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High-Accuracy Numerical Schemes for 2nd-order Forward-Backward Stochastic Differential Equations and its Application

THESIS

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Abstract

The theory of forward backward differential equations have been widely researched among the international researchers, different kinds of FBSDEs have been studied deeply. In 1990, Pardoux and Peng [57] proved the existence and uniqueness of BSDE, and propose firstly the connection between BSDE and PDE, which extends the range of Feynman-Kac formula. From then on, many researchers tends to this research area and BSDE theory became an important branch of stochastic calculus. The theory of stochastic forward differential equations has been greatly developed as the theoretical and numerical development of forward SDE and backward SDE. The research of forward and backward differential equations can be widely used in many areas, such mathematical finance , stochastic control, games, partial differential equations etc.

The connection between semi-linear,quasi-linear PDEs and forward backward stochastic differential equations make an interesting interpretation of partial differential equations. This extends the well-known Faynman-Kac formula. Zhang [87], Bouchard and Touzi [10]. Based on this idea, we can construct the stochastic representation of PDE. However, PDEs corresponding to standard FBSDEs cannot be nonlinear in the second-order derivatives because second-order terms arise only linearly through Itô's formula from the quadratic variation of the underlying state process. Cheridito, Soner, Touzi and Victor [13] introduced an new FBSDEs with second-order dependence in the generator f, they call them second second-order backward stochastic differential equations and show how they are related to fully nonlinear parabolic PDEs. This extends the range of connections between stochastic equations and PDEs.

The second order FBSDE would play an important role in many practical areas. However, there is hardly an closed form solution of the 2FBSDEs. Therefore the numerical solution of 2FBSDE makes great sense.

The fully nonlinear PDE is hardly solvable, and not much research report existed in this area. Moreover, existing work on fully non-linear PDEs aim at design efficient schemes for high dimensional PDEs, however, the convergence

rates are not satisfactory. We adopt firstly in the world an stochastic approach to research the solution of fully nonlinear PDEs, and gain high order convergence result.

Through the property of 2FBSDE, consisted of the theory of scientific computing, We in this thesis research numerical schemes of the decoupled and coupled 2FBSDE. Furthermore, under the connection of 2FBSDE and fully nonlinear PDE, we research the numerical method of 2PDE, and analysis numerically the convergence rate of the scheme. We also apply the schemes to stochastic optimal control problems.

We also introduce a new high accuracy approach for solving the second order stochastic differential equations. which is called stochastic deferred correction method (DC method). What makes this method fascinating is that low-order schemes is need for the initial approximation, while after the iteratively correction, high order schemes will obtain. The approach adopt an one-step low-order method , which reduce the computation shapely as well as the needed information for the terminal time; meanwhile, after the deferred correction, the convergence order can be arised higher.

The contents of the thesis is organized as follows:

In Chapter 1, we introduce the development and research background of the topic concerning in chapter 3 to 5.

In Chapter 2, we introduce the existence and uniqueness of SDE, the generator of SDE. The numerical schemes of SDE are listed in this chapter. We introduce the regularity of 2FBSDE and Gauss integral.

In Chapter 3, We research on the multi-step scheme of 2FBSDE. Transform the 2FBSDE to a relatively deterministic reference equations by the stochastic calculus and theory of 2FBSDE.Obtain the derivatives of the conditional expectation. Approximate it with the multi-step theory and discrete the conditional expectation.Approximate the conditional expectation with Gauss quadrature rule. Interpolate the points which lie on the non-grid place. Solve the equations by the numerical tools, and obtain the high order numerical schemes, and verify the convergence result by several numerical examples.

Feature of this chapter: Propose the high accuracy scheme of 2FBSDE by stochastic analysis and high performance computation

1. Firstly obtain an high order scheme for solve 2FBSDE.
2. The Scheme of this 2FBSDE is high order convergence for $k \leq 6$. The high order property is independent with the choice of the scheme for SDE.
3. Choose the simplest Euler scheme for the forward SDE, reduce sharply the computation.
4. realize the scheme by Fortran, apply to many kinds of 2FBSDEs

This chapter is mainly base on the paper:

Tao Kong, Weidong Zhao and Tao Zhou, *High-Order Multistep Schemes for Solving Second-Order Forward Backward Stochastic Differential Equations*, SIAM on Control and Optimization. arXiv:1502.03206, submitted

In Chapter 4, We mainly research on the stochastic scheme for parabolic fully nonlinear second-order PDE. It shows great interest in the research area about the stochastic representation and scheme of PDE. However it's hard for the nonlinear case. Moreover, existing work on fully non-linear PDEs aim at design efficient schemes for high dimensional PDEs, however, the convergence rates are not satisfactory. We try to research this problem through an stochastic approach.

The feature of this chapter:

1. Firstly solve the 2PDE in an stochastic way.
2. The result we get can gain high order convergence for $k \leq 6$.
3. We realize the scheme by Fortran and verify the convergence result by many numerical examples.

This chapter is mainly based on the paper:

Tao Kong, Weidong Zhao and Tao Zhou, *Probabilistic High Order Numerical Schemes for Fully Nonlinear Parabolic PDEs*, Communications in Computational Physics, 2015, Vol 18, pp 1482-1503.

In Chapter 5, We also introduce a new high accuracy approach for solving the second order stochastic differential equations. The approach, which is called stochastic deferred correction method is shown in this article that can raise the predict scheme from lower convergence rate to higher one.

The feature of this chapter:

1. This is a new high-order approach for solving 2FBSDE.

2. The fist evaluation only needs low-order scheme, however, by means of DC methods, the speed of convergence can be accelerated

3. This method only need the terminal time information, which can be used as initial state for other methods.

This chapter is mainly base on

Weidong Zhao and Tao Kong, *The Deferred Correction Method for 2nd-order FBSDEs with High Accuracy.*

The following is the main results of this thesis.

In Chapter 3, we use the tool of stochastic analysis, base on the adaptiveness of the solution, we change the solving of 2FBSDE to the reference ODE. Construct an new diffusion process X, based on this process, the computation is sharply reduced. By the tool of stochastic analysis and scientific computing, we construct the multistep scheme of 2FBSDE. In the last part of the chapter, we verify the scheme by kinds of examples, and get a k-order convergence scheme of 2FBSDDE

We first consider the decoupled 2FBSDE

$$\begin{cases} \mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \\ -\mathrm{d}Y_t = f(t, X_t, Y_t, Z_t, \Gamma_t)\mathrm{d}t - Z_t\mathrm{d}W_t, \\ \mathrm{d}Z_t = A_t\mathrm{d}t + \Gamma_t\mathrm{d}W_t \end{cases} \tag{0.3}$$

with a given terminal condition $Y_T = g(X_T)$.

we have this property

命题 0.2. *Let $u = u(t,x)$ be the solution of the following fully nonlinear PDE*

$$\mathcal{L}u + f(t,x,u,\nabla_x u\sigma, \nabla_x(\nabla_x u\sigma)\sigma) = 0 \tag{0.4}$$

with the terminal condition $u(T,x) = g(x)$, and let $(X_t, Y_t, Z_t, \Gamma_t, A_t)$ be the solution of the 2FBSDE (3.6). Then we have

$$\begin{aligned} Y_t = u(t,X_t), \quad & Z_t = (\nabla_x u\sigma)(t,X_t), \\ \Gamma_t = (\nabla_x(\nabla_x u\sigma)\sigma)(t,X_t), \quad & A_t = (\mathcal{L}(\nabla_x u\sigma))(t,X_t). \end{aligned}$$

where the associate operator $\mathcal{L}$ is defined by (2.6).

We construct the new diffusion process

$$\bar{X}_t^{t_n,x} = x + \int_{t_n}^t \bar{b}(s,\bar{X}_s^{t_n,x})\mathrm{d}s + \int_{t_n}^t \bar{\sigma}(s,\bar{X}_s^{t_n,x})\mathrm{d}W_s.$$

Based on which we propose the semi-discrete scheme

格式 0.5. *Assume that random variables Y^{N-i} and Z^{N-i}, $i = 0,1,\ldots,k-1$ are known. For $n = N-k,\ldots,0$, with $X_t^{t_n,X^n}$ being the solution of* (4.14), *solve $Y^n = Y^n(X^n)$, $Z^n = Z^n(X^n)$, $A^n = A^n(X^n)$ and $\Gamma^n = \Gamma^n(X^n)$ by*

$$\begin{aligned} Z^n &= \sum_{j=1} \alpha_{k,j}\mathbf{E}_{t_n}^{X^n}\left[\bar{Y}^{n+j}\Delta W_{n,j}^\top\right], \\ \Gamma^n &= \sum_{j=1} \alpha_{k,j}\mathbf{E}_{t_n}^{X^n}\left[(\bar{Z}^{n+j})^\top\Delta W_{n,j}^\top\right], \\ A^n &= \sum_{j=0} \alpha_{k,j}\mathbf{E}_{t_n}^{X^n}\left[\bar{Z}^{n+j}\right],, \\ -\alpha_{k,0}Y^n &= \sum_{j=1} \alpha_{k,j}\mathbf{E}_{t_n}^{X^n}\left[\bar{Y}^{n+j}\right] + f(t_n,X^n,Y^n,Z^n,\Gamma^n), \end{aligned}$$

where $\bar{Y}^{n+j}$ and $\bar{Z}^{n+j}$ are the values of Y^{n+j} and Z^{n+j} at the space points $X_{t_{n+j}}^{t_n,X^n}$.

We then propose the fully-discrete scheme for decoupled 2FBSDE

格式 0.6. *Assume random variables Y^{N-i} and Z^{N-i} defined on $\mathcal{D}_h^{N-i}$, $i = 0,1,\ldots,k-1$, are known. For $n = N-k,\ldots,0$, and for each $x \in \mathcal{D}_h^n$, solve X^n,*

Y^n, Z^n, A^n and Γ^n by

$$
\begin{aligned}
X^{n,j} &= x + b(t_n, x)\Delta t_{n,j} + \sigma(t_n, x)\Delta W_{n,j}, \quad j = 1, \dots, k, \\
Z^n &= \sum \alpha_{k,j} \widehat{\mathbb{E}}^{n,x} \left[\mathbb{I}^{n+j}_{\mathcal{D},\bar{X}^{n,j}} Y^{n+j} \Delta W^{\top}_{n,j} \right], \\
\Gamma^n &= \sum \alpha_{k,j} \widehat{\mathbb{E}}^{n,x} \left[\mathbb{I}^{n+j}_{\mathcal{D},\bar{X}^{n,j}} (Z^{n+j})^{\top} \Delta W^{\top}_{n,j} \right], \\
A^n &= \sum_{j=0} \alpha_{k,j} \widehat{\mathbb{E}}^{n,x} \left[\mathbb{I}^{n+j}_{\mathcal{D},\bar{X}^{n,j}} Z^{n+j} \right], \\
-\alpha_{k,0} Y^n &= \sum_{j=1} \alpha_{k,j} \widehat{\mathbb{E}}^{n,x} \left[\mathbb{I}^{n+j}_{\mathcal{D},\bar{X}^{n,j}} Y^{n+j} \right] + f(t_n, x, Y^n, Z^n, \Gamma^n).
\end{aligned}
$$

We extends the scheme to coupled 2FBSDE

$$
\begin{cases}
X_t = x + \int_0^t b(s, \Theta_s)\mathrm{d}s + \int_0^t \sigma(s, \Theta_s)\mathrm{d}W_s, \\
Y_t = g(X_T) + \int_t^T f(s, \Theta_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \qquad s \in [0, T]. \\
Z_s = Z_0 + \int_0^t A_s \mathrm{d}s + \int_0^t \Gamma_s \mathrm{d}W_s,
\end{cases}
$$

and then propose the scheme

格式 0.7. *Assume random variables Y^{N-i}, Z^{N-i} and Γ^{N-i} defined on $\mathcal{D}_h^{N-i}$, $i = 0, 1, \dots, k-1$, are known. l is the iterative time, $Y^{n,l}$, $Z^{n,l}$, $A^{n,l}$ and $\Gamma^{n,l}$ are the corresponding values at the l-th iterative. For $n = N-k, \dots, 0$, and each $x \in \mathcal{D}_h^n$, solve Y^n, Z^n, A^n and Γ^n by*

1. *Initialize $Y^{n,0}$, $Z^{n,0}$ and $\Gamma^{n,0}$ by Y^{n+1}, Z^{n+1} and Γ^{n+1}, separately, i.e. $Y^{n,0} = Y^{n+1}(x)$, $Z^{n,0} = Z^{n+1}(x)$ and $\Gamma^{n,0} = \Gamma^{n+1}(x)$;*

2. *Calculate* $Y^{n,l+1}$, $Z^{n,l+1}$, $A^{n,l+1}$ *and* $\Gamma^{n,l+1}$ *for* $l=0,1,\ldots$ *by*

$$
\begin{cases}
X^{n,j} = x + b(t_n, x, Y^{n,l}, Z^{n,l}, \Gamma^{n,l})\Delta t_{n,j} + \sigma(t_n, x, Y^{n,l}, Z^{n,l}, \Gamma^{n,l})\Delta W_{n,j}, \\
\qquad j = 1, 2, \ldots, k, \\
Z^{n,l+1} = \sum_{j=1}^{k} \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}^{n+j}_{\mathcal{D},\tilde{X}^{n,j}} Y^{n+j} \Delta W^{\top}_{n,j}\right], \\
\Gamma^{n,l+1} = \sum_{j=1}^{k} \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}^{n+j}_{\mathcal{D},\tilde{X}^{n,j}} (Z^{n+j})^{\top} \Delta W^{\top}_{n,j}\right], \\
A^{n,l+1} = \sum_{j=0}^{k} \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}^{n+j}_{\mathcal{D},\tilde{X}^{n,j}} Z^{n+j}\right], \\
-\alpha_{k,0} Y^{n,l+1} = \sum_{j=1}^{k} \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}^{n+j}_{\mathcal{D},\tilde{X}^{n,j}} Y^{n+j}\right] + f(t_n, x, Y^{n,l+1}, Z^{n,l+1}, \Gamma^{n,l+1}).
\end{cases}
$$

Repeat the above, until

$$
\max\left\{|Y^{n,l+1} - Y^{n,l}|, |Z^{n,l+1} - Z^{n,l}|, |A^{n,l+1} - A^{n,l}|, |\Gamma^{n,l+1} - \Gamma^{n,l}|\right\} < \epsilon_0;
$$

3. 设定 $(Y^n, Z^n, \Gamma^n, A^n)$ 的值为 $(Y^{n,l+1}, Z^{n,l+1}, A^{n,l+1}, \Gamma^{n,l+1})$.

In Chapter 4, Base on the relationship of 2PDE and 2FBSDE, we construct the stochastic scheme for 2PDE. As far as I know, this is the first time for solving 2PDE from an stochastic view. We apply the scheme to many kinds of examples and that the k-step scheme is k-order scheme for our proposed scheme.

We consider the nonlinear parabolic PDEs in the following form:

$$
\begin{cases}
u_t + F(t, x, u, Du, D^2u) = 0, & (t, x) \in [0, T) \times \mathbb{R}^m, \\
u(T, x) = g(x), & x \in \mathbb{R}^m,
\end{cases}
\tag{0.5}
$$

where $u(\cdot,\cdot)$ is a map from $[0,T]\times\mathbb{R}^m \to \mathbb{R}$; $Du(x)$ and $D^2u(x)$ stand for the gradient and the Hessian matrix of u with respect to x, respectively.

We propose our fully discret scheme as follows

格式 0.8. *Given* $\{u^n(x)$ *and* $Z^n(x) = \bar{\sigma}(t_n, x)\nabla u^n(x)$ *for* $n = N, \ldots, N-k+1$ *and* $x \in \cup_{n=N-k+1}^{N} \mathcal{D}^n_{h_n}$, *for* $n = N-k, \ldots, 0$,*and for* $x \in \mathcal{D}^n_{h_n}$, *solve* $u^n = u^n(x)$ *by*

$$
u^n = Y^n, \tag{0.6}
$$

where Y^n is solved by the following procedure

$$X^{n,i} = x + b(t_n, x)\Delta t_{n,i} + \sigma(t_n, x)\Delta W_{n,i}, \quad i = 1, \ldots, k,$$

$$Z^n = \sum \alpha_{k,i} \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}^{n+i}_{\mathcal{D},\bar{X}^{n,i}} Y^{n+i} \Delta W^{\top}_{n,i}\right],$$

$$\Gamma^n = \sum \alpha_{k,i} \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}^{n+i}_{\mathcal{D},\bar{X}^{n,i}} Z^{n+i} \Delta W^{\top}_{n,i}\right],$$

$$-\alpha_{k,0} Y^n = \sum \alpha_{k,i} \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}^{n+i}_{\mathcal{D},\bar{X}^{n,i}} Y^{n+i}\right] + \hat{f}(t_n, x, Y^n_j, Z^n_j, \Gamma^n_j)$$

In Chapter 5, We research on a new high accuracy approach for solving the second order stochastic differential equations. The approach, which is called stochastic deferred correction method can raise the convergence order from the low order of the initial scheme to a higher one. What's more, even the discrete scheme is of a low convergence order, the deferred correction method will strongly accelerate the convergence speed, that's the amazing point for DC method.

First for the decoupled case, we propose the schemes

算法 0.4 (Deferred Correction). *Assume only the terminal value Y_T, Z_T are known. Solve Y^n, Z^n, Γ^n and A^n, region by region under the partition $\mathcal{T}$. For each time region $[\tau_i, \tau_{i+1}]$ with inner partition $\mathcal{T}_i$, do*

1. ***(First evaluation.)*** *Obtain $Y^{n[0]}, Z^{n[0]}, \Gamma^{n[0]}$ and $A^{n[0]}$ at $X^n = x_j$ for all $x_j \in \mathcal{D}^n$ and $n = K-1, \ldots, 0$ in time region $[\tau_i, \tau_{i+1}]$ by* (5.32) *with $Y^{K[0]}$ and $Z^{K[0]}$ set by the values Y^0 and Z^0 in the time region $[\tau_{i+1}, \tau_{i+2}]$. For the last time region, it's just Y_T and Z_T.*

2. ***(Deferred correction.)*** *Correct the numerical values iteratively N_c times by $1 \le l \le N_c$.*

   - *Error processes. Obtain $\delta Y^{n[l]}, \delta Z^{n[l]}, \delta\Gamma^{n[l]}, \delta A^{n[l]}$ for $n = K-1, \ldots, 0$ by* (5.33) *with $\delta Y^{K[l]} = 0, \delta Z^{K[l]} = 0$.*

- *Update the new iterate numerical values* $Y^{n[l]}, Z^{n[l]}, \Gamma^{n[l]}$ *and* $A^{n[l]}$ *by the error processes*

$$\begin{cases} Y^{n[l]} = Y^{n[l-1]} + \delta Y^{n[l]}, & Z^{n[l]} = Z^{n[l-1]} + \delta Z^{n[l]}, \\ \Gamma^{n[l]} = \Gamma^{n[l-1]} + \delta \Gamma^{n[l]}, & A^{n[l]} = A^{n[l-1]} + \delta A^{n[l]}. \end{cases}$$

3. *Finally, set* $Y^n = Y^{n[N_c]}$, $Z^n = Z^{n[N_c]}$, $\Gamma^n = \Gamma^{n[N_c]}$ *and* $A^n = A^{n[N_c]}$.

Then we propose the fully-discrete scheme by

算法 0.5 (Fully-discrete Deferred correction). *Given the time partition* $\mathcal{T}$ (5.2) *of* $[0, T]$. *Assume only the terminal value* Y_T, Z_T *are known. For each time region* $[\tau_i, \tau_{i+1}]$ *with inner partition* $\mathcal{T}_i$ (5.3), *solve* Y^n, Z^n, Γ^n *and* A^n *region by region in time region* $[\tau_i, \tau_{i+1}]$

1. ***(First evaluation.)*** *Obtain* Y^n, Z^n, Γ^n *and* A^n *at* $X^n = x_j$ *for all* $x_j \in \mathcal{D}^n$ *and* $n = K-1, \ldots, 0$ *by*

$$\begin{cases} X^{n,1} = x_j + b(t_n, x_j)\Delta t_n + \sigma(t_n, x_j)\Delta W_n \\ Z_j^n = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} Y^{n+1} \Delta W_n^\top\right]}{\Delta t_n}, \\ \Gamma_j^n = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} Z^{n+1} \Delta W_n^\top\right]}{\Delta t_n}, \\ A_j^n = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} Z^{n+1}\right] - Z^n}{\Delta t_n}, \\ Y_j^n = \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} Y^{n+1}\right] + \Delta t_n f(t_n, x, Y^{n+1}, Z^n, \Gamma^n), \end{cases}$$

with Y^K *and* Z^K *set by the values* Y^0 *and* Z^0 *in the time region* $[\tau_{i+1}, \tau_{i+2}]$. *For the last time region, it's just* Y_T *and* Z_T. *The* Y^n, Z^n, Γ^n *and* A^n *are set as* $Y^{n[0]}, Z^{n[0]}, \Gamma^{n[0]}$ *and* $A^{n[0]}$.

2. ***(Deferred correction.)*** *For the* l *correction,* $1 \le l \le N_c$, *correct the numerical values iteratively by*

- *Error processes. Obtain* $\delta Y^{n[l]}, \delta Z^{n[l]}, \delta \Gamma^{n[l]}, \delta A^{n[l]}$ *for* $n = K-1, \ldots, 0$

by

$$
\begin{cases}
X^{n,1} = x_j + b(t_n, x_j)\Delta t_n + \sigma(t_n, x_j)\Delta W_n \\
\delta Z_j^{n[l]} = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}}\delta Y^{n+1[l]}\Delta W_n^{\top}\right]}{\Delta t_n} \\
\qquad\qquad - Z^{n[l-1]} + \mathcal{L}^1 I^Y(t_n, x_j, \Theta_j^{n+1[l-1]}), \\
\delta \Gamma_j^{n[l]} = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}}\delta Z^{n+1[l]}\Delta W_n^{\top}\right]}{\Delta t_n} \\
\qquad\qquad - \Gamma^{n[l-1]} + \mathcal{L}^1 I^Z(t_n, x_j, \Theta_j^{n+1[l-1]}) \\
\delta A_j^{n[l]} = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}}\delta Z^{n+1[l]}\right] - \delta Z^{n[l]}}{\Delta t_n} \\
\qquad\qquad - A^{n[l-1]} + \mathcal{L}^0 I^Z(t_n, x_j, \Theta_j^{n+1[l-1]}), \\
\delta Y_j^{n[l]} = \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}}\delta Y^{n+1[l]}\right] + \Delta t_n\big(\mathcal{L}^0 I^Y(t_n, x_j, \Theta_j^{n+1[l-1]}) \\
\qquad\qquad + f(t_n, x, Y^{n[l-1]} + \delta Y^{n[l]}, Z^{n[l-1]} + \delta Z^{n[l]}, \Gamma^{n[l-1]} + \delta\Gamma^{n[l]})\big)
\end{cases}
$$

with $\delta Y^{K[l]} = 0, \delta Z^{K[l]} = 0.$

- *Update the new iterate numerical values*$Y^{n[l]}, Z^{n[l]}, \Gamma^{n[l]}$ 和 $A^{n[l]}$ *by the error processes*

$$
\begin{cases}
Y_j^{n[l]} = Y_j^{n[l-1]} + \delta Y_j^{n[l]}, & Z_j^{n[l]} = Z_j^{n[l-1]} + \delta Z_j^{n[l]}, \\
\Gamma_j^{n[l]} = \Gamma_j^{n[l-1]} + \delta \Gamma_j^{n[l]}, & A_j^{n[l]} = A_j^{n[l-1]} + \delta A_j^{n[l]}.
\end{cases}
$$

3. *Finally, set*$Y^n = Y^{n[N_c]}$, $Z^n = Z^{n[N_c]}$, $\Gamma^n = \Gamma^{n[N_c]}$ *and* $A^n = A^{n[N_c]}$.

Finally, we raise schemes for coupled FBSDEs.

算法 0.6 (Coupled ully-discrete Deferred correction). *Given the time partition* $\mathcal{T}$ (5.2) *of* $[0, T]$. *Assume only the terminal value* Y_T, Z_T *are known. For each time region* $[\tau_i, \tau_{i+1}]$ *with inner partition* $\mathcal{T}_i$ (5.3), *solve* Y^n, Z^n, Γ^n *and* A^n *region by region in time region* $[\tau_i, \tau_{i+1}]$

1. ***(First evaluation.)*** *Obtain* Y^n, Z^n, Γ^n *and* A^n *at* $X^n = x_j$ *for all* $x_j \in \mathcal{D}^n$

and $n = K-1, \ldots, 0$ *by*

$$\begin{cases} X^{n,1} = x_j + b(t_n, x_j, \Theta_j^{n+1})\Delta t_n + \sigma(t_n, x_j, \Theta_j^{n+1})\Delta W_n \\ Z_j^n = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} Y^{n+1} \Delta W_n^\top\right]}{\Delta t_n}, \\ \Gamma_j^n = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} Z^{n+1} \Delta W_n^\top\right]}{\Delta t_n}, \\ A_j^n = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} Z^{n+1}\right] - Z^n}{\Delta t_n}, \\ Y_j^n = \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} Y^{n+1}\right] + \Delta t_n f(t_n, x, Y^{n+1}, Z^n, \Gamma^n), \end{cases}$$

with Y^K *and* Z^K *set by the values* Y^0 *and* Z^0 *in the time region* $[\tau_{i+1}, \tau_{i+2}]$. *For the last time region, it's just* Y_T *and* Z_T. *The* Y^n, Z^n, Γ^n *and* A^n *are set as* $Y^{n[0]}, Z^{n[0]}, \Gamma^{n[0]}$ *and* $A^{n[0]}$.

2. ***(Deferred correction.)*** *For the* l *correction,* $1 \leq l \leq N_c$, *correct the numerical values iteratively by*

   - *Error processes. Obtain* $\delta Y^{n[l]}, \delta Z^{n[l]}, \delta\Gamma^{n[l]}, \delta A^{n[l]}$ *for* $n = K-1, \ldots, 0$ *by*

$$\begin{cases} X^{n,1} = x_j + b(t_n, x_j, \Theta_j^{n+1[l-1]})\Delta t_n + \sigma(t_n, x_j, \Theta_j^{n+1[l-1]})\Delta W_n \\ \delta Z_j^{n[l]} = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} \delta Y^{n+1[l]} \Delta W_n^\top\right]}{\Delta t_n} \\ \qquad - Z^{n[l-1]} + \mathcal{L}^1 I^Y(t_n, x_j, \Theta_j^{n+1[l-1]}), \\ \delta\Gamma_j^{n[l]} = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} \delta Z^{n+1[l]} \Delta W_n^\top\right]}{\Delta t_n} \\ \qquad - \Gamma^{n[l-1]} + \mathcal{L}^1 I^Z(t_n, x_j, \Theta_j^{n+1[l-1]}) \\ \delta A_j^{n[l]} = \dfrac{\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} \delta Z^{n+1[l]}\right] - \delta Z^{n[l]}}{\Delta t_n} \\ \qquad - A^{n[l-1]} + \mathcal{L}^0 I^Z(t_n, x_j, \Theta_j^{n+1[l-1]}), \\ \delta Y_j^{n[l]} = \widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}_{\mathcal{D},X^{n,1}} \delta Y^{n+1[l]}\right] + \Delta t_n \big(\mathcal{L}^0 I^Y(t_n, x_j, \Theta_j^{n+1[l-1]}) \\ \qquad + f(t_n, x, Y^{n[l-1]} + \delta Y^{n[l]}, Z^{n[l-1]} + \delta Z^{n[l]}, \Gamma^{n[l-1]} + \delta\Gamma^{n[l]})\big) \end{cases}$$

   with $\delta Y^{K[l]} = 0, \delta Z^{K[l]} = 0$.

   - *Update the new iterate numerical values*$Y^{n[l]}, Z^{n[l]}, \Gamma^{n[l]}$ 和 $A^{n[l]}$ *by the* *error processes*

$$\begin{cases} Y_j^{n[l]} = Y_j^{n[l-1]} + \delta Y_j^{n[l]}, & Z_j^{n[l]} = Z_j^{n[l-1]} + \delta Z_j^{n[l]}, \\ \Gamma_j^{n[l]} = \Gamma_j^{n[l-1]} + \delta \Gamma_j^{n[l]}, & A_j^{n[l]} = A_j^{n[l-1]} + \delta A_j^{n[l]}. \end{cases}$$

3. Finally, set $Y^n = Y^{n[N_c]}$, $Z^n = Z^{n[N_c]}$, $\Gamma^n = \Gamma^{n[N_c]}$ *and* $A^n = A^{n[N_c]}$.

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