Numerical method for Forward-Backward Stochastic Differential Equations

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Introduction

In this paper i introduce three numerical methods for FBSDEs and 2FBSDEs, which includes $\theta - method$, Multistep schemes and Deferred Correction Methods. The important class of the FBSDE and 2FBSDE are the Itôs type equations such as:

1. Backward stochastic differential equation(BSDE):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s$$

2. Decoupled Forward-Backward stochastic differential equation(FBSDE)

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s) \, \mathrm{d}s + \int_0^t \sigma(s, X_s) \, \mathrm{d}W_s \\ Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s) \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s, t \in [0, T] \end{cases}$$

3. Coupled FBSDE:

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s, Y_s, Z_s) \, \mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s, Z_s) \, \mathrm{d}W_s \\ Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s) \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s, t \in [0, T] \end{cases}$$

4. Decoupled second-order FBSDE(2FBSDE):

$$\begin{cases} X_t = x + \int_0^t b(s, X_s) \, \mathrm{d}s + \int_0^t \sigma(s, X_s) \, \mathrm{d}W_s \\ Y_t = g(X_T) + \int_t^T f(s, \Theta_s) \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s, \quad s \in [0, T] \\ Z_s = Z_0 + \int_0^t A_s \, \mathrm{d}s + \int_0^t \Gamma_s \, \mathrm{d}W_s, \end{cases}$$

where $\Theta_s = (X_s, Y_s, Z_s, \Gamma_s)$

5. Coupled 2FBSDE:

$$\begin{cases} X_t = x + \int_0^t b(s, \Theta_s) \, \mathrm{d}s + \int_0^t \sigma(s, \Theta_s) \, \mathrm{d}W_s \\ Y_t = g(X_T) + \int_t^T f(s, \Theta_s) \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s, \quad s \in [0, T] \\ Z_s = Z_0 + \int_0^t A_s \, \mathrm{d}s + \int_0^t \Gamma_s \, \mathrm{d}W_s \end{cases}$$

where $\Theta_s = (X_s, Y_s, Z_s, \Gamma_s)$

1 Stochastic Differential Equations

In order to introduce the backward stochastic differential equation(BSDE), i will briefly introduce the stochastic differential equation(SDE) in the first part, which mainly includes Brownian motion, Martingale approach, Stochastic integrals and Itô rules.

1.1 Brownian motion

Definition (Brownian motion):
$A\ Brownian\ motion\ is\ a\ continuous-time\ stochastic\ process\ W_t,\ t \geqslant 0\ with\ the\ following\ properties:$

(1). $W_0 = 0$

(2). $t \longrightarrow W_t\ is\ continuous\ in\ t\ \ (almost\ surely)$

(3). $W_t\ has\ an\ stationary,\ independent\ increments$

(4). $The\ increment\ follows\ the\ normal\ distribution:$

$$W_t - W_s \sim N\left(0,\ t-s\right),\ 0 \leqslant s < t$$

Definition (Geometric brownian motion)
A geometric Brownian motion (GBM) (also known as exponential Brownian motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift.
A stochastic process St is said to follow a GBM if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$

Where W_t is a Wiener process or Brownian motion, and μ ('the percentage drift') and σ ('the percentage volatility') are constants.

And let S_t be a GBM, it holds that,

(1). $E\left[S_t\right] = S_0 exp\left(\mu t\right)$

(2). $Var\left(S_t\right) = S_0^2 exp\left(2\mu t\right)\left(e^{\sigma^2 t} - 1\right)$

1.2 The martingale approach

Definition (Martingale approach):
An integrabele stochastic process $(X_t)_{t\in\mathbb{R}^+}$ is a martingale (respectively a supermartingale, respectively a submartingale) with respect to a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}^+}$ if it satisfies the property:

$$X_s = E\left[X_t \middle| \mathscr{F}_s\right],\ 0 \leqslant s \leqslant t;$$

respectively

$$X_s \geqslant E\left[X_t \middle| \mathscr{F}_s\right],\ 0 \leqslant s \leqslant t;$$

respectively

$$X_s \leqslant E\left[X_t \middle| \mathscr{F}_s\right],\ 0 \leqslant s \leqslant t;$$

then we have the proposition:

$$\begin{array}{ll} E\left[X_t\right] = E\left[X_s\right],\ 0 \leqslant s \leqslant t & (let\ X_t\ be\ a\ martingale) \\ E\left[X_t\right] \leq E\left[X_s\right],\ 0 \leqslant s \leqslant t & (let\ X_t\ be\ a\ supermartingale) \\ E\left[X_t\right] \geqslant E\left[X_s\right],\ 0 \leqslant s \leqslant t & (let\ X_t\ be\ a\ submartingale) \end{array}$$

1.3 Stochastic integrals and calculus

A general SDE with differential notation reads:

$$dX_t = \underbrace{a\,(t,\ X_t)\,dt}_{drift}\ + \underbrace{b\,(t,\ X_t)\,dW_t}_{diffusion}$$

Which can also be written in integral form as:

$$X_t = \underbrace{X_0}_{known} + \int_0^t a\,(t,\ X_t)\,dt\ +\ \int_0^t b\,(t,\ X_t)\,dW_t$$

The first integral part is the ordinary integral. Since $t \longmapsto W_t$ is of infinite variantion almost surely, $\int_0^t b\,(t,\ X_t)\,dW_t$ can thus not be considered as an ordinary integral!

Stochastic integral with respect to Brownian motion Assumption: [0, *T*] decomposed into subintervals

$$0 = t_0 < t_1 < \ldots < t_N = T,\ \xi_i \in [t_{i-1},\ t_i),\ i = 1,\ \ldots,\ N$$

Suppose that $b\,(t,\ X_t)$ is a constant function:

$$\int_0^T bdW_t = \sum_{i=1}^N b\left(W_{t_i} - W_{t_{i-1}}\right) = b\sum_{i=1}^N \left(W_{t_i} - W_{t_{i-1}}\right) = b\,(W_T - W_0) = bW_T$$

Suppose that $b\,(t,\ X_t)$ is a simple step-function:

$$b\,(t) = \sum_{i=1}^N f\,(\xi_i)\,\mathbb{1}_{\,[t_{i-1}, t_i)}\,(t)\,,\ t \in \mathbb{R}^+$$

i.e. the function $b\,(t)$ takes the value $f\,(\xi_i)$ on the interval $[t_{i-1},\ t_i)$

Definition (L^p spaces)
If $(\Omega, \mathscr{F}, \mathbb{P})$ is a given (σ-finite) probability space, $L^p\,(\Omega, \mathscr{F}, \mathbb{P})$ is defined for $1 \leqslant p \leqslant \infty$ by

$$L^p\,(\Omega, \mathscr{F}, \mathbb{P}) : \ = \{f : \Omega \longrightarrow \{\mathbb{R},\ \mathbb{C}\}\,,\ f\ measurable\ and\ \int_\Omega |f\,(x)|^p\,dP\ < \infty\}$$

With the L^p seminorm

$$\|\cdot\|_{L^p} :\ = \left(\int_\Omega |f\,(x)\,|^p dP\right)^{\frac{1}{p}}$$

Remark: *The set of simple step functions $b\,(t)$ of the form above is a linear space which is dense in $L^2\,(\mathbb{R}^+)$* for the norm

$$\|\ b\,(t)\ \|_{L^2} = \sqrt{\int_0^\infty \mid b\,(t)\mid^2\,dt} < \infty,\ b\,(t) \in L^2\,(\mathbb{R}^+)$$

Classical integral of $b\,(t)$, where c_i could be specific function or constant: :

$$\int_0^\infty b\,(t)\,dt = \sum_{i=1}^N f\,(\xi_i)\,(t_i - t_{i-1}) :\ = \sum_{i=1}^N c_i\,(t_i - t_{i-1})$$

Can be adapted to a stochastic integration with respect to Brownian motion.
Thus, the stochastic integral with respect to the Brownian motion W_t of the simple step function $b\,(t)$ is defined by:

$$\int_0^\infty b\,(t)\,dW_t = \sum_{i=1}^N c_i\left(W_{t_i} - W_{t_{i-1}}\right)$$

Proposition: *The stochastic integral $\int_0^\infty b\,(t)\,dW_t$ defined above has a Gaussian distribution, which only holds if the integrand is deterministic*

$$\int_0^\infty b\,(t)\,dW_t \sim N\left(0, \int_0^\infty \mid b\,(t)\mid^2\,dt\right)$$

With zero mean $E[\int_0^\infty b\,(t)\,dW_t] = 0$ and variance given by the Itô isometry

$$Var\left[\int_0^\infty b\,(t)\,dW_t\right] = E\left[\left(\int_0^\infty b\,(t)\,dW_t\right)^2\right] = \int_0^\infty \mid b\,(t)\mid^2\,dt$$

Suppose that $b\left(t,\ X_t\right)$ now is any function in $L^2\left(\mathbb{R}^+\right)$. Gaussianity of the stochastic integral has nothing to do with the form of $b\left(t, X_t\right)$

Definition ($\mathscr{F}_t - adapted$)
A random variable X is said to be $\mathscr{F}_t - measurable$ if the knowledge of X depends only on the information up to time t. A process $\left(X_t\right)_{t\in\mathbb{R}^+}$ is said to be $\mathscr{F}_t - adapted$ if $\mathscr{F}_t - measurable$ for all $t \in \mathbb{R}^+$

Let $b\left(t,\ X_t\right)$ be a $\mathscr{F}_t - adapted$ process in the space $L^2\left(\Omega, \left[0,\ T\right]\right)$, which denote the space of stochastic process X_t such that

$$\mid X_t \mid_{L^2(\Omega,\ [0,\ T])}:\ = \sqrt{E\left[\int_0^T \mid X_t \mid^2 dt\right]} < \infty$$

1.4 Itô rules

Applying the Taylor expansion we have:

$$df\left(W_t\right) = f'\left(W_t\right)dW_t + \frac{1}{2}f''\left(W_t\right)\left(dW_t\right)^2 + \frac{1}{3!}f'''\left(W_t\right)\left(dW_t\right)^3 + \dots$$

However, from the construction of Brownian motion by it's small increments we see that

$$dW_t = O\left(\sqrt{dt}\right) \Rightarrow \left(dW_t\right)^2 \approx dt$$

Besides,

$$\left(dW_t\right)^3 = O\left(dt^{\frac{3}{2}}\right) \approx 0$$
$$\left(dW_t\right)^4 = O\left(dt^2\right) \approx 0$$

Then we let $\left(X_t\right)_{t\in\mathscr{R}^+}$ be Itô process of the form

$$dX_t = a\left(t,\ X_t\right)dt + b\left(t,\ X_t\right)dW_t$$

or

$$X_t = X_0 + \int_0^t a\left(s,\ X_s\right)ds\ +\ \int_0^t b\left(t,\ X_s\right)dW_s$$

Where $a\left(t,\ X_t\right)$, $b\left(t,\ X_t\right)$ are square-integrable adapted process $\left(L^2\left(\Omega,\ \mathscr{R}^+\right)\right)$. For any $f \in C^2\left(\mathscr{R}^+ \times \mathscr{R}\right)$ we have

$$\begin{aligned} df\left(t,\ X_t\right) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}b^2\left(t,\ X_t\right)dt \\ &= \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}a\left(t,\ X_t\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}b^2(t,\ X_t)\right]dt + \frac{\partial f}{\partial x}b\left(t,\ X_t\right)dW_t \end{aligned}$$

1.5 The numerical methods for SDE

In the time space $[0, T]$, we give a partition δ,

$$0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T$$

and

$$\Delta t_n = t_{n+1} - t_n, \Delta W_n = W_{t_{n+1}} - W_{t_n}, \delta = \max_n \Delta t_n$$

At time t_n, we let X^n be the numerical estimator of the X_{t_n}, the common numerical methods for SDE are blow:

1. Euler Method:

$$X^{n+1} = X^n + b\left(t_n, X^n\right)\Delta t_n + \sigma\left(t_n, X^n\right)\Delta W_n$$

2. $It\hat{o} - Taylor Expansion$

$$
\begin{gathered}
X^{n+1}=X^n+b\left(t_n, X^n\right) \Delta t_n+\sigma\left(t_n, X^n\right) \Delta W_n \\
+\mathcal{L} b\left(t_n, X^n\right) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \mathrm{~d} r \mathrm{~d} s+\mathcal{L}^1 b\left(t_n, X^n\right) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \mathrm{~d} W_r \mathrm{~d} s \\
+\mathcal{L} \sigma\left(t_n, X^n\right) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \mathrm{~d} r \mathrm{~d} W_s+\mathcal{L}^1 \sigma\left(t_n, X^n\right) \int_{t_n}^{t_{n+1}} \int_{t_n}^s \mathrm{~d} W_r \mathrm{~d} W_s
\end{gathered}
$$

where

$$
\mathcal{L}=\frac{\partial}{\partial t}+b \frac{\partial}{\partial x}+\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2}, \quad \mathcal{L}^1=\sigma \frac{\partial}{\partial x}
$$

3. Runge-Kutta Method

$$
\begin{gathered}
X^{n+1}=X^n+b\left(t_n, X^n\right) \Delta t_n+\sigma\left(t_n, X^n\right) \Delta W_n+\tfrac{1}{2}\left(\sigma\left(t_n, \hat{X}_n\right)-\sigma\left(t_n, X^n\right)\right)\left(\Delta W_n^2-\Delta t_n\right) / \sqrt{\Delta t_n} \\
\hat{X}^n=X_n+\sigma\left(t_n, X_n\right) \sqrt{\Delta t_n}
\end{gathered}
$$

Algorithm 1: Euler-Maruyama method

for $i = 0, ..., m-1$ **do**
 $\Delta W = \sqrt{h} Z$, with $Z \sim N(0,1)$
 $y_{j+1} := y_j + a(t_j, y_j)h + b(t_j, y_j)\Delta W$
end

Algorithm 2: Milstein method

for $i = 0, ..., m-1$ **do**
 $\Delta W = \sqrt{h} Z$, with $Z \sim N(0,1)$
 $y_{j+1} = y_j + a\left(t_j, y_j\right) h + b\left(t_j, y_j\right) \Delta W_j(\omega) + \frac{1}{2} b'\left(t_j, y_j\right) b\left(t_j, y_j\right)\left(\left(\Delta W_j(\omega)\right)^2 - h\right)$
end

Algorithm 3: RK method

for $i = 0, ..., m-1$ **do**
 $\Delta W = \sqrt{h} Z$, with $Z \sim N(0,1)$
 $\hat{y} := y_j + a\left(t_j, y_j\right) h + b\left(t_j, y_j\right) \sqrt{h}$
 $y_{j+1} := y_j + a\left(t_j, y_j\right) h + b\left(t_j, y_j\right) \Delta W + \frac{1}{2\sqrt{h}}\left(b\left(t_j, \hat{y}\right) - b\left(t_j, y_j\right)\right)\left((\Delta W)^2 - h\right)\Big)$
end

2 Backward SDE

Primarily motivated by financial problems, backward stochastic differential equations (BSDEs) were developed at high speed during the 1990s. Comparing with Black-Scholes models, Comparing with Black-Scholes models, the BSDEs are more powerful in financial derivative pricing and risk analysis. To solve many problems in mathematical finance, BSDEs have become powerful mathematical tools. Many problems of option pricing and stochastic optimizations are deal with. For stochastic differential equations (SDEs), BSDEs are terminal value problems. A natural time discretization of the BSDEs works backward in time. Yet, the solution must be adapted to deal with the information. It increases forwards in time, and makes numerical solutions to the BSDEs, become a more challenging problem. Thus, numerical solutions of them have had more and more attention in recent years.

2.1 Comparison SDE with ODE

Backward SDE introduces a new equation structure. In order to facilitate the understanding of this theory, I intend to take a look at the well-known ordinary differential equations before discussing it.

Consider these two ordinary differential equation defined on $[0,\ T]$ blow, where $T > 0$ is a given terminal time:

$$\begin{cases} \dot{X}\,(t) = b\,(X\,(t))\,, \quad 0 \leqslant t \leqslant T \\ X\,(0) = x_0 \end{cases} \tag{2.1}$$

$$\begin{cases} \dot{Y}\,(t) = g\,(Y\,(t))\,, \quad 0 \leqslant t \leqslant T \\ Y\,(T) = y_t \end{cases} \tag{2.2}$$

With $b\,(\cdot)\,,\ g\,(\cdot)$ are given function, $x_0,\ y_T$ are given values.

The boundary condition of equation (2.1) given at the initial time t = 0, and we call it a forward ordinary differential equation. The boundary condition of (2.2) given at the terminal time t = T. We call it the backward ordinary differential equations. Mathematically, the processing methods of (2.1) and (2.2) are basically the same. For example, under certain conditions (such as Lipschitz conditions) these two equations have unique solutions. But from the perspective of application In other words, these two equations are already significantly different. In fact, the uniqueness of (2.1) means that as long as the initial state x_0 of the system is known, the values can be calculated at any time $t \in [0,\ T]$. In contrast, (2.2) exists only It means that we can calculate what kind of initial values we should have to make the system reach the predetermined value y_T.

The above two models (2.1) and (2.2) actually assume that the system is a situation without random interference (that is, deterministic System). For stochastic systems, the difference between the two is not only in the sense of application. The mathematical structure of the equation There have been substantial difference.

At first, the forward stochastic differential equation will be:

$$\begin{cases} dX_t = b\,(t,\ X_t)\,dt\ +\ \sigma\,(t,\ X_t)\,dW_t \\ X_0 = x_0 \end{cases} \tag{2.3}$$

Where W_t is the $d-dimentional$ Brownian motion. It represents d independent interference sources. A typical example is the price of stocks: the price of the $i-th$ stock at time t can be described by the $i-th$ component of X_t. We can intuitively explain the solution X_t in this way: the system starts from the given initial value x_0 at time $t = 0$. According to the equation, its state X_T at the future time T is a random variable. At the present time $t = 0$, we cannot determine the value of X_T. Only when T changes to the "present moment" over time We can observe the exact value of X_T (the price of stock is a good example). The stochastic process with the above characteristics is called (relative to the above Brownian motion) adapted process.

We could immediately realize that this adaptability requirement makes the meaning of the existence and uniqueness of the solutions of stochastic differential equations and ordinary differential equations very different: the existence and uniqueness of the solutions in the presence of random interference does not mean that it can be passed. Below we turn to consider the generalization of the ordinary differential equation (2) with random, that is, the backward stochastic differential equation. We still require the solution of the equation to be adaptive. For a better understanding, I will give a very simple example of discrete time, which is very typical in financial mathematics.

Example:
Suppose that there are a bond and a stock. Bonds are risk-free: buy a bond of 1 Euro today. Tomorrow you can get 1.2 Euro with a profit. And stock is risky: buy 1 Euro stock today. Whether you make a profit tomorrow depends on luck: if it is a lucky day it is worth 1.4 Euro; but if it is not a lucky day it is only worth 1 Euro. Suppose an investor sets a "goal" for tomorrow: If tomorrow is a lucky day, he will get a Euro, if tomorrow is not a lucky day, he will get b Euro.

Question: How much does he need to invest today to achieve this "goal"?

Solution: Suppose he needs to invest y Euro today, where z Euro is used to buy stock (thus y-z Euro to buy bonds). Then we have the following euqations

$$\begin{cases} 1.2y + 0.2z = a \\ 1.2y - 0.2z = b \end{cases} \tag{2.4}$$

Obviously this system of equations has a unique solution:

$$\begin{cases} y = \frac{5}{12}(a+b) \\ z = \frac{5}{2}(a-b) \end{cases} \tag{2.5}$$

Notice: The significance of the uniqueness of the solution: if investors want to achieve tomorrow's financial plan ,then their decision must include two parts $(y,\ z)$: he not only has to decide today's total investment y, but also must use the z, which is the risk part (portfolio in finance terms), to buy stocks.

Although the above example is simple, it can also reflect the essence of the problem to be dealt with by the backward stochastic differential equation. Although the investor cannot predict his income tomorrow (it is still a random variable, it will not be known until tomorrow). But despite this, the investor can still calculate exactly what he should do today to achieve tomorrow's uncertain return (note that although the issue of "uncertain return" is dealt with here, in fact the "determined return problem" is also it. The special case of $a = b$. What is interesting is that the risk part of investment is zero at this time.).

2.2 Backward SDE

The important class of the BSDEs are the Itô's type equations such as

$$\begin{cases} dY_t = -f(t,\ Y_t,\ Z_t)\,dt\ +\ Z_t dW_t \\ Y_T = \xi \end{cases}$$

where $t \in [0,\ T]$, W is a Brownian motion and $(\xi,\ f)$ are given. Here Z_t is a predictable process, f is called the generator or the driver, $Y_T = \xi$ is the terminal condition.
Suppose $(Y,\ Z)$ is the solution of standard BSDE. The equations can be interpreted as a stochastic integral equation of the form:

$$Y_t = \xi + \int_t^T f(s,\ Y_s,\ Z_s)\,ds - \int_t^t Z_s dW_s,\ 0 \le t \le T$$

where W is a brownian motion and $(\xi,\ f)$ are given. Then the solution satisfies,

$$Y_t = Y_{t_0} - \int_t^{t_0} f(s,\ Y_s,\ Z_s)\,ds\ +\ \int_t^{t_0} Z_s dW_s,\ 0 \leqslant t,\ t_0 \le T$$

2.3 BSDE application in fiance

In financial markets, there is a large variety of securities whose payoff depends in a nonlinear way on the underlying assets, and usually also on other factors. We call such financial instruments contingent claims, derivative securities, option.

The BSDEs are widely used in many financial problems. Due to the structure of option pricing problems, it allows to be solved by numerical methods. In the same time, we can have confidence intervals, lower and upper bias of numerical solutions. For general nonlinear BSDEs, a similar way, to construct lower and upper approximations, is not available. Nonetheless, the intricate interplay, between the time discretization and the design of the estimation, persists for nonlinear BSDEs in a similar way as for the option problems. The BSDEs appear in many financial problems, for example, the pricing and the hedging of some options. These options include call option, put option, Europe option, American option, lookback options, digital options and compound option, among others.

Example: (*BSDE for option pricing*)

- Risk-free asset (bonds): $dP_t = rP_tdt,\ with\ initial\ value\ P_0 \Rightarrow P_t = P_0exp\left(rt\right)$
- Risky asset (stocks): $dS_t = \mu S_td_t\ +\ \sigma S_tdW_t$
- The number pf stock: a_t
- The number of risk-free asset: b_t

With replication strategy to asset the price of option:$V_t = a_tS_t + b_tP_t$

Self-financing strategy
We assume at time $t_0 = 0$, the price of option is $V_0 = a_0S_0 + b_0P_0$. If we do nothing, at terminal time $t_T = 1$, the price will be $V_T = a_0S_1 + b_0P_1$.However, we could also adjust our investment during the whole time. Then we have $V_T = a_1S_1 + b_1P_1$. Self-financing strategy require that whatever you do, the price of option all the way must be the same.

Therefore, we will get:

$$a_0S_1 + b_0P_1 = a_1S_1 + b_1P_1 \Longrightarrow \underbrace{\left(a_1 - a_0\right)}_{da_t}S_1 + \underbrace{\left(b_1 - b_0\right)}_{db_t}P_1 = 0$$

Then,

$$dV_t = d\left(a_tS_t\right) + d\left(b_tP_t\right) = a_tdS_t + S_tda_t + da_tdS_t + b_tdP_t + P_tdb_t + dP_tdb_t$$

According to the Itô rules:

$$da_tdS_t = 0,\ dP_tdb_t = 0$$

We let,

$$A_1 = S_tda_t,\ A_2 = P_tdb_t$$

According to the Self-financing,

$$S_t\underbrace{\left(a_t - a_{t-}\right)}_{da_t} + P_t\underbrace{\left(b_t - b_{t-}\right)}_{db_t} = 0 \Longrightarrow A_1 + A_2 = 0$$

Then we have,

$$dV_t = \underbrace{a_tdS_t}_{value\ from\ risky\ asset} + \underbrace{b_tdP_t}_{value\ from\ riskless\ asset}$$

$$dV_t = b_trP_td_t + a_tS_t\left(\mu dt + \sigma dW_t\right)$$

We let,

$$a_tS_t :\ = \pi_t,\ b_tP_t = V_t - \pi_t$$
$$\Longrightarrow dV_t = r\left(V_t - \pi_t\right)dt\ +\ \pi_t\left(\mu dt\ +\ \sigma dW_t\right)$$

According to the BSDE's definition, we let $Z_t = \sigma dW_t$
Then we get,

$$dV_t = (rV_t + \tfrac{(\mu - r)Z_t}{\sigma})dt + Z_tdW_t$$

Finally, we have the equation,

$$\begin{cases} -dY_t = \left(-rY_t - \frac{(\mu-r)Z_t}{r}\right)dt - Z_tdW_t \\ Y_T = \underbrace{\left(S_T - K\right)^+}_{European\ call\ option}\quad or \quad \underbrace{\left(K - S_T\right)^+}_{European\ put\ option} \end{cases}$$

$\rightsquigarrow\ to\ find\ the\ solution\left(Y_t,\ Z_t\right)$

- Strike price: K
- Option price: Y_t
- Hedging portfolios: $Z_t = \sigma\pi_t = \sigma a_tS_t\ \left(a_t\ is\ the\ numer\ of\ risky\ asset,\ which\ is\ also\ called\ Delta\right)$

3 Forward-Backward SDE

3.1 Decoupled Forward-Backward SDE

The solution of general form of FBSDEs is complicated, but for the decoupled FBSDEs:

$$\begin{cases} X_t = x_0 + \int_0^t b\,(s,\ X_s)\,ds + \int_0^t \sigma\,(s,\ X_s)\,dW_s \\ Y_t = \varphi\,(Y_T) + \int_t^T f\,(s,\ X_s,\ Y_s,\ Z_s)\,ds - \int_t^T Z_s dW_s \end{cases} \tag{3.1}$$

As we can see, if the function $b = 0,\ \sigma = 1$, the form of FBSDEs will be:

$$\begin{cases} dX_t = dW_t \\ dY_t = f\,(t,\ W_t,\ Y_t,\ Z_t)\,dt - Z_t dW_t \end{cases} \tag{3.2}$$

which is called pure BSDE

Definition Let X_t be a diffusion process in $\mathbb{R}^q$. The generator A_t^x of X_t on $g \in C^{1,2}$ is defined by

$$A_t^x g(t,x) = \lim_{s \downarrow t} \frac{\mathbb{E}_t^x\left[g\,(s, X_s)\right] - g(t,x)}{s-t}, \quad x \in \mathbb{R}^n$$

Concerning the generator A_t^x, we have the following result:

Where $b,\ \sigma,\ \varphi$ and f are deterministic functions, then with some certain smooth conditions, the solution $(Y_t,\ Z_t)$ will be:

$$Y_t = u\,(t,\ X_t)\,,\ Z_t = \nabla_x u\,(t,\ X_t)\,\sigma\,(t,\ X_t)$$

And the $u = u\,(t,\ x)$, is the classical solution of the parabolic PDEs:

$$\tfrac{\partial u}{\partial t} + \textstyle\sum_{i=1}^d b_i \tfrac{\partial u}{\partial x_i} + \tfrac{1}{2}\sum_{i,j=1}^d a_{i,j} \tfrac{\partial^2 u}{\partial x_i \partial x_j} + f\,(t, x, u, \nabla u\sigma) = 0$$

Where, the terminal condition $u\,(T, x) = \varphi\,(x)\,,\ a = \sigma\sigma^\intercal$, and the $\nabla_x u$ is the gradient of x in $u\,(t,x)$

Moreover, we have the following **nonlinear Feynman-Kac formula**:

If the PDE

$$L_{t,x}^0 u(t,x) + f(t,x,u(t,x),\nabla u(t,x)\sigma(t,x)) = 0$$

with termial condition $u(T,x) = \varphi(x)$ has a classical solution $u(t,x) \in C^{1,2}$, then the unique solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ of the Markovian decoupled FBSDEs (3.1) with $\xi = \varphi\left(X_T^{t,x}\right)$ can be represented as

$$Y_s^{t,x} = u\left(s, X_s^{t,x}\right), \quad Z_s^{t,x} = \nabla_x u\left(s, X_s^{t,x}\right)\sigma\left(s, X_s^{t,x}\right), \quad \forall s \in [t,T)$$

where $\nabla_x u$ denotes the gradient of u with respect to the spacial variable x.

3.2 Coupled Forward-Backward SDE

Definition (Forward-Backward SDE) *In the complete filtered probability space $(\Omega,\ \mathscr{F},\ \mathbb{P},\ \{\mathscr{F}_t\}_{0\le t\le T})$ coupled FBSDEs has the following form:*

$$\begin{cases} dX_t = b\,(t,\ X_t,\ Y_t,\ Z_t)\,dt + \sigma\,(t,\ X_t,\ Y_t,\ Z_t)\,dW_t \\ X\,(t_0) = x_0 \\ dY_t = \varphi\,(t,\ X_t,\ Y_t)\,dt - f\,(t,\ X_t,\ Y_t,\ Z_t)\,dt + Z_t dW_t \\ Y_T = \varphi\,(X_T) = \xi \end{cases} \tag{3.3}$$

where $b,\ \sigma,\ \varphi$ and f are deterministic functions
we consider the following integral representation formula of the coupled FBSDEs:

$$\begin{gathered} X_t = x_0 + \int_0^t b\,(s, X_s, Y_s, Z_s)\,ds + \int_0^t \sigma\,(s, X_s, Y_s, Z_s)\,dW_s \\ Y_t = \xi + \int_t^T f\,(s, X_s, Y_s, Z_s)\,ds - \int_t^T Z_s dW_s \end{gathered} \tag{3.4}$$

where, $x_0 \in \mathscr{F}_0,\ \xi \in \mathscr{F}_T$, W_t is d-dimentional standard Brownian motion

- $b : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}^{n \times d}$
- $\sigma : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}^{n \times d}$
- $f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}^m$
- $X_t \in \mathbb{R}^n,\ Y_t \in \mathbb{R}^m,\ Z_t \in \mathbb{R}^{m \times d}$ *are the processes to be sought*
- $L_T^2 (\mathbb{R}^m) :\ space\ of\ \mathscr{F}_t - measurable\ random\ variables,\ \xi\ satisfying\ E\left[\mid \xi \mid^2\right] < \infty$
- $H_T^2 (\mathbb{R}^m) :\ space\ of\ predictable\ process,\ Y_t\ satisfying\ E\left[\int_0^T \mid Y_t \mid^2 dt\right] < \infty$

Here, a triple of processes $(X,\ Y,\ Z) : [0, T] \times \Omega \longrightarrow \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ is called an ordinary adapted solution of the forward-backward SDEs, if it is $\{\mathscr{F}_t\} - adapted$ and square-integrable, such that it satisfies $P - almostsurely.$

3.3 Existence and uniqueness

All the notations of solutions here are actually strong solutions. The FBSDE has a unique solution $(X,\ Y,\ Z)$ in $H_T^2 (\mathbb{R}^n) \times H_T^2 (\mathbb{R}^m) \times H_T^2 \left(\mathbb{R}^{m \times d}\right)$ satisfying

$$\begin{gathered} X_t = x_0 + \int_0^t b\left(s,\ X_s,\ Y_s,\ Z_s\right) ds + \int_0^t \sigma\left(s,\ X_s,\ Y_s,\ Z_s\right) dW_s \\ Y_t = \xi + \int_t^T f\left(s,\ X_s,\ Y_s,\ Z_s\right) ds - \int_t^T Z_s dW_s \end{gathered}$$

if

- $\xi \in L_T^2 (\mathbb{R}^m), \xi \in \mathcal{F}_T\ ,\ E\left[\mid \xi \mid^2\right] < \infty$
- $f \in H_T^2 (\mathbb{R}^m)$
- *for fixed* $X,\ Y,\ Z,\ f\left(t,\ X,\ Y,\ Z\right) \in \mathcal{F}_t$
- f *is uniformly Lipschitz in* (X, Y, Z)

$$\mid f\left(t,\ X,\ Y,\ Z\right) - f\left(t,\ X',\ Y',\ Z'\right) \mid \leq\ L\left(\mid X - X' \mid + \mid Y - Y' \mid + \mid Z - Z' \mid\right)$$

3.4 FBSDEs application in finance

In this section i would like also to discuss option pricing problems in finance and their relationship with FBSDEs. Consider a security market that contains, say, one bond and one stock. suppose that their prices are subject to the following system of stochastic differential equations:

$$\begin{cases} dP_t = r_t P_t dt & ,bond \\ dS_t = S_t b_t dt + P_t \sigma_t dW_t & ,stock \end{cases}$$

Where r_t is the *interest rate* of the bond, b_t and σ_t are the *appreciation rate* and *volatility* of the stock, respectively. Now suppose that the agent sells the option at price y and then invests it in the market, $Y_0 = y$. Assume that at each time t the agent invests a portion of his wealth, say π_t, called portfolio, into the stock, and puts the rest $(Y_t - \pi_t)$ into the bond. It can be shown in the following FBSDE(an *decoupled* FBSDE, to be more precise),

$$\begin{cases} dS_t = S_t b_t dt + S_t \sigma_t dW_t \\ dY_t = \left(r_t Y_t + Z_t \theta_t\right) dt + Z_t dW_t \\ S_0 = s_0\ ,\ Y_T = g\left(S_T\right) \end{cases}$$

Where $Z_t = \pi_t \sigma_t,\ \theta_t = \frac{b_t - r_t}{\sigma_t},\ g(\cdot)$ is the payoff function of the option. The derivation is consistent with self-financing strategy.

4 Θ-method Schemes for the FBSDEs

4.1 Numerical scheme for FBSDEs

For simplicity, i would like to discuss the one-dimensional BSDE. For the real axis $\mathbb{R}$, we introduce a space partition $\Re_h$ as

$$\mathfrak{R}_h=\left\{x_i \mid x_i \in \mathbb{R}, i \in \mathbb{Z}, x_i<x_{i+1}, \lim_{i \rightarrow+\infty}=+\infty, \lim_{i \rightarrow-\infty}=-\infty\right\}$$

where $\mathbb{Z}$ is the set of all integer numbers, and all x_i 's are deterministic. We use $h_i=x_{i+1}-x_i$ to denote the space step and $h=\max_{i\in\mathbb{Z}} h_i$ to denote the maximum space step. For the time interval $[0,T]$, we introduce the time partition

$$\Re_{th}=\{t_i \mid t_i \in [0,T], i=0,1,\ldots,N_T, t_i<t_{i+1}, t_0=0, t_{N_T}=T\}$$

Where all t_n 's are deterministic. Let $\Delta t_n=t_{n+1}-t_n$ be the time step, and let $\Delta t=\max_{0\leq n<N_T}\Delta t_n$ be the maximum time step.
The Backward SDE:

$$y_t=\varphi\left(W_T\right)+\int_t^T f\left(s, y_s, z_s\right) dr-\int_t^T z_s dW_s, \quad t \in[0, T] \tag{4.1}$$

Let (y_t, z_t) be the adapted solution of the BSDE (4.1), then we have

$$y_{t_n}=y_{t_{n+1}}+\int_{t_n}^{t_{n+1}} f(s,\ y_s,\ z_s) ds-\int_{t_n}^{t_{n+1}} z_s dW_s \tag{4.2}$$

Take the mathematical expectation $E_{t_n}^x[\cdot]=E\left[\cdot|\mathscr{F}_{t_n},\ x_{t_n}=x\right]$ we can get,

$$y_{t_n}=E_{t_n}^x\left[y_{t_{n+1}}\right]+\int_{t_n}^{t_{n+1}} E_{t_n}^x\left[f\left(s,\ y_s,\ z_s\right)\right] ds \tag{4.3}$$

Post-multiply $\Delta W_{n,1}$ on the (4.1), and then take the expectation $E_{t_n}^x[\cdot]$

$$0=E_{t_n}\left[y_{t_{n+1}} \Delta W_{n, 1}\right]+\int_{t_n}^{t_{n+1}} E_{t_n}\left[f\left(s, y_s, z_s\right) \Delta W_{n, 1}\right] ds-E_{t_n}\left[\int_{t_n}^{t_{n+1}} z_s dW_s \Delta W_{n, 1}\right] \tag{4.4}$$

According to the properties of Brownian motion and Itô isometry formula,

$$0=E_{t_n}\left[y_{t_{n+1}} \Delta W_{n, 1}\right]+\int_{t_n}^{t_{n+1}} E_{t_n}\left[f\left(s, y_s, z_s\right) \Delta W_{n, 1}\right] ds-\int_{t_n}^{t_{n+1}} E_{t_n}\left[z_s\right] ds \tag{4.5}$$

We can see in the (4.3) and (4.5) the integrand in $\mathscr{F}_{t_n}$ is deterministic function under s. Thus, we could use certain numerical integration formula to do calculation and give the numerical methods for FBSDE at the same time.

With the knowledge of numerical integration theorem we know $g(t)$ on the interval $[0,T]$

$$\int_{t_n}^{t_{n+1}} g(t) dt=\left(\theta g\left(t_n\right)+(1-\theta) g\left(t_{n+1}\right)\right) \Delta t_n+Err_n \tag{4.6}$$

Where,

$$Err_n=\begin{cases}\mathcal{O}((\Delta t_n)^3) & ,\theta=\frac{1}{2}\\ \mathcal{O}((\Delta t_n)^2) & ,\theta\in[0,1]\ and\ \theta\neq\frac{1}{2}\end{cases}$$

Apply (4.6) on (4.3) and (4.5) we can get,

$$\begin{cases}y_{t_n}= & E_{t_n}^x\left[y_{t_{n+1}}\right]+\left(1-\theta_1\right) \Delta t_n E_{t_n}^x\left[f\left(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}\right)\right] \\ & +\theta_1 \Delta t_n f\left(t_n, y_{t_n}, z_{t_n}\right)+R_y^n \\ 0 = & E_{t_n}^x\left[y_{t_{n+1}} \Delta W_{n, 1}\right]+\left(1-\theta_2\right) \Delta t_n E_{t_n}^x\left[f\left(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}\right) \Delta W_{n, 1}\right] \\ & -\{(1-\theta_3) \Delta t_n E_{t_n}^x\left[z_{t_{n+1}}\right]+\theta_3 \Delta t_n z_{t_n}\}+R_z^n\end{cases} \tag{4.7}$$

Where, parameters $\theta_1,\ \theta_2,\ \theta_3\in[0,1]$, and $R_y^n,\ R_z^n$

$$\begin{cases} R_y^n = & \int_{t_n}^{t_{n+1}} \{E_{t_n}^x[f(s, y_s, z_s)] - (1-\theta_1) E_{t_n}^x[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})] \\ & -\theta_1 f(t_n, y_{t_n}, z_{t_n})\} ds \\ R_z^n = & \int_{t_n}^{t_{n+1}} E_{t_n}^x[f(s, y_s, z_s)\Delta W_{n,1}] ds \\ & -(1-\theta_2)\Delta t_n E_{t_n}^x[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})\Delta W_{n,1}] \\ & -\int_{t_n}^{t_{n+1}} \{E_{t_n}^x[z_s] - (1-\theta_3) E_{t_n}^x[z_{t_{n+1}}] - \theta_3 z_{t_n}\} ds \end{cases}$$

θ-method

Given random variable y^N, using the following equations to solve for $y^n,\ z^n,\ n = N-1,\ N-2,\ \dots,\ 1,\ 0$

$$\begin{cases} y_{t_n} & = E_{t_n}^x\left[y_{t_{n+1}}\right] + (1-\theta_1)\Delta t_n E_{t_n}^x\left[f\left(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}\right)\right] \\ & +\theta_1 \Delta t_n f\left(t_n, y_{t_n}, z_{t_n}\right) + R_y^n \\ -E_{t_n}^x\left[y_{t_{n+1}}\Delta W_{n,1}\right] & = (1-\theta_2)\Delta t_n E_{t_n}^x\left[f\left(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}\right)\Delta W_{n,1}\right] \\ & -\{(1-\theta_3)\Delta t_n E_{t_n}^x\left[z_{t_{n+1}}\right] + \theta_3 \Delta t_n z_{t_n}\} + R_z^n \end{cases} \tag{4.8}$$

$\rightsquigarrow (y^n,\ z^n)$ is the numerical solution of $(y_t,\ z_t)$

Notes: If we take different $\theta_i (i = 1, 2, 3)$, there will be different methods. For example, if $\theta_1 = 0,\ \theta_2 = \theta_3 = 1$, it is the **Euler explicit method**; If $\theta_1 = \theta_2 = \theta_3 = 1$ it is the **Euler implicit method**; if $\theta_1 = \theta_2 = \theta_3 = \frac{1}{2}$, it will be**Crank-Nicolson method**

4.2 Monte-Carlo Method to estimate the Conditional Expectation

If we want to calculate $(y^n,\ z^n)$ we need to simulate the conditional expectation and discrete the full space-time to estimate the value of $(y^n,\ z^n)$.

We let $x = x_i \in \mathbb{R}_h$ and $\Delta^k W_{n,1}\,(k = 1, 2, \cdots, N_E)$ as N_E values of $\sqrt{\Delta t_n} N(0,1)$ in (4.9) and (4.10). Where $N(0,\ 1)$ is the standard normal distribution ($\mu = 0,\ \sigma = 1$).
Then at time t_{n+1} we have,

$$x_i^{t_{n+1},k} = x_{t_n} + b\left(t_n,\ x_{t_n},\ y_{t_n},\ z_{t_n}\right)\Delta t_n + \sigma\left(t_n,\ x_{t_n},\ y_{t_n},\ z_{t_n}\right)\Delta_h^k W_{n,1}$$

Where $x_{t_n} = x_i \in \mathbb{R}_h$

We use the Monte-Carlo Method to estimate conditional expectation. The first expectation will be:

$$E_{t_n}^{x_i,t_n}\left[y_{t_{n+1}}\right] = E_{m,t_n}^{t_n,x_i}\left[y_{t_{n+1}}\right] + Err_y \tag{4.9}$$

Where,

$$E_{m,t_n}^{t_n,x_i}\left[y_{t_{n+1}}\right] = \frac{\sum_{k=1}^{N_E} y^{t_{n+1}}\left(x_i^{t_n+1,k}\right)}{N_E}$$

$$Err_y = E_{t_n}^{t_n,x_i}\left[y_{t_{n+1}}\right] - E_{m,t_n}^{t_n,x_i}\left[y_{t_{n+1}}\right]$$

Similarly, we can get,

$$E_{m,t_n}^{t_n,x_i}\left[y_{t_{n+1}}\Delta W_{n,1}\right] = \frac{\sum_{k=1}^{N_E} y^{t_{n+1}}\left(x_i^{t_n+1,k}\right)\Delta_h^k W}{N_E}$$

$$E_{m,t_n}^{t_n,x_i}\left[f(t_{n+1}, x_{t_{n+1}}, y_{t_{n+1}}, z_{t_{n+1}})\right] = \frac{\sum_{k=1}^{N_E} f\left(t_{n+1}, x_{t_{n+1}}, y^{t_{n+1}}\left(x_i^{t_{n+1},k}\right), z^{t_{n+1}}\left(x_i^{t_{n+1},k}\right)\right)}{N_E}$$

$$E_{m,t_n}^{t_n,x_i}\left[f(t_{n+1}, x_{t_{n+1}}, y_{t_{n+1}}, z_{t_{n+1}})\Delta W_{n,1}\right] = \frac{\sum_{k=1}^{N_E} f\left(t_{n+1}, x_{t_{n+1}}, y^{t_{n+1}}\left(x_i^{t_{n+1},k}\right), z^{t_{n+1}}\left(x_i^{t_{n+1},k}\right)\right)\Delta_h^k W_{n,1}}{N_E}$$

And for the error terms,

$$Err_{yw} = E_{t_n}^{t_n,x_i}\left[y_{t_{n+1}}\Delta W_{n,1}\right] - E_{m,t_n}^{t_n,x_i}\left[y_{t_{n+1}}\Delta W_{n,1}\right]$$

$$Err_{fw} = E_{t_n}^{t_n,x_i}\left[f\left(t_{n+1}, x_{t_{n+1}}, y_{t_{n+1}}, z_{t_{n+1}}\right)\Delta W_{n,1}\right] - E_{m,t_n}^{t_n,x_i}\left[f(t_{n+1}, x_{t_{n+1}}, y_{t_{n+1}}, z_{t_{n+1}})\Delta W_{n,1}\right]$$

$$Err_{f} = E_{t_n}^{t_n,x_i}\left[f\left(t_{n+1}, x_{t_{n+1}}, y_{t_{n+1}}, z_{t_{n+1}}\right)\right] - E_{m,t_n}^{t_n,x_i}\left[f(t_{n+1}, x_{t_{n+1}}, y_{t_{n+1}}, z_{t_{n+1}})\right]$$

$$Err_{z} = E_{t_n}^{t_n,x_i}\left[z_{t_{n+1}}\right] - E_{m,t_n}^{t_n,x_i}\left[z_{t_{n+1}}\right]$$

4.3 Gauss-Hermite quadrature Method to estimate the Conditional Expectation

The Gauss-Hermite quadrature rule is an extension of Gaussian quadrature method for approximating the value of intergrals of $\int_{-\infty}^{+\infty} e^{-x^2} g(x)\mathrm{d}x$ by

$$\int_{-\infty}^{+\infty} e^{-x^2} g(x)\mathrm{d}x \approx \sum_{j=1}^{L} \omega_j g\left(a_j\right) \tag{4.10}$$

where L is the number of sample points used in the approximation. The points $\{a_j\}_{j=1}^{L}$ are the roots of the Hermite polynomial $H_L(x)$ of degree L and $\{\omega_j\}_{j=1}^{L}$ are the corresponding weights:

$$\omega_j = \frac{2^{L+1} L! \sqrt{\pi}}{\left(H_L'\left(a_j\right)\right)^2} \tag{4.11}$$

The truncation error $R(g, L)$ of the Gauss-Hermite quadrature formula is

$$R(g, L) = \int_{-\infty}^{+\infty} e^{-x^2} g(x)\mathrm{d}x - \sum_{j=1}^{L} \omega_j g\left(a_j\right) = \frac{L!\sqrt{\pi}}{2^L (2L)!} g^{(2L)}(\bar{x}) \tag{4.12}$$

where $\bar{x}$ is a real number in $\mathbb{R}$. The Gauss-Hermite quadrature formula is exact for polynomial functions g of degree less than $2L-1$.

For a d -dimensional function $g(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d$, the Gauss-Hermite quadrature formula becomes

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(\mathbf{x}) e^{-\mathbf{x}^\tau \mathbf{x}} d\mathbf{x} \approx \sum_{\mathbf{j}=1}^{L} w_{\mathbf{j}} g\left(\mathbf{a}_{\mathbf{j}}\right) \tag{4.13}$$

where $\mathbf{x} = \left(x_1, \ldots, x_d\right)^\tau, \mathbf{x}^\tau \mathbf{x} = \sum_{j=1}^{d} x_j^2$, and

$$\mathbf{j} = \left(j_1, j_2, \ldots, j_d\right), \quad \omega_{\mathbf{j}} = \prod_{i=1}^{d} \omega_{j_i}, \quad \mathbf{a}_{\mathbf{j}} = \left(a_{j_1}, \ldots, a_{j_d}\right) \quad \sum_{\mathbf{j}=1}^{L} = \sum_{j_1=1,\ldots,j_d=1}^{L,\ldots,L} \tag{4.14}$$

It is well known that, for a standard d -dimensional normal random variable $N(0,1)$, it holds that

$$\mathbb{E}[g(N)] = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{-\infty}^{+\infty} g(\mathbf{x}) e^{-\frac{\mathbf{x}^\tau \mathbf{x}}{2}} \mathrm{d}\mathbf{x} = \frac{1}{(\pi)^{\frac{d}{2}}} \int_{-\infty}^{+\infty} g(\sqrt{2}\mathbf{x}) e^{-\mathbf{x}^\tau \mathbf{x}} \mathrm{d}\mathbf{x} \tag{4.15}$$

Then by (4.15), we get

$$\mathbb{E}[g(N)] = \frac{1}{(\pi)^{\frac{d}{2}}} \sum_{\mathbf{j=1}}^{L} w_{\mathbf{j}} g\left(\mathbf{a}_{\mathbf{j}}\right) + R_{\mathbb{E},L}^{GH}(g) \tag{4.16}$$

where $R_{\mathbb{E},L}^{GH}(g) = \mathbb{E}[g(N)] - \frac{1}{(\pi)^{\frac{d}{2}}} \sum_{\mathbf{j=1}}^{L} w_{\mathbf{j}} g\left(\mathbf{a}_{\mathbf{j}}\right)$ is the truncation error of the Gauss- Hermite quadrature rule for g.

Recall that, the conditional expectation $\mathbb{E}_{t_n}^{\mathbf{x}}\left[\bar{Y}^{n+j}\right]$ is approximated by $\mathbb{E}_{t_n}^{\mathbf{x},h}\left[I_{h,X^{n+j}}^{n+j} Y^{n+j}\right]$, where $\mathbb{E}_{t_n}^{\mathbf{x},h}[\cdot]$ is the approximation of $\mathbb{E}_{t_n}^{\mathbf{x}}[\cdot]$, and $I_{h,X^{n+j}}^{n+j} Y^{n+j}$ is the interpolation approximation of $\bar{Y}^{n+j}$. By the nonlinear Feynman-Kac formula $\bar{Y}^{n+j}$ is a function of $X^{n,j}$ and has the following explicit representation. $\bar{Y}^{n+j} = Y^{n+j}\left(X^{n+j}\right) = Y^{n+j}\left(X^{n} + b\left(t_{n}, X^{n}\right)\Delta t_{n,j} + \sigma\left(t_{n}, X^{n}\right)\Delta W_{n,j}\right)$ where $\Delta W_{n,j} \sim \sqrt{\Delta t_{n,j}} N\left(0, I_{d}\right)$ is a d-dimensional Gaussian random variable. Thus we define the approximation $\mathbb{E}_{t_n}^{\mathbf{x},h}\left[\bar{Y}^{n+j}\right]$ by

$$\mathbb{E}_{t_n}^{\mathbf{x},h}\left[\bar{Y}^{n+j}\right] = \frac{1}{(\pi)^{\frac{d}{2}}}\sum_{\mathbf{j}=\mathbf{1}}^{L} w_{\mathbf{j}} FY\left(\mathbf{a}_{\mathbf{j}}\right) \tag{4.17}$$

where $FY = FY(y) = Y^{n+j}\left(\mathbf{x} + b\left(t_{n}, \mathbf{x}\right)\Delta t_{n,j} + \sigma\left(t_{n}, \mathbf{x}\right)\sqrt{2\Delta t_{n,j}}y\right)$. We denote such approximation error by $R_{\mathbb{E}_{t_n}^{\mathbf{x}},L}^{GH}(FY)$ Similarly, we have

$$\mathbb{E}_{t_n}^{\mathbf{x},h}\left[\bar{Y}^{n+j}\left(\Delta W_{t_n,j}\right)^{\tau}\right] = \frac{1}{(\pi)^{\frac{d}{2}}}\sum_{\mathbf{j}=\mathbf{1}}^{L} w_{\mathbf{j}} FYW\left(\mathbf{a}_{\mathbf{j}}\right) \tag{4.18}$$

where $FYW = FYW(y) = Y^{n+j}\left(\mathbf{x} + b\left(t_{n}, \mathbf{x}\right)\Delta t_{n,j} + \sigma\left(t_{n}, \mathbf{x}\right)\sqrt{2\Delta t_{n,j}}y\right)y$. The approximation error is denoted by $R_{\mathbb{E}_{n_n}^{\mathbf{x}},L}^{GH}(FYW)$. By (4.14), we have the following estimates

$$R_{y,n}^{k,\mathbb{E}} = \mathcal{O}\left(\frac{L!}{2^{L}(2L)!}\right), \quad R_{z,n}^{k,\mathbb{E}} = \mathcal{O}\left(\frac{L!}{2^{L}(2L)!}\right) \tag{4.19}$$

4.4 The algorithm for θ method to solve FBSDEs

We could apply them into the (4.8),

$$\begin{cases} y_{t_n} & = E_{m,t_n}^{t_n,x_i}\left[y_{t_{n+1}}\right] + \left(1-\theta_1\right)\Delta t_n E_{m,t_n}^{t_n,x_i}\left[f\left(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}\right)\right] \\ & +\theta_1 \Delta t_n f\left(t_n, y_{t_n}, z_{t_n}\right) + Err_y + Err_f \\ -E_{t_n}^{x}\left[y_{t_{n+1}}\Delta W_{n,1}\right] & = \left(1-\theta_2\right)\Delta t_n E_{m,t_n}^{t_n,x_i}\left[f\left(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}\right)\Delta W_{n,1}\right] \\ & -\{(1-\theta_3)\Delta t_n E_{m,t_n}^{t_n,x_i}\left[z_{t_{n+1}}\right] + \theta_3 \Delta t_n z_{t_n}\} + Err_{fw} + Err_z \end{cases} \tag{4.20}$$

According to the probability theory we can know the error terms of Monte-Carlo simulation Err_y, Err_z, Err_f, Err_{yw}, Err_{fw} will converge to 0 with $\frac{1}{\sqrt{N_E}}\left(N_E \longrightarrow \infty\right)$. Thus, we could control the error by choosing proper N_E.

example for the explicit method: If we let $\theta_1 = 0,\ \theta_2 = 1,\ \theta_3 = 1$ and ignore the error terms,

$$y_{t_n} = E_{m,t_n}^{t_n,x_i}\left[y_{t_{n+1}}\right] + \Delta t_n E_{m,t_n}^{t_n,x_i}\left[f\left(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}\right)\right] \tag{4.21}$$

$$0 = E_{m,t_n}^{t_n,x_i}[y_{t_{n+1}}\Delta W_{n,1}] - \Delta t_n z_{t_n} \tag{4.22}$$

Then we have following steps to find the numerical solution,

Step 1. Let $n = N_T - 1$, then we have $y_{t_{n+1}} = y_{N_T} = y_T = \xi$

Step 2. Discretise the space, then for every points x_i, $y_{t_{n+1}}\left(x_i\right)$ is known

Step 3. Use the Monte-Carlo Method to approximate the conditional expectations.

Step 4. Solve z_{t_n} with (4.24)

Step 5. Solve (4.23) to get the values of y_{t_n}

Step 6. Let $n = n - 1$, repeat the step (2) until $n = 0$ to get the estimator $\left(y_0, z_0\right)$

Warning: In order to easily calculate, we let $\theta_1 = 0,\ \theta_2 = 1,\ \theta_3 = 1$ to eliminate the expectation, but it could increase the error at same time.

Reference: Weidong Zhao, Lifeng Chen, and Shige Peng, New Kind of Accurate Numerical Method for Backward Stochastic Differential Equations, SIAM J. Sci. Comput.,28(4), 1563-1581(19 pages)

5 Multi-step scheme for FBSDE

In this chapter, we are concerned with the high-order numerical methods for coupled forward-backward stochastic differential equations (FBSDEs). Based on the FBSDEs theory, we derive two reference ordinary differential equations (ODEs) from the backward SDE, which contain the conditional expectations and their derivatives. Then, our high-order multistep schemes are obtained by carefully approximating the conditional expectations and the derivatives, in the reference ODEs.

5.1 Derivative Approximation of ODEs

We let $u(t) \in C_b^{k+1}$, $\{t_i\}_{i=0,\ldots,k} \subset \mathbb{R}$ satisfies a partition $t_0 < t_1 < \cdots < t_k$. we denote $\Delta t_{0,i} = t_i - t_0$ for $i = 0, 1, ..., k$. Then, with the Taylor expansion, for arbitrary t_{t_i}, we have:

$$u\left(t_i\right) = \sum_{j=0}^{k} \frac{(\Delta t_{0,i})^j}{j!} \frac{d^j u}{dt^j}\left(t_0\right) + O\left(\Delta t_{0,i}\right)^{k+1}$$

Then we could multiply a real number $\alpha_{k,i}$ to every $u(t_i)$, and sum them up
we could obtain:

$$\sum_{i=0}^{k} \alpha_{k,i} u\left(t_i\right) = \sum_{j=0}^{k} \frac{\sum_{i=0} \alpha_{k,i} (\Delta t_{0,i})^j}{j!} \frac{d^j u}{dt^j}\left(t_0\right) + O\left(\sum_{i=0}^{k} \alpha_{k,i}\left(\Delta t_{0,i}\right)^{k+1}\right)$$

If we ignore the higher order of the derivatives and take some proper $\{\alpha_{k,i}\}_{i=0,\ldots,k}$, which satisfy:

$$\frac{\sum_{i=0}^{k} \alpha_{k,i}\left(\Delta t_{0,i}\right)^j}{j!} = \begin{cases} 1, & j = 1 \\ 0, & j \neq 1 \end{cases} \tag{5.1}$$

Then we get

$$\frac{du}{dt}\left(t_0\right) = \sum_{i=0}^{k} \alpha_{k,i} u\left(t_i\right) + R_D \tag{5.2}$$

Where $R_D = O\left(\sum_{i=0}^{k} \alpha_{k,i}\left(\Delta t_{0,i}\right)^{k+1}\right)$

Especially, if $\{t_i\}_{i=0,\ldots,k}$ is equidistant, i.e. $\Delta t_{0,i} = i\Delta t$, we have a determined equations about $\alpha_{k,i}\Delta t$:

$$\sum_{i=1}^{k} i^j\left[\alpha_{k,i}\Delta t\right] = \begin{cases} 1, & j = 1 \\ 0, & j \neq 1 \end{cases} \tag{5.3}$$

It is easy to calculate the value of $\alpha_{k,i}\Delta t(i = 0, 1, \ldots, k)$ when $k = 1, 2, \ldots, 6$:

$\alpha_{k,i}\Delta t$	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
$k = 1$	-1	1					
$k = 2$	$-\frac{3}{2}$	2	$-\frac{1}{2}$				
$k = 3$	$-\frac{11}{6}$	3	$-\frac{3}{2}$	$\frac{1}{3}$			
$k = 4$	$-\frac{25}{12}$	4	-3	$\frac{4}{3}$	$-\frac{1}{4}$		
$k = 5$	$-\frac{137}{60}$	5	-5	$\frac{10}{3}$	$-\frac{5}{4}$	$\frac{1}{5}$	
$k = 6$	$-\frac{49}{20}$	6	$-\frac{15}{2}$	$\frac{20}{3}$	$-\frac{15}{4}$	$\frac{6}{5}$	$-\frac{1}{6}$

Remark: According to ODEs multiple-step method theory we know it is unstable when $k \geq 7$.

5.2 Two reference ODEs

We first consider the numerical approaches for decoupled FBSDEs (1.1), namely, the functions b and σ are independent of Y_t and Z_t. Let N be a positive integer. For the time interval $[0, T]$, we introduce a regular time partition as

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T$$

We will denote $t_{n+k} - t_n$ by $\Delta t_{n,k}$ and $W_{t_{n+k}} - W_{t_n}$ by $\Delta W_{n,k}$, and use the notaions $\Delta t_{t_n,t} = t - t_n$ and $\Delta W_{t_n,t} = W_t - W_{t_n}$ for $t \geq t_n$. Two reference ODEs. Let (X_t, Y_t, Z_t) be the solution of the decoupled

FBSDEs (3.1) with terminal condition $\xi=\varphi\left(X_T\right)$. By taking conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ on both sides of the BSDE in (3.1), we obtain the integral equation

$$\mathbb{E}_{t_n}^x\left[Y_t\right]=\mathbb{E}_{t_n}^x[\xi]+\int_t^T \mathbb{E}_{t_n}^x\left[f\left(s, X_s, Y_s, Z_s\right)\right] \mathrm{d} s, \quad \forall t \in\left[t_n, T\right] \tag{5.4}$$

The integrand $\mathbb{E}_{t_n}^x\left[f\left(s, X_s, Y_s, Z_s\right)\right]$ is a continuous function of s. Then, by taking derivative with respect to t on both sides of (5.4), we obtain the following reference ordinary differential equation

$$\frac{\mathrm{d} \mathbb{E}_{t_n}^x\left[Y_t\right]}{\mathrm{d} t}=-\mathbb{E}_{t_n}^x\left[f\left(t, X_t, Y_t, Z_t\right)\right], \quad t \in\left[t_n, T\right] \tag{5.5}$$

Note that we also have

$$Y_{t_n}=Y_t+\int_{t_n}^t f\left(s, X_s, Y_s, Z_s\right) \mathrm{d} s-\int_{t_n}^t Z_s \mathrm{~d} W_s, \quad t \in\left[t_n, T\right]$$

By multiplying both sides of the above equation by $\left(\Delta W_{t_n, t}\right)^\tau$ (where $(\cdot)^\tau$ is the transpose of $(\cdot)$), and taking the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ on both sides of the derived equation, we obtain

$$\begin{aligned}
0= & \mathbb{E}_{t_n}^x\left[Y_t\left(\Delta W_{t_n, t}\right)^\tau\right]+\int_{t_n}^t \mathbb{E}_{t_n}^x\left[f\left(s, X_s, Y_s, Z_s\right)\left(\Delta W_{t_n, s}\right)^\tau\right] \mathrm{d} s \\
& -\int_{t_n}^t \mathbb{E}_{t_n}^x\left[Z_s\right] \mathrm{d} s, \quad t \in\left[t_n, T\right]
\end{aligned} \tag{5.6}$$

Again, the two integrands in (5.6) are continuous functions of s. Upon taking derivative with respect to $t \in\left[t_n, T\right)$ in (5.6) one gets the following reference ODE:

$$\frac{\mathrm{d} \mathbb{E}_{t_n}^x\left[Y_t\left(\Delta W_{t_n, t}\right)^\tau\right]}{\mathrm{d} t}=-\mathbb{E}_{t_n}^x\left[f\left(t, X_t, Y_t, Z_t\right)\left(\Delta W_{t_n, t}\right)^\tau\right]+\mathbb{E}_{t_n}^x\left[Z_t\right] \tag{5.7}$$

Remark: *:The two ODEs (5.5) and (5.7) are our reference equations for the FBSDE. Our numerical schemes will be derived by approximating the conditional expectations and the derivatives in* (5.5) *and* (5.7).

5.3 Numerical schemes for decoupled FBSDEs

5.3.1 The time semi-discrete scheme

We choose smooth functions $\bar{b}(t, x)$ and $\bar{\sigma}(t, x)$ for $t \in\left[t_n, T\right]$ and $x \in \mathbb{R}^q$ with constraints $\bar{b}\left(t_n, x\right)= b\left(t_n, x\right)$ and $\bar{\sigma}\left(t_n, x\right)=\sigma\left(t_n, x\right)$. The diffusion process $\bar{X}_t^{t_n, x}$ is defined by

$$\bar{X}_t^{t_n, x}=x+\int_{t_n}^t \bar{b}\left(s, \bar{X}_s^{t_n, x}\right) \mathrm{d} s+\int_{t_n}^t \bar{\sigma}\left(s, \bar{X}_s^{t_n, x}\right) \mathrm{d} W_s \tag{5.8}$$

Let $\left(X_t^{t_n, x}, Y_t^{t_n, x}, Z_t^{t_n, x}\right)$ be the solution of the decoupled FBSDEs. The solutions $Y_t^{t_n, x}$ and $Z_t^{\hat{t}_n, x}$ can be represented as $Y_t^{t_n, x}=u\left(t, X_t^{t_n, x}\right)$ and $Z_t^{t_n, x}=\nabla_x u\left(t, X_t^{t_n, x}\right) \sigma\left(t, X_t^{t_n, x}\right)$, respectively Let $\bar{Y}_t^{t_n, x}= u\left(t, \bar{X}_t^{t_n, x}\right)$ and $\bar{Z}_t^{t_n, x}=\nabla_x u\left(t, \bar{X}_t^{t_n, x}\right) \sigma\left(t, \bar{X}_t^{t_n, x}\right)$.
Then we have

$$\begin{aligned}
\left.\frac{\mathrm{d} \mathbb{E}_{t_n}^x\left[Y_t^{t_n, x}\right]}{\mathrm{d} t}\right|_{t=t_n} & =\left.\frac{\mathrm{d} \mathbb{E}_{t_n}^x\left[\bar{Y}_t^{t_n, x}\right]}{\mathrm{d} t}\right|_{t=t_n} \\
\frac{\mathrm{d} \mathbb{E}_{t_n}^x\left[Y_t^{t_n, x}\left(\Delta W_{t_n, t}\right)^\tau\right]}{\mathrm{d} t} & =\left.\frac{\mathrm{d} \mathbb{E}_{t_n}^x\left[\bar{Y}_t^{t_n, x}\left(\Delta W_{t_n, t}\right)^\tau\right]}{\mathrm{d} t}\right|_{t=t_n}
\end{aligned} \tag{5.9}$$

Now introducing the scheme (5.2) into $\left.\frac{\mathrm{d} \mathbb{E}_{t_n}^x\left[\bar{Y}_t^{t_n, x}\right]}{\mathrm{d} t}\right|_{t=t_n}$ and $\left.\frac{\mathrm{d} \mathbb{E}_{t_n}^x\left[\bar{Y}_t^{t_n, x}\left(\Delta W_{t_n, t}\right)^\tau\right]}{\mathrm{d} t}\right|_{t=t_n}$ we get

$$\begin{aligned}
\left.\frac{\mathrm{d} \mathbb{E}_{t_n}^x\left[Y_t^{t_n, x}\right]}{\mathrm{d} t}\right|_{t=t_n} & =\sum_{i=0}^k \alpha_{k, i} \mathbb{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}^{t_n, x}\right]+\bar{R}_{y, n}^k, \\
\frac{\mathrm{d} \mathbb{E}_{t_n}^x\left[Y_t^{t_n, x}\left(\Delta W_{t_n, t}\right)^\tau\right]}{\mathrm{d} t} & =\sum_{i=1}^k \alpha_{k, i} \mathbb{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}^{t_n, x}\left(\Delta W_{n, i}\right)^\tau\right]+\bar{R}_{z, n}^k
\end{aligned} \tag{5.10}$$

where $\alpha_{k,i}$ are defined by (5.3), and $\bar{R}^k_{y,n}$ and $\bar{R}^k_{z,n}$ are truncation errors, i.e.

$$\bar{R}^k_{y,n} = \left.\frac{\mathrm{d}\mathbb{E}^x_{t_n}\left[Y^{t_n,x}_t\right]}{\mathrm{d}t}\right|_{t=t_n} - \sum_{i=0}^k \alpha_{k,i}\mathbb{E}^x_{t_n}\left[\bar{Y}^{t_n,x}_{t_{n+i}}\right]$$
$$\bar{R}^k_{z,n} = \left.\frac{\mathrm{d}\mathbb{E}^x_{t_n}\left[Y^{t_n,x}_t(\Delta W_{t_n,t})^\tau\right]}{\mathrm{d}t}\right|_{t=t_n} - \sum_{i=1}^k \alpha_{k,i}\mathbb{E}^x_{t_n}\left[\bar{Y}^{t_n,x}_{t_{n+i}}\left(\Delta W_{n,i}\right)^\tau\right]$$

By inserting (5.10) into (5.5) and (5.7), respectively, we obtain

$$\sum_{i=0}^k \alpha_{k,i}\mathbb{E}^x_{t_n}\left[\bar{Y}^{t_n,x}_{t_{n+i}}\right] = -f\left(t_n, x, Y_{t_n}, Z_{t_n}\right) + R^k_{y,n} \tag{5.11}$$

$$\sum_{i=1}^k \alpha_{k,i}\mathbb{E}^x_{t_n}\left[\bar{Y}^{t_n,x}_{t_{n+i}}\left(\Delta W_{n,i}\right)^\tau\right] = Z_{t_n} + R^k_{z,n} \tag{5.12}$$

Where $R^k_{y,n} = -\bar{R}^k_{y,n}$ and $R^k_{z,n} = -\bar{R}^k_{z,n}$ Let Y^n and Z^n be the numerical approximations for the solutions Y_t and Z_t of the BSDE at time t_n, respectively. And we denoted by X^n the numerical solution of the associated forward SDE at t_n. Then, by removing the truncation error terms $R^k_{y,n}$ and $R^k_{z,n}$ from (5.11) and (5.12), we get our time semi-discrete numerical scheme for solving decoupled FBSDEs:

Assume that Y^{N-i} and $Z^{N-i}, i = 0, 1, \ldots, k-1$, are known. For $n = N-k, \ldots, 0$, solve $X^{n,j}(j = 1, 2, \ldots, k), Y^n = Y^n\left(X^n\right)$ and $Z^n = Z^n\left(X^n\right)$ by

$$X^{n,j} = X^n + b\left(t_n, X^n\right)\Delta t_{n,j} + \sigma\left(t_n, X^n\right)\Delta W_{n,j}, \quad j = 1, \ldots, k \tag{5.13}$$

$$\begin{aligned} Z^n &= \sum_{j=1}^k \alpha_{k,j}\mathbb{E}^{X^n}_{t_n}\left[\bar{Y}^{n+j}\left(\Delta W_{n,j}\right)^\tau\right] \\ \alpha_{k,0}Y^n &= -\sum_{j=1}^k \alpha_{k,j}\mathbb{E}^{X^n}_{t_n}\left[\bar{Y}^{n+j}\right] - f\left(t_n, X^n, Y^n, Z^n\right) \end{aligned} \tag{5.14}$$

Where $\bar{Y}^{n+j}$ are the values of Y^{n+j} at the space point $X^{n,j}$, $\bar{R}^k_{y,n}$ and $\bar{R}^k_{z,n}$ are defined in (5.10), respectively.

5.3.2 The fully discrete scheme

To propose a fully discrete scheme, we introduce a general space partition D^n_h of $\mathbb{R}^q$ on each level t_n with parameter $h^n > 0$. The space partition D^n_h is a set of discrete grid points in $\mathbb{R}^q$, i.e $D^n_h = \{x_i \mid x_i \in \mathbb{R}^q\}$. We define the density of the grids in D^n_h by

$$h^n = \max_{x\in\mathbb{R}^q} \min_{x_i \in D^n_h} |x - x_i| = \max_{x\in\mathbb{R}^q} \operatorname{dist}\left(x, D^n_h\right) \tag{5.15}$$

where $\operatorname{dist}\left(x, D^n_h\right)$ is the distance from x to D^n_h. For each $x \in \mathbb{R}^q$, we define a local subset $D^n_{h,x}$ of D^n_h. We call $D^n_{h,x}$ the neighbor grid set in D^n_h at x. Now by (5.14), we will solve Y^n and Z^n at grid points $x \in D^n_h$. That is, for each $x \in D^n_h, n = N-k, \ldots, 0$, we solve $Y^n = Y^n(x)$ and $Z^n = Z^n(x)$ by

$$\begin{aligned} Z^n &= \sum_{j=1}^k \alpha_{k,j}\mathbb{E}^x_{t_n}\left[\bar{Y}^{n+j}\left(\Delta W_{n,j}\right)^\tau\right] \\ \alpha_{k,0}Y^n &= -\sum_{j=1}^k \alpha_{k,j}\mathbb{E}^x_{t_n}\left[\bar{Y}^{n+j}\right] - f\left(t_n, x, Y^n, Z^n\right) \end{aligned} \tag{5.16}$$

where $\bar{Y}^{n+j}$ is the value of Y^{n+j} at the space point $X^{n,j}$ defined by

$$X^{n,j} = X^n + b\left(t_n, X^n\right)\Delta t_{n,j} + \sigma\left(t_n, X^n\right)\Delta W_{n,j}, \quad j = 1, \ldots, k \tag{5.17}$$

Generally, $X^{n,j}$ defined by (5.17) does not belong to D^{n+j}_h on condition of $X^n = x \in D^n_h$. Thus, to solve Y^n and Z^n, interpolation methods are needed to approximate the value of Y^{n+j} at $X^{n,j}$ using the values of

Y^{n+j} on D_h^{n+j}. Here, we will adopt a local interpolation operator $I_{h,X}^n$ such that $I_{h,X}^n g$ is the interpolation value of the function g at space point $X \in \mathbb{R}^q$ by using the values of g only on $D_{h,X}^n$. Note that any interpolation methods can be used here.

In numerical simulations, the conditional expectations $\mathbb{E}_{t_n}^x\left[\bar{Y}^{n+j}\left(\Delta W_{n,j}\right)^\tau\right]$ and $\mathbb{E}_{t_n}^x\left[\bar{Y}^{n+j}\right]$ in (5.16) should also be approximated. The approximation operator of $\mathbb{E}_{t_n}^x[\cdot]$ is denoted by $\mathbb{E}_{t_n}^{x,h}[\cdot]$, which can be any quadrature methods such as the Monte-Carlo methods, the quasi-Monte-Carlo methods, and the Gaussian quadrature methods and so on.

Now using the operators $I_{h,x}^n$ and $\mathbb{E}_{t_n}^{x,h}[\cdot]$, we rewrite (5.11) and (5.12) in the following equivalent form.

$$\begin{aligned} Z_{t_n} =& \sum_{j=1}^k \alpha_{k,j}\mathbb{E}_{t_n}^{x,h}\left[I_{h,\bar{X}_{t_{n+j}}}^{n+j} Y_{t_{n+j}}\left(\Delta W_{n,j}\right)^\tau\right] \\ & - R_{z,n}^k + R_{z,n}^{k,\mathbb{E}} + R_{z,n}^{k,I_n} \end{aligned} \tag{5.18}$$

$$\begin{aligned} \alpha_{k,0}Y_{t_n} =& -\sum_{j=1}^k \alpha_{k,j}\mathbb{E}_{t_n}^{x,h}\left[I_{h,\bar{X}_{t_{n+j}}}^{n+j} Y_{t_{n+j}}\right] - f\left(t_n, x, Y_{t_n}, Z_{t_n}\right) \\ & + R_{y,n}^k + R_{y,n}^{k,\mathbb{E}} + R_{y,n}^{k,I_h} \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} R_{z,n}^{k,\mathbb{E}} &= \sum_{j=1}^k \alpha_{k,j}\left(\mathbb{E}_{t_n}^x - \mathbb{E}_{t_n}^{x,h}\right)\left[\bar{Y}_{t_{n+j}}\left(\Delta W_{n,j}\right)^\tau\right] \\ R_{z,n}^{k,I_h} &= \sum_{j=1}^k \alpha_{k,j}\mathbb{E}_{t_n}^{x,h}\left[\left(\bar{Y}_{t_{n+j}} - I_{h,\bar{X}_{t_{n+j}}}^{n+j} Y_{t_{n+j}}\right)\left(\Delta W_{n,j}\right)^\tau\right] \\ R_{y,n}^{k,\mathbb{E}} &= -\sum_{j=1}^k \alpha_{k,j}\left(\mathbb{E}_{t_n}^x - \mathbb{E}_{t_n}^{x,h}\right)\left[\bar{Y}^{n+j}\right] \\ R_{y,n}^{k,I_h} &= -\sum_{j=1}^k \alpha_{k,j}\mathbb{E}_{t_n}^{x,h}\left[\bar{Y}_{t_{n+j}} - I_{h,\bar{X}_{t_{n+j}}}^{n+j} Y_{t_{n+j}}\right] \end{aligned}$$

The two terms $R_{y,n}^{k,\mathbb{E}}$ and $R_{z,n}^{k,\mathbb{E}}$ are numerical errors introduced by approximating conditional expectations, and the other two terms $R_{y,n}^{k,I_h}$ and $R_{z,n}^{k,I_h}$ are numerical errors caused by numerical interpolations. By removing the six error terms $R_{y,n}^k, R_{z,n}^k, R_{z,n}^{k,\mathbb{E}}, R_{z,n}^{k,I_h}, R_{y,n}^{k,\mathbb{E}}$ and $R_{y,n}^{k,I_h}$ from (5.18) and (5.19), we obtain our fully discrete scheme for solving decoupled FBSDEs as follows:

Assume Y^{N-i} and Z^{N-i} defined on $D_h^{N-i}, i = 0, 1, \ldots, k-1$, are known. For $n = N-k, \ldots, 0$, and for $x \in D_h^n$, solve $Y^n = Y^n(x)$ and $Z^n = Z^n(x)$ by

$$X^{n,j} = X^n + b\left(t_n, X^n\right)\Delta t_{n,j} + \sigma\left(t_n, X^n\right)\Delta W_{n,j}, \quad j = 1, \ldots, k \tag{5.20}$$

$$Z^n = \sum_{j=1}^k \alpha_{k,j}\mathbb{E}_{t_n}^{x,h}\left[I_{h,X^{n,j}}^{n+j} Y^{n+j}\left(\Delta W_{n,j}\right)^\tau\right] \tag{5.21}$$

$$\alpha_{k,0}Y^n = -\sum_{j=1}^k \alpha_{k,j}\mathbb{E}_{t_n}^{x,h}\left[I_{h,X^{n,j}}^{n+j} Y^{n+j}\right] - f\left(t_n, x, Y^n, Z^n\right) \tag{5.22}$$

5.4 Numerical schemes for coupled FBSDEs

Assume Y^{N-i} and Z^{N-i} defined on $D_h^{N-i}, i = 0, 1, \ldots, k-1$, are cnown. For $n = N-k, \ldots, 0$, and for $x \in D_h^n$, solve $Y^n = Y^n(x)$ and $Z^n = Z^n(x)$ by

$$\begin{aligned}
X^{n,j} =& X^n + b\left(t_n, X^n, Y^n, Z^n\right) \Delta t_{n,j} + \sigma\left(t_n, X^n, Y^n, Z^n\right) \Delta W_{n,j} \\
& j = 1, 2, \ldots, k, \\
Z^n =& \sum_{j=1}^{k} \alpha_{k,j} \mathbb{E}_{t_n}^{x,h}\left[I_{h,X^{n,j}}^{n+j} Y^{n+j}\left(\Delta W_{n,j}\right)^{\tau}\right] \\
\alpha_{k,0} Y^n =& -\sum_{j=1}^{k} \alpha_{k,j} \mathbb{E}_{t_n}^{x,h}\left[I_{h,X^{n,j}}^{n+j} Y^{n+j}\right] - f\left(t_n, x, Y^n, Z^n\right)
\end{aligned}$$

Where $X^{n,j}, Y^n$ and Z^n are coupled together.

Reference: Weidong Zhao, Yu Fu, and Tao Zhou, New Kinds of High-Order Multistep Schemes for Coupled Forward Backward Stochastic Differential Equations, SIAM J. Sci. Comput.,36(4),A1731-A1751. (21 pages)

6 Deferred Correction methods for FBSDEs

In this chapter, i will introduce the DC method for ODE and FBSDEs. The deferred correction (DC) method is a classical method for solving ordinary differential equations; one of its key features is to iteratively use lower order numerical methods so that high-order numerical scheme can be obtained. The main advantage of the DC approach is its simplicity and robustness.

6.1 The DC framework for ODEs

We consider this following ODE problems

$$\left\{\begin{array}{l} y'(t) = f(t, y(t)), \quad t \in (0, T], \\ y(0) = y_0 \end{array}\right. \tag{6.1}$$

It aims to create high-order methods from low-order schemes. More precisely, the DC methods begin with a low-order scheme (such as the Euler scheme) and then promote it to a higher-order one by iteratively corrected numerical solutions of residual equations. We simply give the DC procedure for solving ODEs as follows. First, introduce a regular time partition for $[0, T]$ as

$$0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T \tag{6.2}$$

and a finer uniform partition $\mathbb{G}_K^n$ on the time sub-interval $I_n = [t_n, t_{n+1}]$

$$\mathbb{G}_K^n = \left\{t_{n,k} \mid t_n = t_{n,0} < t_{n,1} < \cdots < t_{n,k} < \cdots < t_{n,K} = t_{n+1}\right\} \tag{6.3}$$

with the time sub-step $\delta t = \left(t_{n+1} - t_n\right)/K$, where K is a given positive integer. Let $I_{n,k} = [t_{n,k}, t_{n,k+1}]$ Second, let $\left\{u^{n,k}\right\}_{k=0}^{K}$ be the approximated values of the solution $y(t)$ of (6.1) at time points $\{t_{n,k}\}_{k=0.0}^{K=0} \in \mathbb{G}_K^n$, which are obtained by using a low-order numerical scheme; based on the discrete values $\left\{u^{n,k}\right\}_{k=0}^{K}$, construct a continuous interpolation function such as linear polynomial, cubic polynomial cubic spline, standard Lagrange interpolation operators $Iu(t)$; solve the residual equation

$$\delta'(t) = f(t, \delta(t) + Iu(t)) - \frac{\mathrm{d}}{\mathrm{d}t} Iu(t) \tag{6.4}$$

with $\delta(0) = 0$, where $\delta(t) = y(t) - Iu(t)$ is the error function. Note that this residual equation is of the same form as (6.1), so the same numerical scheme for (6.1) can be used to solve (6.4). This will yield approximated values $\left\{\delta^k\right\}_{i=0}^{K}$. Third, correct the approximation solution $u^{n,k}$ by $u^{n,k,\text{ new }} = u^{n,k} + \delta^k, k = 0, 1, \cdots, K$. The above procedure can be repeated for J times, where J is a positive integer.

$$\mathcal{O}\left((\delta t)^{\min(J,K)+1}\right) \tag{6.5}$$

Algorithm 4: DC method for ODEs

1. Let $u^n = y_0$, for $n = 0$.
2. For $n = 1, 2, \cdots, N-1$, do $(1) - (3)$.

(1). Let $u^{n,0} = u^{n-1}$.

(2). For $j = 1, 2, \cdots, J$, do (i)-(iii).

(i). For $k = 1, 2, \cdots, K$, solve $u^{n,k,[j]}$ by a lower-order numerical method at time points $t_{n,k} \in \mathbb{G}_K^n$

(ii). Let $\delta^{0,[j]} = 0$. For $k = 1, 2, \cdots, K$, solve $\delta^{k,[j]}$ by the same lower-order method at time points $t_{n,k} \in \mathbb{G}_K^n$.

(iii). Update the numerical solutions $u^{n,k,[j]}, k = 1, 2, \cdots, K$, by

$$u^{n,k,[j+1]} = u^{n,k,[j]} + \delta^{k,[j]}$$

(3). Let $u^n = u^{n,K,[J]}$.

6.2 Numerical scheme for decoupled FBSDEs

In this subsection, we shall focus on the DC methods for the following decoupled FBSDEs on the time sub-interval $\mathcal{I}_{n,k} = [\tau_k, t_{n+1}]$:

$$\left\{\begin{array}{ll} X_t = X_{\tau_k} + \int_{\tau_k}^t b(s, X_s)\, \mathrm{d}s + \int_{\tau_k}^t \sigma(s, X_s)\, \mathrm{d}W_s, & t \in \mathcal{I}_{n,k} \\ Y_t = Y_{t_{n+1}} + \int_t^{t_{n+1}} f(s, X_s, Y_s, Z_s)\, \mathrm{d}s - \int_t^{t_{n+1}} Z_s\, \mathrm{d}W_s, & t \in \mathcal{I}_{n,k} \end{array}\right. \tag{6.6}$$

6.2.1 Four reference odes

Suppose that we have obtained a numerical approximation $\left(Y_i^{n,k}, Z_i^{n,k}\right)$ of the solution (Y_t, Z_t) of the BSDE (6.6) at time-space grid points $\left(t_{n,k}, X_{t_{n,k}} = x_i\right), i \in \mathbb{Z}$, by a low order numerical method (denoted as M_l method). Based on these values $\left(Y_i^{n,k}, Z_i^{n,k}\right)$, we can construct an interpolation approximate (I_hY_t, I_hZ_t) of (Y_t, Z_t) for $t \in I_n$. Define the error terms δY_t and δZ_t as

$$\delta Y_t = Y_t - I_hY_t, \quad \delta Z_t = Z_t - I_hZ_t \tag{6.7}$$

It follows (6.6) and (6.7) that the processes δY_t and δZ_t solve the following BSDE:

$$\delta Y_t = \delta Y_{t_{n+1}} + \int_t^{t_{n+1}} F(s, X_s, \delta Y_s, \delta Z_s)\, \mathrm{d}s - \int_t^{t_{n+1}} \delta Z_s\, \mathrm{d}W_s + \mathcal{E}(t) \tag{6.8}$$

where

$$\begin{array}{l} F(s, X_s, \delta Y_s, \delta Z_s) = f(s, X_s, \delta Y_s + I_hY_s, \delta Z_s + I_hZ_s) \\ \mathcal{E}(t) = I_hY_{t_{n+1}} - \int_t^{t_{n+1}} I_hZ_s\, \mathrm{d}W_s - I_hY_t \end{array}$$

After getting the approximated values $\left(\delta Y_i^{n,k}, \delta Z_i^{n,k}\right)$ of $(\delta Y_t, \delta Z_t)$ at the grid points $\left(t_{n,k}, X_{t_{n,k}} = x_i\right), i \in \mathbb{Z}$ by the method M_l for (6.6), we update the approximated solutions $\left(Y_i^{n,k,\text{ new}}, Z_i^{n,k,\text{ new}}\right)$ by

$$\left(Y_i^{n,k,\text{ new}}, Z_i^{n,k,\text{ new}}\right) = \left(Y_i^{n,k} + \delta Y_i^{n,k}, Z_i^{n,k} + \delta Z_i^{n,k}\right)$$

In the section 5.2 we get the following reference ODE:

$$\frac{\mathrm{d}\mathbb{E}_{t_n}^x[Y_t]}{\mathrm{d}t} = -\mathbb{E}_{t_n}^x[f(t, X_t, Y_t, Z_t)], \quad t \in [t_n, T] \tag{6.9}$$

$$\frac{\mathrm{d}\mathbb{E}_{t_n}^x\left[Y_t(\Delta W_{t_n,t})^\tau\right]}{\mathrm{d}t} = -\mathbb{E}_{t_n}^x\left[f(t, X_t, Y_t, Z_t)(\Delta W_{t_n,t})^\tau\right] + \mathbb{E}_{t_n}^x[Z_t] \tag{6.10}$$

For (6.8), by using the same arguments in obtaining (6.9) and (6.10), we derive the following two reference ODEs for the error pair $(\delta Y_t, \delta Z_t)$ for $t \in \mathcal{I}_{n,k}$:

$$
\begin{aligned}
\frac{\mathrm{d}\mathbb{E}_{\tau_k}^x[\delta Y_t]}{\mathrm{d}t} &= -\mathbb{E}_{\tau_k}^x\left[f\left(t, X_t, \delta Y_t + I_h Y_t, \delta Z_t + I_h Z_t\right)\right] - \frac{\mathrm{d}\mathbb{E}_{\tau_k}^x[I_h Y_t]}{\mathrm{d}t} \\
\frac{\mathrm{d}\mathbb{E}_{\tau_k}^x\left[\delta Y_t\left(\Delta W_{\tau_k,t}\right)^\top\right]}{\mathrm{d}t} &= -\mathbb{E}_{\tau_k}^x\left[f\left(t, X_t, \delta Y_t + I_h Y_t, \delta Z_t + I_h Z_t\right)\left(\Delta W_{\tau_k,t}\right)^\top\right] \\
&\quad + \mathbb{E}_{\tau_k}^x\left[\delta Z_t + I_h Z_t\right] - \frac{\mathrm{d}\mathbb{E}_{\tau_k} x_h\left[I_h Y_t(\Delta W_{t_n,t})^\top\right]}{\mathrm{d}t}
\end{aligned}
\tag{6.11}
$$

The Equations give us reference ODEs for solving the BSDE, which will serve as fundamental tools in designing the DC-based numerical schemes. Specifically speaking, our DC schemes will be derived by approximating the conditional expectations and the derivatives in $(6.9) - (6.11)$.

6.2.2 The semi-discrete scheme

We now propose the semi-discrete DC scheme for decoupled FBSDEs on I_n. We choose smooth functions $\bar{b}(t,x)$ and $\bar{\sigma}(t,x)$ for $t \in \mathcal{I}_{n,k}$ and $x \in \mathbb{R}^d$ with constraints $\bar{b}\left(\tau_k, x\right) = b\left(\tau_k, x\right)$ and $\bar{\sigma}\left(\tau_k, x\right) = \sigma\left(\tau_k, x\right)$. Define the diffusion process $\bar{X}_t^{\tau_k,x}$ by

$$
\bar{X}_t^{\tau_k,x} = x + \int_{\tau_k}^t \bar{b}\left(s, \bar{X}_s^{\tau_k,x}\right)\mathrm{d}s + \int_{\tau_k}^t \bar{\sigma}\left(s, \bar{X}_s^{\tau_k,x}\right)\mathrm{d}W_s \tag{6.12}
$$

Let $(X_t^{\tau_k,x}, Y_t^{\tau_k,x}, Z_t^{\tau_k,x})$ be the solution of the decoupled FBSDEs (6.6), and $\left(\bar{Y}_t^{\bar{X}_t^{\tau_k,x} t}, \bar{Z}_t^{\frac{1}{X}_t^{\tau_k},x}\right)$ be the values of $(Y_t^{\tau_k,x}, Z_t^{\tau_k,x})$ at $\left(t, \bar{X}_t^{\tau_k,x}\right)$. Then we have

$$
\begin{aligned}
\left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^x[Y_t]}{\mathrm{d}t}\right|_{t=\tau_k} &= \left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^x\left[\bar{Y}_t\right]}{\mathrm{d}t}\right|_{t=\tau_k} = \frac{\mathbb{E}_{\tau_k}^x\left[\bar{Y}_{\tau_{k+1}}\right]-Y_{\tau_k}}{\delta t} + \tilde{R}_y^k, \\
\left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^x\left[Y_t\left(\Delta W_{\tau_k,t}\right)^\top\right]}{\mathrm{d}t}\right|_{t=\tau_k} &= \left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^x\left[\bar{Y}_t\left(\Delta W_{\tau_k,t}\right)^\top\right]}{\mathrm{d}t}\right|_{t=\tau_k} = \frac{\mathbb{E}_{\tau_k}^x\left[\bar{Y}_{\tau_{k+1}}\left(\Delta W_{\tau_k,t}\right)^\top\right]}{\delta t} + \tilde{R}_z^k
\end{aligned}
\tag{6.13}
$$

where $\tilde{R}_y^k$ and $\tilde{R}_z^k$ are truncation errors, defined by

$$
\tilde{R}_y^k = \left.\frac{d\mathbb{E}_{\tau_k}^x\left[Y_t\right]}{\mathrm{d}t}\right|_{t=\tau_k} - \frac{\mathbb{E}_{\tau_k}^x\left[\bar{Y}_{\tau_{k+1}}\right] - Y_{\tau_k}}{\delta t}
$$

$$
\tilde{R}_y^k = \left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^x\left[Y_t\left(\Delta W_{\tau_k,t}\right)^\top\right]}{\mathrm{d}t}\right|_{t=\tau_k} - \frac{\mathbb{E}_{\tau_k}^x\left[\bar{Y}_{\tau_{k+1}}\left(\Delta W_{\tau_k,t}\right)^\top\right]}{\delta t}
$$

Inserting (6.13) into (6.9) and (6.10), respectively, we obtain the following reference equations for solving BSDE:

$$
Y_{\tau_\nu} = \mathbb{E}_{\tau_i}^x\left[\bar{Y}_{\tau_b 1}\right] + \delta t \cdot f\left(\tau_k, X_{\tau_L}, Y_{\tau_L}, Z_{\tau_L}\right) + R_\mu^k \tag{6.14}
$$

$$
Z_{\tau_k} = \mathbb{E}_{\tau_k}^x\left[\bar{Y}_{\tau_{k+1}}\left(\Delta W_{\tau_k,t}\right)^\top\right]/\delta t + R_z^k \tag{6.15}
$$

where $R_y^k = -\tilde{R}_y^k$ and $R_z^k = \tilde{R}_z^k$. For the forward SDE, we choose the simplest form $\bar{b}\left(t, X_t^{\tau_k,x^y}\right) = b\left(\tau_k, x\right)$ and $\bar{\sigma}\left(t, X_t^{\tau_k,x}\right) = \sigma\left(\tau_k, x\right)$ for $t \in \mathcal{I}_{n,k}$, which results in the Euler scheme for the SDE, i.e.,

$$
X^{k+1} = X^k + b\left(\tau_k, X^k\right)\delta t + \sigma\left(\tau_k, X^k\right)\Delta W_{\tau_k,\tau_{k+1}}
$$

Now let Y^k and Z^k be the numerical approximations for the solutions Y_t and Z_t of the BSDE in (6.6) at time τ_k, respectively. By removing the truncation errors R_y^k and R_z^k from (6.14) and (6.15), respectively, we propose the time semi-discrete numerical scheme.

The Euler scheme. Given Y^K and Z^K on D_h, for $k = K-1, \cdots, 1, 0$, solve $X^{k+1}, Y^k = Y^k\left(X^k\right)$ and $Z^k = Z^k\left(X^k\right)$ for all $X^k \in D_h$ by

$$
\begin{aligned}
&X^{k+1} = X^k + b\left(\tau_k, X^k\right)\delta t + \sigma\left(\tau_k, X^k\right)\Delta W_{\tau_k,\tau_{k+1}} \\
&Z^k = \mathbb{E}_{\tau_k}^{X^k}\left[\bar{Y}^{k+1}\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]/\delta t \\
&Y^k = \mathbb{E}_{\tau_k}^{X^k}\left[\bar{Y}^{k+1}\right] + \delta t \cdot f\left(\tau_k, X^k, Y^k, Z^k\right)
\end{aligned}
\tag{6.16}
$$

where $\bar{Y}^{k+1}$ is the value of Y^{k+1} at the space point X^{k+1}. Similarly, denote by δY^k and δZ^k the approximated solution of δY_t and δZ_t on I_n, respectively. We propose the Euler scheme to solve the solution $(\delta Y_t, \delta Z_t)$ of (6.11):

$$\begin{cases} \delta Z^k = \mathbb{E}_{\tau_k}^{X^k}\left[\delta\bar{Y}^{k+1}\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]/\delta t - Z^k + \left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^{X^k}\left[I_h Y_t\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]}{\mathrm{d}t}\right|_{t=\tau_k} \\ \delta Y^k = \mathbb{E}_{\tau_k}^{X^k}\left[\delta\bar{Y}^{k+1}\right] + \delta t\cdot\left(f\left(\tau_k, X^k, \delta Y^k + Y^k, \delta Z^k + Z^k\right) + \left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^{X}\left[I_h Y_t\right]}{\mathrm{d}t}\right|_{t=\tau_k}\right) \end{cases} \tag{6.17}$$

where $\delta\bar{Y}^{k+1}$ is the value of δY^{k+1} at the space point X^{k+1}. Notice that we have the two identities:

$$\left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^{X^k}\left[I_h Y_t\right]}{\mathrm{d}t}\right|_{t=\tau_k} = L^0_{\tau_k,X^k}\left(I_h Y_{\tau_k}\right) \tag{6.18}$$

$$\left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^{X^k}\left[I_h Y_t\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]}{\mathrm{d}t}\right|_{t=\tau_k} = \nabla\left(I_h Y_{\tau_k}\right)\sigma\left(\tau_k, X^k\right) \tag{6.19}$$

Now combining Equations. (6.17)-(6.19), we propose our time semi-discrete for solving the error pair $(\delta Y_t, \delta Z_t)$ on I_n.

Let $\delta Y^K = 0$ and $\delta Z^K = 0$ on D_h. Then for $k = K-1, \cdots, 1, 0$, solve $X^{n+1}, \delta Y^k = \delta Y^k\left(X^k\right)$ and $\delta Z^k = \delta Z^k\left(X^k\right)$ for all $X^k \in D_h by$

$$X^{k+1} = X^k + b\left(\tau_k, X^k\right)\delta t + \sigma\left(\tau_k, X^k\right)\Delta W_{\tau_k,\tau_{k+1}} \tag{6.20}$$

$$\delta Z^k = \mathbb{E}_{\tau_k}^{X^k}\left[\delta\bar{Y}^{k+1}\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]/\delta t - Z^k + \nabla\left(I_h Y_{\tau_k}\right)\sigma\left(\tau_k, X^k\right) \tag{6.21}$$

$$\delta Y^k = \mathbb{E}_{\tau_k}^{X^k}\left[\delta\bar{Y}^{k+1}\right] + \delta t\left(f\left(\tau_k, X^k, \delta Y^k + Y^k, \delta Z^k + Z^k\right) + L^0_{\tau_k,X^k}\left(I_h Y_{\tau_k}\right)\right) \tag{6.22}$$

where $\delta\bar{Y}^{k+1}$ is the value of δY^{k+1} at the space point X^{k+1}. Note that the above scheme involves the terms $\frac{\mathrm{d}\mathbb{E}_{\tau_k}^{x}[I_h Y_t]}{\mathrm{d}t}$ and $\frac{\mathrm{d}\mathbb{E}_{\tau_k}^{x}\left[I_h Y_t\left(\Delta W_{\tau_k,t}\right)^\top\right]}{\mathrm{d}t}$. By Lemma 3.1 and due to the definition of the operator L^0, we need to pay attention to the derivatives $\frac{\partial(I_h Y_t)}{\partial t}$, $\frac{\partial(I_h Y_t)}{\partial x}$, and $\frac{\partial^2(I_h Y_t)}{\partial x^2}$ of $I_h Y_t$. Thus, high-order accuracy of the DC scheme relies heavily on the approximation quality of $\frac{\partial(I_h Y_t)}{\partial t}$, $\frac{\partial(I_h Y_t)}{\partial x}$, and $\frac{\partial^2(I_h Y_t)}{\partial x^2}$.

6.3 The fully-discrete scheme

The main purpose here is to solve Y^k and Z^k at the grid points $x \in D_h$. Precisely, for each $x \in D_h, k = K-1, \cdots, 1, 0$, we seek to solve $Y^k = Y^k(x)$ and $Z^k = Z^k(x)$ by

$$\begin{cases} Z^k = \mathbb{E}_{\tau_k}^{x}\left[\bar{Y}^{k+1}\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]/\delta t \\ Y^k = \mathbb{E}_{\tau_k}^{x}\left[\bar{Y}^{k+1}\right] + \delta t\cdot f\left(\tau_k, X^k, Y^k, Z^k\right) \end{cases} \tag{6.23}$$

where $\bar{Y}^{k+1}$ are the values of Y^{k+1} at the space point X^{k+1} defined by

$$X^{k+1} = X^k + b\left(\tau_k, X^k\right)\delta t + \sigma\left(\tau_k, X^k\right)\Delta W_{\tau_k,\tau_{k+1}} \tag{6.24}$$

Now by introducing the operators $I^k_{h,x}$ and $\mathbb{E}_{\tau_k}^{x,h}[\cdot]$, we rewrite (6.23) in the equivalent form

$$\begin{cases} Y_{\tau_k} = \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,\bar{X}^{k+1}_{\tau_{k+1}}} Y_{\tau_{k+1}}\right] + \delta t\cdot f\left(\tau_k, x, Y_{\tau_k}, Z_{\tau_k}\right) + R^k_y + R^{k,\mathbb{E}}_y + R^{k,I_h}_y \\ Z_{\tau_k} = \mathbb{E}_{\tau_k}^{x,h}\left[I^{k+1}_{h,\bar{X}_{\tau_{k+1}}} Y_{\tau_{k+1}}\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]/\delta t + R^k_z + R^{k,\mathbb{E}}_z + R^{k,I_h}_z \end{cases} \tag{6.25}$$

where

$$
\begin{aligned}
R_y^{k,\mathbb{E}} &= \left(\mathbb{E}_{\tau_k}^x - \mathbb{E}_{\tau_k}^{x,h}\right)\left[\bar{Y}_{\tau_{k+1}}\right] \\
R_z^{k,\mathbb{E}} &= \left(\mathbb{E}_{\tau_k}^x - \mathbb{E}_{\tau_k}^{x,h}\right)\left[\bar{Y}_{\tau_{k+1}}\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]/\delta t \\
R_y^{k,I_h} &= \mathbb{E}_{\tau_k}^{x,h}\left[\bar{Y}_{\tau_{k+1}} - I_{h,\bar{X}_{\tau_{k+1}}}^{k+1} Y_{\tau_{k+1}}\right] \\
R_z^{k,I_h} &= \mathbb{E}_{\tau_k}^{x,h}\left[\left(\bar{Y}_{\tau_{k+1}} - I_{h,\bar{X}_{\tau_{k+1}}}^{k+1} Y_{\tau_{k+1}}\right)\left(W_{\tau_k,\tau_{k+1}}\right)^\top\right]/\delta t
\end{aligned}
$$

The two terms $R_y^{k,\mathbb{E}}$ and $R_z^{k,\mathbb{E}}$ are numerical errors introduced by approximating conditional expectations, and the other two terms R_y^{k,I_h} and R_z^{k,I_h} are numerical errors caused by numerical interpolations. By removing the six error terms $R_y^k, R_y^{k,\mathbb{E}}, R_y^{k,I_h}, R_z^k, R_z^{k,\mathbb{E}}$, and R_z^{k,I_h} from (6.25), we propose our fully discrete scheme for solving the solution (X_t, Y_t, Z_t) of the decoupled FBSDEs on I_n.

Given Y^K and Z^K on D_h, for $k = K-1, \cdots, 1, 0$, solve $X^{k+1}, Y^k = Y^k\left(X^k\right)$ and $Z^k = Z^k\left(X^k\right)$ for all $X^k \in D_h$ by

$$
\begin{aligned}
X^{k+1} &= X^k + b\left(\tau_k, X^k\right)\delta t + \sigma\left(\tau_k, X^k\right)\Delta W_{\tau_k,\tau_{k+1}} \\
Z^k &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} Y^{k+1}\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]/\delta t \\
Y^k &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} Y^{k+1}\right] + \delta t \cdot f\left(\tau_k, X^k, Y^k, Z^k\right)
\end{aligned}
$$

By using the same arguments, i.e., by approximating the two conditional expectations $\mathbb{E}_{\tau_k}^x\left[\delta\bar{Y}_{\tau_{k+1}}\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]$ and $\mathbb{E}_{\tau_k}^x\left[\delta\bar{Y}_{\tau_{k+1}}\right]$, we propose our fully discrete Euler scheme for solving the error pair $\left(\delta Y_t, \delta\tilde{Z}_t\right)$ on I_n as follows.

Let $\delta Y^K = 0$ and $\delta Z^K = 0$ on D_h, then for $k = K-1, \cdots, 1, 0$, solve $X^{k+1}, \delta Y^k = \delta Y^k\left(X^k\right)$ and $\delta Z^k = \delta Z^k\left(X^k\right)$ for all $X^k \in D_h by$

$$
\begin{aligned}
X^{k+1} &= X^k + b\left(\tau_k, X^k\right)\delta t + \sigma\left(\tau_k, X^k\right)\Delta W_{\tau_k,\tau_{k+1}} \\
\delta Z^k &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1}\delta Y^{k+1}\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^\top\right]/\delta t - Z^k + \nabla\left(I_h Y_{\tau_k}\right)\sigma\left(\tau_k, X^k\right) \\
\delta Y^k &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1}\delta Y^{k+1}\right] + \delta t\left(f\left(\tau_k, X^k, \delta Y^k + Y^k, \delta Z^k + Z^k\right) + L_{\tau_k,X^k}^0\left(I_h Y_{\tau_k}\right)\right)
\end{aligned}
$$

Algorithm 5: DC method for decoupled FBSDEs

---

1. Give Y_i^N and $Z_i^N, i \in \mathbb{Z}$

2. For $n = N-1, \ldots, 1, 0, i \in \mathbb{Z}, \mathrm{do}(1) - (3)$

(1). Let $Y_i^{n,K} = Y_i^{n+1}$ and $Z_i^{n,K} = Z_i^{n+1}$.

(2). For $j = 1, 2, \cdots, J$, do (i)-(iii).

(i). For For $k = K-1, \cdots, 1, 0$, solve $Y_i^{n,k,[j]}$ and $Z_i^{n,k,[j]}$ by a lower-order numerical method at grid points $(t_{n,k}, x_i) \in \mathbb{G}_K^n \times D_h$.

(ii). Let $\delta Y_i^{K,[j]} = 0$ and $\delta Z_i^{K,[j]} = 0$. For $k = K-1, \cdots, 1, 0$, solve $\delta Y_i^{k,[j]}$ and $\delta Z_i^{k,[j]}$ by the same lower-order method at grid points $(t_{n,k}, x_i) \in \mathbb{G}_K^n \times D_h$

(iii). Update the numerical solution pairs $\left(Y_i^{n,k,[j]}, Z_i^{n,k,[j]}\right), k = 0, 1, \cdots, K-1$, by

$$
Y_i^{n,k,[j+1]} = Y_i^{n,k,[j]} + \delta Y_i^{k,[j]}, \quad Z_i^{n,k,[j+1]} = Z_i^{n,k,[j]} + \delta Z_i^{k,[j]}
$$

(3). Let $Y_i^n = Y_i^{n,0,[J]}$ and $Z_i^n = Z_i^{n,0,[J]}$

6.4 Numerical scheme for coupled FBSDEs

In this subsection, i extend our DC schemes for solving fully coupled FBSDEs on I_n.

Assume Y^K and Z^K defined on D_h^K are known. For $k = K-1, \cdots, 1, 0$, solve $X^{k+1}, Y^k = Y^k(x)$ and $Z^k = Z^k(x)$ for all $x \in D_h$ by

$$\begin{aligned}
X^{k+1} &= X^k + b\left(\tau_k, X^k, Y^k, Z^k\right)\delta t + \sigma\left(\tau_k, X^k, Y^k, Z^k\right)\Delta W_{\tau_k,\tau_{k+1}} \\
Z^k &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} Y^{k+1}\left(\Delta W_{\tau_k,t_{k=1}}\right)^{\top}\right]/\delta t \\
Y^k &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} Y^{k+1}\right] - \delta t \cdot f\left(\tau_k, x, Y^k, Z^k\right)
\end{aligned}$$

Let $\delta Y^K = 0$ and $\delta Z^K = 0$ on D_h, then for $k = K-1, \cdots, 1, 0$, solve the errors δY^k and δZ^k by

$$\begin{aligned}
X^{k+1} =& X^k + b\left(\tau_k, X^k, Y^k + \delta Y^k, Z^k + \delta Z^k\right)\delta t \\
&+ \sigma\left(\tau_k, X^k, Y^k + \delta Y^k, Z^k + \delta Z^k\right)\Delta W_{\tau_k,\tau_{k+1}} \\
\delta Z^k =& \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1}\delta Y^{k+1}\left(\Delta W_{\tau_k,\tau_{k+1}}\right)^{\top}\right]/\delta t - Z^k \\
&+ \nabla\left(I_h Y_{\tau_k}\right)\sigma\left(\tau_k, X^k, Y^k + \delta Y^k, Z^k + \delta Z^k\right) \\
\delta Y^k =& \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1}\delta Y^{k+1}\right] + \delta t\left(f\left(\tau_k, X^k, \delta Y^k + Y^k, \delta Z^k + Z^k\right) + L_{\tau_k,X^k}^0\left(I_h Y_{\tau_k}\right)\right)
\end{aligned}$$

Reference:

(1)Tao Tang, Weidong ZhaoandTao Zhou, Deferred Correction Methods for Forward Backward Stochastic Differential Equations,Numerical Mathematics: Theory, Methods and Applications, Volume 10 , Issue 2, May 2017 , pp. 222 - 242

DOI: https://doi.org/10.4208/nmtma.2017.s02

(2)Jie Yang, Weidong Zhao, Tao Zhou, Explicit Deferred Correction Methods for Second-Order Forward Backward Stochastic Differential Equations, Journal of Scientific Computing, Issue 3/2019

7 Multi-step scheme for FBSDE and 2FBSDE

In this chapter, i will introduce the multi-step scheme for 2FBSDE. Transform the 2FBSDE to a relatively deterministic reference equations by the stochastic calculus and theory of 2FBSDE. Obtain the derivatives of the conditional expectation. Approximate it with the multi-step theory and discrete the conditional expectation. Similarly, i use the Monte-Carlo method to get the approximate conditional expectation.

$$\left\{\begin{array}{l}
X_t = x + \int_0^t b\left(s, \Theta_s\right)\mathrm{d}s + \int_0^t \sigma\left(s, \Theta_s\right)\mathrm{d}W_s \\
Y_t = g\left(X_T\right) + \int_t^T f\left(s, \Theta_s\right)\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \quad s \in [0, T] \\
Z_s = Z_0 + \int_0^t A_s\,\mathrm{d}s + \int_0^t \Gamma_s\,\mathrm{d}W_s,
\end{array}\right. \tag{7.1}$$

7.1 Numerical Scheme for Decoupled 2FBSDEs

We consider multi-step scheme for decoupled 2FBSDEs at first, i.e. the forward equation is independent on $(Y_t, Z_t, A_t, \Gamma_t)$

$$\left\{\begin{aligned}
\mathrm{d}X_t &= b\left(t, X_t\right)\mathrm{d}t + \sigma\left(t, X_t\right)\mathrm{d}W_t \\
-\mathrm{d}Y_t &= f\left(t, X_t, Y_t, Z_t, \Gamma_t\right)\mathrm{d}t - Z_t\,\mathrm{d}W_t \\
\mathrm{d}Z_t &= A_t\,\mathrm{d}t + \Gamma_t\,\mathrm{d}W_t
\end{aligned}\right. \tag{7.2}$$

The terminal condition $Y_T = g\left(X_T\right)$ is given.
Let $u = u(t, x)$ be the solution of the following fully nonlinear PDE

$$\mathcal{L}u + f\left(t, x, u, \nabla_x u\sigma, \nabla_x\left(\nabla_x u\sigma\right)\sigma\right) = 0$$

with the terminal condition $u(T,x)=g(x)$, and let $(X_t, Y_t, Z_t, \Gamma_t, A_t)$ be the solution of the 2FBSDE. Then we have

$$Y_t = u(t, X_t), \quad Z_t = (\nabla_x u\sigma)(t, X_t)$$
$$\Gamma_t = (\nabla_x (\nabla_x u\sigma)\sigma)(t, X_t), \quad A_t = (\mathcal{L}(\nabla_x u\sigma))(t, X_t)$$

where the associate operator $\mathcal{L}$ is defined by

$$\mathcal{L}\phi(t,x) := \phi_t(t,x) + \nabla_x\phi(t,x)b(t,x) + \frac{1}{2}\operatorname{tr}\left(\sigma(t,x)\sigma^\top(t,x)\nabla_x^2\phi(t,x)\right)$$

where $\nabla_x\phi = (\partial_{x_1}\phi, \ldots, \partial_{x_m}\phi)$ and $\nabla_x^2\phi$ is the Hessian matrix of x.

7.1.1 Derivatives equations

We give the a partition $\mathcal{T}$ in the time interval $[t_0, T]$:

$$\mathcal{T} : t_0 < t_1 < \cdots < t_N = T$$

For all $n, \in \{1, 2\ldots, N\}, k \in \mathbb{N}$ satisfy $n+k \leq N$. We denote $\Delta t_{t_n,k} = t_{n+k} - t_n$
For $t \geq t_n$, $\Delta W_{t_n,k} = W_{t_{n+k}} - W_{t_n}$, $\Delta t_{t_n,t} = t - t_n$ and $\Delta W_{t_n,t} = W_t - W_{t_n}$ We let $\Theta_t = (X_t, Y_t, Z_t, A_t, \Gamma_t)$ be the adapted solution of the decoupled 2FBSDE(5.5), the terminal condition of the equation is $Y(T) = g(X_T)$. We denote $\mathbf{E}_{t_n}^x[\cdot] = \mathbf{E}\left[\cdot \mid \mathcal{F}_{t_n}^{t_n,x}\right]$, and take expectation $\mathbf{E}_{t_n}^x[\cdot]$ on the both sides of the last equations:

$$\mathbf{E}_{t_n}^x[Y_t] = \mathbf{E}_{t_n}^x[g(X_T)] + \int_t^T \mathbf{E}_{t_n}^x[f(s, \Theta_s)]\,\mathrm{d}s, \quad t \in [t_n, T] \tag{7.3}$$

$$\mathbf{E}_{t_n}^x[Z_t] = \mathbf{E}_{t_n}^x[Z_{t_n}] + \int_{t_n}^t \mathbf{E}_{t_n}^x[A_s]\,ds, \quad t \in [t_n, T] \tag{7.4}$$

Then we take the derivatives with respect to t, we could obtain these two equations:

$$\frac{\mathrm{d}\mathbf{E}_{t_n}^x[Y_t]}{\mathrm{d}t} = -\mathbf{E}_{t_n}^x[f(t, \Theta_t)], \quad t \in [t_n, T] \tag{7.5}$$

$$\frac{\mathrm{d}\mathbf{E}_{t_n}^x[Z_t]}{\mathrm{d}t} = \mathbf{E}_{t_n}^x[A_t], \quad t \in [t_n, T] \tag{7.6}$$

For the Backward equation part, we have:

$$Y_{t_n} = Y_t + \textstyle\int_{t_n}^t f(s, \Theta_s)\,\mathrm{d}s - \int_{t_n}^t Z_s\,\mathrm{d}W_s, \quad t \in [t_n, T]$$

Multiplying $\Delta W_{t_n,,t}^\top$ on both sides, and take the expectation $\mathbf{E}_{t_n}^x[\cdot]$ at the time t_n, when $t \in [t_n, T)$ we could get

$$0 = \mathbf{E}_{t_n}^x\left[Y_t\Delta W_{t_n,t}^\top\right] + \int_{t_n}^t \mathbf{E}_{t_n}^x\left[f(s, \Theta_s)\Delta W_{t_n,s}^\top\right]\mathrm{d}s - \int_{t_n}^t \mathrm{E}_{t_n}^x[Z_s]\,\mathrm{d}s \tag{7.7}$$

Similarly, for the last equation in (5.5), when $t \in [t_n, T)$

$$0 = \mathbf{E}_{t_n}^x\left[Z_t^\top\Delta W_{t_n,t}^\top\right] - \int_{t_n}^t \mathbf{E}_{t_n}^x\left[A_s^\top\Delta W_{t_n,t}^\top\right]\mathrm{d}s - \int_{t_n}^t \mathbf{E}_{t_n}^x[\Gamma_s]\,\mathrm{d}s \tag{7.8}$$

We take the derivatives with respect to :

$$\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t\Delta W_{t_n,t}^\top\right]}{\mathrm{d}t} = -\mathbf{E}_{t_n}^x\left[f(t, \Theta_t)\Delta W_{t_n,t}^\top\right] + \mathbf{E}_{t_n}^x[Z_t], \quad t \in [t_n, T] \tag{7.9}$$

$$\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Z_t^\top\Delta W_{t_n,t}^\top\right]}{\mathrm{d}t.} = \mathrm{E}_{t_n}^x\left[A_t^\top\Delta W_{t_n,t}^\top\right] + \mathrm{E}_{t_n}^x[\Gamma_t], \quad t \in [t_n, T] \tag{7.10}$$

We could approximate the derivatives in (5.8),(5.9),(5.12),(5.13) to construct our multi-step scheme.

7.1.2 Semi-discrete scheme

We construct the new diffusion process:

$$\bar{X}_t^{t_n,x} = x + \int_{t_n}^{t} \bar{b}\left(s, \bar{X}_s^{t_n,x}\right) \mathrm{d}s + \int_{t_n}^{t} \bar{\sigma}\left(s, \bar{X}_s^{t_n,x}\right) \mathrm{d}W_s \tag{7.11}$$

We could see that the variables Y_t, Z_t, Γ_t, A_t are the function related to (t, X_t). we denote $\left(\bar{Y}_t^{t_n,x}, \bar{Z}_t^{t_n,x}\right)$ is the value at the point $\left(t, \bar{X}_t^{t_n,x}\right)$ of the function (Y_t, Z_t).Then we will have:

$$\begin{aligned}
\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t\right]}{\mathrm{d}t}\right|_{t=t_n} &= \left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[\bar{Y}_t^{t_n,x}\right]}{\mathrm{d}t}\right|_{t=t_n} \\
\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Z_t\right]}{\mathrm{d}t}\right|_{t=t_n} &= \left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[\bar{Z}_t^{t_n,x}\right]}{\mathrm{d}t}\right|_{t=t_n} \\
\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t \Delta W_{t_n,t}^\top\right]}{\mathrm{d}t}\right|_{t=t_n} &= \left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[\bar{Y}_t^{t_n,x} \Delta W_{t_n,t}^\top\right]}{\mathrm{d}t}\right|_{t=t_n} \\
\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Z_t^\top \Delta W_{t_n,t}^\top\right]}{\mathrm{d}t}\right|_{t=t_n} &= \left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[\left(\bar{Z}_t^{t_n,x}\right)^\top \Delta W_{t_n,t}^\top\right]}{\mathrm{d}t}\right|_{t=t_n}
\end{aligned}$$

where we could apply (5.3) to get:

$$\left\{\begin{array}{l}
\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t\right]}{\mathrm{d}t}\right|_{t=t_n} = \sum_{i=0}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}^{t_n,x}\right] + \bar{R}_{y,n}^k \\
\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t \Delta W_{t_n,t}^\top\right]}{\mathrm{d}t}\right|_{t=t_n} = \sum_{i=1}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}^{t_n,x} \Delta W_{n,i}^\top\right] + \bar{R}_{z,n}^k \\
\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Z_t\right]}{\mathrm{d}t}\right|_{t=t_n} = \sum_{i=0}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\bar{Z}_{t_{n+i}}^{t_n,x}\right] + \bar{R}_{A,n}^k \\
\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Z_t^\top \Delta W_{t_n,t}^\top\right]}{\mathrm{d}t}\right|_{t=t_n} = \sum_{i=1}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\left(\bar{Z}_{t_{n+i}}^{t_n,x}\right)^\top \Delta W_{n,i}^\top\right] + \bar{R}_{\Gamma,n}^k
\end{array}\right. \tag{7.12}$$

where, the parameters $\alpha_{k,i}$ have been given in (5.2). $\bar{R}_{y,n}^k, \bar{R}_{z,n}^k, \bar{R}_{A,n}^k$ and $\bar{R}_{\Gamma,n}^k$ is the corresponding truncation error:

$$\begin{aligned}
\bar{R}_{y,n}^k &= \left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t\right]}{\mathrm{d}t}\right|_{t=t_n} - \sum_{i=0}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}^{t_n,x}\right] \\
\bar{R}_{z,n}^k &= \left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t \Delta W_{t_n,t}^\top\right]}{\mathrm{d}t}\right|_{t=t_n} - \sum_{i=1}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}^{t_n,x} \Delta W_{n,i}^\top\right] \\
\bar{R}_{A,n}^k &= \left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Z_t\right]}{\mathrm{d}t}\right|_{t=t_n} - \sum_{i=0}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\bar{Z}_{t_{n+i}}^{t_n,x}\right] \\
\bar{R}_{\Gamma,n}^k &= \left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Z_t^\top \Delta W_{t_n,t}^\top\right]}{\mathrm{d}t}\right|_{t=t_n} - \sum_{i=1}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\left(\bar{Z}_{t_{n+i}}^{t_n,x}\right)^\top \Delta W_{n,i}^\top\right]
\end{aligned}$$

Then we could obtain:

$$\left\{\begin{array}{l}
\sum_{i=0}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}\right] = -f\left(t_n, x, Y_{t_n}, Z_{t_n}\right) + R_{y,n}^k \\
\sum_{i=1}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}} \Delta W_{n,i}^\top\right] = Z_{t_n} + R_{z,n}^k \\
\sum_{i=0}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\bar{Z}_{t_{n+i}}\right] = A_{t_n} + R_{A,n}^k \\
\sum_{i=1}^k \alpha_{k,i} \mathbf{E}_{t_n}^x\left[\bar{Z}_{t_{n+i}}^\top \Delta W_{n,i}^\top\right] = \Gamma_{t_n} + R_{\Gamma,n}^k
\end{array}\right. \tag{7.13}$$

where the corresponding truncation error $R_{y,n}^k = -\bar{R}_{y,n}^k, R_{z,n}^k = -\bar{R}_{z,n}^k, R_{\Gamma,n}^k = -\bar{R}_{\Gamma,n}^k$ and $R_{A,n}^k = -\bar{R}_{A,n}^k$ If we ignore the truncation errors, we could get the semi-discrete(time domain) scheme for decoupled 2FBSDEs.

Assume that random variables Y^{N-i} and Z^{N-i}, $i = 0, 1, ..., k-1$ are known. For $n = N-k, ..., 0$, with $X_t^{t_n, X^n}$ being the solution of (5.14). Furthermore, if we choose $\bar{b}$ and $\bar{\sigma}$ in the (5.14) as:

$$\bar{b}\left(s, X_s^{t_n,x}\right) = b\left(t_n, x\right), \quad \bar{\sigma}\left(s, X_s^{t_n,x}\right) = \sigma\left(t_n, x\right), \quad \forall s \in [t_n, T] \tag{7.14}$$

We could solve $Y^n = Y^n(X^n)$, $Z^n = Z^n(X^n)$, $A^n = A^n(X^n)$ and $\Gamma^n = \Gamma^n(X^n)$ by

$$\begin{aligned}
X^{n,j} &= X^n + b(t_n, X^n)\Delta t_{n,j} + \sigma(t_n, X^n)\Delta W_{n,j}, \quad j = 1, \ldots, k \\
Z^n &= \sum_{j=1} \alpha_{k,j} \mathrm{E}_{t_n}^{X^n}\left[\bar{Y}^{n+j}\Delta W_{n,j}^\top\right] \\
A^n &= \sum_{j=0} \alpha_{k,j} \mathbf{E}_{t_n}^{X^n}\left[\bar{Z}^{n+j}\right] \\
\Gamma^n &= \sum \alpha_{k,j} \mathrm{E}_{t_n}^{X^n}\left[\left(\bar{Z}^{n+j}\right)^\top \Delta W_{n,j}^\top\right] \\
-\alpha_{k,0} Y^n &= \sum_{j=1} \alpha_{k,j} \mathbf{E}_{t_n}^{X^n}\left[\bar{Y}^{n+j}\right] + f(t_n, X^n, Y^n, Z^n, \Gamma^n),
\end{aligned} \tag{7.15}$$

where $\bar{Y}^{n+j}$ and $\bar{Z}^{n+j}$ are the values of Y^{n+j} and Z^{n+j} at the space points $X_{t_{n+j}}^{t_n, X^n}$.

7.1.3 Fully-discrete scheme

At first we give the partition $\mathcal{D}_\mathbf{h}$:

$$\mathcal{D}_\mathbf{h} := \left\{\mathcal{D}_{h_n}^n\right\}_{n=0,1,\ldots,N}$$

Similarly, we could get the equation (5.18) in the last section. Since $\bar{Y}, \bar{Z}, \bar{\Gamma}$ and $\bar{A}$ is the functions of $X^{n,j}$. We have:

$$\begin{aligned}
\mathbf{E}_{t_n}^x\left[\bar{Y}^{n+j}\Delta W_{n,j}^\top\right] &= \mathrm{E}_{t_n}^x\left[Y^{n+j}\left(x + b(t_n, x)\Delta t_{n,j} + \sigma(t_n, x)\Delta W_{n,j}\right)\Delta W_{n,j}^\top\right] \\
\mathbf{E}_{t_n}^x\left[\bar{Z}^{n+j}\right] &= \mathbf{E}_{t_n}^x\left[Z_{t_{n+j}}\left(x + b(t_n, x)\Delta t_{n,j} + \sigma(t_n, x)\Delta W_{n,j}\right)\right] \\
\mathbf{E}_{t_n}^x\left[\left(\bar{Z}^{n+j}\right)^\top \Delta W_{n,j}^\top\right] &= \mathrm{E}_{t_n}^x\left[\left(Z^{n+j}\right)^\top\left(x + b(t_n, x)\Delta t_{n,j} + \sigma(t_n, x)\Delta W_{n,j}\right)\Delta W_{n,j}^\top\right] \\
\mathbf{E}_{t_n}^x\left[\bar{Y}^{n+j}\right] &= \mathrm{E}_{t_n}^x\left[Y^{n+j}\left(x + b(t_n, x)\Delta t_{n,j} + \sigma(t_n, x)\Delta W_{n,j}\right)\right]
\end{aligned}$$

Then we could also use the Monte-Carlo Method to estimate the condition Expectation here, just like in section(4.2). In short, I denote the operator as $\widehat{\mathbf{E}}^{n,x}[\cdot]$
we have

$$Z_{t_n} = \sum_{j=1}^k \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[Y_{t_{n+j}}\Delta W_{n,j}^\top\right] - R_{z,n}^k + R_{z,n}^{k,\mathbb{E}}$$

$$\Gamma_{t_n} = \sum_{j=1}^k \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[Z_{t_{n+j}}^\top \Delta W_{n,j}^\top\right] - R_{\Gamma,n}^k + R_{\Gamma,n}^{k,\mathbb{E}}$$

$$A_{t_n} = \sum_{j=0}^k \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[Z_{t_{n+j}}\right] - R_{A,n}^k + R_{A,n}^{k,\mathbb{E}}$$

$$-\alpha_{k,0} Y_{t_n} = \sum_{j=1}^k \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[Y_{t_{n+j}}\right] + f(t_n, x, Y_{t_n}, Z_{t_n}) + R_{y,n}^k + R_{y,n}^{k,\mathbb{E}}$$

$$R_{z,n}^{k,\mathbb{E}} = \sum_{j=1}^k \alpha_{k,j}\left(\mathbb{E}_{t_n}^x - \hat{\mathbb{E}}^{n,x}\right)\left[\bar{Y}_{t_{n+j}}\Delta W_{n,j}^\top\right]$$

$$R_{A,n}^{k,\mathbb{E}} = \sum_{j=1}^k \alpha_{k,j}\left(\mathbb{E}_{t_n}^x - \hat{\mathbb{E}}^{n,x}\right)\left[\bar{Z}_{t_{n+j}}\right],$$

$$R_{\Gamma,n}^{k,\mathbb{E}} = \sum_{j=1}^k \alpha_{k,j}\left(\mathbb{E}_{t_n}^x - \hat{\mathbb{E}}^{n,x}\right)\left[\bar{Z}_{t_{n+j}}^\top \Delta W_{n,j}^\top\right]$$

$$R_{y,n}^{k,\mathbb{E}} = -\sum_{j=1}^k \alpha_{k,j}\left(\mathbb{E}_{t_n}^x - \hat{\mathbb{E}}^{n,x}\right)\left[\bar{Y}^{n+j}\right],$$

Where $R^{k,\mathbb{E}}_{y,n}, R^{k,\mathbb{E}}_{z,n}, R^{k,\mathbb{E}}_{A,n}$ and $R^{k,\mathbb{E}}_{\Gamma,n}$ are approximate conditional expectation errors.

If we ignore these errors, we will have: Assume random variables Y^{N-i} and Z^{N-i} defined on $\mathcal{D}_h^{N-i}$, $i=0,1,\ldots,k-1$, are known. For $n=N-k,\ldots,0$, and for each $x\in\mathcal{D}_h^n$, solve X^n, Y^n, Z^n, A^n and Γ^n by:

$$\begin{aligned}
X^{n,j} &= x + b(t_n,x)\,\Delta t_{n,j} + \sigma(t_n,x)\,\Delta W_{n,j}, \quad j=1,\ldots,k\\
Z^n &= \sum \alpha_{k,j}\widehat{\mathbf{E}}^{n,x}\left[Y^{n+j}\Delta W_{n,j}^\top\right]\\
\Gamma^n &= \sum_{j=1} \alpha_{k,j}\widehat{\mathbf{E}}^{n,x}\left[(Z^{n+j})^\top\Delta W_{n,j}^\top\right]\\
A^n &= \sum_{j=0} \alpha_{k,j}\widehat{\mathbf{E}}^{n,x}\left[Z^{n+j}\right]\\
-\alpha_{k,0}Y^n &= \sum_{j=1} \alpha_{k,j}\widehat{\mathbf{E}}^{n,x}\left[Y^{n+j}\right] + f(t_n,x,Y^n,Z^n,\Gamma^n)
\end{aligned} \tag{7.16}$$

Then we have following steps to find the numerical solution,

Step 1. Use Euler method to calculate $X^{n,j}$

Step 2. Solve Z^n, Γ^n and A^n explicitly

Step 3. Solve Y^n implicitly

$$\alpha_{k,0}Y^{n,l+1} = -\sum \alpha_{k,j}\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}^{n+j}_{\mathcal{D},\overline{\mathbf{X}}^{n,j}}Y^{n+j}\right] - f\left(t_n,x,Y^{n,l};Z^n,\Gamma^n\right)$$

we could use some iteration scheme like newton iteration with terminal condition $\left|Y^{n,l+1}-Y^{n,l}\right| \le \epsilon_0$

$$Y^{n,l+1} = Y^{n,l} - \frac{\alpha_{k,0}Y^{n,l} + \sum_{j=1}^n \alpha_{k,j}\widehat{\mathbf{E}}^{n,x}\left[\mathbb{I}^{n+j}_{\mathcal{D},\bar{X}^{n,j}}Y^{n+j}\right] + f\left(t_n,x,Y^{n,l},Z^n,\Gamma^n\right)}{\alpha_{k,0} + f_y\left(t_n,x,Y^{n,l},Z^n,\Gamma^n\right)}$$

7.1.4 Numerical scheme for coupled 2FBSDE

We extends the scheme in the last section to coupled 2FBSDE

$$\begin{cases}
X_t = x + \int_0^t b(s,\Theta_s)\,\mathrm{d}s + \int_0^t \sigma(s,\Theta_s)\,\mathrm{d}W_s,\\
Y_t = g(X_T) + \int_t^T f(s,\Theta_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \quad s\in[0,T]\\
Z_s = Z_0 + \int_0^t A_s\,\mathrm{d}s + \int_0^t \Gamma_s\,\mathrm{d}W_s,
\end{cases}$$

Similarly, assuming random variables Y^{N-i}, Z^{N-i} and Γ^{N-i} defined on $\mathcal{D}_h^{N-i}$, $i=0,1,\ldots,k-1$ are known. l is the iterative time, Y^{N-i}, Z^{N-i} and Γ^{N-i} are the corresponding values at the $l-th$ iterative. For $n=N-k,\ldots,0$, and each $x\in\mathcal{D}_h^n$, solve Y^{N-i}, Z^{N-i} and Γ^{N-i} by:

Step 1. Initialize $Y^{n,0}, Z^{n,0}$ and $\Gamma^{n,0}$ by Y^{n+1}, Z^{n+1} and Γ^{n+1}, separately, i.e. $Y^{n,0}=Y^{n+1}(x), Z^{n,0}=Z^{n+1}(x)$ and $\Gamma^{n,0}=\Gamma^{n+1}(x)$

Step 2. Calculate $Y^{n,l+1}, Z^{n,l+1}, A^{n,l+1}$ and $\Gamma^{n,l+1}$ for $l=0,1,\ldots$ by

$$\begin{cases}
X^{n,j} = x + b\left(t_n,x,Y^{n,l},Z^{n,l},\Gamma^{n,l}\right)\Delta t_{n,j} + \sigma\left(t_n,x,Y^{n,l},Z^{n,l},\Gamma^{n,l}\right)\Delta W_{n,j},\\
\qquad j=1,2,\ldots,k,\\
Z^{n,l+1} = \sum_{j=1}^k \alpha_{k,j}\widehat{\mathbf{E}}^{n,x}\left[Y^{n+j}\Delta W_{n,j}^\top\right]\\
\Gamma^{n,l+1} = \sum_{j=1}^k \alpha_{k,j}\widehat{\mathbf{E}}^{n,x}\left[\left(Z^{n+j}\right)^\top\Delta W_{n,j}^\top\right]\\
A^{n,l+1} = \sum_{j=0}^k \alpha_{k,j}\widehat{\mathbf{E}}^{n,x}\left[Z^{n+j}\right],\\
-\alpha_{k,0}Y^{n,l+1} = \sum_{j=1}^k \alpha_{k,j}\widehat{\mathbf{E}}^{n,x}\left[Y^{n+j}\right] + f\left(t_n,x,Y^{n,l+1},Z^{n,l+1},\Gamma^{n,l+1}\right)
\end{cases}$$

Repeat the above, until

$$\max\left\{\left|Y^{n,l+1}-Y^{n,l}\right|, \left|Z^{n,l+1}-Z^{n,l}\right|, \left|A^{n,l+1}-A^{n,l}\right|, \left|\Gamma^{n,l+1}-\Gamma^{n,l}\right|\right\} < \epsilon_0$$

Step 3. Set $(Y^n, Z^n, \Gamma^n, A^n)$ to $\left(Y^{n,l+1}, Z^{n,l+1}, A^{n,l+1}, \Gamma^{n,l+1}\right)$

Reference:

(1)W. Zhao, Y. Fu and T. Zhou, SIAM J. Sci. Comput., 36 (2014), pp. A1731-A1751.

(2)Kong Tao, Weidong Zhao, Tao Zhou, High order numerical schemes for second-order FBSDEs with applications to stochastic optimal control, June 2018East Asian Journal on Applied Mathematics 8(3)

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8 The stochastic scheme for Fully Non-linear Parabolic PDEs

In this chapter, i will introduce the stochastic scheme for Fully Nonlinear Parabolic PDEs(2PDE).
We consider the nonlinear parabolic PDEs in the following form:

$$\begin{cases} u_t + F\left(t, x, u, Du, D^2 u\right) = 0, & (t,x) \in [0,T) \times \mathbb{R}^m \\ \qquad u(T,x) = g(x), & x \in \mathbb{R}^m \end{cases} \tag{8.1}$$

Assume $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a continuous function such that $u_t, Dv, D^2u, \mathcal{L}Du$ exist and are continuous on $[0,T) \times \mathbb{R}^d$. and u solves the PDE (6.1). Then it follows directly from $It\hat{o}$ formula that for each pair $(s,x) \in [0,T) \times \mathbb{R}^d$, the process

$$\begin{aligned} Y_t &= v\left(t, X_t^{s,x}\right), & t \in [s,T] \\ Z_t &= Dv\left(t, X_t^{s,x}\right), & t \in [s,T] \\ \Gamma_t &= D^2 v\left(t, X_t^{s,x}\right), & t \in [s,T] \\ A_t &= \mathcal{L}Dv\left(t, X_t^{s,x}\right), & t \in [s,T] \end{aligned}$$

Where $u(\cdot,\cdot)$ is a map from $[0,T] \times \mathbb{R}^m \to \mathbb{R}$; $Du(x)$ and $D^2u(x)$stand for the gradient and the Hessian matrix of u with respect to x, respectively.
Define the function:

$$\hat{f} = \hat{f}\left(t, X_t, Y_t, Z_t, \Gamma_t\right) = f\left(t, X_t, u\left(t, X_t\right), Du\left(t, X_t\right), D^2 u\left(t, X_t\right)\right)$$

$$F\left(t, x, u, Du, D^2u\right) = f\left(t, x, u, Du, D^2u\right) + b^\top(t,x)Du + \frac{1}{2}\sigma^\top\sigma(t,x)D^2u(t,x)$$

We could construct following 2FBSDE:

$$X_t = x + \int_{t_0}^t b\left(s, X_s\right) \mathrm{d}s + \int_{t_0}^t \sigma\left(s, X_s\right) \mathrm{d}W_s \tag{8.2}$$

$$Y_t = g\left(X_T\right) + \int_t^T \hat{f}\left(s, X_s, Y_s, Z_s, \Gamma_s\right) \mathrm{d}s - \int_t^T Z_s^\top \mathrm{d}W_s, \quad t \in [t_0, T] \tag{8.3}$$

$$Z_t = Z_{t_0} + \int_{t_0}^t A_s \,\mathrm{d}s + \int_{t_0}^t \Gamma_s \,\mathrm{d}W_s, \tag{8.4}$$

8.1 Numerical scheme for 2PDE

We give the a partition $\mathcal{T}$ in the time interval $[t_0, T]$:

$$\mathcal{T} : t_0 < t_1 < \cdots < t_N = T$$

For all $n, \in \{1,2\ldots,N\}, k\in\mathbb{N}$ satisfy $n+k\leq N$. We denote $\Delta t_{t_n,k}=t_{n+k}-t_n$
For $t\geq t_n$, $\Delta W_{t_n,k}=W_{t_{n+k}}-W_{t_n}, \Delta t_{t_n,t}=t-t_n$ and $\Delta W_{t_n,t}=W_t-W_{t_n}$ We let $\Theta_t=(X_t,Y_t,Z_t,\Gamma_t)$ be the adapted solution. We denote $\mathbf{E}_{t_n}^x[\cdot]=\mathbf{E}\left[\cdot\mid\mathcal{F}_{t_n}^{t_n,x}\right]$, and take expectation $\mathbf{E}_{t_n}^x[\cdot]$ on the both sides of the equation(6.3):

$$\mathbf{E}_{t_n}^x\left[Y_t\right]=\mathbf{E}_{t_n}^x\left[g\left(X_T\right)\right]+\int_t^T\mathbf{E}_{t_n}^x\left[\hat{f}\left(s,\Theta_s\right)\right]\mathrm{d}s,\quad t\in[t_n,T] \tag{8.5}$$

Then we take the derivatives with respect to t, we could obtain follow equation:

$$\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t\right]}{\mathrm{d}t}=-\mathbf{E}_{t_n}^x\left[\hat{f}\left(t,\Theta_t\right)\right],\quad t\in[t_n,T] \tag{8.6}$$

If the function $\mathbf{E}_{t_n}^x\left[\hat{f}\left(s,\Theta_s\right)\right]$ is continuous at s=t. For the equations(6.3) and (6.4), we have:

$$\begin{array}{ll}Y_{t_n}=Y_t+\int_{t_n}^t\hat{f}\left(s,\Theta_s\right)\mathrm{d}s-\int_{t_n}^tZ_s\,\mathrm{d}W_s, & t\in[t_n,T]\\ Z_{t_n}=Z_t-\int_{t_n}^tA_s\,\mathrm{d}s-\int_{t_n}^t\Gamma_s\,\mathrm{d}W_s, & \end{array} \tag{8.7}$$

Multiplying $\Delta W_{t_n,,t}^{\top}$ on both sides, and take the expectation $\mathbf{E}_{t_n}^x[\cdot]$ at the time t_n, when $t\in[t_n,T)$:

$$\begin{array}{l}0=\mathbf{E}_{t_n}^x\left[Y_t\Delta W_{t_n,t}^{\top}\right]+\int_{t_n}^t\mathbf{E}_{t_n}^x\left[\hat{f}\left(s,\Theta_s\right)\Delta W_{t_n,s}^{\top}\right]\mathrm{d}s-\int_{t_n}^t\mathbf{E}_{t_n}^x\left[Z_s\right]\mathrm{d}s\\ 0=\mathbf{E}_{t_n}^x\left[Z_t\Delta W_{t_n,t}^{\top}\right]-\int_{t_n}^t\mathbf{E}_{t_n}^x\left[A_s\Delta W_{t_n,s}^{\top}\right]\mathrm{d}s-\int_{t_n}^t\mathbf{E}_{t_n}^x\left[\Gamma_s\right]\mathrm{d}s\end{array} \tag{8.8}$$

We assume the functions in (6.8) is continuous, then we have:

$$\begin{array}{ll}\dfrac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t\Delta W_{t_n,t}^{\top}\right]}{\mathrm{d}t}=-\mathrm{E}_{t_n}^x\left[\hat{f}\left(t,\Theta_t\right)\Delta W_{t_n,t}^{\top}\right]+\mathrm{E}_{t_n}^x\left[Z_t\right], & t\in[t_n,T]\\ \dfrac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Z_t\Delta W_{t_n,t}^{\top}\right]}{\mathrm{d}t}=\mathrm{E}_{t_n}^x\left[A_t\Delta W_{t_n,t}^{\top}\right]+\mathbf{E}_{t_n}^x\left[\Gamma_t\right], & t\in[t_n,T]\end{array} \tag{8.9}$$

Fully discrete
At first we give the partition $\mathcal{D}_{\mathbf{h}}$:

$$\mathcal{D}_{\mathbf{h}}:=\left\{\mathcal{D}_{h_n}^n\right\}_{n=0,1,\ldots,N}$$

We could also apply (5.3) to get:

$$\left\{\begin{array}{l}\left.\dfrac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t\right]}{\mathrm{d}t}\right|_{t=t_n}=\displaystyle\sum_{i=0}^k\alpha_{k,i}\mathbf{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}^{t_n,x}\right]+R_{y,n}^k,\\ \left.\dfrac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t\Delta W_{t_n,t}^{\top}\right]}{\mathrm{d}t}\right|_{t=t_n}=\displaystyle\sum_{i=1}^k\alpha_{k,i}\mathbb{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}^{t_n,x}\Delta W_{n,i}^{\top}\right]+R_{z,n}^k,\\ \left.\dfrac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Z_t\Delta W_{t_n,t}^{\top}\right]}{\mathrm{d}t}\right|_{t=t_n}=\displaystyle\sum_{i=1}^k\alpha_{k,i}\mathbf{E}_{t_n}^x\left[\bar{Z}_{t_{n+i}}^{t_n,x}\Delta W_{n,i}^{\top}\right]+R_{\Gamma,n}^k,\end{array}\right. \tag{8.10}$$

where the parameters $\alpha_{k,i}$ have been given in (5.2).$R_{y,n}^k$, $R_{z,n}^k$, and $R_{\Gamma,n}^k$

$$\begin{array}{l}R_{y,n}^k=\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t\right]}{\mathrm{d}t}\right|_{t=t_n}-\sum_{i=0}^{\kappa}\alpha_{k,i}\mathrm{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}^{t_n,x}\right]\\ R_{z,n}^k=\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Y_t\Delta W_{t_n,t}^{\top}\right]}{\mathrm{d}t}\right|_{t=t_n}-\sum_{i=1}^k\alpha_{k,i}\mathbf{E}_{t_n}^x\left[\bar{Y}_{t_{n+i}}^{t_n,x}\Delta W_{n,i}^{\top}\right]\\ R_{\Gamma,n}^k=\left.\frac{\mathrm{d}\mathbf{E}_{t_n}^x\left[Z_t\Delta W_{t_n,t}^{\top}\right]}{\mathrm{d}t}\right|_{t=t_n}-\sum_{i=1}^k\alpha_{k,i}\mathbf{E}_{t_n}^x\left[\bar{Z}_{t_{n+i}}^{t_n,x}\Delta W_{n,i}^{\top}\right]\end{array} \tag{8.11}$$

With the equations(6.6),(6.9), we can get

$$\left\{\begin{aligned} Z_{t_n} &= \sum_{i=1}^{k} \alpha_{k,i} \mathbf{E}_{t_n}^{x} \left[\bar{Y}_{t_{n+i}}^{t_n,x} \Delta W_{n,i}^{\top}\right] + R_{z,n}^{k} \\ \Gamma_{t_n} &= \sum_{i=1}^{k} \alpha_{k,i} \mathbf{E}_{t_n}^{x} \left[\bar{Z}_{t_n+i}^{t_n,x} \Delta W_{n,i}^{\top}\right] + R_{\Gamma,n}^{k} \\ -\alpha_{k,0} Y_{t_n} &= \sum_{i=1}^{k} \alpha_{k,i} \mathbf{E}_{t_n}^{x} \left[\bar{Y}_{t_{n+i}}^{t_n,x}\right] + \hat{f}\left(t_n, x, Y_{t_n}, Z_{t_n}, \Gamma_{t_n}\right) + R_{y,n}^{k} \end{aligned}\right. \tag{8.12}$$

where $Y_{t_n} = \bar{Y}_{t_n}^{t_n,x}, Z_{t_n} = \bar{Z}_{t_n}^{t_n,x}$

Since $\bar{Y}, \bar{Z}$, are the functions of $X^{n,j}$.We have

$$\begin{aligned} \mathbf{E}_{t_n}^{x}\left[\bar{Y}_{t_{n+i}} \Delta W_{n,i}^{\top}\right] &= \mathbf{E}_{t_n}^{x}\left[Y_{t_{n+i}}\left(x + b\left(t_n, x\right) \Delta t_{n,i} + \sigma\left(t_n, x\right) \Delta W_{n,i}\right) \Delta W_{n,i}^{\top}\right] \\ \mathbf{E}_{t_n}^{x}\left[\bar{Z}_{t_{n+i}} \Delta W_{n,i}^{\top}\right] &= \mathbf{E}_{t_n}^{x}\left[Z_{t_{n+i}}\left(x + b\left(t_n, x\right) \Delta t_{n,i} + \sigma\left(t_n, x\right) \Delta W_{n,i}\right) \Delta W_{n,i}^{\top}\right] \\ \mathbf{E}_{t_n}^{x}\left[\bar{Y}_{t_{n+i}}\right] &= \mathbf{E}_{t_n}^{x}\left[Y_{t_{n+i}}\left(x + b\left(t_n, x\right) \Delta t_{n,i} + \sigma\left(t_n, x\right) \Delta W_{n,i}\right)\right] \end{aligned} \tag{8.13}$$

Then we could also use the Monte-Carlo Method to estimate the condition Expectation here, just like in section(4.2). In short, I denote the operator as $\widehat{\mathbf{E}}^{n,x}[\cdot]$
we have

$$Z_{t_n} = \sum_{j=1}^{k} \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[Y_{t_{n+j}} \Delta W_{n,j}^{\top}\right] - R_{z,n}^{k} + R_{z,n}^{k,\mathbb{E}}$$

$$\Gamma_{t_n} = \sum_{j=1}^{k} \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[Z_{t_{n+j}}^{\top} \Delta W_{n,j}^{\top}\right] - R_{\Gamma,n}^{k} + R_{\Gamma,n}^{k,\mathbb{E}}$$

$$-\alpha_{k,0} Y_{t_n} = \sum_{j=1}^{k} \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[Y_{t_{n+j}}\right] + \hat{f}\left(t_n, x, Y_{t_n}, Z_{t_n}\right) + R_{y,n}^{k} + R_{y,n}^{k,\mathbb{E}}$$

$$R_{z,n}^{k,\mathbb{E}} = \sum_{j=1}^{k} \alpha_{k,j}\left(\mathbb{E}_{t_n}^{x} - \hat{\mathbb{E}}^{n,x}\right)\left[\bar{Y}_{t_{n+j}} \Delta W_{n,j}^{\top}\right]$$

$$R_{\Gamma,n}^{k,\mathbb{E}} = \sum_{j=1}^{k} \alpha_{k,j}\left(\mathbb{E}_{t_n}^{x} - \hat{\mathbb{E}}^{n,x}\right)\left[\bar{Z}_{t_{n+j}}^{\top} \Delta W_{n,j}^{\top}\right]$$

$$R_{y,n}^{k,\mathbb{E}} = -\sum_{j=1}^{k} \alpha_{k,j}\left(\mathbb{E}_{t_n}^{x} - \hat{\mathbb{E}}^{n,x}\right)\left[\bar{Y}^{n+j}\right],$$

Where $R_{y,n}^{k,\mathbb{E}}, R_{z,n}^{k,\mathbb{E}}$ and $R_{\Gamma,n}^{k,\mathbb{E}}$ are approximate conditional expectation errors.

If we ignore these errors, we will have: Given $\{u^n(x)$ and $Z^n(x) = \bar{\sigma}\left(t_n, x\right) \nabla u^n(x)$ for $n = N, \ldots, N-k+1$ and $x \in \cup_{n=N-k+1}^{N} \mathcal{D}_{h_n}^{n}$, and for $n = N-k, \ldots, 0$, solve $u^n = u^n(x)$ by

$$u^n = Y^n$$

Where Y^n is solved by the following procedure

$$\begin{aligned} X^{n,j} &= x + b\left(t_n, x\right) \Delta t_{n,j} + \sigma\left(t_n, x\right) \Delta W_{n,j}, \quad j = 1, \ldots, k \\ Z^n &= \sum \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[Y^{n+j} \Delta W_{n,j}^{\top}\right] \\ \Gamma^n &= \sum_{j=1} \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[\left(Z^{n+j}\right)^{\top} \Delta W_{n,j}^{\top}\right] \\ -\alpha_{k,0} Y^n &= \sum_{j=1} \alpha_{k,j} \widehat{\mathbf{E}}^{n,x}\left[Y^{n+j}\right] + \hat{f}\left(t_n, x, Y^n, Z^n, \Gamma^n\right) \end{aligned} \tag{8.14}$$

Reference: Kong Tao, Weidong Zhao , Shige Peng, High-Accuracy Numerical Schemes for 2nd-order Forward-Backward Stochastic Differential Equations and its Application, Doctor of Philosophy Thesis at ShanDong university,2015