Example:

a) non-perturbate ETP ⟹ a') perturbate ETP

	S_1	S_2	...	S_n	
W_1	c_{11}	c_{12}	...	c_{1n}	a_1
W_2	c_{21}	c_{22}	...	c_{2n}	a_2
⋮	⋮	⋮	⋮	⋮	⋮
W_m	c_{m1}	c_{m2}	...	c_{mn}	a_m
	b_1	b_2	...	b_n	

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

	S_1	S_2	...	S_n	
W_1	c_{11}	c_{12}	...	c_{1n}	$a_1+\varepsilon$
W_2	c_{21}	c_{22}	...	c_{2n}	$a_2+\varepsilon$
⋮	⋮	⋮	⋮	⋮	⋮
W_m	c_{m1}	c_{m2}	...	c_{mn}	$a_m+\varepsilon$
	b_1	b_2	...	$b_n+m\cdot\varepsilon$	

$$\sum_{i=1}^{m}(a_i+\varepsilon) \equiv \sum_{i=1}^{m} a_i + m\varepsilon = \underline{\sum_{j=1}^{n} b_j + m\cdot\varepsilon}$$

Examples:

Solve the next TP. if the second B.A. solution which we will obtain it isn't the optimal solution don't continue the algorithm:

a)

	S_1	S_2	S_3	
W_1	3	2	1	30
W_2	1	4	2	20
W_3	2	1	3	20
	20	15	20	

Because, $\sum_{i=1}^{3} a_i = 70 > \sum_{j=1}^{3} b_j = 55$

we have a NeTP, so we must to equilibrate.

⟹

①

	S_1	S_2	S_3	S_4^f	
W_1	[3] 20	[2] 10	[1] *	[0] *	$\cancel{30}, \cancel{10}, 0$
W_2	[1] *	[4] 5	[2] 15	[0] *	$\cancel{20}, \cancel{15}, 0$
W_3	[2] *	[1] *	[3] 5	[0] 15	$\cancel{20}, \cancel{15}, 0$
	$\cancel{20}$, 0	$\cancel{15}$, $\cancel{5}$, 0	$\cancel{20}$, $\cancel{5}$, 0	$\cancel{15}$, 0	

For ETP we applied the diagonal method to find the IBAS: $\bar{X}_0$ $\begin{cases} \bar{x}_{11}=20;\ \bar{x}_{12}=10;\ \bar{x}_{22}=5; \\ \bar{x}_{23}=15;\ \bar{x}_{33}=5;\ \bar{x}_{34}=15 \\ \text{the other } \bar{x}_{ij} = \text{"}*\text{"} = 0 \end{cases}$

$\underline{f(\bar{X}_0) = 145}$

② We check if $\bar{X}_0$ it's optimal solution:

$$\left.\begin{aligned} \delta_{13} &= -1+2-4+2 = -1 \\ \delta_{14} &= -0+0-3+2-4+2 = -3 \\ \delta_{21} &= -1+3-2+4 = \underline{4>0} \\ \delta_{24} &= -0+0-3+2 = -1 \\ \delta_{31} &= -2+3-2+4-2+3 = \underline{4>0} \\ \delta_{32} &= -1+4-2+3 = \underline{4>0} \end{aligned}\right\} \Rightarrow (\exists)\,\delta_{ij}>0 \Rightarrow \bar{X}_0 \text{ it isn't optimal solution!!}$$

③ We apply the "input criteria": $\delta_{ij} = \max\{\delta_{kl}>0\} = \max\{\underbrace{\delta_{21}}_{=4}, \underbrace{\delta_{31}}_{=4}, \underbrace{\delta_{32}}_{=4}\} \overset{(!)}{=} \delta_{21} \Rightarrow x_{21}\downarrow$

I choose "δ_{21}"

④ We apply the "output criteria":

$$\begin{tikzcd} ② \overset{20}{(1,1)} \arrow[r] & \overset{10}{(1,2)}\ ③ \arrow[d] \\ ① \underset{\theta=5}{(2,1)} \arrow[u] & \underset{5}{(2,2)}\ ④ \arrow[l] \end{tikzcd}$$

$\theta = \min\{x_{ij} / x_{ij}$ been into even no. cells$\} = \min\{\underbrace{x_{11}}_{=20}, \underbrace{x_{22}}_{=5}\} = 5 \Rightarrow x_{22} \to 0$

⑤ We determine the new solution : $\overline{X}_1$

	S_1	S_2	S_3	S_4^f	
W_1	15 [3]	15 [2]	* [1]	* [0]	30
W_2	5 [1]	* [4]	5 [2]	* [0]	20
W_3	* [2]	* [1]	5 [3]	15 [0]	20
	20	15	20	15	

$$\begin{tikzcd} \overset{=15}{\overbrace{20-\theta}} \atop (1,1) \arrow[r] & \overset{=15}{\overbrace{10+\theta}} \atop (1,2) \arrow[d] \\ "\downarrow" \ \underset{\theta=5}{(2,1)} \arrow[u] & \underset{\underbrace{5-\theta}_{=0 = \text{not } "*"}}{(2,2)} \ "\to" \arrow[l] \end{tikzcd}$$

$\overline{X}_1 : \begin{cases} \overline{x}_{11}=15, \overline{x}_{12}=15, \overline{x}_{21}=5, \overline{x}_{23}=5, \overline{x}_{33}=5, \overline{x}_{34}=15 \to \text{basic components} \\ \overline{x}_{13}=\overline{x}_{14}=\overline{x}_{22}=\overline{x}_{24}=\overline{x}_{31}=\overline{x}_{32}=0 \,(="*") - \text{non-basic (free) components} \end{cases}$

$f(\overline{X}_1) = 115 \; (< f(\overline{X}_0))$

② We check if $\overline{X}_1$ it is a optimal solution:

$\delta_{13} = -1+2-1+3 = 3 > 0 \Rightarrow (\exists)\delta_{ij} > 0 \Leftrightarrow \overline{X}_1$ - it isn't optimal solution $\Rightarrow$ the algorithm must to continue ... !!

$\vdots$

b)

	S_1	S_2	S_3	
W_1	20 [2]	0 [2]	* [1]	~~20~~, 0
W_2	* [3]	15 [4]	* [3]	~~15~~, 0
W_3	* [1]	5 [3]	5 [2]	~~10~~, ~~5~~, 0
	~~20~~ 0	~~20~~ ~~5~~ 0	~~5~~ 0	

$\left(\sum_{i=1}^{3} a_i = 45 = \sum_{j=1}^{3} b_j\right)$

$\Leftrightarrow \overline{X}_0 : \begin{cases} \overline{x}_{11}=20; \overline{x}_{12}=0; \overline{x}_{22}=15, \overline{x}_{32}=5; \overline{x}_{34}=5 - \text{basic components } (=m+n-1 = 3+3-1=\ldots) \\ \overline{x}_{13}=\overline{x}_{21}=\overline{x}_{23}=\overline{x}_{31}=0 \,(="*") \to \text{non basic components} \end{cases}$

$f(\overline{X}_0) = 125$

it's degenerate BAS(!!!)

(so, it's possible to appear the cycle phenom.)

we apply the perturbation method

①

	S_1	S_2	S_3	
W_1	20 [2]	ε [2]	* [1]	20+ε, ε, 0,
W_2	* [3]	15+ε [4]	* [3]	15+ε, 0
W_3	* [1]	5−2ε [3]	5+3ε [2]	10+ε, 5+3ε, 0
	~~20~~ 0	~~20~~, ~~20−ε~~, ~~5−2ε~~, 0	5+3ε	

$\Rightarrow \overline{X}_0 = \begin{cases} \overline{x}_{11}=20; \overline{x}_{12}=\varepsilon; \overline{x}_{22}=15+\varepsilon; \overline{x}_{32}=5-2\varepsilon; \overline{x}_{33}=5+3\varepsilon - \text{basic com}\ldots \\ \overline{x}_{13}=\overline{x}_{21}=\overline{x}_{23}=\overline{x}_{31}=0 \,(="*") - \text{non basic components} \end{cases}$

$f(\overline{X}_0) = 125 + 6\varepsilon$

non-degenerate iBAS(!!!)

② We apply the "optimal criteria":

$$\left.\begin{array}{l} \delta_{13} = -1+2-3+2 = 0 \\ \underline{\delta_{21}} = -3+2-2+4 = \underline{1>0} \\ \delta_{23} = -3+2-3+4 = 0 \\ \underline{\delta_{31}} = -1+2-2+3 = \underline{2>0} \end{array}\right\} \Rightarrow (\exists)\,\delta_{ij} > 0 \Rightarrow \overline{X}_0 \text{ it isn't the optimal solution}$$

③ $\delta_{ij} = \max\{\delta_{kl} > 0\} = \max\{\underbrace{\delta_{21}}_{=1}, \underbrace{\delta_{31}}_{=2}\} = \delta_{31} \Rightarrow$ "$x_{31} \downarrow$" (becomes a basic component)

④
$$\begin{tikzcd} \overset{20-\theta}{②\,(1,1)} \arrow[r] & \overset{\varepsilon+\theta}{(1,2)\,③} \arrow[d] \\ \text{"}\downarrow\text{"}\,①\,\underset{\theta=5-2\varepsilon}{\underline{(3,1)}} \arrow[u] & \underset{5-2\varepsilon-\theta}{(3,2)\,④}\ \text{"}\rightarrow\text{"} \arrow[l] \end{tikzcd}$$

$\Rightarrow \underline{\theta} = \min\{\underbrace{x_{11}}_{=20}, \underbrace{x_{32}}_{=5-2\varepsilon}\} = \underline{5-2\varepsilon} \Rightarrow$ "$x_{32} \rightarrow$" (becomes a non-basic variable)

⑤

	S_1	S_2	S_3	
W_1	$\boxed{2}$ $15+2\varepsilon$	$\boxed{2}$ $5-\varepsilon$	$\boxed{1}$ *	$20+\varepsilon$
W_2	$\boxed{3}$ *	$\boxed{4}$ $15+\varepsilon$	$\boxed{3}$ *	$15+\varepsilon$
W_3	$\boxed{1}$ $5-2\varepsilon$	$\boxed{3}$ *	$\boxed{2}$ $5+3\varepsilon$	$10+\varepsilon$
	20	20	$5+3\varepsilon$	

$$\Rightarrow \overline{X}_1 \begin{cases} \overline{x}_{11} = 15+2\varepsilon;\ \overline{x}_{12} = 5-\varepsilon;\ \overline{x}_{22} = 15+\varepsilon;\ \overline{x}_{31} = 5-2\varepsilon;\ \overline{x}_{33} = 5+3\varepsilon \\ \overline{x}_{13} = \overline{x}_{21} = \overline{x}_{23} = \overline{x}_{32} = 0 \ (= \text{"*"}) \end{cases}$$

$\underline{f(\overline{X}_1) = 115+10\varepsilon}$ $(< f(\overline{X}_0))$

We repeat the steps of the algorithm:

② $\begin{cases} \delta_{13} = -1+2-1+2 = 2>0 \\ \vdots \end{cases} \Rightarrow (\exists)\,\delta_{ij} > 0 \Rightarrow \overline{X}_1$ - it isn't the optimal solution $\Rightarrow$ the algorithm continued...