7-3. Two circular coils. (a) Using the result of Example 6-6, the magnetic flux density produced on the axis of coil 1 with radius $a_1 = 5$ cm, current $I_1 = 100\cos(2000\pi t)$ A, and number of turns $N_1 = 10$, at a distance of $z = 1$ m away from its center can be calculated as

$$B_{1z} = \frac{\mu_0 N_1 I_1 a_1{}^2}{2(a_1{}^2 + z^2)^{3/2}}$$
$$= \frac{(4\pi \times 10^{-7})(10)[100\cos(2000\pi t)](0.05)^2}{2[(0.05)^2 + 1^2]^{3/2}} \simeq 1.56\cos(2000\pi t)\ \mu\text{T}$$

The voltage induced across the terminals of coil 2 ($a_2 = 5$ cm and $N_2 = 100$) due to the time-varying flux Ψ_{12} produced by coil 1 linking coil 2 can be evaluated as

$$\mathcal{V}_{\text{ind2}} = -\frac{d\Psi_{12}}{dt} = -N_2(\pi a_2{}^2)\frac{dB_{1z}}{dt}$$
$$\simeq -(100)[\pi(0.05)^2][-(1.56 \times 10^{-6})(2000\pi)\sin(2000\pi t)]$$
$$\simeq 7.72 \times 10^{-3}\sin(2000\pi t)\ \text{V}$$

Therefore, the amplitude of the induced voltage $\mathcal{V}_{\text{ind2}}$ is ~ 7.72 mV.

(b) Repeating the same calculations for the same current I_1 oscillating at 10 kHz [i.e., $I_1(t) = 100\cos(20000\pi t)$ A] results in an induced voltage $\mathcal{V}_{\text{ind2}}$ of amplitude ~ 77.2 mV.

7-5. Triangular loop and long wire. Let $I = I_0\cos(2\pi f_0 t)$, where $f_0 = 60$ Hz. Based on the result of Example 6-3, the **B**-field due to the current in the long wire is $B = \mu_0 I/(2\pi r)$, where r is the radial distance from the wire. By symmetry, we expect the magnetic fluxes linked by the two halves of the triangular loop to be the same. Thus,

$$\Psi_\triangle = 2\Psi_{S2} = 2\int_d^{d+a\sqrt{3}/2}\int_0^{a/2+d/\sqrt{3}-y/\sqrt{3}} \frac{\mu_0 I_0 \cos(2\pi f_0 t)}{2\pi y}\,dx\,dy$$
$$= \frac{\mu_0 I_0\cos(2\pi f_0 t)}{\pi}\int_d^{d+a\sqrt{3}/2}\frac{1}{y}\left(\frac{a}{2} + \frac{d}{\sqrt{3}} - y\sqrt{3}\right)dy$$
$$= \frac{\mu_0 I_0\cos(2\pi f_0 t)}{\pi}\left[\left(\frac{a}{2} + \frac{d}{\sqrt{3}}\right)\ln\left(\frac{2d + a\sqrt{3}}{2d}\right) - \frac{a}{2}\right]$$

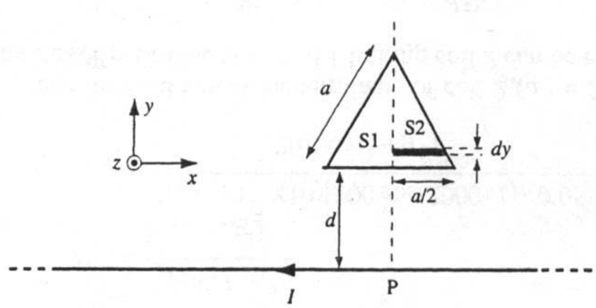

Fig. 7.1. Figure for Problem 7-5.

The induced emf is then given by

$$\mathcal{V}_{\text{ind}} = -\frac{d\Psi_{\triangle}}{dt} = 2f_0\mu_0 I_0 \sin(2\pi f_0 t)\left[\left(\frac{a}{2}+\frac{d}{\sqrt{3}}\right)\ln\left(\frac{2d+a\sqrt{3}}{2d}\right)-\frac{a}{2}\right]$$

so that the peak value of induced emf (which is also equal to the product of the measured 1 mA loop current and 0.01Ω loop resistance) is

$$[\mathcal{V}_{\text{ind}}]_{\text{peak}} = 2f_0\mu_0 I_0\left[\left(\frac{a}{2}+\frac{d}{\sqrt{3}}\right)\ln\left(\frac{2d+a\sqrt{3}}{2d}\right)-\frac{a}{2}\right] = (1\text{ mA})(0.01\Omega) = 10\mu\text{V}$$

from which we can find $I_0 \simeq 6.96$ A.

7-7. Current transformer. The voltage induced in the toroidal coil due to the current flowing in the long wire is determined by the mutual inductance between the wire and the circular toroidal coil. To find the mutual inductance, we assume current I to flow in the wire and determine the magnetic flux produced by this current and linked by the toroid. The magnetic field which is produced by the long wire is given by

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\frac{\mu I}{2\pi r}$$

where we have taken the z axis to be along the wire, so that the $\hat{\boldsymbol{\phi}}$ direction encircles the wire along the toroid. Using $\mathbf{B}$ given above along with the definitions shown in Figure 7.2, the mutual inductance between the circular toroid and the wire can be written as

$$\begin{aligned} L_{12} &= \frac{\Lambda}{I} = \frac{N}{I}\int_S \mathbf{B}\cdot d\mathbf{s} = \frac{N}{I}\int_S \frac{\mu I}{2\pi r}\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}}ds_\phi \\ &= \frac{\mu N}{2\pi}\int_0^{2\pi}\int_0^a \frac{RdRd\theta}{r_{\text{m}} - R\cos\theta} \end{aligned}$$

where r_{m} is the mean radius of the toroid. From integral tables†, we have

$$\int_0^\pi \frac{du}{1\pm k\cos u} = \frac{\pi}{\sqrt{1-k^2}}$$

Using this integral in the inductance expression obtained above, we find

$$L_{12} = \frac{\mu N}{2\pi}\int_0^a\left[\frac{2\pi}{\sqrt{r_{\text{m}}^2 - R^2}}\right]RdR = \mu_r\mu_0 N\left[-\sqrt{r_{\text{m}}^2-R^2}\right]\Big|_0^a = \mu_r\mu_0 N\left[r_{\text{m}} - \sqrt{r_{\text{m}}^2 - a^2}\right]$$

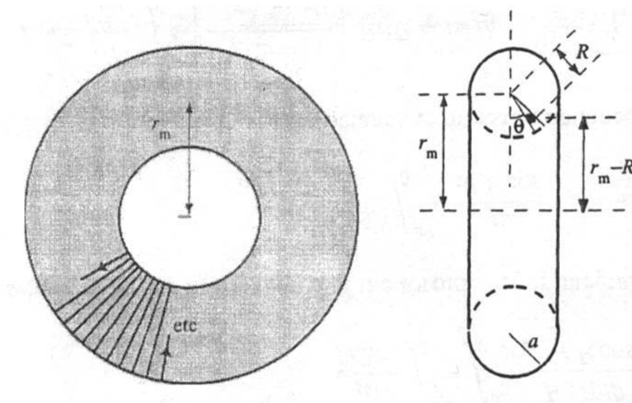

Fig. 7.2. Figure for problem 7-7. (a) Top view of the toroid. (b)Cross-sectional view.

where μ_r is the relative permeability of the core material used.
The magnetic flux linkage through the circular toroidal coil with $N = 300$, $r_m = 3$ cm, cross-sectional radius $a = 0.5$ cm, and core material with $\mu_r = 200$ due to the 1000-A, 60-Hz current flowing through the high-voltage line is thus given by

$$\Psi_{12} = L_{12}I = \mu_r \mu_0 N \left[r_m - \sqrt{r_m^2 - a^2} \right] I(t)$$

Substituting values, we have

$$\Psi_{12} = (200)(4\pi \times 10^{-7})(300) \left[(0.03) - \sqrt{(0.03)^2 - (0.005)^2} \right] I(t)$$
$$\simeq 3.16 \times 10^{-5}[1000 \cos(120\pi t)] \simeq 3.16 \times 10^{-2} \cos(120\pi t) \text{ Wb}$$

From Faraday's law, the rms value of the induced voltage across the terminals of the toroid can be calculated as

$$\mathcal{V}_{\text{ind}_{\text{rms}}} = -\frac{1}{\sqrt{2}} \frac{d\Psi_{12}}{dt} \simeq \frac{11.9}{\sqrt{2}} \sin(120\pi t) \simeq 8.43 \sin(120\pi t) \text{ V}$$

7-11. Rectangular loop and wire. Since the **B**-field produced by the wire is into the paper, its is convenient to choose the area element (for the loop) to also be into the paper, in which case the associated (via the right hand rule) $d\mathbf{l}$ element is outward at the terminal of the loop farther away from the wire. Based on the discussion on pages 568 to 570 of the text in connection with Figure 7.4 of the text the general rule here is that, in order for [7.1] to be valid, $\mathcal{V}_{\text{ind}}$ must be defined to be positive on the terminal at which the $d\mathbf{l}$ element, the polarity of which is associated with that of $d\mathbf{s}$ via the right hand rule, points outward.

With the loop moving to the right with a velocity v_0, the distance between the left edge of the loop and the long wire varies as $r(t) = v_0 t + r_1$, where r_1 is the initial distance between the loop and the wire. The flux linked by the loop is given by

$$\Psi = \frac{\mu_0 I b}{2\pi}\int_{v_0 t + r_1}^{v_0 t + r_1 + a}\frac{dr}{r} = \frac{\mu_0 I b}{2\pi}\left(1 + \frac{a}{v_0 t + r_1}\right)$$

The induced voltage is then given by

$$\mathcal{V}_{\text{ind}} = -\frac{d\Psi}{dt} = -\frac{\mu_0 I b}{2\pi}\left(\frac{v_0 t + r_1}{v_0 t + r_1 + a}\right)\left[\frac{-v_0 a}{(v_0 t + r_1)^2}\right] = \frac{\mu_0 I a b v_0}{2\pi(v_0 t + r_1)(v_0 t + r_1 + a)} \qquad t \geq 0$$

7-15. Extracting power from a power line. The constraint due to the length of the wire in hand can be expressed as

$$2a + 2b = 200 \text{ m} \qquad \longrightarrow \qquad a = 100 - b$$

Let the current flowing in the power line be $I = I_0 \sin(2\pi f t)$, where $I_0 = 4000$ A. The magnetic flux linked by the pick-up loop is

$$\Psi = b\int_{20}^{20+a}\frac{\mu_0 I}{2\pi r}dr = \frac{\mu_0 I_0 \sin(2\pi f t)}{2\pi} b \ln\left(\frac{20 + a}{20}\right) = \frac{\mu_0 I_0 \sin(2\pi f t)}{2\pi} b \ln\left(\frac{120 - b}{20}\right)$$

The induced voltage is then

$$\mathcal{V}_{\text{ind}} = -\frac{d\Psi}{dt} = f\mu_0 I_0 \cos(2\pi f t) b \ln\left(\frac{120 - b}{20}\right)$$

Power extracted is maximized if $\mathcal{V}_{\text{ind}}$ is maximized. Hence we need to find the value of b that maximizes $b\ln[(120 - b)/20]$. This can be done by differentiating with respect to b and setting equal to zero or by simple plotting or by trial and error. The result is $b \simeq 62$ m, which means in turn that $a \simeq 38$ m.

The choice of the optimum loop dimensions maximizes the induced voltage. Since the wire has nonzero resistance R_{wire}, we can view the induced voltage and the wire as a voltage source with voltage $V_s = \mathcal{V}_{\text{ind}}$ and source resistance $R_s = R_{\text{wire}}$. To extract maximum amount of power from such a source, we must use a load 'matched' to the source, i.e., $R_L = R_s$. The wire resistance can be calculated simply as

$$R_{\text{wire}} = \frac{l}{\sigma A} = \frac{l}{\sigma_{\text{copper}}\pi(d/2)^2} = \frac{200 \text{ m}}{(5.8 \times 10^7 \text{S-m}^{-1})\pi(4.1/10^{-3}/2)^2} \simeq 0.261\Omega$$

Thus we choose $r_L \simeq 0.261\Omega$. We then have

$$\mathcal{V}_{\text{ind}} = f\mu_0 I_0 \cos(2\pi f t)\left[62 \ln\left(\frac{120-62}{62}\right)\right] \simeq 19.9\cos(2\pi f t)$$

The power extracted by the load under conditions of maximum power transfer (i.e., $R_L = R_s$) is

$$P_L = \left(\frac{1}{2}\right)\frac{\mathcal{V}_{\text{ind}}^2}{R_L + R_s} \simeq 379.3[\cos^2(2\pi f t)] \text{ W} \quad \rightarrow \quad (P_L)_{\text{avg}} = 190 \text{ W}$$

(b) Try $N = 2$. Now our constraint is $a + b = 50$, or $a = 50 - b$. The flux linked is

$$\Psi = Nb\int_{20}^{20+a}\frac{\mu_0 I}{2\pi r}dr = \frac{\mu_0 I_0 \sin(2\pi f t)}{2\pi}2b\ln\left(\frac{20-b}{20}\right)$$

Thus, the induced voltage $\mathcal{V}_{\text{ind}}$ is maximized if $2b\ln[(20-b)/b]$ is maximized, which occurs for $b \simeq 29.16$. Using this value of b to calculate $\mathcal{V}_{\text{ind}}$ as before, we find a smaller value for P_L. Same holds true for other values of $N \geq 2$. Therefore, we conclude that using more turns does not allow us to extract higher power.

7-24. Displacement current in a capacitor. (a) Following an approach similar to Example 7-11, the displacement current through the capacitor can be found as

$$I_d = J_d A = A\frac{dD}{dt} = \epsilon A\frac{dE}{dt} = \frac{\epsilon_r \epsilon_0 A}{a}\frac{dV(t)}{dt} = -\frac{10\epsilon_r\epsilon_0 A(2\pi f)}{a}\sin(2\pi f t)$$

Substituting $A = 1 \text{ cm}^2$, $\epsilon_r = 6$, $a = 1$ mm, and $f = 10$ kHz yields $I_d \simeq -3.34\sin(2\pi \times 10^4 t)$ μA. Note that the displacement current is 90° out of phase with the applied voltage across (or the electric field inside) the capacitor.

(b) Repeating the calculations in part (a) for $f = 1$ MHz results in $I_d \simeq -0.334\sin(2\pi \times 10^6 t)$ mA. Note that the displacement current increases in amplitude with increase in frequency, i.e., displacement current becomes more and more important at higher frequencies.

7-26. Sea water. (a) Similar to Example 7-12, the ratio of the magnitudes of the conduction-current density to the displacement-current density is given by

$$\frac{|J_c|_{\max}}{|J_d|_{\max}} = \frac{\sigma}{\omega\epsilon} = \frac{\sigma}{(2\pi f)(\epsilon_r\epsilon_0)}$$

For sea water with $\sigma = 4$ S-m^{-1}, $\epsilon_r = 81$ and $\mu_r = 1$, we find

$$\frac{|J_c|_{\max}}{|J_d|_{\max}} \simeq \frac{8.88 \times 10^8}{f(\text{Hz})}$$

So, using this expression, the ratio of the magnitudes $|J_c|_{\max}/|J_d|_{\max}$ at 10 kHz, 1 MHz, 100 MHz, and 10 GHz are $\sim 8.88 \times 10^4$, ~ 888, ~ 8.88, and 8.88×10^{-2} respectively.

(b) The frequency at which the magnitude of the conduction-current density is equal to the magnitude of the displacement current density can be found as

$$\frac{|J_c|_{\max}}{|J_d|_{\max}} \simeq \frac{8.88 \times 10^8}{f(\text{Hz})} = 1 \quad \to, \quad f \simeq 888 \text{ MHz}$$

7-29. AM radio waves. Substituting

$$\overline{\mathscr{E}} = \hat{\mathbf{x}}\mathscr{E}_x(z,t) = \hat{\mathbf{x}}E_0\cos(7.5 \times 10^6 t - \beta z)$$

and

$$\overline{\mathscr{H}} = \hat{\mathbf{y}}\mathscr{H}_y(z,t) = \hat{\mathbf{y}}(E_0/\eta)\cos(7.5 \times 10^6 t - \beta z)$$

into [7.18a] yields

$$\nabla \times \overline{\mathscr{E}} = \frac{\partial \overline{\mathscr{B}}}{\partial t} \quad \to \quad \hat{\mathbf{y}}\frac{\partial \mathscr{E}_x}{\partial z} = -\hat{\mathbf{y}}\mu_0\frac{\partial \mathscr{H}_y}{\partial t}$$

$$\to (-\beta)E_0[-\sin(7.5 \times 10^6 t - \beta z)] = -\mu_0\frac{E_0}{\eta}(7.5 \times 10^6)[-\sin(7.5 \times 10^6 t - \beta z)]$$

resulting in

$$\to \quad \beta = \frac{\mu_0(7.5 \times 10^6)}{\eta}$$

Similarly, substituting $\overline{\mathscr{E}}$ and $\overline{\mathscr{H}}$ into [7.18c], we have

$$\nabla \times \overline{\mathscr{H}} = \frac{\partial \overline{\mathscr{D}}}{\partial t} = \epsilon_0\frac{\partial \overline{\mathscr{E}}}{\partial t} \quad \to \quad -\hat{\mathbf{x}}\frac{\partial \mathscr{H}_y}{\partial z} = \hat{\mathbf{x}}\epsilon_0\frac{\partial \mathscr{E}_x}{\partial t}$$

$$\to -(-\beta)\frac{E_0}{\eta}[-\sin(7.5 \times 10^6 t - \beta z)] = \epsilon_0 E_0(7.5 \times 10^6)[-\sin(7.5 \times 10^6 t - \beta z)]$$

resulting in

$$\longrightarrow \quad \frac{\beta}{\eta} = \epsilon_0 (7.5 \times 10^6)$$

Solving these two equations simultaneously, the values for β and η can be calculated as

$$\beta^2 = (7.5 \times 10^6)^2 \mu_0 \epsilon_0 \quad \longrightarrow \quad \beta = \frac{7.5 \times 10^6}{3 \times 10^8} = 0.025 \text{ rad-m}^{-1}$$

$$\eta^2 = \frac{\mu_0}{\epsilon_0} \quad \longrightarrow \quad \eta = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 377\Omega$$

Note that the $\overline{\mathscr{E}}$ and $\overline{\mathscr{H}}$ expressions also satisfy [7.18b] and [7.18d] as

$$\nabla \cdot \overline{\mathscr{D}} = \epsilon_0 \nabla \cdot \overline{\mathscr{E}} = \epsilon_0 \frac{\partial \mathscr{E}_x(z)}{\partial x} = 0$$

$$\nabla \cdot \overline{\mathscr{B}} = \mu_0 \nabla \cdot \overline{\mathscr{H}} = \mu_0 \frac{\partial \mathscr{H}_y(z)}{\partial y} = 0$$

7-36. Microwave heating of beef products. (a) >From loss tangent $\tan \delta_c = \epsilon_r''/\epsilon_r' = 0.33$ with $\epsilon_r' = 52.4$, we find $\epsilon_r'' \simeq 17.3$.

(b) The dissipated power density in beef with $E_{\text{peak}} = 25$ kV-m^{-1} can be calculated as

$$\begin{aligned} \frac{1}{2}\omega \epsilon_r'' \epsilon_0 E_{\text{peak}}^2 &\simeq \frac{1}{2}(2\pi \times 2.45 \times 10^9)(17.3)(8.85 \times 10^{-12})(2.5 \times 10^4)^2 \\ &\simeq 7.37 \times 10^8 \text{ W-m}^{-3} = 0.737 \text{ W-mm}^{-3} \end{aligned}$$