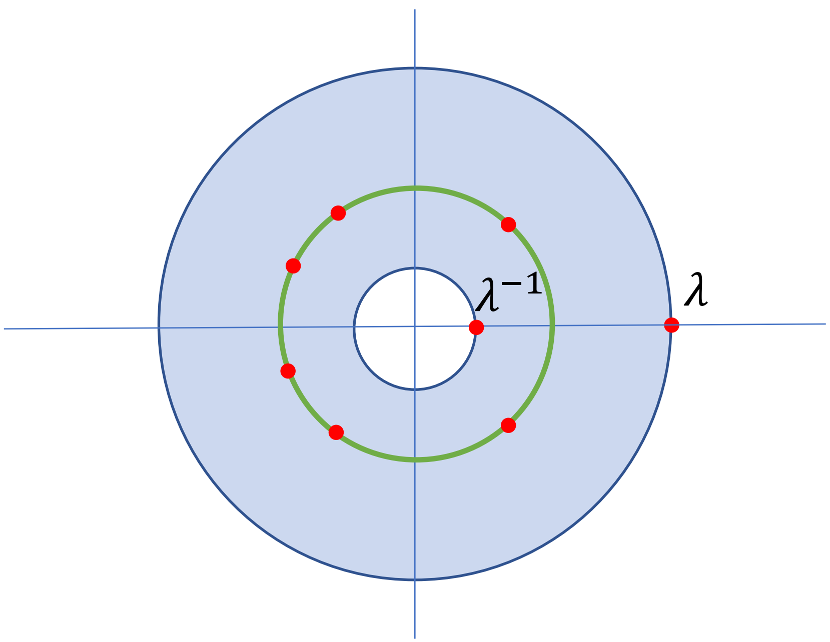
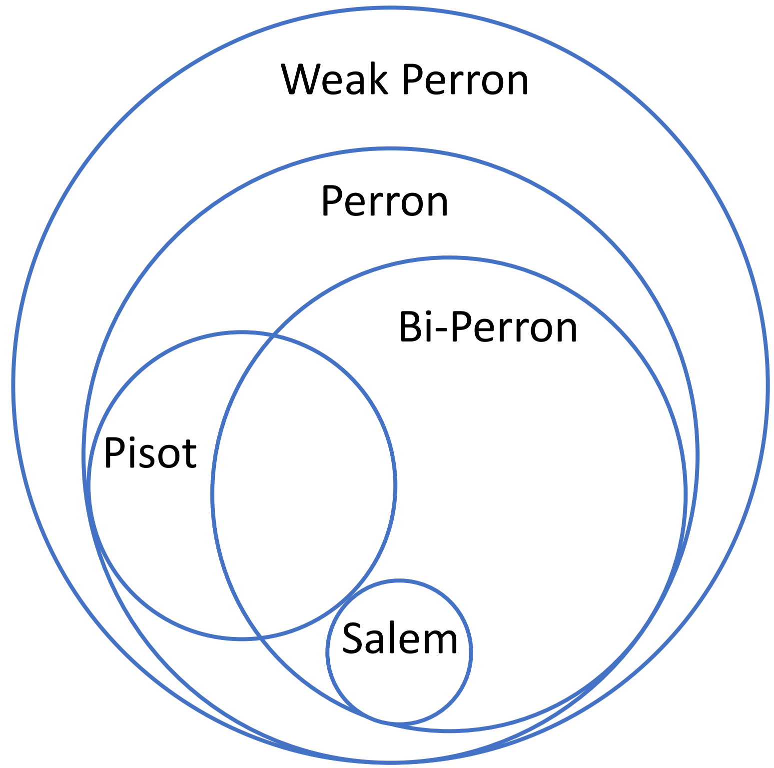
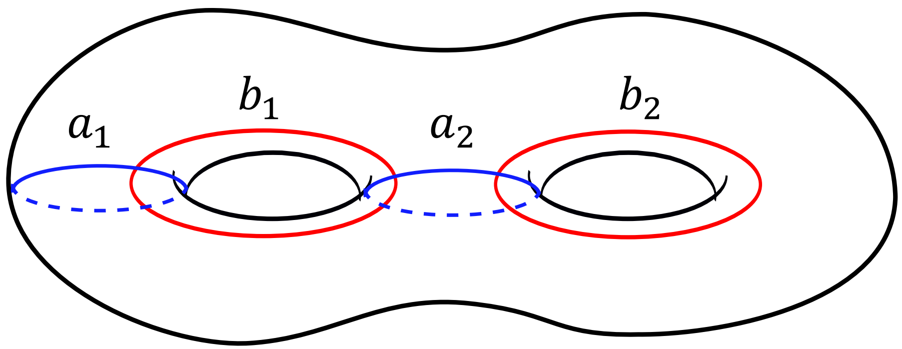




Penner’s construction

Penner's construction

$A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_m\}$ filling multicurves.
Any product of T_{a_i} and $T_{b_j}^{-1}$ containing each of the Dehn twists at least once is pseudo-Anosov.



E.g.: $T_{b_1}^{-4}\, T_{a_2}^{2}\, T_{a_1}^{6}\, T_{b_1}^{-1}\, T_{b_2}^{-10}$ is pseudo-Anosov.

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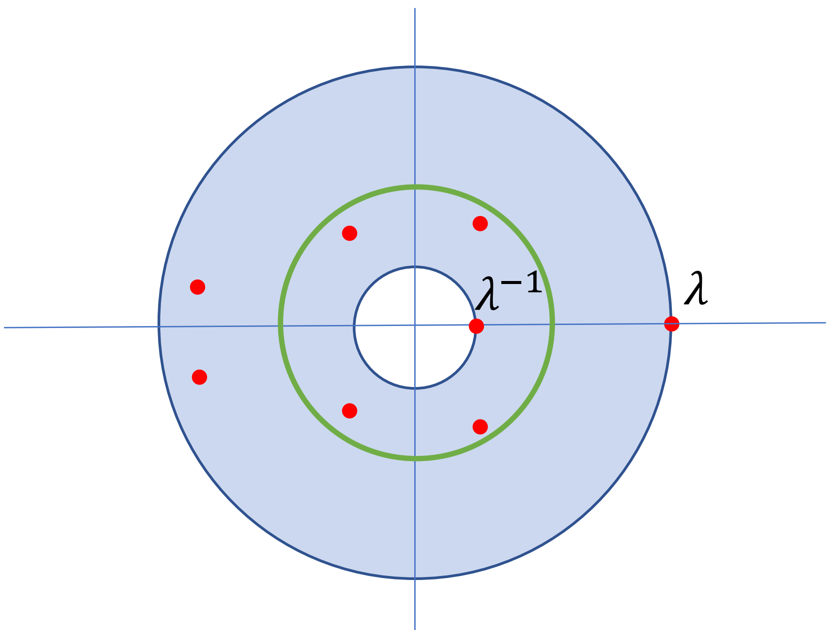
Shin-Strenner ('15):

1. Galois conjugates of Penner stretch factors lie off the unit circle.



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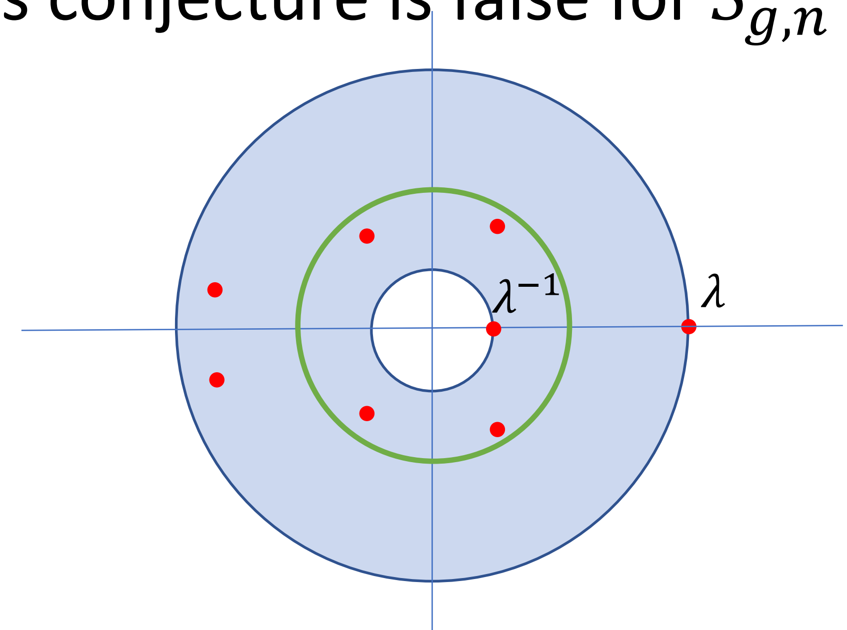
Shin-Strenner ('15):

1. Galois conjugates of Penner stretch factors lie off the unit circle.
2. Penner's conjecture is false for $S_{g,n}$ if $g + n \geq 5$.



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Strenner (16'): Galois conjugates of Penner stretch factors are dense in $\mathbb{C}$.

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Shin ('14): construction for $\deg(\lambda) = 2g$.
Strenner ('16): All integers subject to Thurston's and Long's constraints arise from Penner's construction.

Challenge (Agol):

Use the ideas in Kenyon's paper (embedding $\mathbb{Z}[\lambda]$ into $\mathbb{R}^d$ as a lattice) to realize a given λ as a pseudo-Anosov stretch factor.

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Show: these matrices do not have eigenvalues on the unit circle.

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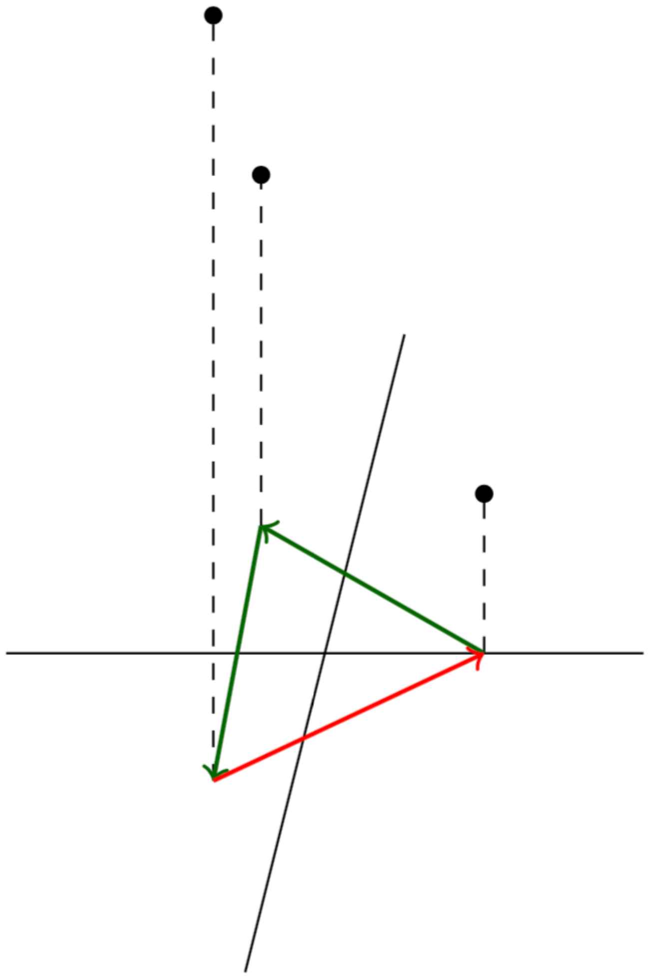
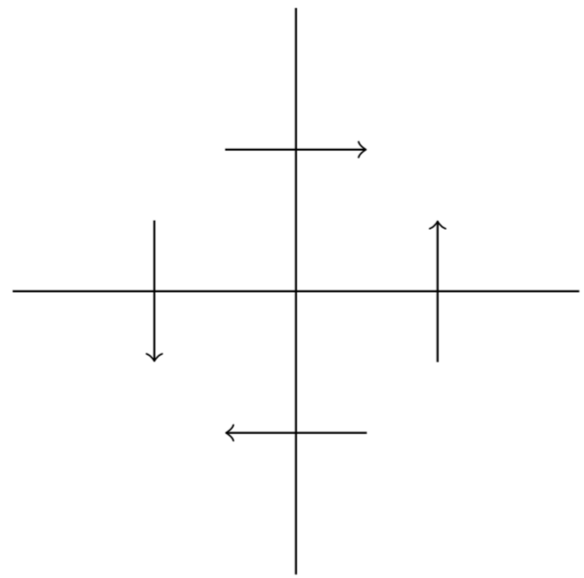
$\updownarrow$

rotation of invariant plane

Idea: define height that increases after every iteration



$$Q_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$Q_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



An increasing height function: $h(x, y) = xy$.