Fibrations of 3-manifolds and nowhere continuous functions

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Geometric Topology Seminar
Columbia University
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Calculus exercise 1

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Solution:

For example, $\mu\left(\frac{p}{q}\right) = \frac{1}{q}$.

Then $\bar{\mu} = 0$.

Calculus exercise 2

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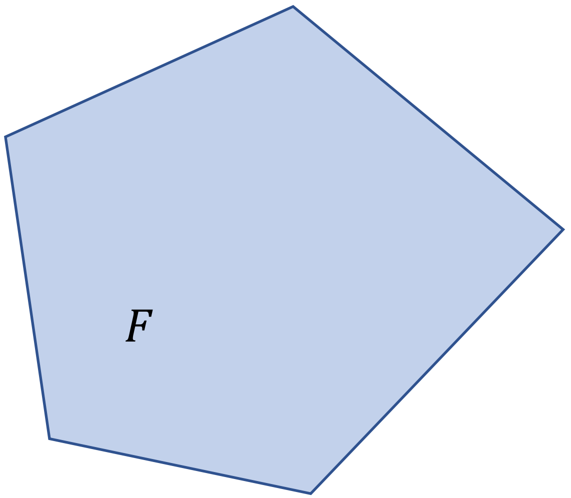
Let $F \subseteq \mathbb{R}^n$ be a compact, convex polyhedron whose faces are determined by rational equations.



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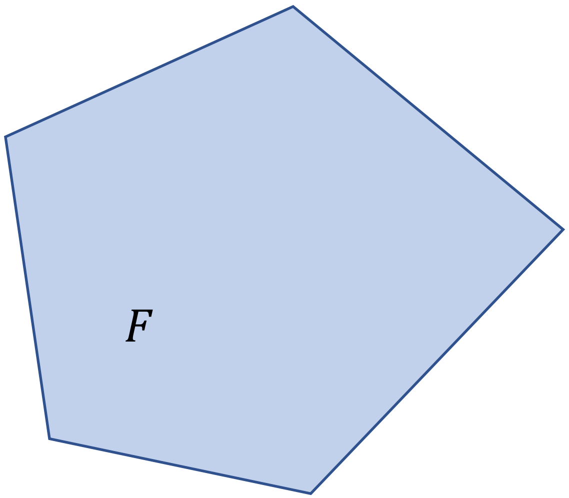

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Consider a function $\mu: int(F)_{\mathbb{Q}} \to \mathbb{R}$.

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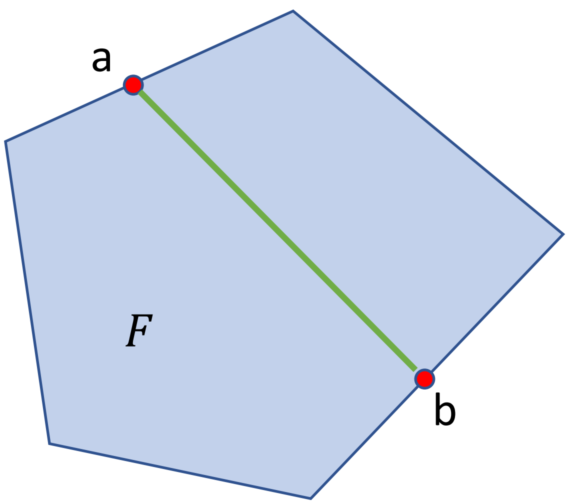

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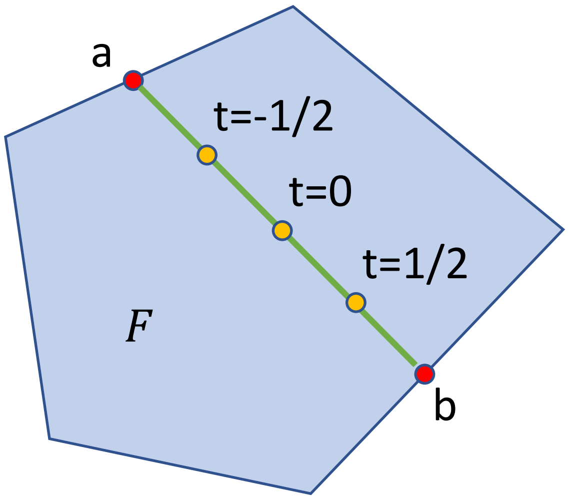

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For $a, b \in \partial F_{\mathbb{Q}}$ where the line segment ab meets $int(F)$, consider $v_{a,b}(t) = \mu\left(\frac{a+b}{2} + t\,\frac{b-a}{2}\right)$, defined on $\mathbb{Q} \cap (-1,1)$. This is a slice of μ.

Devil’s bowl

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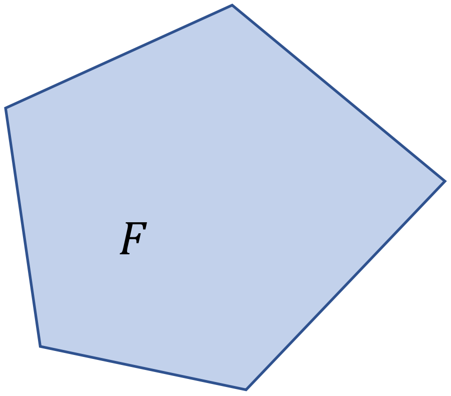

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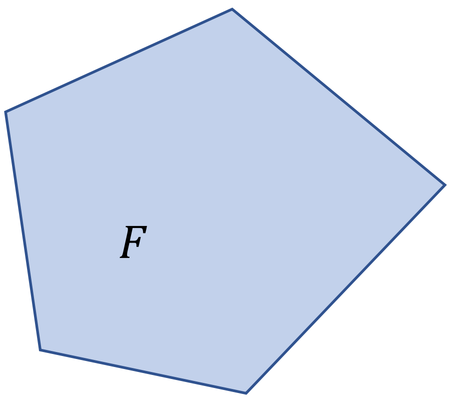

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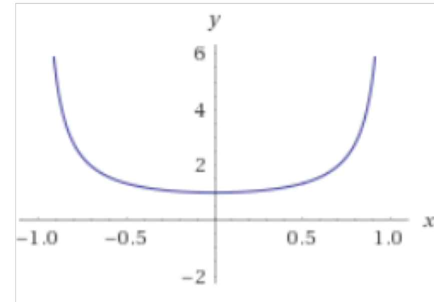

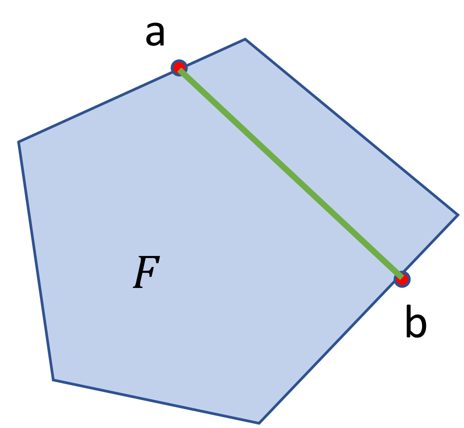

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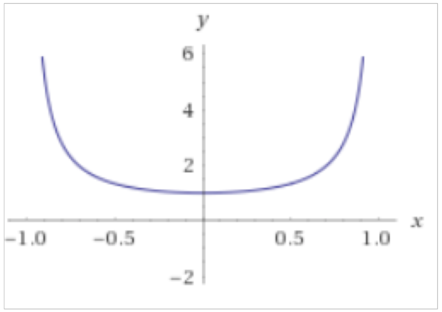
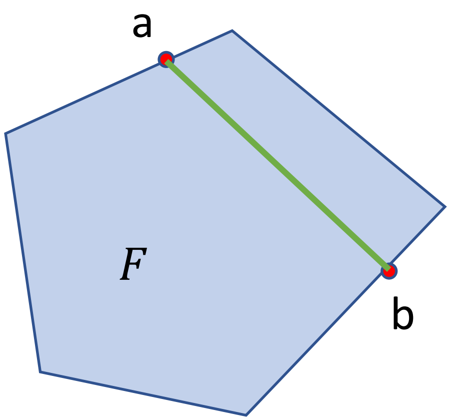


Exercise: Construct a devil's bowl.



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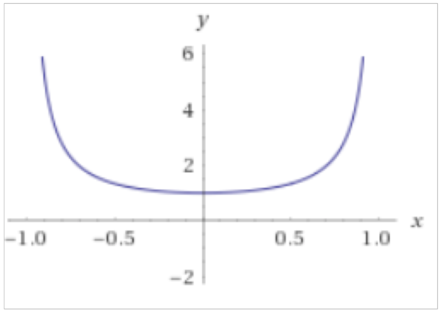

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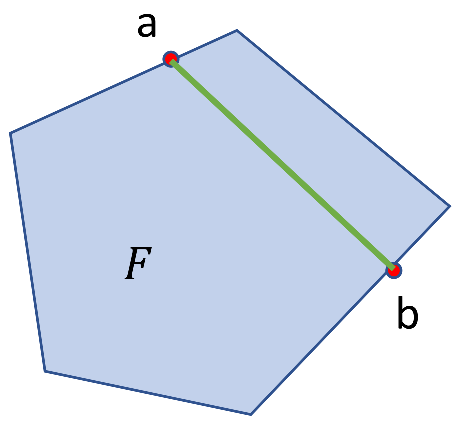

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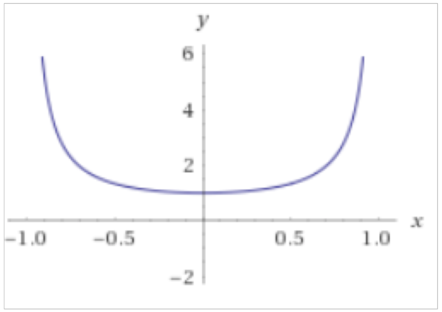

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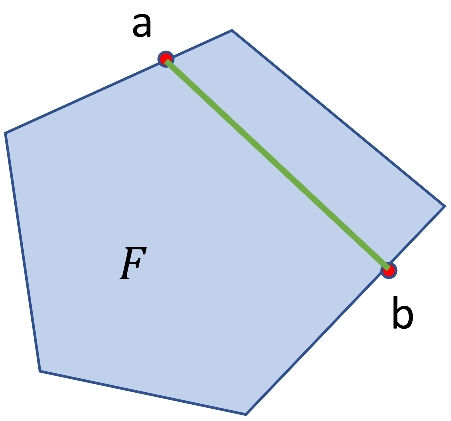

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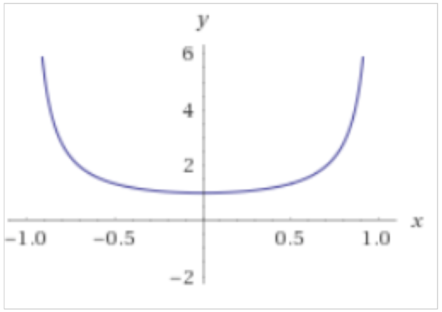

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Solution 2: Study fibrations of 3-manifolds.

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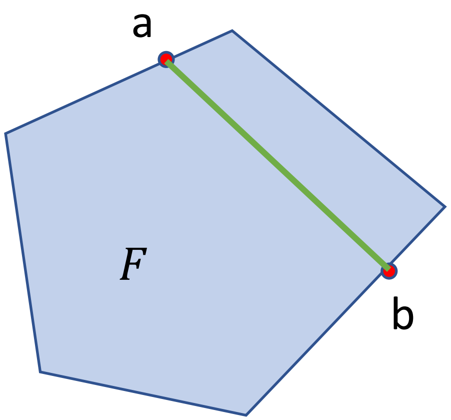
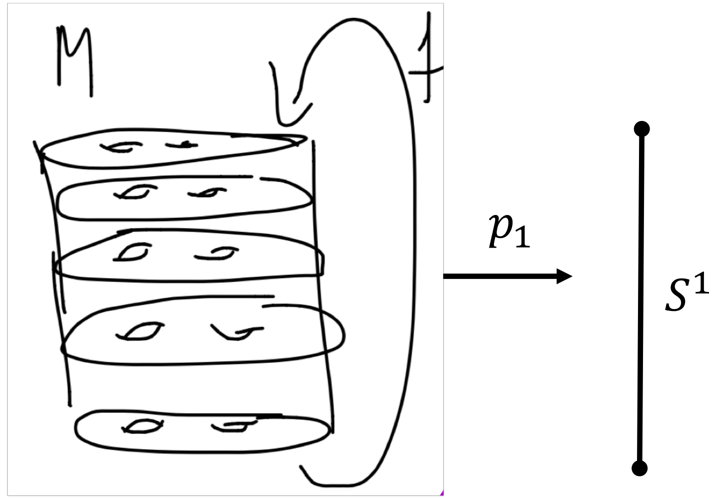


3-manifolds

3-manifold fibering over the circle.



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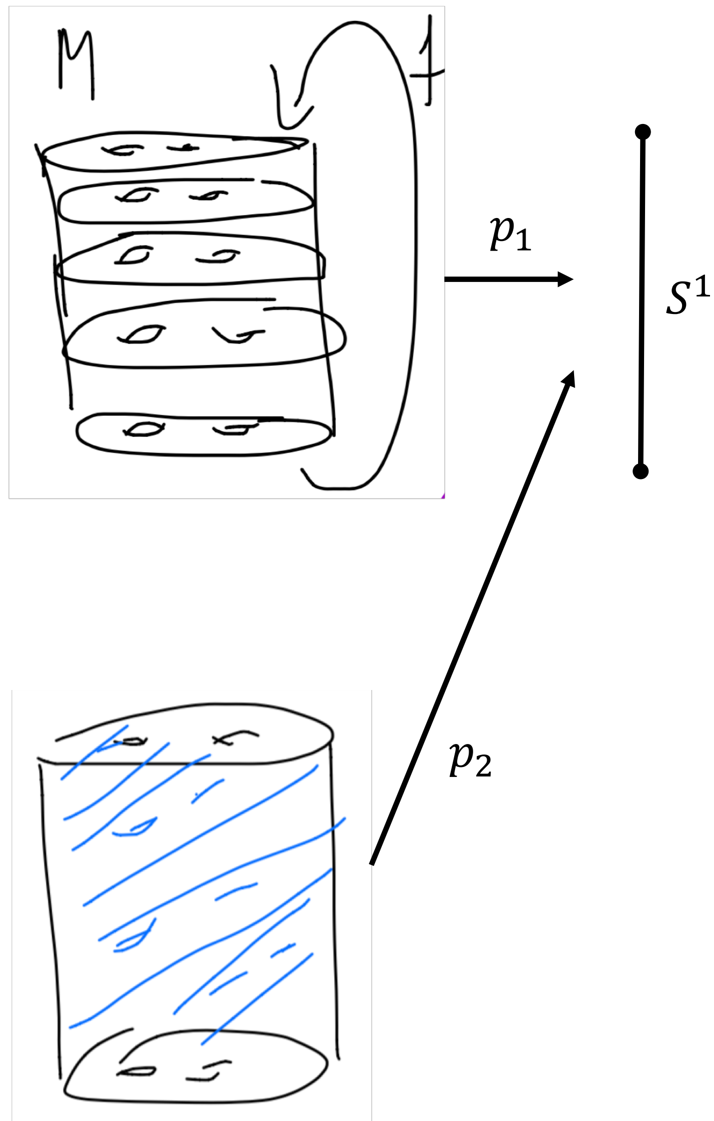

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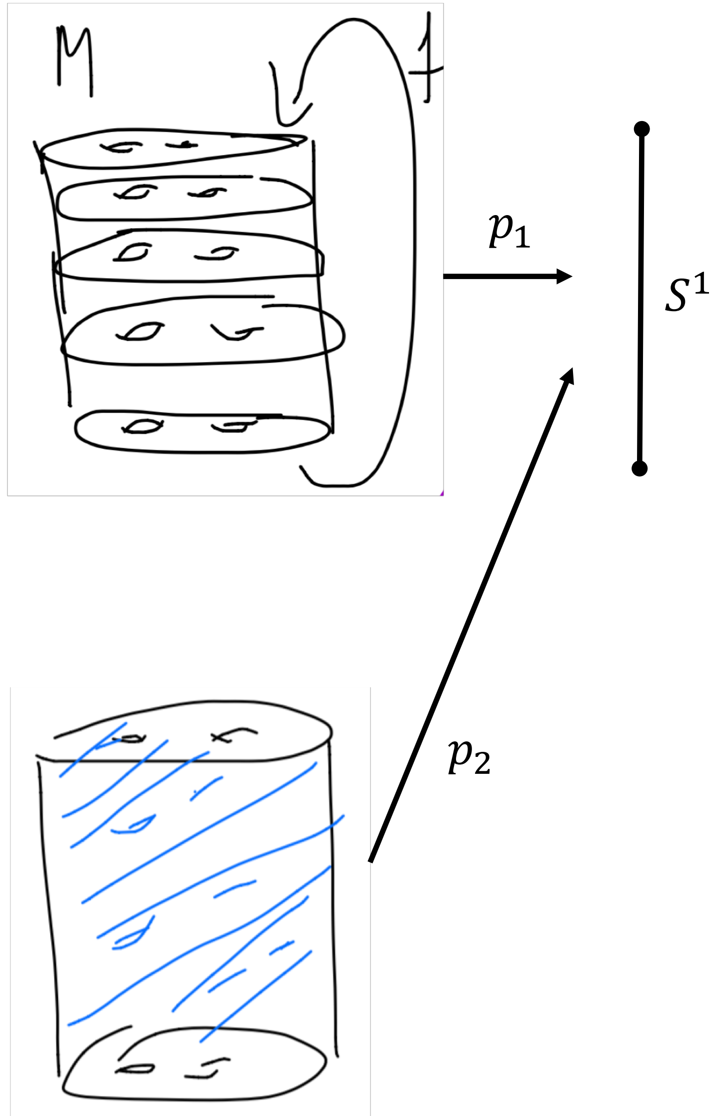

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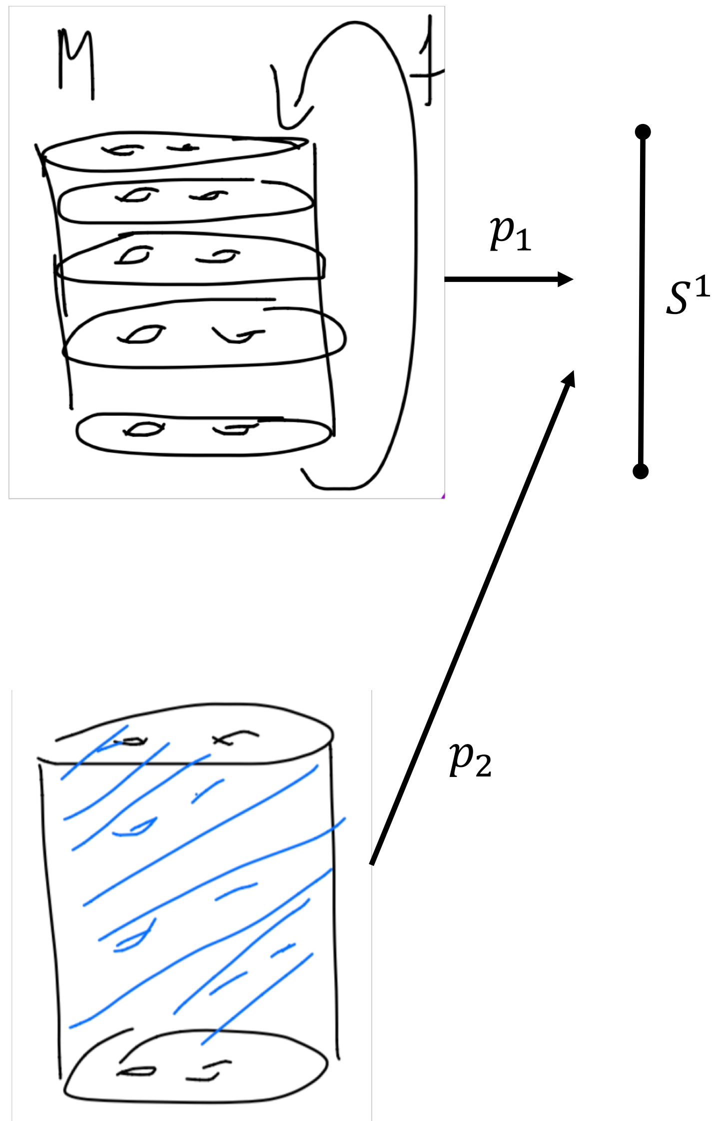

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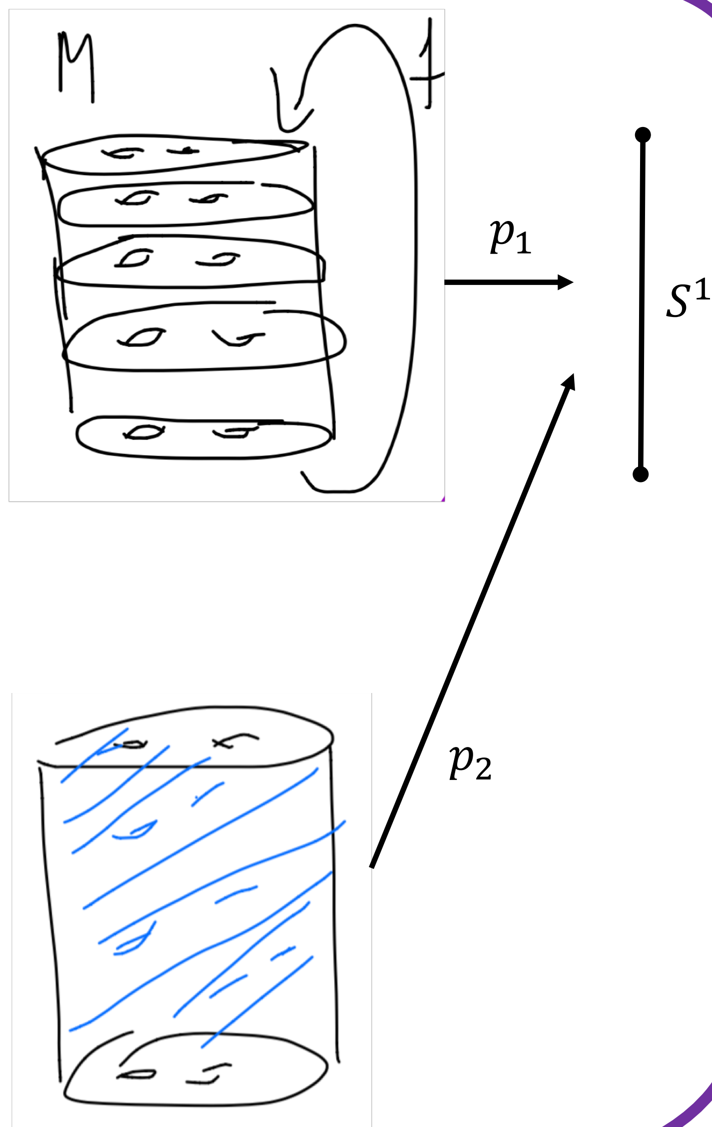

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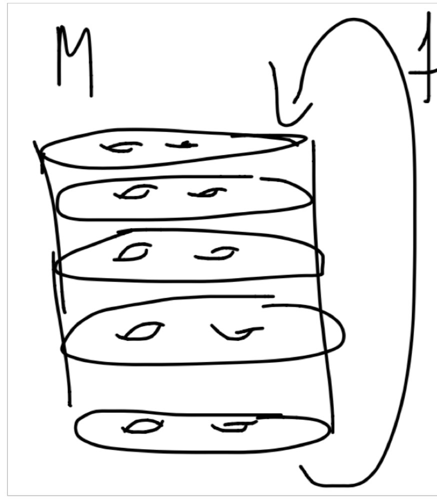

p_1

S^1

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p_2

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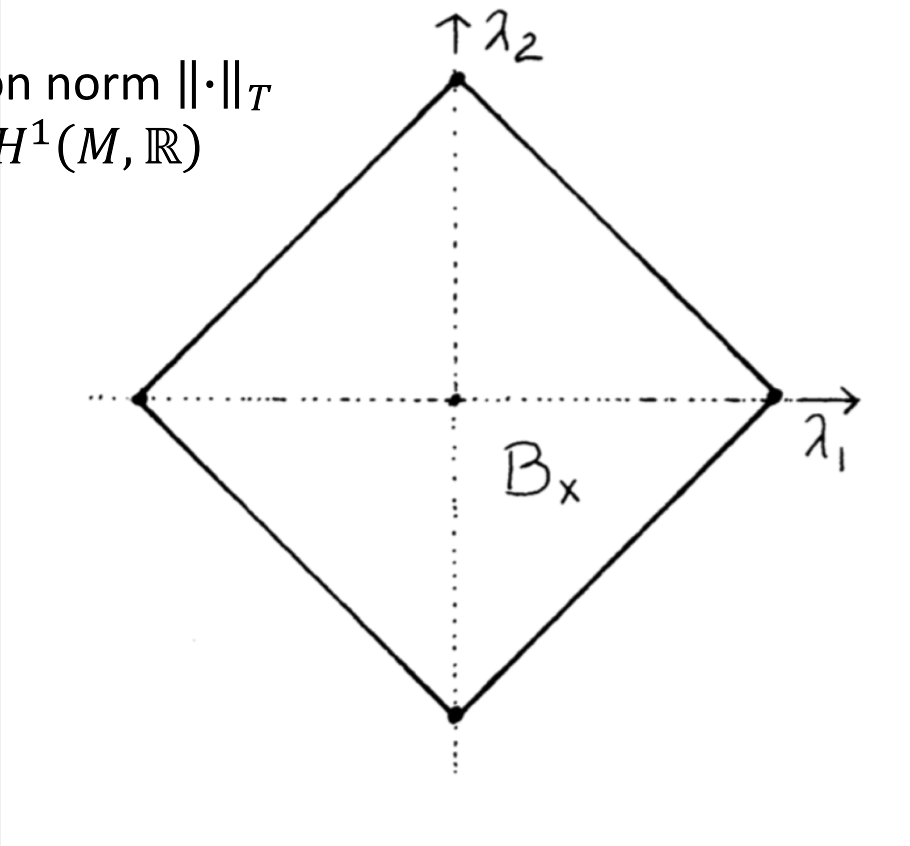
Thurston norm $\|\cdot\|_T$
on $H^1(M, \mathbb{R})$



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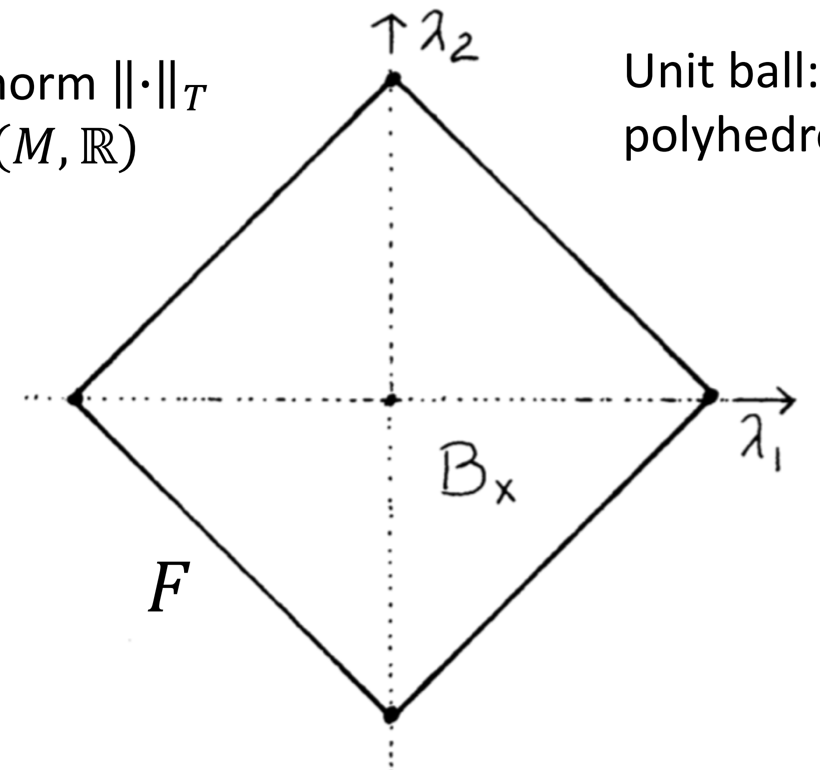
Unit ball: polyhedron



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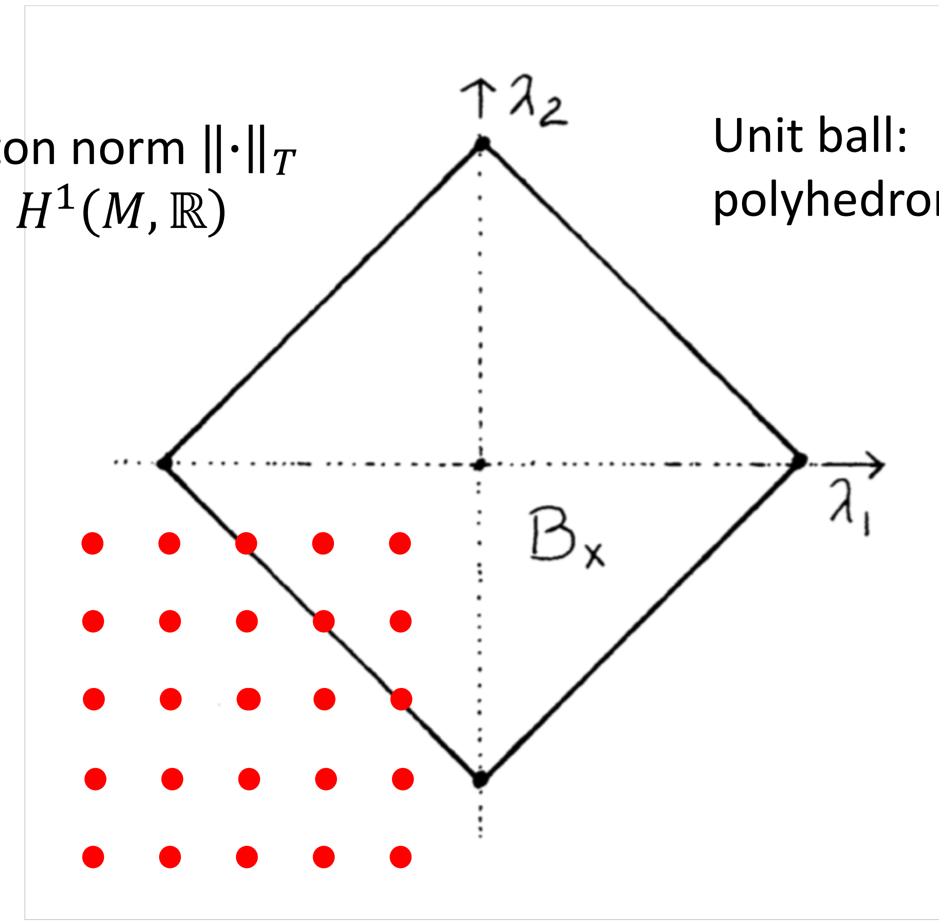

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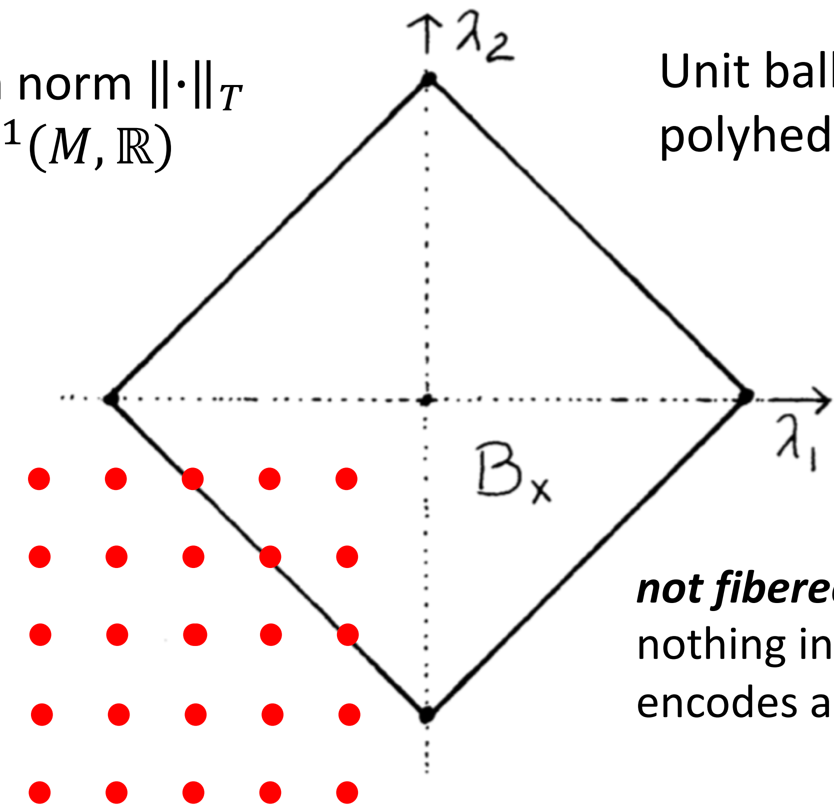

fibered face: every integral class in this cone encodes a fibration

not fibered face: nothing in this cone encodes a fibration

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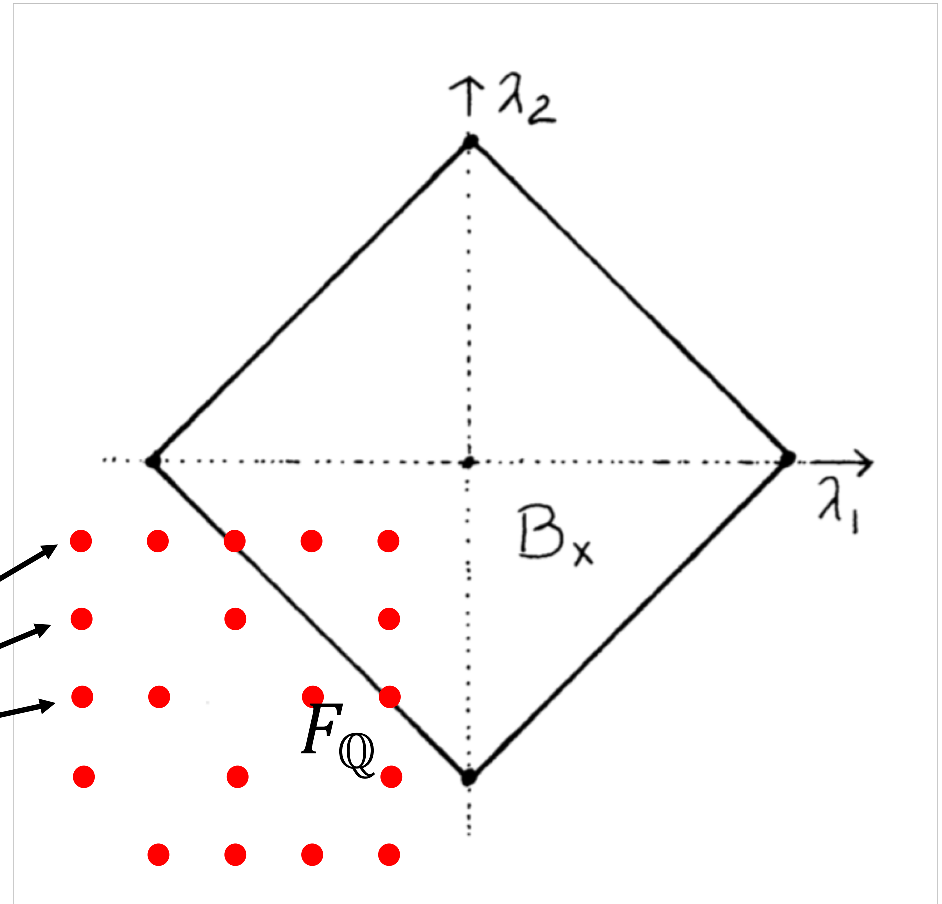

pseudo-Anosov mapping classes

How does the monodromy change as we vary the fibration?

Pseudo-Anosov mapping classes

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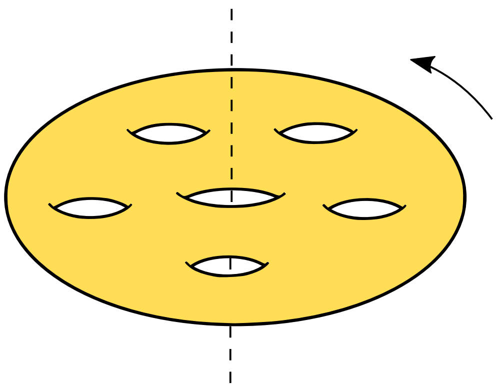
Nielsen-Thurston Classification



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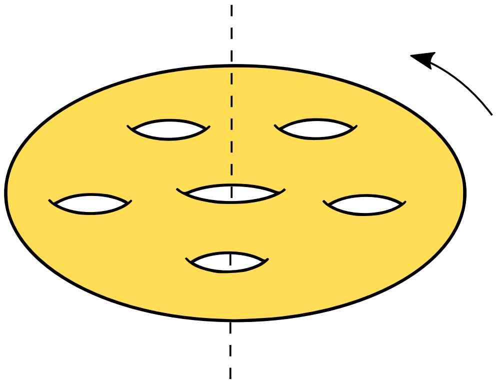

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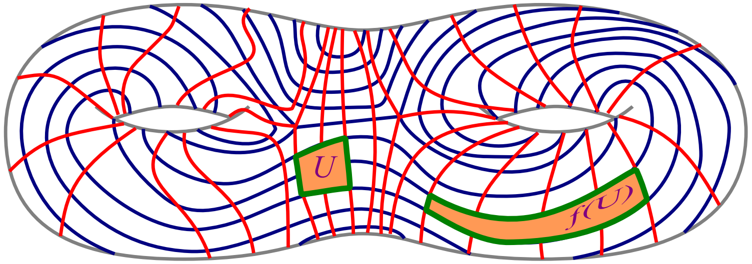

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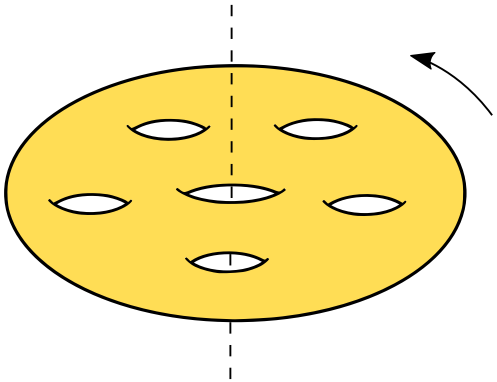

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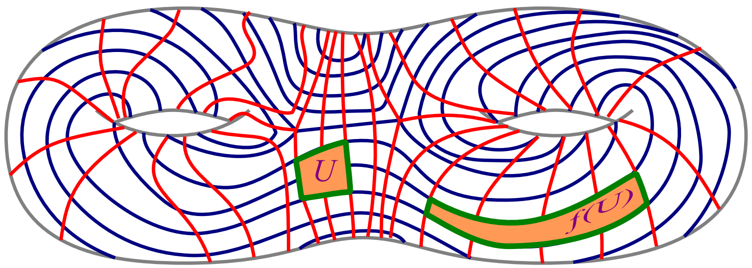

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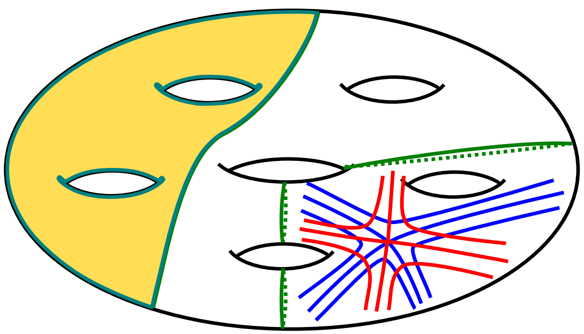

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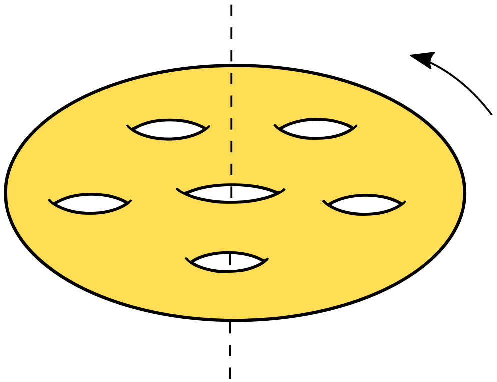
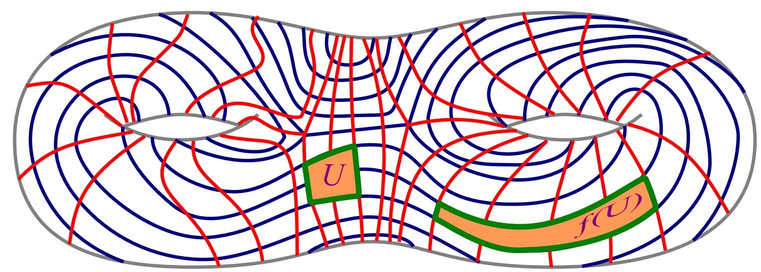


Finite order

Stretch factor: λ



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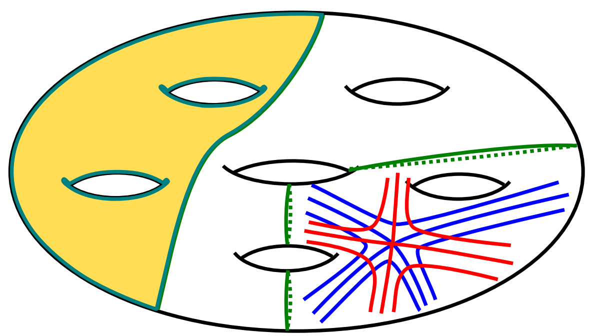

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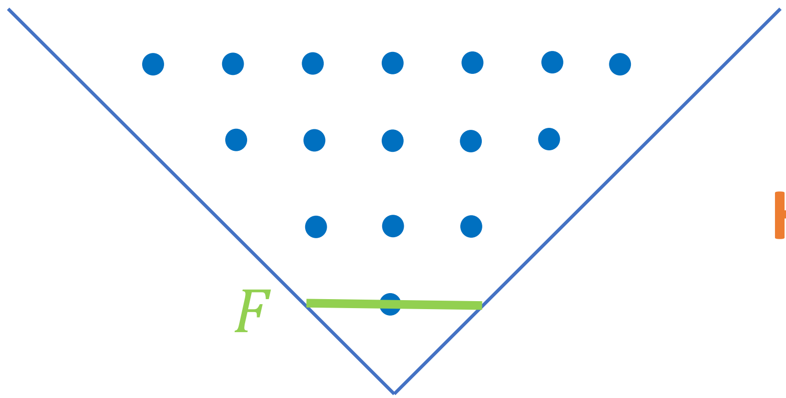
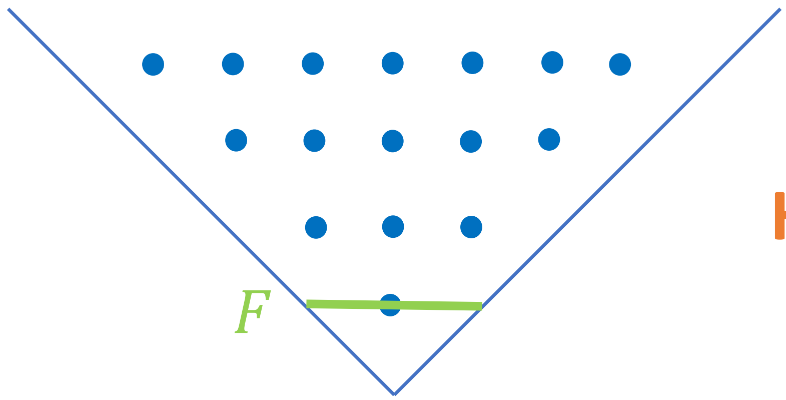


How does λ change?

Fried ('82): Let M be a hyperbolic 3-manifold with $b_1(M) \geq 2$ and fibered face F.



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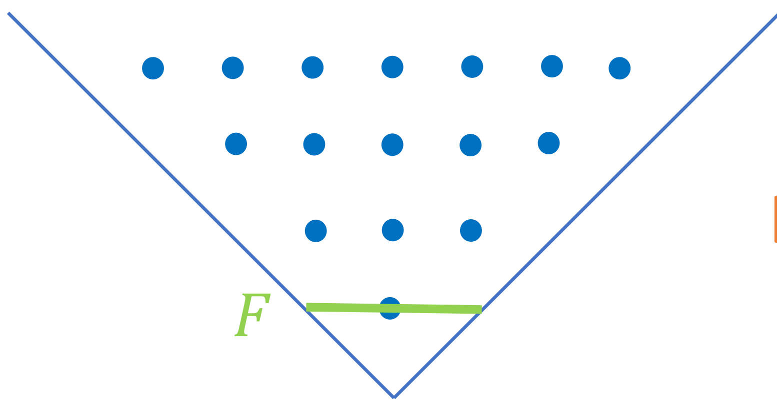

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The normalized stretch factor function

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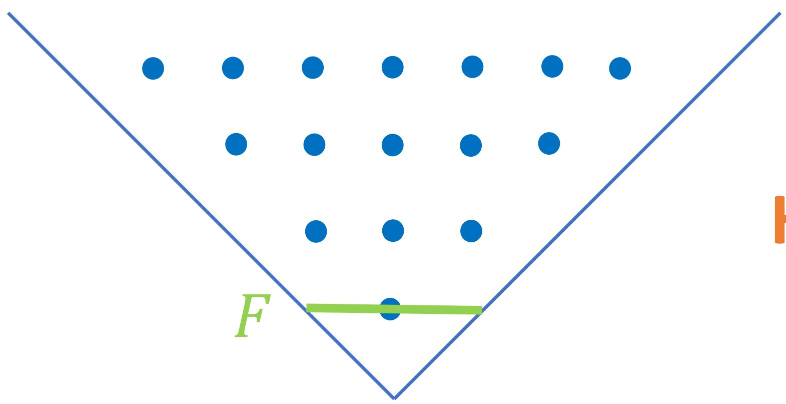

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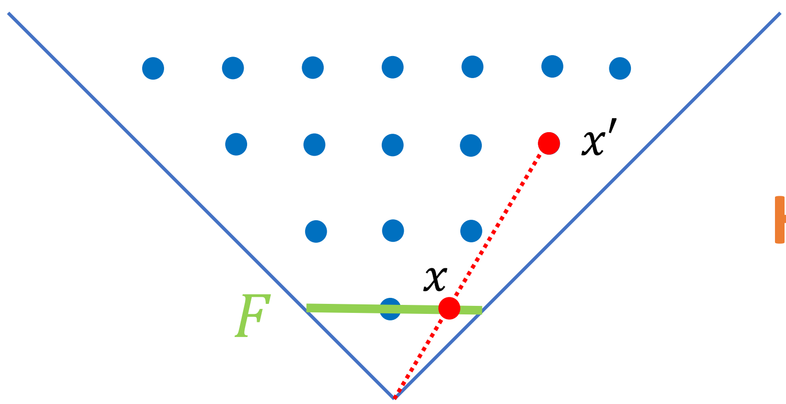

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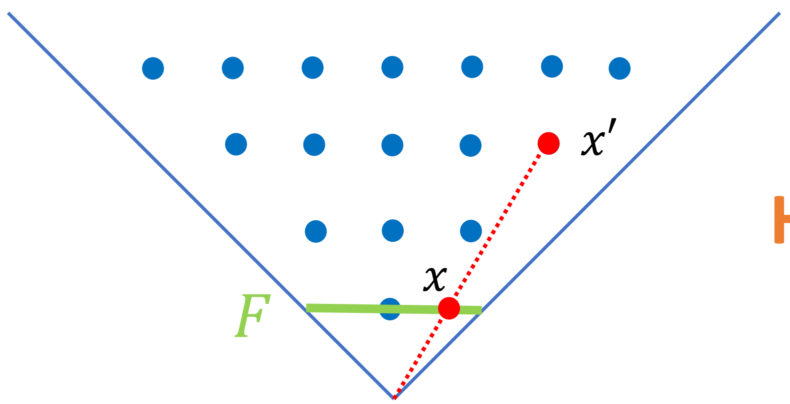

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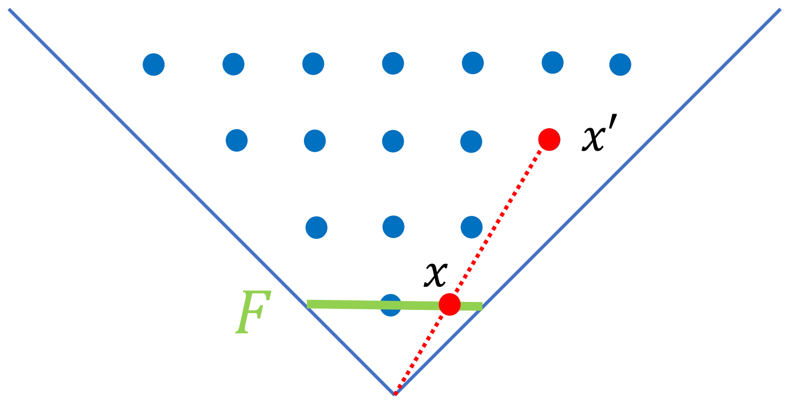
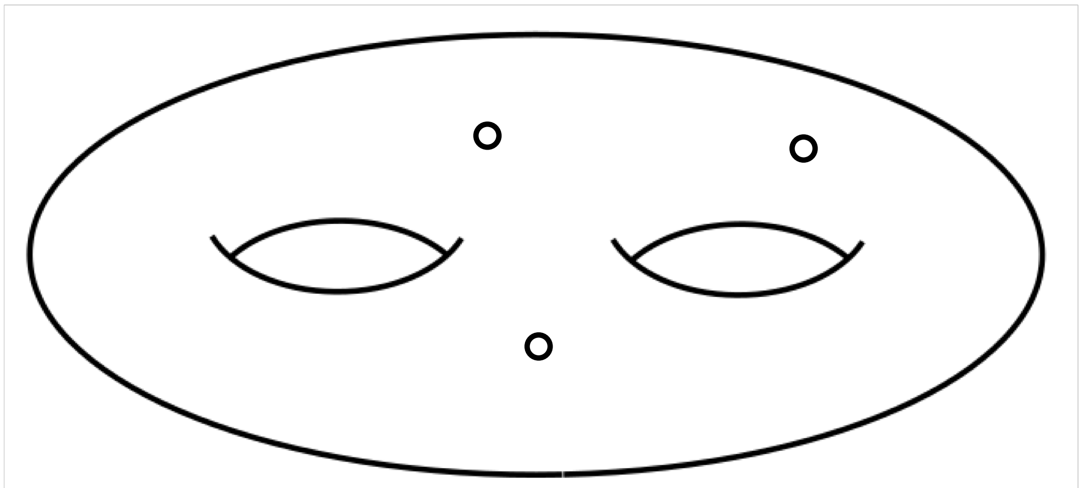


Application (McMullen):

$$\boldsymbol{\log(\lambda_{min,g}) \leq \frac{c}{g}}$$

Arc graph $\mathcal{A}(S)$



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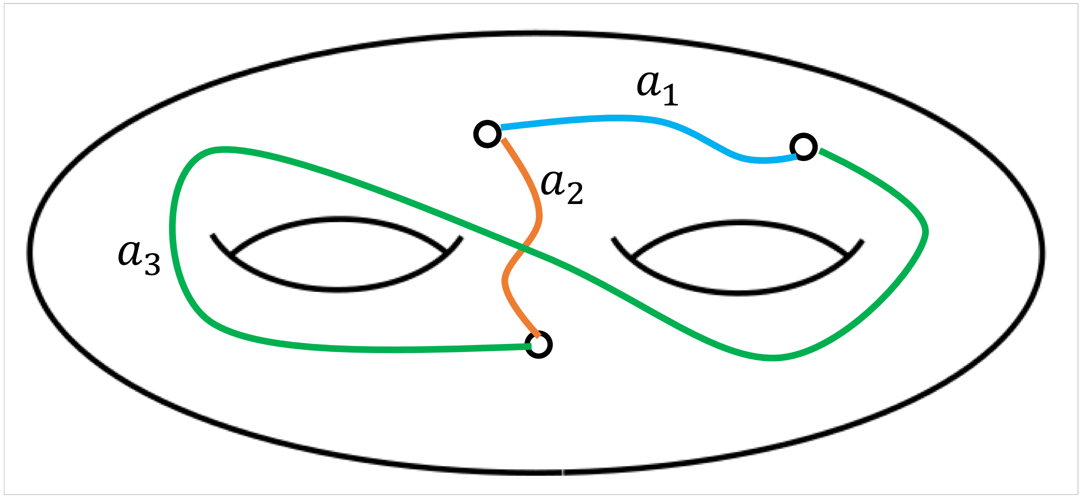

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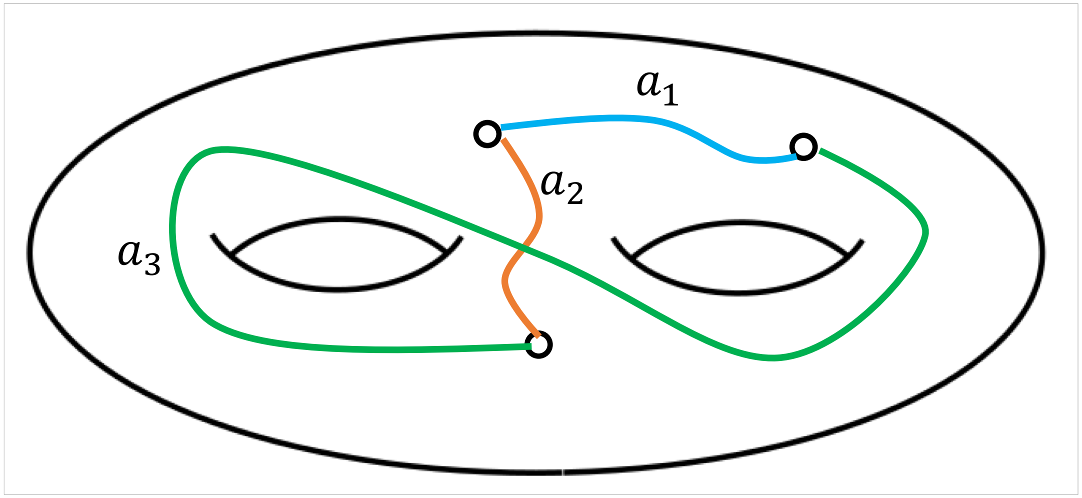

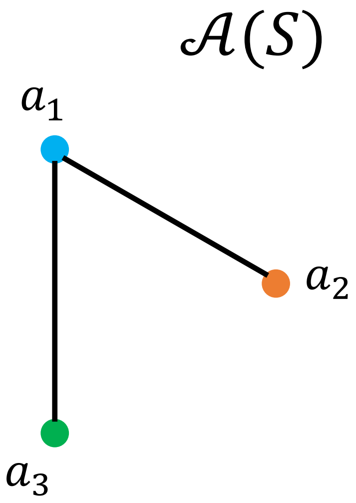

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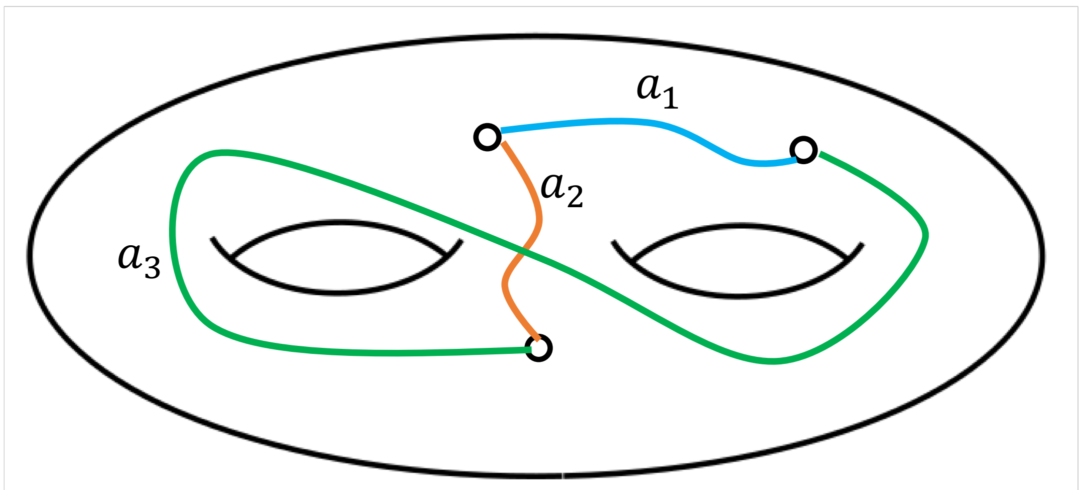

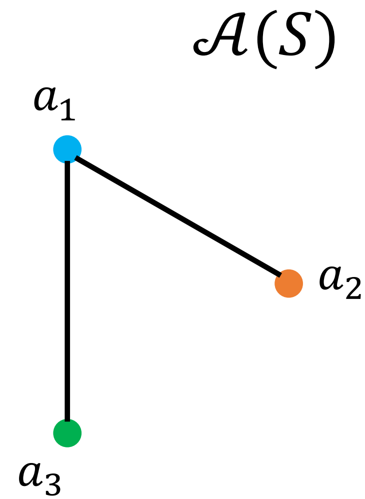

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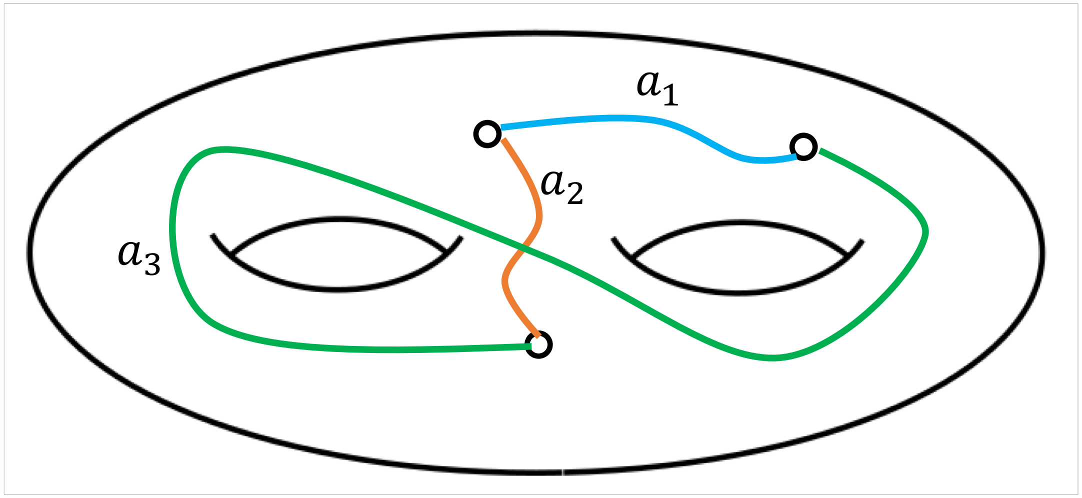

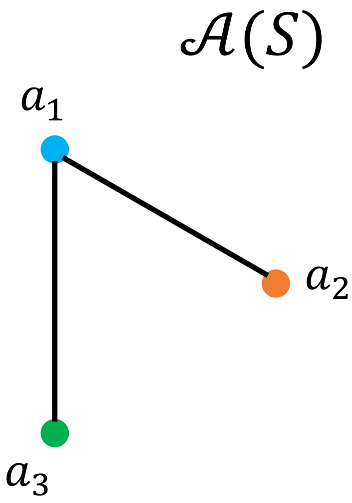

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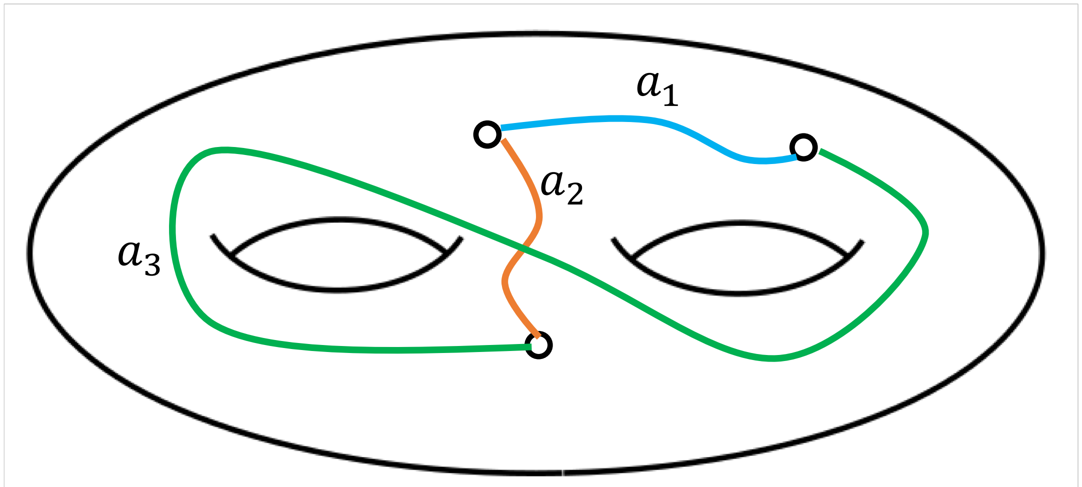

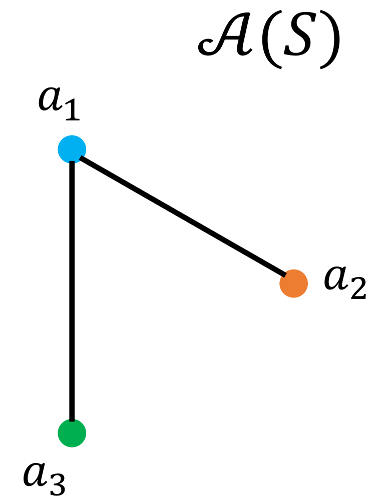

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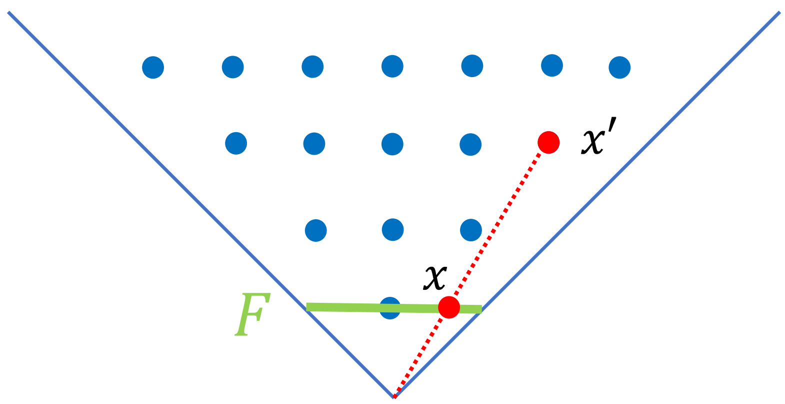

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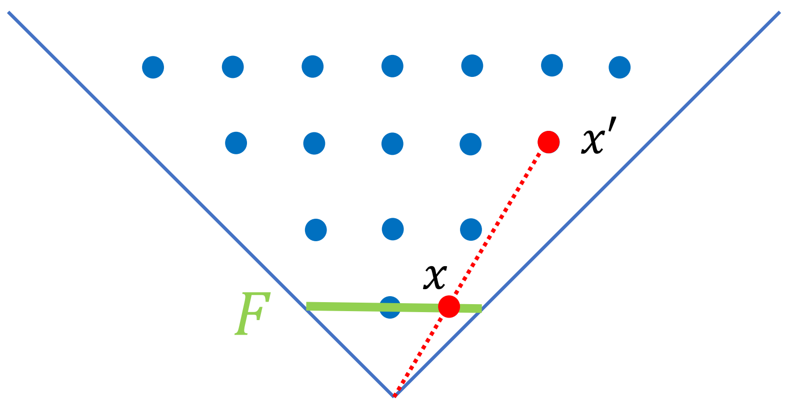

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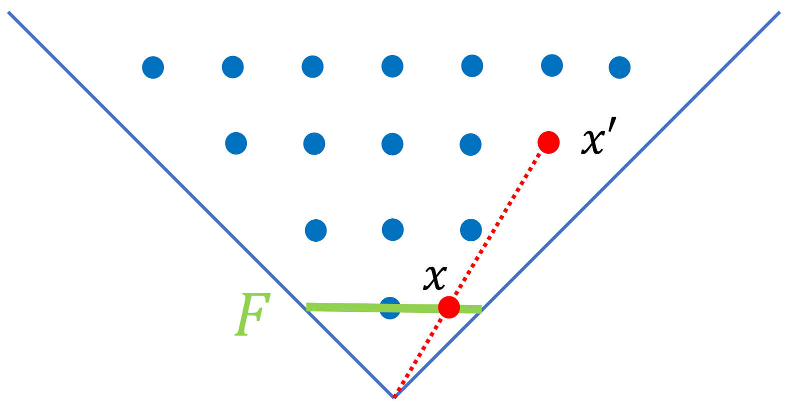

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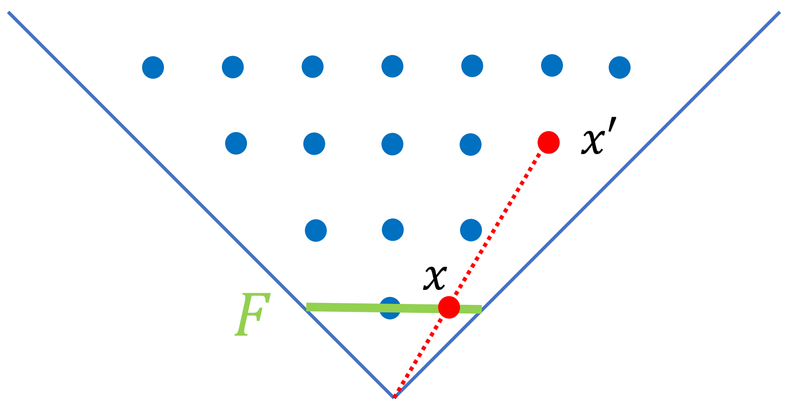

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The general theorem

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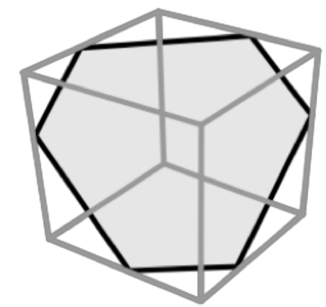

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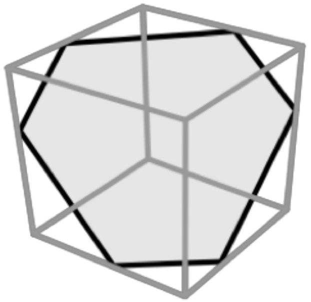

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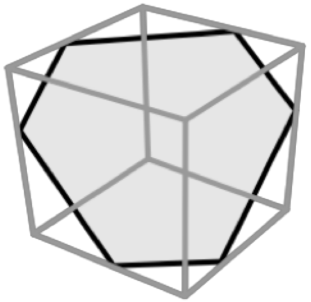

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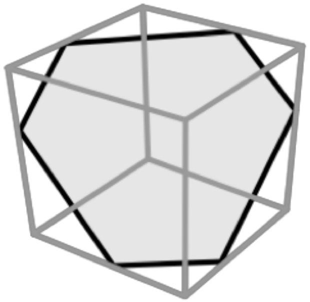

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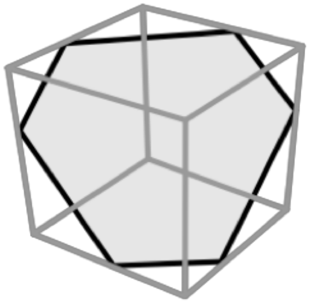

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Question: Is the bounding function g always convex?

Theorem 3 (S.): Let M, F, d, Ω and g as in the previous theorem.

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$$g^*(\alpha_1, \ldots, \alpha_{d+1}) = \sqrt[d]{\frac{vol(\Sigma/\Gamma)}{O_d \cdot d! \cdot vol(\Sigma/\langle \omega_1, \ldots, \omega_{d+1} \rangle_{\mathbb{Z}}) \cdot \prod_{i=1}^{d+1} \alpha_i}}$$

where O_d is a constant depending only on d.

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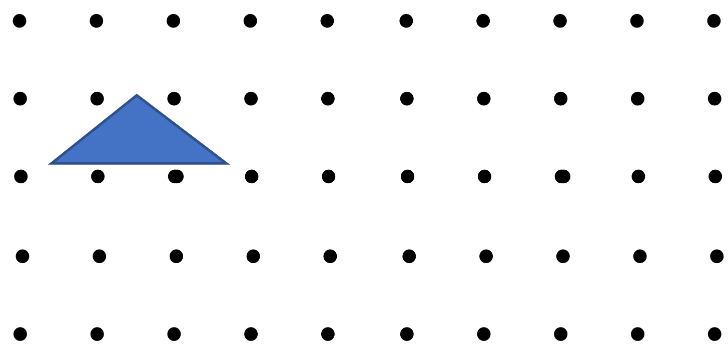

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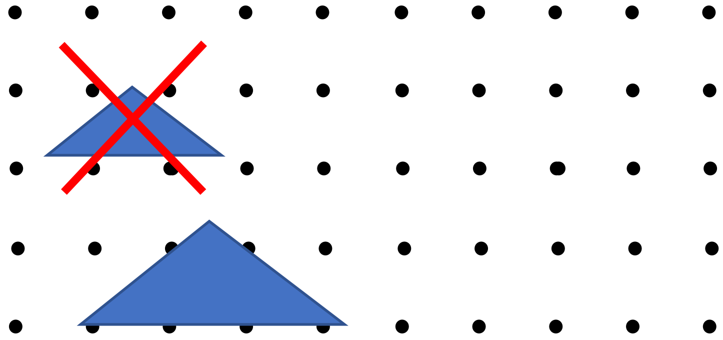

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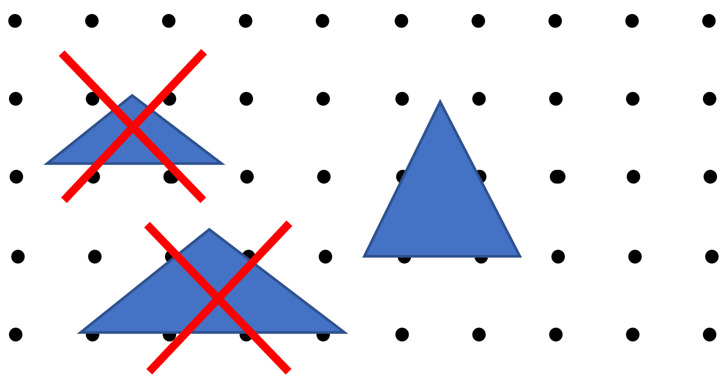

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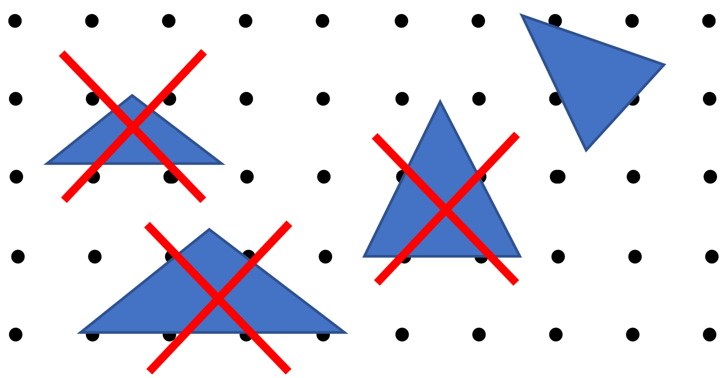

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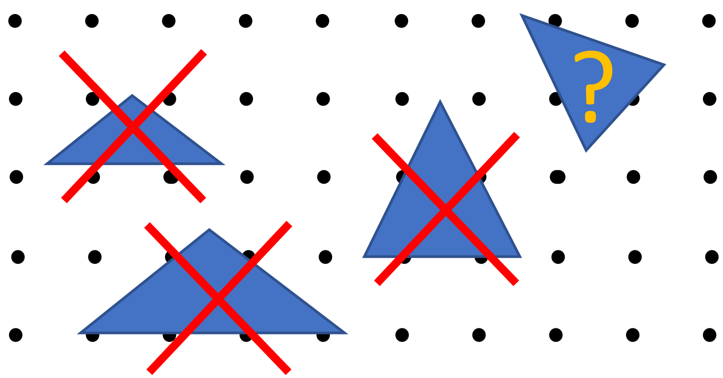

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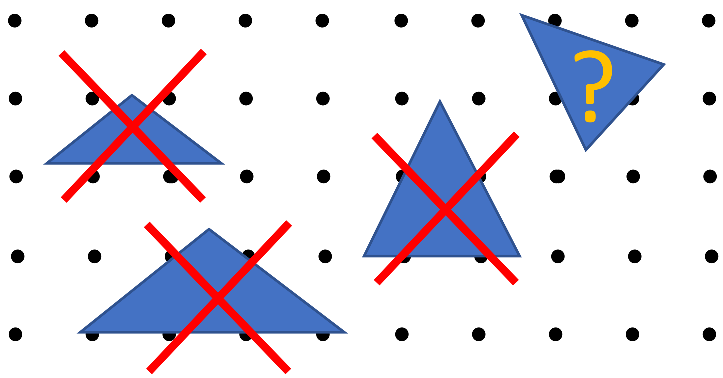

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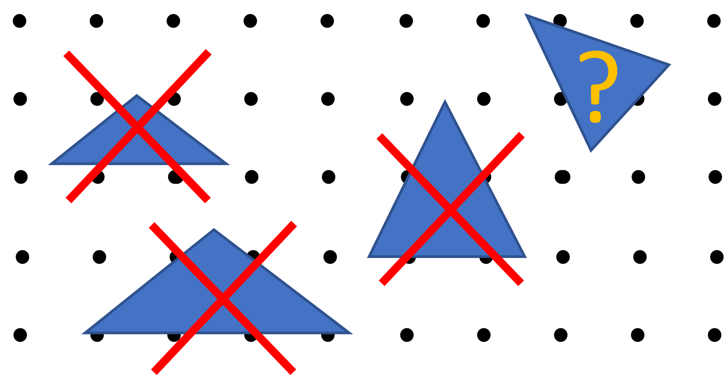

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Fact 2: $O_1 = 1$.

Proof of the the main theorem

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Let $\mathrm{C} = \langle b_1, \dots, b_k \rangle_{\mathbb{R}_+} \subset \mathbb{R}^n$ be a cone with nonempty interior. Let $\phi \colon \mathbb{Z}^n \to \mathbb{Z}$ be a surjective homomorphism.

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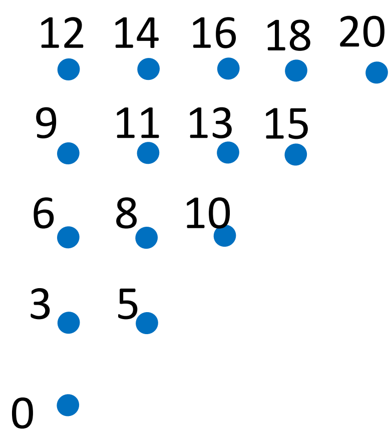

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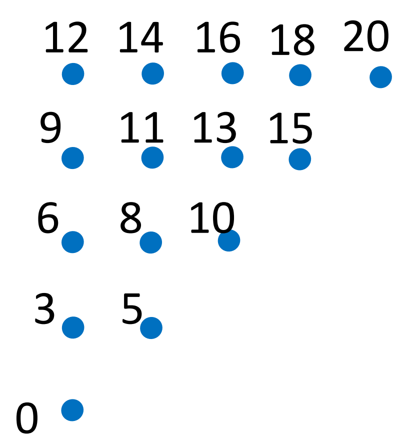

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2. Weights $\approx$ Frobenius numbers of a cone C in $\mathrm{H}_1(M;\mathbb{Z})$ with respect to $\mathrm{x}' \in H^1(M;\mathbb{Z})$.
3. Understand the way these Frobenius numbers change when $\mathrm{x}' \in H^1(M;\mathbb{Z})$ is varied.

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Questions

1. Is the bounding function g convex?
2. What is the value of O_d when $d \geq 2$?
3. Do our results generalize to the curve complex?
4. Is the fully-punctured hypothesis necessary?