

IIT CS536: Science of Programming

Homework 4: Proofs, WP and SP

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1 Substitution

Task 1.1 (Written, 8 points).

- a) $[y + x/x](2x + y \geq z) = 2(y + x) + y \geq z$
- b) $[z/x](x \geq 0 \rightarrow (\forall x. x * z > y) \wedge x > -1) = [z/x](x \geq 0 \rightarrow (\forall i. i * z > y) \wedge x > -1)$
 $= z \geq 0 \rightarrow (\forall i. i * z > y) \wedge z > -1$
- c) $[x/y]\forall x.(y > 0 \rightarrow \exists y. y = x) = [x/y]\forall i.(y > 0 \rightarrow \exists y. y = i)$
 $= [x/y]\forall i.(y > 0 \rightarrow \exists j. j = i)$
 $= \forall i.(x > 0 \rightarrow \exists j. j = i)$
- d) $[x + 2/y]\exists x.\forall y. x > y = [x + 2/y]\exists i.\forall y. i > y$
 $= [x + 2/y]\exists i.\forall j. i > j$
 $= \exists i.\forall j. i > j$

2 Proofs and Proof Outlines

Task 2.1 (Written, 10 points).

Proof outline for $\{x \neq y\}$ if $y > x$ then $\{t := x; x := y; y := t\}$ else $\{\text{skip}\} \{x > y\}$

$$\begin{array}{ll} \{x \neq y\} & \\ \text{if}(y > x) \{ & \{x \neq y \wedge y > x\} \Rightarrow \{y > x\} \\ \quad t := x; & \{y > x \wedge t = x\} \Rightarrow \{y > t\} \\ \quad x := y; & \{y > t \wedge x = y\} \Rightarrow \{x > t\} \\ \quad y := t & \{x > t \wedge y = t\} \Rightarrow \{x > y\} \\ \} \text{ else } \{ & \{x \neq y \wedge y \leq x\} \Rightarrow \{x > y\} \\ \quad \text{skip} & \{x > y\} \\ \} & \{x > y\} \end{array}$$

Task 2.2 (Written, 7 points).

1. $\{x = 0 \wedge 0 = 0\} s := \bar{0} \{x = 0 \wedge s = 0\}$ Assign
2. $\{T \wedge x = 0\} s := \bar{0} \{x = 0 \wedge s = 0\}$ Weaken 1
3. $\{x < 0 \wedge -1 = -1\} s := \overline{-1} \{x < 0 \wedge s = -1\}$ Assign
4. $\{x < 0 \wedge -1 = -1\} s := \overline{-1} \{s = \frac{x}{|x|}\}$ Weaken 3
5. $\{x \neq 0 \wedge x < 0\} s := \overline{-1} \{s = \frac{x}{|x|}\}$ Weaken 4
6. $\{x > 0 \wedge 1 = 1\} s := \bar{1} \{x > 0 \wedge s = 1\}$ Assign
7. $\{x \neq 0 \wedge x \geq 0\} s := \bar{1} \{x > 0 \wedge s = 1\}$ Weaken 6
8. $\{x \neq 0 \wedge x \geq 0\} s := \bar{1} \{s = \frac{x}{|x|}\}$ Weaken 7

$$9. \{x \neq 0\} \text{ if } x < 0 \text{ then } \{s := \overline{-1}\} \text{ else } \{s := \overline{1}\} \{s = \frac{x}{|x|}\} \quad \text{If 5, 8}$$

$$10. \{T \wedge x \neq 0\} \text{ if } x < 0 \text{ then } \{s := \overline{-1}\} \text{ else } \{s := \overline{1}\} \{s := \frac{x}{|x|}\} \quad \text{Weaken 9}$$

$$11. \{T\} \text{ if } x = \overline{0} \text{ then } \{s := \overline{0}\} \text{ else } \{\text{if } x < 0 \text{ then } \{s := \overline{-1}\} \text{ else } \{s := \overline{1}\}\} \{(x = 0 \wedge s = 0) \vee s = \frac{x}{|x|}\} \quad \text{If 2, 10}$$

3 Weakest Preconditions and Strongest Postconditions

Task 3.1 (Written, 15 points).

a) $wlp(x := x + y; n := x * z, n = 0)$

$$\begin{aligned} & wlp(x := x + y; n := x * z, n = 0) \\ &= wlp(x := x + y, wlp(n := x * z, n = 0)) \\ &= wlp(x := x + y, [x * z / n](n = 0)) \\ &= wlp(x := x + y, x * z = 0) \\ &= [x + y / x](x * z = 0) \\ &= ((x + y) * z = 0) \end{aligned}$$

b) $wp(\text{if } x = y \text{ then } \{z := \overline{1}\} \text{ else } \{z := x / y\}, z = 1)$

$$\begin{aligned} & wp(\text{if } x = y \text{ then } \{z := \overline{1}\} \text{ else } \{z := x / y\}, z = 1) \\ &= (x = y \rightarrow wlp(z := \overline{1}, z = 1)) \wedge (x \neq y \rightarrow wlp(z := x / y, z = 1)) \wedge D(\text{if } x = y \text{ then } \{z := \overline{1}\} \text{ else } \{z := \overline{x / y}\}) \\ &= (x = y \rightarrow [1 / z](z = 1)) \wedge (x \neq y \rightarrow [(x / y) / z](z = 1)) \wedge D(x = y) \wedge (x = y \rightarrow D(z := \overline{1})) \wedge (x \neq y \rightarrow D(z := x / y)) \\ &= (x = y \rightarrow 1 = 1) \wedge (x \neq y \rightarrow x / y = 1) \wedge D(x) \wedge D(y) \wedge (x = y \rightarrow D(\overline{1})) \wedge (x \neq y \rightarrow D(x / y)) \\ &= (x = y \rightarrow T) \wedge (x \neq y \rightarrow x / y = 1) \wedge T \wedge T \wedge (x = y \rightarrow T) \wedge (x \neq y \rightarrow D(x) \wedge D(y) \wedge y \neq 0) \\ &= (x = y \rightarrow T) \wedge (x \neq y \rightarrow x / y = 1) \wedge (x \neq y \rightarrow T \wedge T \wedge y \neq 0) \\ &= (x = y \rightarrow T) \wedge (x \neq y \rightarrow x / y = 1) \wedge (x \neq y \rightarrow y \neq 0) \end{aligned}$$

c) $sp(x = 1, \text{if } y > 0 \text{ then } \{x := x + 1\} \text{ else } \{\text{skip}\})$

$$\begin{aligned} & sp(x = 1, \text{if } y > 0 \text{ then } \{x := x + 1\} \text{ else } \{\text{skip}\}) \\ &= (y > 0 \rightarrow sp(x = 1, x := x + 1)) \vee (y \leq 0 \rightarrow sp(x = 1, \text{skip})) \\ &= (y > 0 \rightarrow [x_0 / x](x = 1) \wedge x = [x_0 / x](x + 1)) \vee (y \leq 0 \rightarrow x = 1) \\ &= (y > 0 \rightarrow x_0 = 1 \wedge x = x_0 + 1) \vee (y \leq 0 \rightarrow x = 1) \\ &= (y > 0 \rightarrow x = 2) \vee (y \leq 0 \rightarrow x = 1) \end{aligned}$$

d) $sp(x \geq 0, x := 1; \text{if } x > 0 \text{ then } \{x := x - 1\} \text{ else } \{x := 0\})$

$$\begin{aligned} & sp(x \geq 0, x := 1; \text{if } x > 0 \text{ then } \{x := x - 1\} \text{ else } \{x := 0\}) \\ &= sp(sp(x \geq 0, x := 1), \text{if } x > 0 \text{ then } \{x := x - 1\} \text{ else } \{x := 0\}) \\ &= sp(x_0 \geq 0 \wedge x = 1, \text{if } x > 0 \text{ then } \{x := x - 1\} \text{ else } \{x := 0\}) \\ &= sp(x = 1, \text{if } x > 0 \text{ then } \{x := x - 1\} \text{ else } \{x := 0\}) \\ &= (x > 0 \rightarrow sp(x = 1, x := x - 1)) \vee (x \leq 0 \rightarrow sp(x = 1, x := 0)) \\ &= (x > 0 \rightarrow x_0 = 1 \wedge x = x_0 - 1) \vee (x \leq 0 \rightarrow x_0 = 1 \wedge x = 0) \\ &= (x > 0 \rightarrow x = 0) \vee (x \leq 0 \rightarrow x = 0) \end{aligned}$$

4 One more wrap-up question

Task 4.1 (Written, 0 points).

I spent about 5 hours on this homework, in total 1-2 hours of actual working time.