Classification of algebraic tangles – supplement 1/5

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S1. The Kauffman bracket of a tangle

Let us first recall the definition of the Kauffman bracket for unoriented links.

Definition 1 ([1]). The Kauffman bracket is a polynomial invariant of unoriented framed links in $\mathbb{R}^3$. For a link L, the Kauffman bracket $\langle L \rangle$ is defined recursively on the link diagram D of L using the following rules:

- The skein relation:
$$\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle, \tag{1}$$
where $\langle \times \rangle$, $\langle)(\rangle$, and $\langle \asymp \rangle$ are diagrams that are everywhere the same as L, except inside a small disk, where they look like the figures indicated.
- The bracket of a disjoint union of a link diagram L and a simple closed curve (unknot) is:
$$\langle D \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle \tag{2}$$
- The bracket of an unknot is normalized to be:
$$\langle \bigcirc \rangle = 1 \tag{3}$$

The Kauffman bracket is an invariant of framed links, meaning it is invariant under the second and the third Reidemeister moves.

A crossing in an oriented diagram D has a sign of $+1$ or -1 according to the right-hand rule. The sum of signs of the crossings for a given diagram D is called the *writhe* of D and is denoted by $wr(D)$. If we appropriately normalize the Kauffman bracket, it becomes an invariant also under the first Reidemeister move.

Theorem 1 ([1])**.** *Let L be an oriented link and D its diagram. The X polynomial, also known as the normalized Kauffman bracket or the normalized bracket polynomial, defined by*

$$X(D) = (-A^3)^{-wr(D)}\langle D\rangle$$

is an oriented link invariant.

We can obtain the Jones polynomial V if we make a substitution $A = t^{-1/4}$ in X, i.e. $V(L)(t) = X(L)(t^{-1/4})$.

We can naturally extend the normalised Kauffman bracket to an invariant of tangles – in [2] Turaev extended the Jones polynomial to define a Jones-type invariant for tangles . In this context, we describe the invariant using a recursive definition in terms of the variable A, rather than the equivalent state-sum definition in terms of the variable t as presented in the original paper [2]:

Definition 2. The Kauffman bracket $\langle T\rangle$ for a tangle diagram T is defined using the following rules:

$$\langle \times \rangle = A\langle)\,(\rangle + A^{-1}\langle \asymp \rangle, \tag{4}$$

If the tangle consists of a closed component without crossings, we have:

$$\langle T \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle T\rangle \tag{5}$$

After removing all crossings and closed components, we end up with an expression of the form

$$\langle T\rangle = P_1(A)\cdot\langle)\,(\rangle + P_2(A)\cdot\langle \asymp \rangle,$$

where $P_1(A)$ and $P_2(A)$ are Laurent polynomials in the variable A.

Remark. We can think of such an expression as an element in the free R-module generated by $\{\langle 0\rangle, \langle\infty\rangle\}$ where $R = \mathbb{Z}[A, A^{-1}]$.

Theorem 2 (follows from [2])**.** *Let T be an oriented tangle. The normalised Kauffman bracket for tangles defined by*

$$X(T) = (-A^3)^{-wr(T)}\langle T\rangle.$$

is an isotopy tangle invariant.

Note that $X(T)$ requires the tangle to be oriented. In the case of unoriented tangles, we have two choices: in the case T does not contain closed components, we can assign a tangle a natural top-down orientation, otherwise the invariant is defined up to multiplication by $(-A^3)^{\pm 1}$.

References

[1] Kauffman, L.H.: State models and the jones polynomial. Topology **26**(3), 395–407 (1987)

[2] Turaev, V.: Jones-type invariants of tangles. Journal of Soviet Mathematics **52**, 2806–2807 (1990)

Table 1 Number of all distinct tangle orbits of X-type, divided into subgroups by number of crossings ("#cross") and number of extra closed components ("#closed components"). Tangles in each column/row are added up ("Orbit" column/row). Additionally tangles are counted and added up to equivalence ("Equiv.") and isotopy ("Isotopy").

#cross	#closed components							Total		
	0	1	2	3	4	5	6	Orbit	Equiv.	Isotopy
0	-	-	-	-	-	-	-	-	-	-
1	1	-	-	-	-	-	-	1	1	2
2	-	-	-	-	-	-	-	-	-	-
3	1	-	-	-	-	-	-	1	2	4
4	1	-	-	-	-	-	-	1	2	4
5	3	2	-	-	-	-	-	5	10	20
6	11	1	-	-	-	-	-	12	24	68
7	22	17	3	-	-	-	-	42	82	236
8	88	39	1	-	-	-	-	128	256	976
9	247	158	53	3	-	-	-	461	920	3784
10	851	670	99	1	-	-	-	1621	3242	15860
11	3076	2212	774	120	4	-	-	6186	12360	66414
12	10572	10071	2991	210	1	-	-	23845	47690	283736
13	41638	38142	13502	2786	246	4	-	96318	192602	1228024
14	155814	162954	66716	10135	404	1	-	396024	792048	5361364
Orbit	212325	214266	84139	13255	655	5	-	524645		
Equiv.	424644	428515	168264	26499	1307	10	-		1049239	
Isotopy	2935790	2849382	1029454	140382	5464	20	-			6960492

Table 2 Same as Table 1 but for V-type and H-type tangles. Each V-type tangle has an equivalent, non-isotopic H-type tangle.

#cross	#closed components							Total		
	0	1	2	3	4	5	6	Orbit	Equiv.	Isotopy
0	1	-	-	-	-	-	-	1	1	2
1	-	-	-	-	-	-	-	-	-	-
2	1	-	-	-	-	-	-	1	2	4
3	1	-	-	-	-	-	-	1	2	4
4	3	2	-	-	-	-	-	5	9	18
5	8	1	-	-	-	-	-	9	18	48
6	19	11	2	-	-	-	-	32	63	168
7	64	29	1	-	-	-	-	94	188	644
8	179	113	36	3	-	-	-	331	656	2466
9	597	450	76	1	-	-	-	1124	2248	10116
10	2059	1532	512	80	3	-	-	4186	8353	41628
11	7026	6530	2029	166	1	-	-	15752	31504	175792
12	26543	24782	8922	1811	177	4	-	62239	124401	751960
13	98140	102960	42571	7045	327	1	-	251044	502088	3254932
14	384058	429673	182655	40844	5557	327	4	1043118	2085933	14241600
Orbit	518699	566083	236804	49950	6065	332	4	1377937		
Equiv.	1037278	1132082	473489	99848	12101	660	8		2755466	
Isotopy	7256706	7630826	2965420	565208	58584	2622	16			18479382