EECS 117 Homework Assignment 7 **Spring 2004**

6-11. An N-sided regular loop. (a) We consider each side of the N-sided polygon-shaped loop separately and apply the superposition principle. Note that due to symmetry, the magnetic field due to the current of each side is the same. Using the result of Example 6-3 with the figure shown, the magnitude of the total **B** field produced at the center of the loop can be written as

$$B = N\frac{\mu_0 I a}{2\pi r\sqrt{r^2+a^2}} = \frac{\mu_0 I}{2\pi}\frac{2}{d}\tan\frac{\pi}{N} = \frac{\mu_0 N I}{\pi d}\tan\frac{\pi}{N}$$

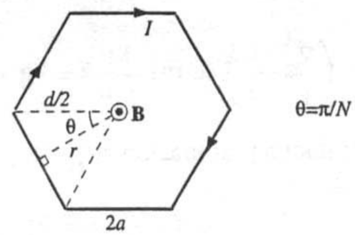

Fig. 6.4. Figure for Problem 6-11. An N-sided regular polygon-shaped loop.

6-13. A wire with two circular arcs. We consider each portion of the loop separately and apply the superposition principle. The straight portions do not produce any **B** field at point P since P lies along their axis. Assuming the z direction to be out of the page, the **B** field at point P produced by the current in the circular-arc portion with radius a is

$$\mathbf{B}_1 = \int_0^{\phi_0}\frac{\mu_0}{4\pi}\frac{Iad\phi'(-\hat{\boldsymbol{\phi}})\times(-\hat{\mathbf{r}})}{a^2} = -\hat{\mathbf{z}}\frac{\mu_0 I\phi_0}{4\pi a}$$

Similarly, the **B** field produced by the current in the circular arc with radius b is

$$\mathbf{B}_2 = \int_0^{\phi_0}\frac{\mu_0}{4\pi}\frac{Ibd\phi'(\hat{\boldsymbol{\phi}})\times(-\hat{\mathbf{r}})}{b^2} = \hat{\mathbf{z}}\frac{\mu_0 I\phi_0}{4\pi b}$$

Therefore, the total **B** produced by the wire loop is

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = \hat{\mathbf{z}}\frac{\mu_0 I\phi_0}{4\pi}\left(\frac{1}{b}-\frac{1}{a}\right)$$

6-24. B inside a solenoid. (a) From Section 6.2.2, the **B** field at the center of a long solenoid made of a magnetic core material with $\mu_r = 300$ and having a current of $I = 10$ A can be written as

$$B_{\text{ctr}} \simeq \frac{\mu_r\mu_0 N I}{l} = \frac{(300)(4\pi\times10^{-7})(10)}{l} = 1.5\ \text{T}$$

from which we find the number of turns of wire needed to produce 1.5 T field at the center as $N/l \simeq 398$ turns/m or $\sim$4 turns/cm.

(b) Repeating the same calculation in part (a) for an air-core solenoid yields $N/l \simeq 1194$ turns/cm. It is clear from these results that an air-core solenoid requires a much larger number of turns to produce the same field as produced by a solenoid of same geometry which is made of a magnetic core with a high μ_r.

6-32. Two infinitely long wires. Let us first start with two equal finite length parallel wires (with currents I and $-I$) oriented in the z direction and extending from $z = -d$ to $z = +d$, with their symmetry axis being the z axis. We choose the observation point on the xy plane since this choice maintains its generality in the case when the wires are infinitely long, i.e., when we let $d \to \infty$. Using the result of Example 6-22 along with the superposition principle, the magnetic vector potential at point P$(x, y, 0)$ due to both finite wires can be written as

$$\begin{aligned}\mathbf{A} =& \hat{\mathbf{z}}\frac{\mu_0 I}{2\pi}\left[\ln\left(\frac{d}{a}+\sqrt{\frac{d^2}{a^2}+1}\right) - \ln\left(\frac{d}{b}+\sqrt{\frac{d^2}{b^2}+1}\right)\right] \\ =& \hat{\mathbf{z}}\frac{\mu_0 I}{2\pi}\ln\left[\frac{b}{a}\frac{d+\sqrt{d^2+a^2}}{d+\sqrt{d^2+b^2}}\right]\end{aligned}$$

Next, letting $d \to \infty$, and noting that

$$\lim_{d\to\infty}\frac{d+\sqrt{d^2+a^2}}{d+\sqrt{d^2+b^2}} = 1$$

the magnetic vector potential at point P due to two infinitely long parallel wires can be found as

$$\mathbf{A} = \hat{\mathbf{z}}\frac{\mu_0 I}{2\pi}\ln\frac{b}{a}$$

6-41. Inductance of a thin circular loop of wire. (a) Since $d = 0.2$ cm $\ll 5$ cm $= a$, thin wire approximation can be used. From Section 6.7.3, the inductance of the single loop can be calculated as

$$L \simeq \mu_0 a\left(\ln\frac{8a}{d} - 2\right) = 4\pi\times 10^{-7}(0.05)\left(\ln\frac{8\times 5}{0.2} - 2\right)\ \text{H} \simeq 207\ \text{nH}$$

(b) For $d = 0.05$ cm $\ll 10$ cm $= a$, we have

$$L \simeq 4\pi\times 10^{-7}(0.1)\left(\ln\frac{8\times 10}{0.05} - 2\right)\ \text{H} \simeq 676\ \text{nH}$$

6-43. Mutual inductance between a wire and a circular loop. Assuming a current I to flow in the wire, the $\mathbf{B}$ field at a point P(r, θ) as shown in Figure 6.15 is given by

$$\mathbf{B}_{\text{P}} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I}{2\pi(d + r\cos\phi)}$$

The flux linked by the circular loop can be found by integrating over the area of the loop which we sweep if we vary ϕ between 0 and 2π and r between 0 and a. We thus have

$$\Psi_{12} = \frac{\mu_0 I}{2\pi}\int_0^a\int_0^{2\pi}\frac{r\,dr\,d\phi}{d + r\cos\phi} = \frac{\mu_0 I}{2\pi}\int_0^a\frac{2\pi r\,dr}{\sqrt{d^2 - r^2}} = \mu_0 I(d - \sqrt{d^2 - a^2})$$

so that the mutual inductance is given by

$$L_{12} = \mu_0(d - \sqrt{d^2 - a^2})$$

6-46. Two square coils. (a) For $d \gg a$, we can treat the square loop as a magnetic dipole with $|\mathbf{m}| = Ia^2$. From equation [6.14] we have

$$\mathbf{B} = \hat{\mathbf{r}}\frac{\mu_0|\mathbf{m}|\cos\theta}{2\pi r^3} + \hat{\boldsymbol{\theta}}\frac{\mu_0|\mathbf{m}|\sin\theta}{4\pi r^3}$$

For all points on the xy plane, we have $\theta = \pi/2$, so that

$$\mathbf{B} = \hat{\boldsymbol{\theta}}\frac{\mu_0 I a^2}{4\pi r^3} = -\hat{\mathbf{z}}\,\frac{\mu_0 I a^2}{4\pi r^3}$$

which only depends on radial distance r. The flux produced by coil 1 and linked by coil 2 is then

$$\Psi_{12} = \int_{d-a/2}^{d+a/2} \mathbf{B}\cdot(\hat{\mathbf{z}}\,a dr) = \frac{\mu_0 I a^3}{4\pi}\int_{d-a/2}^{d+a/2}\frac{1}{r^3}dr$$

$$= \frac{\mu_0 I a^3}{4\pi}\left[\frac{-r^{-2}}{2}\right]_{d-a/2}^{d+a/2} = \frac{\mu_0 I a^3}{8\pi}\left[\frac{1}{(d-a/2)^2} - \frac{1}{(d+a/2)^2}\right]$$

and the mutual inductance is given by

$$L_{12} = \frac{\Psi_{12}}{I} = \frac{\mu_0 I a^3}{8\pi}\left[\frac{1}{(d-a/2)^2} - \frac{1}{(d+a/2)^2}\right]$$

(b) We now have $d = 4a$, which means that we cannot assume $d \gg a$. What we have to do in this case is to use the exact expression for the **B**-field produced by a fixed length wire of length L at a general point P(r, ϕ, z) at a distance r from the center of the wire and at a distance z from its axis, as shown in Figure 6.16b. The general expression for the **B**-field at such a point P was found in Example 6-7 as

$$\mathbf{B}_{\mathrm{P}} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi r}\left[\frac{z+L/2}{\sqrt{r^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{r^2+(z-L/2)^2}}\right]$$

We now separately evaluate the magnetic flux produced by each side of coil 1 and linking coil 2. We start with side A, selecting our origin ($z = 0$) to be at its center as shown in Figure 6.16d. Noting that the wire length is $L = a$, we have

$$\Psi_{\mathrm{A}} = \int_0^a \int_{3.5a}^{4.5a} \frac{\mu_0 I}{4\pi r}\left[\frac{z+a/2}{\sqrt{r^2+(z+a/2)^2}} - \frac{z-a/2}{\sqrt{r^2+(z-a/2)^2}}\right] dzdr$$

$$= \int_0^a \frac{\mu_0 I}{4\pi r}\int_{3.5a}^{4.5a}\left[\frac{z+a/2}{\sqrt{r^2+(z+a/2)^2}} - \frac{z-a/2}{\sqrt{r^2+(z-a/2)^2}}\right] dzdr$$

$$= \frac{\mu_0 I}{4\pi}\int_0^a \frac{1}{r}\left[\int_{3.5a}^{4.5a}\frac{z+a/2}{\sqrt{r^2+(z+a/2)^2}}dz - \int_{3.5a}^{4.5a}\frac{z-a/2}{\sqrt{r^2+(z-a/2)^2}}dz\right] dr$$

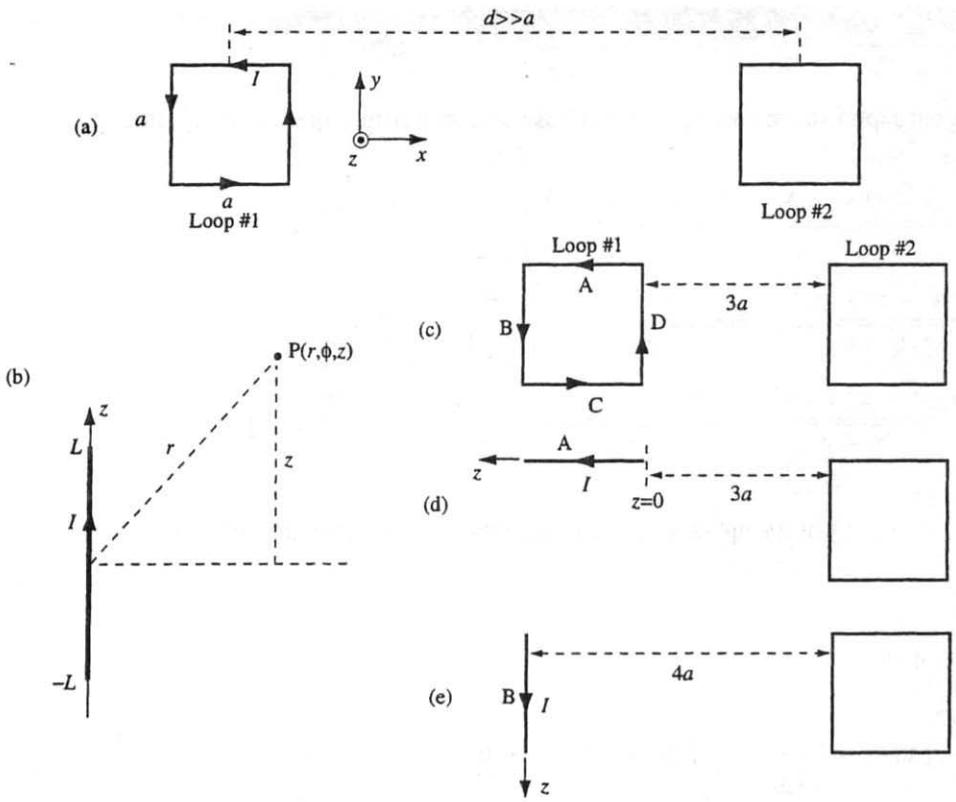

Fig. 6.16. Figure for Problem 6-46.

To evaluate the first integral we let $u = \sqrt{r^2 + (z + +a/2)^2}$, in which case we have

$$du = \frac{1}{2}\frac{1}{\sqrt{r^2 + (z + a/2)^2}}2(z + a/2)dz = \frac{z + a/2}{\sqrt{r^2 + (z + a/2)^2}}\,dz$$

so that

$$\int_{\sqrt{r^2+9a^2}}^{\sqrt{r^2+16a^2}} du = \sqrt{r^2 + 16a^2} - \sqrt{r^2 + 9a^2}$$

Similarly, we have

$$\int_{3.5a}^{4.5a} \frac{z - a/2}{\sqrt{r^2 + (z - a/2)^2}}dz = \sqrt{r^2 + 25a^2} - \sqrt{r^2 + 16a^2}$$

Thus,

$$\begin{aligned}\Psi_A &= \frac{\mu_0 I}{4\pi}\int_0^a \frac{1}{r}\left(\sqrt{r^2 + 16a^2} - \sqrt{r^2 + 9a^2} - \sqrt{r^2 + 25a^2} + \sqrt{r^2 + 16a^2}\right) dr \\ &= \frac{\mu_0 I}{4\pi}\int_0^a \frac{1}{r}\left(2\sqrt{r^2 + 16a^2} - \sqrt{r^2 + 9a^2} - \sqrt{r^2 + 25a^2}\right) dr\end{aligned}$$

Looking up a Table of Integrals we find

$$\int \frac{\sqrt{r^2 + \zeta^2}}{r}\,dr = \sqrt{\zeta^2 + r^2} + \zeta \ln r - \zeta \ln[\zeta^2 + \zeta\sqrt{\zeta^2 + r^2}]$$

Thus we have

$$\int_0^a \frac{\sqrt{r^2+16a^2}}{r}\,dr = \left\{\sqrt{16a^2+r^2} + 4a\ln r - 4a\ln[16a^2 + 4a\sqrt{16a^2+r^2}]\right\}_0^a$$
$$= a(\sqrt{17}-\sqrt{16}) + 4a\ln a - 4a\ln(0) - 4a\ln[16a^2+4a^2\sqrt{17}]$$
$$+ 4a\ln[16a^2+4a^2\sqrt{16}]$$

Note that the ln(0) term does not cause a problem since this term cancels out when we integrate the other two terms in the expression for Ψ_A. Integrating the other two terms we find

$$\Psi_A = \frac{\mu_0 I}{4\pi}\left\{2\left[a\sqrt{17} - 4a + 4a\ln a - 4a\ln 0 - 4a\ln[16a^2 + a^2\sqrt{17}] + 4a\ln(32a^2)\right]\right\}$$
$$- \left[a\sqrt{10} - 3a + 3a\ln a - 3a\ln 0 - 3a\ln[9a^2 + 3a^2\sqrt{10}] + 3a\ln(18a^2)\right]$$
$$- \left[a\sqrt{26} - 5a + 5a\ln a - 5a\ln 0 - 5a\ln[25a^2 + 5a^2\sqrt{26}] + 5a\ln(50a^2)\right]$$
$$= \frac{\mu_0 I}{4\pi}\left[a(2\sqrt{17}-\sqrt{10}-\sqrt{26}) - 8a\ln(16a^2 4\sqrt{17}a^2) + 3a\ln(9a^2+3\sqrt{10}a^2)\right.$$
$$\left. + 5a\ln(25a^2+5\sqrt{16}a^2) + 8a\ln(32a^2) - 3a\ln(18a^2) - 5a\ln(50a^2)\right]$$
$$\Psi_A = \frac{\mu_0 I}{4\pi}\left[a(2\sqrt{17}-\sqrt{10}-\sqrt{26}) - 8a\ln\left(\frac{4+\sqrt{17}}{8}\right) + 3a\ln\left(\frac{3+\sqrt{10}}{6}\right)\right.$$
$$\left. + 5a\ln\left(\frac{5+\sqrt{26}}{10}\right)\right]$$

We now proceed with side B, selecting our z axis as shown in Figure 6.16e. We have

$$\Psi_B = \int_{4a}^{5a}\int_{-a/2}^{a/2}\frac{\mu_0 I}{4\pi r}\left[\frac{z+a/2}{\sqrt{r^2+(z+a/2)^2}} - \frac{z-a/2}{\sqrt{r^2+(z-a/2)^2}}\right]dz\,dr$$
$$= \frac{\mu_0 I}{4\pi}\int_{4a}^{5a}\frac{1}{r}\int_{-a/2}^{a/2}\left[\frac{z+a/2}{\sqrt{r^2+(z+a/2)^2}} - \frac{z-a/2}{\sqrt{r^2+(z-a/2)^2}}\right]dz\,dr$$
$$= \frac{\mu_0 I}{4\pi}\int_{4a}^{5a}\frac{2}{r}\int_{0}^{a/2}\left[\frac{z+a/2}{\sqrt{r^2+(z+a/2)^2}} - \frac{z-a/2}{\sqrt{r^2+(z-a/2)^2}}\right]dz\,dr$$

Using the same substitution as was used for Ψ_A, the two terms under the integral over z yield

$$\int_0^{a/2}\frac{z+a/2}{\sqrt{r^2+(z+a/2)^2}}\,dz = \sqrt{r^2+a^2} - \sqrt{r^2+(a/2)^2}$$
$$\int_0^{a/2}\frac{z-a/2}{\sqrt{r^2+(z-a/2)^2}}\,dz = \sqrt{r^2} - \sqrt{r^2+(a/2)^2}$$

Thus we have

$$\Psi_{\rm B} = \frac{\mu_0 I}{2\pi}\int_{4a}^{5a}\frac{1}{r}\left(\sqrt{r^2+a^2}-\sqrt{r^2}\right)dr$$

$$= \frac{\mu_0 I}{2\pi}\left\{\sqrt{a^2+r^2}+a\ln r - a\ln\left[a^2+a\sqrt{a^2+r^2}\right]\right\}_{4a}^{5a} - a$$

$$= \frac{\mu_0 I}{2\pi}\left[a\sqrt{26}+a\ln(5a)-a\ln(a^2+a^2\sqrt{26})-a\sqrt{17}-a\ln(4a)+a\ln(a^2+a^2\sqrt{17})-a\right]$$

$$\Psi_{\rm B} = \frac{\mu_0 I}{2\pi}\left[a(\sqrt{26}-\sqrt{17}-1)+a\ln\left(\frac{5}{4}\right)-a\ln\left(\frac{1+\sqrt{26}}{1+\sqrt{17}}\right)\right]$$

It is clear that the contribution from side D should be quite similar to that from side B except for all distances being smaller by an amount a. We have

$$\Psi_{\rm D} = -\int_{3a}^{4a}\int_{-a/2}^{a/2}\frac{\mu_0 I}{4\pi r}\left[\frac{z+a/2}{\sqrt{r^2+(z+a/2)^2}}-\frac{z-a/2}{\sqrt{r^2+(z-a/2)^2}}\right]dz\,dr$$

$$= \frac{-\mu_0 I}{2\pi}\left[a(\sqrt{17}-\sqrt{10}-1)+a\ln\left(\frac{4}{3}\right)-a\ln\left(\frac{1+\sqrt{17}}{1+\sqrt{10}}\right)\right]$$

Also noting that by symmetry we have $\Psi_{\rm C} = \Psi_{\rm A}$, we are now ready to find the total magnetic flux produced by coil 1 and linked by coil 2. We have

$$\Psi_{12} = \Psi_{\rm A}+\Psi_{\rm B}+\Psi_{\rm C}+\Psi_{\rm D} = 2\Psi_{\rm A}+\Psi_{\rm B}++\Psi_{\rm D}$$

Evaluating the numerical values of the various terms gives

$$\Psi_{12} \simeq -0.024\frac{\mu_0 I a}{2\pi}$$

where the minus sign simply indicates the fact that the net flux is into the paper. The mutual inductance is then found to be

$$L_{12} = \frac{|\Psi_{12}|}{I} = 0.024\frac{\mu_0 a}{2\pi} = 4.8\times10^{-9}\,a\ \text{H}$$

(c) With coil 1 having $N_1 = 10$ turns and coil 2 having $N_2 = 100$ turns, the mutual inductance is larger by a factor of N_1N_2, or $L_{12} = 4.8a\ \mu$H, with a in meters. For $I = \sin(2\pi\times10^4 t)$, the induced voltage is

$$\mathcal{V}_{\rm ind} = L_{12}\frac{dI}{dt} = -(4.8\times10^{-6}a)(2\pi\times10^4)\cos(2\pi\times10^4 t) = -0.3016a\cos(2\pi\times10^4 t)\ \text{V}$$