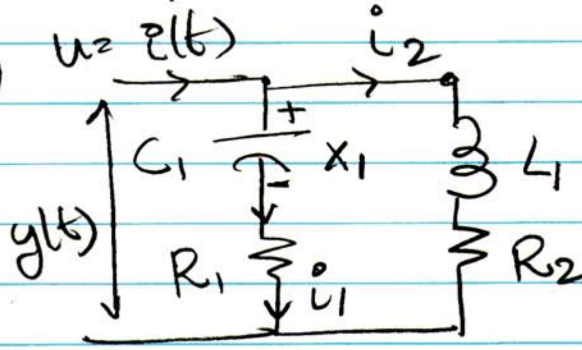
EE128/ME134 PS10 SOLUTIONS.

1 (a)



$$\boxed{\dot{x}_1 = \dot{V}_{C_1} = \frac{i_1}{C_1} = \frac{u - i_2}{C_1} = \frac{u - x_2}{C_1}}$$ — (1)

$$y(t) = \overbrace{x_1}^{V_{C_1}} + \overbrace{(u - x_2)}^{i_1} R_1$$

$$\boxed{y(t) = x_1 - x_2 R_1 + u R_1} \quad — (2)$$

Also, $$\overbrace{L_1 \frac{di_2}{dt}}^{\dot{x}_2} + \overbrace{i_2}^{x_2} R_2 = x_1 + (u - x_2) R_1 = y(t)$$

$$\Rightarrow \quad L_1 \dot{x}_2 = -x_2 R_2 + x_1 + (u - x_2) R_1$$

$$\Rightarrow \quad \boxed{\dot{x}_2 = \frac{-(R_1 + R_2) x_2}{L_1} + \frac{x_1}{L} + u \frac{R_1}{L}}$$

— (3)

(1), (2), (3) $\Rightarrow$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/C_1 \\ \frac{1}{L_1} & \frac{-(R_1+R_2)}{L_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/C_1 \\ R_1/L_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -R_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + R_1 (u)$$

(b)(i) $[B \quad AB] = \begin{bmatrix} 1/C_1 & -R_1/L_1C_1 \\ (R_1/L_1) & \frac{1}{L_1C_1} - \frac{(R_1+R_2)R_1}{L_1^2} \end{bmatrix}$

$$\Rightarrow \det\left(\begin{bmatrix} B & AB \end{bmatrix}\right) = \frac{1}{L_1C_1^2} - \frac{(R_1+R_2)R_1}{L_1^2C_1}$$

$$+ \frac{R_1}{L_1C_1}\left(\frac{R_1}{L_1}\right)$$

$$= \frac{L_1 - C_1R_1R_2}{(L_1^2C_1^2)}$$

controllability $\Rightarrow \det([B \quad AB]) \neq 0$

$\Leftrightarrow L_1 \neq C_1R_1R_2$.

$\Leftrightarrow \boxed{\left(\frac{L_1}{R_2}\right) \neq C_1R_1}$ — 4

(ii) $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & -R_1 \\ \frac{-R_1}{L_1} & \frac{-1}{C_1} + \frac{R_1(R_1+R_2)}{L_1} \end{bmatrix}$

observability $\Rightarrow \det\left(\begin{bmatrix} C \\ CA \end{bmatrix}\right) \neq 0.$

$$\Leftrightarrow \begin{vmatrix} 1 & -R_1 \\ \left(-\frac{R_1}{L_1}\right) & -\frac{1}{C_1} + \frac{R_1(R_1+R_2)}{L_1} \end{vmatrix} \neq 0.$$

$$\Leftrightarrow \frac{-1}{C_1} + \frac{R_1(R_1+R_2)}{L_1} - \frac{R_1^2}{L_1} \neq 0.$$

$$\Leftrightarrow \frac{(-L_1) + C_1R_1R_2}{(L_1C_1)} \neq 0.$$

$$\Leftrightarrow L_1 \neq C_1R_1R_2$$

$$\Leftrightarrow \boxed{\frac{L_1}{R_2} \neq C_1R_1} \quad (5)$$

(c) (4) & (5) $\Rightarrow$ system is controllable and observable when the two time constants are not equal.

2. $\dot{z} = Az + Bu = \begin{bmatrix} -3 & 1 \\ -1 & 2 \end{bmatrix} z + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t) \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} z$

$\dot{x} = P^{-1}APx + P^{-1}Bu \qquad \bar{A} = P^{-1}AP \qquad \bar{B} = P^{-1}B$
$y = CPx \qquad \bar{C} = CP$

$$C_{M_x} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} \end{bmatrix} = \begin{bmatrix} P^{-1}B & P^{-1}APP^{-1}B \end{bmatrix} = \begin{bmatrix} P^{-1}B & P^{-1}AB \end{bmatrix}$$
$$= P^{-1}\begin{bmatrix} B & AB \end{bmatrix} = P^{-1}C_{M_z}$$

$$C_{M_x} = P^{-1}C_{M_z}$$
$$PC_{M_x} = PP^{-1}C_{M_z}$$
$$PC_{M_x} = C_{M_z}$$
$$P = C_{M_z}C_{M_x}^{-1}$$

$$C_{M_z} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & \begin{bmatrix} -3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ 3 & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}$$

Only need $\bar{A}$ and $\bar{B}$ in phase variable form, so only the denominator of transfer function is needed.

$$\det(sI - A) = \det\left(\begin{bmatrix} s+3 & -1 \\ 1 & s-2 \end{bmatrix}\right) = (s+3)(s-2) + 1$$
$$= s^2 + s - 5$$

$$\dot{x} = \bar{A}x + \bar{B}u = \begin{bmatrix} 0 & 1 \\ 5 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$C_{M_x} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} \end{bmatrix} = \begin{bmatrix} 0 & \begin{bmatrix} 0 & 1 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 & \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$C_{M_x}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\right)}\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P = C_{M_z} C_{M_x}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 8 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 8 & 3 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 5 & -1 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\bar{C} = CP = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Prob 3 starts

(3)

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{(\underset{\sim}{x})} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

(a) with feed back s.t. the poles are at $s_1 = -4$, $s_2 = -5$ the closed loop system equations are

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

$$y = [10 \quad 0]\underset{\sim}{x}$$

(a) <u>steady state</u> :

$\dot{y} = 0$ (because system is stable).

$$\Rightarrow \begin{bmatrix} 0 & -1 \\ +20 & +9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1)$$

$$\Rightarrow \quad x_2 = 0 \quad \& \quad x_1 = (1/20)$$

$$\Rightarrow y = 10 x_1 = 1/2 \Rightarrow \lim_{t \to \infty} e = \boxed{1/2}$$

(b) $X_3 = \int (10x_1 - r)\,dt$ (defn).

$\Rightarrow \dot{X}_3 = 10x_1 - r$

Let $u = -k_1 x_1 - k_2 x_2 - k_3 \int (10x_1 - r)\,dt + r$

then,

$$\begin{bmatrix} \dot{X}_3 \\ \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 1 \\ -k_3 & -6-k_1 & (-5-k_2) \end{bmatrix} \begin{bmatrix} X_3 \\ x_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} (r) \quad (1)$$

Let's assume that k_3, k_1, k_2 are s.t.

closed loop poles are at $-4, -5, -25$

$\Rightarrow$ characteristic polynomial:

$$(s+4)(s+5)(s+25) = (s^3 + \underbrace{34}_{a_2} s^2 + \underbrace{245}_{a_1} s + \underbrace{500}_{a_0}) \quad (2)$$

$\Rightarrow$ closed loop system state equations are:

$$\begin{bmatrix} \dot{X}_3 \\ \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 1 \\ -k_3 & -6-k_1 & -5-k_2 \end{bmatrix} \begin{bmatrix} X_3 \\ X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} (r)$$

where k_1, k_2, k_3 are given by (5) in next page.

$\Rightarrow$ @ steady state:

$\dot{\underset{\sim}{x}} = 0$

$\Rightarrow \dot{x}_3 = 0 \Rightarrow \boxed{10x_1 = r}$

$$\Rightarrow \boxed{\lim_{t \to \infty} e(t) = 0}$$

Corresponding gains:

Characteristic equations corresponding to system (1):

$$\begin{vmatrix} s & -10 & 0 \\ 0 & s & -1 \\ k_3 & 6+k_1 & s+5+k_2 \end{vmatrix} = s^3 + (5+k_2)s^2 \ldots + (6+k_1)s + 10k_3$$

(3)

Equating (3) and (2):

$10k_3 = 500 \Rightarrow k_3 = 50$

$(6+k_1) = 245 \Rightarrow k_1 = 239$

$5+k_2 = 34 \Rightarrow k_2 = 29.$

(4)

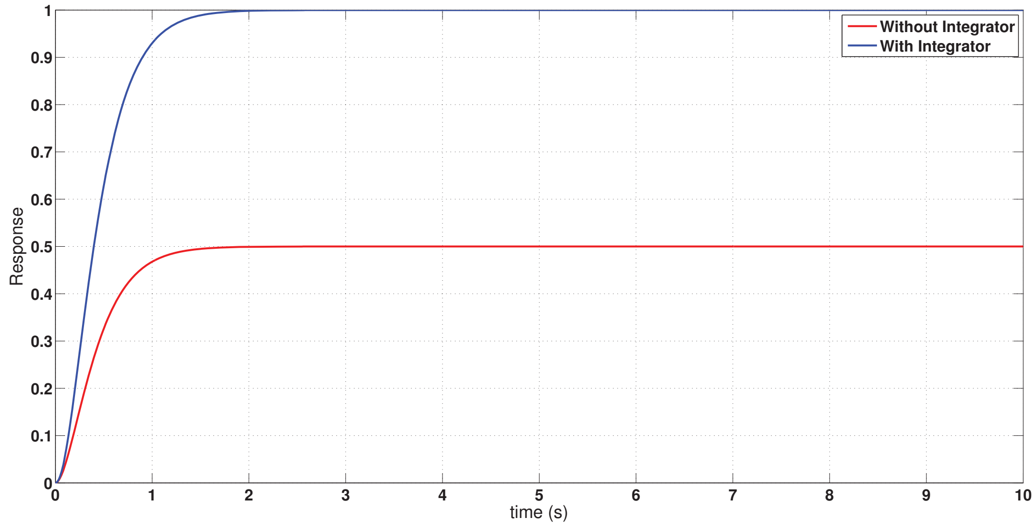


Figure 1: Step Response - Both Systems

4. $\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \quad x(0) = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix}$

a. Design $u = -Kx$ such that C.L. system has $\zeta = 0.5$, $\omega_n = 6$

$$P_{1,2} = -\omega_n \zeta \pm \omega_n \sqrt{\zeta^2 - 1} = -6(0.5) \pm 6\sqrt{(0.5)^2 - 1}$$
$$= -3 \pm 6j\frac{\sqrt{3}}{2}$$
$$= -3 \pm 3\sqrt{3}\, j$$

Desired characteristic polynomial P_D:

$$P_D(s) = (s + 3 - 3\sqrt{3}j)(s + 3 + 3\sqrt{3}j)$$
$$P_D(s) = s^2 + 6s + 36$$

Designing K:

$$A_K = A - BK = \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 - k_1 & -k_2 \end{bmatrix}$$

$$P(s) = \det(sI - A_K) = \det\left(\begin{bmatrix} s & -1 \\ k_1 - 6 & s + k_2 \end{bmatrix}\right)$$
$$= s(s + k_2) + (k_1 - 6)$$
$$= s^2 + k_2 s + (k_1 - 6)$$

Equating $P(s)$ and $P_D(s)$ coefficients:

$k_2 = 6$

$(k_1 - 6) = 36 \Rightarrow k_1 = 42$

So $K = \begin{bmatrix} 42 & 6 \end{bmatrix}$

b. Matlab plots attached.

c. $\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu - L(y - \hat{y})$

$$= A(x - \hat{x}) - L(Cx - C\hat{x})$$

$$= A(x - \hat{x}) - LC(x - \hat{x})$$

$$= (A - LC)(x - \hat{x})$$

$$\dot{e} = A_c e$$

$$A_c = A - LC = \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -l_1 & 1 \\ 6 - l_2 & 0 \end{bmatrix}$$

$$P(s) = \det(sI - A_c) = \det\left(\begin{bmatrix} s + l_1 & -1 \\ l_2 - 6 & s \end{bmatrix}\right) = s(s + l_1) + (l_2 - 6)$$

$$= s^2 + l_1 s + (l_2 - 6)$$

Desired characteristic polynomial:

$$P_D(s) = (s + 10)(s + 10) = s^2 + 20s + 100$$

Designing L by matching terms:

$$l_1 = 20$$

$$(l_2 - 6) = 100 \Rightarrow l_2 = 106$$

$$L = \begin{bmatrix} 20 \\ 106 \end{bmatrix}$$

d. $\dot{x} = Ax + Bu \qquad y = Cx$

$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \qquad \hat{y} = C\hat{x}$

Let $u = -K\hat{x}$:

$\dot{x} = Ax - BK\hat{x}$

$\dot{\hat{x}} = A\hat{x} - BK\hat{x} + LCx - LC\hat{x}$

$= (A - BK - LC)\hat{x} + LCx$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6 & 0 & -42 & -6 \\ 20 & 0 & -20 & 1 \\ 106 & 0 & -142 & -6 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Using separation principle:

Apply transformation $\bar{x} = Pz$ where $\bar{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$, $z = \begin{bmatrix} x \\ e \end{bmatrix}$

$$\dot{\bar{x}} = P\dot{z} = \begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix} Pz \qquad Pz = P\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} x \\ x-e \end{bmatrix}$$

$$\text{So: } \dot{z} = P^{-1}\begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix} Pz \qquad = \begin{bmatrix} x \\ x-(x-\hat{x}) \end{bmatrix} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \bar{x}$$

Proof that eignenvalues of A equal eisenvalues of $P^{-1}AP$ for a similarity transform P

$$\chi(A) = A^n + \alpha_1 A^{n-1} + \ldots + \alpha_{n-1} A + \alpha_n I = 0$$

For characteristic polynomial $\chi(s)$ and by Cayley-Hamilton

$$\chi(P^{-1}AP) = (P^{-1}AP)^n + \alpha_1 (P^{-1}AP)^{n-1} + \ldots + \alpha_{n-1}(P^{-1}AP) + \alpha_n I = 0$$

$$(P^{-1}AP)^n = (P^{-1}AP)(P^{-1}AP)\ldots(P^{-1}AP) = P^{-1}A^nP$$

$$\text{So: } \chi(P^{-1}AP) = P^{-1}\underbrace{\left(A^n + \alpha_1 A^{n-1} + \ldots + \alpha_{n-1}A + \alpha_n I\right)}_{0}P$$

$$= 0$$

$\therefore$ $P^{-1}AP$ and A have the same characteristic polynomial and eigen values.

$$\text{Let } P = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$$

$$P^{-1}\begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix}P = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}\begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix}\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}\begin{bmatrix} A-BK & BK \\ A-BK & -A+BK+LC \end{bmatrix}$$

$$= \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}$$

Since $P^{-1}AP$ has the same eigenvalues as A,

the eigenvalues of $\begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix}$ are the eigenvalues

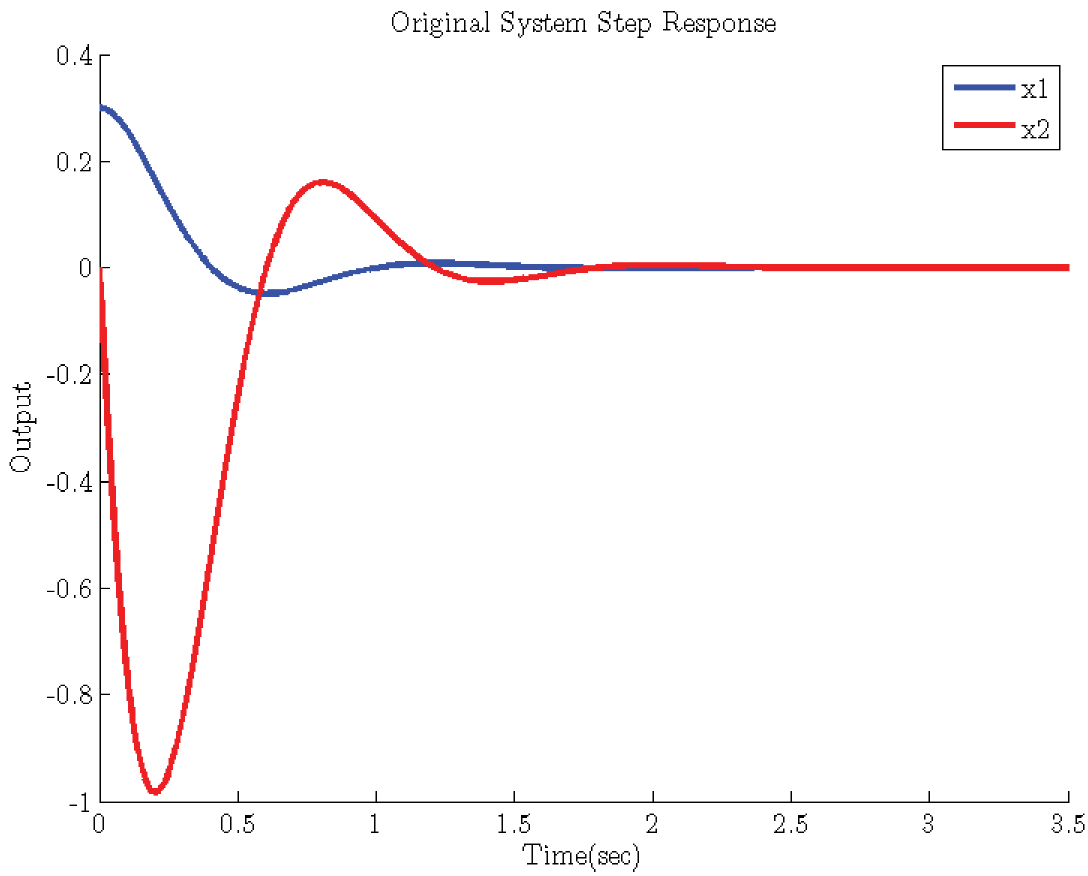
of $\begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}$, which are the eigenvalues of

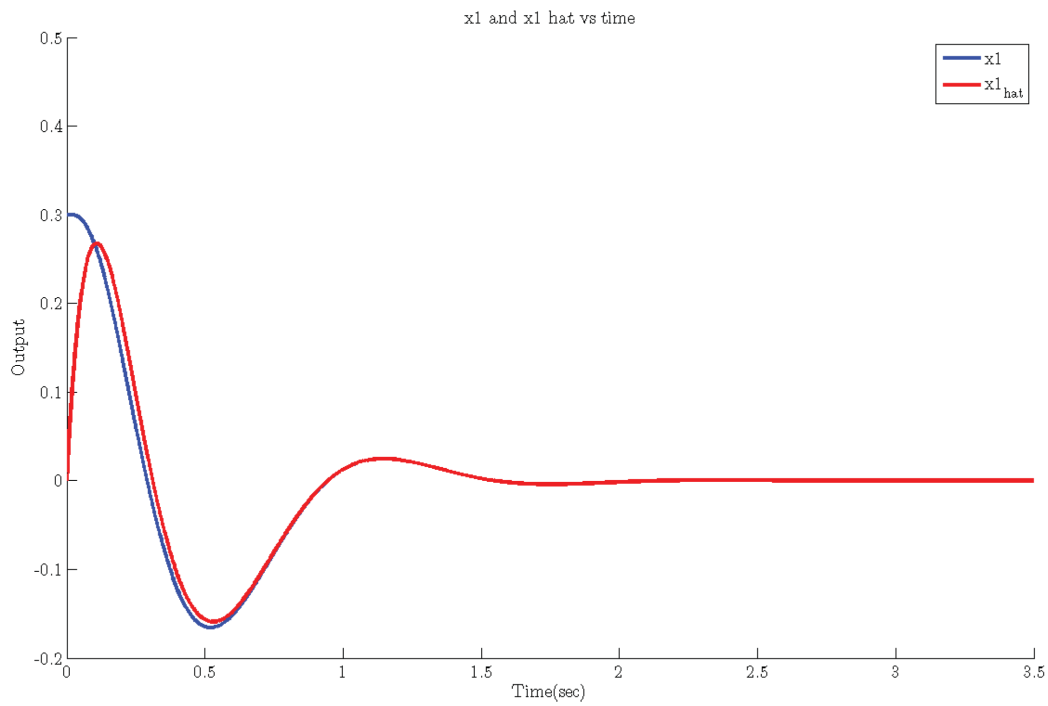
$A-BK$ and $A-LC$, which are the poles that we designed

K and L to obtain.

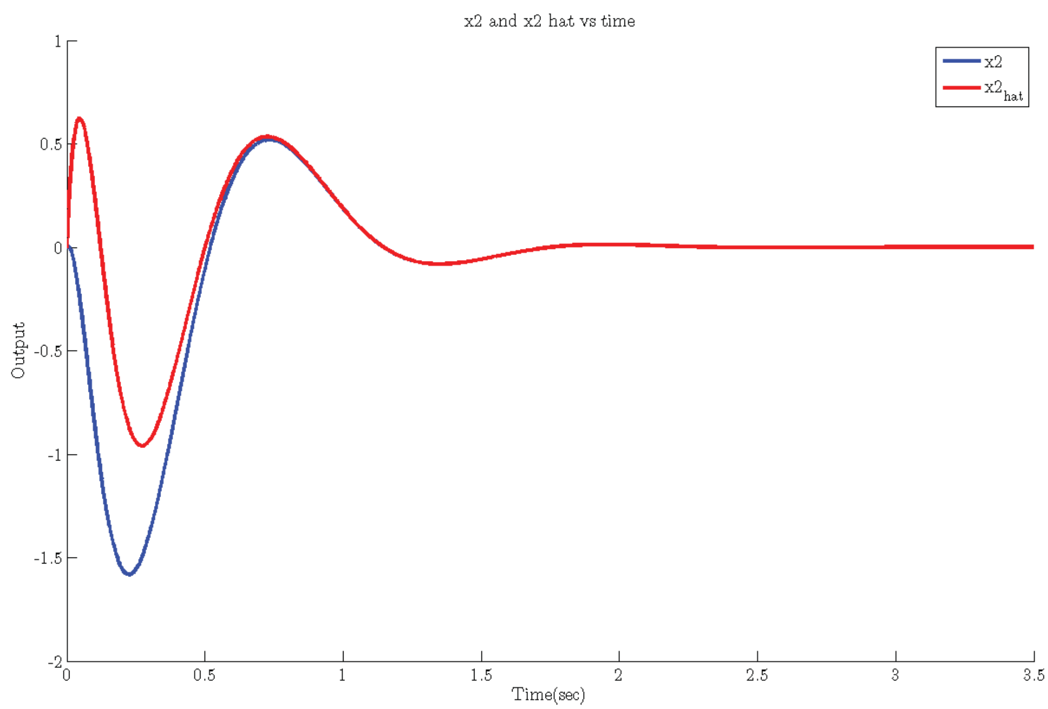
$$\therefore \quad \lambda_1 = -3 + 3\sqrt{3}j$$
$$\lambda_2 = -3 - 3\sqrt{3}j$$
$$\lambda_3 = -10$$
$$\lambda_4 = -10$$

e. Matlab plots attached

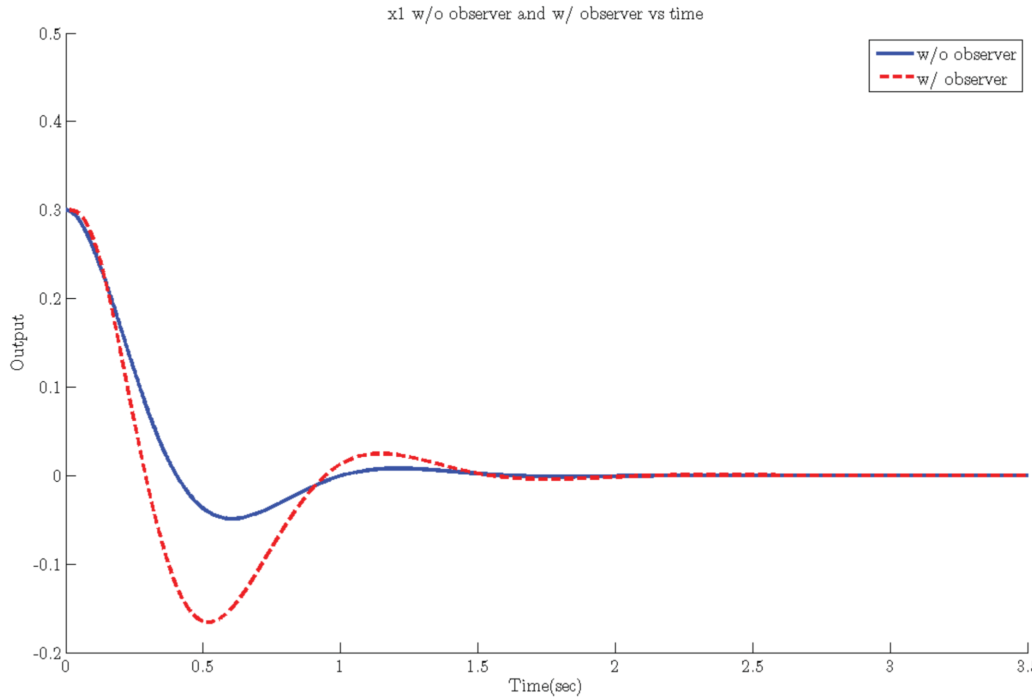

Original System Step Response
x1
x2
Output
Time(sec)
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.5
1
1.5
2
2.5
3
3.5

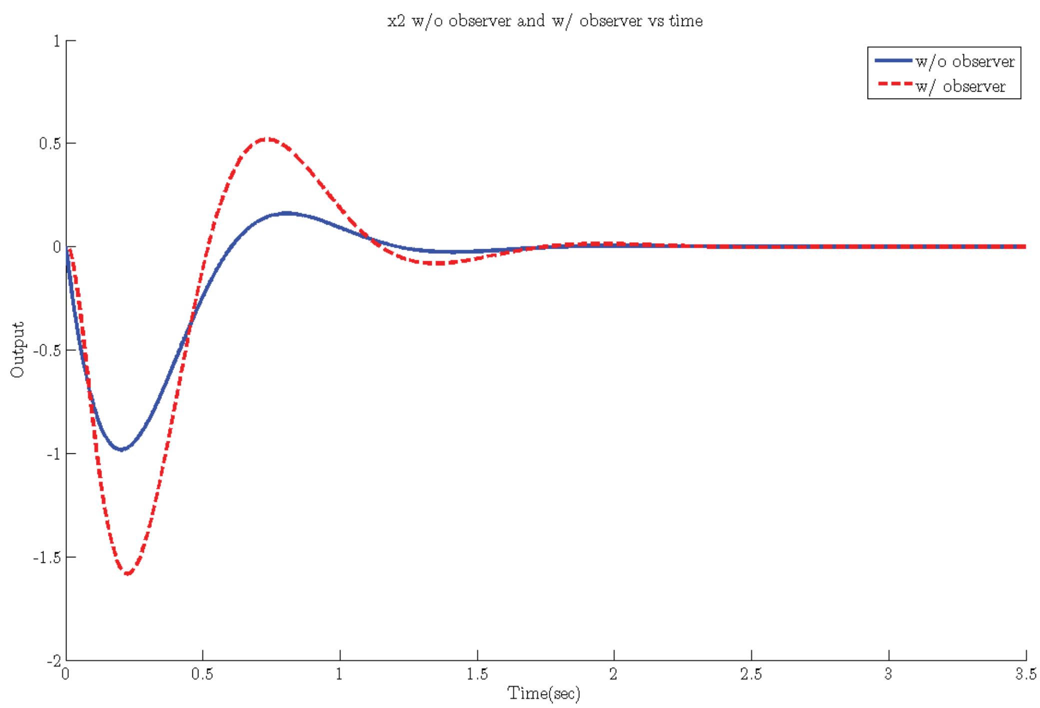


x1 and x1 hat vs time
Output
Time(sec)
x1
x1 hat
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0
0.5
1
1.5
2
2.5
3
3.5



x2 and x2 hat vs time
Output
Time(sec)
x2
x2 hat
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5
3
3.5


x1 w/o observer and w/ observer vs time
w/o observer
w/ observer
Output
Time(sec)
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0
0.5
1
1.5
2
2.5
3
3.5


x2 w/o observer and w/ observer vs time
w/o observer
w/ observer
Output
Time(sec)
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5
3
3.5

Prob 5 ...

5

A satisfies.

$$A^2+2A+3=0$$

$$\Rightarrow A^3+2A^2+3A=0.$$

$$\Rightarrow A^3 = -2A^2-3A \quad -(1).$$

$$\Rightarrow A^4 = -2A^3-3A^2$$

$$\Rightarrow A^4 = -2(-2A^2-3A) -3A^2$$

$$\boxed{A^4 = A^2+6A} \quad -(2).$$

Subs from (1) and (2) in.

$A^4+3A^3+2A^2+A+2I$ we get,

$$(A^2+6A)+3(-2A^2-3A)+2A^2+A+2I.$$

$$=(-3A^2-2A+2I)$$

= 4A + 11 I (using A^2 + 2A + 3I = 0)