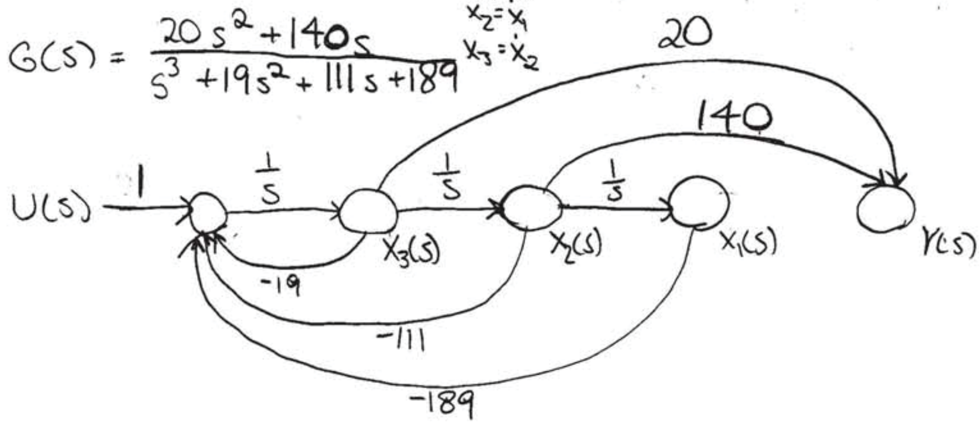
1. $G(s) = \dfrac{20\, s(s+7)}{(s+3)(s+7)(s+9)} = \dfrac{Y(s)}{U(s)}$

a. $G(s) = \dfrac{20 s^2 + 140 s}{s^3 + 19 s^2 + 111 s + 189}$

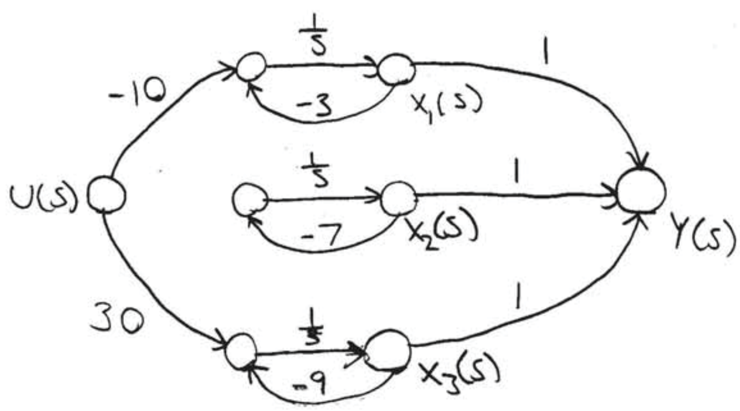
$x_1 = x_1$
$x_2 = \dot{x}_1$
$x_3 = \dot{x}_2$



$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -189 & -111 & -19 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 140 & 20 \end{bmatrix} x$$

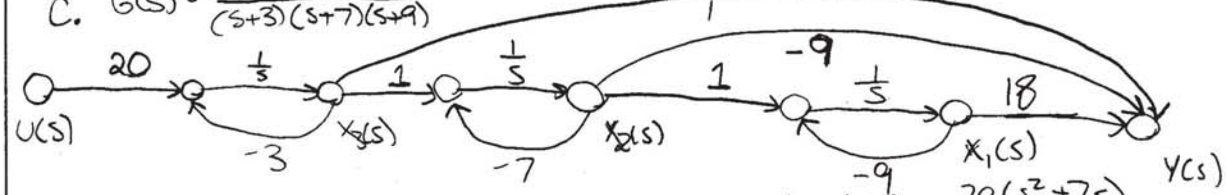
b. $G(s) = \dfrac{20\, s(s+7)}{(s+3)(s+7)(s+9)} = \dfrac{-10}{s+3} + \dfrac{0}{s+7} + \dfrac{30}{s+9}$



$$\dot{x} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -9 \end{bmatrix} + \begin{bmatrix} -10 \\ 0 \\ 30 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

c. $G(s) = \dfrac{20(s^2+7s)}{(s+3)(s+7)(s+9)}$



$$\dot{x} = \begin{bmatrix} -9 & 1 & 0 \\ 0 & -7 & 1 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} u$$

$$y = \begin{bmatrix} 18 & -9 & 1 \end{bmatrix} x$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{20(s^2+7s)}{(s+3)(s+7)(s+9)}$$

$$Y(s) = (s^2+7s)X(s)$$

$$= \ddot{x}_1 + 7\dot{x}_1$$

$$= 81x_1 - 16x_2 + x_3 + 7(-9x_1 + x_2)$$

$$\boxed{y(s) = 18x_1 - 9x_2 + x_3}$$

$$\dot{x}_1 = -9x_1 + x_2$$

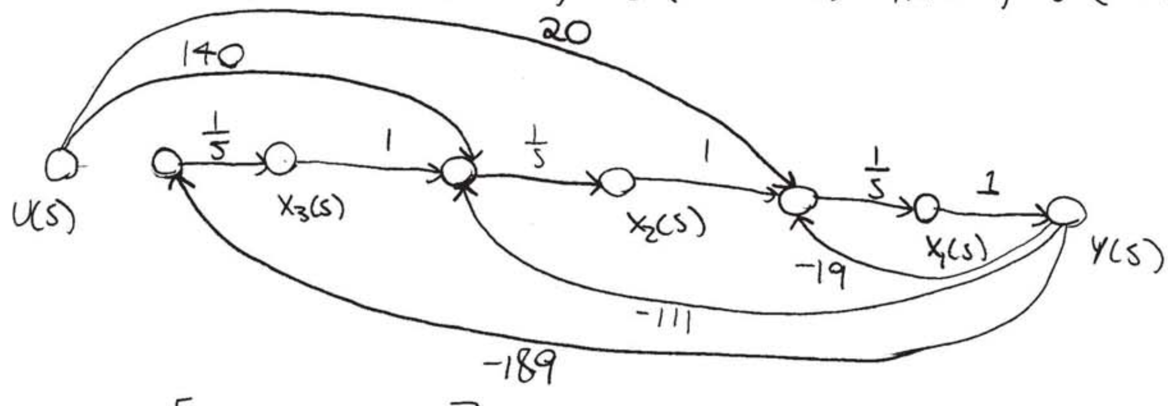
$$\ddot{x}_1 = -9\dot{x}_1 + \dot{x}_2 = -9(-9x_1 + x_2) + (-7x_2 + x_3)$$

$$= 81x_1 - 16x_2 + \dot{x}_3$$

d. $G(s) = \dfrac{\frac{20}{s} + \frac{140}{s^2}}{1 + \frac{19}{s} + \frac{111}{s^2} + \frac{189}{s^3}}$ (divided by s^3 on top and bottom)

$$\left(\frac{20}{s} + \frac{140}{s^2}\right)U(s) = \left(1 + \frac{19}{s} + \frac{111}{s^2} + \frac{189}{s^3}\right)Y(s)$$

$$Y(s) = \frac{1}{s}\left(20U(s) - 19Y(s)\right) + \frac{1}{s^2}\left(140U(s) - 111Y(s)\right) + \frac{1}{s^3}\left(-189Y(s)\right)$$



$$\dot{x} = \begin{bmatrix} -19 & 1 & 0 \\ -111 & 0 & 1 \\ -189 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 20 \\ 140 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Prob 2 starts.

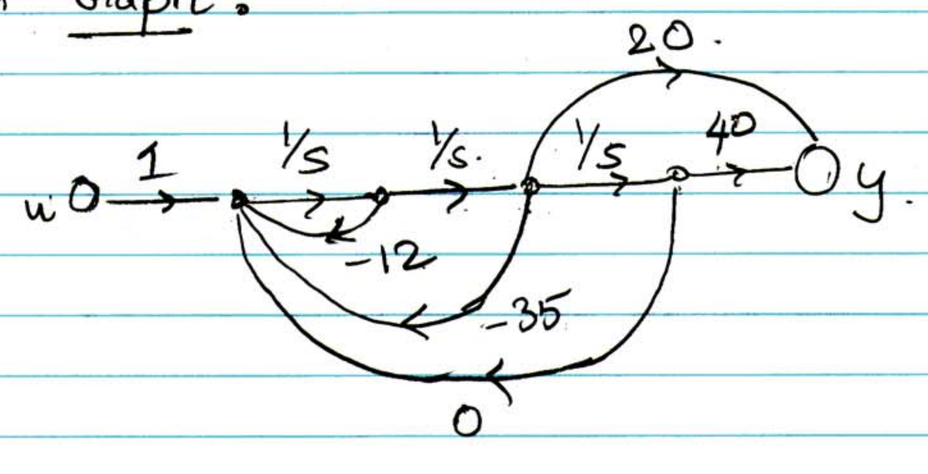
(2) $G(s) = \dfrac{20(s+2)}{s(s+5)(s+7)}$.

(a) $G(s) = \dfrac{20s+40}{s(s^2+12s+35)} = \dfrac{20s+40}{(s^3+12s^2+35s)}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -35 & -12 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 40 & 20 & 0 \end{bmatrix} x.$$

Signal Graph:



(b) $OS\% = 10\%$

$\Rightarrow 0.10 = e^{-\xi\pi/\sqrt{1-\xi^2}}$

$\Rightarrow +\ln 10 = \xi\pi/\sqrt{1-\xi^2}$

$(\ln 10)^2 = \xi^2\pi^2/(1-\xi^2)$

$\Rightarrow (\ln 10)^2 = (\pi^2+(\ln 10)^2)\,\xi^2$

$\Rightarrow \xi^2 = (\ln 10)^2/(\pi^2+(\ln 10)^2)$

$\Rightarrow \boxed{\xi = 0.6}$

$T_s = 2 = \left(\dfrac{4}{\xi\omega_n}\right)$

$\Rightarrow \omega_n = \dfrac{4}{2\xi} = \dfrac{4}{1.2} = 3.33.$

$\Rightarrow P_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

$= \boxed{-2 \pm j\,2.66}$

Choose $P_3 = 10\,(\,Re\,(P_{1,2}))$

$= \boxed{-20.}$

<u>Characteristic polynomial:</u> $(s-P_1)(s-P_2)(s-P_3)$

$$= (s+2+j2.66)(s+2-j2.66)(s+20)$$

$$= s^3 + \underbrace{24}_{(a_2)} s + \underbrace{91.08}_{(a_1)} s + \underbrace{221.51}_{(a_0)} \quad — (1)$$

Let $u = -kx = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow A - Bk = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2-35 & -12-k_3 \end{bmatrix}$$

$\Rightarrow$ characteristic Polynomial:

$$s^3 + (12+k_3)s^2 + (k_2+35)s + k_1 \quad — (2)$$

equating (1) and (2) we get

$$\boxed{k_1 = 221.51, \quad k_2 = 56.08, \quad k_3 = 12}$$

3. $\dot{x} = Ax = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad x_0(t=0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

a. $\bar{A} = P^{-1}AP$

Find eigenvalues:

$$\det(\lambda I - A) = \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 24 & 26 & \lambda+9 \end{bmatrix} \right|$$

$$= \lambda^3 + 9\lambda^2 + 26\lambda + 24$$

$$\lambda = -2, -3, -4$$

Find eigenvectors:

$\underline{\lambda = -2}$

$Av_1 = \lambda v_1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$v_2 = -2v_1$

$v_3 = -2v_2$

$-24v_1 - 26v_2 - 9v_3 = -2v_3$

$\Rightarrow$

$v_1 = \frac{1}{4}v_3$

$v_2 = -\frac{1}{2}v_3$

$\left[-24\left(\frac{1}{4}v_3\right) - 26\left(-\frac{1}{2}v_3\right) - 9v_3 = -2v_3 \right.$

$-2v_3 = -2v_3$

Let $v_3 = 1$

$v_2 = -\frac{1}{2}$

$v_1 = \frac{1}{4}$

$$\lambda_1 = -2 \Rightarrow v_1 = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Using Matlab to find the rest:

$$\lambda_2 = -3 \Rightarrow v_2 = \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \qquad \lambda_3 = -4 \Rightarrow v_3 = \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{9} & \frac{1}{16} \\ -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{9} & \frac{1}{16} \\ -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 24 & 14 & 2 \\ -72 & -54 & -9 \\ 48 & 40 & 8 \end{bmatrix}$$

So:

$$\bar{A} = P^{-1} A P$$

$$\underbrace{\begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}}_{\bar{A}} = \underbrace{\begin{bmatrix} 24 & 14 & 2 \\ -72 & -54 & -9 \\ 48 & 40 & 8 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \frac{1}{4} & \frac{1}{9} & \frac{1}{16} \\ -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix}}_{P}$$

b. $e^{At} = P e^{\bar{A}t} P^{-1}$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{9} & \frac{1}{16} \\ -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 24 & 14 & 2 \\ -72 & -54 & -9 \\ 48 & 40 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 6e^{-2t} - 8e^{-3t} + 3e^{-4t} & \frac{7}{2}e^{-2t} - 6e^{-3t} + \frac{5}{2}e^{-4t} & \frac{1}{2}e^{-2t} - e^{-3t} + \frac{1}{2}e^{-4t} \\ -12e^{-2t} + 24e^{-3t} - 12e^{-4t} & -7e^{-2t} + 18e^{-3t} - 10e^{-4t} & -e^{-2t} + 3e^{-3t} - 2e^{-4t} \\ 24e^{-2t} - 72e^{-3t} + 48e^{-4t} & 14e^{-2t} - 54e^{-3t} + 40e^{-4t} & 2e^{-2t} - 9e^{-3t} + 8e^{-4t} \end{bmatrix}$$

$$x(t) = e^{At}x_0 + \int_0^t B e^{A(t-\tau)} u(\tau)\, d\tau \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So:

$$x(t) = e^{At} x_0$$

$$x(t) = e^{At} x_e$$

$$= e^{At} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \frac{13}{2} e^{-2t} - 9 e^{-3t} + \frac{7}{2} e^{-4t} \\ -13 e^{-2t} + 27 e^{-3t} - 14 e^{-4t} \\ 26 e^{-2t} - 81 e^{-3t} + 56 e^{-4t} \end{bmatrix}$$

5. $\dot{x} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} u(t)$

For which values of k_1 and k_2 is system completely controllable?

$$C_M = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} k_1 & k_1 - 2k_2 \\ k_2 & -k_1 + k_2 \end{bmatrix}$$

Full rank unless:

$$\alpha \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 - 2k_2 \\ -k_1 + k_2 \end{bmatrix}$$

$$\frac{k_1 - 2k_2}{k_1} = \frac{-k_1 + k_2}{k_2}$$

$$k_1 k_2 - 2k_2^2 = -k_1^2 + k_1 k_2$$

$$k_1^2 = 2k_2^2$$

$$|k_1| = \sqrt{2} |k_2|$$

System controllable $\forall k_1, k_2$ such that $|k_1| \neq \sqrt{2} |k_2|$

Prob 4 starts

(4) $$G(s) = \frac{1}{s(s+3)(s+7)} = \frac{1}{(s^3+10s^2+21s+0)} \quad (1)$$

(a) (1) $\Rightarrow$ $a_0 = 0$, $a_1 = -21$, $a_2 = -10$.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -21 \\ 0 & 1 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

(b) Let $L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$

$$\Rightarrow A-LC = \begin{bmatrix} 0 & 0 & -L_1 \\ 1 & 0 & -21-L_2 \\ 0 & 1 & -10-L_3 \end{bmatrix} \quad (1)$$

Desired pole locations:

$\xi = 0.4$, $\omega_n = 75 \Rightarrow P_{1,2} = -30 \pm j68.74$.

$\Rightarrow P_3 = -300.$

Desired characteristic Polynomial:

$(s-P_1)(s-P_2)(s-P_3)$

$= s^3 + 360s^2 + 23625.19s + 1687556.28.$ — ②

characteristic polynomial corresponding to

① : $s^3 + (10+L_3)s^2 + (21+L_2)s + L_1$

Desired characteristic polynomial can be achieved by setting:

$$\boxed{L_1 = 1687556.28 \; ; \; L_2 = 23604.19 \quad L_3 = 350.}$$

[by equating coefficients of ① and ②]

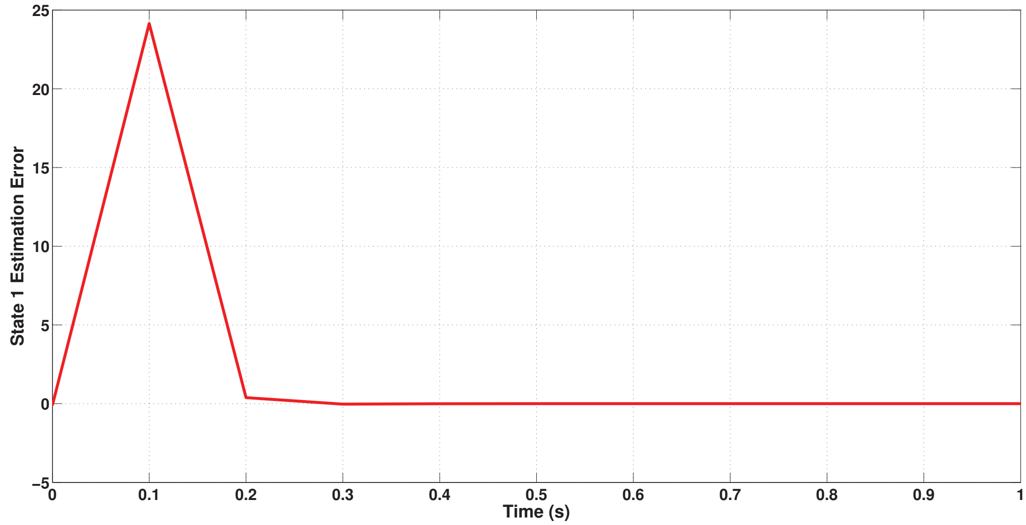


Figure 1: State 1 Estimation Error

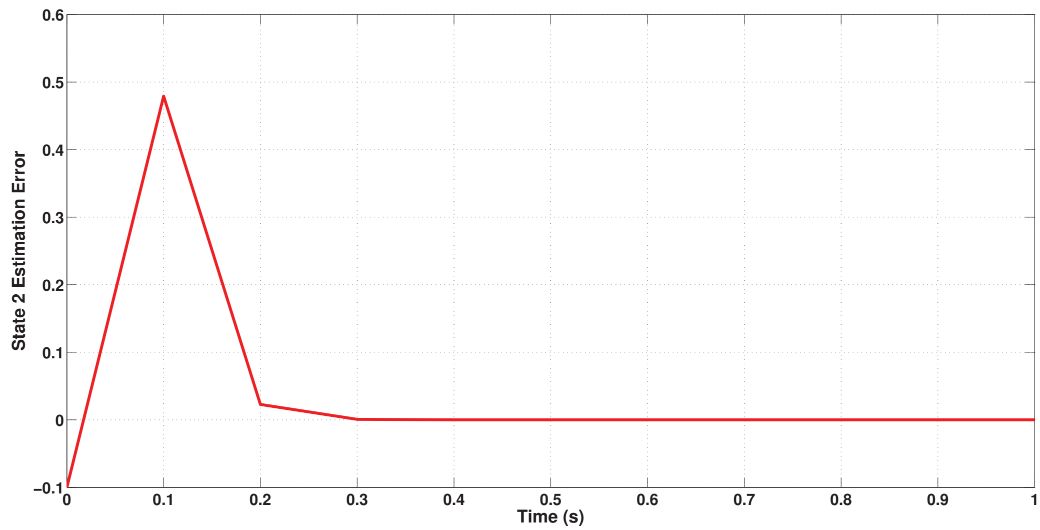


Figure 2: State 2 Estimation Error

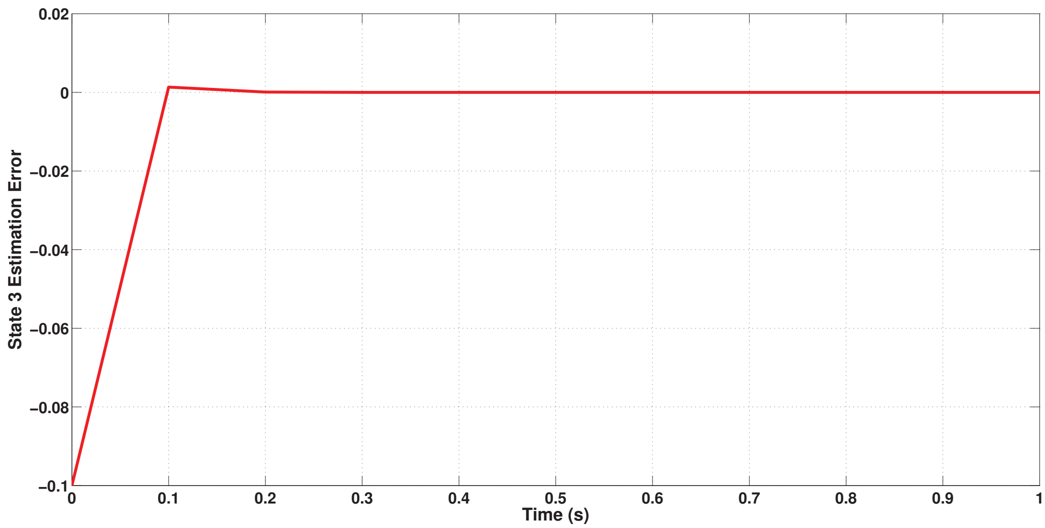


Figure 3: State 3 Estimation Error

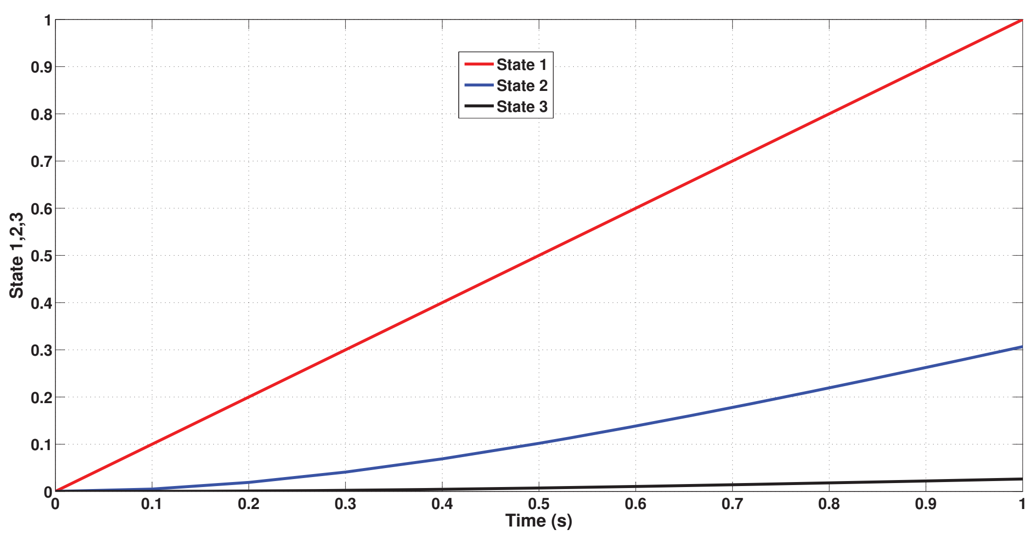


Figure 4: State Response