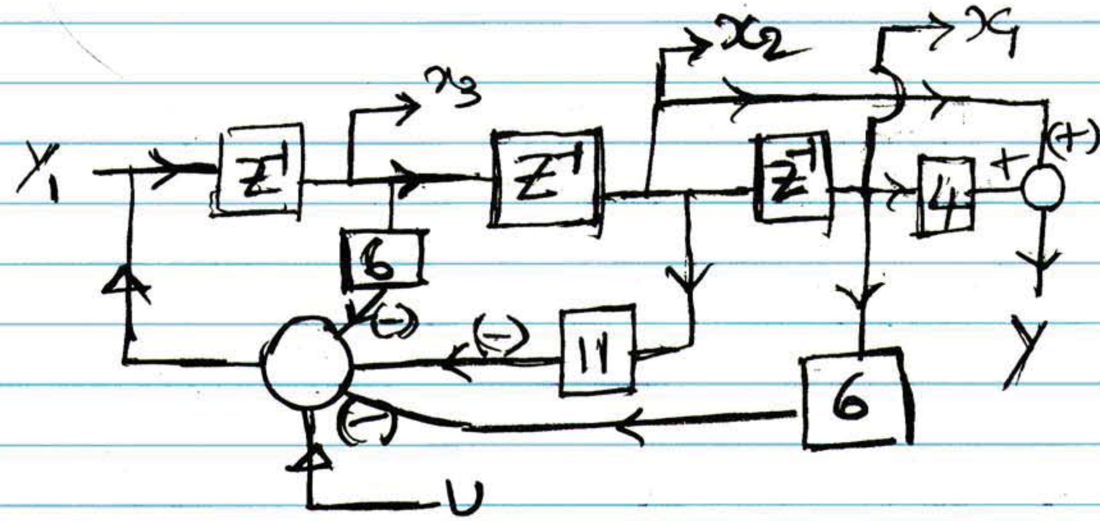
1 a $$\frac{Y(z)}{U(z)} = \frac{z^{-2} + 4z^{-3}}{1 + 6z^{-1} + 11z^{-2} + 6z^{-3}}$$

Call,

$$\frac{Y_1}{U} = \frac{1}{1 + 6z^{-1} + 11z^{-2} + 6z^{-3}}$$



b **Phase Variable Form**

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2. Given: $x[k+1] = Gx[k] + Hu[k] = \begin{bmatrix} 1 & 0.2 \\ 0.5 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 0.4 \\ -0.5 \end{bmatrix} u[k]$

a. Find transfer function $\frac{X[z]}{U[z]}$

$$\frac{X[z]}{U[z]} = (zI - G)^{-1} H = \begin{bmatrix} z-1 & -0.2 \\ -0.5 & z \end{bmatrix}^{-1} \begin{bmatrix} 0.4 \\ -0.5 \end{bmatrix}$$

$$= \frac{1}{z(z-1) - (0.5)(0.2)} \begin{bmatrix} z & 0.2 \\ 0.5 & z-1 \end{bmatrix} \begin{bmatrix} 0.4 \\ -0.5 \end{bmatrix} = \begin{bmatrix} \frac{0.4z - (0.5)(0.2)}{z(z-1)-(0.5)(0.2)} \\ \frac{(0.4)(0.5) + (z-1)(-0.5)}{z(z-1)-(0.5)(0.2)} \end{bmatrix}$$

$$\frac{X[z]}{U[z]} = \begin{bmatrix} \frac{0.4z - 0.1}{z^2 - z - 0.1} \\ \frac{-0.5z + 0.7}{z^2 - z - 0.1} \end{bmatrix}$$

b. Is the system BIBO stable?

Poles of TF: $z^2 - z - 0.1 = 0$

$z = 1.092, -0.092$

$z = 1.092$ is outside the unit circle, so the system is

$$\boxed{\text{NOT BIBO stable}}$$

Prob 3 starts:

(3) $H(s) = \dfrac{1}{(s+\alpha)^2}$

$$\Rightarrow h(t) = \boxed{e^{-\alpha t}\, t} \quad -(1)$$

$$\Rightarrow h(kT) = e^{-\alpha kT}\, kT \quad -(2)$$

$$\Rightarrow H(z) = \sum_{k=0}^{\infty} \left(e^{-\alpha kT}\, kT\right) z^{-k}$$

$$\Rightarrow H(z) = \sum_{k=0}^{\infty} (kT) \left(z e^{\alpha T}\right)^{-k} \quad -(1)$$

$$\Rightarrow zH(z) - zh(0) = \sum_{k=0}^{\infty} (k+1)T \left(z e^{\alpha T}\right)^{-k} e^{-\alpha T} \quad -(2)$$

$$\Rightarrow \left(\frac{H(z)}{T}\right) = \sum_{k=0}^{\infty} k \left(z e^{\alpha T}\right)^{-k} \quad -(3)$$

$$\left(\frac{zH(z) - zh(0)}{Te^{-\alpha T}}\right) = \sum_{k=0}^{\infty} (k+1)\left(z e^{\alpha T}\right)^{-k} \quad -(4)$$

(4) - (3):

Prob 3 continues:

④ - ③ :

$$\frac{zH(z) - zh(0)}{Te^{-\alpha T}} - \frac{H(z)}{T} = \sum_{k=0}^{\infty} \left(ze^{\alpha T}\right)^{-k}$$

$h(0) = 0 \Rightarrow$

$$\left[\frac{zH(z)}{e^{-\alpha T}} - H(z)\right]\left(\frac{1}{T}\right) = \frac{1}{1-\left(ze^{\alpha T}\right)^{-1}}$$

$$= \frac{z}{z - e^{-\alpha T}}$$

$$\Rightarrow H(z)\left[z - e^{-\alpha T}\right] = \frac{Te^{-\alpha T} z}{\left(z - e^{-\alpha T}\right)}$$

$$\boxed{H(z) = \frac{Te^{-\alpha T} z}{\left(z - e^{-\alpha T}\right)^2}}$$

4. $F(s) = \frac{k}{s+10}$

a. $e(\infty) = \frac{1}{1+K_p} < 0.1$

$K_p > 9$

$K_p = \lim_{s\to 0} F(s) = \lim_{s\to 0} \frac{k}{s+10} = \frac{k}{10} > 9$

$\boxed{K_p > 90}$

$F_{CL}(s) = \frac{K_p}{s+10+K_p}$

Closed loop pole:

$\boxed{S < -100}$

since $y=x$

b. $\dot{x} = -10x + u \qquad y = x \qquad u = K_p(r - y) = K_p(r - x)$

$x[k+1] = e^{AT}x[k] + \left(\int_0^T e^{A\lambda}\,d\lambda\right)B\,u[k]$

$G = e^{AT} = e^{-10T}$

$H = \left(\int_0^T e^{A\lambda}\,d\lambda\right)B = \left(\int_0^T e^{-10\lambda}\,d\lambda\right)(1) = -\frac{1}{10}e^{-10\lambda}\Big|_{\lambda=0}^{T} = \frac{1}{10}\left(1 - e^{-10T}\right)$

$$\boxed{x[k+1] = e^{-10T}x[k] + \frac{1}{10}\left(1-e^{-10T}\right)u[k]}$$

Let $u[k] = K_p(r[k] - x[k])$

$$x[k+1] = e^{-10T}x[k] + \frac{1}{10}\left(1-e^{-10T}\right)K_p\left(r[k] - x[k]\right)$$

$$= \left(e^{-10T} - \frac{K_p}{10}\left(1-e^{-10T}\right)\right)x[k] + \frac{K_p}{10}\left(1-e^{-10T}\right)r[k]$$

$$\boxed{x[k+1] = \left(\left(\frac{K_p}{10}+1\right)e^{-10T} - \frac{K_p}{10}\right)x[k] + \frac{K_p}{10}\left(1-e^{-10T}\right)r[k]}$$

c. For stability $G_k = (G - K_p H)$ must be inside unit circle

$$G = \left(\left(\frac{K_p}{10}+1\right)e^{-10T} - \frac{K_p}{10}\right) \Rightarrow \left|\left(\frac{K_p}{10}+1\right)e^{-10T} - \frac{K_p}{10}\right| < 1$$

$$\left(\frac{K_p}{10}+1\right)e^{-10T} - \frac{K_p}{10} < 1 \qquad -1 < \left(\frac{K_p}{10}+1\right)e^{-10T} - \frac{K_p}{10}$$

$$\left(\frac{K_p}{10}+1\right)e^{-10T} < 1 + \frac{K_p}{10} \qquad \frac{\left(\frac{K_p}{10}-1\right)}{\left(\frac{K_p}{10}+1\right)} < e^{-10T}$$

$$e^{-10T} < 1$$

$$-10T < \ln(1) \qquad \ln\left(\frac{K_p - 10}{K_p + 10}\right) < -10T$$

$$T > \frac{-\ln(1)}{10}$$

$$T > 0 \qquad \frac{\ln\left(\frac{K_p-10}{K_p+10}\right)}{-10} > T$$

For SSE < 0.1:

$x[k+1] = x[k]$ in steady state $r[k] = 1$ since input is a step

$$x[k] = (G - K_p H)\, x[k] + K_p H$$

$$x[k] = (1 - (G - K_p H))^{-1} K_p H \quad \text{at steady state}$$

$$x[k] = \frac{K_p\left(\frac{1}{10}\right)(1-e^{-10T})}{\left(1 - \left[\left(\frac{K_p}{10}+1\right)e^{-10T} - \frac{K_p}{10}\right]\right)} = \frac{\frac{K_p}{10}(1-e^{-10T})}{\left(1 - \left(\frac{K_p}{10}+1\right)e^{-10T} + \frac{K_p}{10}\right)}$$

$$= \frac{\frac{K_p}{10}(1-e^{-10T})}{\left(\frac{K_p}{10}+1\right) - \left(\frac{K_p}{10}+1\right)e^{-10T}} = \frac{\frac{K_p}{10}(1-e^{-10T})}{\left(\frac{K_p}{10}+1\right)(1-e^{-10T})} = \frac{\frac{K_p}{10}}{\frac{K_p}{10}+1}$$

$$x[k] = \frac{K_p}{K_p + 10}$$

So $x[k]$ at steady state only depends on K_p, not T.

For SSE < 0.1, $0.9 < x[k] < 1.1 \Rightarrow 0.9 < \frac{K_p}{K_p+10} < 1.1$

So $90 < K_p$ for this condition to hold

For SSE: $90 < K_p$

For stability: $0 < T < \dfrac{\ln\left(\frac{K_p - 10}{K_p + 10}\right)}{-10}$

Since $\ln\left(\frac{K_p - 10}{K_p + 10}\right)$ decreases with increasing K_p, maximal T is at $K_p = 90$

So maximal $T = \dfrac{\ln\left(\frac{90-10}{90+10}\right)}{-10} = \dfrac{\ln(0.8)}{-10} = 0.0223$

$$\boxed{T_{max} = 0.0223}$$

Prob 5 starts . . .

(5) $$G(s) = \frac{k}{s(s+1)}$$

(a) $$G(s) = k\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\Rightarrow g(t) = (k) - ke^{-t} \quad -(1)$$

$$\Rightarrow g(mT) = k - ke^{-mT} \quad -(2)$$

$$G(z) = \sum_{m=0}^{\infty} g(mT)\, z^{-m}$$

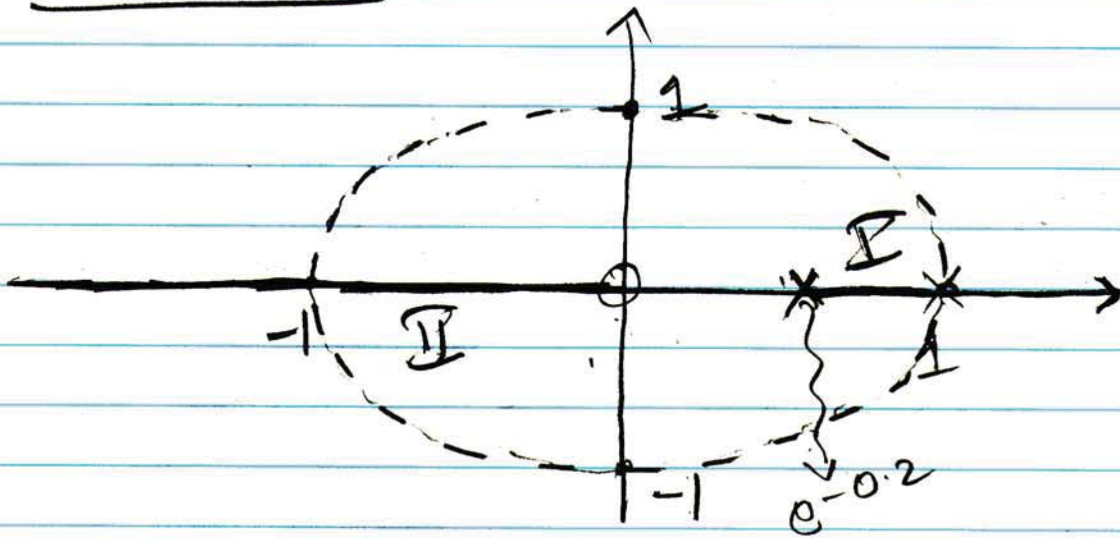
$$= k\sum_{m=0}^{\infty} z^{-m} - k\sum_{m=0}^{\infty} (ze^{T})^{-m}$$

$$\Rightarrow \boxed{G(z) = \left(\frac{kz}{z-1}\right) - k\frac{z}{z-e^{-T}}}$$

$$T = 0.2 \Rightarrow G(z) = \left(\frac{kz}{z-1}\right) - \frac{kz}{z-e^{-0.2}}$$

(b) $$\boxed{G(z) = \frac{kz(1-e^{-0.2})}{(z-1)(z-e^{-0.2})}}$$

Root Locus:



(i) RAS = {I, II}.

(ii) Breakaway / Break in points:

$$\sum \frac{1}{\sigma_B + p_i} = \sum \frac{1}{\sigma_B + z_i}$$

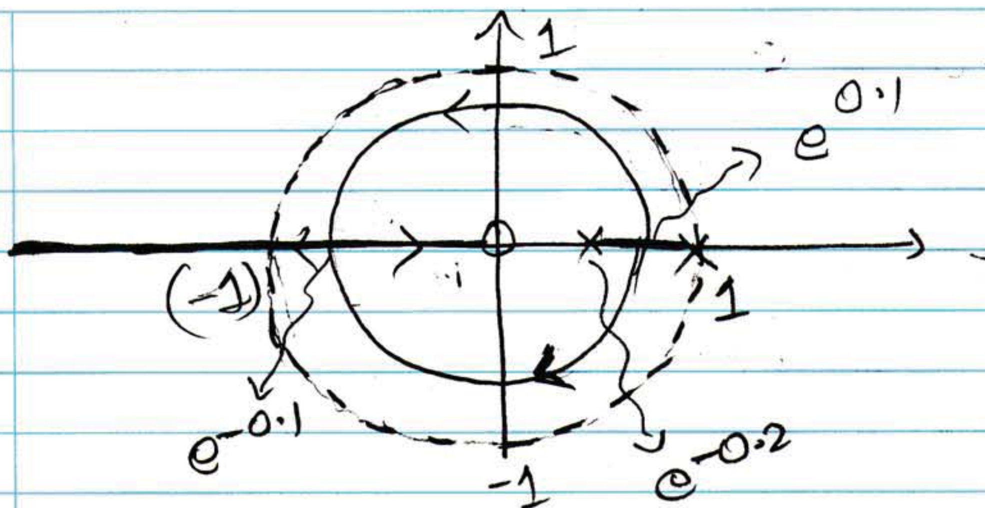
$$\Rightarrow \frac{1}{(\sigma_B - 1)} + \frac{1}{\sigma_B - e^{-0.2}} = \frac{1}{\sigma_B}.$$

$$\Rightarrow (\sigma_B - e^{-0.2})\sigma_B + \sigma_B(\sigma_B - 1) = (\sigma_B - 1)(\sigma_B - e^{-0.2})$$

$$\Rightarrow 2\sigma_B^2 - \sigma_B(1 + e^{-0.2}) = \sigma_B^2 - \sigma_B(1 + e^{-0.2}) + e^{-0.2}$$

$$\Rightarrow \sigma_B^2 = e^{-0.2}$$

$$\boxed{\sigma_B = \pm e^{-0.1}}$$



Closed loop characteristic polynomial:

$$z^2 + \left[(k-1) - e^{-0.2}(k+1)\right] z + e^{-0.2} = CP(z)$$

When $z = -1$ is a pole,

$$CP(-1) = 0 \Rightarrow 1 - \left[(k-1) - (k+1)e^{-0.2}\right] + e^{-0.2} = 0$$

$$\Rightarrow k(1 - e^{-0.2}) = 2(1 + e^{-0.2})$$

$$\Rightarrow \boxed{k = \frac{2(1+e^{-0.2})}{(1-e^{-0.2})}}$$

$\Rightarrow$ Range of k for stability:

$$\boxed{0 < k < \frac{2(1+e^{-0.2})}{1-e^{-0.2}}}$$

(c) $0.20 = e^{-\xi\pi/\sqrt{1-\xi^2}}$

$$+\frac{\xi\pi}{\sqrt{1-\xi^2}} = -\ln(0.2) = \ln 5$$

$$\frac{\xi^2\pi^2}{(1-\xi^2)} = (\ln 5)^2 \Rightarrow \xi_1^2 = \frac{(\ln 5)^2}{(\ln 5)^2+\pi^2}$$

$$\Rightarrow \xi_1 = \sqrt{\frac{(\ln 5)^2}{(\ln 5)^2+\pi^2}} = \frac{\ln 5}{\sqrt{\ln 5^2+\pi^2}}$$

$$\Rightarrow \boxed{\xi_1 \approx 0.46}$$

Using matlab :

$$\boxed{k \approx 0.23}$$

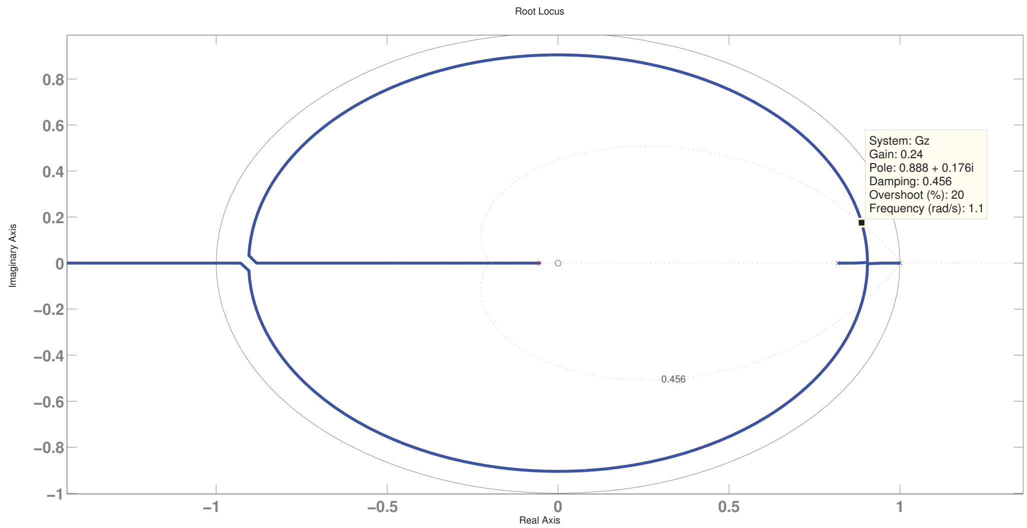


Figure 1: Root Locus showing Gain at OS % = 20%

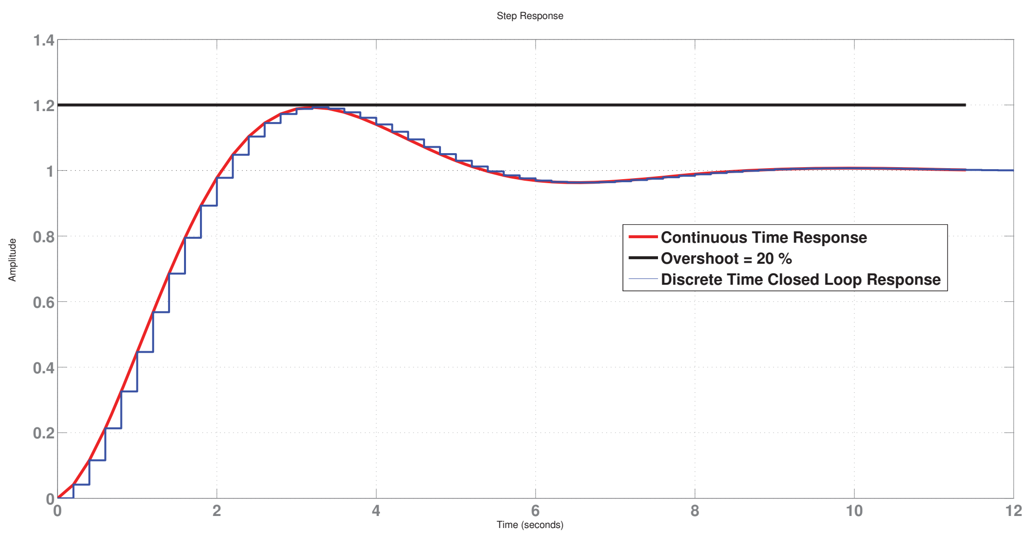


Figure 2: Step Response of Closed Loop System

6. $x[k+1] = G_1 x[k] + H_1 u[k] = \begin{bmatrix} 0 & 1 \\ -0.64 & 1.6 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k] \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x[k]$

$T_s = 1$

a. $e[k] = r[k] - y[k]$

$u[k] = -K_1 x[k] + r[k]$

$x[k+1] = [G_1 - H_1 K] x[k] + H_1 r[k]$

Characteristic equation: $\det\left(zI - (G_1 - H_1 K)\right)$

$$= \det\left(zI - \begin{bmatrix} 0 & 1 \\ -0.64 - k_1 & 1.6 - k_2 \end{bmatrix}\right)$$

$$= \det\begin{pmatrix} z & -1 \\ 0.64 + k_1 & z - 1.6 + k_2 \end{pmatrix}$$

$$= z^2 + (-1.6 + k_2)z + (0.64 + k_1)$$

desired characteristic equation:

$$\chi(z) = (z - (0.3 + 0.3j))(z - (0.3 - 0.3j))$$

$$= z^2 - 0.6z + 0.18$$

$-1.6 + k_2 = -0.6 \qquad 0.64 + k_1 = 0.18$

$k_2 = 1 \qquad k_1 = -0.46$

$K_1 = \begin{bmatrix} -0.46 & 1 \end{bmatrix}$

$G_k = (G_1 - H_1 K) = \begin{bmatrix} 0 & 1 \\ -0.64 - (-0.46) & 1.6 - 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.18 & 0.6 \end{bmatrix}$

Closed loop system:

$x[k+1] = G_k x[k] + H_1 r[k] = \begin{bmatrix} 0 & 1 \\ -0.18 & 0.6 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r[k], \; y = \begin{bmatrix} 1 & 0 \end{bmatrix} x[k]$

At steady state: $x[k+1] = x[k]$, $r[k]=1$

$$x[k] = G_k x[k] + H_1 r[k] = G_k x[k] + H_1$$

$$x[k] = (I - G_k)^{-1} H_1$$

$$e[k] = r[k] - y[k] = 1 - Cx[k] = 1 - C(I - G_k)^{-1} H_1$$

$$= 1 - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0.18 & 0.4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 1 - \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{0.58} \begin{bmatrix} 0.4 & 1 \\ -0.18 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 1 - \frac{1}{0.58} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 1 - \frac{1}{0.58}(1)$$

$$e[k] = -0.724$$

b. Add integrator: $x_N[k+1] = x_N[k] + e[k]$

$$= x_N[k] + r[k] - Cx[k]$$

$$x[k+1] = \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ x_N[k+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0.64 & 1.6 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_N[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r[k]$$

$$y[k] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_N[k] \end{bmatrix}$$

This representation:

$$\bar{x}[k+1] = \bar{G}\,\bar{x}[k] + \bar{H}\,u[k] + \bar{F}\,r[k]$$

$$y[k] = \bar{C}\,x[k]$$

c. $u[k] = -Kx[k] + K_e x_w[k] = -\overbrace{\begin{bmatrix} K & -K_e \end{bmatrix}}^{\bar{K}} \begin{bmatrix} x[k] \\ x_w[k] \end{bmatrix}$

$$x[k+1] = (\bar{G} - \bar{H}\bar{K})x[k] + \bar{F}r[k]$$

$$x[k+1] = \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ x_w[k+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0.64-k_1 & 1.6-k_2 & k_e \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_w[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r[k]$$

$$y[k] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_w[k] \end{bmatrix}$$

Characteristic polynomial:

$$\det(zI - (\bar{G} - \bar{H}\bar{K})) = z^3 + \left(k_2 - \frac{13}{5}\right)z^2 + \left(k_1 - k_2 + \frac{56}{25}\right)z + \left(-k_1 + k_e - \frac{16}{25}\right)$$

Desired characteristic polynomial:

$$\chi(z) = (z - (0.3 + 0.3j))(z - (0.3 - 0.3j))(z - 0.1)$$

$$= z^3 - 0.7z^2 + 0.24z - 0.018$$

$k_2 - \frac{13}{5} = -0.7$ $\quad\left(k_1 - k_2 + \frac{56}{25}\right) = 0.24$ $\quad -k_1 + k_e - \frac{16}{25} = -0.018$

$k_2 = 1.9$ $\quad k_1 = -0.1$ $\quad k_e = 0.522$

$$\bar{K} = \begin{bmatrix} -0.1 & 1.9 & -0.522 \end{bmatrix}$$

At steady state: $x[k+1] = x[k]$, $r[k] = 1$

$$x[k] = (\bar{G} - \bar{H}\bar{K})x[k] + \bar{F}r[k]$$

$$x[k] = (I - (\bar{G} - \bar{H}\bar{K}))^{-1}\bar{F}$$

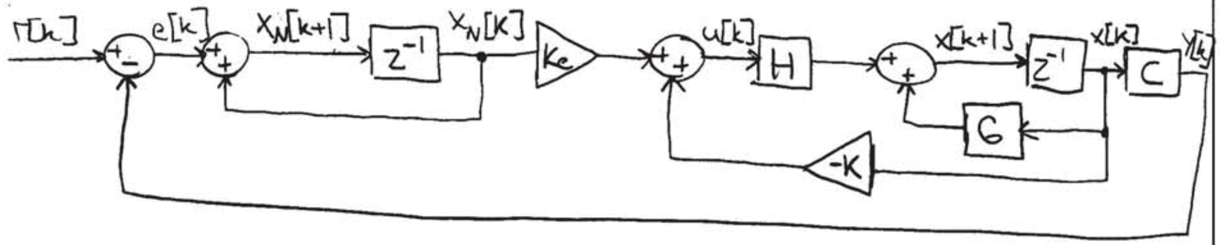
$$e[k] = r[k] - y[k] = 1 - \bar{C}x[k] = 1 - \bar{C}(I - (\bar{G} - \bar{H}\bar{K}))^{-1}\bar{F}$$

$$= 1 - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0.54 & 0.7 & -0.522 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= 1 - 1$$

$$e[k] = 0$$

Block diagram of system with integrator



Step Response