

Noțiuni teoretice

a) $B \subseteq \mathbb{R}^n \Leftrightarrow \begin{cases} \text{i) card } B = n = \dim \mathbb{R}^n \\ \text{ii) } B \text{ l.i.} \end{cases} \Leftrightarrow r_A = n \text{ (nr. vect.)}$, unde A - matricea componentelor vectorilor din B

b) Dacă $B = \{u_1, u_2, \dots, u_n\} \subseteq \mathbb{R}^n \rightarrow (\exists!) \lambda_i \in \mathbb{R}, i = \overline{1, n}$ a.ș. $y = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_n u_n$
 $y \in \mathbb{R}^n$ - cunoscut
 $y_B = [\lambda_1, \lambda_2, \dots, \lambda_n]$ - coordonatele lui y în baza B

Ex:

① Fie mulțimea $B = \{u_1, u_2, u_3\} \subseteq \mathbb{R}^3$ cu $\begin{cases} u_1 = (1, 0, -1)^T \\ u_2 = (-2, 1, 3)^T \\ u_3 = (0, 2, 1)^T \end{cases}$. Analizăm că:

a) mulțimea de vectori B formează o bază în sp. lin. \mathbb{R}^3 ($B \subseteq \mathbb{R}^3$);

b) determinăm coordonatele vectorului $v = (3, -1, 4)^T$ în baza B (not: $v = (3, -1, 4)^T \Rightarrow v_B = ?$);

c) fiind \hat{v} coordonatele vectorului $w \in \mathbb{R}^3$ în baza B sunt $2, -1, -1$ (not: $w_B = [2, -1, -1]$) determinăm componentele vectorului w ($w = ?$ ($\Rightarrow w_B = ?$)).

Dăm:

a) $B \subseteq \mathbb{R}^3 \Leftrightarrow \begin{cases} \text{i) card } B = 3 = \dim \mathbb{R}^3 \text{ (A)} \\ \text{ii) } B \text{ l.i.} \end{cases} \Leftrightarrow r_A = 3 \text{ (nr. vect.)}$ cu $A = \begin{pmatrix} u_1 & u_2 & u_3 \\ 1 & -2 & 0 \\ 0 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix}$ matricea componentelor vectorilor

Vom determina r_A cu t.e.:

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_1 + 2R_2} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_1 - 4R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_3 \cdot (-1)} A_{\text{diag}} = I_3 \Rightarrow r_A = 3 \Leftrightarrow \text{ii) (A)}$$

$$b) v = (3, -1, 4)^T \Rightarrow v_B = [\lambda_1, \lambda_2, \lambda_3]$$

Notăm cu $\lambda_1, \lambda_2, \lambda_3$ coordonatele vectorului v în baza B , adică:

$$v_B = [\lambda_1, \lambda_2, \lambda_3] \Leftrightarrow v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \Leftrightarrow (3, -1, 4)^T = \lambda_1 (1, 0, -1)^T + \lambda_2 (-2, 1, 3)^T + \lambda_3 (0, 2, 1)^T \Leftrightarrow$$

$$(M) \begin{cases} \lambda_1 - 2\lambda_2 = 3 \\ \lambda_2 + 2\lambda_3 = -1 \\ -\lambda_1 + 3\lambda_2 + \lambda_3 = 4 \end{cases} \xrightarrow{\text{metoda Gauss}} \bar{A} = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ -1 & 3 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 7 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 8 \end{pmatrix} \xrightarrow{R_1 + 2R_2} \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 8 \end{pmatrix} \xrightarrow{R_1 - 4R_3} \begin{pmatrix} 1 & 0 & 0 & 33 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 8 \end{pmatrix} \xrightarrow{R_2 + 2R_3} \begin{pmatrix} 1 & 0 & 0 & 33 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & -1 & 8 \end{pmatrix} \xrightarrow{R_3 \cdot (-1)} \begin{pmatrix} 1 & 0 & 0 & 33 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & -8 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \lambda_1 = 33 \\ \lambda_2 = 15 \\ \lambda_3 = -8 \end{cases} \Leftrightarrow v = 33u_1 + 15u_2 - 8u_3 \Leftrightarrow v_B = [33, 15, -8]$$

$$\text{Verificare (calcul): } v = 33(1, 0, -1)^T + 15(-2, 1, 3)^T - 8(0, 2, 1)^T = (33, 0, -33)^T + (-30, 15, 45)^T + (0, -16, -8)^T = (3, -1, 4)^T$$

$$c) \text{ Avem } w_B = [2, -1, -1] \Leftrightarrow w = 2u_1 - u_2 - u_3 = 2(1, 0, -1)^T - (-2, 1, 3)^T - (0, 2, 1)^T = (4, -3, -6)^T$$

$$\text{deci } w = (4, -3, -6)^T \Leftrightarrow w = 4e_1 - 3e_2 - 6e_3 \Leftrightarrow w_{\mathcal{B}_e} = [4, -3, -6] \text{ unde } \begin{cases} e_1 = (1, 0, 0)^T \\ e_2 = (0, 1, 0)^T \\ e_3 = (0, 0, 1)^T \end{cases}$$

q.e.d.

② Fie (B) $\begin{cases} u_1 = (3, -1)^T \in \mathbb{R}^2 \\ u_2 = (-4, 1)^T \in \mathbb{R}^2 \end{cases}$ în (B') $\begin{cases} v_1 = (5, 2)^T \in \mathbb{R}^2 \\ v_2 = (-3, -1)^T \in \mathbb{R}^2 \end{cases}$, se cere:

- a) $B; B' \leq \mathbb{R}^2$
- b) $x = (-5, 4)^T \Rightarrow \begin{cases} x_B = ? \\ x_{B'} = ? \end{cases}$
- c) $y = [1, 2] \Rightarrow y_{B'} = ?$
- d) $z_B = [-1, -3] \Rightarrow z_{B'} = ?$
- e) $S_{B'B} = ?$ și $S_{B'B'} = ?$

Solu:

a) $B \in \mathbb{R}^2(\mathbb{C}) \Rightarrow \begin{cases} i) \text{ card } B = 2 = \dim \mathbb{R}^2 \text{ (adun.)} \\ ii) B - L.i \Leftrightarrow r_A = 2 (= \text{nr. vect.}) \text{ cu } A = \begin{pmatrix} u_1 & u_2 \\ 3 & -4 \\ -1 & 1 \end{pmatrix} \text{ (evident adun.)} \end{cases}$

$B' \in \mathbb{R}^2(\mathbb{C}) \Rightarrow \begin{cases} i) \text{ card } B' = 2 = \dim \mathbb{R}^2 \text{ (A)} \\ ii) B' - L.i \Leftrightarrow r_{A'} = 2 \Rightarrow A' = \begin{pmatrix} v_1 & v_2 \\ 5 & -3 \\ 2 & -1 \end{pmatrix} \text{ (A)} \end{cases}$

Rez: pentru altele puncte, voi folosi dear lemma substitutiei (nu m. not. lui Gauss)

b) $x_B = [u_1, u_2]$

Solu:

$$\begin{array}{c|cc} B & x & u_1 & u_2 \\ \hline e_1 & -5 & 3 & -4 \\ e_2 & 4 & -1 & 1 \end{array} \begin{array}{l} \leftarrow + \\ \leftarrow + \end{array}$$

$$\Leftrightarrow \begin{array}{c|cc} e_1 & 7 & 0 & -1 \\ u_1 & -4 & 1 & -1 \\ u_2 & -7 & 0 & 1 \\ u_1 & -11 & 1 & 0 \end{array} \begin{array}{l} \leftarrow + \\ \leftarrow + \\ \leftarrow + \end{array}$$

Solu:

$$\begin{array}{c|cc} B & x & u_1 & u_2 \\ \hline e_1 & -5 & 3 & -4 \\ e_2 & 4 & -1 & 1 \\ e_1 & 11 & -1 & 0 \end{array} \begin{array}{l} \leftarrow + \\ \leftarrow + \\ \leftarrow + \end{array}$$

$$\begin{array}{c|cc} u_1 & -4 & 1 & -1 \\ u_2 & -7 & 0 & 1 \\ u_1 & -11 & 1 & 0 \\ u_2 & -7 & 0 & 1 \end{array} \begin{array}{l} \leftarrow + \\ \leftarrow + \\ \leftarrow + \end{array}$$

Solu:

$$\begin{array}{c|cc} B & x & u_1 & u_2 \\ \hline e_1 & -5 & 3 & -4 \\ e_2 & 4 & -1 & 1 \\ u_1 & -5 & 1 & -1/3 \\ e_2 & 7/3 & 0 & -1/3 \end{array} \begin{array}{l} \leftarrow + \\ \leftarrow + \\ \leftarrow + \end{array}$$

$$\begin{array}{c|cc} u_1 & -11 & 1 & 0 \\ u_2 & -7 & 0 & 1 \end{array} \begin{array}{l} \leftarrow + \\ \leftarrow + \end{array}$$

$x = -11u_1 - 7u_2 \Leftrightarrow x_B = [-11, -7]$

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Verificare: $x_B = [-11, -7] \Leftrightarrow x = -11u_1 - 7u_2 = -11(3, -1)^T - 7(-4, 1)^T = (-33+28, 11-7)^T = (-5, 4)^T \text{ (adun.)}$

b2) $x_{B'} = [v_1, v_2]$

$$\begin{array}{c|cc} B & x & v_1 & v_2 \\ \hline e_1 & -5 & 5 & -3 \\ e_2 & 4 & 2 & -1 \\ v_1 & -1 & 1 & -3/5 \\ e_2 & 6 & 0 & 1/5 \end{array} \begin{array}{l} \leftarrow + \\ \leftarrow + \\ \leftarrow + \end{array}$$

$$\Leftrightarrow \begin{array}{c|cc} v_1 & 17 & 1 & 0 \\ v_2 & 30 & 0 & 1 \end{array} \begin{array}{l} \leftarrow + \\ \leftarrow + \end{array}$$

$\begin{cases} x = -5e_1 + 4e_2 \\ v_1 = 5e_1 + 2e_2 \\ v_2 = -3e_1 - e_2 \end{cases}$

$\begin{cases} x = -v_1 + 6e_2 \\ v_1 = v_1 + 0e_2 \\ v_2 = -3/5v_1 + 1/5e_2 \end{cases}$

Verif: $x_{B'} = [v_1, v_2] \Rightarrow$

$x = 17v_1 + 30v_2 \Leftrightarrow$

$x = 17(5, 2)^T + 30(-3, -1)^T$

$= (85, 34)^T + (-90, -30)^T$

$= (-5, 4)^T \text{ (adun.)}$

Solu:

$$\begin{array}{c|cc} B & x & v_1 & v_2 \\ \hline e_1 & -5 & 5 & -3 \\ e_2 & 4 & 2 & -1 \\ e_1 & 17 & 1 & 0 \\ v_2 & -4 & -2 & 1 \\ v_1 & 17 & 1 & 0 \\ v_2 & 30 & 0 & 1 \end{array} \begin{array}{l} \leftarrow + \\ \leftarrow + \\ \leftarrow + \\ \leftarrow + \end{array}$$

$\rightarrow x = -5e_1 + 4e_2$

$\rightarrow x = -17v_1 + 4v_2$

$\rightarrow x = 17v_1 + 4v_2$

$x = 17v_1 + 30v_2$

$x_{B'} = [17, 30]$

c) $y_B = [p_1, p_2]$ $\{ p_1, p_2 = ? \text{ a.s. } y = p_1 u_1 + p_2 u_2 \}$

Donnée $y_B = [1, 2]$ $\Leftrightarrow y = u_1 + 2u_2 \Leftrightarrow y = (3, -1)^T + 2(-4, 1)^T \Leftrightarrow y = (-5, 1)^T \Leftrightarrow y = -5e_1 + e_2$

B \ V	u_1	u_2
e_1	-5	5
e_2	1	2
e_1	-8	-1
u_2	-1	-2
u_1	8	1
u_2	15	0

Verification

$y = 8u_1 + 15u_2 = 8(5, 2)^T + 15(-3, -1)^T = (-5, 1)^T$ (adieu)

$y = 8u_1 + 15u_2$
 $y_B = [8, 15]$

d) $z_B = [\alpha_1, \alpha_2]$ $\{ \alpha_1, \alpha_2 = ? \text{ a.s. } z = \alpha_1 u_1 + \alpha_2 u_2 \}$

Don: $z_B = [-1, 3]$ $\Leftrightarrow z = -u_1 - 3u_2 = -(5, 2)^T - 3(-3, -1)^T = (4, 1)^T$

B \ V	u_1	u_2
e_1	4	3
e_2	1	-1
e_1	8	-1
u_2	1	-1
u_1	-8	1
u_2	-7	0

$z = 4e_1 + e_2$

$z = 8e_1 + u_2$

$z = -8u_1 - 7u_2$

$z = -8u_1 - 7u_2 \Leftrightarrow z_B = [-8, -7]$, vérif: $z = -8(3, -1)^T - 7(-4, 1)^T = (-24, 8)^T + (28, -7)^T = (4, 1)^T$

e) e) $S_{B' \mid B} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ a.s. $\begin{cases} u_1 = s_{11}u_1 + s_{12}u_2 \\ u_2 = s_{21}u_1 + s_{22}u_2 \end{cases}$ $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

B \ V	u_1	u_2
e_1	5	-3
e_2	2	-1
e_1	13	-7
u_2	2	-1
u_1	-13	7
u_2	-11	6

$\begin{cases} u_1 = -13u_1 - 11u_2 \\ u_2 = 7u_1 + 6u_2 \end{cases} \Leftrightarrow S_{B' \mid B} = \begin{pmatrix} -13 & -11 \\ 7 & 6 \end{pmatrix}$

$$e_2) S'_{B|B} = \begin{pmatrix} s'_{11} & s'_{12} \\ s'_{21} & s'_{22} \end{pmatrix} \stackrel{\text{def}}{=} \begin{cases} u_1 = s'_{11} v_1 + s'_{12} v_2 \\ u_2 = s'_{21} v_1 + s'_{22} v_2 \end{cases} \Leftrightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} s'_{11} & s'_{12} \\ s'_{21} & s'_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Leftrightarrow B = S'^{-1}_{B|B} B'$$

	u_1	u_2	v_1	v_2
e_1	3	-4	5	-3
e_2	-1	1	2	-1
e_1	6	-7	-1	0
e_2	1	-1	-2	1
v_1	-6	7	1	0
v_2	-11	13	0	1

$$\begin{cases} u_1 = -6v_1 + 7v_2 \\ u_2 = -11v_1 + 13v_2 \end{cases} \Rightarrow S'_{B|B} = \begin{pmatrix} -6 & 7 \\ -11 & 13 \end{pmatrix}$$

Conclusion finale:

$$b) x = (-5, 4)^T \Rightarrow \begin{cases} x_B = [-11, -7] \\ x_{B'} = [17, 30] \end{cases}$$

$$c) y_B = [1, 2]^T \Rightarrow y_{B'} = [8, 15]^T$$

$$d) z_{B'} = [-1, -3]^T \Rightarrow z_B = [-8, -7]^T$$

$$e) S_{B|B} = \begin{pmatrix} -13 & -11 \\ 7 & 6 \end{pmatrix}; S'_{B|B} = \begin{pmatrix} -6 & 7 \\ -11 & 13 \end{pmatrix}$$

Obs: vous vérifiez formule démontrée la cours $\begin{cases} u_B = S'^T u_{B'} \\ u_{B'} = (S')^{-1} u_B \end{cases}$; $S' = S^{-1}$

finale $S \equiv S_{B|B}$ et $S' \equiv S'_{B|B}$. Avec:

$$b) \begin{cases} x_B = S'^T x_{B'} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -13 & 7 \\ -11 & 6 \end{pmatrix} \begin{pmatrix} 17 \\ 30 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -13 \cdot 17 + 7 \cdot 30 \\ -11 \cdot 17 + 6 \cdot 30 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -11 \\ -7 \end{pmatrix} \\ x_B = (S')^{-1} x_{B'} = (S^{-1})^T x_{B'} = S'^T x_{B'} = \begin{pmatrix} -6 & 7 \\ -11 & 13 \end{pmatrix} \begin{pmatrix} -11 \\ -7 \end{pmatrix} = \begin{pmatrix} 66 - 49 \\ 121 - 91 \end{pmatrix} = \begin{pmatrix} 17 \\ 30 \end{pmatrix} \end{cases}$$

$$c) y_{B'} = (S')^{-1} y_B = (S^{-1})^T y_B = S'^T y_B = \begin{pmatrix} -6 & 7 \\ -11 & 13 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 15 \end{pmatrix}$$

$$d) z_B = S'^T z_{B'} = \begin{pmatrix} -13 & 7 \\ -11 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ -7 \end{pmatrix}$$

et calculer S^{-1}

$$S = \begin{pmatrix} -13 & -11 & | & 1 & 0 \\ 7 & 6 & | & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 7 & 6 & | & 0 & 1 \\ -13 & -11 & | & 1 & 0 \end{pmatrix} \xrightarrow{\substack{R_1 \times \frac{1}{7} \\ R_2 \times \frac{1}{6}}} \begin{pmatrix} 1 & \frac{6}{7} & | & 0 & \frac{1}{7} \\ -13 & -11 & | & 1 & 0 \end{pmatrix} \xrightarrow{R_2 + 13R_1} \begin{pmatrix} 1 & \frac{6}{7} & | & 0 & \frac{1}{7} \\ 0 & -\frac{11}{7} & | & 1 & \frac{13}{7} \end{pmatrix} \xrightarrow{R_2 \times \frac{7}{-11}} \begin{pmatrix} 1 & \frac{6}{7} & | & 0 & \frac{1}{7} \\ 0 & 1 & | & -\frac{7}{11} & -\frac{13}{11} \end{pmatrix} \xrightarrow{R_1 - \frac{6}{7}R_2} \begin{pmatrix} 1 & 0 & | & \frac{6}{11} & \frac{1}{11} \\ 0 & 1 & | & -\frac{7}{11} & -\frac{13}{11} \end{pmatrix} \Rightarrow S^{-1} = \begin{pmatrix} \frac{6}{11} & \frac{1}{11} \\ -\frac{7}{11} & -\frac{13}{11} \end{pmatrix} = S'$$