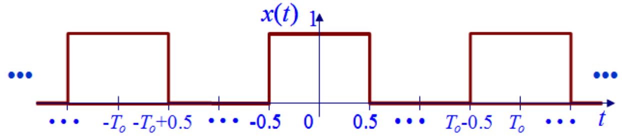
Fourier Transform

Rectangular Pulse



$$c_0 = \frac{1}{T}\int_{-0.5}^{0.5} 1\cdot e^{-j\cdot 0\cdot \omega_0 \cdot t}dt = \frac{1}{T}$$

$$c_k = \frac{1}{T}\int_{-0.5}^{0.5} 1e^{-jk\omega_0 t}dt = \frac{1}{-Tjk\omega_0}\left[e^{-jk\omega_0 0.5} - e^{jk\omega_0 0.5}\right]$$

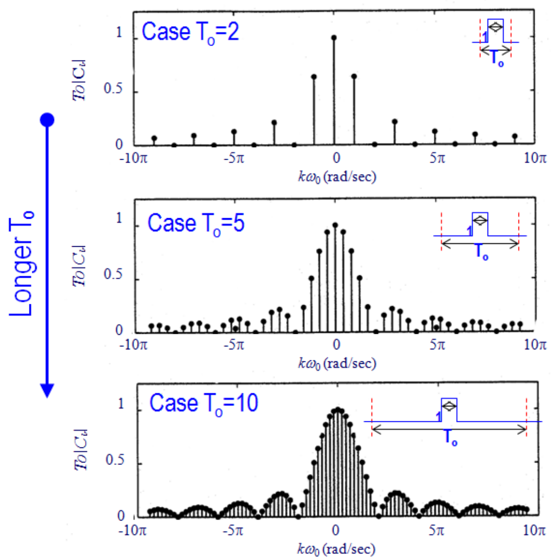
$$c_k = \frac{1}{Tjk\omega_0}\left[e^{jk\omega_0 0.5} - e^{-jk\omega_0 0.5}\right] = \frac{2}{Tk\omega_0}\sin\frac{k\omega_0}{2}$$

$$\Rightarrow$$

$$T = \frac{2\pi}{\omega_0},$$

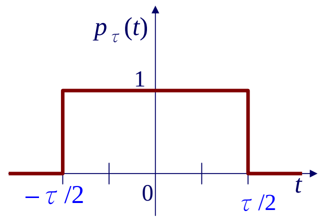
$$c_0 T = 1$$

$$c_k T = \frac{2}{k\omega_0}\sin\frac{k\omega_0}{2}$$



As T→ ∞,
Converges to a
continuous spectrum

Example : rectangular pulse



$$x(t) = \begin{cases} 1 & -\frac{\tau}{2} \le t < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} 1e^{-j\omega t}dt$$

$$X(\omega) = \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{-1}{j\omega}\left[e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}\right]$$

$$X(\omega) = \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right) = \tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

Exercise: Exponential function

Exponential signal: $x(t)=e^{-bt}u(t)$

- Time-domain representation
- If b>0, exp(-bt) → 0

Frequency domain

$$X(\omega)=\int_{-\infty}^{\infty} e^{-bt}u(t)e^{-j\omega t}dt=\int_{0}^{\infty} e^{-bt}e^{-j\omega t}dt$$

$$X(\omega)=\frac{-1}{b+j\omega}\left[e^{-(b+j\omega)t}\right]_0^{\infty}$$

Exercise: Exponential function

Exponential signal: $x(t)=e^{-bt}u(t)$

Frequency domain

If b ≤0, the limit cannot be evaluated

If b>0, exp(-bt) → 0 as t approaches infinity

$$X(\omega)=\frac{-1}{b+j\omega}[0-1]=\frac{1}{b+j\omega}$$

$$|X(\omega)|=\frac{1}{\sqrt{b^2+\omega^2}}$$

$$\angle X(\omega)=-\tan^{-1}\frac{\omega}{b}$$

Exercise: Exponential function

Exponential signal: $x(t)=e^{-bt}u(t)$

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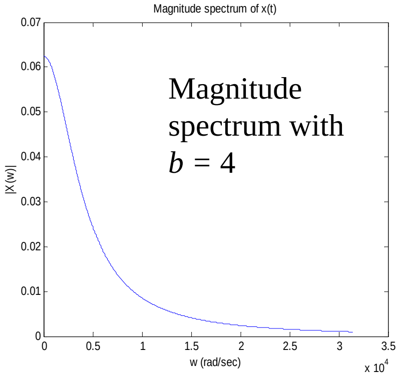
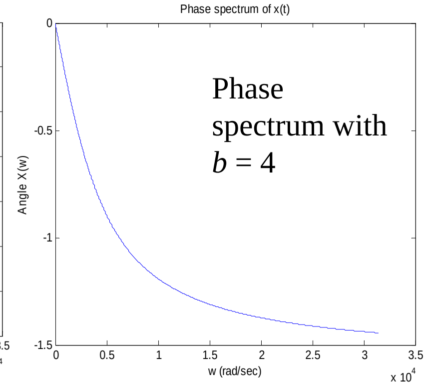
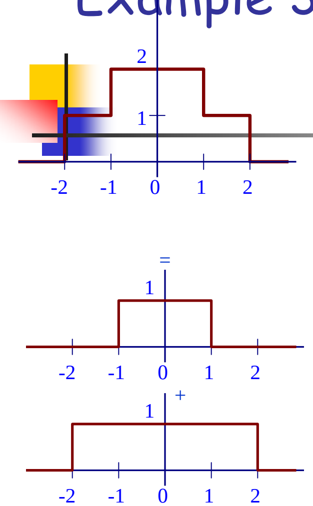




Example 3: Linearity:



$$x(t)=p_\tau(t)=\begin{cases}1 & -\frac{\tau}{2}\le t<\frac{\tau}{2}\\ 0 & \text{otherwise}\end{cases}$$

$$\leftrightarrow X(\omega)=\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

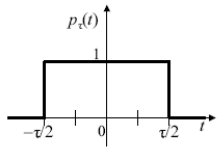
$$y(t)=p_2(t)+p_4(t)$$

$$\leftrightarrow Y(\omega)=2\,\mathrm{sinc}\left(\frac{\omega 2}{2\pi}\right)+4\,\mathrm{sinc}\left(\frac{\omega 4}{2\pi}\right)$$

$$\leftrightarrow Y(\omega)=2\,\mathrm{sinc}\left(\frac{\omega}{\pi}\right)+4\,\mathrm{sinc}\left(\frac{\omega 2}{\pi}\right)$$

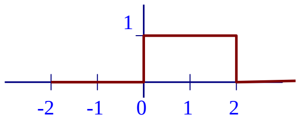
Example : Time-Shift

Previous result:



$$x(t) = p_\tau(t) = \begin{cases} 1 & -\frac{\tau}{2} \le t < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases} \leftrightarrow X(\omega) = \tau \,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

New function:



$$y(t) = p_2(t-1)$$

$$\leftrightarrow Y(\omega) = e^{-j\omega} 2\,\text{sinc}\left(\frac{\omega 2}{2\pi}\right) = 2e^{-j\omega}\,\text{sinc}\left(\frac{\omega}{\pi}\right)$$

Exercise

- Consider $FT\{x(t)\} = X(\omega) = \dfrac{1}{b + j\omega}$
- Find the FT of the following signals

1. $v(t) = x(5t - 4)$
   (Hint: Use the time delay and time scaling properties)
2. $v(t) = e^{j2t}x(t)$
   (Hint: Use the time shift and duality properties)
3. $v(t) = x(t)\cos(4t)$
   (Hint: $\cos(x) = \frac{e^{x}+e^{-x}}{2}$)

Solution

1. $v(t) = x(5t - 4)$

Hence $FT\{v(t)\} = V(\omega) = e^{-j4\frac{\omega}{5}} \frac{1}{5} X\left(\frac{\omega}{5}\right)$

$$= e^{-j\frac{\omega}{5}4}\left(\frac{1}{5}\right)\frac{1}{b + j\omega/5}$$

$$= e^{-j\frac{\omega}{5}4}\frac{1}{5b + j\omega}$$

Solution

2. $v(t) = e^{j2t}x(t)$

Assume we have a signal $X(t)$ and has the FT $x(\omega)$. By time shift property, we have

$$FT\{X(t - \omega_o)\} = e^{j\omega\omega_o}x(\omega)$$

By duality property, we have

$$FT\left\{e^{jt\omega_o}x(t)\right\} = X(\omega - \omega_o)$$

Hence $FT\{v(t)\} = V(\omega) = \frac{1}{b+j(\omega-2)}$

Solution

- 3. $v(t) = x(t)\cos(4t)$

  we know that

  $$FT\{e^{jt\omega_o}x(t)\} = X(\omega - \omega_o)$$

  Hence

  $$FT\{e^{-jt\omega_o}x(t)\} = X(\omega + \omega_o)$$

  $$FT\left\{\frac{e^{jt\omega_o} + e^{-jt\omega_o}}{2}x(t) = x(t)\cos(\omega_o t)\right\}$$

  $$= \frac{1}{2}\{X(\omega - \omega_o) + X(\omega + \omega_o)\}$$

Solution

$$V(\omega) = FT\{x(t)\cos(4t)\}$$

$$= \frac{1}{2}\{X(\omega - 4) + X(\omega + 4)\}$$

$$= \frac{0.5}{b + j(\omega - 4)} + \frac{0.5}{b + j(\omega + 4)}$$

$$= \frac{b + j\omega}{b^2 + bj2\omega - (\omega^2 - 16)}$$