Seminar 4

Coordonatele unui vector într-o bază. Schimbarea coordonatelor la schimbarea bazei

Noțiuni teoretice

a) $B \underset{\text{"bază"}}{\leq} \mathbb{R}^n \iff \begin{cases} \text{(i) card } B = n = \dim \mathbb{R}^n \\ \text{(ii) } B - \text{L.i} \iff r_A = n \text{ (nr. vect.)} \end{cases}$, unde A - matricea componentelor vectorilor din B

b) Dacă ~~B~~ $\left.\begin{array}{l} B = \{u_1, u_2, \ldots, u_n\} \leq \mathbb{R}^n \\ y \in \mathbb{R}^n \text{ - oarecare} \end{array}\right| \Rightarrow (\exists!)\ \lambda_i \in \mathbb{R},\ i = \overline{1,n}$ a.î.: $y = \lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_n u_n$

$\Updownarrow$

$y_B = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ → coordonatele lui y în baza B

Ex:

① Fie mulțimea $B = \{u_1, u_2, u_3\} \subset \mathbb{R}^3$ cu $\begin{cases} u_1 = (1, 0, -1)^T \\ u_2 = (-2, 1, 3)^T \\ u_3 = (0, 2, 1)^T \end{cases}$. Arătați că:

a) mulțimea de vectori B formează o bază în sp. lin. $\mathbb{R}^3$ ($\overset{\text{not}}{\Leftrightarrow} B \leq \mathbb{R}^3$);

b) determinați coordonatele vectorului $v = (3, -1, 4)^T$ în baza B (not: $v = (3,-1,4)^T \Rightarrow v_B = ?$)

c) știind că coordonatele vectorului $w \in \mathbb{R}^3$ în baza B sunt $2, -1, -1$ ($\overset{\text{not}}{=}$: $w_B = [2, -1, -1]$) determinați componentele vectorului w ($w = ? \iff w_{B_c} = ?$)

Dem:

a) $B \leq \mathbb{R}^3 \iff \begin{cases} \text{i) card } B = 3 = \dim \mathbb{R}^3 \ (A) \\ \text{(ii) } B - \text{L.i} \iff r_A = 3 \text{ (=nr. vect.)} \end{cases}$ cu $A = \begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \\ \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix} \end{array}$ matricea componentelor vectorilor

Vom determina r_A cu t.e.:

$$A = \begin{pmatrix} \boxed{1} & -2 & 0 \\ 0 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & \boxed{1} & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ |\cdot 2 \ |\cdot(-1) \\ \\ \end{matrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & \boxed{-1} \end{pmatrix} \begin{matrix} \\ \\ |\cdot 2 \ |\cdot 4 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv A_{f.g.} = I_3 \Rightarrow r_A = 3 \iff \text{ii)} (A)$$

$\Downarrow$

$\underline{B \leq \mathbb{R}^3}$

b) $\underline{v = (3, -1, 4)^T \overset{?}{\Longrightarrow} v_B = [\alpha_1, \alpha_2, \alpha_3]}$

Notăm cu $\alpha_1, \alpha_2, \alpha_3$ coordonatele vectorului v în baza B, adică:

$v_B = [\alpha_1, \alpha_2, \alpha_3] \iff v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \iff (3, -1, 4)^T = \alpha_1 (1, 0, -1)^T + \alpha_2 (-2, 1, 3)^T + \alpha_3 (0, 2, 1)^T \iff$

$$(*) \begin{cases} \alpha_1 - 2\alpha_2 = 3 \\ \alpha_2 + 2\alpha_3 = -1 \\ -\alpha_1 + 3\alpha_2 + \alpha_3 = 4 \end{cases} \xrightarrow{\text{rezolvăm}} \text{cu met. lui Gauss} \quad \bar{A} = \begin{array}{c} \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \end{matrix} \\ \left(\begin{array}{ccc|c} \boxed{1} & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ -1 & 3 & 1 & 4 \end{array}\right) \\ \begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow \\ u_1 & u_2 & u_3 & v \end{matrix} \end{array} \sim \left(\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 1 & 1 & 7 \end{array}\right) \begin{matrix} \\ |\cdot 2 \ |\cdot(-1) \\ \\ \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & \boxed{-1} & 8 \end{array}\right) \begin{matrix} \\ \\ |\cdot(-1) \ |\cdot 2 \ |\cdot 4 \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 33 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & -8 \end{array}\right)$$

$\Rightarrow \begin{cases} \alpha_1 = 33 \\ \alpha_2 = 15 \\ \alpha_3 = -8 \end{cases} \iff \underline{v = 33u_1 + 15u_2 - 8u_3} \iff \underline{v_B = [33, 15, -8]}$

Verificare (calcule): $v = 33(1,0,-1)^T + 15(-2,1,3)^T - 8(0,2,1)^T = (33, 0, -33)^T + (-30, 15, 45)^T + (0, -16, -8)^T = \underline{(3, -1, 4)^T}$ adevărat.

c) Avem $w_B = [2, -1, -1] \iff \underline{w} = 2u_1 - u_2 - u_3 = 2(1,0,-1)^T - (-2,1,3)^T - (0,2,1)^T = \underline{(4, -3, -6)^T}$

deci $\underline{w = (4, -3, -6)^T} \iff w = 4e_1 - 3e_2 - 6e_3 \iff w_{B_c} = [4, -3, -6]$ unde $(B_c) \begin{cases} e_1 = (1, 0, 0)^T \\ e_2 = (0, 1, 0)^T \\ e_3 = (0, 0, 1)^T \end{cases}$

q.e.d.

② Fie $(B)\begin{cases} u_1=(3,-1)^T \\ u_2=(-4,1)^T \end{cases} \in \mathbb{R}^2$ și $(B')\begin{cases} v_1=(5,2)^T \\ v_2=(-3,-1)^T \end{cases} \in \mathbb{R}^2$, Se cere:

a) $B;\ B' \leq \mathbb{R}^2$

b) $x=(-5,4)^T \Rightarrow \begin{cases} x_B=? \\ x_{B'}=? \end{cases}$

c) $y_B=[1,2] \Rightarrow y_{B'}=?$

d) $z_{B'}=[-1,-3] \Rightarrow z_B=?$

e) $S_{B'|B}=?$ și $S'_{B|B'}=?$

Dem:

a) $B \leq \mathbb{R}^2 \Leftrightarrow \begin{cases} \text{i) card } B = 2 = \dim \mathbb{R}^2 \text{ (adev.)} \\ \text{ii) } B - \text{L.i} \Leftrightarrow r_A = 2 \text{ (= nr. vect)} \end{cases}$ cu $A=\begin{array}{cc} u_1 & u_2 \\ \end{array}\begin{pmatrix} 3 & -4 \\ -1 & 1 \end{pmatrix}$ (evident adev.)

$B' \leq \mathbb{R}^2 \Leftrightarrow \begin{cases} \text{i) card } B' = 2 = \dim \mathbb{R}^2 \text{ (A)} \\ \text{ii) } B' - \text{L.i} \Leftrightarrow r_{A'} = 2 \end{cases}$, $A' = \begin{array}{cc} v_1 & v_2 \\ \end{array}\begin{pmatrix} 5 & -3 \\ 2 & -1 \end{pmatrix}$ (A)

Obs: pentru celelalte puncte, voi folosi doar lema substituției (nu și met. lui Gauss)

b)

b₁) $x_B \overset{?}{=} [\alpha_1, \alpha_2]$

B	x	u_1	u_2	
e_1	-5	3	-4	
e_2	4	$\boxed{-1}$	1	/(-1)/·3
e_1	7	0	$\boxed{-1}$	/(-1)/(-1)
u_1	-4	1	-1	
u_2	-7	0	1	
u_1	-11	1	0	

$x=-11u_1+7u_2 \Leftrightarrow x_B=[-11,7]$

sau:

B	x	u_1	u_2	
e_1	-5	3	-4	
e_2	4	-1	$\boxed{1}$	/·4
e_1	11	$\boxed{-1}$	0	/(-1)/·(-1)
u_2	4	-1	1	
u_1	-11	1	0	
u_2	-7	0	1	

$x=-11u_1-7u_2 \Leftrightarrow x_B=[-11,-7]$

sau:

B	x	u_1	u_2	
e_1	-5	$\boxed{3}$	-4	/·1/3 /·1/3
e_2	4	-1	1	
u_1	-5/3	1	-4/3	
e_2	7/3	0	$\boxed{-1/3}$	/·(-3)/(-4)
u_1	-11	1	0	
u_2	-7	0	1	

$x=-11u_1-7u_2 \Leftrightarrow x_B=[-11,-7]$

Verificare: $x_B=[-11,-7] \Leftrightarrow x=-11u_1-7u_2=-11(3,-1)^T-7(-4,1)^T=(-33+28,\ 11-7)^T=(-5,4)^T$ (adev.).

b₂) $x_{B'} \overset{?}{=} [\beta_1, \beta_2]$

B	x	v_1	v_2	
e_1	-5	$\boxed{5}$	-3	/·1/5 /(-2/5)
e_2	4	2	-1	
v_1	-1	1	-3/5	
e_2	6	0	$\boxed{1/5}$	/·5/·3
v_1	17	1	0	
v_2	30	0	1	

$\begin{cases} x=-5e_1+4e_2 \\ v_1=5e_1+2e_2 \\ v_2=-3e_1-e_2 \end{cases}$

$\begin{cases} x=-v_1+6e_2 \\ v_1=v_1+0e_2 \\ v_2=-3/5\,v_1+\frac{1}{5}e_2 \end{cases}$

$x=17v_1+30v_2$

$\Updownarrow$

$x_{B'}=[17,30]$

Verif.: $x_{B'}=[17,30] \Leftrightarrow$

$x=17v_1+30v_2 \Leftrightarrow$

$x=17(5,2)^T+30(-3,-1)^T$

$=(85,34)^T+(-90,-30)^T$

$=(-5,4)^T$ (adev.)

sau:

B	x	v_1	v_2	
e_1	-5	5	-3	
e_2	4	2	$\boxed{-1}$	/(-1)/(-3)
e_1	-17	$\boxed{-1}$	0	/(-1)/(-2)
v_2	-4	-2	1	
v_1	17	1	0	
v_2	30	0	1	

$\mapsto x=-5e_1+4e_2$

$\mapsto x=-17e_1-4v_2$

$\mapsto x=17v_1+4v_2$

$x=17v_1+30v_2$

$\Updownarrow$

$x_{B'}=[17,30]$

c) $\underline{y_{B'} \stackrel{?}{=} [\beta_1, \beta_2]}$ $\{\exists\, \beta_1, \beta_2 = ?$ a.î.: $y = \beta_1 v_1 + \beta_2 v_2\}$

Deoarece $y_B = [1,2] \iff y = u_1 + 2u_2 \iff y = (3,-1)^T + 2(-4,1)^T \iff \underline{y = (-5,1)^T}$ $(\iff y = -5e_1 + e_2)$

B	y	v_1	v_2		
e_1	-5	5	-3		
e_2	1	2	$\boxed{-1}$	$\cdot(-1)$	$\cdot(-3)$
e_1	-8	$\boxed{-1}$	0	$\cdot(-1)$	$\cdot(-2)$
v_2	-1	-2	1		
v_1	8	1	0		
v_2	15	0	1		

Verificare

$\underline{y} = 8v_1 + 15v_2 = 8(5,2)^T + 15(-3,-1)^T = \underline{(-5,1)^T}$ (adev.)

$\Downarrow$

$y = 8v_1 + 15v_2$

$\Updownarrow$

$\underline{y_{B'} = [8, 15]}$

d) $\underline{z_B \stackrel{?}{=} [\alpha_1, \alpha_2]}$ $\{\exists\, \alpha_1, \alpha_2 = ?$ a.î. $z = \alpha_1 u_1 + \alpha_2 u_2\}$

Din: $z_{B'} = [-1,-3] \iff \underline{z} = -v_1 - 3v_2 = -(5,2)^T - 3(-3,-1)^T = \underline{(4,1)^T}$

B	z	u_1	u_2		
e_1	4	3	-4		
e_2	1	-1	$\boxed{1}$	$\cdot 4$	
e_1	8	$\boxed{-1}$	0	$\cdot(-1)$	$\cdot(-1)$
u_2	1	-1	1		
u_1	-8	1	0		
u_2	-7	0	1		

$\longmapsto z = 4e_1 + e_2$

$\longmapsto z = 8e_1 + u_2$

$\longmapsto z = -8u_1 - 7u_2$

$\Downarrow$

$\underline{z = -8u_1 - 7u_2 \iff z_B = [-8,-7]}$; verif: $z = -8(3,-1)^T - 7(-4,1)^T = (-24,+8)^T + (28,-7)^T = \underline{(4,1)^T}$

e) e_1) $S_{B'|B} \stackrel{def}{=} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ a.î. $(*)\begin{cases} v_1 = s_{11}u_1 + s_{12}u_2 \\ v_2 = s_{21}u_1 + s_{22}u_2 \end{cases}$ $\left[\underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{B'} = \underbrace{\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}}_{S_{B'|B}} \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{B} \right]$

B	v_1	v_2	u_1	u_2		
e_1	5	-3	3	-4		
e_2	2	-1	-1	$\boxed{1}$	$\cdot 4$	
e_1	13	-7	$\boxed{-1}$	0	$\cdot(-1)$	$\cdot(-1)$
u_2	2	-1	-1	1		
u_1	-13	7	1	0		
u_2	-11	6	0	1		

$\Downarrow$

$\begin{cases} v_1 = -13u_1 - 11u_2 \\ v_2 = 7u_1 + 6u_2 \end{cases} \stackrel{(*)}{\Longrightarrow} S_{B'|B} = \begin{pmatrix} -13 & -11 \\ 7 & 6 \end{pmatrix}$

e2) $S'_{B|B'} \stackrel{def}{=} \begin{pmatrix} s'_{11} & s'_{12} \\ s'_{21} & s'_{22} \end{pmatrix}$ a.î. $\stackrel{(*)}{} \begin{cases} u_1 = s'_{11} v_1 + s'_{12} v_2 \\ u_2 = s'_{21} v_1 + s'_{22} v_2 \end{cases}$ $\left[\Leftrightarrow \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} s'_{11} & s'_{12} \\ s'_{21} & s'_{22} \end{pmatrix}}_{S'_{B|B'}} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{B'} \Leftrightarrow B = S'_{B|B'} \cdot B' \right]$

B	u_1	u_2	v_1	v_2	
e_1	3	-4	5	-3	
e_2	-1	1	2	$\boxed{-1}$	$/(-1)/\cdot(-3)$
e_1	6	-7	$\boxed{-1}$	0	$/(-1)/\cdot(-2)$
v_2	1	-1	-2	1	+
v_1	-6	7	1	0	
v_2	-11	13	0	1	

$\begin{cases} u_1 = -6 v_1 - 11 v_2 \\ u_2 = +7 v_1 + 13 v_2 \end{cases} \stackrel{(*)}{\Longrightarrow} S'_{B|B'} = \begin{pmatrix} -6 & -11 \\ 7 & 13 \end{pmatrix}$

Concluzie finală:

b) $x = (-5, 4)^T \Rightarrow \begin{cases} x_B = [-11, -7] \\ x_{B'} = [17, 30] \end{cases}$

c) $y_B = [1, 2] \Rightarrow y_{B'} = [8, 15]$

d) $z_{B'} = [-1, -3] \Rightarrow z_B = [-8, -7]$

e) $S_{B'|B} = \begin{pmatrix} -13 & -11 \\ 7 & 6 \end{pmatrix}$; $S'_{B|B'} = \begin{pmatrix} -6 & -11 \\ 7 & 13 \end{pmatrix}$

Obs: vom verifica formulele demonstrate la curs $\begin{cases} u_B = S^T \cdot u_{B'} \\ u_{B'} = (S^T)^{-1} \cdot u_B \end{cases}$ și $S' = S^{-1}$

[unde $S \equiv S_{B'|B}$ și $S' \equiv S'_{B|B'}$]. Avem:

b) $\begin{cases} x_B = S^T \cdot x_{B'} \Leftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -13 & 7 \\ -11 & 6 \end{pmatrix}}_{= S^T} \begin{pmatrix} 17 \\ 30 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -13 \cdot 17 + 7 \cdot 30 \\ -11 \cdot 17 + 6 \cdot 30 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -11 \\ -7 \end{pmatrix} \\ x_{B'} = (S^T)^{-1} \cdot x_B \equiv (S^{-1})^T x_B \equiv S'^T \cdot x_B = \begin{pmatrix} -6 & +7 \\ -11 & 13 \end{pmatrix} \begin{pmatrix} -11 \\ -7 \end{pmatrix} = \begin{pmatrix} 66 - 49 \\ 121 - 91 \end{pmatrix} = \begin{pmatrix} 17 \\ 30 \end{pmatrix} \end{cases}$

c) $y_{B'} = (S^T)^{-1} \cdot y_B = (S^{-1})^T y_B = S'^T \cdot y_B = \begin{pmatrix} -6 & 7 \\ -11 & 13 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 15 \end{pmatrix}$

d) $z_B = S^T \cdot z_{B'} = \begin{pmatrix} -13 & 7 \\ -11 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ -7 \end{pmatrix}$

e) calculăm S^{-1}

$\tilde{S} = \left(\underbrace{\begin{matrix} -13 & -11 \\ 7 & \boxed{6} \end{matrix}}_{= S} \middle| \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{= I_2} \right) /\cdot\frac{1}{6} /\cdot\frac{11}{6} \sim \left(\begin{matrix} \boxed{-\frac{1}{6}} & 0 \\ \frac{7}{6} & 1 \end{matrix} \middle| \begin{matrix} 1 & \frac{11}{6} \\ 0 & \frac{1}{6} \end{matrix} \right) /\cdot(-6) /\cdot 7 \sim \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \middle| \underbrace{\begin{matrix} -6 & -11 \\ 7 & 13 \end{matrix}}_{S^{-1}} \right) \Rightarrow S^{-1} = \begin{pmatrix} -6 & -11 \\ 7 & 13 \end{pmatrix} = S'$

q.e.d.