

Derivate parțiale de ordinul I și II pentru funcții de n variabile

Obs. reguli de derivare uzuale și necesare:

$$\begin{cases} (f+g)' = f' + g' \\ (\lambda f)' = \lambda f' \end{cases} \quad \begin{cases} (x^n)' = n x^{n-1} \\ x' = 1 \\ c' = 0 \text{ (derivata unei constante)} \end{cases}$$

Să se determine, derivatele parțiale de ord. I și II, diferențialele de ord. I și II și hessiană următoarelor funcții în punctele indicate:

I) $\begin{cases} f: \mathbb{R}^2 \rightarrow \mathbb{R} \\ f(x,y) = 3x^2y + xy^2 - 2xy + 1 \end{cases} ; P_0(-1,2)$

1) calculăm derivatele parțiale de ord. I $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$:

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 6xy + y^2 - 2y \\ \frac{\partial f}{\partial y}(x,y) = 3x^2 + 2xy - 2x \end{cases} \begin{array}{l} \rightarrow \text{se derivatează } f(x,y) \text{ în raport cu "x" ca "n",} \\ \text{cum "y" este o constantă} \\ \rightarrow \text{se derivatează } f(x,y) \text{ în raport cu "y" ca "n",} \\ \text{cum "x" este o constantă} \end{array}$$

$$\begin{cases} \frac{\partial f}{\partial x}(P_0) = \frac{\partial f}{\partial x}(-1,2) = 6(-1)^2 \cdot 2 + 2^2 - 2 \cdot 2 = 30 \\ \frac{\partial f}{\partial y}(P_0) = \frac{\partial f}{\partial y}(-1,2) = 3(-1)^2 + 2(-1) \cdot 2 - 2(-1) = -9 \end{cases}$$

2) scriem diferențiala de ord. I $(df(x,y))$:

$$\begin{cases} df(x,y) \stackrel{\text{def}}{=} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = (6xy + y^2 - 2y)dx + (3x^2 + 2xy - 2x)dy \\ df(P_0) = df(-1,2) = 30dx - 9dy \end{cases}$$

3) calculăm derivatele parțiale de ordinul II $\left(\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y} \right)$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\left\{ \begin{aligned} \frac{\partial^2 f}{\partial x^2} &\stackrel{\text{def}}{=} \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (18xy - 4y) = 18x - 4 \\ \frac{\partial^2 f}{\partial y^2} &\stackrel{\text{def}}{=} \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (3x^2 + 2xy - 2x^2) = 2x \\ \frac{\partial^2 f}{\partial x \partial y} &\stackrel{\text{def}}{=} \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (3x^2 + 2xy - 2x^2) = 9x^2 + 2y - 4x \\ \frac{\partial^2 f}{\partial y \partial x} &\stackrel{\text{def}}{=} \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (18xy - 4y) = 18x^2 + 2y - 4x \end{aligned} \right.$$

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$$\left\{ \begin{aligned} \frac{\partial^2 f}{\partial x^2}(P_0) &= 18 \cdot (-1) \cdot 2 - 4 \cdot 2 = 28 \\ \frac{\partial^2 f}{\partial y^2}(P_0) &= -2 \\ \frac{\partial^2 f}{\partial x \partial y}(P_0) &= \frac{\partial^2 f}{\partial y \partial x}(P_0) = 17 \end{aligned} \right.$$

$\frac{\partial}{\partial x}(\dots) \rightarrow$ derivata (\dots) în raport
cu "x"
 $\frac{\partial}{\partial y}(\dots) \rightarrow$ derivata (\dots) în raport
cu "y"

4) scriem diferențiala de ordinul II ($d^2f(x,y)$)

$$\left\{ \begin{aligned} d^2f(x,y) &\stackrel{\text{def}}{=} \frac{\partial^2 f}{\partial x^2} dx^2 + \frac{\partial^2 f}{\partial y^2} dy^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy = \frac{(18xy - 4y) dx^2 + 2x dy^2 + 2(9x^2 + 2y - 4x) dx dy}{dx dy} \\ d^2f(P_0) &\equiv d^2f(-1,2) = \frac{28 dx^2 - 2 dy^2 + 34 dx dy}{dx dy} \end{aligned} \right.$$

5) scriem matricea hessiană ($H(x,y)$)

$$H(x,y) \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 18xy & 9x^2 + 2y - 4x \\ 9x^2 + 2y - 4x & 2x \end{pmatrix}$$

$$H(P_0) \equiv H(-1,2) = \begin{pmatrix} 28 & 17 \\ 17 & -2 \end{pmatrix}$$

II) $f: \mathbb{R}^3 \rightarrow \mathbb{R}$
 $f(x, y, z) = x^3 y^2 z + 2xy^2 - 3yz + 4$; $P_0(1, 1, 1)$

1)
$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 y^2 z + 2y^2 \\ \frac{\partial f}{\partial y} = 2x^3 y z + 4xy - 3z \\ \frac{\partial f}{\partial z} = x^3 y^2 - 3y \end{cases} \Rightarrow \begin{cases} \frac{\partial f}{\partial x}(P_0) = 5 \\ \frac{\partial f}{\partial y}(P_0) = 3 \\ \frac{\partial f}{\partial z}(P_0) = -2 \end{cases}$$

2)
$$df(x, y, z) \stackrel{\text{def}}{=} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = (3x^2 y^2 z + 2y^2) dx + (2x^3 y z + 4xy - 3z) dy + (x^3 y^2 - 3y) dz$$

$$df(P_0) = 5dx + 3dy - 2dz$$

3)
$$\begin{cases} \frac{\partial^2 f}{\partial x^2} = 6xy^2 z \\ \frac{\partial^2 f}{\partial y^2} = 2x^3 z + 4x \\ \frac{\partial^2 f}{\partial z^2} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 f}{\partial x^2}(P_0) = 6 \\ \frac{\partial^2 f}{\partial y^2}(P_0) = 6 \\ \frac{\partial^2 f}{\partial z^2}(P_0) = 0 \end{cases}$$

$$\begin{cases} \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6x^2 y z + 4y \\ \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = 3x^2 y \\ \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = 2x^3 y - 3 \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 f}{\partial x \partial y}(P_0) = \frac{\partial^2 f}{\partial y \partial x}(P_0) = 10 \\ \frac{\partial^2 f}{\partial x \partial z}(P_0) = \frac{\partial^2 f}{\partial z \partial x}(P_0) = 3 \\ \frac{\partial^2 f}{\partial y \partial z}(P_0) = \frac{\partial^2 f}{\partial z \partial y}(P_0) = -1 \end{cases}$$

4)
$$d^2 f(x, y, z) \stackrel{\text{def}}{=} \frac{\partial^2 f}{\partial x^2} dx^2 + \frac{\partial^2 f}{\partial y^2} dy^2 + \frac{\partial^2 f}{\partial z^2} dz^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + 2 \frac{\partial^2 f}{\partial x \partial z} dx dz + 2 \frac{\partial^2 f}{\partial y \partial z} dy dz$$

$$= \cancel{(6x^2 y z + 4y) dx^2} + 6xy^2 z dx^2 + (2x^3 z + 4x) dy^2 + 2(6x^2 y z + 4y) dx dy + 6x^2 y^2 dx dz + 2(2x^3 y - 3) dy dz$$

$$d^2 f(P_0) = 6dx^2 + 6dy^2 + 20dx dy + 6dx dz - 2dy dz$$

5)

$$H(x,y,z) \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 6xy^2z & 6x^2yz+4y & 3x^2yz \\ 6x^2yz+4y & 2x^3z+4x & 2x^3y-3 \\ 3x^2yz & 2x^3y-3 & 0 \end{pmatrix} \quad (9)$$

$$H(P_0) = \begin{pmatrix} 6 & 10 & 3 \\ 10 & 6 & -1 \\ 3 & -1 & 0 \end{pmatrix}$$

q.e.d

Obs: ① Din suportul de curs (cap. 5) aveți alte două exemple rezolvate

② tot acolo aveți și două exemple pentru determinarea punctelor de extrem local $\begin{cases} f(x,y) \\ f(x,y,z) \end{cases}$