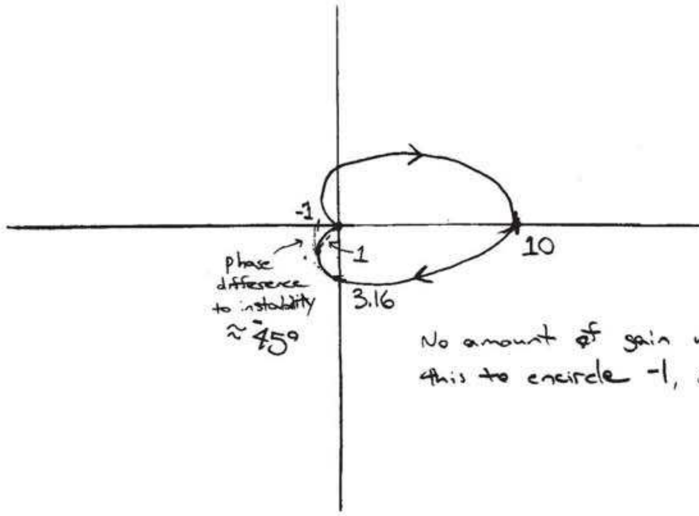
$$G_1(s) = \frac{100}{(s+1)(s+10)} = \left(\frac{100}{(1)(10)}\right)\frac{1}{(s+1)(\frac{s}{10}+1)}$$

1. a. i. Bode attached

Phase margin $\approx 45^\circ$

Gain margin $\approx \infty$

ii.



No amount of gain will cause this to encircle -1, so Gain Margin = ∞

iii. Stability: $Z = P - N = 0 - 0 = 0$

So no unstable closed loop poles

System is stable

Phase margin $\approx -45^\circ$

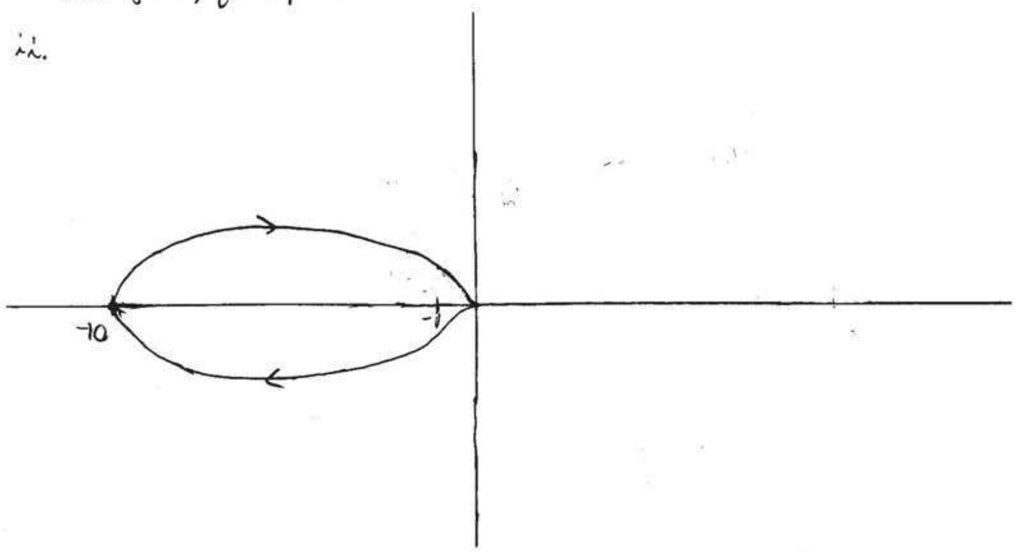
Gain margin $\approx \infty$

$$G_2(s) = \frac{100}{(s+1)(s-10)} = \left(\frac{100}{(1)(10)}\right)\left(\frac{1}{(s+1)(\frac{s}{10}-1)}\right)$$

b. i. Bode attached

Phase margin and gain margin do not exist because the system is always unstable as shown through Nyquist plot.

ii.



iii. Stability: $Z = P - N = 1 - (-1) = 2$ ← clock wise encircle ment makes this negative

So 2 unstable closed loop poles

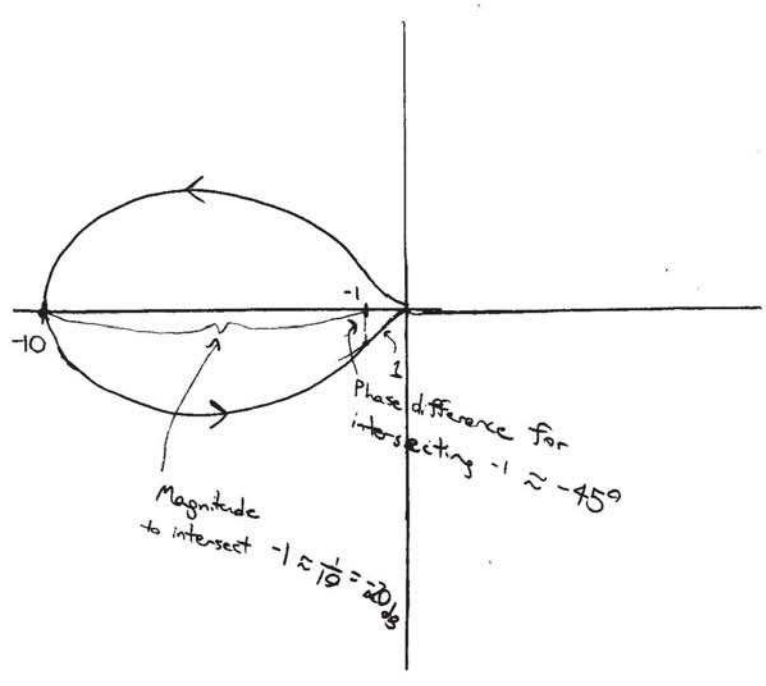
Closed loop system is unstable

Even if the gain is adjusted so that there are no encirclements of -1, $Z = P - N = 1 - 0 = 1$, so there will still be 1 unstable closed loop pole.

C. i. Bode attached $G_2(s) = \frac{100}{(s-1)(s+10)} = \frac{100}{(1)(10)}\left(\frac{1}{(s-1)(\frac{s}{10}+1)}\right)$

Cannot determine stability from bode plot because of the RHP pole.

ii.



iii. Stability: $Z = P - N = 1 - 1 = 0$

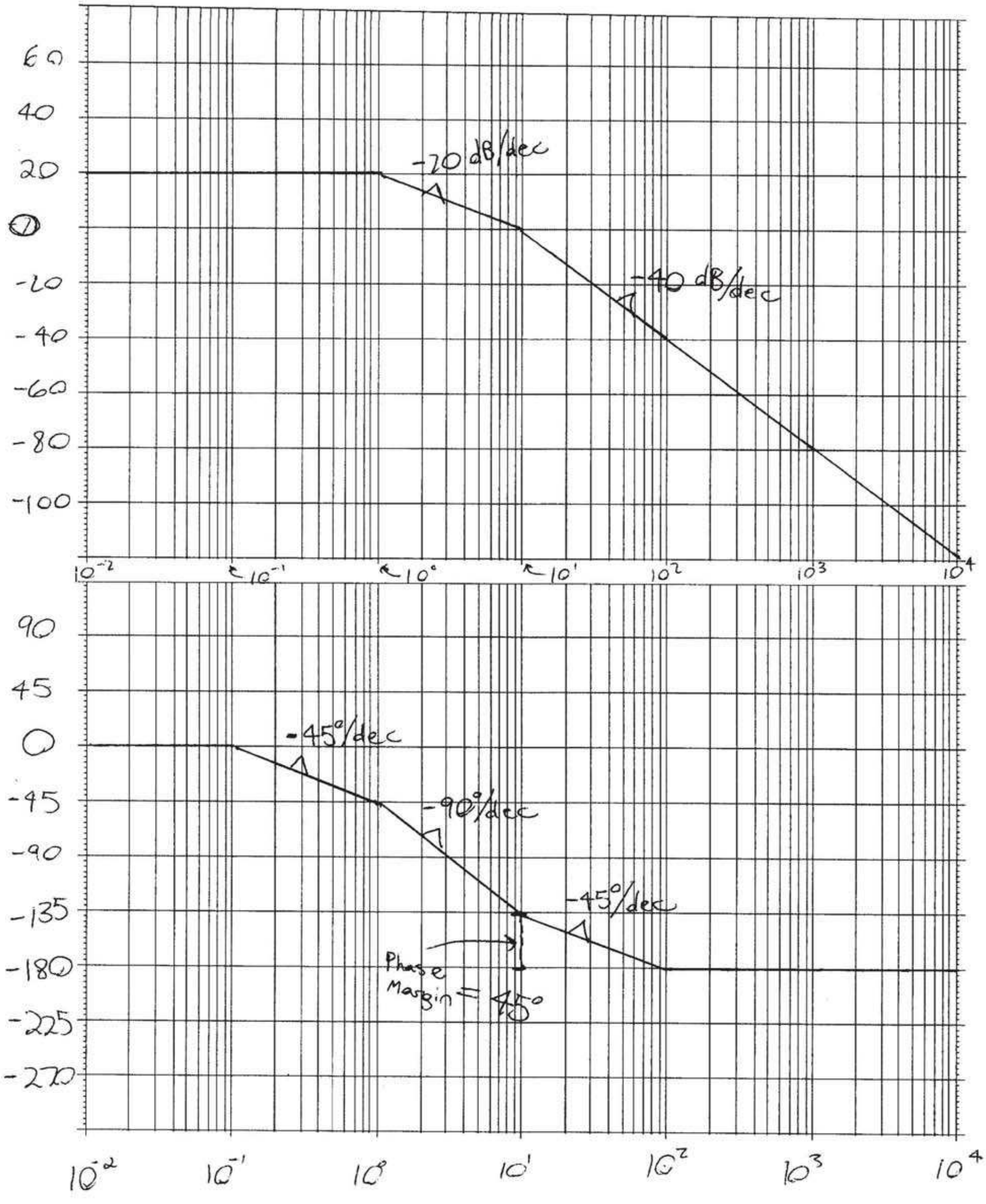
So no unstable closed loop poles

System is stable

Phase margin $\approx -45^\circ$

Gain margin > -20 dB since for all gains greater than -20 dB, there will always be one encirclement of -1, and the system will be stable.

a.

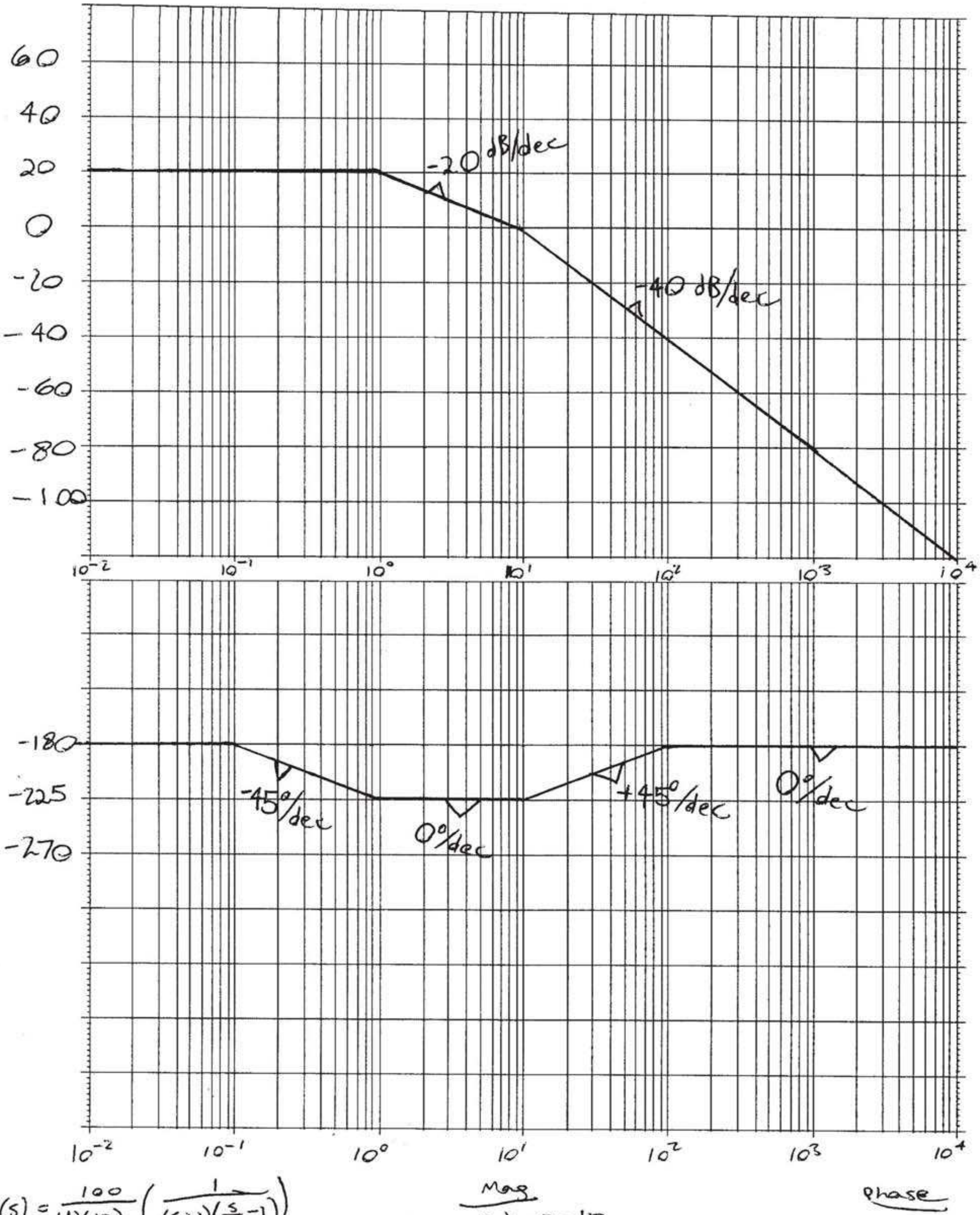


$G(s) = \frac{100}{(1)(10)}\left(\frac{1}{(s+1)(\frac{s}{10}+1)}\right)$ DC: Mag $20\log(10) = 20$ dB Phase $0°$

$\frac{1}{s+1}$: -20 dB/dec from $\omega = 1$ $-45°$/dec from $\omega = 0.1$ to 10

$\frac{1}{\frac{s}{10}+1}$: -20 dB/dec from $\omega = 10$ $-45°$/dec from $\omega = 1$ to 100

b.



$$G(s) = \frac{100}{(1)(10)}\left(\frac{1}{(s+1)\left(\frac{s}{10}-1\right)}\right)$$

Mag

DC: $20\log(10) = 20\,dB$

$\frac{1}{s+1}$: -20 dB/dec from $\omega = 1$

$\frac{1}{\frac{s}{10}-1}$: -20 dB/dec from $\omega = 10$

Phase

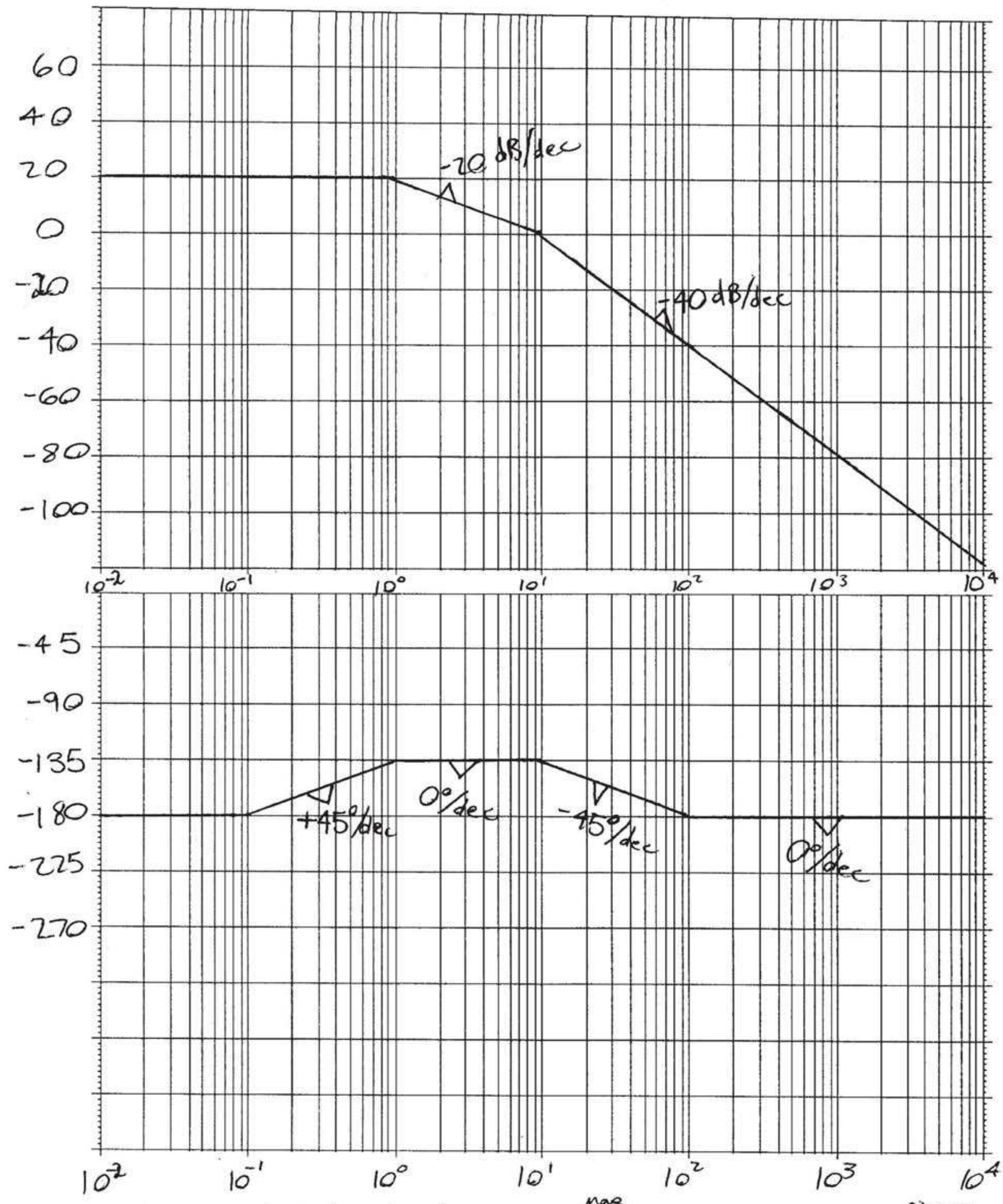
0°

-45°/dec from $\omega = 0.1$ to 10

-180° starting phase

$+45^\circ$/dec from $\omega = 1$ to 100

c.



$$G(s) = \frac{100}{(s-1)(s+10)} = \frac{100}{(1)(100)}\left(\frac{1}{(s-1)(\frac{s}{10}+1)}\right)$$

Mag

DC: $20\log(10) = 20\,\text{dB}$

$\frac{1}{s-1}$: -20 dB/dec from $\omega=1$

$\frac{1}{\frac{s}{10}+1}$: -20 dB/dec from $\omega=10$

Phase

0°

$-180°$ starting phase

$+45°$/dec from $\omega=0.1$ to 10

$-45°$/dec from $\omega=1$ to 100

Prob 2 starts

(2) <u>Car 1 equation:</u>

$$\dot{x}_1 = x_2 \;,\; \dot{x}_2 = 2u_1 \;,\; y_1 = x_1$$

<u>Car 2 equation:</u>

$$\dot{x}_3 = x_4 \;,\; \dot{x}_4 = 0.5u_2 \;;\; y_2 = x_3 - x_1$$

(a) <u>State Space Equations:</u>

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(b) Using LQR we get,

$$K_b = \begin{bmatrix} 1.58 & 3.4 & 0 & 0 \\ 0 & 0 & 1.58 & 4.04 \end{bmatrix}$$

- time 't' by which both $|x_1(t)| \leq 1$ and $|x_3(t)| \leq 1$ are within 1m :

$\boxed{t = 9.28}$ {using matlab}.

- $x_2(10) = \theta_1(10) = 0.344$

  $x_4(10) = \theta_2(10) = 0.34$.

- At a particular time; x_3 crosses x_4 which is physically impossible.

c) $Q = diag([2.5 ; 10, 25, 100])$

d) b) $J = 1.0566 \times 10^5$

c) $J = 5.8083 \times 10^5$

e) From the plots it is clear that the control effort required in the second case (c part) is higher.

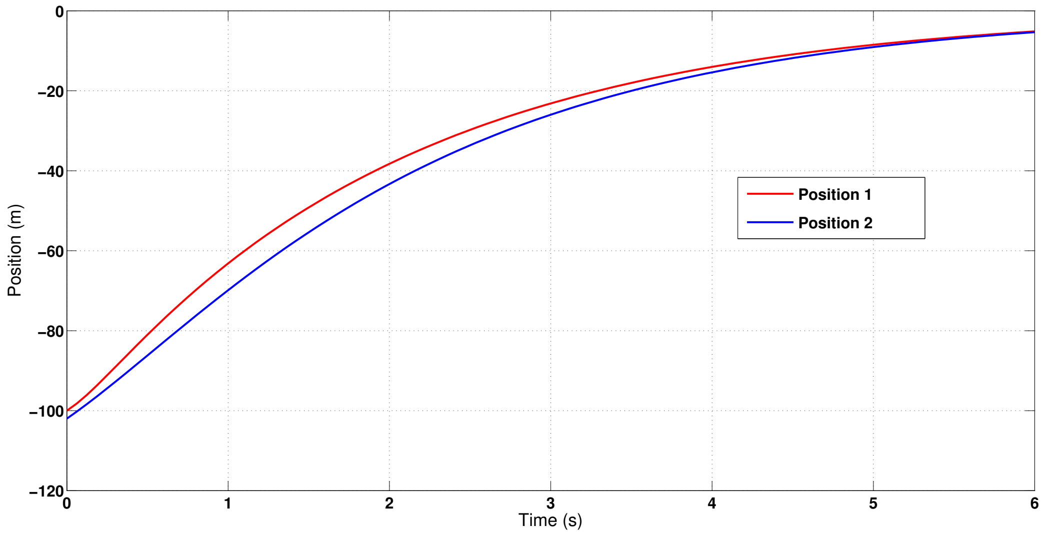


Figure 1: Position of cart 1 and cart 2 when $Q = \mathbf{diag}([2.5\ 10\ 2.5\ 10])$

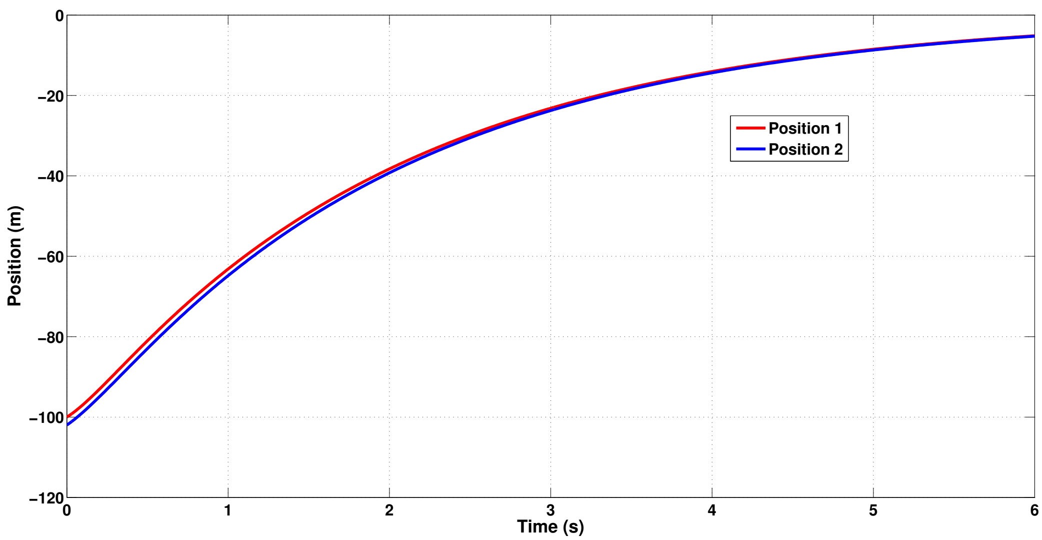


Figure 2: Position of cart 1 and cart 2 when $Q = \mathbf{diag}([2.5\ 10\ 25\ 100])$

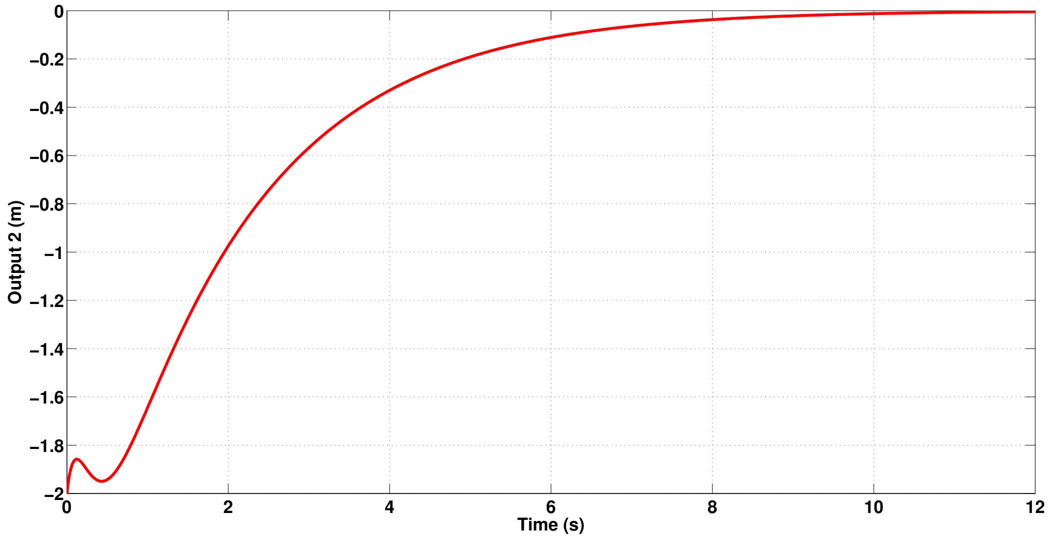


Figure 3: $y_2(t) = x_3(t) - x_1(t)$ when $Q = \mathbf{diag}([2.5\ 10\ 25\ 100])$

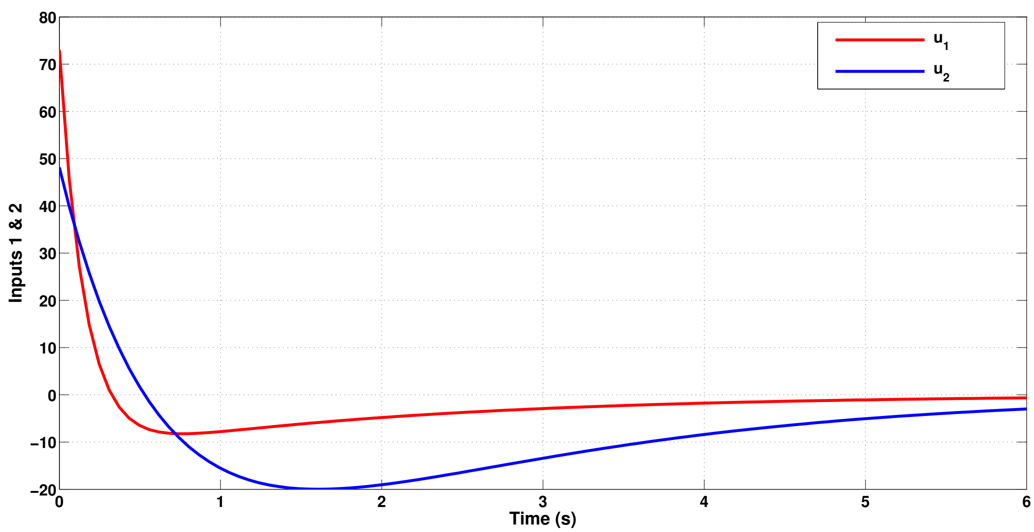


Figure 4: u_1 and u_2 when $Q = \mathbf{diag}([2.5\ 10\ 2.5\ 10])$

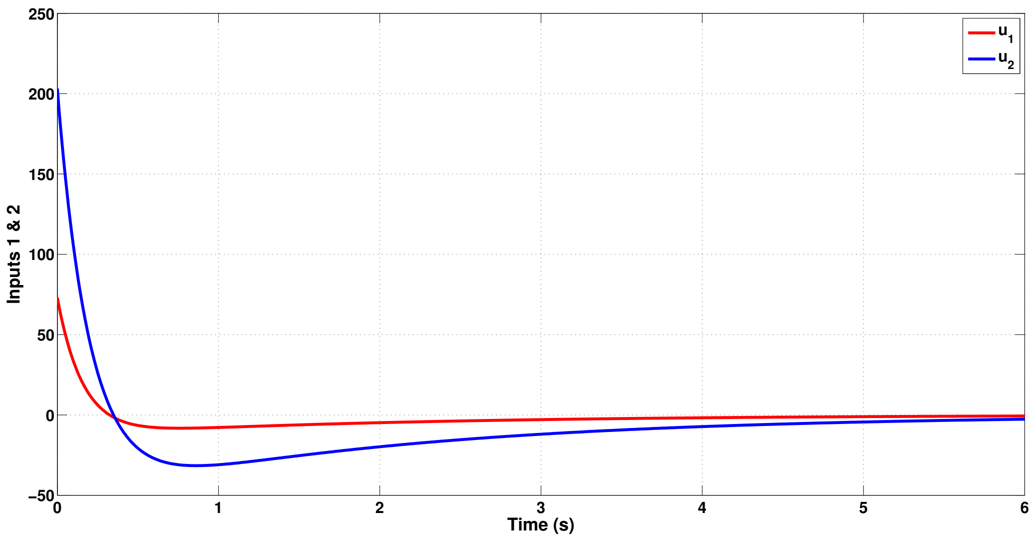


Figure 5: u_1 and u_2 when $Q = \mathbf{diag}([2.5\ 10\ 25\ 100])$

3. (35 pts) Discrete Time Control (Handout)

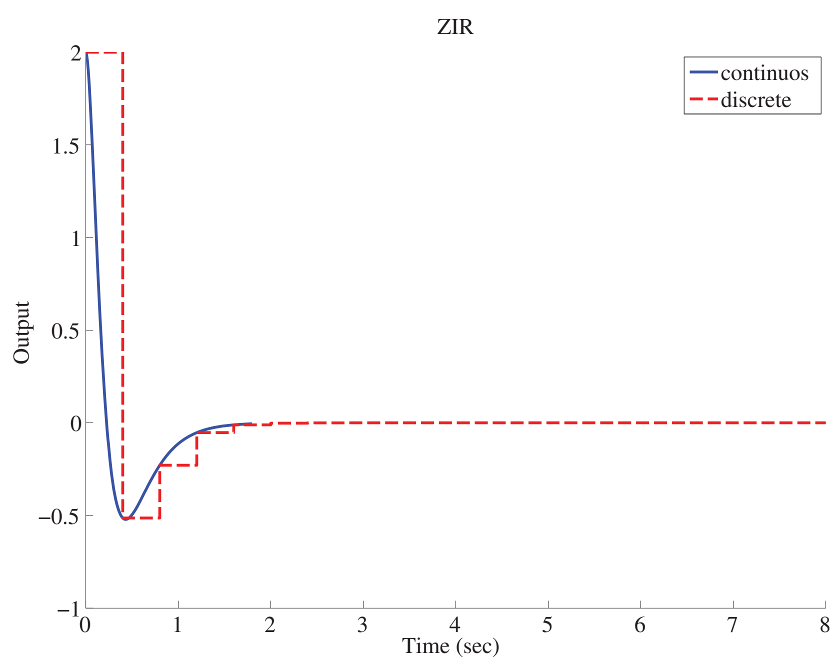
Given the following continuous time (CT) system

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -320 & -152 & -22 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \quad y = [2\ 1\ 0]\ \mathbf{x}$$

the corresponding discrete time (DT) system is

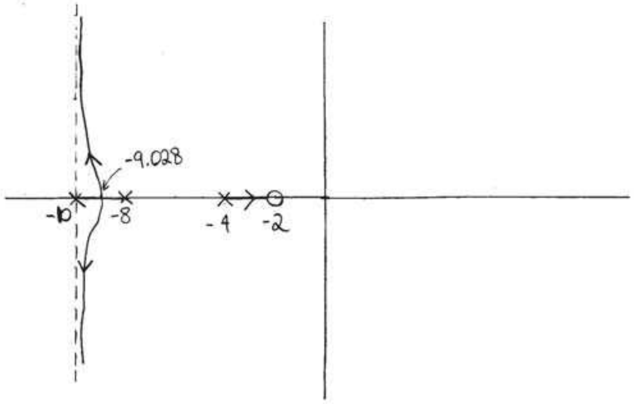
$$\begin{aligned} x[n+1] &= Gx[n] + Hu[n] \\ &= \begin{bmatrix} 0.518 & 0.0984 & 0.004843 \\ -1.55 & -0.2182 & -0.00815 \\ 2.608 & -0.311 & -0.03887 \end{bmatrix} \mathbf{x}[\mathbf{n}] + \begin{bmatrix} 0.001506 \\ 0.004843 \\ -0.00815 \end{bmatrix} u[n] \\ y[n] &= [2\ 1\ 0]\ \mathbf{x}[\mathbf{n}] \end{aligned}$$

a) With initial condition $x_0 = [1\ \ 0\ \ 0]'$, plot the ZIR using Matlab function `initial()` for the CT system and the DT system (with $T = 0.4$ sec).



b) For output feedback control $u = k(r - y)$, sketch the root locus for the equivalent transfer function for the continuous time (CT) system.
Transfer function:

$$G_(s) = \frac{k(s+2)}{(s+4)(s+8)(s+10)}$$



poles: $-4, -8, -10$ zeros: -2

real axis segments: $[-2, -4]$; $[-8, -10]$

asymptotes: $\sigma = \dfrac{\Sigma p - \Sigma z}{p - z} = \dfrac{(-4 + -8 + -10) - (-2)}{3 - 1} = \dfrac{-20}{2} = -10$

$\theta = \dfrac{\pi(2k+1)}{p - z} = \dfrac{\pi(2k+1)}{2} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

break-away points: $N(s)D'(s) - N'(s)D(s)$

$= (s+2)(3s^2 + 44s + 152) - (1)(s^3 + 22s^2 + 152s + 320)$

$= 0$

$s = -9.028$

c) Determine the closed loop pole locations for the CT system for $k = 20$ and plot the closed-loop step response using Matlab.

Closed loop pole locations: $-9.3828 + j4.8237, -9.3828 - j4.8237, -3.2343$
This is from the closed loop transfer function from k=20:

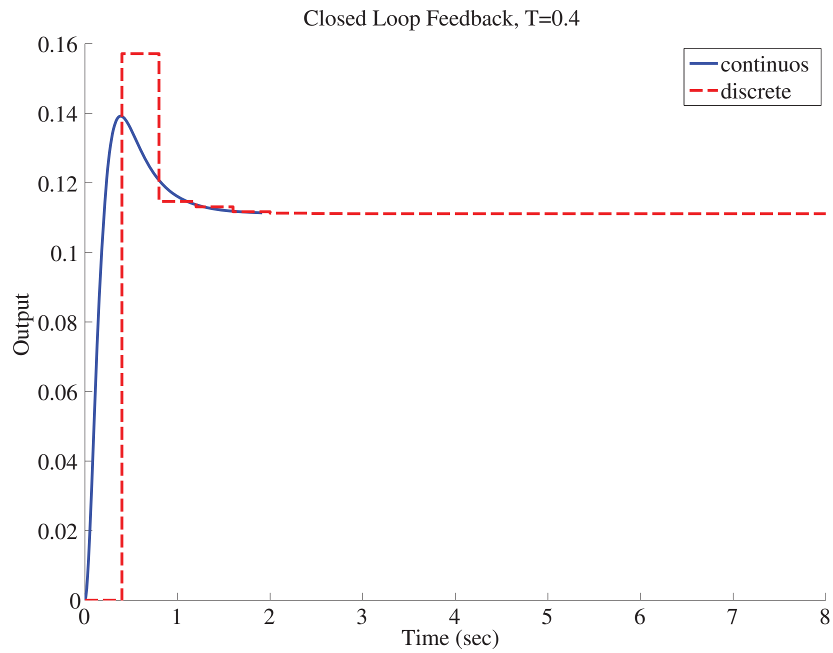
$$G_{CL}(s) = \frac{k(s+2)}{(s+4)(s+8)(s+10) + k(s+2)}$$

d) The closed loop DT system has state equation
$x[n+1] = (G - kHC)x[n] + kHr[n]$, $y[n] = Cx[n]$
(which can be found using the Matlab `feedback` function). Using Matlab, determine the closed loop pole locations for the DT system for $k = 20$ and sampling period $T = 0.4$ sec and plot the step response.

The closed loop discrete time system equations are:

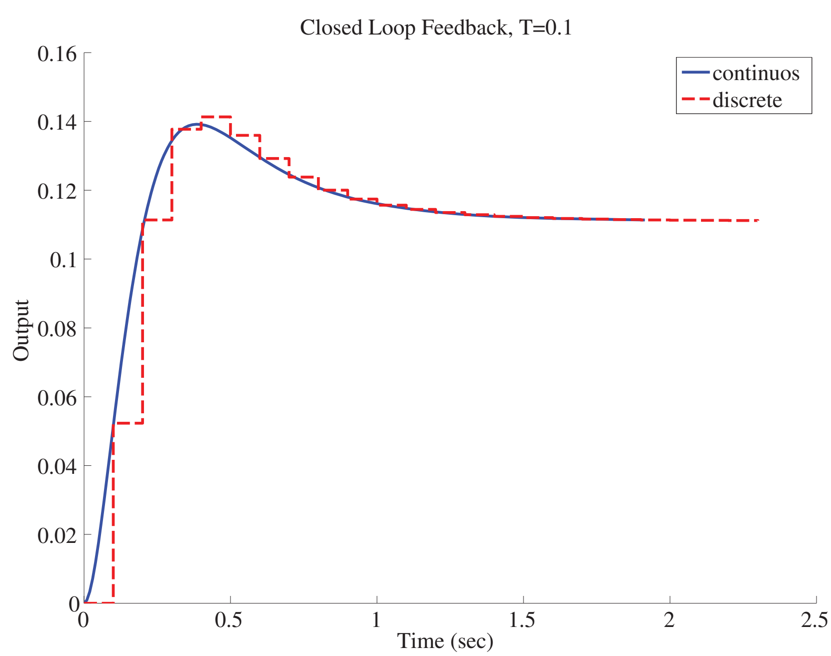
$$\begin{aligned} x[n+1] &= (G - kHC)x[n] + kHr[n] \\ &= \begin{bmatrix} 0.4578 & 0.06828 & 0.004843 \\ -1.744 & -0.315 & -0.00815 \\ 2.934 & -0.148 & -0.03887 \end{bmatrix} \mathbf{x[n]} + \begin{bmatrix} 0.03012 \\ 0.09687 \\ -0.163 \end{bmatrix} u[n] \\ y[n] &= [2\ 1\ 0]\ \mathbf{x[n]} \end{aligned}$$

Closed loop poles: $0.3043, -0.1002 + j0.0694, -0.1002 - j0.0694$



e) Use Matlab (iteratively if necessary) to find a sampling period T which gives a closed-loop step response that is "reasonably close" to the CT closed-loop step response. Determine closed-loop pole locations, and plot the DT step response.

The overshoot in the discrete time case is currently about 0.02 higher than the overshoot in the continuous time case. To remedy this, a shorter time step, $T = 0.1$ was chosen. The results are in the figure below. The closed loop poles are:
Closed loop poles: $0.729, 0.35 + j0.24, 0.35 - j0.24$



f) Briefly explain why the CT and DT ZIR responses from a) above are reasonably close, but the closed loop responses from c) and d) (with $T = 0.4$ sec) do not agree at all. (Hint, consider e^{AT}.)

The ZIR responses are close because in the discrete case $x[k+1] = e^{AT}$, while in the continuous case $x(t) = e^{At}$. So, the discrete and continuous time ZIR responses are exactly the same at each time step T.

On the other hand, the discrete time closed loop system matrix is $G_k = (G - HK)$, and the continuous time closed loop system matrix is $(A - BK)$. Where the continuous closed loop response will be $x(t) = e^{(A-BK)t}$, the discrete time closed loop response $x[k+1] = (G - HK)x[k]$ is not equal to $e^{(A-BK)T}$, so the two responses will be different.

Prob 4 starts...

4) a) $F(z) = \dfrac{(z+4)(z+1)}{(z-0.3)(z-0.6)}$

$$= \frac{z^2+5z+4}{(z^2-0.9z+0.18)}.$$

$$\Rightarrow \frac{F(z)}{z} = \frac{z^2+5z+4}{z(z^2-0.9z+0.18)}$$

$$= \frac{40.89}{z-0.6} - \frac{62.11}{z-0.3} + \frac{22.22}{z}$$

$$\Rightarrow F(z) = \frac{40.89\,z}{(z-0.6)} - \frac{62.11\,z}{(z-0.3)} + 22.22$$

$$\Rightarrow f(kT) = 40.89(0.6)^k - 62.11(0.3)^k + 22.22\,\delta(k).$$

b) $F(z) = \dfrac{(z+0.2)(z+1)}{z(z-0.1)(z-0.2)(z-0.3)}.$

$$\Rightarrow \frac{F(z)}{z} = \frac{(z+0.2)(z+1)}{z^2(z-0.1)(z-0.2)(z-0.3)}$$

$$= \frac{361.1}{(z-0.3)} - \frac{1200}{(z-0.2)} + \frac{1650}{(z-0.1)}$$

-811.11/z - 33.3/z^2

$$F(z) = \frac{361.1z}{(z-0.3)} - \frac{1200z}{(z-0.2)} + \frac{1650z}{(z-0.1)} - 844.4$$

$$\Rightarrow \boxed{f(kT) = 361.1(0.3)^k - 1200(0.2)^k}$$

+ 1650 0.1^k - 811.1 (u(k) - u(k-1)) - 33.3(u(k-1) - u(k-2))