Sheet 7

In [ ]:
```
import numpy as np
from matplotlib import pyplot as plt
import torch
import torch.nn as nn
import scipy
```

1) Log-sum-exp and softmax

(a)

In [ ]:
```
s1 = np.array([1,2,3])
s2 = np.array([11,12,13])
s3 = np.array([10,20,30])

print(scipy.special.softmax(s1))
print(scipy.special.softmax(s2))
print(scipy.special.softmax(s3))
```

```
[0.09003057 0.24472847 0.66524096]
[0.09003057 0.24472847 0.66524096]
[2.06106005e-09 4.53978686e-05 9.99954600e-01]
```

(i) constant offset

$$
\begin{aligned}
\text{softmax}(\sigma, \lambda)_k &= \frac{exp(\lambda\sigma_k)}{\sum_{j=0}^{K} exp(\lambda\sigma_j)} \\
\text{softmax}(\sigma + c, \lambda)_k &= \frac{exp(\lambda\sigma_k + c)}{\sum_{j=0}^{K} exp(\lambda\sigma_j + c)} \\
&= \frac{exp(\lambda\sigma_k)exp(c)}{\sum_{j=0}^{K} exp(\lambda\sigma_j)exp(c)} \\
&= \frac{exp(c)exp(\lambda\sigma_k)}{exp(c)\left(\sum_{j=0}^{K} exp(\lambda\sigma_j)\right)} \\
&= \frac{exp(\lambda\sigma_k)}{\sum_{j=0}^{K} exp(\lambda\sigma_j)} \\
&= \text{softmax}(\sigma, \lambda)_k
\end{aligned}
$$

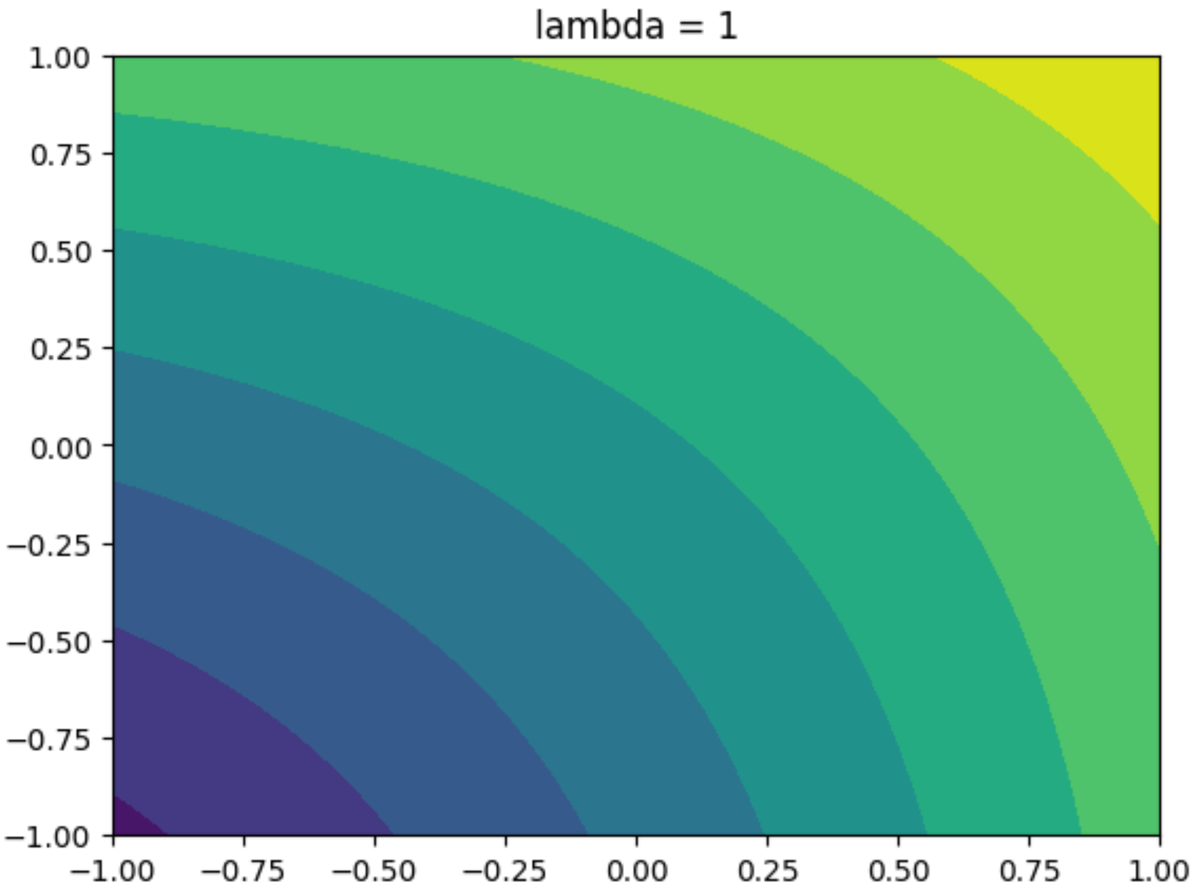
(ii) rescaling

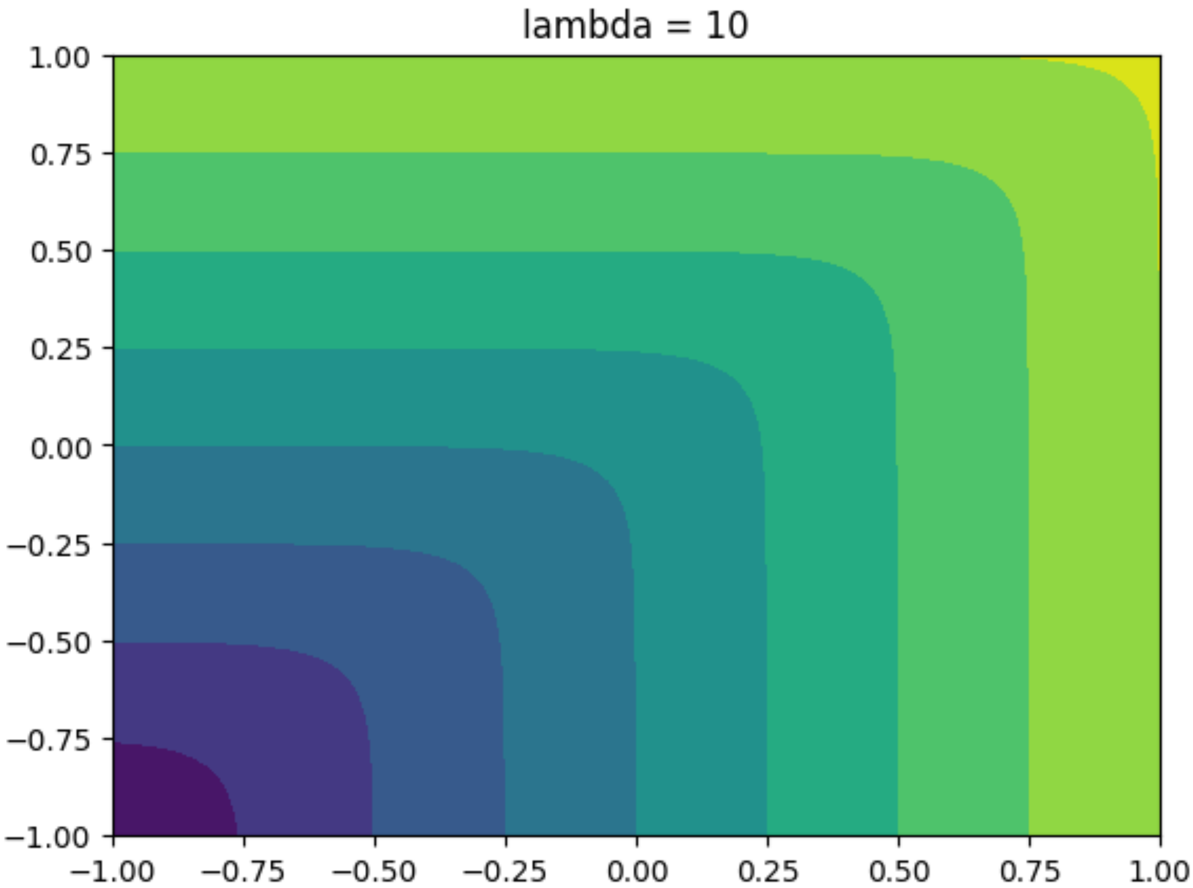
As seen in the calculations above, the softmax is not invariant for rescaling.

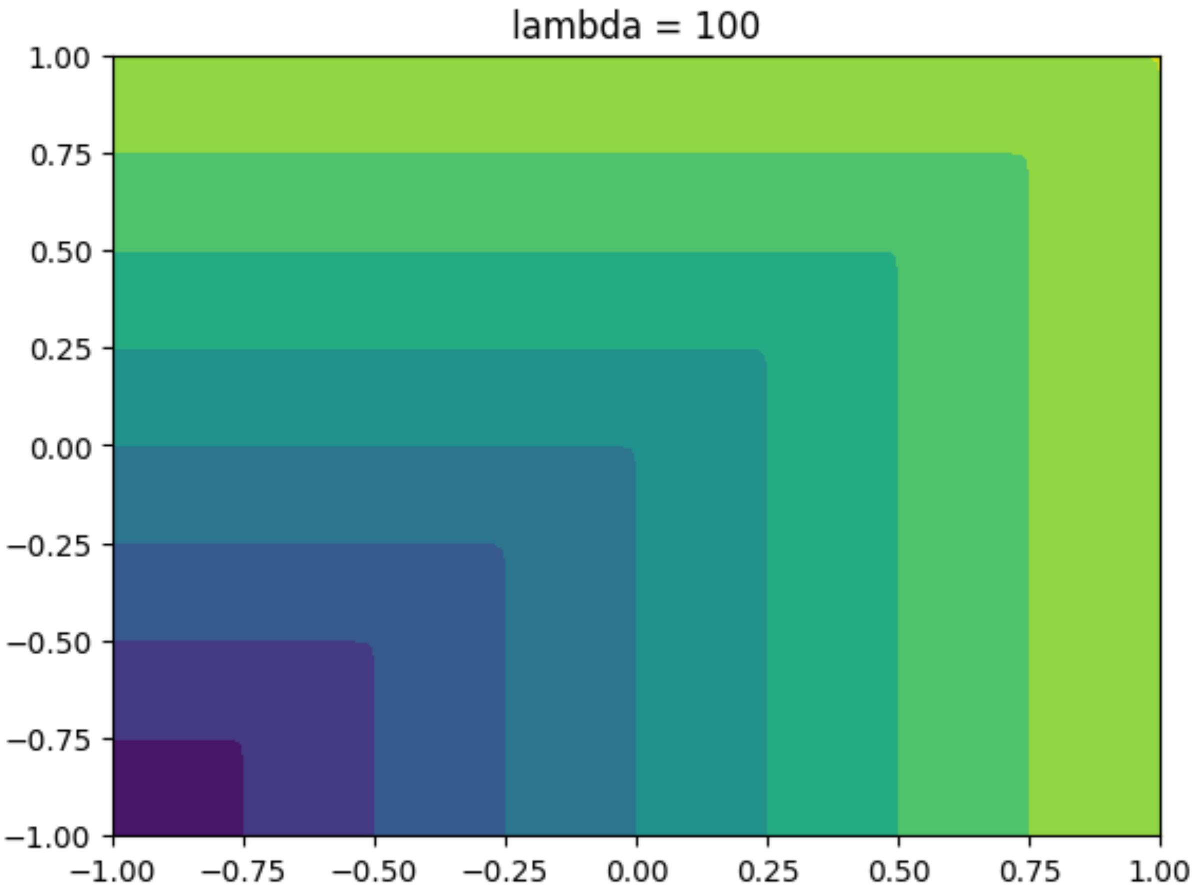
(b)

In [ ]:

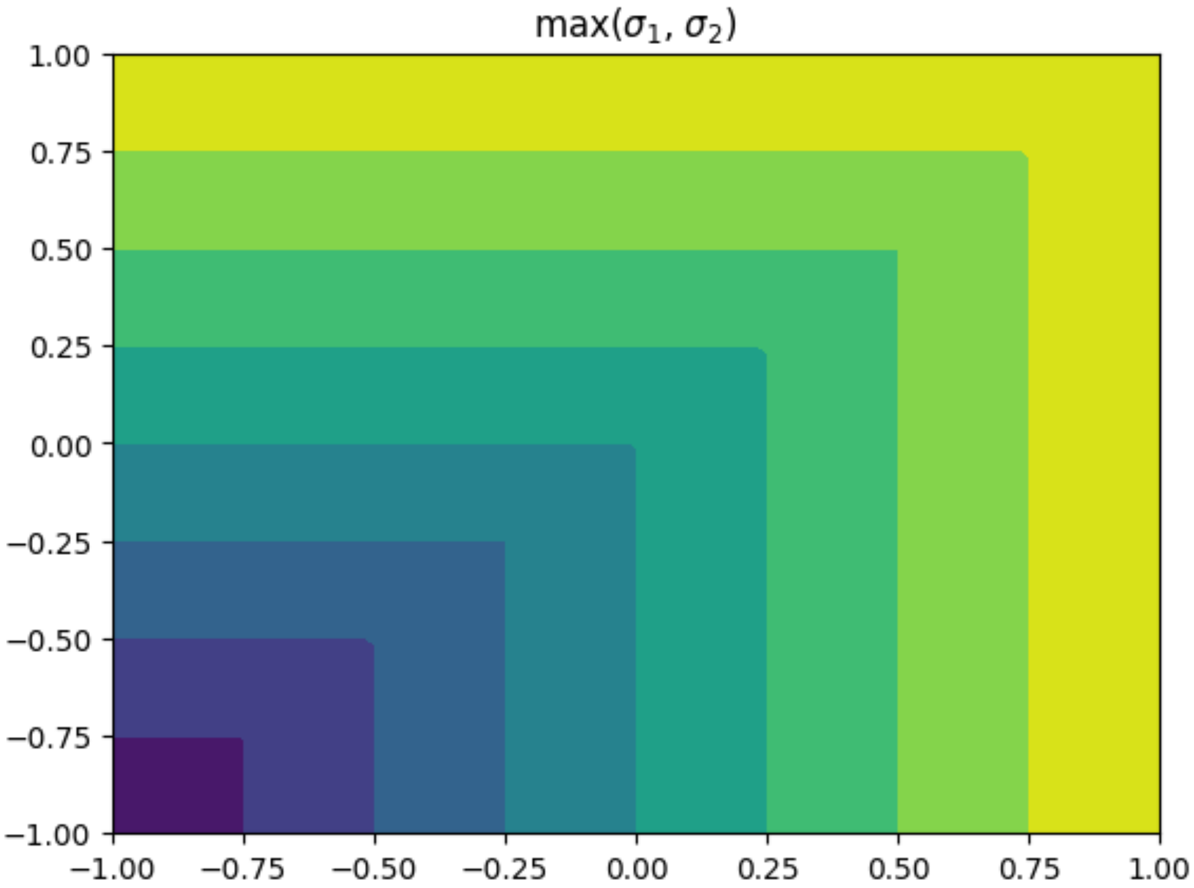
```
def logsumexp(x, lamb=1):
    # TODO: implement the logsumexp
    out = np.zeros(x[1].shape)
    for i in range(len(out[0])):
        for j in range(len(out[1])):
            out[i][j] = np.exp(lamb * x[0][i][j]) + np.exp(lamb * x[1][i]

    return 1/lamb * np.log(out)

# TODO: set up a grid of points in [-1, 1] x [-1, 1]
x = np.linspace(-1, 1, 100)
y = np.linspace(-1, 1, 100)
X, Y = np.meshgrid(x, y)

# TODO: calculate and plot the functions as specified in the task
plt.contourf(X,Y, logsumexp(np.array([X,Y]), 1))
plt.title("lambda = 1")
plt.show()

plt.contourf(X,Y, logsumexp(np.array([X,Y]), 10))
plt.title("lambda = 10")
plt.show()

plt.contourf(X,Y, logsumexp(np.array([X,Y]), 100))
plt.title("lambda = 100")
plt.show()
```




lambda = 10
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00


lambda = 100
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00

In [ ]:
```
def max(x):
    # TODO: implement the logsumexp
    out = np.zeros(x[1].shape)
    for i in range(len(out[0])):
        for j in range(len(out[1])):
            out[i][j] = np.maximum(x[0][i][j], x[1][i][j])

    return out

plt.contourf(X,Y, max(np.array([X,Y])))
plt.title("max($\sigma_1$, $\sigma_2$)")
plt.show()
```

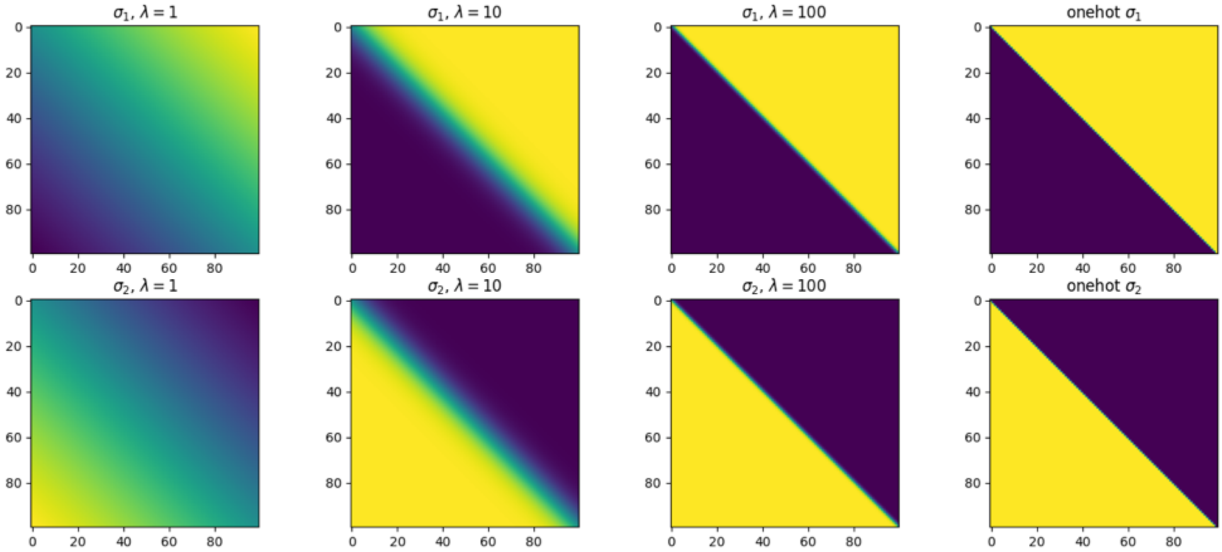


For bigger λ the lse will converge the max function.

(c)

```
In [ ]: def softmax(x, axis, lamb=1):
            # TODO: implement the softmax function. Axis should specify along whi
            out = np.zeros(x[1].shape)
            for i in range(len(out[0])):
                for j in range(len(out[1])):
                    out[i][j] = np.exp(lamb * x[0][i][j]) + np.exp(lamb * x[1][i]

            return np.exp(lamb * x) / out

        xy = softmax(np.array([X,Y]), axis=-1, lamb=1)

        # TODO: compute the argmax of each gridpoint in one-hot form
        def to_onehot(x):
            a = np.zeros(x.shape)
            b = np.zeros(x.shape)

            for i in range(x.shape[0]):
                for j in range(x.shape[1]):
                    if x[i][j] == 0:
                        a[i][j] = 1
                    elif x[i][j] == 1:
                        b[i][j] = 1

            return np.array([a, b])


        one_hotted = to_onehot(np.argmax(xy, axis=0))

        # TODO: make the plots as specified on the sheet (nicest is in a grid whi

        # plot the softmax
        fig, axs = plt.subplots(2, 4, figsize=(17, 7))
        axs[0, 0].imshow( xy[0])
        axs[0, 0].set_title("$\sigma_1$, $\lambda = 1$")
        axs[1, 0].imshow( xy[1])
        axs[1, 0].set_title("$\sigma_2$, $\lambda = 1$")
        axs[0, 1].imshow( softmax(np.array([X,Y]), axis=-1, lamb=10)[0])
        axs[0, 1].set_title("$\sigma_1$, $\lambda = 10$")
        axs[1, 1].imshow( softmax(np.array([X,Y]), axis=-1, lamb=10)[1])
        axs[1, 1].set_title("$\sigma_2$, $\lambda = 10$")
        axs[0, 2].imshow( softmax(np.array([X,Y]), axis=-1, lamb=100)[0])
        axs[0, 2].set_title("$\sigma_1$, $\lambda = 100$")
        axs[1, 2].imshow( softmax(np.array([X,Y]), axis=-1, lamb=100)[1])
        axs[1, 2].set_title("$\sigma_2$, $\lambda = 100$")
        axs[0, 3].imshow( one_hotted[0])
        axs[0, 3].set_title("onehot $\sigma_1$")
        axs[1, 3].imshow( one_hotted[1])
        axs[1, 3].set_title("onehot $\sigma_2$")


        # plot the onehot argmax
```

```
Out[ ]: Text(0.5, 1.0, 'onehot $\\sigma_2$')
```



(c)

$$
\begin{aligned}
\frac{\partial}{\partial \sigma_i} \mathrm{lse}(\sigma, \lambda) &= \frac{\partial}{\partial \sigma_i} \left(\frac{1}{\lambda} \log \sum_{j=0}^{K} exp(\lambda \sigma_j) \right) \\
&= \frac{1}{\lambda} \frac{1}{\sum_{j=0}^{K} exp(\lambda \sigma_j)} \frac{\partial}{\partial \sigma_i} \left(\sum_{j=0}^{K} exp(\lambda \sigma_j) \right) \\
&= \frac{1}{\lambda} \frac{1}{\sum_{j=0}^{K} exp(\lambda \sigma_j)} \lambda exp(\lambda \sigma_i) \\
&= \frac{exp(\lambda \sigma_i)}{\sum_{j=0}^{K} exp(\lambda \sigma_j)} \\
&= \mathrm{softmax}(\sigma, \lambda)_i
\end{aligned}
$$

(d)

$$
\lim_{\lambda \to \infty} \mathrm{lse}(\sigma, \lambda) = \lim_{\lambda \to \infty} \frac{1}{\lambda} \log \sum_{j=0}^{K} exp(\lambda \sigma_j)
$$

For $\lambda \to \infty$ the term $exp(\lambda \sigma_{max})$ will dominate the sum, which then can be rewritten as:

$$
\begin{aligned}
&= \frac{1}{\lambda} \log \left(exp(\lambda \sigma_{max}) \right) \\
&= \frac{1}{\lambda} \left(\lambda \sigma_{max} \right) \\
&= \sigma_{max}
\end{aligned}
$$

Which than equals the max function.