Lecture #23 Linear Quadratic Regulator (Apr. 14, 2015) v. 1.0 (R. Fearing)
Ref: K. Ogata, *Modern Control Engineering* 2002.

Here the infinite horizon, continuous time, Linear Quadratic Regulator is derived. A cost function which is *Quadratic* in control and error is used. The considered system is *Linear* and uses a linear state feedback control $u = -Kx$. The *Regulator* problem considers $\mathbf{x}(t) \to \mathbf{0}$. Infinite horizon means considering the total cost as $t \to \infty$.

Given a system in state-space form:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \qquad y = C\mathbf{x} \tag{1}$$

Define instantaneous cost of control $\mathbf{u}(t)^T R\mathbf{u}(t)$, instantaneous cost of output error $\mathbf{y}(t)^T Q\mathbf{y}(t)$. Q, R are positive semidefinite, that is $\mathbf{x}^T Q\mathbf{x} \geq 0 \ \ \forall \mathbf{x}$. Assume $Q = Q^T$ and $R = R^T$.

Define Quadratic cost

$$J = \int_0^\infty (y^T Qy + u^T Ru)dt. \tag{2}$$

To combine (1) and (2), we introduce a "cost of state x", $x^T Px$, where P is positive semidefinite (also $P = P^T$) and where P will be derived. The difference between "cost" of final and initial states is given by

$$x^T(\infty)Px(\infty) - x^T(0)Px(0) = -x^T(0)Px(0) = \int_0^\infty \frac{d}{dt}(x^T Px)dt, \tag{3}$$

using the assumption of a stable regulator, thus $x^T(\infty)Px(\infty) \to 0$.

Now adding Eq.(2) and Eq.(3 gives

$$J - x^T(0)Px(0) = \int_0^\infty (y^T Qy + u^T Ru) + \frac{d}{dt}(x^T Px)dt = \int_0^\infty (y^T Qy + u^T Ru) + \dot{x}^T Px + x^T P\dot{x}dt \tag{4}$$

Using (1) we get:

$$J - x^T(0)Px(0) = \int_0^\infty x^T C^T QCx + u^T Ru + (x^T A^T + u^T B^T)Px + x^T P(Ax + Bu)dt \tag{5}$$

Grouping quadratic terms

$$J - x^T(0)Px(0) = \int_0^\infty x^T[C^T QC + A^T P + PA]x + u^T Ru + u^T B^T Px + x^T PBu \ dt \tag{6}$$

Algebraic Riccati Equation (A.R.E.)
For a given N and since A, C, Q are known, the A.R.E. $PNP = C^T QC + PA + A^T P$ can be solved for P, e.g. using Matlab `care()`. (N is to be determined below.)

Substituting PNP in (6), we get:

$$J - x^T(0)Px(0) = \int_0^\infty x^T[PNP]x + u^T Ru + u^T B^T Px + x^T PBu \ dt \tag{7}$$

Considering that for optimal linear state feedback, $u(t) \to -Kx$, the instantaneous "cost" of the *difference* between u and Kx can be given by $(u + Kx)^T R(u + Kx) = 0$. Let $K = R^{-1}B^T P$. Then

$$\begin{aligned}
(u + Kx)^T R(u + Kx) &= x^T K^T RK^T x + u^T Ru + u^T RKx + x^T K^T Ru && (8)\\
&= x^T P^T BR^{-1}B^T Px + u^T Ru + u^T B^T Px + x^T PBu && (9)\\
&= x^T P^T NPx + u^T Ru + u^T B^T Px + x^T PBu && (10)
\end{aligned}$$

where $N = BR^{-1}B^T$. Noting that Eq.(10) is identical to the term inside the integral in Eq.(7), which is 0 for $u = -Kx$, then the minimum cost is:

$$J = x^T(0)Px(0) \tag{11}$$

where P is solution of $C^T QC + PA + A^T P - PBR^{-1}B^T P = 0$, the algebraic Riccati equation.