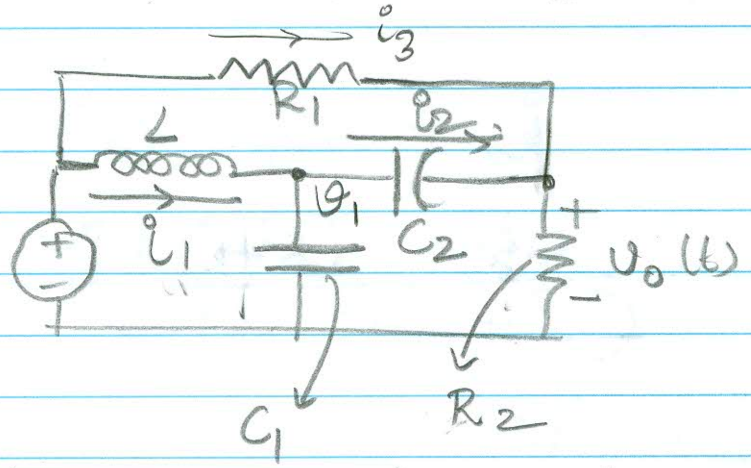
EE128 PS2 SOLUTIONS

1) a



$R_1 = 1\Omega$; $R_2 = 1\Omega$; $C_1 = 1F$; $C_2 = 1F$

$L = 1H$

R_1 : $v_i(t) - v_0(t) = i_3(t)$ — (1)

L : $v_i(t) - v_1(t) = \dfrac{di_1}{dt}$ — (2)

C_1 : $i_1(t) - i_2(t) = \dfrac{dv_1}{dt}$ — (3)

C_2 : $i_2(t) = \dfrac{dv_1}{dt} - \dfrac{dv_0}{dt}$ — (4)

R_2 : $v_0(t) = i_2(t) + i_3(t)$ — (5)

(5) & (1) $\Rightarrow$ $2 i_3(s) = v_i(s) - i_2(s)$ — (6)

(4) $\Rightarrow$ $i_2(s) = s v_1 - s v_0 \xRightarrow{(5)}$

$(1+s)\, i_2(s) + s\, i_3 = s v_1$ — (7)

(3) $\Rightarrow$ $i_1 - i_2 = s v_1 \xRightarrow{(7)} (2+s)\, i_2 + s\, i_3 = i_1$ — (8)

(2) $\Rightarrow$ $v_i - v_1 = s i_1 \Rightarrow s v_i - s v_1 = s^2 i_1$ — (9)

9 $-(s^2+1)\times$ 8 :

$$(s^2+1)(2+s)\,i_2 - i_2 = s v_i - (s^2+1)\,s\,i_3 .$$

$$\Rightarrow \quad i_2 = \frac{s v_i - (s^2+1)\,s\,i_3}{(s^3+2s^2+s+1)} \quad -\ 10$$

10 in 6 $\Rightarrow$

$$\boxed{i_3 = \frac{(s^3+2s^2+1)\ v_i}{(s^3+4s^2+s+2)}} \quad -\ 11$$

b Phase Variable Form:

11 is of the form

$$i_3 = \frac{(s^3+2s^2+1)\ v_i}{(s^3+4s^2+s+2)}$$

$$\Rightarrow i_3 = v_i - \frac{(2s^2+s+1)}{(s^3+4s^2+s+2)}(v_i)$$

Phase variable form for the system:

$$y = \frac{(\beta_2 s^2 + \beta_1 s + \beta_0)}{(s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0)}\ u$$

$$\left.\begin{aligned}\tilde{A} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix} ; \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \tilde{C} &= [\beta_0 \quad \beta_1 \quad \beta_2] ; \quad \tilde{D} = 0 \end{aligned}\right\} \quad (12)$$

Call $y = \dfrac{(2s^2+s+1)}{(s^3+4s^2+s+2)} v_i \quad (13)$

<u>Phase, variable form</u> for (13):

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -4 \end{bmatrix} ; \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

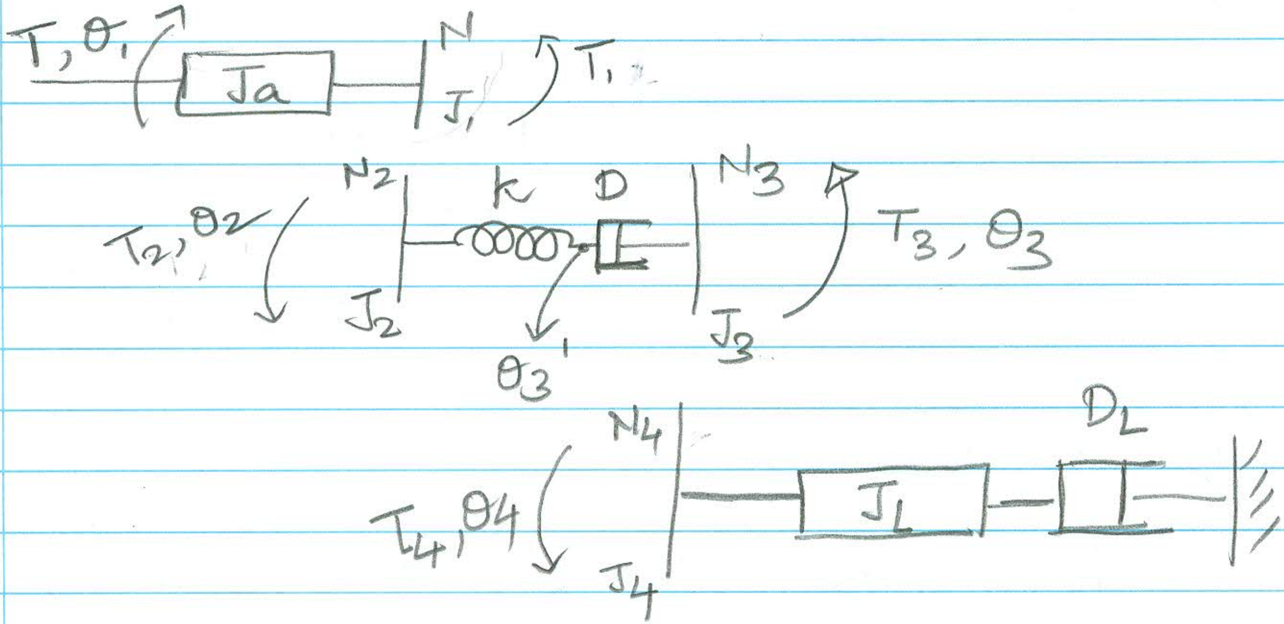
$$\tilde{C} = [1 \quad 1 \quad 2] ; \quad \tilde{D} = 0$$

<u>Phase variable form</u> for (11):

Note $i_3 = v_i - y$

$$\Rightarrow \boxed{\begin{aligned} A &= \tilde{A} ; & B &= \tilde{B} \\ C &= -\tilde{C} ; & D &= \tilde{D}+1 = 1 \end{aligned}}$$

2



From Diagram:

$$T - T_1 = (J_a + J_1)\,\ddot{\theta}_1 \quad — 1$$

$$\frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} \; ; \quad \frac{T_2}{T_1} = \frac{N_2}{N_1} \quad — 2$$

$$T_2 = J_2\,\ddot{\theta}_2 + k(\theta_2 - \theta_3'') \quad — 3$$

$$k(\theta_2 - \theta_3') = D(\dot{\theta}_3 - \dot{\theta}_3) \quad — 4$$

$$D(\dot{\theta}_3' - \dot{\theta}_3) - T_3 = J_3\,\ddot{\theta}_3 \quad — 5$$

$$\frac{\theta_3}{\theta_4} = \frac{N_4}{N_3} \; ; \quad \frac{T_4}{T_3} = \frac{N_4}{N_3} \quad — 6$$

$$T_4 = (J_4 + J_L)\,\ddot{\theta}_4 + D_L\,\dot{\theta}_4 \quad — 7$$

5

Combining 1, 2, 3 :

$$\left(J_a + J_1 + J_2 \frac{N_1^2}{N_2^2}\right)\ddot{\theta}_1 + \frac{kN_1^2}{N_2^2}\left(\theta_1 - \frac{N_2}{N_1}\theta_3'\right) = T \quad -(8)$$

2 & 4 $\Rightarrow$

$$k\frac{N_1}{N_2}\left(\theta - \frac{N_2}{N_1}\theta_3'\right) = D(\dot{\theta}_3' - \dot{\theta}_3) \quad -(9)$$

5, 6, 7 $\Rightarrow$

$$\left(J_3 + (J_4 + J_L)\frac{N_3^2}{N_4^2}\right)\ddot{\theta}_3 + \left(D + D_L\frac{N_3^2}{N_4^2}\right)\dot{\theta}_3 = D\dot{\theta}_3' \quad -(10)$$

10 $\Rightarrow$ $\theta_3'(s) = \dfrac{\left[\left(J_3 + (J_4 + J_L)\frac{N_3^2}{N_4^2}\right)s + \left(D + D_L\frac{N_3^2}{N_4^2}\right)\right]\theta_3}{D}$

Set $J' = J_3 + (J_4 + J_L)\dfrac{N_3^2}{N_4^2}$; $D' = D + D_L\dfrac{N_3^2}{N_4^2}$

$\Rightarrow$ $\theta_3'(s) = (J's + D')\theta_3 / D$

9 $\Rightarrow$ $(k + Ds)\theta_3' - Ds\,\theta_3 = \dfrac{kN_1}{N_2}\theta_1$ $-(11)$

Substituting $\theta_3'(s)$ above 11

$\theta_3(s)$

$$= \frac{(k D N_1/N_2)\,\theta_1(s)}{\left[D J' s^2 + (kJ' + D D_L N_3^2/N_4^2)s + kD'\right]}$$

$$\Rightarrow \theta_3' = \frac{(J's + D')\,kN_1/N_2\ \theta_1}{\left(DJ's^2 + (kJ' + DD_L N_3^2/N_4^2)s + kD'\right]}$$

Call $J'' = J_a + J_1 + J_2 \frac{N_1^2}{N_2^2}$

then (8) $\Rightarrow$.

$$J''s^2\theta_1 + \frac{kN_1^2}{N_2^2}\theta_1 - \frac{kN_1}{N_2}\theta_3'(s) = T.$$

Subs for $\theta_3'(s)$ we get

$$\theta_1(s) = \frac{\left[DJ's^2 + \left(kJ' + DD_L \frac{N_3^2}{N_4^2}\right)s + kD'\right] T}{\left[DJ'J''s^4 + (DD_L N_3^2/N_4^2 + kJ')J''s^3 + k\left(D'J'' + \frac{N_1^2}{N_2^2} DJ'\right)s^2 + kDD_L \frac{N_1^2 N_3^2}{N_2^2 N_4^2} s\right]}$$

Using the parameter values:

$J' = 2$; $D' = 3$; $J'' = 9/4$.

$$\Rightarrow \boxed{\theta_1 = \frac{(4s^2+10s+12)\,T}{\left[9s^4+\frac{45}{2}s^3+31s+28\right]}}$$

(b) For the system:

$$\theta_1 = \frac{[\beta_3 s^3+\beta_2 s^2+\beta_1 s+\beta_0]\,T}{[s^4+\alpha_3 s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0]}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = [\beta_0 \;\; \beta_1 \;\; \beta_2 \;\; \beta_3]; \quad D = 0$$

$\Rightarrow$

$$\boxed{\begin{gathered} A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2}{9} & -\frac{31}{9} & -\frac{5}{2} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ C = \left[\frac{4}{3} \;\; \frac{10}{9} \;\; \frac{4}{9} \;\; 0\right]; \quad D = 0. \end{gathered}}$$

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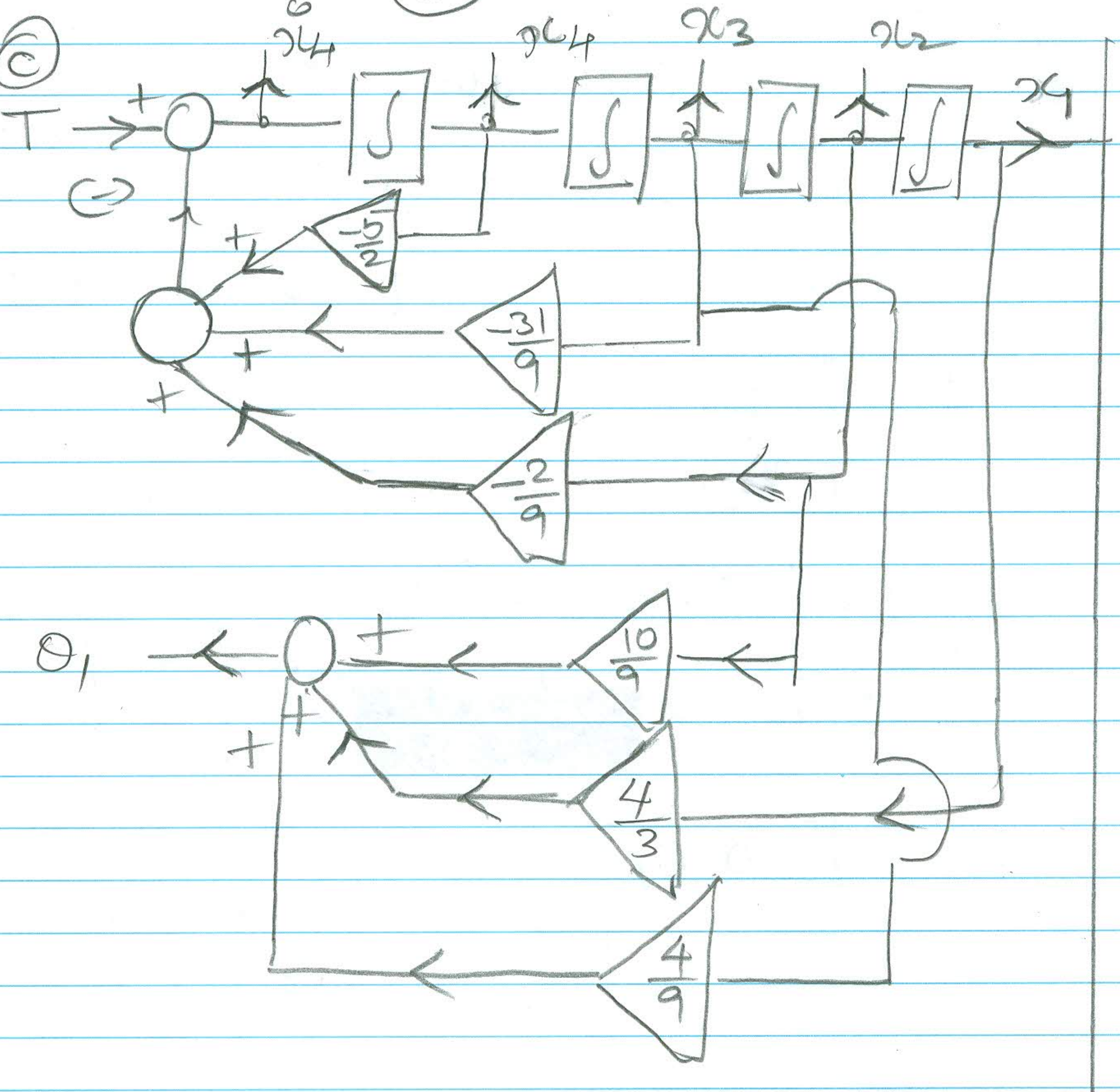
(c)



T
+
(−)
$\dot{x}_4$
x_4
x_3
x_2
x_1
$\frac{-5}{2}$
$\frac{-31}{9}$
$\frac{-2}{9}$
$\frac{10}{9}$
$\frac{4}{3}$
$\frac{4}{9}$
θ_1

3 (a) For the given system:

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} ; \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -6 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}.$$

$$G(s) = C(sI-A)^{-1}B$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (sI-A)^{-1} \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3 & 6 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{\begin{bmatrix} s^2+5s+6 & s+5 & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}{\left[s^3+5s^2+6s+3 \right]} \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

these values don't contribute

$$= \frac{4}{\left[s^3+5s^2+6s+3 \right]}$$

(b) $C = \begin{bmatrix} -3 & -9 & -7 \end{bmatrix}$

$$A = \begin{bmatrix} -6 & -12 & -9 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \quad D = 1.$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D.$$

$$= \begin{bmatrix} -3 & -9 & -7 \end{bmatrix} \begin{bmatrix} s+6 & 12 & 9 \\ -1 & s & 0 \\ 0 & -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1.$$

$$= \frac{\begin{bmatrix} -3 & -9 & -7 \end{bmatrix} \begin{bmatrix} s^2 & 0 & 0 \\ s & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1.}{[s^3 + 6s^2 + 12s + 9]}.$$

$$= \boxed{\frac{-3s^2 - 9s - 7}{[s^3 + 6s^2 + 12s + 9]} + 1}$$

$$= \boxed{\frac{s^3 + 3s^2 + 3s + 2}{s^3 + 6s^2 + 12s + 9}}$$

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4 $$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \; ; \; \dot{\underset{\sim}{x}} = \begin{bmatrix} x_2 \\ \beta(x_3 + kx_1) - \frac{\alpha}{x_4^2} u \\ \beta x_1^2 \end{bmatrix} = f(\underset{\sim}{x}, u)$$

$$\delta \dot{\underset{\sim}{x}} = \frac{\partial f}{\partial \underset{\sim}{x}}(\underset{\sim}{x}, u)\, \delta \underset{\sim}{x} + \frac{\partial f}{\partial u}(\underset{\sim}{x}, u)\, \delta u$$

$$\frac{\partial f}{\partial \underset{\sim}{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix} \Bigg|_{\substack{@\, x_1 = 1,\, x_2 = 1 \\ x_3 = 1,\, x_4 = 2 \\ u = 0}}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \beta k & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 \\ 2\beta & 0 & 0 & 0 \end{bmatrix}$$

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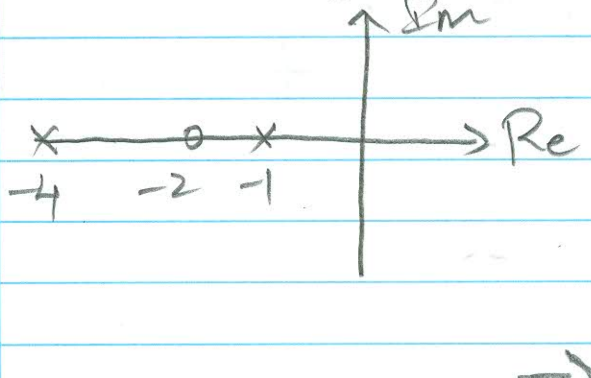
$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \\ \frac{\partial f_4}{\partial u} \end{bmatrix} @\ x_1=1, x_2=1, x_3=1, x_4=2, u=0$$

$$= \begin{bmatrix} 0 \\ 0 \\ -\frac{\alpha}{4} \\ 0 \end{bmatrix}$$

5 a $$y_1(s) = \frac{s+2}{(s+1)(s+4)}\left(\frac{1}{s}\right) = H(s)u(s)$$

step.

P-Z Diagram:



Response:

$P_1 = -1, \quad P_2 = -4.$

$$\lim_{t\to\infty} y_1(t) = 1/2$$

$$\Rightarrow \boxed{y_1(t) = \frac{1}{2} - A_1 e^{-t} - B_1 e^{-4t}}$$

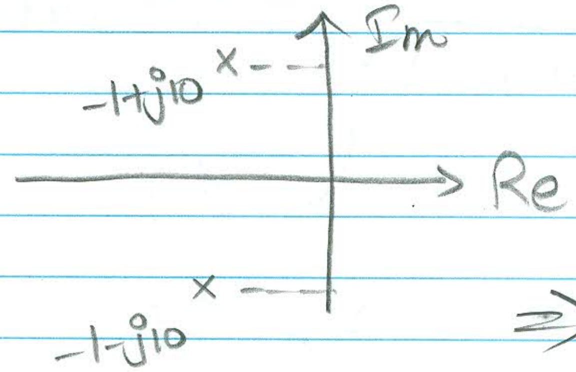
(b) $H(s) = \left(\dfrac{101}{s^2+2s+101}\right)$

$$H_2(s) = \left(\frac{101}{s^2+2s+101}\right)\left(\frac{1}{s}\right) = Y_2(s).$$

Zeros: no zeros (zeros at ∞)

Poles: $$P_{1,2} = \frac{-2 \pm \sqrt{-400}}{2}$$
$$= -1 \pm j10.$$

P-Z diagram:



Response:

$$\lim_{t\to\infty} y_2(t) = \lim_{s\to 0} s\,Y_2(s)$$
$$= 1$$

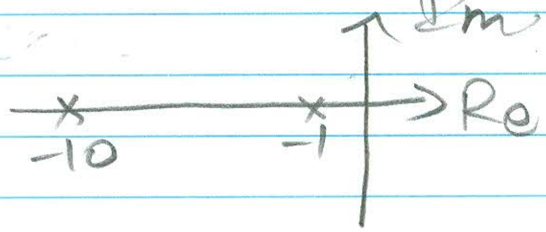
$$\Rightarrow \boxed{y_2(t) = 1 - e^{-t}\left[A_2\cos(10t) + B_2\sin(10t)\right]}$$

c) $y_3(s) = \frac{10}{(s+1)(s+10)} \cdot \frac{1}{s} \quad \Big| \quad H(s) = \frac{10}{(s+1)(s+10)}$

Poles : $P_1 = -1, \quad P_2 = -10.$

Zeros : Zeros at ∞

P-Z diagram :

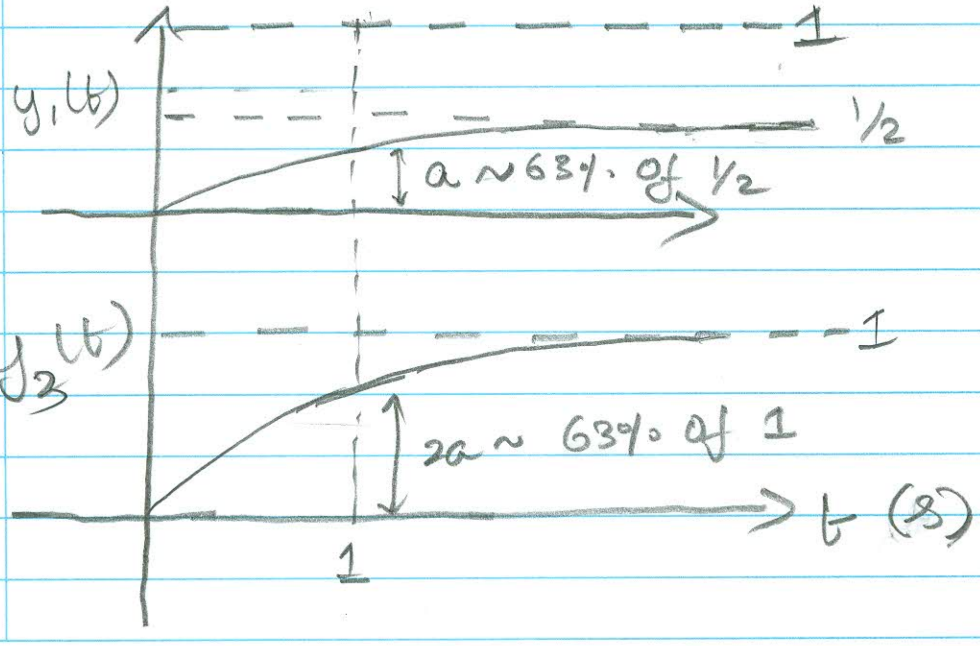


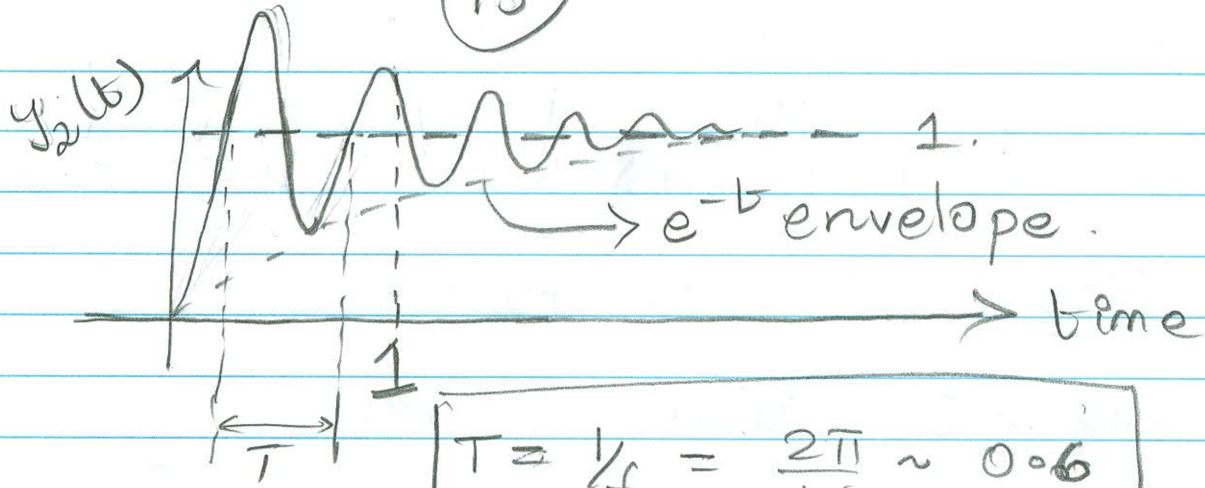
Response :

$\lim_{t \to \infty} y_3(t) = \lim_{s \to 0} s\, y_3(s) = (1)$

$\Rightarrow \boxed{y_3(t) = 1 - A_3 e^{-t} - B_3 e^{-10t}.}$

Response Plot :





$$\boxed{T = 1/f = \frac{2\pi}{\omega} \sim 0.6}$$