Problem 1

The auto-correlation function for a real-valued stable sequence x[n] is defined as

$$c_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot x[n+k]$$

1. Show that the z-transform of $c_{xx}[n]$ is

$$C_{xx}(z) = X(z) \cdot X(z^{-1})$$

Determine the ROC of $C_{xx}(z)$.

2. Suppose that $x[n] = a_n u[n]$. Sketch the pole-zero plot for $C_{xx}(z)$, including the region of convergence. Also find $c_{xx}[n]$ by evaluating the inverse z-transform of $C_{xx}(z)$.

Solution.
Definition of Z-transform is

$$Z[x[n]] = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} \tag{1}$$

Thus $Z[c_{xx}[n]]$ is

$$\begin{aligned}
C_{xx}(z) &= Z[x[n] \otimes x[n]] \\
&= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x[k]x[k+n] \right] z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x[k] \left[\sum_{n=-\infty}^{\infty} x[k+n] z^{-(k+n)} \right] z^{k} \\
&= \left[\sum_{k=-\infty}^{\infty} x[k] z^{-(-k)} \right] \left[\sum_{n=-\infty}^{\infty} x[k+n] z^{-(k+n)} \right] \\
&= X(z^{-1}) \cdot X(z)
\end{aligned}$$

ROC doesn't include poles.
ROC of $C_{xx}(z)$ depends upon the nature of $X(z)$. Assuming $X(z)$ is a rational expression, and $x[n]$ is assumed to be right-sided signal, there exists a such that ROC of $X(z) \in z > a \ni a < 1$ for any finite z.
Replacing z in $X(z)$ with z^{-1}, the minimum distance from origin pole, let q becomes the max distance pole $1/q$ and because we have assumed a right sided signal, ROC is either greater than $|a|$ or $|1/q|$, whichever being the greater quantity.

Now given $x[n] = a^n u[n]$, is a step function. Thus $\mathcal{Z}\{x[n]\} = X(z) = \frac{z}{z-a}$.

$$C_{xx}(z) = X(z)X(z^{-1}) = \frac{z}{(z-a)(1-az)}$$

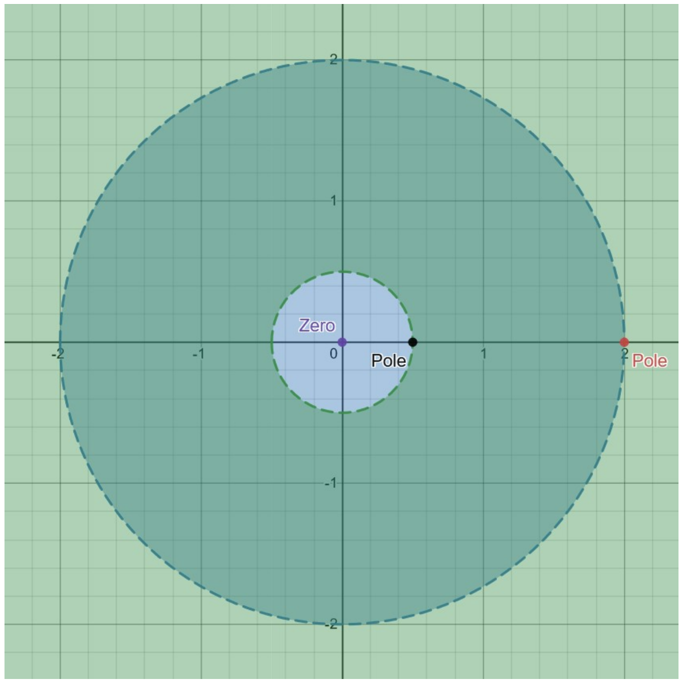


Figure 1: ROC represented by darker region

Now using partial fractions,

$$C_{xx}(z) = \frac{z}{(z-a)(1-az)} = \frac{a}{(1-a^2)(z-a)} + \frac{1}{(1-a^2)(1-za)}$$
$$\implies c_{xx}[n] = (a^n + a^{-n})\frac{u[n-1]}{1-a^2}$$

■

Problem 2

Suppose

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kP]$$

Show that the Fourier transform of $x[n]$ is given by

$$X(\omega) = \mathcal{F}\{x[n]\} = \sum_{k=-\infty}^{\infty} \frac{2\pi}{P}\delta(\omega - \frac{2\pi}{P}k)$$

Solution.
Given P to be the fixed period of the impulse train $x[n]$. Now any discrete periodic signal can be decomposed to its periodic fourier series. Taking the period of $[0, P-1]$ and extending it later, also assuming $\omega_0 = \frac{2\pi}{P}$

$$x[n] = \sum_{k=0}^{P-1} X_k e^{jk\omega_0 n} \tag{2}$$

$$\begin{aligned} X_k &= \sum_{n=0}^{P-1} \frac{x[n]}{P} e^{-jk\omega_0 n} \\ &= \sum_{n=0}^{P-1} \frac{\delta[n]}{P} e^{-jk\omega_0 n} \\ &= \frac{1}{P} \end{aligned} \tag{3}$$

Now replacing in (3),

$$x[n] = \sum_{k=0}^{P-1} \frac{1}{P} e^{jk\omega_0 n}$$

$$e^{j(k\omega_0)n} \Longleftrightarrow 2\pi\delta(\omega - k\omega_0)$$

The above by Time shifting property. Therefore Discrete fourier transform of $x[n]$ is

$$X[\omega] = \sum_{k=0}^{P-1} \frac{2\pi}{P}\delta(\omega - k\omega_0)$$

and because this is periodic,

$$X[\omega] = \sum_{k=-\infty}^{\infty} \frac{2\pi}{P}\delta(\omega - k\omega_0) \tag{4}$$

■

Problem 3

Consider a vocal fold oscillation in a vocal fry or diplophonic state, where a secondary glottal flow occurs within a glottal cycle. We model this condition over one pitch period as

$$\tilde{g}[n] = g[n] + \alpha g[n - n_0]$$

where n_0 is the time delay between the primary and secondary pulse. The resulting periodic glottal flow waveform is given by $u[n] = \sum_{k=-\infty}^{\infty} \tilde{g}[n - kP]$, where P is the pitch period.

1. Determine the Fourier transform of $\tilde{g}[n]$ in terms of the Fourier transform of $g[n]$. With this result, write the Fourier transform of the periodic glottal flow waveform $u[n]$, i.e., $U(\omega)$.

2. Suppose, in a diplophonic state, that $n_0 = P/2$, where P is the glottal pitch period. Describe how the presence of $g[n-n_0]$ affects, at the harmonic frequencies, the squared magnitude of Fourier transform $U(\omega)$ derived in (1), i.e. Fourier transform of the glottal flow waveform $u[n] = \sum_{k=-\infty}^{\infty} \tilde{g}[n - kP]$. Describe the effect as α changes from 0 to 1.

3. Repeat part (2) with $n_0 = 3P/4$ corresponding to a vocal fry state where the secondary glottal pulse is close to the primary pulse.

Solution.

Consider $g[n] \Longleftrightarrow G(\omega)$. Thus,

$$\tilde{g}[n] = g[n] + \alpha g[n - n_0]$$
$$\tilde{G}(\omega) = G(\omega) + \alpha G(\omega) e^{-j\omega n_0}$$

also, $u[n] = \sum_{k=-\infty}^{\infty} \tilde{g}[n - kP] \Longleftrightarrow U(\omega) = \sum_{k=-\infty}^{\infty} \tilde{G}(\omega) e^{-j\omega kP}$,

$$U(\omega) = \sum_{k=-\infty}^{\infty} G(\omega)(1 + \alpha e^{-j\omega n_0}) e^{-j\omega kP}$$

Now taking $n_0 = \frac{P}{2}$ and harmonic frequencies, i.e. $\omega = \frac{2\pi m}{P}$

$$\begin{aligned} U(\omega) &= \sum_{k=-\infty}^{\infty} G(\omega)(1 + \alpha e^{-j\frac{2\pi m}{P}\frac{P}{2}}) e^{-j\frac{2\pi m}{P}kP} \\ &= G(\omega)(1 + \alpha e^{-j\pi m}) \sum_{k=-\infty}^{\infty} e^{-j2\pi mk} \\ &= G(\omega)(1 + \alpha cos(m\pi)) \sum_{k=-\infty}^{\infty} e^{-j2\pi mk} \end{aligned}$$

The variation of α depends upon the parity of m, if m is even, $U(\omega)$ increases linearly as α goes from 0 to 1, while an odd m would lead to the opposite.

If $n_0 = \frac{3P}{4}$ and again harmonic frequencies are considered,

$$U(\omega) = \sum_{k=-\infty}^{\infty} G(\omega)(1 + \alpha e^{-j\frac{2\pi m}{P}\frac{3P}{4}})e^{-j\frac{2\pi m}{P}kP}$$
$$= G(\omega)(1 + \alpha e^{-j\frac{3\pi m}{2}}) \sum_{k=-\infty}^{\infty} e^{-j2\pi mk}$$

For even parity of m, this acts like $n_0 = \frac{P}{2}$, i.e. if $m/2$ is even then $U(\omega)$ increases as α goes from 0 to 1, while an odd $m/2$ would lead to $U(\omega)$ decreasing with α going from 0 to 1.
For odd parity, $U(\omega)$ is independent of α. ■

Problem 4

Programming Assignment: In this MATLAB exercise, you design a number of glottal pulse trains with different shapes and pitch, and analyze their spectral and perceptual properties.

1. Create in MATLAB the following glottal pulse, which is a time-reversed decaying exponential convolved with itself:

$$g[n] = (\alpha - nu[-n]) * (\alpha - nu[-n])$$

   Set the length of $g[n]$ to where it has effectively decayed to zero. Experiment with the different values of $\alpha = 0.9, 0.95, 0.99, 0.998$ and compute the resulting Fourier transform magnitude, using a FFT length sufficiently long to avoid significant truncation of the pulse.

2. Convolve $g[n]$ from part (a) (with $\alpha = 0.99$) with a periodic impulse train of pitch 100 Hz and 200 Hz. Assume an underlying sampling rate of 10000 samples/s. Using the MATLAB *sound.m* function, listen to the two pulse trains and make a perceptual comparison. Window a 20-ms piece (with a Hamming window) of the waveform and compare the spectral envelope and harmonic structure for the two pitch selections.

Solution.
Question Cancelled. ■

Problem 5

Programming Assignment: In this MATLAB exercise, you design a number of glottal pulse trains with different shapes and pitch, and analyze their spectral and perceptual properties.

1. Create in MATLAB the following glottal pulse, which is a time-reversed decaying exponential convolved with itself:

$$g[n] = (\alpha^{-n}u[-n]) * (\alpha^{-n}u[-n])$$

   Set the length of $g[n]$ to where it has effectively decayed to zero. Experiment with the different values of $\alpha = 0.9, 0.95, 0.99, 0.998$ and compute the resulting Fourier transform magnitude, using a FFT length sufficiently long to avoid significant truncation of the pulse.

2. Convolve g[n] from part (a) (with $\alpha = 0.99$) with a periodic impulse train of pitch 100 Hz and 200 Hz. Assume an underlying sampling rate of 10000 samples/s. Using the MATLAB *sound.m* function, listen to the two pulse trains and make a perceptual comparison. Window a 20-ms piece (with a Hamming window) of the waveform and compare the spectral envelope and harmonic structure for the two pitch selections.

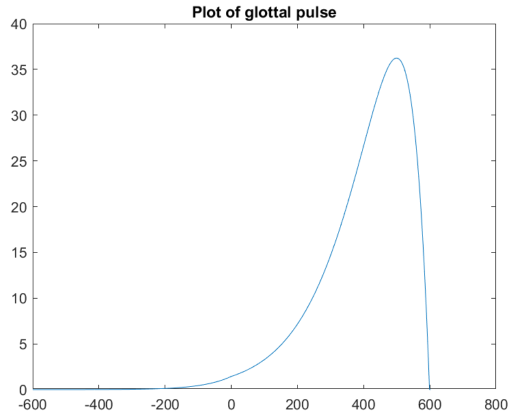
3. We can consider the glottal pulse in part (a) as arising from vocal folds with an abrupt closure and thus corresponding to a "hard" voice. Modify the pulse shape ($\alpha = 0.99$) so that the sharp edge at the closure is rounded off, corresponding to vocal folds that close more gradually. Compute the resulting Fourier transform magnitude. Repeat part (b) with your new glottal pulse. How does your result compare perceptually with the pulse train of part (b)?

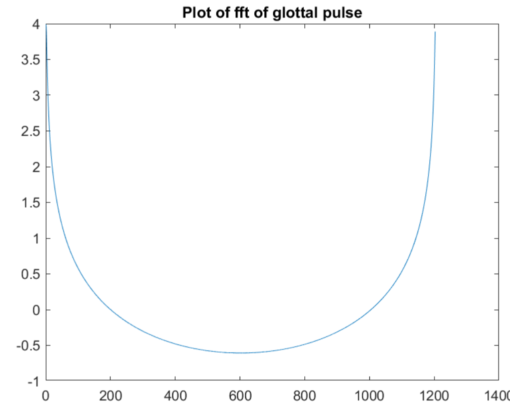
Solution.
Changing α leads to change in the shape of the pulse, as α gets closer to 1, $g[n]$ gets smoother with lesser abrupt and longer time-period pulse. The actual time of a glottal pulse with $\alpha = 0.99$ is 10ms.

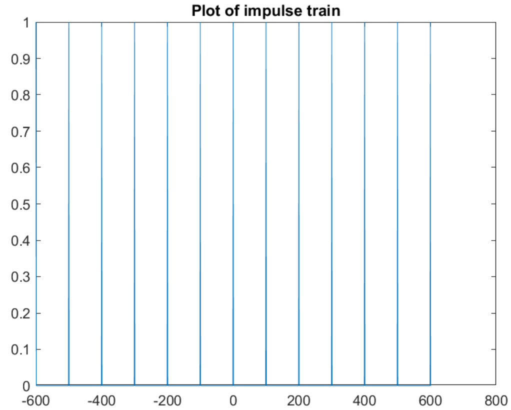
All relevant pictures are attached below. ■

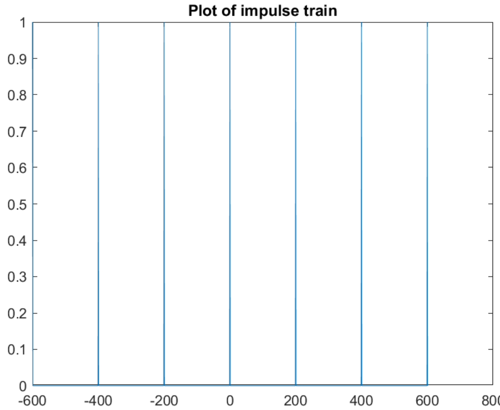
MATLAB Code

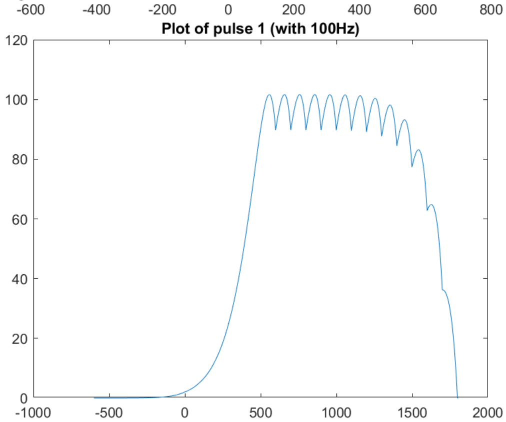
```
% a = 0.9;
% a = 0.95;
a = 0.99;
% a = 0.998;

% n = -100:80; % for a=0.90
% n = -200:80; % for a=0.95
```

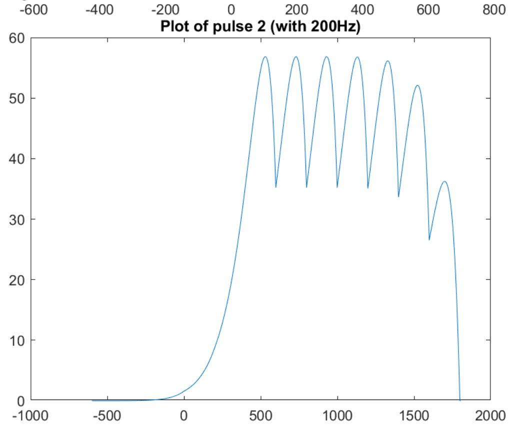

Plot of glottal pulse
0 5 10 15 20 25 30 35 40
-600 -400 -200 0 200 400 600 800

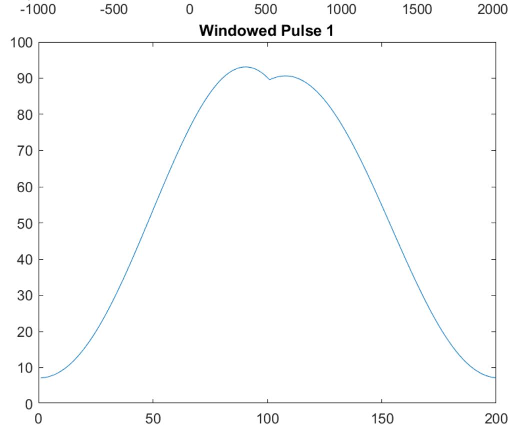

Plot of fft of glottal pulse
-1 -0.5 0 0.5 1 1.5 2 2.5 3 3.5 4
0 200 400 600 800 1000 1200 1400

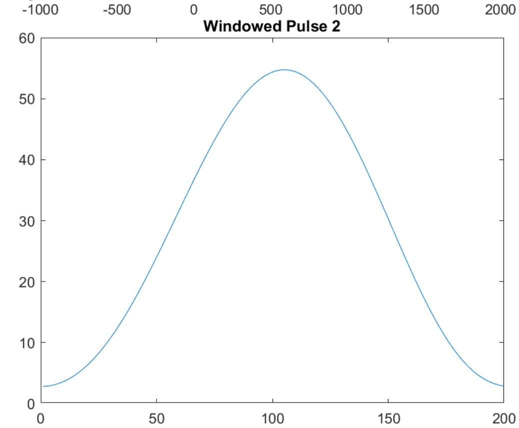

Plot of impulse train
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
-600 -400 -200 0 200 400 600 800

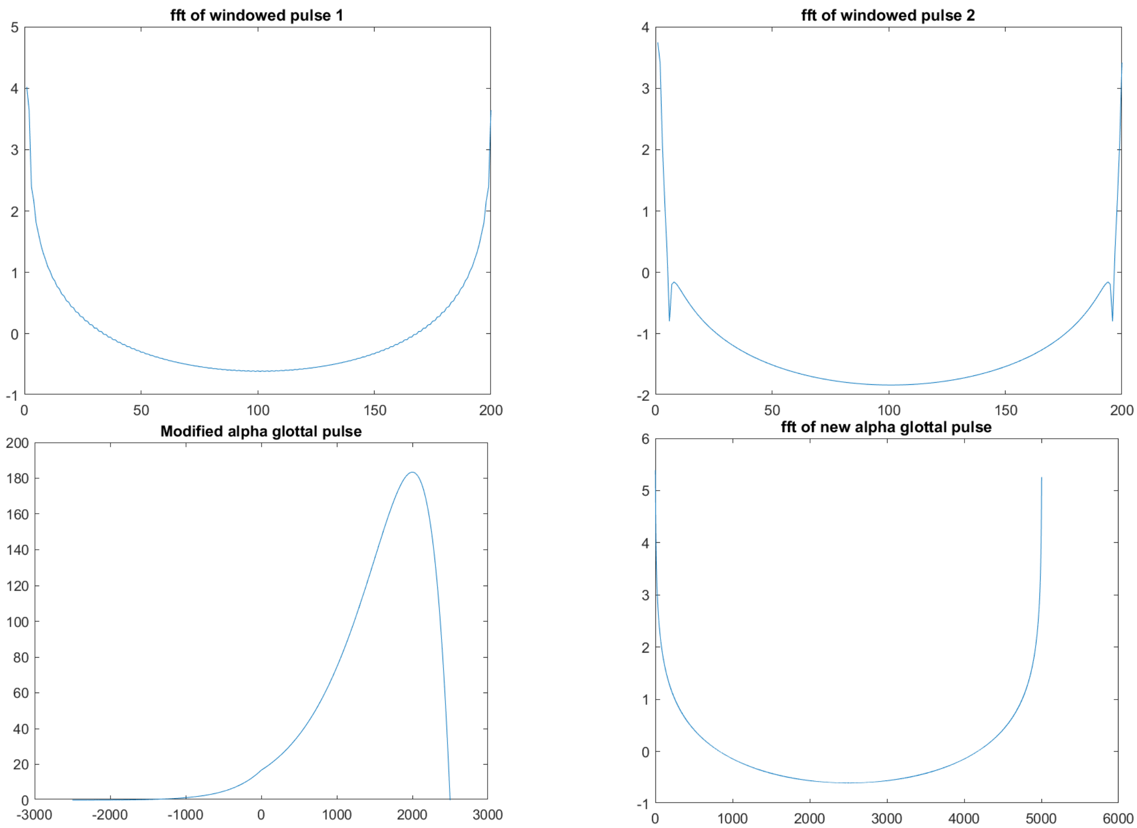

Plot of impulse train
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
-600 -400 -200 0 200 400 600 800


Plot of pulse 1 (with 100Hz)
0 20 40 60 80 100 120
-1000 -500 0 500 1000 1500 2000


Plot of pulse 2 (with 200Hz)
0 10 20 30 40 50 60
-1000 -500 0 500 1000 1500 2000


Windowed Pulse 1
0 10 20 30 40 50 60 70 80 90 100
0 50 100 150 200


Windowed Pulse 2
0 10 20 30 40 50 60
0 50 100 150 200




```
n = -600:1; % for a=0.99
g = conv(a.^(-n) .* h(-n), a.^(-n) .* h(-n), 'full');
soundsc(g);

figure;
plot(n, a.^(-n) .* h(-n));
title('Plot of time reverse decaying exponential')
figure;
n2 = -600:602;
plot(n2, g);
title('Plot of glottal pulse')

f = fft(g);
figure;
plot(log10(abs(f)));
title('Plot of fft of glottal pulse')

f1 = 100;
f2 = 200;
fs = 10000;
y1 = zeros(size(n2));
y1(mod(n2, f1) == 0) = 1;
```

```
figure;
plot(n2,y1);
title('Plot of impulse train')
pulse1 = conv(y1, g, 'full');
soundsc(pulse1);

figure;
plot(-600:1804, pulse1);
title('Plot of pulse 1 (with 100Hz)')

y2 = zeros(size(n2));
y2(mod(n2, f2) == 0) = 1;
figure;
plot(n2,y2);
title('Plot of impulse train');

pulse2 = conv(y2, g, 'full');
length(pulse2);
soundsc(pulse2);
figure;
plot(-600:1804, pulse2);
title('Plot of pulse 2 (with 200Hz)')

hamming_len = 0.02*fs;
ham = hamming(hamming_len, "symmetric");
figure;
plot(ham);
title('Hamming (200 in discrete time scale)')

window1 = (ham' .* pulse1(1600:1799));
figure;
plot(pulse1(1600:1799))
title('200 length Pulse 1')
figure;
plot(window1)
title('Windowed Pulse 1')
fwindow1 = fft(window1);
figure;
plot(log10(abs(fwindow1)))
title('fft of windowed pulse 1')

window2 = (ham' .* pulse2(1600:1799));
figure;
plot(pulse2(1600:1799))
title('200 length Pulse 2')
```

```
figure;
plot(window2)
title('Windowed Pulse 2')
fwindow2 = fft(window2);
figure;
plot(log10(abs(fwindow2)))
title('fft of windowed pulse 2')

plot(n2, g)

% Now for a glottal pulse to be more rounded off,
a = 0.998;
n = -2500:1;
n2 = -2500:2502;
gNew = conv(a.^(-n) .* h(-n), a.^(-n) .* h(-n), 'full');
length(gNew)
figure;
plot(n2, gNew)
title('Modified alpha glottal pulse')

fgNew = fft(gNew);
figure;
plot(log10(abs(fgNew)))
title('fft of new alpha glottal pulse')
soundsc(gNew)
soundsc(g)
```