EECS 117 Homework Assignment 3 **Spring 2004**

1. Suppose the resonant frequency ω_0 is equal to $(LC)^{-0.5}$. The load impedance Z_L is

$$Z_L = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

If $\omega = \omega_0 + \Delta\omega$, Z_L is equal to

$$Z_L = R + j(\omega_0 + \Delta\omega)L + \frac{1}{j(\omega_0 + \Delta\omega)C} = R + j\left[\omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0}\right) - \frac{1}{\omega_0\left(1 + \frac{\Delta\omega}{\omega_0}\right)C}\right]$$

$$\approx R + j\left[\omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0}\right) - \frac{1}{\omega_0 C}\left(1 - \frac{\Delta\omega}{\omega_0}\right)\right] = R + j\left[\Delta\omega L + \frac{\Delta\omega}{\omega_0^{\,2} C}\right]$$

The last equality holds because $\omega_0 = (LC)^{-0.5}$. Furthermore,

$$R + j\left[\Delta\omega L + \frac{\Delta\omega}{\omega_0^{\,2} C}\right] = R + j\frac{\Delta\omega}{\omega_0}\left[\omega_0 L + \frac{1}{\omega_0 C}\right] = R + j\frac{\Delta\omega}{\omega_0}(2\omega_0 L) = R + j2\Delta\omega L$$

Using the values of the inductance and capacitance, the length of 2 cm corresponds 1.5π.

$$\beta_0 l = \frac{\omega_0}{v} l = \frac{1}{v\sqrt{LC}} l = \frac{2cm}{10^8\, m/s\sqrt{2\times10^{-9}\times 9.01\times10^{-13}}} = 1.5\pi$$

In general, $\beta l = (\beta_0 + \Delta\beta)l = \frac{\omega_0 + \Delta\omega}{v} l = \beta_0 l(1 + \frac{\Delta\omega}{\omega_0})$. Thus Z_{in} is

$$Z_{in} = Z_0 \frac{R + j2\Delta\omega L + jZ_0 \tan(1.5\pi(1 + \Delta\omega/\omega_0))}{Z_0 + j(R + j2\Delta\omega L)\tan(1.5\pi(1 + \Delta\omega/\omega_0))} \approx Z_0 \frac{jZ_0 \tan(1.5\pi(1 + \Delta\omega/\omega_0))}{j(R + j2\Delta\omega L)\tan(1.5\pi(1 + \Delta\omega/\omega_0))}$$

$$= \frac{Z_0^{\,2}}{(R + j2\Delta\omega L)}$$

The approximation holds because $\tan(1.5\pi(1 + \Delta\omega/\omega_0)) >> R, \Delta\omega L, Z_0$. This expression has the same form as a parallel RLC circuit, with

$$R_{eq} = \frac{Z_0^{\,2}}{R},\; C_{eq} = \frac{L}{Z_0^{\,2}},\; L_{eq} = \frac{1}{\omega_0^{\,2}\frac{L}{Z_0^{\,2}}} = Z_0^{\,2} C$$

Therefore, the input impedance Z_{in} is that of a second order circuit. Also, $L_{eq}C_{eq} = LC = \omega_0^{-2}$, so our assumption is correct, i.e., $\omega_0 = (LC)^{-0.5}$. The Q factor is

$$Q = \omega_0 R_{eq} C_{eq} = \frac{L/R}{\sqrt{LC}} = \sqrt{\frac{L}{C}} R = 94$$

The equivalent circuit is $L_{eq} = Z_0^2 C = 5$ μF, $C_{eq} = L/Z_0^2 = 0.3604$ fH, and $R_{eq} = Z_0^2/R = 1250$ in parallel.

2. For a lossy line, the input impedance has a form

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tanh(\alpha l + j\beta l)}{Z_0 + jZ_L \tanh(\alpha l + j\beta l)}$$

where $Z_L = 2j\Delta\omega L$. Also,

$$\tanh(\alpha l + j\beta l) = \frac{\sinh(2\alpha l)}{\cos(2\beta l) + \cosh(2\alpha l)} + j\frac{\sin(2\beta l)}{\cos(2\beta l) + \cosh(2\alpha l)}$$

$\sin(2\beta l) = \sin(3\pi(1 + \Delta\omega/\omega_0)) \approx \sin(3\pi) + \cos(3\pi) * (3\pi\Delta\omega/\omega_0) = -3\pi\Delta\omega/\omega_0$. Likewise, $\cos(2\beta l) \approx \cos(3\pi) - \sin(3\pi) * (3\pi\Delta\omega/\omega_0) = -1$. So,

$$\tanh(\alpha l + j\beta l) = \frac{\sinh(2\alpha l)}{\cosh(2\alpha l) - 1} - j\frac{3\pi\Delta\omega/\omega_0}{\cosh(2\alpha l) - 1}$$

The input impedance is then equal to

$$\begin{aligned}
Z_{in} &= Z_0 \frac{2j\Delta\omega L + jZ_0\left(\frac{\sinh(2\alpha l)}{\cosh(2\alpha l) - 1} - j\frac{3\pi\Delta\omega/\omega_0}{\cosh(2\alpha l) - 1}\right)}{Z_0 + j2j\Delta\omega L\left(\frac{\sinh(2\alpha l)}{\cosh(2\alpha l) - 1} - j\frac{3\pi\Delta\omega/\omega_0}{\cosh(2\alpha l) - 1}\right)} \\
&= Z_0 \frac{Z_0 \frac{3\pi\Delta\omega/\omega_0}{\cosh(2\alpha l) - 1} + j\left(2\Delta\omega L + Z_0 \frac{\sinh(2\alpha l)}{\cosh(2\alpha l) - 1}\right)}{Z_0 - 2\Delta\omega L \frac{\sinh(2\alpha l)}{\cosh(2\alpha l) - 1} + j2\Delta\omega L \frac{3\pi\Delta\omega/\omega_0}{\cosh(2\alpha l) - 1}} \\
&\approx Z_0 \frac{Z_0 3\pi\Delta\omega/\omega_0 + j\left(2\Delta\omega L(\cosh(2\alpha l) - 1) + Z_0 \sinh(2\alpha l)\right)}{Z_0(\cosh(2\alpha l) - 1) - 2\Delta\omega L \sinh(2\alpha l)}
\end{aligned}$$

The third term in the denominator has $\Delta\omega^2$ dependence and is thus negligible.

3-31. Quarter-wave matching. (a) For the first circuit with the single quarter-wave transformer, we find

$$Z_Q = \sqrt{Z_0 R_L} = \sqrt{(50)(400)} \simeq 141.4\Omega$$

For the second circuit involving two quarter-wave sections cascaded together, the input impedance of the two transformers can be written as

$$Z_{in2} = \frac{Z_{Q2}^2}{R_L} \quad \text{and}$$

$$Z_{in1} = \frac{Z_{Q1}^2}{Z_{in2}} = \frac{Z_{Q1}^2}{Z_{Q2}^2} R_L = Z_0$$

resulting in $Z_{Q1}/Z_{Q2} = \sqrt{Z_0/R_L}$. But it is also given that $Z_{Q1}Z_{Q2} = Z_0 R_L$. Therefore, solving these two equations simultaneously, we have

$$Z_{Q1}^2 = \sqrt{\frac{Z_0}{R_L}}(Z_0 R_L) = \sqrt{Z_0^3 R_L} = \sqrt{(50)^3(400)}$$

yielding $Z_{Q1} \simeq 84.1\Omega$ and

$$Z_{Q2} = \frac{Z_0 R_L}{Z_{Q1}} \simeq \frac{(50)(400)}{84.1} \simeq 238\Omega$$

respectively.

(b) At 15% above the design frequency we have for the first circuit:

$$Z_{in} = Z_Q \frac{R_L + jZ_Q \tan[(2\pi)(1/4)(1.15)]}{Z_Q + jR_L \tan[(2\pi)(1/4)(1.15)]} \simeq 60.21e^{j29.33°}$$

and

$$|\Gamma_{in}| = \left|\frac{Z_{in} - 50}{Z_{in} + 50}\right| \simeq 0.278 \quad \rightarrow \quad S = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} \simeq 1.768$$

and for the second circuit we have

$$Z_{in} = Z_{Q1} \frac{Z'_{in} + jZ_{Q1} \tan[(2\pi)(1/4)(1.15)]}{Z_{Q1} + jZ'_{in} \tan[(2\pi)(1/4)(1.15)]}$$

where

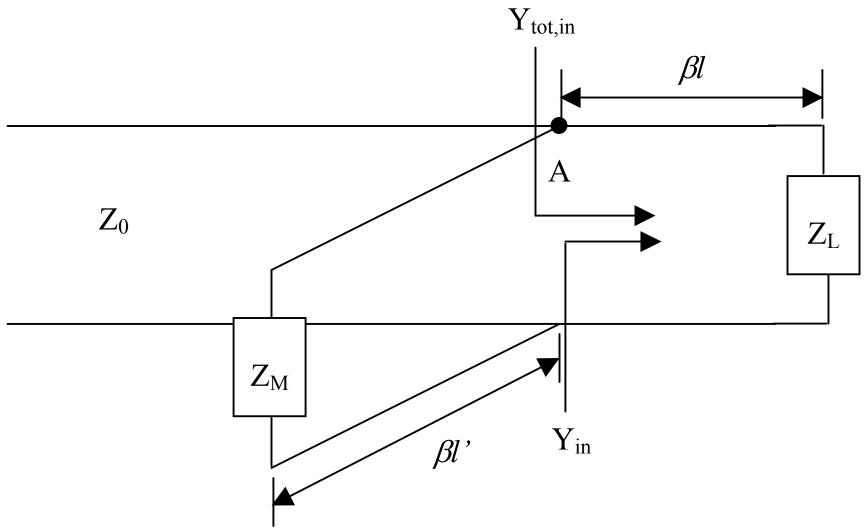
$$Z'_{in} = Z_{Q2} \frac{R_L + jZ_{Q2} \tan[(2\pi)(1/4)(1.15)]}{Z_{Q2} + jR_L \tan[(2\pi)(1/4)(1.15)]}$$

substituting values we find $Z_{in} \simeq 44.55e^{j3.98°}$ and

$$|\Gamma_{in}| = \left|\frac{Z_{in} - 50}{Z_{in} + 50}\right| \simeq 0.0673 \quad \rightarrow \quad S = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} \simeq 1.144$$

Similar analysis for 15% below the design frequency gives $S \simeq 1.22$ for the first circuit and $S \simeq 1.02$ for the second circuit.

4. This problem is similar to the example 3-16 in the textbook. For matching of the load with either a short or an open shunt stub, we have the following circuit:



Where Z_M is either 0 or infinity for a short or an open stub, respectively.

Since we are dealing with shunt stub, admittance would simplify the calculation. The equivalent admittance at point A (excluding the stub for a moment) is given by

$$Y_{in} = Y_0 \frac{Y_L + jY_0 \tan(\beta l)}{Y_0 + jY_L \tan(\beta l)}$$

where $Y_0 = [Z_0]^{-1} = 1/50$, and $Y_L = [Z_0]^{-1} = 1/100 + j3/100$.

Separating the expression into the real and imaginary parts gives

$$Y_{in} = \frac{1+x^2}{(10+15x)^2 + 25x^2} + j\frac{3x^2 - 3x - 3}{(10+15x)^2 + 25x^2}$$

where $x = \tan(\beta l)$.

Since a short or an open stub can behave as a purely reactive element, the real part of *Yin* above should be equal to Z_0 in order for the matching to take place. Thus, we have,

$$\frac{1+x^2}{(10+15x)^2 + 25x^2} = Y_0 = \frac{1}{50}$$

The solution of this equation is: $\tan(\beta l) = \dfrac{-3-\sqrt{5}}{4}$ or $\dfrac{-3+\sqrt{5}}{4}$. Both of them are negative. This means that βl is larger than $\pi/2$. The more negative value represents a point closer to the load and this value will be used in the following calculation.

$$\tan(\beta l) = \frac{-3-\sqrt{5}}{4} \Leftrightarrow \beta l = 2.22315.$$

Substitute this value into the imaginary part of *Yin* and get

$$j\left.\frac{3x^2-3x-3}{(10+15x)^2+25x^2}\right|_{x=\frac{-3-\sqrt{5}}{4}} = 0.044721\,j$$

The stub needs to have an impedance opposite to the value above in order to cancel the reactive part of the *Yin* for matching. For a short stub, $Y = -jY_0\cot(\beta l')$. Therefore, we want

$$\frac{0.044721}{Y_0} = \cot(\beta l') \quad \Leftrightarrow \beta l' = 0.420534.$$

The input impedance at point A including the stub is

$$Y_{tot,in} = \frac{1+\tan^2(2.22315)}{(10+15\tan(2.22315))^2+25\tan^2(2.22315)} + j\left(\frac{3\tan^2(2.22315)-3\tan(2.22315)-3}{(10+15\tan(2.22315))^2+25\tan^2(2.22315)} - \frac{Y_0}{\tan(0.420534)}\right)$$

The imaginary part is equal to zero. The SWR is given by

$$S = \frac{1+\left|\frac{Y_0-Y_{in}}{Y_0+Y_{in}}\right|}{1-\left|\frac{Y_0-Y_{in}}{Y_0+Y_{in}}\right|}$$

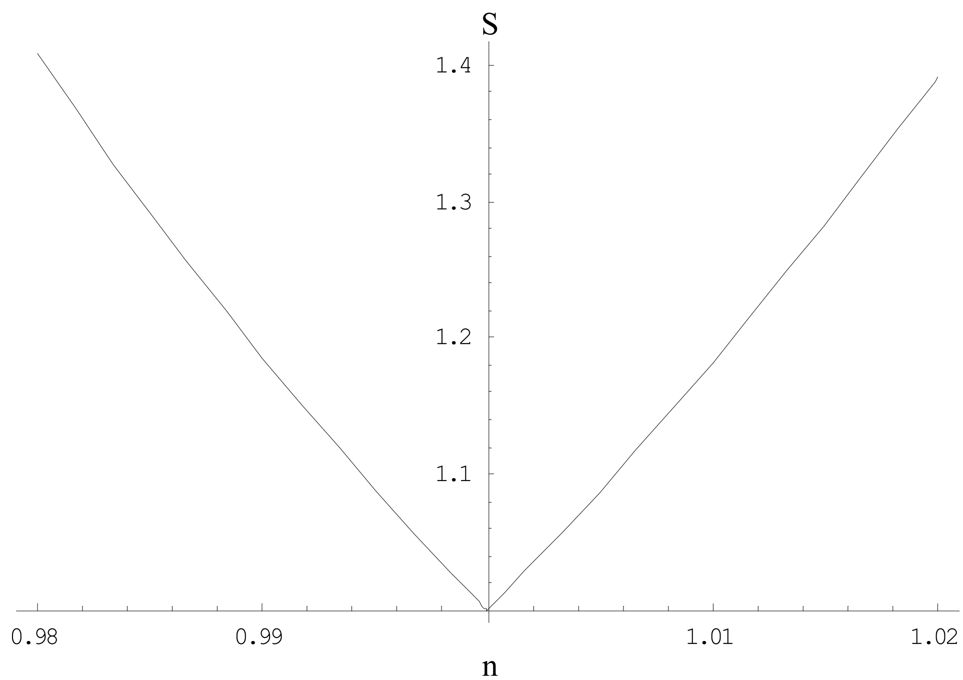
In general, for $\beta' \neq \beta$, i.e., $\beta' l = \beta l * n$, where n is the ratio of β' to β, the input impedance shown above is

$$Y_{tot,in}(n) = \frac{1+\tan^2(2.22315n)}{\left(10+15\tan(2.22315n)\right)^2 + 25\tan^2(2.22315n)} + j\left(\frac{3\tan^2(2.22315n) - 3\tan(2.22315n) - 3}{\left(10+15\tan(2.22315n)\right)^2 + 25\tan^2(2.22315n)} - \frac{Y_0}{\tan(0.420534 * n)}\right)$$

and the SWR is equal to

$$S = \frac{1 + \left|\dfrac{Y_0 - Y_{in}(n)}{Y_0 + Y_{in}(n)}\right|}{1 - \left|\dfrac{Y_0 - Y_{in}(n)}{Y_0 + Y_{in}(n)}\right|}$$

SWR can be plotted as a function of n, representing the amount of shift in the signal frequency / phase constant from the ones used in calculating the numbers above.



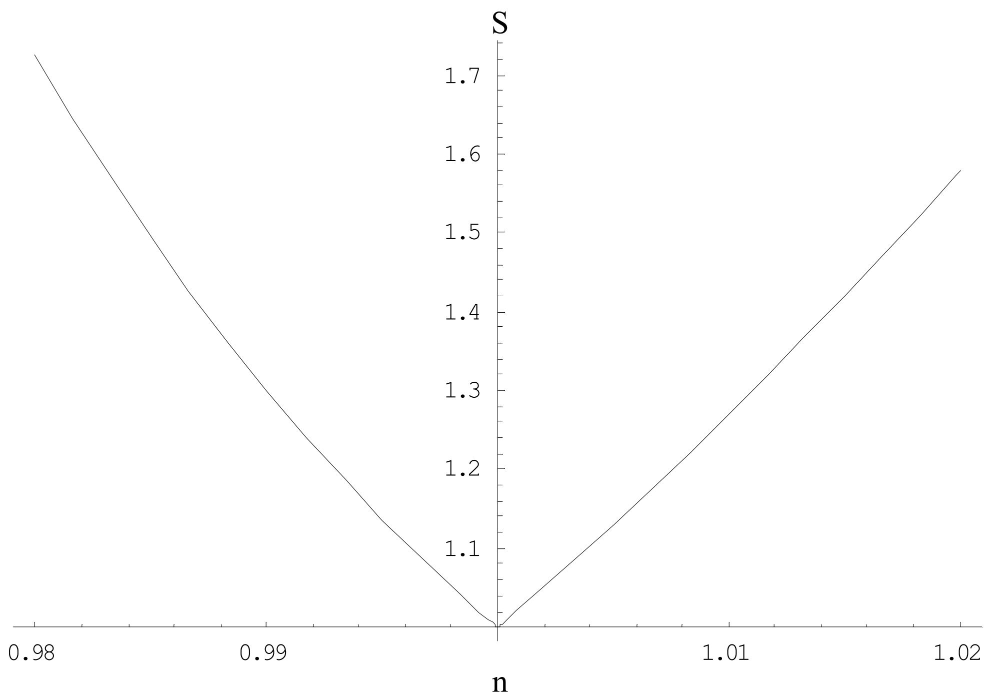
The values of n where S = 1.2 are 0.989 and 1.01, for a short stub.

Following the same procedure, we have these for an open stub

$$Y = jY_0\tan(\beta l') \Leftrightarrow \frac{0.044721}{Y_0} = -\tan(\beta l') \Leftrightarrow \beta l' = 1.99133.$$

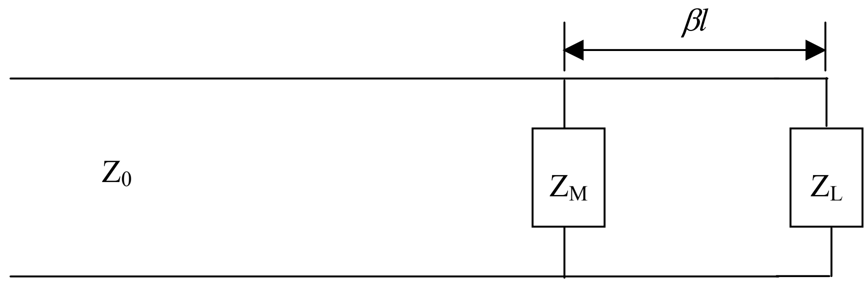
$$Y_{tot,in}(n) = \frac{1+\tan^2(2.22315n)}{(10+15\tan(2.22315n))^2 + 25\tan^2(2.22315n)}$$
$$+ j\left(\frac{3\tan^2(2.22315n) - 3\tan(2.22315n) - 3}{(10+15\tan(2.22315n))^2 + 25\tan^2(2.22315n)} - Y_0 \tan(1.99133 * n\right)$$

The plot of S vs. n:



The values of n where S = 1.2 are 0.993 and 1.01.

For impedance matching with a lumped element, we have the following circuit:



To keep the problem simple, let's make the lumped element Z_M a purely reactive component. With this constraint, the real part of Y_{in} needs to match the characteristic impedance of the transmission line, just like the cases of short and open stubs above. The previous calculation gives $\beta l = 2.22315$ or 2.95288. Just as before, we pick the shortest distance $\beta l = 2.22315$. At this length, Y_{in} is equal to

$$Y_{in} = 0.02 + 0.0447214j$$

The positive imaginary part implies that the lumped element needs to be an inductor in order to make the impedance matching work. So,

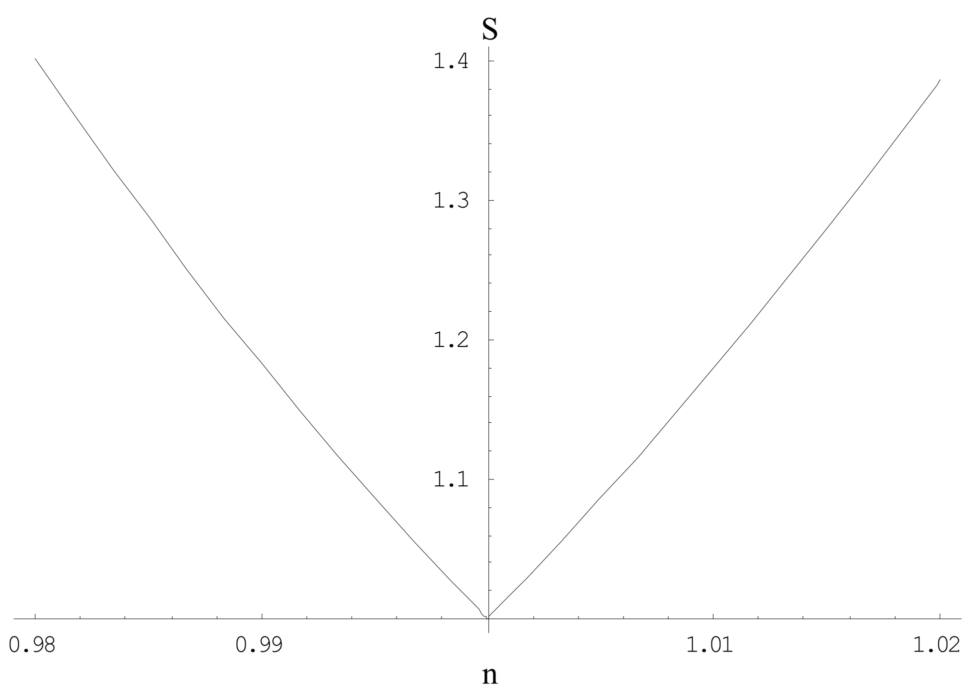
$$Y_M = 1/Z_M = -j/(\omega L) = -j/(\beta v L) = -0.0447214j$$

where v is the propagation velocity of the transmission line. The velocity is property of the transmission line, and does not depend on the signal frequency.

In general, $Y_{tot,in}$ is

$$Y_{tot,in}(n) = \frac{1+\tan^2(2.22315n)}{(10+15\tan(2.22315n))^2 + 25\tan^2(2.22315n)} + j\left(\frac{3\tan^2(2.22315n) - 3\tan(2.22315n) - 3}{(10+15\tan(2.22315n))^2 + 25\tan^2(2.22315n)} - \frac{0.04427214}{n}\right)$$

The plot of S vs. n:



The values of n where S = 1.2 are 0.989 and 1.01.

In summary,

	Bandwidth for $S \leq 1.2$
Short	$0.989\ \omega_0$ to $1.01\ \omega_0$
Open	$0.993\ \omega_0$ to $1.01\ \omega_0$
Lumped	$0.989\ \omega_0$ to $1.01\ \omega_0$

3-38. Quarter-wave matching. (a) We start by writing the input impedance of the 50Ω transmission line of length l on the right looking toward the load $Z_L = 40 + j30\ \Omega$ as

$$
\begin{aligned}
Z_1 =& (50)\frac{(40 + j30) + j50T}{50 + j(40 + j30)T} = (50)\frac{40 + j(30 + 50T)}{(50 - 30T) + j40T} \\
=& (50)\frac{[40 + j(30 + 50T)][(50 - 30T) - j40T]}{(50 - 30T)^2 + (40T)^2}
\end{aligned}
$$

where $T = \tan(\beta l)$ and $\beta = 2\pi/\lambda$. Note that the input impedance of the 50Ω line at the location of the quarter-wave transformer must be purely real so that a match can be achieved. Therefore, to find the location of the quarter-wave transformer with respect to the load, we equate the imaginary part of Z_1 to zero, i.e.,

$$
\mathcal{I}m\{Z_1\} = 0 \quad \rightarrow \quad (30 + 50T)(50 - 30T) - (40)^2 T = 0
$$

$$
\rightarrow \quad 15T^2 = 15 \quad \rightarrow \quad T = \tan(\beta l) = \pm 1
$$

From $\tan(\beta l) = +1$, we find $l/\lambda = 0.125$ and from $\tan(\beta l) = -1$, we find $l/\lambda = 0.375$. If we choose the nearest location (i.e., $l = 0.125\lambda$) for the quarter-wave transformer design, then $T = +1$, and substituting this value into the Z_1 expression above yields

$$Z_1 = (50)\frac{(40 + j30) + j50}{50 + j(40 + j30)} = (50)\frac{40 + j80}{20 + j40} = 100\Omega$$

which is a purely resistive impedance, as expected. Using this value of Z_1, we can now determine the characteristic impedance of the quarter-wave transformer inserted at a distance of $l = 0.125\lambda$ away from the load to match the load impedance $Z_L = 40 + j30\ \Omega$ to the $Z_0 = 50\Omega$ line as

$$Z_Q = \sqrt{Z_0 Z_1} = \sqrt{(50)(100)} \simeq 70.7\Omega$$

(Note that if the other location was chosen for the design, then $T = -1$, and the input impedance of the 50Ω line at that location is $Z_1 = 25\Omega$ and therefore for a quarter-wave transformer introduced at that position, the characteristic impedance would be $Z_Q = \sqrt{(50)(25)} \simeq 35.4\Omega$.)

(b) Following the same steps with $Z_L = 80 - j60\ \Omega$, we have

$$\begin{aligned} Z_1 =& (50)\frac{(80 - j60) + j50T}{50 + j(80 - j60)T} = (50)\frac{80 + j(50T - 60)}{(50 + 60T) + j80T} \\ =& (50)\frac{[80 + j(50T - 60)][(50 + 60T) - j80T]}{(50 + 60T)^2 + (80T)^2} \end{aligned}$$

Equating the imaginary part of Z_1 to zero yields

$$\mathcal{I}m\{Z_1\} = 0 \quad \rightarrow \quad (50T - 60)(50 + 60T) - (80)^2 T = 0$$

$$\rightarrow \quad 6T^2 - 15T - 6 = 0 \quad \rightarrow \quad T \simeq 2.85, -0.351$$

From $\tan(\beta l) \simeq 2.85$, we find $l/\lambda \simeq 0.196$ and from $\tan(\beta l) \simeq -0.351$, we find $l/\lambda \simeq 0.446$. Choosing $l \simeq 0.196\lambda$ (i.e., the nearest location with respect to the position of the load) for the design, the value of Z_1 at that position is

$$Z_1(l \simeq 0.196\lambda) \simeq (50)\left.\frac{80(50 + 60T) + 80T(50T - 60)}{(50 + 60T)^2 + (80T)^2}\right|_{T \simeq 2.85} \simeq 18.1\Omega$$

To match $Z_1 \simeq 18.1\Omega$ to 50Ω, we need a quarter-wave transformer with characteristic impedance given by

$$Z_Q = \sqrt{Z_0 Z_1} \simeq \sqrt{(50)(18.1)} \simeq 30.1\Omega$$