EECS 117 Homework Assignment 6 — Spring 2004

5-16. Diffused IC resistor. (a) The sheet resistance of the diffused p-type IC resistor with differential thickness dx and conductivity $\sigma(x)$ can be written as

$$dR_{\text{sq}} = \frac{1}{\sigma(x)dx}$$

Noting that these differential resistances are connected in parallel, it is more convenient to find the total sheet resistance using differential sheet conductance given by

$$dG_{\text{sq}} = \frac{1}{dR_{\text{sq}}} = \sigma(x)dx$$

The total sheet conductance can be found by adding (i.e., integrating) the sheet conductances of the differential resistors as

$$G_{\text{sq}} = \int_0^t \sigma(x)dx$$

Therefore, the total sheet resistance follows as

$$R_{\text{sq}} = \frac{1}{G_{\text{sq}}} = \left[\int_0^t \sigma(x)dx\right]^{-1}$$

(b) Since the conductivity of the p-type layer decreases linearly from $\sigma(x = 0) = \sigma_0$ and $\sigma(x = t) = \sigma_1 \ll \sigma_0$, the conductivity expression can be written as

$$\sigma(x) = \left(\frac{\sigma_1 - \sigma_0}{t}\right)x + \sigma_0 = Kx + \sigma_0$$

where $K = (\sigma_1 - \sigma_0)/t$. The sheet resistance can be found by integrating this expression as

$$\begin{aligned}
R_{\text{sq}} &= \left[\int_0^t (Kx + \sigma_0)dx\right]^{-1} \\
&= \left[\left(\frac{Kx^2}{2} + \sigma_0 x\right)\Bigg|_0^t\right]^{-1} \\
&= \left[\frac{\sigma_1 - \sigma_0}{2t}t^2 + \sigma_0 t\right]^{-1} \\
&= \frac{2}{(\sigma_0 + \sigma_1)t} \simeq \frac{2}{\sigma_0 t}
\end{aligned}$$

since $\sigma_1 \ll \sigma_0$.

(c) Assuming the conductivity expression to be $\sigma(x) = \sigma_0 e^{-ax}$ and noting that $\sigma(t) = \sigma_0 e^{-at} = \sigma_1 \ll \sigma_0$, the sheet resistance for this case can be evaluated as

$$
\begin{aligned}
R_{\text{sq}} &= \left[\int_0^t \sigma_0 e^{-ax} dx\right]^{-1} \\
&= \left[\left(-\frac{\sigma_0 e^{-ax}}{a}\right)\Bigg|_0^t\right]^{-1} \\
&= \left[\frac{-\sigma_0 e^{-at} + \sigma_0}{a}\right]^{-1} = \frac{a}{\sigma_0 - \sigma_1} \simeq \frac{a}{\sigma_0}
\end{aligned}
$$

5-20. Resistance of a semicircular ring. (a) Using the result of Example 5-3, the resistance of the semicircular ring shown in Figure 5.20 can be written as

$$R = \frac{\pi a}{\sigma t a \ln(b/a)} = \frac{\pi}{\sigma t \ln(b/a)}$$

(b) For the straight conductor with length $l = \pi(a+b)/2$ and cross-sectional area $A = (b-a)t$, resistance is given by

$$R = \frac{l}{\sigma A} = \frac{\pi(a+b)}{2\sigma(b-a)t}$$

(c) Substituting $\sigma = 7.4 \times 10^4$ S-m^{-1}, $b = 1.5a = 10t = 3$ cm into the R expression in part (a) yields $R \simeq 0.0349\Omega$. Similarly, substituting the same values in the R expression in part (b) yields $R \simeq 0.0354\Omega$.

(d) When $a \gg b - a = \zeta$, the natural logarithm term in the resistance expression in part (a) can be approximated as

$$\ln\left(\frac{a+\zeta}{a}\right) = \ln\left(1 + \frac{\zeta}{a}\right) = \frac{\zeta}{a} - \frac{\zeta^2}{2a^2} + \frac{\zeta^3}{3a^3} - \ldots \simeq \frac{\zeta}{a}$$

Using this approximation, the resistance expression in part (a) can be written as

$$R \simeq \frac{\pi a}{\sigma t \zeta}$$

The resistance expression in part (b) can also be approximated as

$$R = \frac{\pi(a+b)}{2\sigma(b-a)t} \simeq \frac{\pi(2a)}{2\sigma\zeta t} = \frac{\pi a}{\sigma\zeta t}$$

which is the same expression obtained from part (a).

5-23. Leakage resistance. This problem can be solved in two ways. The first method involves the simple realization that the geometry of the copper wire with an insulating sheath is identical to that of the coaxial shell in Example 5-6. Thus, the resistance per unit length is given by

$$R = \frac{1}{2\pi\sigma_{\text{sheath}}} \ln\left(\frac{b}{a}\right)$$

where a and b are the inner and outer radii respectively. Alternatively, we can integrate over concentric cylindrical shells to find the resistance directly:

$$dR = \frac{dr}{\sigma_{\text{sheath}} 2\pi r} \quad \rightarrow \quad R = \int_a^b \frac{dr}{\sigma_{\text{sheath}} 2\pi r} = \frac{1}{2\pi\sigma_{\text{sheath}}} \ln\left(\frac{b}{a}\right)$$

For $2a = 2$ mm, $2b = 4$ mm, and $\sigma_{\text{sheath}} = 10^{-8}$ S-m^{-1}, the leakage resistance per kilometer is found to be $R = 11.032$ k Ω.

5-27. Leaky capacitor. As discussed in Section 5.6, at the interface between lossy dielectrics, the current boundary condition [5.7] is in general incompatible with the electrostatic boundary condition concerning the continuity of the normal component of electric flux density, unless a surface charge layer is assumed to exist. (a) At steady-state, the electric field between the plates must satisfy the following conditions:

$$E_1 d + E_2 d = V_0$$

$$\nabla \cdot \mathbf{J} = 0 \quad \rightarrow \quad J_{1n} = J_{2n} \quad \rightarrow \quad \sigma_1 E_1 = \sigma_2 E_2$$

$$\nabla \cdot \mathbf{D} = \rho \quad \rightarrow \quad \rho_s = \epsilon_2 E_2 - \epsilon_1 E_1$$

so that we have

$$E_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}\frac{V_0}{d} \qquad \text{and} \qquad E_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}\frac{V_0}{d}$$

(b) The answer to this part is actually worked out at the end of Section 5.6.2; we have

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right) E_2 = \left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2}\right) E_1$$

or

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right) \frac{\sigma_1}{\sigma_1 + \sigma_2}\frac{V_0}{d} = \frac{\epsilon_2\sigma_1 - \epsilon_1\sigma_2}{\sigma_1 + \sigma_2}\frac{V_0}{d}$$

5-29. Leakage resistance. In Chapter 4, the capacitance per unit length of a very long wire of radius a parallel to a perfectly conducting ground plane at a height of $d/2$ above it was given by [4.50] as

$$C = \frac{2\pi\epsilon_0}{\ln(d/a)}$$

Using the duality relationship $RC = \epsilon/\sigma$, we can write the resistance per unit length between the wire of radius a submerged in a deep lake of conductivity σ at a height $h \gg a$ above its planar bottom as

$$R = \frac{\epsilon}{\sigma C} = \frac{\epsilon}{\sigma}\frac{\ln(2h/a)}{2\pi\epsilon} = \frac{\ln(2h/a)}{2\pi\sigma}$$