EECS 117

Lecture 24: Oblique Incidence on Dielectrics

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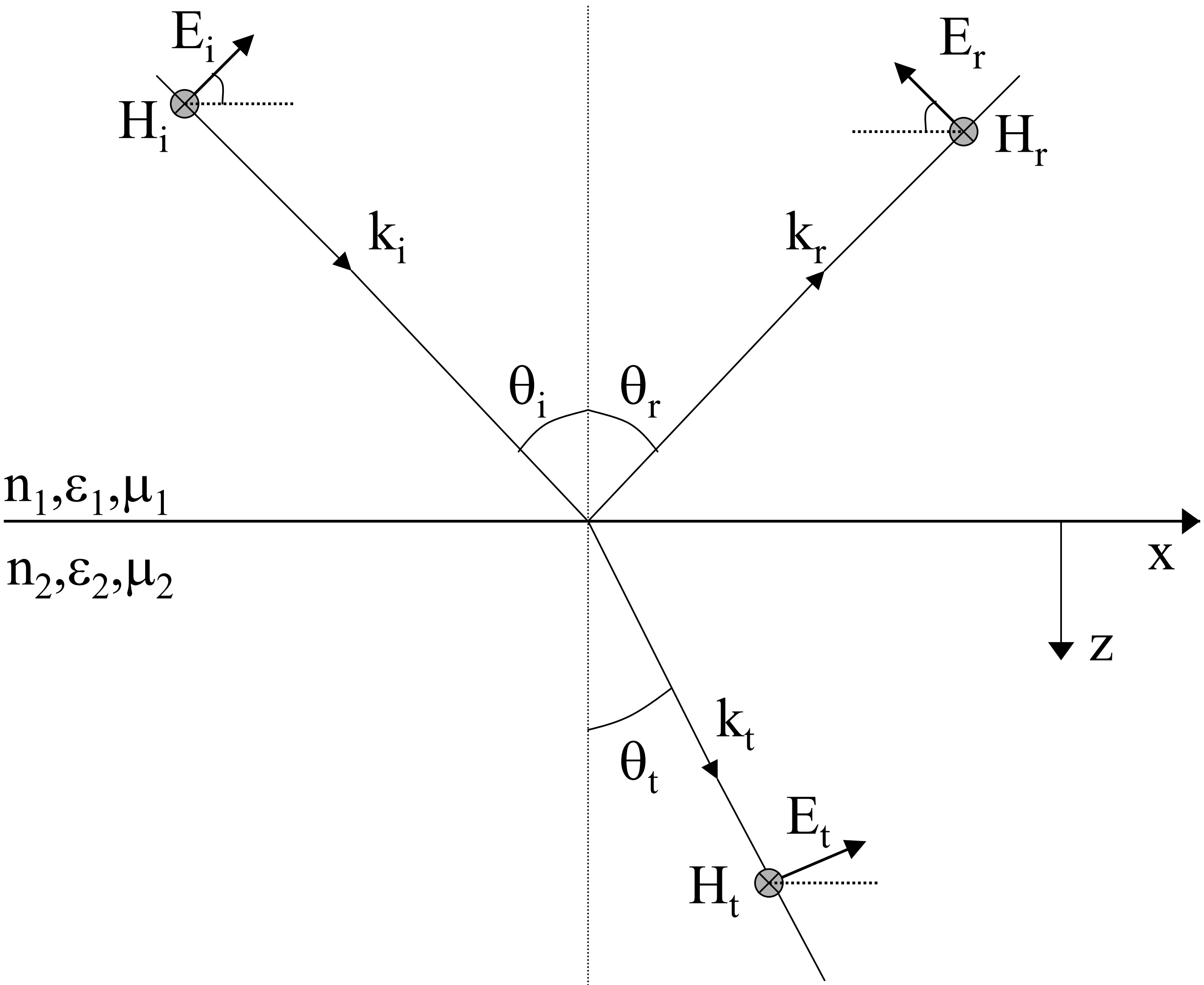
Oblique Incidence (TM waves)



TE Waves (cont.)

- A transverse magnetic (TM) wave, also so called the P-polarized wave, has the electric field oscillating in the plane of incidence and the magnetic field out of the plane. The plane of incidence is defined to be the one made by the incident wave vector (propagation direction vector) and the line normal to the surface of discontinuity.

- In general, a field (either electric or magnetic) of a wave can be described by

$$\vec{F}(x,z) = |F| e^{j\vec{k}\bullet\vec{r}}\,\hat{f}$$

where $\vec{k}$ is the wave vector, $\hat{f}$ is a unit vector along the field $\vec{F}$, and $\vec{r}$ is a general space vector, i.e., $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

Incident, Reflected and Transmitted Waves

- For the incident electric field,

$$\vec{E}_i(x,z) = |E_i| e^{j\vec{k}_i \bullet \vec{r}} \hat{e}_i = |E_i| e^{jn_1 k \hat{k}_i \bullet \vec{r}} \hat{e}_i$$
$$= |E_i| e^{jn_1 k \sin\theta_i x} e^{jn_1 k \cos\theta_i z} (\cos\theta_i \hat{x} - \sin\theta_i \hat{z})$$
$$= E_{i_x} \hat{x} + E_{i_z} \hat{z}$$

k is the wave number of the field in vacuum, and $k_1 = \dfrac{\omega}{v} = \dfrac{\omega}{c/n_1} = n_1 k$

- For the reflected and transmitted fields,

$$\vec{E}_r(x,z) = |E_r| e^{jn_1 k \sin\theta_r x} e^{-jn_1 k \cos\theta_r z} \left(-\cos\theta_r \hat{x} - \sin\theta_r \hat{z}\right)$$
$$\vec{E}_t(x,z) = |E_t| e^{jn_2 k \sin\theta_t x} e^{jn_2 k \cos\theta_t z} \left(\cos\theta_t \hat{x} - \sin\theta_t \hat{z}\right)$$

Boundary Conditions

- Applying the boundary condition for tangential electric field at $z = 0$, we have

$$E_{i_x} + E_{r_x} = E_{t_x}$$

$$\left|E_i\right| \cos\theta_i e^{jn_1 k \sin\theta_i x} - \left|E_r\right| \cos\theta_r e^{jn_1 k \sin\theta_r x} = \left|E_t\right| \cos\theta_t e^{jn_2 k \sin\theta_t x}$$

- Just like in the case of oblique incidence on a perfect conductor, this condition needs to be satisfied independent of x. This is possible by equating the exponents and coefficients separately:

$$n_1 k \sin\theta_i = n_1 k \sin\theta_r = n_2 k \sin\theta_t \qquad (1)$$

$$\left|E_i\right| \cos\theta_i - \left|E_r\right| \cos\theta_r = \left|E_t\right| \cos\theta_t \qquad (2)$$

Law of Reflection and Refraction

- The first equality in (1) gives the law of reflection, i.e., $\theta_i = \theta_r$ and the second equality in (2) gives the Snell's law, namely,

$$n_1 \sin\theta_i = n_1 \sin\theta_r = n_2 \sin\theta_t$$

- For most of the materials, which are non-magnetic, $\mu_1 = \mu_2 = \mu_0$. Then,

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} = \sqrt{\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}} \approx \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \approx \sqrt{\frac{\varepsilon_1\mu_2}{\varepsilon_2\mu_1}} = \frac{\eta_2}{\eta_1}$$

Total Reflection

- If $n_1 > n_2$ or $\eta_1 < \eta_2$, we have total reflection for incident angle larger than the critical angle θ_c , where

$$\theta_c = \sin^{-1}\left(\frac{n_2 \sin \pi / 2}{n_1}\right) \approx \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \approx \sin^{-1}\frac{\eta_1}{\eta_2}$$

- For $\theta_i > \theta_c$

$$\sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i > 1$$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \pm j\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}$$

Total Reflection (cont.)

- The transmitted electric and magnetic fields are then equal to:

$$\vec{E}_t(x,z) = |E_t| e^{jn_2 k \sin\theta_t x} e^{jn_2 k \cos\theta_t z} \left(\cos\theta_t \hat{x} - \sin\theta_t \hat{z}\right)$$
$$= |E_t| e^{j\beta_{tx} x} e^{j\alpha_{tz} z} \left(\cos\theta_t \hat{x} - \sin\theta_t \hat{z}\right)$$

$$\vec{H}_t(x,z) = |H_t| e^{jn_2 k \sin\theta_t x} e^{jn_2 k \cos\theta_t z} \hat{y} = |H_t| e^{j\beta_{tx} x} e^{j\alpha_{tz} z} \hat{y}$$

where

$$\beta_{tx} = n_2 k \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \sin\theta_i, \;\; \alpha_{tz} = n_2 k \sqrt{\frac{\varepsilon_1}{\varepsilon_2} \sin^2\theta_i - 1}$$

- The wave propagates in x direction only!! It does *penetrate* into medium 2, with its amplitude decaying exponentially inside the medium. This surface wave is called an evanescent wave.

Reflection and Transmission Coefficients

- The equation (2) obtained from applying the boundary condition to the tangential electric field can be used to find the reflection and transmission coefficients.

- But we need one more piece of information, which we can get from applying the tangential magnetic field boundary condition:

$$\left|H_i\right|e^{jn_1 k\sin\theta_i x} + \left|H_r\right|e^{jn_1 k\sin\theta_r x} = \left|H_t\right|e^{jn_2 k\sin\theta_t x}$$

$$\left|H_i\right| + \left|H_r\right| = \left|H_t\right|$$

- We know that *H* and *E* are related to each other through impedance of the medium. Thus, we have

$$\left|H_i\right| = -\left|E_i\right|/\eta_1,\ \left|H_r\right| = -\left|E_r\right|/\eta_1,\ \left|H_t\right| = -\left|E_t\right|/\eta_2$$

Coefficients (cont.)

- We need the negative signs in the equations because with our definition of direction of $\vec{E}, \vec{H}$ must be negative according to the right hand rule so that $\vec{E} \times \vec{H}$ is pointing the same direction as does.

- Now we have two simultaneous equations:

$$\left|E_i\right| \cos\theta_i - \left|E_r\right| \cos\theta_r = \left|E_t\right| \cos\theta_t$$

$$\left|E_i\right| / \eta_1 + \left|E_r\right| / \eta_1 = \left|E_t\right| / \eta_2$$

- Solving these two equations, we can expressions for the reflection and transmission coefficients.

$$\Gamma_{TM} = \frac{\left|E_r\right|}{\left|E_i\right|} = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$$

Coefficients (cont.)

$$\tau_{TM} = \frac{|E_t|}{|E_i|} = \frac{\eta_2}{\eta_2}(1+\Gamma_{TM}) = \frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}$$

- We can also define the coefficients in terms of the tangential components of the electric fields. They have expressions:

$$\Gamma_{Z_{TM}} = \frac{|E_{r_x}|}{|E_{i_x}|} = \frac{-|E_r|\cos\theta_r}{|E_i|\cos\theta_i} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}$$

$$\tau_{Z_{TM}} = \frac{|E_{t_x}|}{|E_{i_x}|} = \frac{|E_t|\cos\theta_t}{|E_i|\cos\theta_i} = \frac{2\eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}$$

Brewster Angle (TM Waves)

- There is a situation that we don't have reflection off the boundary, when the numerator of Γ_{TM} is equal to 0, or $\eta_1 \cos\theta_i = \eta_2 \cos\theta_t$

- The incident angle that this happens is called Brewster angle. It can be shown that

$$\sin^2\theta_{B_{TM}} = \frac{1}{1+\left(\eta_2/\eta_1\right)^2} = \frac{1-\mu_2\varepsilon_1/\mu_1\varepsilon_2}{1-\left(\varepsilon_1/\varepsilon_2\right)^2} \approx \frac{1}{1+\varepsilon_1/\varepsilon_2}$$

or

$$\theta_{B_{TM}} \approx \tan^{-1}\sqrt{\varepsilon_1/\varepsilon_2} \approx \tan^{-1}\sqrt{n_1/n_2}$$

Oblique Incidence (TE Waves)

- For a TE wave, which has the electric field oscillating out of the plane of incidence and magnetic field parallel to the plane, we can obtain the same set of quantities as in TM waves quickly by swapping E and H: $E \to H, H \to E$ with some *cautions*.

- In the TM case, we applied the boundary condition to the tangential electric field. Here we need to apply the condition to the tangential magnetic field. Because both conditions require the field to be continuous across the boundary, we have

$$\left|H_i\right| \cos\theta_i e^{jn_1 k \sin\theta_i x} - \left|H_r\right| \cos\theta_r e^{jn_1 k \sin\theta_r x} = \left|H_t\right| \cos\theta_t e^{jn_2 k \sin\theta_t x}$$

which also yield the laws of reflection and refraction. If the incident angle is larger than the critical angle (of course $n_1 > n_2$), the transmitted wave is evanescent.

Boundary Conditions for TE Waves

- Analogous to the TM case, we obtain this from the boundary condition for tangential magnetic fields:

$$\left|H_i\right|\cos\theta_i - \left|H_r\right|\cos\theta_r = \left|H_t\right|\cos\theta_t$$

- Just like what we had in TM case, we also have this equality due to the boundary condition for tangential electric fields:

$$\left|E_i\right|e^{jn_1 k\sin\theta_i x} + \left|E_r\right|e^{jn_1 k\sin\theta_r x} = \left|E_t\right|e^{jn_2 k\sin\theta_t x}$$

$$\left|E_i\right| + \left|E_r\right| = \left|E_t\right|$$

- The electric and magnetic fields are related:

$$\left|H_i\right| = -\left|E_i\right|/\eta_1,\ \left|H_r\right| = -\left|E_r\right|/\eta_1,\ \left|H_t\right| = -\left|E_t\right|/\eta_2$$

Boundary Conditions (cont.)

- So we have these two simultaneous equations in the TE case:

$$\left(\left|E_i\right|\cos\theta_i - \left|E_r\right|\cos\theta_r\right)/\eta_1 = \left|E_t\right|/\eta_2\cos\theta_t$$

$$\left|E_i\right| + \left|E_r\right| = \left|E_t\right|$$

- These two equations yield the expressions of reflection and transmission coefficients:

$$\Gamma_{TE} = \frac{\left|E_r\right|}{\left|E_i\right|} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\tau_{TE} = \frac{\left|E_t\right|}{\left|E_i\right|} = 1 + \Gamma_{TE} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

- Since the electric fields are tangential to the boundary,

$$\Gamma_{Z_{TE}} = \Gamma_{TE}\,,\, \tau_{Z_{TE}} = \tau_{TE}$$

Brewster Angle (TE Waves)

- We can also define Brewster angle just like before. In this case, we have

$$\sin^2 \theta_{B_{TM}} = \frac{1-\mu_1\varepsilon_2/\mu_2\varepsilon_1}{1-\left(\mu_1/\mu_2\right)^2}$$

- For most of the materials which have $\mu_1 = \mu_2 = \mu_0$, this angle usually does not exist.

Impedance for Waves at Oblique Incidence

- Since in many cases the tangential components of the fields are continuous across the boundary, it is useful to define a characteristic wave impedance referred to the z direction (as in our setup) in terms of the components in planes transverse to that direction. as the ratio of electric to magnetic field components in planes parallel to the boundary. In other words, in the TM case,

$$\eta_{z_{TM}} = \frac{E_{x+}}{H_{y+}} = -\frac{E_{x-}}{H_{y-}} = \eta\cos\theta$$

$$\eta_{z_{TE}} = -\frac{E_{y+}}{H_{x+}} = \frac{E_{y-}}{H_{x-}} = \eta\sec\theta$$

+ and – refer, respectively, to incident and reflected wave. The sign of the ratio is chosen for each wave to yield a positive impedance.

Reflection and Transmission Revisit

- If we substitute these new definitions into the reflection and transmission coefficient expressions, we can see that these expressions are just what we learned in the transmission line.

$$\Gamma_{Z_{TM}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} = \frac{\eta_{2,Z_{TM}} - \eta_{1,Z_{TM}}}{\eta_{1,Z_{TM}} + \eta_{2,Z_{TM}}}$$

$$\tau_{Z_{TM}} = \frac{2\eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} = \frac{2\eta_{2,Z_{TM}}}{\eta_{1,Z_{TM}} + \eta_{2,Z_{TM}}}$$

$$\Gamma_{Z_{TE}} = \Gamma_{TE} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{\eta 2_{1,Z_{TE}} - \eta_{1,Z_{TE}}}{\eta_{1,Z_{TE}} + \eta_{2,Z_{TE}}}$$

$$\tau_{Z_{TE}} = \tau_{TE} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{2\eta_{2,Z_{TE}}}{\eta_{2,Z_{TE}} + \eta_{1,Z_{TE}}}$$

Reflection and Transmission Revisit (cont.)

In general,

$$\Gamma_Z = \frac{\eta_{2,Z} - \eta_{1,Z}}{\eta_{1,Z} + \eta_{2,Z}} \qquad \tau_Z = \frac{2\eta_{2,Z}}{\eta_{1,Z} + \eta_{2,Z}}$$

The impedance defined by the fields' tangential components is analogous to the impedance in transmission lines.