

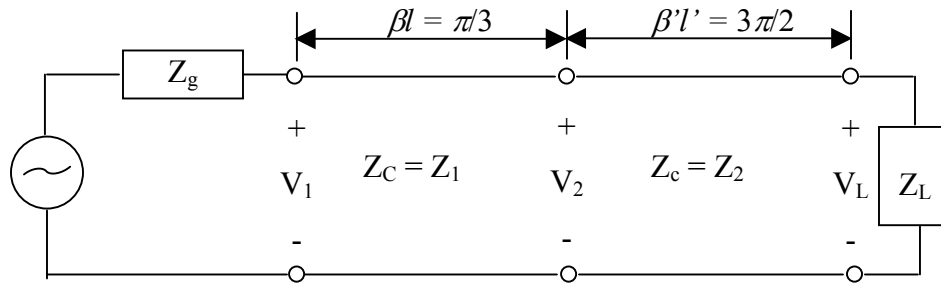
1(a). To find the voltage in each transmission line, we need to calculate the V_1 and propagate it along the transmission lines (figure (a).)

We first calculate the equivalent input impedance looking down from the end of transmission line 1 (figure (b).)

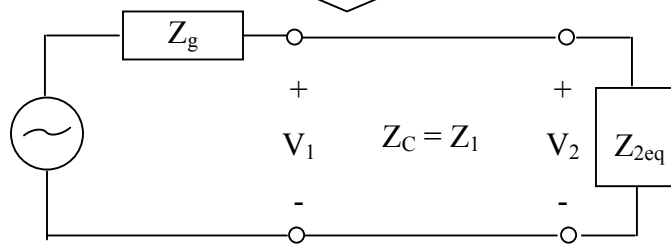
$$Z_{2eq} = Z_2 \frac{Z_L \cos \beta' l' + j Z_2 \sin \beta' l'}{Z_2 \cos \beta' l' + j Z_L \sin \beta' l'} = Z_2 \frac{j Z_2}{j Z_L} = \frac{Z_2^2}{Z_L} = \frac{Z_1^2}{9 Z_L}$$

$$\text{where } Z_L = R // 1/j\omega C = \frac{R / j\omega C}{R + 1/j\omega C} = \frac{R}{1 + j\omega RC} = \frac{100\Omega}{1 + j(2\pi * 10^9 \text{ Hz} * 100\Omega * 10^{-11} \text{ F})}$$

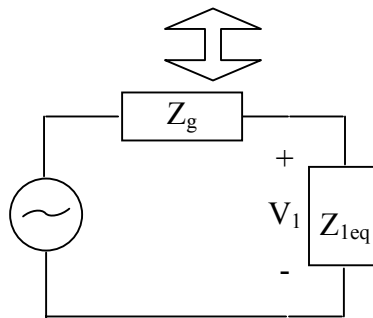
$$= 2.47 - 15.52j$$



(a)



(b)



(c)

Figure (b) can be simplified to become figure (c), where Z_{1eq} is the equivalent impedance looking from the beginning of the transmission line 1.

$$Z_{1eq} = Z_1 \frac{Z_{2eq} \cos \beta l + j Z_1 \sin \beta l}{Z_1 \cos \beta l + j Z_{2eq} \sin \beta l} = Z_1 \frac{Z_{2eq} / 2 + j \sqrt{3} / 2 Z_1}{Z_1 / 2 + j \sqrt{3} / 2 Z_{2eq}} = Z_1 \frac{Z_1^2 + j 9 \sqrt{3} Z_1 Z_L}{9 Z_1 Z_L + j \sqrt{3} Z_1^2}$$

$$= 55.3 - 425j$$

Then,

$$V_1 = \frac{Z_{1eq}}{Z_{1eq} + Z_g} V_g = 6.52 - 4.16j$$

To calculate the voltage, V_{TL1} , along the transmission line 1, let's define the origin of z at Z_{2eq} .

$$V_{TL1}(z) = V_1^+ e^{-j\beta z} + V_1^- e^{j\beta z} = V_1^+ (e^{-j\beta z} + \Gamma_1 e^{j\beta z})$$

and,

$$V_1 = V_{TL1}\left(-\frac{\pi}{3\beta}\right) = V_1^+ (e^{j\pi/3} + \Gamma_1 e^{-j\pi/3})$$

Γ_1 is the reflection coefficient in the figure (b), and is equal to:

$$\Gamma_1 = \frac{Z_{2eq} - Z_1}{Z_{2eq} + Z_1} = 0.567 + 0.680j$$

Using the known information, we can get

$$V_1^+ = \frac{V_1}{e^{j\pi/3} + \Gamma_1 e^{-j\pi/3}} = 2.50 - 4.33j$$

Thus, we have calculated the steady state voltage along the transmission line 1 (for $z=0$ at the end of the transmission line 1):

$$V_{TL1}(z) = V_1^+ (e^{-j\beta z} + \Gamma_1 e^{j\beta z}) = (2.50 - 4.33j) * ((e^{-j\beta z} + (0.567 + 0.680j) * e^{j\beta z}))$$

The corresponding current:

$$I_{TL1}(z) = (0.0083 - 0.0144j) * ((e^{-j\beta z} - (0.567 + 0.680j) * e^{j\beta z}))$$

For getting the steady state voltage along the transmission line 2, one can repeat the above calculation using the corresponding parameters. For instance, V_2 should replace V_1 , the reflection coefficient Γ_2 replaces Γ_1 , and Z_2 replaces Z_1 .

$$V_2 = V_{TL1}(0) = V_1^+ (1 + \Gamma) = 6.86 - 5.09j$$

$$\Gamma_2 = \frac{Z_L - Z_2}{Z_L + Z_2} = -0.908 - 0.289j$$

$$V_2^+ = \frac{V_2}{e^{j3\pi/2} + \Gamma_2 e^{-j3\pi/2}} = 3.14 + 3.12j$$

For $z' = 0$ at the load,

$$V_{TL2}(z') = V_2^+ (e^{-j\beta z'} + \Gamma_2 e^{j\beta z'}) = (3.14 + 3.12j) * ((e^{-j\beta z'} - (0.908 + 0.289j) * e^{j\beta z'})$$

$$I_{TL2}(z') = (0.0314 + 0.0312j) * ((e^{-j\beta z'} + (0.908 + 0.289j) * e^{j\beta z'})$$

(b) For line 1, $VSWR = \frac{1 + |\Gamma_1|}{1 - |\Gamma_1|} = 16.4$

For line 2, $VSWR = \frac{1 + |\Gamma_2|}{1 - |\Gamma_2|} = 41.5$

(c). The average power delivered to the load = $P_{av,L} = \frac{1}{2} \text{Re}\{V_L [I_L]^*\}$, where V_L and I_L are the voltage and current at the load, respectively.

$$P_{av,L} = \frac{1}{2} \text{Re}\{V_L [I_L]^*\} = \frac{1}{2} \text{Re}\{V_{TL2}(0) [I_{TL2}(0)]^*\} = 9 \text{ mW}$$

(d). The average power delivered by the source to the transmission lines =

$$P_{av,L} = \frac{1}{2} \text{Re}\{V_1 [I_1]^*\} = \frac{1}{2} \text{Re}\left\{V_1 \left(\frac{V_1}{Z_{1eq}}\right)^*\right\} = \frac{|V_1|^2}{2} \text{Re}\left\{\frac{1}{Z_{1eq}}\right\} = 9 \text{ mW}.$$

This is the same as the one in part (c), which is expected as the transmission lines are lossless.

2. (Two typos in the question: ... desired to deliver 10W into Z_{L2} and the remaining power into Z_{L3} .)

Since $VSWR = 0 \text{ dB}$ is required for each line and $|\Gamma| = \frac{VSWR - 1}{VSWR + 1}$, $|\Gamma| = 0$. Thus, $Z_{L2} = Z_2$,

$Z_{L3} = Z_3$.

Because it is a perfect match, the length of the transmission lines does not change the impedance of the transmission lines seen from the junction downwards.

The voltages at the junction of the two transmission lines are the same, for they are measured at the same junction.

Thus, the currents going into individual lines determine the amount of power flowing into the respective loads.

$$\frac{P_{L2}}{P_{L3}} = \frac{\left(|V_1|^2 / 2\right) \operatorname{Re}\{1 / Z_2\}}{\left(|V_1|^2 / 2\right) \operatorname{Re}\{1 / Z_3\}} = \frac{Z_3}{Z_2} = \frac{1}{4}$$

So $Z_2 = 4 Z_3$ would make 10 W going into Z_{L2} and 40 W into Z_{L3} .

3. Given $VSWR = 3.2$, $|\Gamma| = \frac{VSWR - 1}{VSWR + 1} = 0.524$

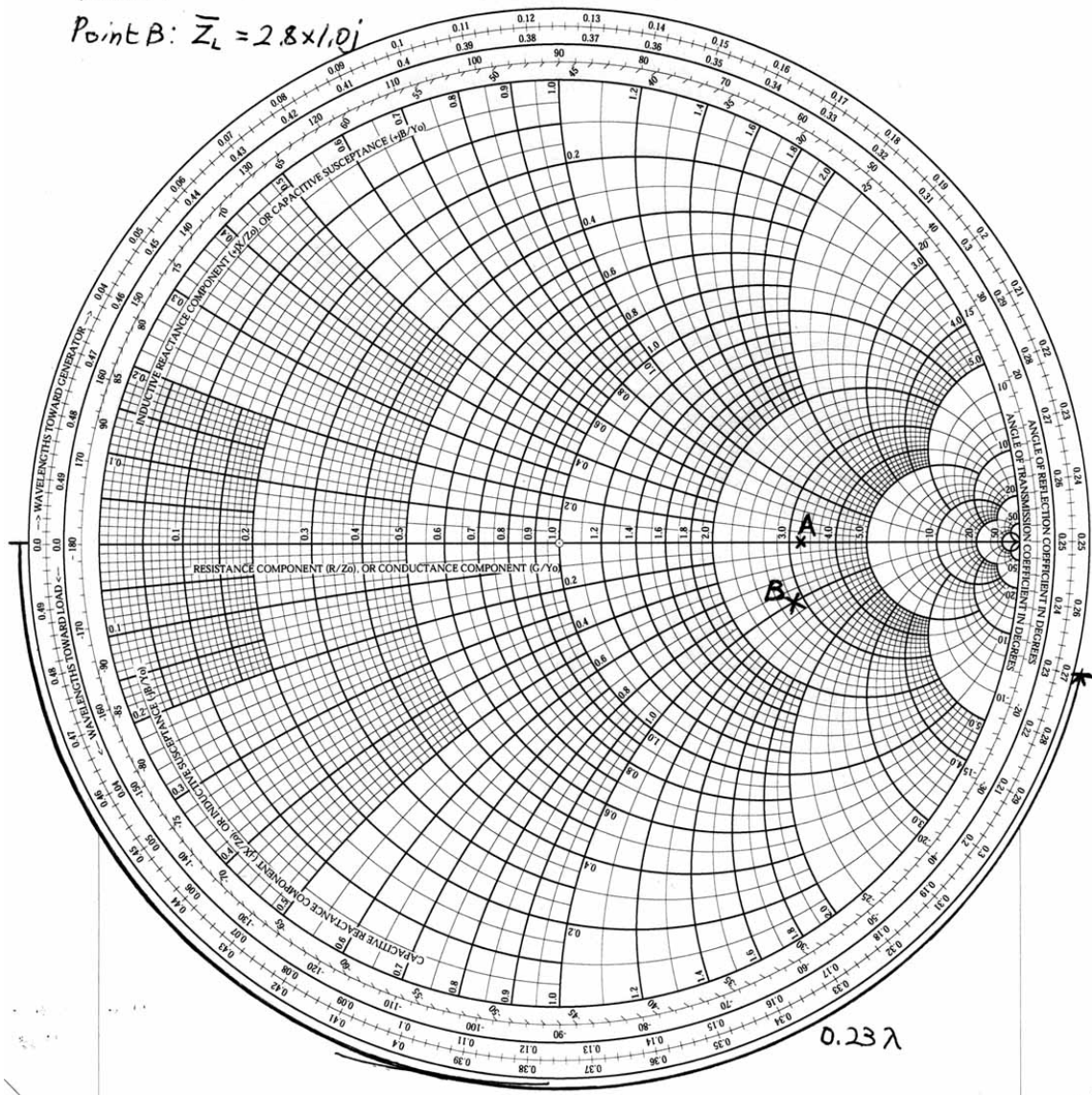
At the position in the transmission line where the voltage is minimum, the current is maximum, so the impedance and the impedance normalized to Z_0 at that point is minimum. The impedance is also real. Thus the normalized impedance of this position has a point lying on the left hand part along the horizontal axis in the Smith chart.

Since this position is 0.23 wavelength away from the load, you need to move the point counterclockwise 0.23λ to find the normalized impedance of the load. The new point has an angle subtended at the origin of -14.4° .

To find the distance of the new point from the origin of the chart, one needs to make use of the fact that propagation along the transmission line just rotates the impedance point about the origin. The radius of the circular locus does not change. At the position of the line where the voltage is maximum, the normalized impedance is maximum and real, and is equal to $VSWR = 3.2$. Thus, the point corresponding to the normalized load impedance can be obtained by moving this point about the origin of the chart, by -14.4° . This point has a reading of $2.8 - 1.0j$. Therefore, the load impedance is $196 - 70j \Omega$.

Point A: $VSWR = S = 3.2$

Point B: $\bar{Z}_L = 2.8 + j1.0j$



4. This is somewhat similar to the problem 1. We need to find the input impedance of the transmission network.

$$Z_{in} = Z_1 \frac{Z_{Leq} \cos \pi/4 + jZ_1 \sin \pi/4}{Z_1 \cos \pi/4 + jZ_{Leq} \sin \pi/4}$$

where $Z_{Leq} = jX_1 + (jX_2 // Z_{2eq}) = jZ_1 + (jZ_1 // Z_{2eq})$

$$Z_{2eq} = Z_1 \frac{Z_L \cos 3\pi/4 + jZ_1 \sin 3\pi/4}{Z_1 \cos 3\pi/4 + jZ_L \sin 3\pi/4} = Z_1 \frac{2Z_1 \cos 3\pi/4 + jZ_1 \sin 3\pi/4}{Z_1 \cos 3\pi/4 + j2Z_1 \sin 3\pi/4}$$

Simplifying the above expressions,

$$\begin{aligned}
Z_{2eq} &= (0.8 + 0.6j)Z_1 \\
Z_{L1eq} &= jZ_1 + \frac{jZ_1 * Z_1(0.8 + 0.6j)}{jZ_1 + Z_1(0.8 + 0.6j)} = (0.25 + 1.5j)Z_1 \\
Z_{in} &= Z_1 \frac{(0.25 + 1.5j)Z_1 \cos \pi/4 + jZ_1 \sin \pi/4}{Z_1 \cos \pi/4 + j(0.25 + 1.5j)Z_1 \sin \pi/4} = (1.6 - 4.2j)Z_1
\end{aligned}$$

Thus, voltage across the beginning of the line 1, V_1 is equal to:

$$V_1 = \frac{Z_{in}}{Z_{in} + Z_g} V_g = 4.47 - 0.861j$$

Since the transmission lines are lossless, the power delivered into the line is equal to the power consumed by the load.

Hence, power transmitted to the load =

$$P_{av,L} = \frac{1}{2} \operatorname{Re} \left\{ V_1 \left(\frac{V_1}{Z_{1eq}} \right)^* \right\} = \frac{|V_1|^2}{2} \operatorname{Re} \left\{ \frac{1}{Z_{1eq}} \right\} = \frac{0.820}{Z_1} \text{ W}$$

$$\Gamma_1 \text{ for transmission line 1} = \frac{(0.25 + 1.5j - 1)}{(0.25 + 1.5j + 1)} = 0.344 + 0.787j$$

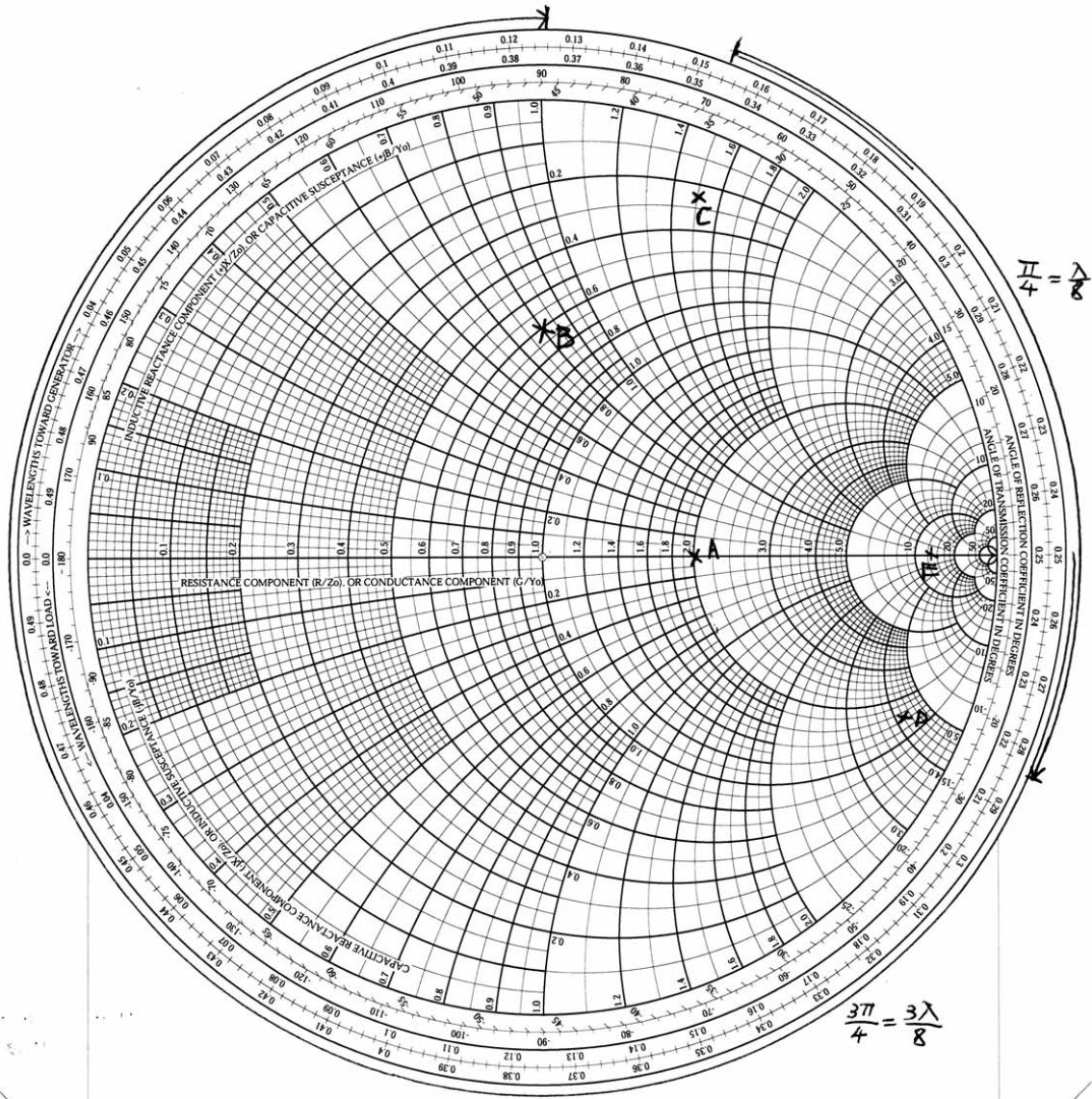
$$VSWR = \frac{1 + |\Gamma_1|}{1 - |\Gamma_1|} = 13.2$$

$$\Gamma_2 \text{ for transmission line 2} = \frac{(2 - 1)}{(2 + 1)} = 1/3$$

$$VSWR = \frac{1 + |\Gamma_2|}{1 - |\Gamma_2|} = 2$$

(Using the Smith chart)

The normalized load impedance is 2, which is represented by point A in the figure below. To find the normalized impedance at the beginning of line 2, one would move clockwise through 270° and get to point B. The reading of this point is $(0.8 + 0.6j) Z_1$. This impedance needs to combine with the load “ jX_2 ” in parallel and subsequently the load “ jX_1 ” in series. The result is $(0.25 + 1.5j) Z_1$. This impedance is shown as point C in the figure. The input impedance of the transmission line network can be found by moving clockwise through 90° . The new point has a reading of $(1.6 - 4.2j) Z_1$. This result is the same as the one found above. Using the method indicated above, one can find the average power delivered to the load.



The VSWR of a transmission line is equal to the maximum normalized impedance along the line, i.e.,

$$VSWR = \frac{Z_{\max}}{Z_0}$$

It is a real number, and lies on the right side along the horizontal axis (u axis). So for transmission line 1, we can find its VSWR by moving point C about a circle and reading the intersection at the right u axis. The intersection is denoted by point D, and has a value of ~ 13 .

For the line 2, one may note that point A is already the intersection on the right u axis. So VSWR for this line is 2.

5(a). In general, voltage and current along a transmission line have forms:

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$I(z) = \frac{1}{Z_0} [V^+ e^{-\gamma z} - V^- e^{\gamma z}]$$

where $\gamma = \sqrt{ZY}$

$$Z_0 = \sqrt{\frac{Z}{Y}}$$

$$Z = R' + j\omega L', Y = G' + j\omega C'$$

In this case, because $R' \gg \omega L'$, $Z \sim R'$. $Y \sim j\omega C'$. Thus, the propagation constant $\gamma = \sqrt{j\omega R' C'} = \sqrt{\omega R' C' / 2} (1 + j)$. Since $\gamma = \alpha + j\beta$, $\beta = \sqrt{\omega R' C' / 2}$, and propagation velocity $= \omega / \beta = \sqrt{2\omega / (R' C')}$.

(b) Using the voltage and current expression given above, the input impedance is equal to be ($z = -l$) :

$$Z_i(l) = Z_0 \frac{V^+ e^{\gamma l} + V^- e^{-\gamma l}}{V^+ e^{\gamma l} - V^- e^{-\gamma l}} = Z_0 \frac{1 + \Gamma e^{-2\gamma l}}{1 - \Gamma e^{-2\gamma l}} = Z_0 \frac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l}$$

where Z_L is the load impedance, $Z_0 = \sqrt{R' / (j\omega C')} = \sqrt{R' / (2\omega C')} (1 - j)$, and l is the distance of the point of interest to the load.

(c) For a shorted termination,

$$Z_i(l) = Z_0 \frac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l} = Z_0 \tanh \gamma l = \sqrt{\frac{R'}{2\omega C'}} (1 - j) \tanh \left[\sqrt{\frac{\omega R' C'}{2}} (1 + j) l \right]$$

Expansion of Z_0 around $\omega = 0$ gives:

$$Z_0 = \sqrt{-\frac{jR'}{\omega C'}} + O(\omega)$$

$O(\omega)$ represents the higher-order terms in the expansion.

Expansion of $\tanh(\gamma l)$ around $\omega = 0$ gives:

$$\tanh(\gamma l) = \sqrt{j\omega R' C'} l - \frac{1}{3} jRC \sqrt{jR' C'} l^3 \omega^{3/2} + O(\omega)$$

So,

$$Z_i \sim R - j\frac{1}{3}\omega R^2 C$$

R and C are the sums of the resistance and capacitance of the components in the transmission model. At low frequency, the shorted transmission line acts like a resistor. This is true until the frequency is so large that the imaginary part, which is the reactive component of the impedance, is comparable to the real part. In particular, the parts become equal when

$$R = \omega R^2 C / 3$$

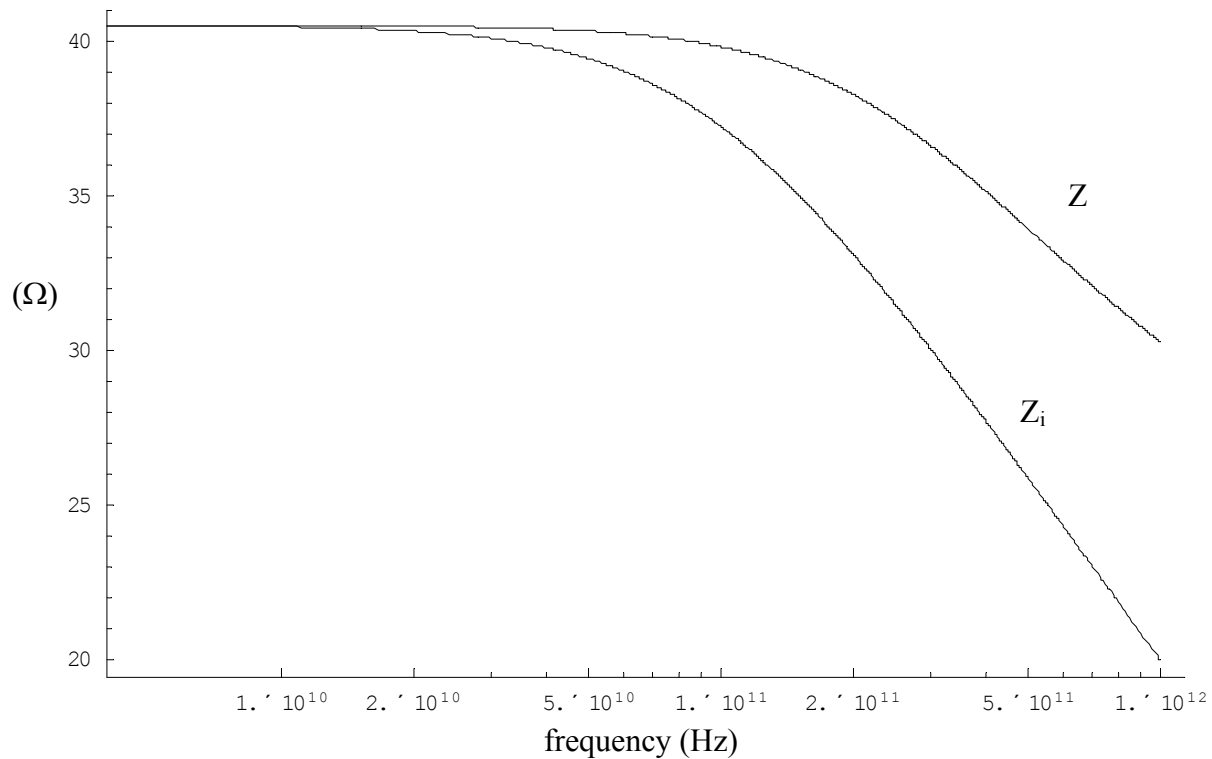
or $\omega = 3 / RC$

Thus the pole frequency is $3/(2\pi RC)$.

1-section lumped approximation to the line is a resistor of resistance R in parallel with a capacitor of capacitance C . Expansion of the impedance of the lumped circuit gives:

$$Z = \frac{R}{1 + j\omega RC} \sim R + j\omega RC$$

Using the values given in the question, plots of the input impedance of the transmission line and the impedance of the lumped circuit are shown below.



(d) For an open termination,

$$Z_i(l) = Z_0 \frac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l} = Z_0 \tanh \gamma l = \sqrt{\frac{R'}{2\omega C'}}(1-j) \bigg/ \tanh \left[\sqrt{\frac{\omega R' C'}{2}}(1+j)l \right]$$

Just like in part (c), expansion of Z_0 around $\omega = 0$ gives:

$$Z_0 = \sqrt{-\frac{jR'}{\omega C'}} + O(\omega)$$

Expansion of $1/\tanh(\gamma l)$ around $\omega = 0$ gives:

$$1/\tanh(\gamma l) = \frac{1}{\sqrt{j\omega R' C'}l} + \frac{j}{3} \sqrt{\frac{\omega R' C'}{j}}l + \frac{(\omega R' C')^{\frac{3}{2}}}{45\sqrt{j}}l^3 + O(\omega)$$

Combining the two results gives:

$$Z_i \sim -\frac{R}{3} + j \left(\frac{1}{\omega C} + \frac{\omega R^2 C}{45} \right)$$

At low frequency, $(\omega C)^{-1} \gg \omega R^2 C$ because of the values of R and C in this problem. Thus,

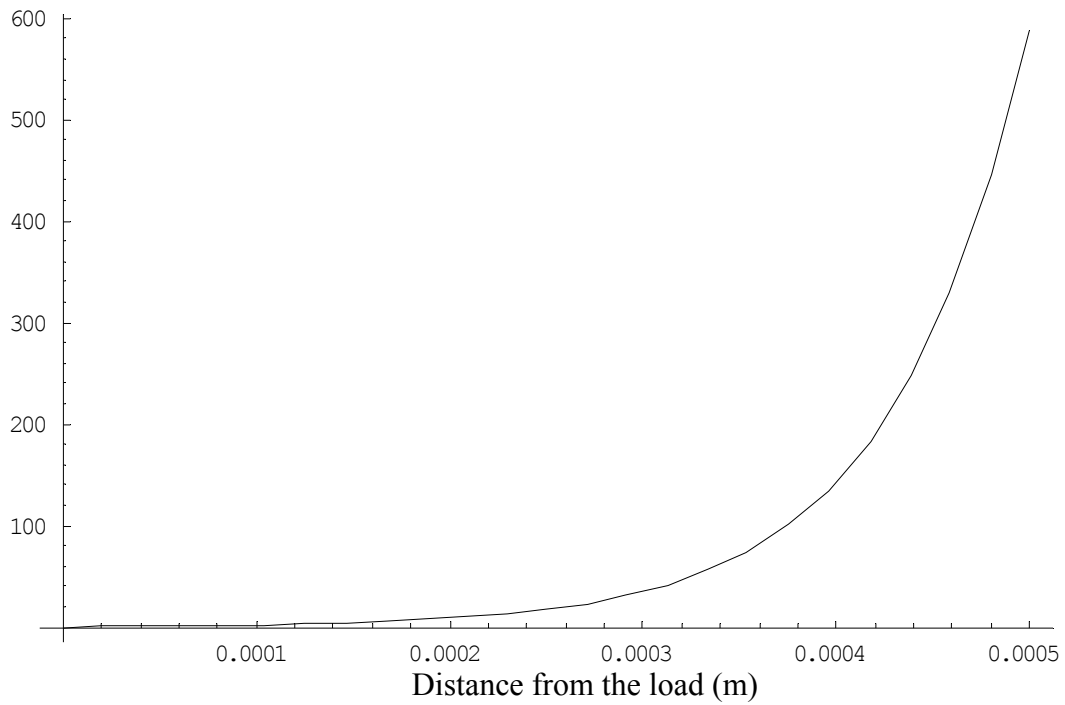
$$Z_i \sim -\frac{R}{3} + j \frac{1}{\omega C}$$

It is a capacitor of capacitance C in series to a resistor of resistance R/3. An 1-section lumped approximation to the line gives R in series with C, i.e., $Z_i = R - j/(\omega C)$.

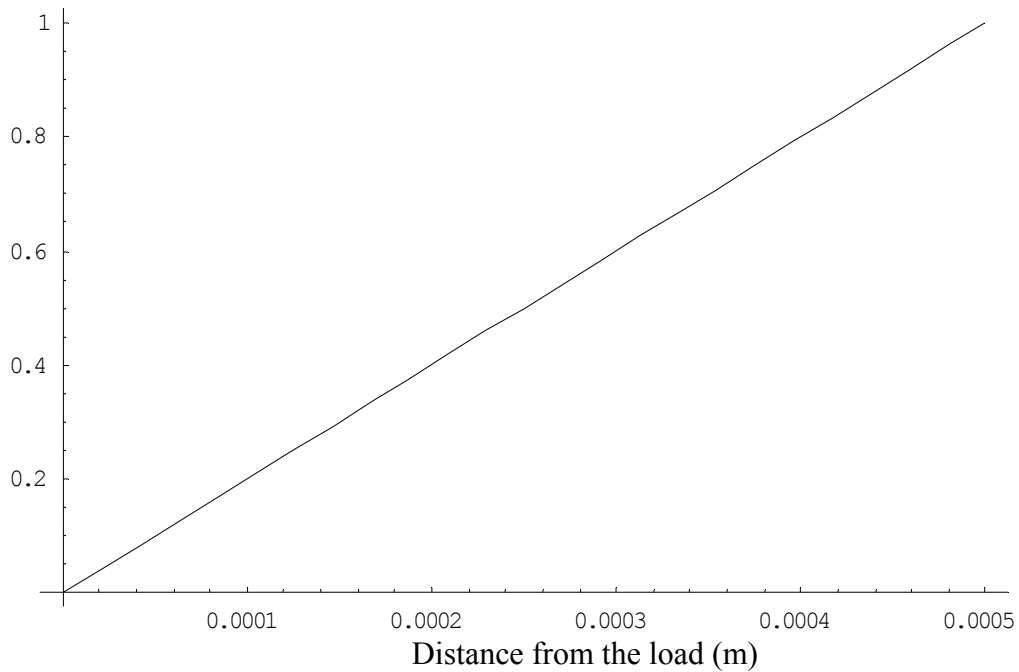
(e) The voltage along the line is equal to:

$$V(l) = V^+ (e^{\gamma l} - e^{-\gamma l}) = V^+ \sinh(\gamma l)$$

So for $R = 10^6 \Omega / 500 \mu\text{m} = 2 \cdot 10^9 \Omega$, $C = 100 \cdot 10^{-15} \text{F} / 500 \mu\text{m} = 2 \cdot 10^{10} \text{F}$, we can plot the V/V^+ vs. the distance from the load.



For an ideal resistor, the plot would look like this:



The plot for the transmission line is not linear like the one for a resistor because the frequency is 3 orders of magnitude higher than the pole frequency.