EECS Homework Assignment 4 **Spring 2004**

4-5. Three charges. The force on one of the charges due to the other two (identical each having a charge Q) at the corners of an equilateral triangle of side a is given by

$$F = 2\left(\frac{kQ^2}{a^2}\right)\cos 30^\circ = \frac{\sqrt{3}kQ^2}{a^2}$$

4-13. Two straight-line charges. (a) With $\rho_l = 100\,\text{nC}/(1\text{ m}) = 10^{-7}\text{ C-m}^{-1}$ the potential at point P is given by

$$\Phi_{\text{P}} = \frac{1}{4\pi\epsilon_0}\left[\underbrace{\int_0^1 \frac{\rho_l\,dy}{\sqrt{(0.5)^2+y^2}}}_{\text{Vertical line}} + \underbrace{\int_1^2 \frac{\rho_l\,dx}{(x-0.5)}}_{\text{Horizontal line}}\right]$$
$$= \frac{\rho_l}{4\pi\epsilon_0}\left\{\left[\ln(y+\sqrt{0.25+y^2})\right]_0^1 + \left[\ln(x-0.5)\right]_1^2\right\}$$
$$= 9\times10^9\times10^{-7}\times\left[\ln\frac{1+\sqrt{1.25}}{\sqrt{0.25}} + \ln\frac{1.5}{0.5}\right] \simeq 2.29\times10^3\text{ V}$$

(b)

$$\mathbf{E}_{\text{P}} = \frac{1}{4\pi\epsilon_0}\left[\int_0^1 \frac{(0.5\hat{\mathbf{x}} - y\hat{\mathbf{y}})\rho_l\,dy}{[(0.5)^2+y^2]^{3/2}} + \int_1^2 \frac{(0.5-x)\hat{\mathbf{x}}\,dx}{(x-0.5)^3}\right]$$
$$= \frac{\rho_l}{4\pi\epsilon_0}\left[\hat{\mathbf{x}}\int_0^1 \frac{0.5\,dy}{(y^2+0.25)^{3/2}} - \hat{\mathbf{y}}\int_0^1 \frac{y\,dy}{(y^2+0.25)^{3/2}} - \hat{\mathbf{x}}\int_1^2 \frac{dx}{(x-0.5)^2}\right]$$
$$= \frac{\rho_l}{4\pi\epsilon_0}\left[\hat{\mathbf{x}}\frac{4}{\sqrt{5}} - \hat{\mathbf{y}}\left(2-\frac{2}{\sqrt{5}}\right) - \hat{\mathbf{x}}\frac{4}{3}\right]$$
$$\simeq 10^{-7}\times9\times10^9\times\left[\left(\frac{4}{\sqrt{5}}-\frac{4}{3}\right)\hat{\mathbf{x}} + \left(\frac{2}{\sqrt{5}}-2\right)\hat{\mathbf{y}}\right]$$
$$\simeq (410\,\hat{\mathbf{x}} - 995\,\hat{\mathbf{y}})\ \text{V-m}^{-1}$$

4-13. Two straight-line charges. (a) With $\rho_l = 100\,\text{nC}/(1\text{ m}) = 10^{-7}\text{ C-m}^{-1}$ the potential at point P is given by

$$\Phi_{\text{P}} = \frac{1}{4\pi\epsilon_0}\left[\underbrace{\int_0^1 \frac{\rho_l\,dy}{\sqrt{(0.5)^2+y^2}}}_{\text{Vertical line}} + \underbrace{\int_1^2 \frac{\rho_l\,dx}{(x-0.5)}}_{\text{Horizontal line}}\right]$$
$$= \frac{\rho_l}{4\pi\epsilon_0}\left\{\left[\ln(y+\sqrt{0.25+y^2})\right]_0^1 + \left[\ln(x-0.5)\right]_1^2\right\}$$
$$= 9\times10^9\times10^{-7}\times\left[\ln\frac{1+\sqrt{1.25}}{\sqrt{0.25}} + \ln\frac{1.5}{0.5}\right] \simeq 2.29\times10^3\text{ V}$$

(b)

$$
\begin{aligned}
\mathbf{E}_\mathrm{P} &= \frac{1}{4\pi\epsilon_0}\left[\int_0^1 \frac{(0.5\hat{\mathbf{x}} - y\hat{\mathbf{y}})\rho_l\, dy}{[(0.5)^2 + y^2]^{3/2}} + \int_1^2 \frac{(0.5 - x)\hat{\mathbf{x}}\, dx}{(x - 0.5)^3}\right] \\
&= \frac{\rho_l}{4\pi\epsilon_0}\left[\hat{\mathbf{x}}\int_0^1 \frac{0.5\, dy}{(y^2 + 0.25)^{3/2}} - \hat{\mathbf{y}}\int_0^1 \frac{y\, dy}{(y^2 + 0.25)^{3/2}} - \hat{\mathbf{x}}\int_1^2 \frac{dx}{(x - 0.5)^2}\right] \\
&= \frac{\rho_l}{4\pi\epsilon_0}\left[\hat{\mathbf{x}}\frac{4}{\sqrt{5}} - \hat{\mathbf{y}}\left(2 - \frac{2}{\sqrt{5}}\right) - \hat{\mathbf{x}}\frac{4}{3}\right] \\
&\simeq 10^{-7} \times 9 \times 10^9 \times \left[\left(\frac{4}{\sqrt{5}} - \frac{4}{3}\right)\hat{\mathbf{x}} + \left(\frac{2}{\sqrt{5}} - 2\right)\hat{\mathbf{y}}\right] \\
&\simeq (410\,\hat{\mathbf{x}} - 995\,\hat{\mathbf{y}})\ \text{V-m}^{-1}
\end{aligned}
$$

4-17. Semicircular line charge. (a) We choose an elemental line charge element $\rho_l dl' = a d\phi$ along the semicircle. The electric field at the origin is given by [4.14], namely

$$
\mathbf{E}(0,0,0) = \int_C \frac{1}{4\pi\epsilon_0}\frac{(\mathbf{r} - \mathbf{r}'\rho_l(\mathbf{r}')dl'}{|\mathbf{r} - \mathbf{r}'|^3}
$$

where $\mathbf{r} = 0$, $\rho_l(\mathbf{r}') = \rho_l = \text{const.}$, $dl' = a d\phi'$, $(\mathbf{r} - \mathbf{r}') = -(a\cos\phi'\,\hat{\mathbf{x}} + a\sin\phi'\,\hat{\mathbf{y}})$, and $|\mathbf{r} - \mathbf{r}'|^3 = a^3$. Thus,

$$
\begin{aligned}
\mathbf{E}(0,0,0) &= \frac{\rho_l}{4\pi\epsilon_0}\int_0^\pi \frac{-(a\cos\phi'\,\hat{\mathbf{x}} + a\sin\phi'\,\hat{\mathbf{y}}) a d\phi'}{a^3} \\
&= \frac{-\rho_l\,\hat{\mathbf{y}}}{4\pi\epsilon_0 a}\int_0^\pi \sin\phi' d\phi' = -\frac{\rho_l}{2\pi\epsilon_0 a}\,\hat{\mathbf{y}}
\end{aligned}
$$

(b) With $\rho_l(\mathbf{r}') = \rho_0 \sin\phi'$, we have

$$
\begin{aligned}
\mathbf{E}(0,0,0) &= \frac{\rho_l}{4\pi\epsilon_0}\int_0^\pi \frac{-(a\cos\phi'\,\hat{\mathbf{x}} + a\sin\phi'\,\hat{\mathbf{y}})(\rho_0 \sin\phi') a d\phi'}{a^3} \\
&= \frac{-\rho_0}{4\pi\epsilon_0 a}\left[\underbrace{\int_0^\pi (\cos\phi' \sin\phi'\,\hat{\mathbf{x}} d\phi'}_{=0} + \int_0^\pi \sin^2\phi'\,\hat{\mathbf{y}}\, d\phi'\right] \\
&= \frac{-\rho_0}{4\pi\epsilon_0 a}\left(\frac{\pi}{2}\hat{\mathbf{y}}\right) = \frac{-\rho_0}{8\epsilon_0 a}\,\hat{\mathbf{y}}
\end{aligned}
$$

4-20. Spherical charge distribution. (a) The total charge Q in the spherical region $0 < r < a$ is given by

$$Q = \int_{V'} \rho(r')dv' = K \int_0^{2\pi} \int_0^{\pi} \int_0^{a} e^{-br'} r'^2 \sin\theta' dr' d\theta' d\phi'$$
$$= K(2\pi)(2) \left[\frac{2}{b^3} - e^{-ba} \left(\frac{a^2}{b} + \frac{2a}{b^2} + \frac{2}{b^3} \right) \right]$$

where for the r' integral, we used

$$\int u^2 e^{\alpha u} du = e^{\alpha u} \left[\frac{u^2}{\alpha} - \frac{2u}{\alpha^2} + \frac{2}{\alpha^3} \right]$$

which can easily be shown using integration by parts twice.

(b) Since there is spherical symmetry, we consider a spherical Gaussian surface S with radius r and apply Gauss's law for the cases when $r < a$ and $r > a$ with the result of part (a) as

$$\int_S \epsilon_0 \mathbf{E} \cdot ds = \epsilon_0 E_r (4\pi r^2) = Q_{\text{enc}} = (4\pi K) \left[-\frac{r^2}{b} - \frac{2r}{b^2} - \frac{2}{b^3} \right]$$

resulting in

$$E_r = \begin{cases} \dfrac{K}{\epsilon_0 r^2} \left[\dfrac{2}{b^3} - e^{-br} \left(\dfrac{r^2}{b} + \dfrac{2r}{b^2} + \dfrac{2}{b^3} \right) \right] & r \leq a \\ \dfrac{K}{\epsilon_0 r^2} \left[\dfrac{2}{b^3} - e^{-ba} \left(\dfrac{a^2}{b} + \dfrac{2a}{b^2} + \dfrac{2}{b^3} \right) \right] & r > a \end{cases}$$

(c) The electric potential can simple be found by integrating the electric field, namely

$$\Phi(r) = -\int_{\infty}^{2} E_r(r) dr$$

For $r > a$ we simply find

$$\Phi(r) = -\frac{K}{\epsilon_0} \left[\frac{2}{b^3} - e^{-ba} \left(\frac{a^2}{b} + \frac{2a}{b^2} + \frac{2}{b^3} \right) \right] \int_{\infty}^{2} \frac{dr}{r^2}$$
$$= \frac{K}{\epsilon_0 r} \left[\frac{2}{b^3} - e^{-ba} \left(\frac{a^2}{b} + \frac{2a}{b^2} + \frac{2}{b^3} \right) \right]$$

For $r \leq a$ we have

$$\Phi(r) = \frac{K}{\epsilon_0 r} \left(\frac{2}{b^3} \right) + \left(\frac{K}{\epsilon_0 b^2} \right) e^{-br} + \frac{2K}{\epsilon_0 b} \int_{\infty}^{r} \frac{e^{-br}}{r} dr + \frac{2K}{\epsilon_0 b^3} \int_{\infty}^{r} \frac{e^{-br}}{r^2} dr$$

where the last two terms are the so-called Exponential Integrals for which there is no closed form solution but which are well tabulated†.

(d) The result can be shown by simple substitution into the cylindrical coordinate version of Poisson's equation, namely

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}}_{\text{No variation in } \phi} + \underbrace{\frac{\partial^2\Phi}{\partial z^2}}_{\text{No variation in } z} = \frac{-\rho(r)}{\epsilon_0}$$

$$\longrightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = \frac{-Ke^{-br}}{\epsilon_0}$$

Note that differentiation of the Exponential Integral terms simply yield the integrands.

† See Chapter 5 of M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs, and mathematical Tables*, National Bureau of Standards, Tenth Printing, 1972.

4-22. Spherical shell of charge. (a) The total charge in the spherical shell region specified by $a \leq r \leq b$ is given by

$$Q = \int_{V'} \rho(r') dv' = \int_0^{2\pi} \int_0^{\pi} \int_a^b \rho(r') r'^2 \sin\theta' dr' d\theta' d\phi'$$
$$= (2\pi)(2) \int_a^b \frac{K}{r'^2} r'^2 dr' = 4\pi K \int_a^b dr' = 4\pi K(b-a)$$

(b) Due to spherical symmetry, the electric field has the form $\mathbf{E} = \hat{\mathbf{r}} E_r(r)$. We consider a spherical Gaussian surface with radius r and apply Gauss's law:

$$\oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{s} = \epsilon_0 E_r (4\pi r^2) = Q_{\text{enc}} = \begin{cases} 0 & r < a \\ 4\pi K(r-a) & a \leq r \leq b \\ 4\pi K(b-a) & r > b \end{cases}$$

from which the electric field can be found as

$$E_r(r) = \begin{cases} 0 & r < a \\ \dfrac{K(r-a)}{\epsilon_0 r^2} & a \leq r \leq b \\ \dfrac{K(b-a)}{\epsilon_0 r^2} & r > b \end{cases}$$

(c) The electric potential Φ can be evaluated from the electric field as

$$\Phi(r) = -\int_\infty^r \mathbf{E} \cdot d\mathbf{l}$$

For $r > b$, this integral results in

$$\Phi(r) = -\int_\infty^{r>b} \frac{K(b-a)}{\epsilon_0 r^2} dr = \left. \frac{K(b-a)}{\epsilon_0 r} \right|_\infty^{r>b} = \frac{K(b-a)}{\epsilon_0 r}$$

For $a \leq r \leq b$, this integral yields

$$\Phi(r) = -\int_\infty^b \frac{K(b-a)}{\epsilon_0 r^2} dr - \int_b^r \frac{K(r-a)}{\epsilon_0 r^2} dr$$
$$= \frac{K(b-a)}{\epsilon_0 b} - \left[\frac{K}{\epsilon_0} \ln r + \frac{Ka}{\epsilon_0} \frac{1}{r} \right]_b^r$$
$$= \frac{K(b-a)}{\epsilon_0 b} - \frac{K}{\epsilon_0} \ln \frac{r}{b} + \frac{Ka}{\epsilon_0} \left[\frac{1}{b} - \frac{1}{r} \right]$$

For $r < a$, the electric field is zero, which means that the potential is constant, equal to the value it has at $r = a$, namely,

$$\Phi(a) = \frac{K(b-a)}{\epsilon_0 b} - \frac{K}{\epsilon_0} \ln \frac{a}{b} + \frac{Ka}{\epsilon_0} \left[\frac{1}{b} - \frac{1}{a} \right]$$

(d) When $b \to a$, all of the charge resides on a spherical surface of radius $b = a$. Thus, the electric field inside (i.e., $r < a$) is zero, while that outside (i.e., $r > a$) is identical to that due to a point charge.