4-31. A parallel-plate capacitor with variable ϵ_r. We can model the variation of permittivity as

$$\epsilon_r(z) = 1 + \frac{9z}{d}$$

with z pointing away from the bottom capacitor plate and toward the top plate. Consider a cylindrical Gaussian surface in the form of a pillbox with top and bottom surface areas equal to A, and with is top surface inside the lower capacitor plate, assumed to be positively charged with a surface charge density ρ_s. From Gauss's law we have

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \rho_s A \quad \rightarrow \quad \epsilon_0 \epsilon_r(z) E(z) A = \rho_s A \quad \rightarrow \quad E(z) = \frac{\rho_s}{\epsilon_0 \left(1 + \dfrac{9z}{d}\right)}$$

where we have noted that the the electric flux density $\mathbf{D}$ is zero on the lower plate and parallel to the side surface so that the total contribution comes from only the top surface. The total charge enclosed by the closed pill-box surface is simply the surface charge density on the lower plate (i.e., ρ_s) times the area A.

The potential difference between the top (t) and bottom (b) plates of the capacitor is then given by

$$\Phi_{\text{bt}} = -\int_d^0 E(z)dz = \int_0^d E(z)dz = \int_0^d \frac{\rho_s dz}{\epsilon_0 \left(1 + \dfrac{9z}{d}\right)} = \frac{\ln(10)}{9} \frac{\rho_s d}{\epsilon_0}$$

and

$$C = \frac{A}{\Phi_{\text{bt}}} = \frac{\rho_s A}{\Phi_{\text{bt}}} = \left[\frac{9}{\ln(10)}\right] \left(\frac{\epsilon_0 A}{d}\right) = \frac{9}{\ln(10)} C_{\text{air filled}} \simeq 3.9 C_{\text{air filled}}$$

4-33. Coaxial capacitor with two dielectrics. (a) This problem can be solved in two different ways. In the first method, we can take advantage of the formula for capacitance per unit length of the coaxial line, as determined in Example 4-27, namely

$$C_{\text{coax}} = \frac{2\pi\epsilon}{\ln(b/a)}$$

where ϵ is the permittivity of the dielectric filling. The configuration shown in Figure 4.70 can be considered to be two 'half-coaxial' capacitors of length l connected in parallel, with capacitances respectively of

$$C_1 = \frac{1}{2} \frac{2\pi\epsilon_{1r} l}{\ln(b/a)} \quad \text{and} \quad C_2 = \frac{1}{2} \frac{2\pi\epsilon_{2r} l}{\ln(b/a)}$$

which combine additively (capacitors in parallel) to give the total capacitance per unit length of

$$C = \frac{\pi\epsilon_0(\epsilon_{1r} + \epsilon_{2r})l}{\ln(b/a)}$$

The same result can also be obtained using Gauss's law and noting that the electric field has to be radial (due to symmetry) and must have the same value in both dielectrics, due to the electrostatic boundary condition [4.66] dictating the continuity of the tangential component of the electric field across a dielectric boundary. Considering a cylindrical Gaussian surface of length l and radius $a < r < b$ we can then write

$$\oint \mathbf{D} \cdot d\mathbf{s} = \int_0^l \int_0^\pi \epsilon_0 \epsilon_{1r} E_r r d\phi dz + \int_0^l \int_\pi^{2\pi} \epsilon_0 \epsilon_{2r} E_r r d\phi dz = \rho_l l$$

$$\rightarrow \quad \pi r l(\epsilon_{1r} + \epsilon_{2r})\epsilon_0 E_r = \rho_l l \quad \rightarrow \quad E_r = \frac{\rho_l}{\pi r \epsilon_0(\epsilon_{1r} + \epsilon_{2r})}$$

This electric field is radially outward if the inner conductor is positively charged, so that $\rho_l > 0$. The potential difference between the inner and outer cylinders of the coaxial line is then

$$\Phi_{ba} = \Phi(a) - \Phi(b) = -\int_b^a E_r dr = \int_b^a \frac{\rho_l}{\pi \epsilon_0(\epsilon_{1r} + \epsilon_{2r})} \frac{dr}{r} = \frac{\rho_l}{\pi \epsilon_0(\epsilon_{1r} + \epsilon_{2r})} \ln\left(\frac{b}{a}\right)$$

so that we have

$$C = \frac{Q}{\Phi_{ba}} = \frac{\rho_l l}{\Phi_{ba}} = \frac{\pi \epsilon_0(\epsilon_{1r} + \epsilon_{2r})l}{\ln(b/a)}$$

(b) From Table 4.1 we have $\epsilon_{1r} = 2.3$ for oil and $\epsilon_{2r} = 5.4$ for mica, which gives

$$C = \frac{\pi(8.85 \times 10^{-12})(2.3 + 5.4)(5 \times 10^{-2})}{\ln(3)} \simeq 9.75 \text{ nF}$$

(c) With only oil used throughout we have $\epsilon_{1r} = \epsilon_{2r} = 2.3$ so that

$$C = \frac{\pi(8.85 \times 10^{-12})(2.3 + 2.3)(5 \times 10^{-2})}{\ln(3)} \simeq 5.82 \text{ nF}$$

4-37. Planar charge. We test the validity of the different potential functions by requiring that

$$\nabla^2 \Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \Phi = 0 \quad \text{for} \quad |y| > 0 \ \text{(i.e.,}\rho = 0)$$

We have

$$\nabla^2 \Phi_1 = e^{-y} \cosh x + \cosh x \, e^{-y} = 2 \cosh x e^{-y} \neq 0$$
$$\nabla^2 \Phi_3 = -2 \sin(2x) e^{y\sqrt{2}} + \sin(2x) e^{-y\sqrt{2}} = -\sin(2x) e^{-y\sqrt{2}} \neq 0$$
$$\nabla^2 \Phi_4 = -3 \sin x \sin y \sin z \neq 0$$

so that Φ_1, Φ_3 and Φ_4 are not valid solutions. We now consider Φ_2:

$$\nabla^2 \Phi_2 = -\cos x e^{-y} + e^{-y} \cos x = 0 \quad \rightarrow \quad \text{valid solution}$$

(b) The corresponding electric field is given by

$$\mathbf{E}_2(x, y, z) = -\nabla\Phi_2 = (e^{-y}\sin x)\hat{\mathbf{x}} + (e^{-y}\cos x)\hat{\mathbf{y}} \quad \text{for} \quad y > 0$$

By symmetry we must have $\mathbf{E}_2(x, -y, z) = -\mathbf{E}_2(x, y, z)$. Also, since Φ_2 and $\mathbf{E}_2$ do not depend on z, the surface charge density ρ_s also does not depend on z. Thus, we must have $\rho_s = \rho_s(x)$. To find $\rho_s(x)$, consider a Gaussian surface as shown and let $\Delta y \rightarrow 0$. Using Gauss's law, we have

$$\int_{S_1} \epsilon_0 \mathbf{E}_2 \cdot d\mathbf{s}_1 + \int_{S_2} \epsilon_0 \mathbf{E}_2 \cdot d\mathbf{s}_2 = \int_A \rho_s dA$$

where we can ignore the other sides of the cube since we let $\Delta y \rightarrow 0$. Since $d\mathbf{s}_1 = -d\mathbf{s}_2$, and $\mathbf{E}_2(x, y) = -\mathbf{E}_2(x, y)$ we have

$$\int_{S_1} \epsilon_0 \mathbf{E}_2 \cdot d\mathbf{s}_1 = \int_{S_2} \epsilon_0 \mathbf{E}_2 \cdot d\mathbf{s}_2$$

and thus

$$\int_A \rho_s(x) dx dz = 2\epsilon_0 \int_{S_1} E_{2y}(x, y = 0) dx dz$$

$$\int_{S_1} [\rho_s(x) - 2\epsilon_0 \cos x] dx dz = 0 \quad (\text{since } E_{2y}(x, y = 0) = \cos x$$

$$\rightarrow \quad \rho_s(x) = 2\epsilon_0 \cos x$$

4. At the boundary of the two perfect dielectrics, there is not free charge. So the normal component of $\boldsymbol{D}$ is continuous across the boundary, i.e., $D_{1n} = D_{2n}$. So the scalar potential is continuous across the boundary. The tangential component of $\boldsymbol{E}$ is also continuous, i.e., $E_{1t} = E_{2t}$.

5. Using Gauss's law, one can easily show that the electric field outside of the Earth is equal to $E = Q/(4\pi\varepsilon_0 r^2)\hat{r}$, where Q is the charge present on the Earth's "conducting" surface. The scalar potential is $\Phi = Q/(4\pi\varepsilon_0 r)$. The capacitance is equal to

$C = Q / \Phi(r = a) = 4\pi\varepsilon_0 a$, where a is the radius of the Earth. Evaluation of the expression yields C = 709 μF.

(b) The air typically can sustain an electric field of 3 x 10^6 V/m before breakdown. This would imply that the maximum charge that can be put onto the Earth is governed by

$$E = Q_{\max} / (4\pi\varepsilon_0 a^2) = 3\times10^6 \Rightarrow Q_{\max} = 3\times10^6 (4\pi\varepsilon_0 a^2) = 1.35\times10^{10}\, C$$

6 (a). Since there is no free charge in the dielectric regions, we have the Laplace equation to describe the electrostatic potential inside the regions. Exploiting the spherical symmetry of the problem, we have:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

Because of the symmetry, Φ only varies in r. Thus only the radial term in the above expression remains.

Thus, we have

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = 0$$

$$r^2\frac{\partial\Phi}{\partial r} = C$$

$$\frac{\partial\Phi}{\partial r} = \frac{C}{r^2}$$

$$\Phi = -\frac{C}{r} + D$$

where, C and D are constants. To account for the different dielectric constants in the two regions, we set up the following expressions:

$$\Phi_1(r) = -\frac{C_1}{r} + D_1 \qquad \text{for } R_i \le r \le R_b$$

$$\Phi_2(r) = -\frac{C_2}{r} + D_2 \qquad \text{for } R_b \le r \le R_o$$

The subscript 1 and 2 denotes the dielectric region closer to the inner shell and to the outside shell, respectively. With an assumption that the inner shell has a voltage V_0 and the outer one is grounded, the boundary conditions are:

$$\Phi_1(R_i) = V_0 \qquad (1)$$

$$\Phi_2(R_o) = 0 \qquad (2)$$

$$\Phi_1(R_b) = \Phi_1(R_b) \qquad (3)$$

$$D_{1n}(R_b) = D_{2n}(R_b) \qquad (4)$$

The last condition is true because there is no free charge on the boundary surface between the two dielectric.

Applying the conditions towards this problem, we have

$$-\frac{C_1}{R_i} + D_1 = V_0$$
$$-\frac{C_2}{R_o} + D_2 = 0$$
$$-\frac{C_1}{R_b} + D_1 = -\frac{C_2}{R_b} + D_2$$
$$-\varepsilon_1 \frac{C_1}{R_b{}^2} = -\varepsilon_2 \frac{C_2}{R_b{}^2}$$

Expressing D_1 and D_2 in terms of C_1 and C_2 using the first and second equations, and substituting the results into the third equation yield:

$$V_0 + \left(\frac{1}{R_i} - \frac{1}{R_b}\right) C_1 = \left(\frac{1}{R_o} - \frac{1}{R_b}\right) C_2$$

Using the fourth equation, one can find

$$C_1 = \frac{V_0}{\frac{\varepsilon_1}{\varepsilon_2}\frac{1}{R_o} + \left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right)\frac{1}{R_b} - \frac{1}{R_i}}$$

Therefore,

$$\Phi_1 = -\frac{C_1}{r} + D_1 = -\frac{C_1}{r} + V_0 + \frac{C_1}{R_i} = V_0 + \frac{V_0}{\frac{\varepsilon_1}{\varepsilon_2}\frac{1}{R_o} + \left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right)\frac{1}{R_b} - \frac{1}{R_i}}\left(\frac{1}{R_i} - \frac{1}{r}\right)$$

$$\Phi_2 = -\frac{C_2}{r} + D_2 = -\frac{\varepsilon_1}{\varepsilon_2}\frac{C_1}{r} + \frac{\varepsilon_1}{\varepsilon_2}\frac{C_1}{R_o} = \frac{\varepsilon_1}{\varepsilon_2}\frac{V_0}{\frac{\varepsilon_1}{\varepsilon_2}\frac{1}{R_o} + \left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right)\frac{1}{R_b} - \frac{1}{R_i}}\left(\frac{1}{R_0} - \frac{1}{r}\right)$$

Then, use the fact that $\varepsilon_2 = 2\,\varepsilon_1 = 2\,\varepsilon_r$, we get

$$E_{1r}\,\hat{r} = -\nabla\Phi_1 = \frac{V_0}{\frac{1}{R_i} - \frac{1}{2R_o} - \frac{1}{2R_b}}\frac{1}{r^2}\hat{r} \text{ and } D_r\,\hat{r} = \frac{\varepsilon_r V_0}{\frac{1}{R_i} - \frac{1}{2R_o} - \frac{1}{2R_b}}\frac{1}{r^2}\hat{r}$$

$$E_{2r}\hat{r} = -\nabla\Phi_1 = \frac{V_0}{\frac{2}{R_i} - \frac{1}{R_o} - \frac{1}{R_b}}\frac{1}{r^2}\hat{r} \text{ and } D_r\hat{r} = \frac{\varepsilon_r V_0}{\frac{1}{R_i} - \frac{1}{2R_o} - \frac{1}{2R_b}}\frac{1}{r^2}\hat{r}$$

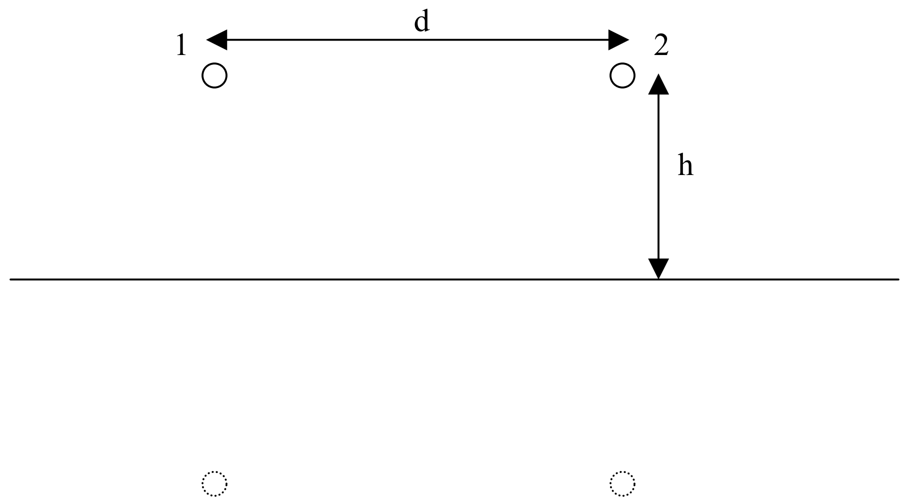
(b) The surface density of the inner shell is given by

$$\rho_{si} = \hat{r} \cdot D_{1r}(r = R_i)\hat{r} = \frac{\varepsilon_r V_0}{\frac{1}{R_i} - \frac{1}{2R_o} - \frac{1}{2R_b}}\frac{1}{R_i^2}$$

The capacitance is then equal to

$$C = \frac{Q}{V_0} = \frac{4\pi R_i^2 \rho_{si}}{V_0} = \frac{4\pi\varepsilon_r}{\frac{1}{R_i} - \frac{1}{2R_o} - \frac{1}{2R_b}}$$

7. The problem can be analyzed using method of image, namely the ground plane can be replaced by two image wires distance h below the ground.



In general, the voltage on the surface of the wire 1 is given by

$$V_1 = V_{11} + V_{12} = P_{11}Q_1 + P_{12}Q_2 \qquad (1)$$

V_{11} is the voltage on the surface of the wire 1 due to a charge Q_1 in the wire 1 (and $-Q_1$ in its image wire), and V_{12} is the voltage of wire 1 due to a charge Q_2 in the wire 2 (and $-Q_2$ in its image wire).

To find V_{11}, we neglect wire 2 and calculate the potential on the wire surface due to a charge Q_1 inside the wire and an image charge $-Q_1$. This potential is same as the one due to a dipole evaluated on the surface of the positive charge.

$$V_{11} = \frac{Q_1}{4\pi\varepsilon_0 a} - \frac{Q_1}{8\pi\varepsilon_0 h}$$

or

$$P_{11} = \frac{1}{4\pi\varepsilon_0 a} - \frac{1}{8\pi\varepsilon_0 h}$$

Likewise, for wire 2, we have

$$V_2 = V_{22} + V_{21} = P_{22}Q_2 + P_{21}Q_2 \qquad (2)$$

Because of the symmetry, $P_{11} = P_{22}$, and $P_{12} = P_{21}$.

To calculate P_{21} (and thus P_{12}), we need find V_{21}, the voltage on the surface of wire 2 due to a charge Q_1 inside wire 1 (and -Q_1 in its image wire). The potential is

$$V_{21} = \frac{Q_1}{4\pi\varepsilon_0 d} - \frac{Q_1}{4\pi\varepsilon_0 (4h^2 + d^2)^{1/2}}$$

P_{21} and P_{12} are equal to

$$P_{21} = P_{12} = \frac{1}{4\pi\varepsilon_0 d} - \frac{1}{4\pi\varepsilon_0 (4h^2 + d^2)^{1/2}}$$

The equation (1) and (2) can be expressed as V = P Q, or $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$. We can express the Q vector in terms of V vector by inverting P matrix, i.e.

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{P_{11}P_{22} - P_{21}P_{12}} \begin{bmatrix} P_{22} & -P_{12} \\ -P_{21} & P_{11} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{P_{11}^{\ 2} - P_{21}^{\ 2}} \begin{bmatrix} P_{11} & -P_{12} \\ -P_{12} & P_{11} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

In the equation form, we have

$$Q_1 = c_{11}V_1 + c_{12}V_2$$
$$Q_2 = c_{12}V_1 + c_{11}V_2$$

where $c_{11} = {}^{P_{11}}\!/_{(P_{11}^{\ 2} - P_{21}^{\ 2})}$, $c_{12} = -{}^{P_{12}}\!/_{(P_{11}^{\ 2} - P_{21}^{\ 2})}$. These two equations can be expressed by self capacitance C_{11} and mutual capacitance C_{12}, such that

$$Q_1 = c_{11}V_1 + c_{12}V_2 = C_{11}V_1 + C_{12}(V_1 - V_2)$$
$$Q_2 = c_{12}V_1 + c_{11}V_2 = C_{11}V_2 + C_{12}(V_2 - V_1)$$

One can see that $C_{11} = c_{11} + c_{12}$, and $C_{12} = -\, c_{12}$. So

$$C_{11} = 4\pi\varepsilon_0 \frac{\frac{1}{a} - \frac{1}{2h} - \frac{1}{d} + \frac{1}{\sqrt{4h^2+d^2}}}{\left(\frac{1}{a} - \frac{1}{2h}\right)^2 - \left(\frac{1}{d} - \frac{1}{\sqrt{4h^2+d^2}}\right)^2} \text{ and } C_{12} = 4\pi\varepsilon_0 \frac{\frac{1}{d} - \frac{1}{\sqrt{4h^2+d^2}}}{\left(\frac{1}{a} - \frac{1}{2h}\right)^2 - \left(\frac{1}{d} - \frac{1}{\sqrt{4h^2+d^2}}\right)^2}.$$

8(a). In both dielectric-filled and vacuum-filled capacitor areas, the electric field is equal to $-V_0/d\,\hat{j}$, where y axis is pointing up normal to the capacitor plates. But in the dielectric region, $\boldsymbol{D} = \varepsilon_r \boldsymbol{E} = -\varepsilon_r V_0/d\,\hat{j}$ and $\rho_s = -\varepsilon_r V_0/d$. The vacuum region has $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} = -\varepsilon_0 V_0/d\,\hat{j}$ and $\rho_s = -\varepsilon_0 V_0/d$.

(b). The electrostatic energy U_d stored inside the dielectric region is equal to $U_d = \frac{\varepsilon_r |E|^2}{2} xdw = \frac{\varepsilon_r V_0^{\ 2}}{2d} xw$. By the same token, The electrostatic energy U_d stored inside the vacuum region is described by $U_d = \frac{\varepsilon_0 |E|^2}{2}(l-x)dw = \frac{\varepsilon_0 V_0^{\ 2}}{2d}(l-x)w$. The value of x for equal amount of energy stored in each region is given by $(l-x)/x = \varepsilon_r/\varepsilon_0$, or $x = l\varepsilon_0/(\varepsilon_0+\varepsilon_r)$.