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A NEW KIND OF ACCURATE NUMERICAL METHOD FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract. In this paper, we propose a new kind of numerical simulation method for backward stochastic differential equations (BSDEs). We discretize the continuous BSDEs on time-space discrete grids, use the Monte Carlo method to approximate mathematical expectations, and use space interpolations to compute values at nongrid points. To demonstrate the accuracy and the effectiveness of our method, several numerical examples are given.

Key words. backward stochastic differential equations, Monte Carlo method, time-space discretization

AMS subject classifications. 60H35, 65C05

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1. Introduction.

In this paper, we consider the numerical solutions for the backward stochastic differential equations (BSDEs)

$$\begin{aligned} -dy_t &= f(t, y_t, z_t)dt - z_t \mathrm{d}W_t, \quad t \in [0, T), \\ y_T &= \xi, \end{aligned} \tag{1.1}$$

where $W_t = (W_t^1, \ldots, W_t^d)^*$ is a d-dimensional Brownian motion defined on some complete, filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0 \le t \le T})$, and $\xi \in \mathcal{F}_T$, and the notation * is the transpose operator for matrix or vector.

In 1990, Pardoux and Peng in [26] obtained the unique solvability results for the nonlinear BSDEs under some standard conditions. Since then, the BSDEs have been extensively studied [28, 29, 30, 18, 1, 11, 33, 24, 25] by many researchers. In [18], the properties of the solutions of the BSDEs and their important applications in mathematical finance were discussed. Some fine properties of the BSDEs by using Malliavin derivatives [14, 15] under weaker conditions were studied in the Ph.D. thesis [33] of Zhang. The applications of the BSDEs now cover many scientific fields, such as stochastic control, turbulence fluid flow, biology, stock markets, radiation, chemical reactions, and PDEs. However, it is often difficult to solve the BSDEs analytically, even the linear BSDEs.

Many efforts have been made in the numerical methods for BSDEs [2, 3, 4, 6, 9, 12, 22, 23, 24, 25, 30, 33, 34, 35]. By using the relations between the BSDEs and the PDEs [28, 17], a four step scheme for solving the BSDEs was developed in [23, 24]. Based on the four step scheme, a characteristic difference method was proposed to solve forward-backward SDEs (FBSDEs) in [12]. These two methods depend on the numerical schemes for PDEs and for the related ordinary SDEs. In [3], Bally and Pagès studied an optimal discrete quantization method designed to compute the solution of a certain kind of reflected BSDE, and the error analysis was given in [4]. In his Ph.D. thesis [9], Chevance proposed a numerical method for BSDEs by using binomial approach to approximate the process y_t. Peng proposed an iterative

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linearization procedure to solve BSDEs in [30]. In [6], the time discrete approximation and the Monte Carlo method for solving BSDEs were studied by Bounchard and Touzi. In [34], Zhang studied a numerical approximation to the solution of the BSDEs under some weaker conditions.

Compared with the development of the numerical methods for the ordinary stochastic differential equations (SDEs) [5, 19, 20, 21, 32], the numerical methods for solving the solution of the BSDEs are far undeveloped, and effective and efficient methods are urgently needed.

In this paper, we study the numerical methods for the BSDEs (1.1). Based on the properties of the BSDEs, we propose a new accurate numerical method for the BSDEs. We fully discretize the time-space continuous problems and use the Monte Carlo method to approximate the mathematical expectations in our method. The space interpolation is used when the values at nongrid points are needed. The unknowns on each time level are totally uncoupled among the space grids, and the used data can be local for each space grid. Thus our method can be used to solve the general BSDEs in high performance parallel computers. To demonstrate the efficiency of our method, we use our method to solve some model BSDEs, and the results show that our method is efficient and is of high order.

We organize this paper as follows. In section 2, some basic notation and definitions are given. In section 3, we outline our solution procedure for one-dimensional BSDEs. Then, we generalize it and propose the numerical scheme for the general BSDEs (1.1) in section 4. Numerical experiments for some BSDE models are given in section 5. Finally, we draw some conclusions in section 6.

2. Notation and definitions. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0\le t\le T})$ be a complete, filtered probability space on which a standard d-dimensional Brownian motion $W_t (0 \le t \le T)$ is defined with a finite terminal time T. Let the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ be the σ-field generated by the Brownian motion W_t, i.e., $\mathcal{F}_t = \sigma\{W_s, 0 \le s \le t\}$. All the P-null sets are augmented to each σ-field $\mathcal{F}_t$. Denote $L^2 = L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$ to be the set of all $\mathcal{F}_t$-adapted and mean-square-integrable processes valued in $\mathbb{R}^d$.

A process $(y_t, z_t) : [0, T] \times \Omega \to \mathbb{R}^m \times \mathbb{R}^{m\times d}$ is called an L^2-adapted solution of the BSDEs (1.1) if it is $\{\mathcal{F}_t\}$-adapted and square integrable and satisfies (1.1) in the sense of

$$y_t = \xi + \int_t^T f(s, y_s, z_s)ds - \int_t^T z_s \mathrm{d}W_s, \quad t \in [0, T), \tag{2.1}$$

where $f(t, y, z) : [0, T] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$, and $f(t, y, z)$ is an adapted stochastic process with respect to $\{\mathcal{F}_t\}(0 \le t \le T)$ for each (y, z).

Under some conditions for $f(t, y, z)$, Pardoux and Peng in [26] obtained the uniqueness solvability results for the BSDE (2.1). Properties and applications of BSDEs can be found in [18]. In our paper, we are interested in the numerical solution of the BSDE (2.1). We assume that the BSDE (2.1) admits a unique L^2-adapted solution (y_t, z_t).

Let $\mathcal{F}^{t,x}_s (t \le s \le T)$ be a σ-field generated by the Brownian motion $\{x + W_v - W_t, t \le v \le s\}$ starting from the time-space point (t, x), and let $\mathcal{F}^{t,x} = \mathcal{F}^{t,x}_T$. We use $\mathbb{E}[X]$ to denote the mathematical expectation of the random variable X and $\mathbb{E}^{t,x}_s[X]$ to denote the conditional mathematical expectation of the random variable X under the σ-field $\mathcal{F}^{t,x}_s (t \le s \le T)$; that is, $\mathbb{E}^{t,x}_s[X] = \mathbb{E}[X|\mathcal{F}^{t,x}_s]$. If $X = (X_{ij})_{m\times n}$ is a matrix or vector stochastic variable, we define the conditional mathematical expectation $\mathbb{E}^{t,x}_s[X]$ by $\mathbb{E}^{t,x}_s[X] = (\mathbb{E}^{t,x}_s[X_{ij}])_{m\times n}$.

3. Development of the numerical method. For simplicity, in this section, we start with the one-dimensional BSDE to develop our numerical method for the BSDE (2.1)

For the real axis $\mathbb{R}$, we introduce a space partition $\mathfrak{R}_h$ as

$$\mathfrak{R}_h = \{x_i | x_i \in \mathbb{R}, i \in \mathbb{Z}, x_i < x_{i+1}, \lim_{i\to+\infty} = +\infty, \lim_{i\to-\infty} = -\infty\}, \tag{3.1}$$

where $\mathbb{Z}$ is the set of all integer numbers, and all x_i's are deterministic. We use $h_i = x_{i+1} - x_i$ to denote the space step and $h = \max_{i\in\mathbb{Z}} h_i$ to denote the maximum space step.

For the time interval $[0, T]$, we introduce the time partition

$$\mathfrak{R}_{th} = \{t_i | t_i \in [0, T], i = 0, 1, \ldots, N_T, t_i < t_{i+1}, t_0 = 0, t_{N_T} = T\}, \tag{3.2}$$

where all t_n's are deterministic. Let $\Delta t_n = t_{n+1} - t_n$ be the time step, and let $\Delta t = \max_{0\le n<N_T} \Delta t_n$ be the maximum time step.

3.1. Reference equations. Let (y_t, z_t) be the adapted solution of the BSDE (2.1); then we have

$$y_t = \quad y_{t+\delta} + \textstyle\int_t^{t+\delta} f(s, y_s, z_s)ds - \int_t^{t+\delta} z_s dW_s, \quad t \in [0, T), \tag{3.3}$$

where δ is a deterministic nonnegative number with $t+\delta \le T$. Taking the conditional mathematical expectation $\mathbb{E}_t^{t,x}$ on the two sides of (3.3), we deduce

$$y_t^{t,x} = \quad \mathbb{E}_t^{t,x}[y_{t+\delta}] + \int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s, y_s, z_s)]ds, \tag{3.4}$$

where $y_t^{t,x} = \mathbb{E}[y_t|\mathcal{F}_t^{t,x}]$; that is, $y_t^{t,x}$ is the value of y_t at the time-space point (t, x). With respect to the filtration $\mathcal{F}_t$, the integrand on the right-hand side of (3.4) is deterministic. We write the integral term in (3.4) in the form

$$\begin{aligned}\int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s, y_s, z_s)]ds = \quad & \delta\{(1-\theta_1)\mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})] \\ & +\theta_1 f(t, y_t^{t,x}, z_t^{t,x})\} + R_y,\end{aligned} \tag{3.5}$$

where R_y is defined by

$$R_y = \quad \int_t^{t+\delta} \{\mathbb{E}_t^{t,x}[f(s, y_s, z_s)] - (1-\theta_1)\mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})] - \theta_1 f(t, y_t^{t,x}, z_t^{t,x})\}ds$$

with the parameter $\theta_1 \in [0, 1]$. Inserting (3.5) into (3.4) leads to

$$\begin{aligned}y_t^{t,x} = \quad & \mathbb{E}_t^{t,x}[y_{t+\delta}] + \delta\{(1-\theta_1)\mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})] \\ & +\theta_1 f(t, y_t^{t,x}, z_t^{t,x})\} + R_y.\end{aligned} \tag{3.6}$$

Let $\Delta W_s = W_s - W_t$ for $t \le s \le t+\delta$; then ΔW_s is a standard Brownian motion with mean zero and variance $\sqrt{s-t}$. Now multiplying the two sides of (3.3) by $\Delta W_{t+\delta}$, and taking the mathematical expectation $\mathbb{E}_t^{t,x}$ to the two sides of the derived equation, by using Itô isometry formula we obtain

$$-\mathbb{E}_t^{t,x}[y_{t+\delta}\Delta W_{t+\delta}] = \quad \int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s, y_s, z_s)\Delta W_s]ds - \int_t^{t+\delta} \mathbb{E}_t^{t,x}[z_s]ds. \tag{3.7}$$

The two integral terms on the right-hand side of (3.7) are two standard integrals with respect to time under the filtration $\mathcal{F}_t$. We decompose the two terms on the right-hand side of (3.7) by

$$\int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s, y_s, z_s)\Delta W_s]ds$$
$$= \delta(1-\theta_2)\mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta w_{t+\delta}] + R_{z1}, \tag{3.8}$$

$$-\int_t^{t+\delta} \mathbb{E}_t^{t,x}[z_s]ds = -\delta\left\{(1-\theta_2)\mathbb{E}_t^{t,x}[z_{t+\delta})] + \theta_2 z_t^{t,x}\right\} + R_{z2}, \tag{3.9}$$

where $z_t^{t,x}$ is the value of z_t at the time-space point (t,x) under the filtration $\mathcal{F}_t$, and

$$\begin{aligned} R_{z1} = & \int_t^{t+\delta} \{\mathbb{E}_t^{t,x}[f(s, y_s, z_s)\Delta W_s] - (1-\theta_2)\mathbb{E}[f(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta W_{t+\delta}]\}ds, \\ R_{z2} = & -\int_t^{t+\delta} \{\mathbb{E}_t^{t,x}[z_s] - (1-\theta_2)\mathbb{E}_t^{t,x}[z_{t+\delta}] - \theta_2 z_t^{t,x}\}\}ds \end{aligned}$$

with $\theta_2 \in [0,1]$. Combining the (3.7), (3.8), and (3.9), we obtain

$$\begin{aligned} -\mathbb{E}_t^{t,x}[y_{t+\delta}\Delta W_{t+\delta}] = & \ \delta(1-\theta_2)\mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta w_{t+\delta}] \\ & -\delta\left\{(1-\theta_2)\mathbb{E}_t^{t,x}[z_{t+\delta})] + \theta_2 z_t^{t,x}\right\} + R_z \end{aligned} \tag{3.10}$$

with $R_z = R_{z1} + R_{z2}$.

The equations (3.6) and (3.10) are the two fundamental reference equations obtained from (2.1). They are held for the deterministic time variable t and the deterministic parameters δ, θ_1, and θ_2. Based on these two reference equations, we then propose our numerical method in the following sections.

3.2. Discrete scheme.

3.2.1. Semidiscrete $\boldsymbol{\theta}$-scheme. In the reference equations (3.6) and (3.10), let $t = t_n$, $\delta = \Delta t_n$. Take $(y^n, z^n)(n = N_T, N_T - 1, \ldots, 0)$ as the approximations to the analytic solution (y_t, z_t) of the BSDE (2.1) at time levels $t_n(n = N_T, N_T - 1, \ldots, 0)$. Let $(y^n, z^n)(n = N_T, N_T - 1, \ldots, 0)$ satisfy (3.6) and (3.10) without terms R_y and R_z; that is, (y^n, z^n) satisfies

$$\begin{aligned} y^n = & \ \mathbb{E}_{t_n}^{t_n,x}[y^{n+1}] + \Delta t_n[(1-\theta_1^n)\mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y^{n+1}, z^{n+1})] \\ & +\theta_1^n f(t_n, y^n, z^n)], \end{aligned} \tag{3.11}$$

$$\begin{aligned} 0 = & \ \mathbb{E}_{t_n}^{t_n,x}[y^{n+1}\Delta W_{t_{n+1}}] + \Delta t_n(1-\theta_2^n)\mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}] \\ & -\Delta t_n\{(1-\theta_2^n)\mathbb{E}_{t_n}^{t_n,x}[z^{n+1}] + \theta_2^n z^n\} \end{aligned} \tag{3.12}$$

for $n = N_T - 1, N_T - 2, \ldots, 2, 1, 0$ with the parameters θ_1^n and θ_2^n in $[0,1]$.

The equations (3.11) and (3.12) are a semidiscrete θ-scheme for the BSDE (2.1). The semidiscrete θ-scheme is nonlinear in the random variables y^n and z^n. So for the general nonlinear BSDE problems, some methods for finding (y^n, z^n) are needed. In this paper, we use Newton's method to solve (y^n, z^n). The procedure to solve (y^n, z^n) is as follows:

1. Let $y^{N_T} = \xi$.
2. If $n = N_T - 1$, let $\theta_2^n = 1$, and use (3.11) and (3.12) to solve (y^n, z^n).
3. If $0 \le n < N_T - 1$, use (3.11) and (3.12) to solve (y^n, z^n).

Remark 1. By choosing θ_1^n and θ_2^n in (3.11) and (3.12), we can obtain different methods. When we choose $\theta_1^n = \theta_2^n = 0$, we get the explicit Euler scheme. When $\theta_1^n = \theta_2^n \neq 0$, the θ-scheme is implicit. The implicit Euler scheme is the case $\theta_1^n = \theta_2^n = 1$, and the implicit Crank–Nicolson (CN) scheme is the case $\theta_1^n = \theta_2^n = \frac{1}{2}$. Our implicit θ-scheme for the BSDE (2.1) is different from the implicit schemes for solving ODEs and PDEs in that our θ-scheme is always explicit for solving z^n.

3.2.2. Fully discrete $\boldsymbol{\theta}$-scheme. To simulate (y^n, z^n) based on the semidiscrete θ-scheme (3.11) and (3.12), two more numerical approximations must be done; these are the approximation of the conditional mathematical expectation and the space approximation of (y^n, z^n). To do these, we take $x = x_i \in \mathfrak{R}_h$ in (3.11) and (3.12), and let $\hat{x}_i = x_i + \Delta W_{t_{n+1}}$. Then there is an integer $\hat{i} \in \mathbb{Z}$ such that $\hat{x}_i \in [x_{\hat{i}}, x_{\hat{i}+1}]$. Let $I_h u(\hat{x}_i)$ be an interpolation approximation of the function $u(x)$ at the space point $\hat{x}_i$ by using the values of $u(x)$ at a finite number of the space points x_j's in $\mathfrak{R}_h$ near the space point $\hat{x}_i$. The more accurate the simulations are required to be, the more points in $\mathfrak{R}_h$ near $\hat{x}_i$ need be used. For some complicated problems, some special interpolations may be taken.

For each $x = x_i \in \mathfrak{R}_h$ in (3.11) and (3.12), let $\Delta_h^k W_{t_{n+1}} (k = 1, 2, \ldots, N_E)$ be the N_E times realizations of $\sqrt{\Delta t_n} N(0,1)$, where $N(0,1)$ is a standard normal distribution with mean zero and variance 1, and let $\hat{x}_i^k = x_i + \Delta_h^k W_{t_{n+1}}$.

To approximate the conditional mathematical expectations, we use the Monte Carlo method. We write the first term on the right-hand side of (3.11) in the following form:

$$\mathbb{E}_{t_n}^{t_n,x_i}[y^{n+1}] = \mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}] + Err_y, \tag{3.13}$$

where

$$\mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}] = \frac{\sum_{k=1}^{N_E} I_h y^{n+1}(\hat{x}_i^k)}{N_E}, \quad Err_y = \mathbb{E}_{t_n}^{t_n,x_i}[y^{n+1}] - \frac{\sum_{k=1}^{N_E} I_h y^{n+1}(\hat{x}_i^k)}{N_E}.$$

The first term on the right-hand side of (3.13) is the Monte Carlo approximation to the mathematical expectation $\mathbb{E}_{t_n}^{t_n,x_i}[y^{n+1}]$, and the second term is the Monte Carlo approximation error.

Similarly we write the second term on the right-hand side of (3.11) as

$$\mathbb{E}_{t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})] = \mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})] + Err_f \tag{3.14}$$

with the Monte Carlo approximation

$$\mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})] = \frac{\sum_{k=1}^{N_E} f(t_{n+1}, I_h y^{n+1}(\hat{x}_i^k), I_h z^{n+1}(\hat{x}_i^k))}{N_E}$$

and the corresponding approximation error

$$Err_f = \mathbb{E}_{t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})] - \mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})].$$

Combining (3.11), (3.13), and (3.14), we obtain

$$\begin{aligned} y_i^n = \; & \mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}] + \Delta t_n[(1-\theta_1^n)\mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})] \\ & + \theta_1^n f(t_n, y_i^n, z_i^n)] + \Delta t_n(1-\theta_1^n) Err_f + Err_y, \end{aligned} \tag{3.15}$$

where (y_i^x, z_i^n) is the value of (y^n, z^n) at the space point x_i with respect to the filtration $\mathcal{F}_{t_n}$, and $\Delta t_n(1-\theta_1^n)Err_f + Err_y$ is the Monte Carlo approximation error.

Now, let us turn to approximate the three conditional mathematical expectations in (3.12). We write the first term on the right-hand side of (3.12) as

$$\mathbb{E}_{t_n}^{t_n,x_i}[y^{n+1}\Delta W_{t_{n+1}}] = \mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}\Delta W_{t_{n+1}}] + Err_{yw}, \tag{3.16}$$

where

$$\begin{aligned}\mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}\Delta W_{t_{n+1}}] &= \frac{\sum_{k=1}^{N_E} I_h y^{n+1}(\hat{x}_i^k)\Delta_h^k W_{t_{n+1}}}{N_E},\\ Err_{yw} &= \mathbb{E}_{t_n}^{t_n,x_i}[y^{n+1}\Delta W_{t_{n+1}}] - \mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}\Delta W_{t_{n+1}}],\end{aligned}$$

we write the second term on the right-hand side of (3.12) as

$$\mathbb{E}_{t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}] = \mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}] + Err_{fw}, \tag{3.17}$$

where

$$\begin{aligned}\mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}] = &\ \frac{\sum_{k=1}^{N_E} f(t_{n+1}, I_h y^{n+1}(\hat{x}_i^k), I_h z^{n+1}(\hat{x}_i^k))\Delta_h^k W_{t_{n+1}}}{N_E},\\ Err_{fw} = &\ \mathbb{E}_{t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}]\\ &-\mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}],\end{aligned}$$

and we write the third term on the right-hand side of (3.12) as

$$\mathbb{E}_{t_n}^{t_n,x_i}[z^{n+1}] = \mathbb{E}_{h,t_n}^{t_n,x_i}[z^{n+1}] + Err_z, \tag{3.18}$$

where

$$\mathbb{E}_{h,t_n}^{t_n,x_i}[z^{n+1}] = \frac{\sum_{k=1}^{N_E} I_h z^{n+1}(\hat{x}_i^k)}{N_E}, \quad Err_z = \mathbb{E}_{t_n}^{t_n,x_i}[z^{n+1}] - \mathbb{E}_{h,t_n}^{t_n,x_i}[z^{n+1}].$$

Now combining (3.12), (3.16), (3.17), and (3.18), we get

$$\begin{aligned}&\mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}\Delta W_{t_{n+1}}] + \Delta t_n(1-\theta_2^n)\mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}]\\ &-\Delta t_n\{(1-\theta_2^n)\mathbb{E}_{h,t_n}^{t_n,x_i}[z^{n+1}] + \theta_2^n z_i^n\}\\ &\quad+Err_{yw} + \Delta t_n(1-\theta_2^n)Err_{fw} - \Delta t_n(1-\theta_2^n)Err_z = 0,\end{aligned} \tag{3.19}$$

where $Err_{yw}+\Delta t_n(1-\theta_2^n)Err_{fw}-\Delta t_n(1-\theta_2^n)Err_z$ is the Monte Carlo approximation error.

From probability theory, we know that the Monte Carlo approximation errors Err_y in (3.13), Err_z in (3.18), Err_f in (3.14), Err_{yw} in (3.16), and Err_{fw} in (3.17) converge to 0 with the convergence rate $\frac{1}{\sqrt{N_E}}$ as $N_E \to \infty$. So we can choose a proper integer N_E in the Monte Carlo method to approximate the mathematical expectations with the required accuracy.

Now we take (y_i^n, z_i^n) as the approximation of the solution (y_t, z_t) at the time-space point (t_n, x_i) with $t_n \in \mathfrak{R}_{th}$ and $x_i \in \mathfrak{R}_h$. Based on the two equations (3.15) and (3.19), by omitting the Monte Carlo error terms, we propose a fully time-space discrete θ-scheme to solve (y_t, z_t), the solution of the BSDE (2.1), as for the given terminal values $y_i^{N_T} (i \in \mathfrak{R}_h)$, select θ_1^n and θ_2^n in [0,1], and find (y_i^n, z_i^n) that satisfies

$$
\begin{aligned}
y_i^n = \mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}] + \Delta t_n\{(1-\theta_1^n)\mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})] \\
+\theta_1^n f(t_n, y_i^n, z_i^n)\},
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
0 = \mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}\Delta W_{t_{n+1}}] + \Delta t_n(1-\theta_2^n)\mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}] \\
-\Delta t_n\{(1-\theta_2^n)\mathbb{E}_{h,t_n}^{t_n,x_i}[z^{n+1}] + \theta_2^n z_i^n\}
\end{aligned} \tag{3.21}
$$

for $i \in \mathbb{Z}$ and $n = N_T - 1, N_T - 2, \ldots, 2, 1, 0$ with $\theta_2^{N_T-1} = 1$.

In the fully discrete θ-scheme (3.20) and (3.21), for each point (t_n, x_i), the (y_i^n, z_i^n) is unknown. Here we suggest a procedure to solve (y_i^n, z_i^n) from the two equations (3.20) and (3.21) for $i \in \mathbb{Z}$ and $n = N_T - 1, N_T - 2, \ldots, 2, 1, 0$. The procedure is as follows:

1. Given $y_i^{N_T}, \quad i \in \mathbb{Z}$.
2. For $n = N_T - 1$, let $\theta_2^n = 1$ in (3.21), and solve (y_i^n, z_i^n) from the two equations (3.20) and (3.21) for each $i \in \mathbb{Z}$.
3. For $0 \le n < N_T - 1$, solve (y_i^n, z_i^n) from the two equations (3.20) and (3.21) for each $i \in \mathbb{Z}$.

Remark 2.

1. In the fully discrete θ-scheme (3.20) and (3.21), in order to solve z_i^n, it is necessary for θ_2^n not to be zero. And solving z_i^n is always explicit from (3.21). For the linear BSDE, solving y_i^n is explicit. For the nonlinear BSDE, some numerical methods to solve y_i^n are needed if $\theta_1^n \neq 0$.
2. Solving (y_i^n, z_i^n) with different $i \in \mathbb{Z}$ is totally uncoupled. We can solve $(y_i^n, z_i^n)(i \in \mathbb{Z})$ in parallel.
3. In practical applications, we are often interested in the values of (y_0, z_0) in a bounded domain. Thus from the Brownian motion theory, the data of $(y_i^{N_T}, z_i^{N_T})$ on some bounded domain is needed under some required accuracy.
4. The terminal condition is y_T. In the first computing step, we set $\theta_2^{N_T-1} = 1$ for solving $z_i^{N_T-1}$. For $\theta_2^{N_T-1} = 1$, the accuracy of the scheme is not high. To achieve the high accuracy, a special time partition step Δt_{N_T-1} is needed. For example, we can choose $\theta_1^n = \theta_2^n = \frac{1}{2}$ for $n < N_T - 1$ and choose $\theta_2^n = 1$ and let $\Delta t_n = (\Delta t_{n-1})^2$ for $n = N_T - 1$.

4. Discrete θ-scheme for general BSDEs. Based on the method of constructing the discrete θ-scheme (3.20) and (3.21) in section 3, in this section, we generalize the scheme and propose a new numerical scheme for the general BSDEs (2.1) driven by a standard d-dimensional Brownian motion $W_t = (W_t^1, W_t^2, \ldots, W_t^d)^*$, where $W_t^i (i = 1, 2, \ldots, d)$ are the independent standard one-dimensional Brownian motions, and the notation * is the transpose operator for matrix or vector.

We use $\mathbb{R}^d$ to denote the d-dimensional Euclidean space. $x \in \mathbb{R}^d$ means $x = (x^1, x^2, \ldots, x^d)$ with $x^i \in \mathbb{R}(i = 1, 2, \ldots, d)$. Let $\mathbb{Z}^d = \{i | i = (i_1, i_2, \ldots, i_d), i_k \in \mathbb{Z}, k = 1, 2, \ldots, d\}$ and $x_i = (x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_d}^d)$ for $i = (i_1, i_2, \ldots, i_d) \in \mathbb{Z}^d$.

Let $(y_t, z_t) : (t, \Omega) \to \mathbb{R}^m \times \mathbb{R}^{m \times d}$ be the solution of the BSDE (2.1), where

$y_t = (y_t^1, y_t^2, \ldots, y_t^m)^*$ and $z_t = (z_t^{i,j})_{m\times d}$. Similarly to obtaining (3.3), we have

$$y_t^k = y_{t+\delta}^k + \int_t^{t+\delta} f^k(s, y_s, z_s)ds - \int_t^{t+\delta} \sum_{j=1}^{d} z_s^{k,j} dW_s^j, \quad t \in [0,T), \tag{4.1}$$

where f^k is the kth component of the vector function f. Taking the conditional mathematical expectation $\mathbb{E}_t^{t,x}$ on the two sides of (4.1), we obtain

$$y_t^{k,t,x} = \mathbb{E}_t^{t,x}[y_{t+\delta}^k] + \int_t^{t+\delta} \mathbb{E}_t^{t,x}[f^k(s, y_s, z_s)]ds, \quad t \in [0,T), t+\delta \in [0,T] \tag{4.2}$$

for $k = 1, 2, \ldots, m$. We rewrite (4.2) in the following vector form:

$$y_t^{t,x} = \mathbb{E}_t^{t,x}[y_{t+\delta}] + \int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s, y_s, z_s)]ds, \quad t \in [0,T), t+\delta \in [0,T]. \tag{4.3}$$

As we did for $m = d = 1$ in section 3, for $t \le s \le t+\delta$, we let $\Delta W_s^l = W_s^l - W_t^l (l = 1, 2, \ldots, d)$, which are the independent standard one-dimensional Brownian motions with mean zero and variance $\sqrt{s-t}$. Multiplying the two sides of (4.1) by $\Delta W_{t+\delta}^l$ and taking the conditional mathematical expectation $\mathbb{E}_t^{t,x}$ on the two sides of the derived equation, by the Itô isometry formula we then obtain

$$-\mathbb{E}_t^{t,x}[y_{t+\delta}^k \Delta W_{t+\delta}^l] = \int_t^{t+\delta} \mathbb{E}_t^{t,x}[f^k(s, y_s, z_s)\Delta W_s^l]ds - \int_t^{t+\delta} \mathbb{E}_t^{t,x}[z_s^{k,l}]ds \tag{4.4}$$

for $k = 1, 2, \ldots, m$ and $l = 1, 2, \ldots, d$, or in the matrix form

$$-\mathbb{E}_t^{t,x}[y_{t+\delta}\Delta W_{t+\delta}^*] = \int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s, y_s, z_s)\Delta W_s^*]ds - \int_t^{t+\delta} \mathbb{E}_t^{t,x}[z_s]ds. \tag{4.5}$$

Here for a matrix random variable $X = (X_{lk})_{m\times d}$, the conditional mathematical expectation $\mathbb{E}_t^{t,x}[X]$ is defined by

$$\mathbb{E}_t^{t,x}[X] = (\mathbb{E}_t^{t,x}[X^{l,j}])_{m\times d}. \tag{4.6}$$

We now introduce the partitions of the Euclidean space $\mathbb{R}^d$ and the time interval $[0,T]$. The partition of the Euclidean space $\mathbb{R}^d$ is

$$\mathfrak{R}_h^d = \overbrace{\mathfrak{R}_h \times \mathfrak{R}_h \times \cdots \times \mathfrak{R}_h}^{d \ \ times}, \tag{4.7}$$

where $\mathfrak{R}_h$ is defined in section 3. The partition of the time interval $[0,T]$ is $\mathfrak{R}_{th}$ defined in section 3.

Let the approximations of the conditional mathematical expectations in (4.3) and (4.5) be defined as those in (3.13), (3.14), (3.16), (3.18), and (3.17); that is,

$$\mathbb{E}_{h,t_n}^{t_n,x_i}[X] = (\mathbb{E}_{h,t_n}^{t_n,x_i}[X^{l,j}]). \tag{4.8}$$

In the computation of $\mathbb{E}_{h,t_n}^{t_n,x_i}[X^{l,j}]$, the random variable $\hat{x}_i = x_i + \Delta W_{t_{n+1}}$ for $x_i \in \mathbb{R}_h^d$ and the d-dimensional interpolation operator I_h are used.

The number of the d-dimensional interpolation points near the space point $\hat{x}_i$ depends on the required accuracy. For the high-dimensional problems, the space interpolations may be expensive. It is still a challenging problem to propose an efficient high-dimensional interpolation method. In this paper, we focus our attention on the numerical methods for the BSDE (2.1) and use the bilinear polynomial, bicubic polynomial, and bicubic spline interpolations to solve the two-dimensional BSDE (2.1) in Example 3 in section 5.

Now let (y_i^n, z_i^n) be an approximation of (y_t, z_t) at the time-space point (t_n, x_i) with $t_n \in \mathfrak{R}_{th}$ and $x_i \in \mathfrak{R}_h^d$. Based on the two equations (4.2) and (4.5), by using the same procedure of getting the fully discrete θ-scheme (3.20) and (3.21) in section 3, we propose a new fully discrete θ-scheme for the general BSDE (2.1) as follows: Given $y_i^{N_T} (i \in \mathbb{Z}^d)$, find $(y_i^n, z_i^n) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}$ that satisfies

$$\begin{aligned} y_i^n = {} & \mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}] + \Delta t_n\{(1-\theta_1^n)\mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})] \\ & + \theta_1^n f(t_n, y_i^n, z_i^n)\}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} 0 = {} & \mathbb{E}_{h,t_n}^{t_n,x_i}[y^{n+1}(\Delta W_{t_{n+1}})^*] + \Delta t_n(1-\theta_2^n)\mathbb{E}_{h,t_n}^{t_n,x_i}[f(t_{n+1}, y^{n+1}, z^{n+1})(\Delta W_{t_{n+1}})^*] \\ & - \Delta t_n\{(1-\theta_2^n)\mathbb{E}_{h,t_n}^{t_n,x_i}[z^{n+1}] + \theta_2^n z_i^n\} = 0 \end{aligned} \tag{4.10}$$

for $i \in \mathbb{Z}^d$ and $n = N_T-1, N_T-2, \ldots, 2, 1, 0$, where θ_1^n and θ_2^n are two parameters in [0,1] with $\theta_2^{N_T-1} = 1$.

A solution procedure for the scheme (4.9) and (4.10) can be described as follows:

1. given $y_i^{N_T}$, $\ i \in \mathbb{Z}^d$, $n = N_T - 1$;
2. solve $z_i^n (i \in \mathbb{Z}^{m\times d})$ from (4.10);
3. solve $y_i^n (i \in \mathbb{Z}^d)$ from (4.9);
4. if $n = 0$, stop; else let $n \leftarrow n-1$; goto step 2.

Remark 3. Like the one-dimensional fully discrete θ-scheme, in the fully discrete θ-scheme (4.9) and (4.10), the unknowns are y_i^n and z_i^n. Solving z_i^n is explicit from (4.10). For $\theta_1^n \neq 0$, solving y_i^n is implicit. At each time level $t = t_n$, $y_l^n (l \in \mathbb{Z}^d)$ are totally uncoupled among the space grids and can be solved parallelly.

5. Numerical experiments. The aim of this section is to use some numerical examples to demonstrate the effectiveness and accuracy of our method for solving the BSDE (2.1). In the examples, the time and space step sizes are Δt and h, respectively. The positive integer N_T is the total discrete time steps $N_T = \frac{T}{\Delta t}$ with the finite terminal time T. In the tables, the notations of $|\cdot|$, CR, EMC, HS, and Scheme$^\alpha (\alpha = L, C, S)$ represent, respectively, the standard Euclidean norm in the Euclidean space $\mathbb{R}^m$ or $\mathbb{R}^{m\times d}$, the convergence rate with respect to the time step sizes, the implicit Euler Monte Carlo method related to our scheme with $\theta_1^n = \theta_2^n = 1$, the Heun's method in which the cubic polynomial interpolation operator I_h is used, and our method in which the linear polynomial, cubic polynomial, and cubic spline interpolation operators I_h are used with $\theta_1^n = \theta_2^n = \frac{1}{2}$ for $\alpha = L, C, S$, respectively.

To obtain the convergence rates for the different interpolations, we adjust the space partition step h to the time partition step Δt. As we know, for ordinary differential equations, the CN scheme is a second order method. In our method, the time discretization is based on the integral approximations with the deterministic functions, which are the conditional mathematical expectations. The details of the approximations are described in section 3. If the functions $f(t, y, z)$ and $\Phi(x)$ are sufficiently smooth, the discrete approximation errors in the time direction are $C_t\Delta t$ for $\theta_1^n \neq \frac{1}{2}$ or $\theta_2^n \neq \frac{1}{2}$, and $C_t(\Delta t)^2$ for $\theta_1^n = \theta_2^n = \frac{1}{2}$. The space interpolation error is $I_h u - u = C_h h^r$ with a positive number r, which depends on the interpolation operator I_h and the regularity of the function u with respect to the space variables. For the linear polynomial interpolation operator I_h, $r = 2$. For the cubic polynomial and cubic spline interpolation operators I_h, $r = 4$. In numerical simulations, the time errors and the space errors should be balanced, so the space discrete step h and the time discrete step Δt should be chosen to satisfy the equality $C_h h^r = C_t(\Delta t)^2$. In our numerical experiments, we use $h = \Delta t$ when the linear polynomial interpolation is used, and use $h = (\Delta t)^{\frac{1}{2}}$ when the cubic polynomial and cubic spline interpolations are used. If the order of the accuracy in the time direction is α, then the order of the accuracy in the space is α for the linear interpolation and 2α for the cubic polynomial and cubic spline interpolations.

In the Monte Carlo approximations, we use different random sampling numbers N_E for different time partitions. The sampling numbers N_E are listed in the corresponding tables.

It is well known that the random number generator is very important for Monte Carlo method and that the accuracy of the Monte Carlo method depends not only on the sampling number N_E but also on the quality of the random number generator. In this paper, we choose the software SPRNG (Scalable Parallel Random Number Generators Library) as our random number generator. We use SPRNG to generate uniform distribution samples and then use the well-known Box–Muller method [7] to transform these samples into the ones with normal distributions. The software SPRNG and its details can be found on the website http://sprng.cs.fsu.edu/.

To compare our method with other ones, we also use the Euler method and the Heun's method [8] combined with the Monte Carlo method to solve the BSDEs in the following numerical examples, and list their errors and their convergence rates in the corresponding tables.

Example 1. Consider the following nonlinear BSDE:

$$\left\{ \begin{array}{l} -dY_t = \lambda(Y_t)|Z_t|^2 dt - Z_t dW_t, 0 \le t \le T, \\ Y_T = \Phi(W_T), \end{array} \right. \tag{5.1}$$

where $\Phi(x)$ and $\lambda(x)$ are two deterministic functions of x, and W_T is the Brownian motion at time T. The analytic solution of the problem (5.1) is in the form

$$Y_0 = h^{-1}(\mathbb{E}(h(\Phi(X_T)))) \tag{5.2}$$

where the function $h(x)$ is the solution of the following ordinary differential equation:

$$-h'(x)\lambda(x) + \frac{1}{2}h''(x) = 0. \tag{5.3}$$

When $\lambda(x) = ax$, the analytic solution of (5.1) is

$$Y_0 = \frac{-C_2\sqrt{C_1 a} + \tan^{-1}(\mathbb{E}[\tan(\Phi(X_T)\sqrt{C_1 a} + C_2\sqrt{C_1 a})])}{\sqrt{C_1 a}}, \tag{5.4}$$

where a, C_1, and C_2 are parameters, $\tan(\cdot)$ is the triangle function *tangent*, and $\tan^{-1}(\cdot)$ is the inverse of the function $\tan(\cdot)$.

In this numerical example, we take $a = 0.01$, $C_1 = 1$, and $C_2 = 0$ in (5.4). We choose the terminal time $T = 1$ and the terminal function $\Phi(x) = |\cos(x)|$. The solution of (5.1) at time $t = 0$ is $Y_0 = 0.6943$ from (5.4). We list the absolute errors between the analytic solution Y_0 and the numerical solution y_0^0 and list the convergence rates in Table 5.1. The plots of $\log_2(|Y_0 - y_0^0|)$ with respect to $\log_2(N_T)$ are drawn in Figure 5.1.

TABLE 5.1
Errors and convergence rates.

	$\lvert Y_0 - y_0^0\rvert$						CR
N_T	2	4	8	16	32	64	
N_E	400	1000	4000	10000	20000	40000	
EMC	0.1259	0.0872	0.0510	0.0237	0.0177	0.0075	0.80
HS	0.0206	0.0093	0.0039	0.0015	0.0010	0.0006	0.99
Scheme^L	0.0312	0.0086	0.0021	0.0009	0.0003	0.0001	1.66
Scheme^C	0.0302	0.0060	0.0017	0.0005	0.0002	0.00008	1.71
Scheme^S	0.0242	0.0048	0.0033	0.0013	0.0002	0.00004	1.98

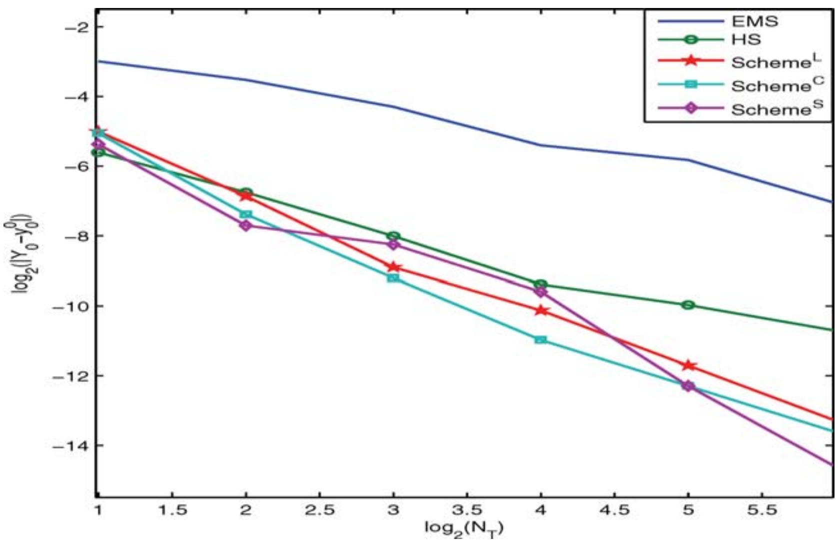

FIG. 5.1. *The plots of* $\log_2(|Y_0 - y_0^0|)$ *with respect to* $\log_2(N_T)$.

Example 2. The BSDE is

$$\begin{cases} -dY_t = \left(\dfrac{Y_t}{2} - Z_t\right) dt - Z_t dW_t, \ 0 \le t \le T, \\ Y_T = \sin(W_T + T). \end{cases} \tag{5.5}$$

The analytic solution of (5.5) is

$$(Y_t, Z_t) = (\sin(W_t + t), \cos(W_t + t)), \tag{5.6}$$

where W_t is a standard Brownian motion. In this example, we take $T = 1$. The solution at $t = 0$ is $(Y_t, Z_t)|_{t=0} = (Y_0, Z_0) = (0, 1)$. The errors $|Y_0 - y_0^0|$ and $|Z_0 - z_0^0|$ and their convergence rates are listed in Table 5.2. We plot the lines of $\log_2(|Y_0 - y_0^0|)$ with respect to $\log_2(N_T)$ in Figure 5.2.

Example 3. Let $\mathbf{W}_t = (W_t^1, W_t^2)^*$ be a two-dimensional Brownian motion, *where* W_t^1 *and* W_t^2 *are two* independent standard one-dimensional Brownian motions. The BSDE driven by $\mathbf{W}_t$ is

$$-dY_t = \left(\frac{Y_t - d}{2}(b^2 + c^2) - Z_t \cdot \mathbf{A}\right) dt - Z_t \cdot d\mathbf{W}_t, \quad 0 \le t \le T, \tag{5.7}$$

TABLE 5.2
Errors and convergence rates.

	$\lvert Y_0 - y_0^0\rvert$						CR
N_T	2	4	8	16	32	64	
N_E	400	1000	4000	10000	20000	40000	
EMC	0.0735	0.0327	0.0179	0.0086	0.0035	0.0015	1.13
HS	0.1334	0.0348	0.0198	0.0112	0.0053	0.0020	0.82
Scheme^L	0.0395	0.0114	0.0034	0.0010	0.0004	0.00012	1.7
Scheme^C	0.0357	0.0095	0.0094	0.0020	0.0004	0.0001	1.91
Scheme^S	0.0351	0.0072	0.0023	0.0008	0.0003	0.00008	2.03
	$\lvert Z_0 - z_0^0\rvert$						CR
N_T	2	4	8	16	32	64	
N_E	400	1000	4000	10000	20000	40000	
EMC	0.2356	0.0587	0.0065	0.0502	0.0402	0.0565	0.39
HS	0.2316	0.1002	0.0720	0.0318	0.0432	0.0251	0.64
Scheme^L	0.2874	0.1056	0.0371	0.0136	0.0102	0.0035	1.27
Scheme^C	0.2706	0.0752	0.0253	0.0121	0.0059	0.0016	1.48
Scheme^S	0.2911	0.0553	0.0281	0.0033	0.0042	0.0011	1.61

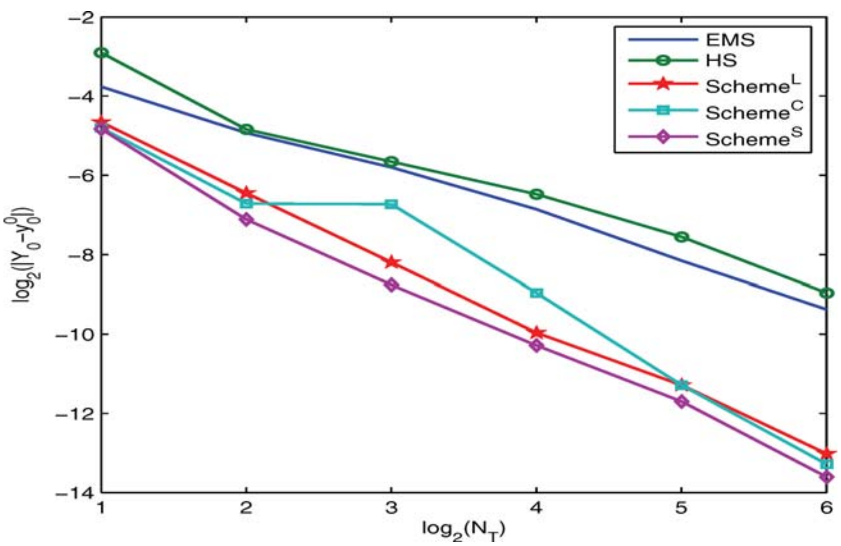

FIG. 5.2. *The plots of* $\log_2(|Y_0 - y_0^0|)$ *with respect to* $\log_2(N_T)$.

with the terminal condition $Y_T = a\sin(\mathbf{M}\cdot\mathbf{W}_T + T) + d$, where $\mathbf{A} = (\frac{1}{2b}, \frac{1}{2c})^*$, $\mathbf{M} = (b, c)$, $Z_t \cdot \mathbf{A} = \frac{Z_t^1}{2b} + \frac{Z_t^2}{2c}$, and $Z_t \cdot d\mathbf{W}_t = Z_t^1 dW_t^1 + Z_t^2 dW_t^2$ with the constant parameters a, b, c, and d. The analytic solution (Y_t, Z_t) is in the following form:

$$\begin{cases} Y_t = a\sin(\mathbf{M}\cdot\mathbf{W_t} + t) + d, \\ Z_t = (ab\cos(\mathbf{M}\cdot\mathbf{W}_t + t), ac\cos(\mathbf{M}\cdot\mathbf{W}_t + t)), \end{cases} \tag{5.8}$$

where $\mathbf{M}\cdot\mathbf{W}_t = bW_t^1 + cW_t^2$.

TABLE 5.3
Errors and convergence rates.

	$\lvert Y_0 - y_0^0\rvert$						CR
N_T	2	4	8	16	32	64	
N_E	1000	2000	6000	10000	20000	40000	
EMC	0.3698	0.2504	0.1385	0.0617	0.0276	0.0129	0.97
HS	0.2687	0.1592	0.0598	0.0329	0.0115	0.0070	1.05
Scheme^L	0.1483	0.0419	0.0106	0.0037	0.0008	0.0003	1.78
Scheme^C	0.0996	0.0174	0.0058	0.0023	0.0006	0.0002	1.80
Scheme^S	0.0353	0.0122	0.0039	0.0011	0.0003	0.00003	2.04
	$\lvert Z_0 - z_0^0\rvert$						CR
N_T	2	4	8	16	32	64	
N_E	1000	2000	6000	10000	20000	40000	
EMC	0.6987	0.3561	0.2010	0.0782	0.335	0.0129	1.14
HS	0.6239	0.3517	0.1862	0.1021	0.0435	0.0390	0.80
Scheme^L	0.3561	0.1181	0.0462	0.0113	0.0031	0.0012	1.64
Scheme^C	0.2826	0.1032	0.0394	0.0098	0.0027	0.0008	1.70
Scheme^S	0.3125	0.1092	0.0435	0.0103	0.0029	0.0007	1.76

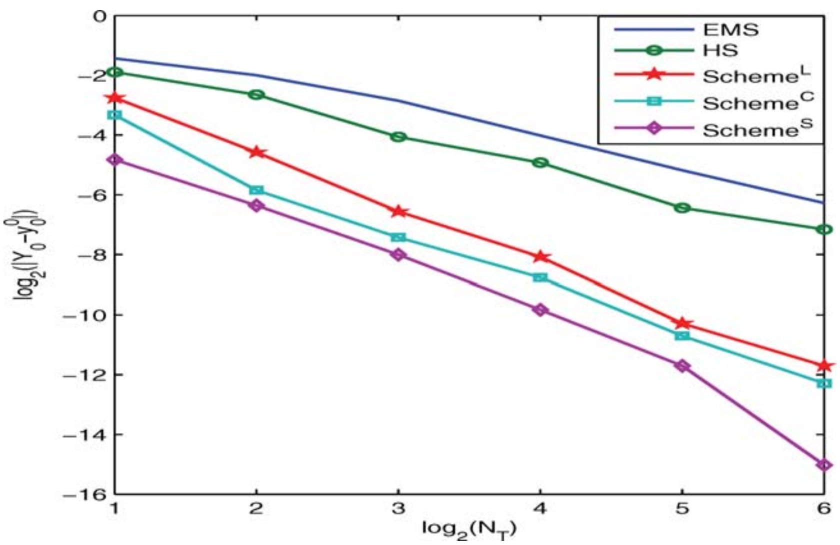

FIG. 5.3. *The plots of* $\log_2(|Y_0 - y_0^0|)$ *with respect to* $\log_2(N_T)$.

In this example, we take $T = 1$, $a = b = c = 1$, and $d = 0$. The solution at $t = 0$ is $(y_0, z_0) = (0, [1, 1])$. The absolute errors $|Y_0 - y_0^0|$ and $|Z_0 - z_0^0|$ and their convergence rates are listed in Table 5.3. The plots of $\log_2(|Y_0 - y_0^0|)$ with respect to $\log_2(N_T)$ are in Figure 5.3.

Example 4. In this example, we use our method to solve one-dimensional evaluation problems in the stock market. Here we briefly describe the mathematical models in the BSDE form for these problems.

Consider a bond and a stock traded in a stock market. The bond price p_t and the stock price S_t, respectively, obey

$$\begin{cases} dp_t = & r_t p_t dt, \quad t \geq 0, \\ p_0 = & p, \end{cases} \tag{5.9}$$

and

$$\left\{\begin{array}{rl} dS_t = & b_tS_tdt + \sigma_tS_tdW_t, \quad t \geq 0, \\ S_0 = & x, \end{array}\right. \tag{5.10}$$

where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion, r_t is the return rate of the bond, b_t is the expected return rate of the stock, $\sigma_t > 0$ is the volatility of the stock, and r_t, b_t, σ_t, and ${\sigma_t}^{-1}$ are all bounded.

An investor with wealth Y_t at time t uses π_t money to buy stock and uses $Y_t - \pi_t$ to buy bond. Suppose that the stock pays dividends continuously with a bounded dividend rate $d(t, S_t)$ at the time t. Then the processes Y_t and π_t satisfy the following SDE [26, 18, 24, 25]:

$$-dY_t = -(r_tY_t + (b_t - r_t + d(t, S_t))\pi_t)dt - \sigma_t\pi_tdW_t. \tag{5.11}$$

Let $Z_t = \sigma_t\pi_t$. Then (Y_t, Z_t) satisfies

$$-dY_t = -(r_tY_t + (b_t - r_t + d(t, S_t))\sigma_t^{-1}Z_t)dt - Z_tdW_t. \tag{5.12}$$

When the money $Y_t - \pi_t \geq 0$, the investor invests $Y_t - \pi_t$ in the bond with the rate r_t. Otherwise, he will borrow $(Y_t - \pi_t)^- = \min\{0, Y_t - \pi_t\}$ money at the rate R_t. As we obtained (5.12), we have that (Y_t, Z_t) satisfies

$$\begin{array}{rl} -dY_t = & -(r_tY_t + (b_t - r_t + d(t, S_t)){\sigma_t}^{-1}Z_t \\ & \quad +(R_t - r_t)(Y_t - \sigma_t^{-1}Z_t)^-)dt - Z_tdW_t \end{array} \tag{5.13}$$

for $t \in [0, T)$ with Z_t defined as the one in (5.12).

For the European call option, the terminal condition is given at the mature time T for (5.12) and (5.13) by

$$Y_T = \max\{S_T - K, 0\}, \tag{5.14}$$

where S_T is the solution S_t of the problem (5.10) at the mature time T, and K is the strike price.

When $r_t = r$, $b_t = b$, and $\sigma_t = \sigma$, and $d(t, S_t) = d$ are constants in (5.12) and (5.13), we can obtain their analytic solutions in closed forms. The analytic solution of the BSDE (5.12) with the terminal condition (5.14) is

$$\left\{\begin{array}{l} Y_t = V(t, S_t) = S_te^{-d(T-t)}N(d_1(S_t)) - Ke^{-r(T-t)}N(d_0(S_t)), \\ Z_t = \dfrac{\partial V}{\partial S}\sigma = S_te^{-d(T-t)}N(d_1(S_t))\sigma, \\ d_0(S_t) = \dfrac{1}{\sigma\sqrt{T-t}}\ln\left(\dfrac{S_t}{Ke^{(r-d)(T-t)}}\right) - \dfrac{1}{2}\sigma\sqrt{T-t}, \\ d_1(S_t) = d_0(S_t) + \sigma\sqrt{T-t}, \end{array}\right. \tag{5.15}$$

and the analytic solution of the BSDE (5.13) with the terminal condition (5.14) is

$$\left\{\begin{array}{l} Y_t = S_te^{-d(T-t)}N(d_1(S_t)) - Ke^{-R(T-t)}N(d_0(S_t)), \\ Z_t = S_te^{-d(T-t)}N(d_1(S_t))\sigma, \\ d_0(S_t) = \dfrac{1}{\sigma\sqrt{T-t}}\ln\left(\dfrac{S_t}{Ke^{(R-d)(T-t)}}\right) - \dfrac{1}{2}\sigma\sqrt{T-t}, \\ d_1(S_t) = d_0(S_t) + \sigma\sqrt{T-t}, \end{array}\right. \tag{5.16}$$

TABLE 5.4
The errors and convergence rates.

	$\lvert Y_0 - y_0^0 \rvert$						CR
N_T	2	4	8	16	32	64	
N_E	400	1000	4000	10000	20000	40000	
EMC	0.1038	0.0872	0.0473	0.0310	0.0223	0.0108	0.65
HS	0.0632	0.0295	0.0109	0.0112	0.0087	0.0015	1.03
Scheme^L	0.1572	0.0451	0.0105	0.0027	0.0002	0.00005	2.30
Scheme^C	0.1201	0.0310	0.0089	0.0025	0.0009	0.0002	1.84
Scheme^S	0.0973	0.0263	0.0062	0.0020	0.0006	0.0002	1.79
	$\lvert Z_0 - z_0^0 \rvert$						CR
N_T	2	4	8	16	32	64	
N_E	400	1000	4000	10000	20000	40000	
EMC	1.1769	0.2431	0.7025	0.4432	0.4607	0.5826	0.20
HS	1.8355	1.4859	0.7811	0.0756	0.1247	0.0939	0.85
Scheme^L	0.7817	0.4466	0.2712	0.0632	0.0162	0.0127	1.18
Scheme^C	0.9834	0.5382	0.2437	0.1084	0.0722	0.0384	0.94
Scheme^S	1.1326	0.8730	0.5218	0.2421	0.0989	0.0492	0.90

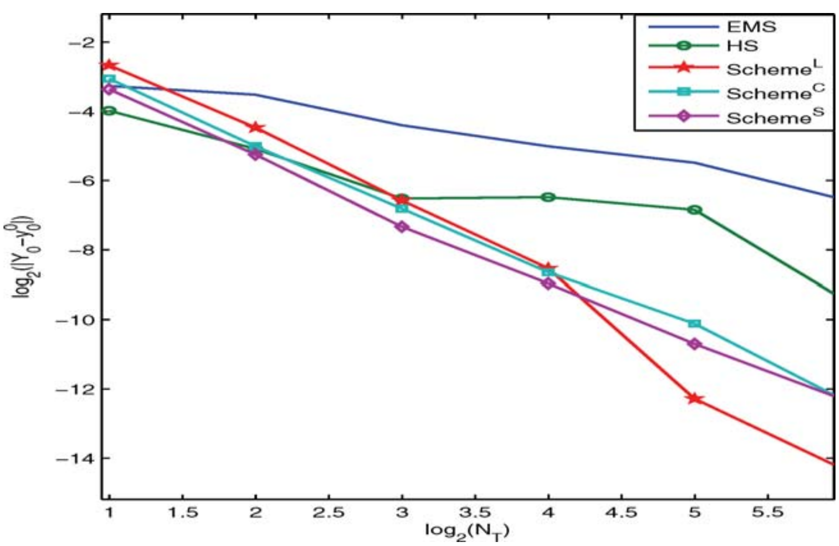

FIG. 5.4. *The plots of* $\log_2(\lvert Y_0 - y_0^0 \rvert)$ *with respect to* $\log_2(N_T)$.

where N is the cumulative normal distribution function.

For the BSDE problem (5.12) and (5.14), we take $T = 0.33$, $K = S_0 = 100$, $r = 0.03$, $\mu = 0.05$, $d = 0.04$, and $\sigma = 0.2$. From (5.15), we know $(Y_0, Z_0) = (4.3671, 10.0950)$. We list the absolution errors $\lvert Y_0 - y_0^0 \rvert$ and $\lvert Z_0 - z_0^0 \rvert$ and their convergence rates in Table 5.4, and plot the errors $\log_2(\lvert Y_0 - y_0^0 \rvert)$ with respect to $\log_2(N_T)$ in Figure 5.4.

For the BSDE problem (5.13) and (5.14), we take $T = 0.33$, $K = S_0 = 100$, $r = 0.03$, $\mu = 0.05$, $d = 0$, $\sigma = 0.2$, and $R = 0.09$. From (5.16), we have $(Y_0, Z_0) = (6.1269, 12.4796)$. We list the absolute errors $\lvert Y_0 - y_0^0 \rvert$ and $\lvert Z_0 - z_0^0 \rvert$ and their convergence rates in Table 5.5. We plot the absolute errors $\log_2(\lvert Y_0 - y_0^0 \rvert)$ with respect to $\log_2(N_T)$ in Figure 5.5.

TABLE 5.5
Errors and convergence rates.

	$\lvert Y_0 - y_0^0\rvert$						CR
N_T	2	4	8	16	32	64	
N_E	400	1000	4000	10000	20000	40000	
EMC	0.1259	0.0847	0.0539	0.0332	0.0173	0.0073	0.82
HS	0.3383	0.1450	0.0446	0.0286	0.0141	0.0050	1.30
Scheme^L	0.0792	0.0406	0.0163	0.0047	0.0009	0.0001	1.89
Scheme^C	0.0895	0.0356	0.0143	0.0052	0.0013	0.0002	1.77
Scheme^S	0.0742	0.0242	0.0046	0.0022	0.0010	0.0002	1.71
	$\lvert Z_0 - z_0^0\rvert$						CR
N_T	2	4	8	16	32	64	
N_E	400	1000	4000	10000	20000	40000	
EMC	0.4382	0.7426	0.8235	0.4092	0.5211	0.4763	-0.02
HS	2.1918	1.0871	0.5013	0.4147	0.2729	0.1224	0.83
Scheme^L	0.8762	0.4042	0.2381	0.1069	0.0428	0.0105	1.28
Scheme^C	1.5382	0.7320	0.3647	0.2201	0.1052	0.0387	1.06
Scheme^S	1.8993	1.0830	0.4247	0.1835	0.0859	0.0322	1.17

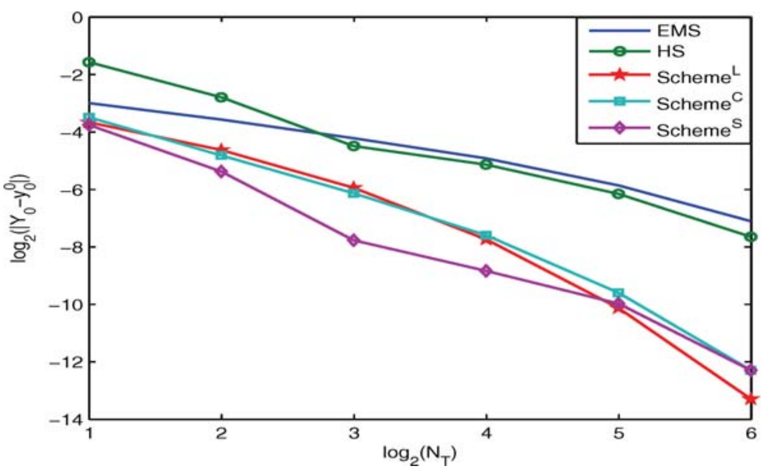

FIG. 5.5. *The plots of* $\log_2(|Y_0 - y_0^0|)$ *with respect to* $\log_2(N_T)$.

We summarize all numerical experiments as follows:

1. The accuracy of our method depends on the following:
   (a) Parameters θ_n^1 and θ_n^2. The accuracy of our method depends on the parameters θ_n^1 and θ_n^2, and if we choose them properly, the accuracy of our method can be high. Our numerical results show that the accuracy of our method is high when we choose $\theta_n^1 = \theta_n^2 = 0.5$.
   (b) Random sampling number N_E. It is known that the approximation error of the Monte Carlo method is approximately $\frac{1}{\sqrt{N_E}}$. Theoretically, if we want to obtain the accuracy ϵ, the random sampling number should approximately be $\left(\frac{1}{\epsilon}\right)^2$. For instance, when $\epsilon = (\Delta t)^2 = (\frac{T}{64})^2$, the sampling number is approximately $(64)^4$, which is a very big integer. In our simulations, for the time step $\Delta t = \frac{T}{64}$, the sampling number we used is 40000, which is much less than $(64)^4$. Even though we use the random samples much less than those needed theoretically, the accuracy

of our method is still high for solving (Y_t, Z_t) with $\theta_n^1 = \theta_n^2 = 0.5$.

(c) Space interpolations. In our method, we use the Monte Carlo method to approximate the conditional mathematical expectations and the space interpolations to compute the values at nongrid points. The accuracy of our method in space depends on the space interpolation operators and the regularity of the solutions. Our numerical results show that the higher the accuracy of the interpolation operators is, the higher the accuracy of our method is. The equidistant grid is far from optimal, especially for nonsmooth problems. To solve practical problems, some adaptive methods and some more efficient special grid interpolation operators may be used. All these will be our study projects on numerical methods for BSDEs in our future works.

(d) The regularity of the functions $f(t, y, z)$ and $\Phi(x)$. From the relation between BSDEs and PDEs [28], we know $Y_t = V(t, X_t)$ and $Z_t = \frac{\partial V(t,X_t)}{\partial x}$, where $V(t, x)$ is the solution of the PDEs, X_t is W_t in Examples 1–3, and $X_t = S_t$ in Example 4. The regularity of the solution (Y_t, Z_t) of the BSDE (2.1) depends on the regularity of the functions $f(t, y, z)$ and $\Phi(x)$. From PDE theory [31, 13], we know that the solutions of Examples 2 and 3 have smooth fourth order derivatives, and in Example 4, the solution Y_t has continuous second order derivatives with respect to S_t, and Z_t has only continuous first order derivatives. Our numerical results in Example 4 show that for nonsmooth data, the high-order interpolations cannot improve the approximate accuracy.

2. By comparison, we also solve the BSDEs in the examples by using Heun's method, in which the cubic polynomial interpolation is used in space. It is well known that Heun's method is a second order method for solving ODEs. But when used to solve the BSDEs, its accuracy is at most first order from our numerical results.

   It is well known that the accuracy of the Monte Carlo method depends not only on the sampling number but also on the random number generator. We have tried several random number generators in C, Fortran, MATLAB, and the software SPRNG which is free and can be downloaded from the website http://sprng.cs.fsu.edu.

   We found from our numerical experiments that the accuracy of our method depends on the random number generators, especially for solving Z_t. The high order accuracy was obtained for solving Y_t by using all the random number generators mentioned above in our numerical examples, but for solving Z_t, only by using the SPRNG random number generator was high order accuracy obtained. Since, in this paper, our aim is on the accuracy of our method, we list only the numerical results for EMC, HS and $Scheme^{\alpha}(\alpha = L, C, S)$ by using the software SPRNG to generate the random numbers. Better results may be obtained if some other better random number generators are used.

6. Conclusions. In this paper, we have proposed a new accurate numerical method to solve BSDEs at time-space grids. We use the Monte Carlo method to approximate the conditional mathematical expectations, and use local space interpolation to approximate the values of the solutions at nongrid points. Our numerical method is uncoupled not only between y_t and z_t, but also among the space grids at each time level. The data used to simulate the numerical solutions at each time-space point can be local. At the same time, the accuracy of our method can be improved by better choosing the space interpolation operator and the parameters θ_1^n and θ_2^n. So there are large capacities of our method to solve the high-dimensional BSDEs with high accuracy by using parallel computers. To demonstrate the robustness of our methods, we performed some numerical experiments, and the results showed that our method is very effective and accurate for solving the BSDEs.

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