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A novel fitted finite volume method for the Black–Scholes equation governing option pricing

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In this paper we present a novel numerical method for a degenerate partial differential equation, called the Black–Scholes equation, governing option pricing. The method is based on a fitted finite volume spatial discretization and an implicit time stepping technique. To derive the error bounds for the spatial discretization of the method, we formulate it as a Petrov–Galerkin finite element method with each basis function of the trial space being determined by a set of two-point boundary value problems defined on element edges. Stability of the discretization is proved and an error bound for the spatial discretization is established. It is also shown that the system matrix of the discretization is an M-matrix so that the discrete maximum principle is satisfied by the discretization. Numerical experiments are performed to demonstrate the effectiveness of the method.

1. Introduction

Financial derivatives markets form an enormous global sector of our economy. In general, derivative securities consist of three major parts: *Forwards and Future* (obligation to buy or sell), *Options* (right to buy or sell) and *Swaps* (simultaneous selling and purchasing). In particular, the first two form the basis of derivative securities. It is known that a traded option is a contract which gives to its owner the right to buy (*call option*) or to sell (*put option*) a fixed quantity of assets of a specified stock at a fixed price (*exercise* or *strike price*) on (European option) or before (American option) a given date (*expiry date*). The market prices of the rights to buy and to sell are called *call prices* and *put prices*, respectively. It was shown by Black & Scholes (1973) that these option prices satisfy a second-order partial differential equation with respect to the time horizon t and the underlying asset price x. This equation is now known as the Black–Scholes equation, and can be solved exactly when the coefficients are constants. However, in many practical situations, numerical solutions are normally sought. Therefore, efficient and accurate numerical algorithms are essential for solving this problem accurately. The first numerical approach to the Black–Scholes equations was the lattice technique proposed in Cox *et al.* (1979) and improved in Hull & White (1988). That approach is equivalent to an explicit time-stepping scheme. Other numerical schemes based on classical finite difference methods applied to constant-coefficient heat equations have also been developed (cf. Barles, 1997; Barles *et al.*, 1995; Courtadon, 1982; Hull & White, 1996; Rogers

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& Talay, 1997; Schwartz, 1977; Vazquez, 1998; Wilmott *et al.*, 1993). The reason for this is that when the coefficients of the Black–Scholes equation are constant or space-independent, the equation can be transformed into a diffusion equation. In this case the problem is said to be path-independent. However, when a problem is path-dependent, this transformation is impossible, and thus the Black–Scholes equation in the original form needs to be solved. Since the Black–Scholes equation becomes degenerate at the underlying asset price $x = 0$, classical finite difference methods may fail to give accurate approximations when x is small. To overcome this difficulty, some authors suggest to solve the differential equation in a truncated space interval excluding the point singularity $x = 0$ (cf. for example, Vazquez, 1998). Others tend to use a transformation technique which transforms the space interval $(0, X]$ into a semi-infinite interval (cf. for example, Barles, 1997). Obviously, neither of these approaches resolves the singularity. Furthermore, if the final condition of the problem is chosen to be a step or delta function, an interior layer appears in the solution due to the singularity in the final condition. In this case, the gradient near the layer is very large so that classical methods may fail to yield accurate approximations.

In this paper we present a novel discretization method for the Black–Scholes equation. The method is based on a finite volume formulation of the problem coupled with a fitted local approximation to the solution and on an implicit time-stepping technique. The local approximation is determined by a set of two-point boundary value problems defined on the element edges. This fitting technique is based on the idea proposed by Allen & Southwell (1955) for convection-diffusion equations, and has been extended to triangular and tetrahedral elements by several authors (for example, Miller & Wang, 1994a,b; Wang, 1997; Angermann & Wang, 2003). We shall show that the system matrix of the discretization scheme is an M-matrix, so that the discretization is monotonic. In this case the discrete maximum principle is satisfied and thus the discrete solution is always non-negative. To analyse the method, we formulate it as a Petrov–Galerkin finite element method in which each of the basis functions of the trial space is determined by a set of ordinary differential equations defined on element edges. Using this formulation, we establish the stability of the method with respect to a discrete energy norm, and show that the error of the numerical solution in the energy norm is bounded by $\mathcal{O}(h)$, where h denotes the mesh parameter. Without loss of generality, we shall discuss the method using the model for European options in our paper. Naturally, the method is applicable to American options if it is used together with a technique for free boundary problems. We will discuss this in a forthcoming paper. The rest of the paper is organized as follows.

In the next section we discuss the continuous model of the Black–Scholes equations and rewrite the spatial derivatives in a self-adjoint form. The discretization method is described in Section 3. In Section 4, we present a stability and error analysis for the finite volume method within a Petrov–Galerkin finite element framework. It is shown that the finite element solution converges to the exact solution at the rate of $\mathcal{O}(h)$. Numerical experiments for various test problems will be presented in Section 5.

2. The continuous problem

Let V denote the value of a European call or put option and let x denote the price of the underlying asset. It is known that V satisfies the following Black–Scholes equation (see, for example, Wilmott *et al.*, 1993):

$$LV := -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 V}{\partial x^2} - (r(t)x - D(x,t))\frac{\partial V}{\partial x} + rV = 0, \tag{2.1a}$$

for $(x, t) \in I \times [0, T)$, with the boundary and final (or payoff) conditions

$$V(0,t) = g_1(t) \quad t \in [0, T), \tag{2.1b}$$

$$V(X,t) = g_2(t) \quad t \in [0, T), \tag{2.1c}$$

$$V(x,T) = g_3(x) \quad x \in \bar{I}, \tag{2.1d}$$

where $I = (0, X) \in \mathbb{R}$, $\sigma > 0$ denotes the volatility of the asset, $T > 0$ the expiry date, r the interest rate and D the dividend. We assume that these given functions g_1, g_2 and g_3 defining the above boundary and final conditions satisfy the following compatibility conditions:

$$g_3(0) = g_1(T) \quad \text{and} \quad g_3(X) = g_2(T). \tag{2.2}$$

Let us introduce $d(x, t)$ such that $D(x, t) = xd(x, t)$. We assume that $d(x, t)$ is continuously differentiable. Then, (2.1a) becomes

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 x^2\frac{\partial^2 V}{\partial x^2} - (r(t) - d(x,t))x\frac{\partial V}{\partial x} + rV = 0, \quad (x,t) \in (0, X) \times [0, T). \tag{2.3}$$

We assume that $r > d$. When $d(x, t) = \mathrm{d}(t)$, the problem is said to be path-independent. Otherwise, it is path-dependent. There are various choices of final/payoff conditions depending on models. For example, for a call option, the most common final condition is the ramp payoff given by

$$V(x,T) = \max(0, x - E), \quad x \in \bar{I} \tag{2.4}$$

where E denotes the exercise price of the option satisfying $0 < E < X$. A second choice is the *cash-or-nothing* payoff given by

$$V(x,T) = B\mathcal{H}(x - E), \quad x \in \bar{I}, \tag{2.5}$$

where $B > 0$ is a constant and $\mathcal{H}$ denotes the Heaviside function. Obviously, this final condition is a step function which is zero if $x < E$ and X if $x \geqslant E$. Another choice is the *bullish vertical spread* payoff defined by

$$V(x,T) = \max(x - E_1) - \max(x - E_2), \quad x \in \bar{I}, \tag{2.6}$$

where E_1 and E_2 are two exercise prices satisfying $0 < E_1 < E_2 < X$. This represents a portfolio of buying one call option with the exercise price E_1 and writing (or issuing) one call option with the same expiry date but a larger exercise price (i.e. E_2). For a detailed discussion on this, we refer to Wilmott *et al.* (1993).

The simplest way to determine the boundary conditions for call options is to choose $V(0, t) = 0$ and $V(X, t) = V(X, T)$. We may also calculate the present value of an amount received at time T. For example, the boundary condition corresponding to (2.4) can be calculated by (see, for example, Vazquez, 1998)

$$V(0,t) = 0, \tag{2.7a}$$

$$V(X,t) = X\exp\left(-\int_t^T d(X,\tau)\mathrm{d}\tau\right) - E\exp\left(-\int_t^T r(\tau)\mathrm{d}\tau\right). \tag{2.7b}$$

We comment that final and boundary conditions for put options can be defined analogously.

Before we discuss the discretization method in the next section, we first transform (2.3) with the non-homogeneous Dirichlet boundary conditions in (2.1b) and (2.1c) into one with the homogeneous boundary condition. This can be achieved by adding $f(x,t) = -LV_0$ to both sides of (2.3) and introducing a new variable $u = V - V_0$, where

$$V_0(x,t) = g_1(t) + \frac{g_2(t) - g_1(t)}{X}x \tag{2.8}$$

and L is the differential operator in (2.1a). The resulting problem can be written in the following self-adjoint form:

$$-\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[a(t)x^2\frac{\partial u}{\partial x} + b(x,t)xu\right] + c(x,t)u = f(x,t), \tag{2.9a}$$

where

$$a = \frac{1}{2}\sigma^2, \tag{2.9b}$$

$$b = r - d - \sigma^2, \tag{2.9c}$$

$$c = r + b - x\frac{\partial d}{\partial x} = 2r - \sigma^2 - \frac{\partial D}{\partial x}. \tag{2.9d}$$

From (2.1b)–(2.1d) we see that the boundary and final conditions for (2.9a) now become

$$u(0,t) = 0 = u(X,t), \quad t \in [0,T), \tag{2.10a}$$

$$u(x,T) = g_3(x) - V_0(x,T), \quad x \in \bar{I}. \tag{2.10b}$$

We note that, using (2.8) and (2.2), it is easy to verify that the boundary and final conditions in (2.10a) and (2.10b) satisfy the compatibility conditions that $g_3(0) - V_0(0,T) = 0 = g_3(X) - V_0(X,T)$.

3. The finite volume method

Let the interval $I = (0,X)$ be divided into N sub-intervals

$$I_i := (x_i, x_{i+1}), \quad i = 0, 1, \ldots, N-1,$$

with $0 = x_0 < x_1 < \cdots < x_N = X$. For each $i = 0, 1, \ldots, N-1$, we put $h_i = x_{i+1} - x_i$ and $h = \max_{0\leqslant i\leqslant N-1} h_i$. We also let $x_{i-1/2} = (x_{i-1} + x_i)/2$ and $x_{i+1/2} = (x_i + x_{i+1})/2$ for each $i = 1, 2, \ldots, N-1$. These mid-points form a second partition of $(0,X)$ if we define $x_{-1/2} = x_0$ and $x_{N+1/2} = x_N$. Integrating both sides of (2.9a) over $(x_{i-1/2}, x_{i+1/2})$ we have

$$-\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial u}{\partial t}\,\mathrm{d}x - \left[x\left(ax\frac{\partial u}{\partial x} + bu\right)\right]_{x_{i-1/2}}^{x_{i+1/2}} + \int_{x_{i-1/2}}^{x_{i+1/2}} cu\mathrm{d}x = \int_{x_{i-1/2}}^{x_{i+1/2}} f\,\mathrm{d}x,$$

for $i = 1, 2, \ldots, N-1$. Applying the mid-point quadrature rule to the first, third and last terms we obtain from the above

$$-\frac{\partial u_i}{\partial t}l_i - \left[x_{i+1/2}\rho(u)|_{x_{i+1/2}} - x_{i-1/2}\rho(u)|_{x_{i-1/2}}\right] + c_i u_i l_i = f_i l_i \tag{3.1}$$

for $i = 1, 2, \ldots, N-1$, where $l_i = x_{i+1/2} - x_{i-1/2}$, $c_i = c(x_i, t)$, $f_i = f(x_i, t)$, u_i denotes the nodal approximation to $u(x_i, t)$ to be determined and $\rho(u)$ is the flux associated with u defined by

$$\rho(u) := ax\frac{\partial u}{\partial x} + bu. \tag{3.2}$$

Clearly, we now need to derive approximations of the continuous flux $\rho(u)$ defined above at the mid-point, $x_{i+1/2}$, of the interval I_i for all $i = 0, 1, \ldots, N-1$. This discussion is divided into two cases for $i \geqslant 1$ and $i = 0$, respectively.

Case I. Approximation of ρ at $x_{i+1/2}$ for $i \geqslant 1$.

Let us consider the following two-point boundary value problem:

$$(axv' + b_{i+1/2}v)' = 0, \quad x \in I_i, \tag{3.3a}$$

$$v(x_i) = u_i, \quad v(x_{i+1}) = u_{i+1}, \tag{3.3b}$$

where $b_{i+1/2} = b(x_{i+1/2}, t)$. Integrating (3.3a) yields the first-order linear equation

$$\rho_i(v) := axv' + b_{i+1/2}v = C_1, \tag{3.4}$$

where C_1 denotes an additive constant. The integrating factor of this linear equation is $\mu = x^{b_{i+1/2}/a}$ and the analytic solution to (3.4) is

$$v = x^{-b_{i+1/2}/a}\left(\int x^{b_{i+1/2}/a}\frac{C_1}{ax}\,\mathrm{d}x + C_2\right) = \frac{C_1}{b_{i+1/2}} + C_2 x^{-b_{i+1/2}/a}, \tag{3.5}$$

where C_2 is also an additive constant. Note that in this deduction we assume that $b_{i+1/2} \neq 0$. But, as will be seen below, this restriction can be lifted as it is the limiting case of the above when $b_{i+1/2} \to 0$. Applying the boundary conditions in (3.3b) to (3.5) we obtain

$$u_i = \frac{C_1}{b_{i+1/2}} + C_2 x_i^{-\alpha_i}, \quad \text{and} \quad u_{i+1} = \frac{C_1}{b_{i+1/2}} + C_2 x_{i+1}^{-\alpha_i}, \tag{3.6}$$

where $\alpha_i = b_{i+1/2}/a$. Solving this linear system gives

$$\rho_i(u) = C_1 = b_{i+1/2}\frac{x_{i+1}^{\alpha_i}u_{i+1} - x_i^{\alpha_i}u_i}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}}. \tag{3.7}$$

This gives a representation for the flux on the right-hand side of (3.4). Note that (3.7) also holds when $\alpha_i \to 0$. This is because

$$\lim_{\alpha_i \to 0} \frac{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}}{b_{i+1/2}} = \frac{1}{a} \lim_{\alpha_i \to 0} \frac{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}}{\alpha_i} = \frac{1}{a}(\ln x_{i+1} - \ln x_i) > 0 \tag{3.8}$$

since $x_i < x_{i+1}$ and $a > 0$. Obviously, $\rho_i(u)$ in (3.7) provides an approximation to the flux $\rho(u)$ at $x_{i+1/2}$.

Case II. Approximation of ρ at $x_{1/2}$.

Note that the analysis in Case I does not apply to the approximation of the flux on $(0, x_1)$ because (3.3a) is degenerate. This can be seen from the expression (3.5). When $\alpha_0 > 0$, we have to choose $C_2 = 0$ as, otherwise, v blows up as $x \to 0$. However, the resulting solution $v = C_1/b_{1/2}$ can never satisfy both of the conditions in (3.3b) unless $u_0 = u_1$. To solve this difficulty, let us re-consider (3.3a)–(3.3b) with an extra degree of freedom in the following form:

$$(axv' + b_{1/2}v)' = C_2, \quad \text{in} \quad (0, x_1), \tag{3.9a}$$

$$v(0) = u_0, \quad v(x_1) = u_1, \tag{3.9b}$$

where C_2 is an unknown constant to be determined. Integrating (3.9a) once we have

$$axv' + b_{1/2}v = C_2 x + C_3.$$

Using the condition $v(0) = u_0$ we have $C_3 = b_{1/2}u_0$, and so the above equation becomes

$$\rho_0(u) := axv' + b_{1/2}v = b_{1/2}u_0 + C_2 x. \tag{3.10}$$

Solving this problem analytically gives

$$v = \begin{cases} u_0 + \frac{C_2 x}{a + b_{1/2}} + C_4 x^{-\alpha_0}, & \alpha_0 \neq -1, \\ u_0 + \frac{C_2}{a} x \ln x + C_4 x, & \alpha_0 = -1, \end{cases} \tag{3.11}$$

where $\alpha_0 = b_{1/2}/a$ as defined before and C_4 is an additive constant (depending on t).

To determine the constants C_2 and C_4, we first consider the case when $\alpha_0 \neq -1$. When $\alpha_0 \geqslant 0$, $v(0) = u_0$ implies that $C_4 = 0$. If $\alpha_0 < 0$, C_4 is arbitrary, so we also choose $C_4 = 0$. Using $v(x_1) = u_1$ in (3.9b) we obtain $C_2 = \frac{1}{x_1}(a + b_{1/2})(u_1 - u_0)$.

When $\alpha_0 = -1$, from (3.11) we see that $v(0) = u_0$ is satisfied for any C_2 and C_4. Therefore, solutions with such C_2 and C_4 are not unique. We choose $C_2 = 0$, and $v(x_1) = u_1$ in (3.9b) gives $C_4 = (u_1 - u_0)/x_1$. Therefore, from (3.10) we have that

$$\rho_0(u) = (axv' + b_{1/2}v)_{x_{1/2}} = \frac{1}{2}[(a + b_{1/2})u_1 - (a - b_{1/2})u_0] \tag{3.12}$$

for both $\alpha_0 = -1$ and $\alpha_0 \neq -1$. Furthermore, (3.11) reduces to

$$v = u_0 + (u_1 - u_0)x/x_1, \qquad x \in [0, x_1]. \tag{3.13}$$

Now, using (3.7) and (3.12) obtained in Case I and Case II respectively, we define a global piecewise constant approximation to $\rho(u)$ by $\rho_h(u)$ satisfying

$$\rho_h(u) = \rho_i(u) \qquad \text{if} \quad x \in I_i \tag{3.14}$$

for $i = 0, 1, \ldots, N-1$.

Substituting (3.7) or (3.12) into (3.14), depending on the value of i, and then the result into (3.1) we obtain

$$-\frac{\partial u_i}{\partial t} l_i + e_{i,i-1}u_{i-1} + e_{i,i}u_i + e_{i,i+1}u_{i+1} = f_i l_i, \tag{3.15a}$$

where

$$e_{1,0} = -\frac{x_1}{4}(a - b_{1/2}) \tag{3.15b}$$

$$e_{1,1} = \frac{x_1}{4}(a + b_{1+1/2}) + \frac{b_{1+1/2}x_{1+1/2}x_1^{\alpha_1}}{x_2^{\alpha_1} - x_1^{\alpha_1}} + c_1 l_1, \tag{3.15c}$$

$$e_{1,2} = -\frac{b_{1+1/2}x_{1+1/2}x_2^{\alpha_1}}{x_2^{\alpha_1} - x_1^{\alpha_1}}, \tag{3.15d}$$

and

$$e_{i,i-1} = -\frac{b_{i-1/2}x_{i-1/2}x_{i-1}^{\alpha_{i-1}}}{x_i^{\alpha_{i-1}} - x_{i-1}^{\alpha_{i-1}}}, \tag{3.15e}$$

$$e_{i,i} = \frac{b_{i-1/2}x_{i-1/2}x_i^{\alpha_{i-1}}}{x_i^{\alpha_{i-1}} - x_{i-1}^{\alpha_{i-1}}} + \frac{b_{i+1/2}x_{i+1/2}x_i^{\alpha_i}}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}} + c_i l_i, \tag{3.15f}$$

$$e_{i,i+1} = -\frac{b_{i+1/2}x_{i+1/2}x_{i+1}^{\alpha_i}}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}}, \tag{3.15g}$$

for $i = 2, 3, \ldots, N-1$. These form an $(N-1) \times (N-1)$ linear system for $\boldsymbol{u} := (u_1(t), \ldots, u_N(t))^\top$ with $u_0(t)$ and $u_N(t)$ in (3.15a) being equal to the given homogeneous boundary conditions in (2.10a).

We now discuss the time discretization of the linear ODE system (3.15a). Let E_i, $i = 1, 2, \ldots, N-1$, be $1 \times (N-1)$ row vectors defined by

$$\begin{aligned} E_1 &= (e_{11}(t), e_{12}(t), 0, \ldots, 0), \\ E_i &= (0, \ldots, 0, e_{i,i-1}(t), e_{i,i}(t), e_{i,i+1}(t), 0, \ldots, 0), \quad i = 2, 3, \ldots, N-2, \\ E_{N-1} &= (0, \ldots, 0, e_{N-1,N}(t), e_{N-1,N-1}(t)), \end{aligned}$$

where $e_{i,i-1}$, $e_{i,i}$ and $e_{i,i+1}$ are defined in (3.15b)–(3.15g) and those entries which are not defined are considered to be zero. Obviously, using E_i, (3.15a) can be rewritten as

$$-\frac{\partial u_i(t)}{\partial t} l_i + E_i(t)\boldsymbol{u}(t) = f_i(t) l_i, \tag{3.16}$$

for $i = 1, 2, \ldots, N-1$. This is a first-order linear ODE system. To discretize this system, we let t_i $(i = 0, 1, \ldots, K)$ be a set of partition points in $[0, T]$ satisfying $T = t_0 > t_1 >$

$\cdots > t_K = 0$. Then, we apply the two-level implicit time-stepping method with a splitting parameter $\theta \in [1/2, 1]$ to (3.16) to yield

$$\frac{u_i^{k+1} - u_i^k}{-\Delta t_k} l_i + \theta E_i^{k+1} \boldsymbol{u}^{k+1} + (1-\theta) E_i^k \boldsymbol{u}^k = (\theta f_i^{k+1} + (1-\theta) f_i^k) l_i$$

for $k = 0, 1, \ldots, K-1$, where $\Delta t_k = t_{k+1} - t_k < 0$, $E_i^k = E_i(t_k)$, $f_i^k = f(x_i, t_k)$ and $\boldsymbol{u}^k$ denotes the approximation of $\boldsymbol{u}$ at $t = t_k$. Let E^k be the $(N-1) \times (N-1)$ matrix given by $E^k = (E_1^k, E_2^k, \ldots, E_{N-1}^k)^\top$. Then, the above linear system can be rewritten as

$$(\theta E^{k+1} + G^k) \boldsymbol{u}^{k+1} = \boldsymbol{f}^k + [G^k - (1-\theta) E^k] \boldsymbol{u}^k \tag{3.17}$$

for $k = 0, 1, \ldots, K-1$, where $G^k = \text{diag}(l_1/(-\Delta t_k), \ldots, l_{N-1}/(-\Delta t_k))$ is an $(N-1) \times (N-1)$ diagonal matrix and $\boldsymbol{f}^k = \theta(f_1^{k+1} l_1, \ldots, f_{N-1}^{k+1} l_{N-1})^\top + (1 - \theta)(f_1^k l_1, \ldots, f_{N-1}^k l_{N-1})^\top$. When $\theta = 1/2$, the time-stepping scheme becomes the Crank–Nicolson scheme and when $\theta = 1$ it is the backward Euler scheme. Both of these schemes are unconditionally stable, and they are of second- and first-order accuracy, respectively.

We now show that, when $|\Delta t_k|$ is sufficiently small, the system matrix of (3.17) is an M-matrix.

THEOREM 3.1 For any given $k = 1, 2, \ldots, K-1$, if $|\Delta t_k|$ is sufficiently small, then the system matrix of (3.17) is an M-matrix.

Proof. Let us first investigate the off-diagonal entries of E^{k+1} in (3.17). From (3.15b)–(3.15g) we see that $e_{i,j} \leqslant 0$ for all $i, j = 1, 2, \ldots, N-1$, $j \neq i$. This is because

$$\frac{b_{i+1/2}}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}} = \frac{a\alpha_i}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}} > 0$$

for all $i = 1, 2, \ldots, N-1$ and all $b_{i+1/2} \neq 0$. From (3.8) we see that this also holds when $b_{i+1/2} \to 0$. This proves that all of the off-diagonal elements of the system matrix of (3.17) are non-positive.

Furthermore, using (3.15b)–(3.15g) and the definitions of E_i^{k+1}, $i = 1, 2, \ldots, N-1$, it is easy to check that the diagonal entries of $(\theta E^{k+1} + G^k)$ are given by

$$\frac{l_1}{-\Delta t_k} + \theta e_{1,1}^{k+1} = \theta \left(\sum_{j=1}^{N-1} |e_{1,j}^{k+1}| \right) + \theta \frac{x_1}{4}(a^{k+1} + b_{1/2}^{k+1}) + \left(\theta c_1^{k+1} + \frac{1}{|\Delta t_k|} \right) l_1,$$

$$\frac{l_j}{-\Delta t_k} + \theta e_{i,i}^{k+1} = \theta \left(\sum_{j=1}^{N-1} |e_{i,j}^{k+1}| \right) + \left(\theta c_j^{k+1} + \frac{1}{|\Delta t_k|} \right) l_j$$

for $i = 2, 3, \ldots, N-1$. Thus, when $|\Delta t_k|$ is sufficiently small, $\theta E^{k+1} + G^k$ is (strictly) diagonally dominant. Therefore, it is an M-matrix (cf. Varga, 1962, p. 85).

REMARK 3.3.1 We remark that $e_{1,0}$, defined in (3.15b), may not be negative. However, this will not cause any difficulties in the method because $e_{1,0}$ does not appear in the system matrix of (3.17).

REMARK 3.3.2 Theorem 3.1 shows that the fully discretized system (3.17) satisfies the discrete maximum principle and thus the above discretization is monotonic. This guarantees that for a non-negative final condition, numerical solutions obtained by this method are non-negative, as option prices should be.

To conclude this section, we note that local approximations to the first and second partial derivatives, $\partial u/\partial x$ and $\partial^2 u/\partial x^2$, can be easily obtained from (3.13) and (3.5). These two quantities, known respectively as the Δ and Γ of an option, are important in practice. In particular, the former is used by financial engineers for constructing portfolios that hedge against risk (or portfolios that are *delta neutral*). This is also known as *delta hedging*.

4. Error estimates for the spatial discretization

We first introduce some standard and special notation to be used in the analysis presented in this section. In what follows, we will use the standard notation for common function spaces such as $C^m(I)$ and $C^m(\bar{I})$, where m is a non-negative integer, and the space of square-integrable functions $L^2(I)$ with the usual L^2-norm $||\cdot||_0$ and the inner product $(\cdot,\cdot)$. We will also use the function space space $L^\infty(S)$ with the norm $||\cdot||_{\infty,S}$ for a given open and measurable set $S \subset \mathbb{R}$. To handle the degeneracy in the Black–Scholes equation, we introduce the following weighted L^2-norm:

$$||v||_{0,w} := \left(\int_0^X x^2 v^2 \mathrm{d}x\right)^{1/2}.$$

The space of all weighted square-integrable functions is defined as

$$L^2_w(I) := \{v : ||v||_{0,w} < \infty\}.$$

We also define a weighted inner product on $L^2_w(I)$ by $(u,v)_w := \int_0^X x^2 uv \mathrm{d}x$. Using a standard argument (see, for example, Brenner & Scott, 1994, Chapters 1 and 2) it is easy to show that the pair $(L^2_w(I), (\cdot,\cdot)_w)$ is a Hilbert space. For brevity, we omit this discussion. Using $L^2(I)$ and $L^2_w(I)$, we define the following weighted Sobolev space:

$$H^1_{0,w}(I) := \{v : v \in L^2(I), v' \in L^2_w(I) \quad \text{and} \quad v(0) = v(X) = 0\},$$

where v' denotes the weak derivative of v. Let $||\cdot||_{1,w}$ be the functional on $H^1_{0,w}(I)$ defined by $||v||_{1,w} = (||v||_0^2 + ||v'||_{0,w}^2)^{1/2}$. Then, it is easy to see that $||\cdot||_{1,w}$ is a norm on $H^1_{0,w}(I)$; it is called the weighted H^1-norm on $H^1_{0,w}(I)$. Furthermore, using the inner products on $L^2(I)$ and $L^2_w(I)$, we define a weighted inner product on $H^1_{0,w}(I)$ by $(\cdot,\cdot)_H := (\cdot,\cdot) + (\cdot,\cdot)_w$. For the pair $(H^1_{0,w}(I), (\cdot,\cdot)_H)$, we have the following lemma.

LEMMA 4.1 The pair $(H^1_{0,w}(I), (\cdot,\cdot)_H)$ is a Hilbert space.

The result is obvious since both pairs $(L^2(I), (\cdot,\cdot))$ and $(L^2_w(I), (\cdot,\cdot)_w)$ are Hilbert spaces. For brevity, we omit a formal proof of this lemma.

REMARK 4.4.1 We remark that it is easy to show that $H^1_{0,w}(I)$ contains the conventional Sobolev space $H^1_0(I)$ as a proper subspace.

From the discussion in the previous section we see that the time discretization of (3.17) is of first-order accuracy if $\theta = 1$ (backward Euler scheme) and of second-order accuracy if $\theta = 1/2$ (Crank–Nicolson scheme). Both of these two cases are unconditionally stable as time-stepping schemes. We now consider the errors in the spatial discretization. To this end, we assume that $\frac{\partial u}{\partial t} = 0$ and consider the finite volume method for the resulting steady-state Black–Scholes equation. This steady-state case is called the *perpetual* form of the Black–Scholes equation. The weak formulation corresponding to the steady-state form of (2.9a) and (2.10a) is as follows.

PROBLEM 4.1 Find $u \in H^1_{0,w}(I)$ such that, for all $v \in H^1_{0,w}(I)$,

$$(ax^2u' + bxu, v') + (cu, v) = (f, v). \tag{4.1}$$

In what follows we first formulate the previous finite volume method as a Petrov–Galerkin finite element method. We will then establish the coercivity of the corresponding bilinear form and use this to derive an upper bound for the approximation error in a proper norm.

First, however, we make the following assumption on the coefficient r, σ and $D = xd$ in (2.9a).

ASSUMPTION 4.1 Assume that the coefficients r, σ and D satisfy

$$3r - D' - \sigma^2 \geqslant \beta > 0$$

for all $x \in \bar{I}$, where β is a positive constant.

REMARK 4.4.2 We remark that the above is just a sufficient condition which guarantees that the bilinear form on the right-hand side of (4.1) is coercive, as shown below. This condition is usually satisfied when the volatility is not too high and the interest rate is not too low. The condition may be far from necessary. Necessary and sufficient conditions under which the bilinear form is coercive are still unknown for this perpetual or time-independent case. However, from Theorem 3.1 we see that the system matrix of the fully discretized time-dependent equation is always an M-matrix. Therefore, it is possible that Assumption 4.1 can be removed if $\frac{\partial u}{\partial t} \neq 0$. This, of course, is a more challenging problem than the stationary one.

The following theorem shows that Problem 4.1 is uniquely solvable under Assumption 4.1.

THEOREM 4.1 Let Assumption 4.1 be satisfied. Then, there exists a unique solution to Problem 4.1.

Proof. We first show that the bilinear form on the right-hand side of (4.1) is coercive. Integrating by parts, we have, for any $v \in H^1_{0,w}(I)$,

$$\int_0^X bxvv'\mathrm{d}x = -\int_0^X (b + xb')v^2\mathrm{d}x - \int_0^X bxvv'\mathrm{d}x$$

which gives

$$\int_0^X bxvv'\mathrm{d}x = -\frac{1}{2}\int_0^X (b + xb')v^2\mathrm{d}x = -\frac{1}{2}\int_0^X (b - xd')v^2\mathrm{d}x.$$

In the above we used (2.9c) and the fact that r and σ are constants. Substituting the above into the bilinear form on the right-hand side of (4.1) and using (2.9c) and (2.9d) we have

$$\begin{aligned}(ax^2v' + bxv, v') + (cv, v) &= (ax^2v', v') + \left(\left(r + \frac{b}{2} - \frac{xd'}{2}\right)v, v\right)\\ &= (ax^2v', v') + \frac{1}{2}((3r - \sigma^2 - D')v, v)\\ &\geqslant \frac{\beta}{2}\left[(ax^2v', v') + (v, v)\right]\\ &\geqslant C||v||^2_{1,w}\end{aligned}$$

by Assumption 4.1, where C denotes a positive constant, independent of v. The above inequality shows that the bilinear form is coercive. Furthermore, using a standard argument it is easy to show that the bilinear form $(ax^2v' + bxv, w') + (cv, w)$ is continuous with respect to the norm $||\cdot||_{1,w}$. Furthermore, $H^1_{0,w}(I)$ is a Hilbert space by Lemma 4.1. Therefore, by the Lax–Milgram lemma, Problem 4.1 is uniquely solvable.

We now discuss the finite element formulation of the discretization scheme. For any $i = 1, 2, \ldots, N-1$, let ψ_i denote the characteristic function given by

$$\psi_i = \begin{cases} 1 & x \in (x_{i-1/2}, x_{i+1/2}) \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

We choose the test space to be $T_h = \mathrm{span}\{\psi_i\}_1^{N-1}$.

To define the trial space, we choose the hat function ϕ_i associated with x_i in the following way. On (x_i, x_{i+1}) we choose ϕ_i so that it satisfies (3.3a) with $\phi_i(x_i) = 1$ and $\phi_i(x_{i+1}) = 0$. Naturally, the solution to this two-point boundary value problem is given in (3.5) where C_1 and C_2 are determined by (3.6) with $u_i = 1$ and $u_{i+1} = 0$. Similarly, we can define $\phi_i(x)$ on the interval (x_{i-1}, x_i) so that $\phi_i(x_{i-1}) = 0$ and $\phi_i(x_i) = 1$. Combining these two solutions and extending $\phi_i(x)$ to the rest of the solution interval we have

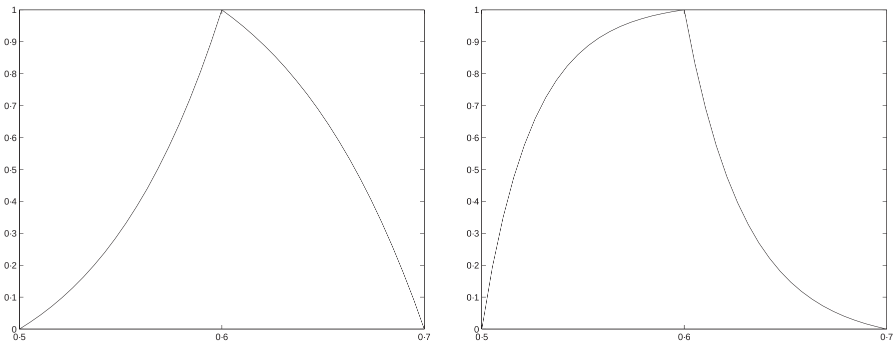
$$\phi_i(x) = \begin{cases} \left[1 - \left(\frac{x_i}{x_{i-1}}\right)^{\alpha_{i-1}}\right]^{-1}\left[1 - \left(\frac{x}{x_{i-1}}\right)^{\alpha_{i-1}}\right], & x \in (x_{i-1}, x_i), \\ \left[1 - \left(\frac{x_i}{x_{i+1}}\right)^{\alpha_i}\right]^{-1}\left[1 - \left(\frac{x}{x_{i+1}}\right)^{\alpha_i}\right], & x \in [x_i, x_{i+1}), \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

The only exception is that $\phi_1(x)$ in the subinterval $[0, x_1]$ is given by the linear function in (3.13) with $u_0 = 0$ and $u_1 = 1$. For this set of basis functions we have the following result.

THEOREM 4.2 For each $i = 1, \ldots, N-1$, the function ϕ_i is monotonically increasing and decreasing on (x_{i-1}, x_i) and (x_i, x_{i+1}) respectively. Furthermore, ϕ_i and ϕ_{i+1} satisfy $\phi_i(x) + \phi_{i+1}(x) = 1$ for all $x \in (x_i, x_{i+1})$ and $i = 1, 2, \ldots, N-1$.

Proof. Differentiating ϕ_i on $I_i = (x_i, x_{i+1})$ we have

$$\phi_i'(x) = \frac{-\alpha_i}{1 - \left(\frac{x_i}{x_{i+1}}\right)^{\alpha_i}} \frac{x^{\alpha_i}}{x_{i+1}^{\alpha_i}}, \qquad x \in (x_i, x_{i+1}).$$



$\alpha = 10$ $\alpha = -20$

FIG. 1. Examples of the hat functions for different values of α.

Since $x_{i+1} - x_i > 0$, we have $\alpha_i/[1 - (x_i/x_{i+1})^{\alpha_i}] > 0$ for all α_i. Therefore, $\phi'(x) < 0$, and so ϕ is monotonically decreasing in I_i. Similarly, ϕ_i is monotonically increasing in (x_{i-1}, x_i). The proof of the last part of this theorem is trivial and is omitted here.

Two examples of these hat functions with constant α are plotted in Fig. 1. Note that these basis functions are not explicitly used in the discretization discussed in the previous section.

Now the finite element trial space is chosen to be $S_h = \operatorname{span}\{\phi_i\}_1^{N-1}$. Let P denote the mass lumping operator from $C^0(I)$ to T_h such that for any $v \in C^0(I)$, $P(v) = \sum_{i=0}^{N} v(x_i)\psi_i(x)$ where ψ_i is defined in (4.2). Using S_h and T_h, we define the following Petrov–Galerkin problem.

PROBLEM 4.2 Find $u_h \in S_h$ such that for all $v_h \in T_h$

$$A(u_h, v_h) := -\sum_{j=1}^{N-1}\left[x\left(ax\frac{\partial u_h}{\partial x} + \hat{b}u_h\right)\right]_{x_{j-1/2}}^{x_{j+1/2}} v_h + (P(cu_h), v_h) = (P(f), v_h), \tag{4.4}$$

where $\hat{b}$ denotes the piecewise constant approximation of b on I satisfying $\hat{b} = b_{i+1/2}$ if $x \in I_i$.

Obviously, $A(\cdot,\cdot)$ is a bilinear form on $S_h \times T_h$. Using the construction of the finite volume method in the previous section and the definitions of the basis functions ϕ_i and ψ_j, it is easy to verify that if $u_h = \sum_{j=1}^{N-1} u_j\phi_j$ and $v_h = \psi_i$ for $i = 1, 2, \ldots, N-1$, (4.4) becomes (3.16) with $\partial u_i/\partial t = 0$.

Note that, when restricted to S_h, the lumping operator P is surjective from S_h to T_h. Using this P, we rewrite Problem 4.2 as the following equivalent Galerkin finite element formulation.

PROBLEM 4.3 Find $u_h \in S_h$ such that, for all $v_h \in S_h$,

$$B(u_h, v_h) = (P(f), P(v_h)), \tag{4.5}$$

where $B(u_h, v_h) = A(u_h, P(v_h))$.

Before further discussion, we first define a functional $||\cdot||_{1,h}$ on S_h by

$$||v_h||_{1,h}^2 = \sum_{j=1}^{N-1} b_{j+1/2}x_{j+1/2}\frac{x_{j+1}^{\alpha_j}+x_j^{\alpha_j}}{x_{j+1}^{\alpha_j}-x_j^{\alpha_j}}(v_{j+1}-v_j)^2 \tag{4.6}$$

for any $v_h = \sum_{j=0}^{N} v_j\phi_j \in S_h$, with $v_N = 0$. It is easy to show that $||\cdot||_{1,h}$ is a norm on S_h, because $(x_{j+1}^{\alpha_j}+x_j^{\alpha_j})/b_{j+1/2} > 0$ for any α_j $(= b_{j+1/2}/a)$. (The limiting case of $\alpha_j \to 0$ is given in (3.8)). $||\cdot||_{1,h}$ is a weighted discrete energy norm on S_h. Using this norm we define the following weighted discrete H^1-norm on S_h:

$$||v_h||_h^2 = ||v_h||_{1,h}^2 + \sum_{j=1}^{N-1} v_j^2 l_j \tag{4.7}$$

with the convention that $v_0 = v_N = 0$. Using this norm, we have the following theorem.

THEOREM 4.3 Let Assumption 4.1 be fulfilled. If h is sufficiently small, then, for all $v_h \in S_h$, we have

$$B(v_h, v_h) \geqslant C||v_h||_h^2, \tag{4.8}$$

where C denotes a positive constant, independent of h and v_h.

Proof. Let $v_h = \sum_{j=1}^{N-1} v_i\phi_i \in S_h$. Then, we have

$$\begin{aligned}
B(v_h, v_h) &= -\sum_{j=1}^{N-1}\left[x\left(ax\frac{\partial v_h}{\partial x}+\hat{b}v_h\right)\right]_{x_{j-1/2}}^{x_{j+1/2}} P(v_h) + (P(cv_h), P(v_h)) \\
&= -\sum_{j=1}^{N-1}\left(x_{j+\frac{1}{2}}\left(ax\frac{\partial v_h}{\partial x}+\hat{b}v_h\right)_{x_{j+\frac{1}{2}}} - x_{j-\frac{1}{2}}\left(ax\frac{\partial v_h}{\partial x}+\hat{b}v_h\right)_{x_{j-\frac{1}{2}}}\right) \\
&\quad \times v_j + \sum_{j=1}^{N-1} c_j v_j^2 l_j.
\end{aligned}$$

Re-arranging the first sum and using the flux representations (3.14) (with ρ_i given by (3.7)) and (3.12) (with $u_0 = 0$) we have

$$\begin{aligned}
B(v_h, v_h) &= \frac{x_{1/2}(a+b_{\frac{1}{2}})}{2}v_1^2 + \sum_{j=1}^{N-1} b_{j+\frac{1}{2}}x_{j+\frac{1}{2}}\frac{x_{j+1}^{\alpha_j}v_{j+1}-x_j^{\alpha_j}v_j}{x_{j+1}^{\alpha_j}-x_j^{\alpha_j}} \\
&\quad \times (v_{j+1}-v_j) + \sum_{j=1}^{N-1} c_j v_j^2 l_j \\
&=: x_{1/2}\frac{(a+b_{1/2})}{2}v_1^2 + I + \sum_{j=1}^{N-1} c_j v_j^2 l_j,
\end{aligned} \tag{4.9}$$

since $x_1 = 2x_{1/2}$ and $v_0 = 0$. For the term I, we have

$$
\begin{aligned}
I &= \sum_{j=1}^{N-1} b_{j+1/2} x_{j+1/2} \frac{x_{j+1}^{\alpha_j}(v_{j+1} - v_j) + (x_{j+1}^{\alpha_j} - x_j^{\alpha_j}) v_j}{x_{j+1}^{\alpha_j} - x_j^{\alpha_j}} (v_{j+1} - v_j) \\
&= \sum_{j=1}^{N-1} b_{j+1/2} x_{j+1/2} \frac{x_{j+1}^{\alpha_j}}{x_{j+1}^{\alpha_j} - x_j^{\alpha_j}} (v_{j+1} - v_j)^2 + \sum_{j=1}^{N-1} b_{j+1/2} x_{j+1/2} v_j (v_{j+1} - v_j) \\
&= \sum_{j=1}^{N-1} b_{j+1/2} x_{j+1/2} \frac{x_{j+1}^{\alpha_j}}{x_{j+1}^{\alpha_j} - x_j^{\alpha_j}} (v_{j+1} - v_j)^2 \\
&\quad + \sum_{j=1}^{N-1} b_{j+1/2} x_{j+1/2} \left[-\frac{1}{2} (v_{j+1} - v_j)^2 + \frac{1}{2} (v_{j+1}^2 - v_j^2) \right] \\
&= \frac{1}{2} \sum_{j=1}^{N-1} b_{j+1/2} x_{j+1/2} \frac{x_{j+1}^{\alpha_j} + x_j^{\alpha_j}}{x_{j+1}^{\alpha_j} - x_j^{\alpha_j}} (v_{j+1} - v_j)^2 + \frac{1}{2} [b_{1+1/2} x_{1+1/2} (v_2^2 - v_1^2) \\
&\quad + b_{2+1/2} x_{2+1/2} (v_3^2 - v_2^2) + \cdots + b_{(N-1)+1/2} x_{(N-1)+1/2} (v_N^2 - v_{N-1}^2)] \\
&= \frac{1}{2} ||v_h||_{1,h}^2 + \frac{1}{2} \sum_{j=1}^{N-1} \left(x_{j-\frac{1}{2}} b_{j-\frac{1}{2}} - x_{j+\frac{1}{2}} b_{j+\frac{1}{2}} \right) v_j^2 - \frac{1}{2} x_{1/2} b_{1/2} v_1^2 \\
&= \frac{1}{2} ||v_h||_{1,h}^2 - \frac{1}{2} \sum_{j=1}^{N-1} \left(x_{j-\frac{1}{2}} \frac{b_{j+\frac{1}{2}} - b_{j-\frac{1}{2}}}{l_j} + b_{j+\frac{1}{2}} \right) v_j^2 l_j - \frac{1}{2} x_{1/2} b_{1/2} v_1^2,
\end{aligned}
$$

since $v_N = 0$. Note that r and σ in (2.9c) are constants. (They are functions of t only in the unsteady-state case.) Therefore, substituting the above into (4.9) and using (4.6), (2.9c) and (2.9d) we have

$$
\begin{aligned}
B(v_h, v_h) &= x_{1/2} \frac{(a + b_{\frac{1}{2}})}{2} v_1^2 + \frac{1}{2} ||v_h||_{1,h}^2 + \sum_{j=1}^{N-1} \left(c_j - \frac{x_{j-\frac{1}{2}}}{2} \frac{b_{j+\frac{1}{2}} - b_{j-\frac{1}{2}}}{l_j} - \frac{b_{j+\frac{1}{2}}}{2} \right) \\
&\qquad \times v_j^2 l_j - \frac{1}{2} x_{\frac{1}{2}} b_{\frac{1}{2}} v_1^2 \\
&= x_{1/2} \frac{a}{2} v_1^2 + \frac{1}{2} ||v_h||_{1,h}^2 \\
&\qquad + \sum_{j=1}^{N-1} \left(r + b_j - x_j d'(x_j) - \frac{x_{j-\frac{1}{2}}}{2} \frac{b_{j+\frac{1}{2}} - b_{j-\frac{1}{2}}}{l_j} - \frac{b_{j+\frac{1}{2}}}{2} \right) v_j^2 l_j \\
&= x_{1/2} \frac{a}{2} v_1^2 + \frac{1}{2} ||v_h||_{1,h}^2 \\
&\qquad + \sum_{j=1}^{N-1} \left[r + \frac{b_j}{2} + \frac{d_j - d_{j+\frac{1}{2}}}{2} - \frac{x_j}{2} d'(x_j) \right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\left(x_j d'(x_j) - x_{j-\frac{1}{2}}\frac{d_{j+\frac{1}{2}} - d_{j-\frac{1}{2}}}{l_j}\right)\Bigg] v_j^2 l_j \\
&\geqslant \frac{1}{2}||v_h||_{1,h}^2 + \frac{1}{2}\sum_{j=1}^{N-1}(3r - \sigma^2 - D'(x_j))v_j^2 l_j \\
&\quad + \sum_{j=1}^{N-1}\left[\frac{d_j - d_{j+\frac{1}{2}}}{2} - \frac{1}{2}\left(x_j d'(x_j) - x_{j-\frac{1}{2}}\frac{d_{j+\frac{1}{2}} - d_{j-\frac{1}{2}}}{l_j}\right)\right] v_j^2 l_j.
\end{aligned}
$$

Note that $d_j - d_{j+1/2}$ and $x_j d'(x_j) - x_{j-1/2}(d_{j+1/2} - d_{j-1/2})/l_j$ are of $\mathcal{O}(h)$. When h is sufficiently small, the absolute value of each of these terms is smaller than, say, $\beta/12$. Therefore, (4.8) follows from the above estimate and Assumption 4.1, because $(3r - D'(x_j) - \sigma^2)/2 \geqslant \beta/2$. This completes the proof.

As it turns out, the lower bound on $B(v_h, v_h)$ is just a discrete analogue of that in the continuous case in Theorem 4.1. The continuous and the discrete bilinear form is coercive with respect to the continuous and discrete norm, respectively, under Assumption 4.1.

COROLLARY 4.1 Problem 4.3 has a unique solution.

Proof. This follows from Theorem 4.3 and the Lax–Migram theorem.

We remark, as in the continuous case, that Assumption 4.1 is a sufficient condition for the unique solvability of Problem 4.3, but it is not necessary. As proved in the previous section, the system matrix of the fully discrete problem is always an M-matrix, and thus the fully discretized problem is uniquely solvable even when Assumption 4.1 does not hold. In fact, the method works well in practice without Assumption 4.1, as will be seen in Section 5. However, necessary and sufficient conditions under which (4.8) holds still remain unknown.

We now derive an upper bound on the difference between the approximate and the exact solution in the norm $||\cdot||_h$. The following theorem establishes an estimate for the error in the flux of the S_h-interpolant of a given function.

LEMMA 4.2 Let w be a sufficiently smooth function and let w_I be the S_h-interpolant of w. Then, we have

$$
||\rho(w) - \rho_h(w_I)||_{\infty,I_i} \leqslant C\left(||\rho'(w)||_{\infty,I_i} + ||b'||_{\infty,I_i}||w||_{\infty,I_i}\right) h_i \tag{4.10}
$$

for $i = 0, 1, \ldots, N-1$ where ρ and ρ_h are the fluxes defined in (3.2) and (3.14), respectively, and C is a positive constant, independent of h_i and w.

Proof. Let C denote a generic positive constant, independent of h_i and w. From (3.3a) and (3.3b) we see that the mapping from $\rho(w)$ to $\rho_h(w_I)$ preserves constants. Therefore, by a standard argument we have

$$
||\rho_h(w_I) - \rho_h(w)||_{\infty,I_i} \leqslant Ch_i||\rho'(w)||_{\infty,I_i}.
$$

Hence, from (3.2), (3.14) and the above estimate we have

$$
\begin{aligned}
||\rho(w) - \rho_h(w_I)||_{\infty,I_i} &\leqslant ||\rho_h(w) - \rho_h(w_I) + (b - b_{i+1/2})w||_{\infty,I_i} \\
&\leqslant C\left(h_i||\rho'(w)||_{\infty,I_i} + ||b - b_{i+1/2}||_{\infty,I_i}||w||_{\infty,I_i}\right). \qquad (4.11)
\end{aligned}
$$

Since $b_{i+1/2} = b(x_{i+1/2})$, we have $||b - b_{i+1/2}||_{\infty,I_i} \leqslant Ch_i||b'||_{\infty,I_i}$. Combining this estimate and (4.11) we obtain (4.10).

Using Theorems 4.3 and 4.2 we have the following theorem which establishes the error bound for the solution to Problem 4.3.

THEOREM 4.4 Let Assumption 4.1 be fulfilled, and let u and u_h be respectively the solutions to Problems 4.1 and 4.3. Then, the following bound holds:

$$||u - u_h||_h \leqslant Ch\left(||\rho'(u)||_\infty + ||u'||_\infty + ||b'||_\infty + ||c'||_\infty + ||f'||_\infty\right) \tag{4.12}$$

for some positive constants C, independent of h and u.

Proof. Let u_I be the S_h-interpolant of u and C a generic positive constant, independent of h and u. For any $v_h \in S_h$, multiplying (2.9a) (with $\partial u/\partial t = 0$) by $P(v_h)$ and by integration by parts in the first term we have

$$-\sum_{j=1}^{N-1}[x\rho(u)]_{x_{j-1/2}}^{x_{j+1/2}} P(v_h) + (cu, v_h) = (f, v_h).$$

From this equality, (4.5) and (4.4), and using an argument similar to that in (4.9) we have

$$\begin{aligned} |B(u_h - u_I, v_h)| &= \Bigg|(P(f) - f, P(v_h)) + (cu - P(cu), P(v_h)) \\ &\qquad - \sum_{j=1}^{N-1}[x(\rho(u) - \rho_h(u_I))]_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} v_j\Bigg| \\ &\leqslant \sum_{j=1}^{N-1}[|f_i - f| + |cu - c_i u_i|]\, v_j l_j \\ &\qquad + \left|\sum_{j=0}^{N-1} x_{j+\frac{1}{2}}(\rho(u) - \rho_h(u_I))_{x_{j+\frac{1}{2}}}(v_{j+1} - v_j)\right| \\ &=: R_1 + R_2. \end{aligned} \tag{4.13}$$

Since the lumping operator P preserves constants, we have

$$\begin{aligned} R_1 &\leqslant \sum_{j=1}^{N-1}[|f_i - f| + |(c - c_i)u| + |c_i(u - u_i)|]\, v_j l_j \\ &\leqslant Ch(||f'||_\infty + ||c'||_\infty + ||u'||_\infty)\sum_{j=1}^{N-1} v_j l_j \\ &\leqslant Ch(||f'||_\infty + ||c'||_\infty + ||u'||_\infty)\left(\sum_{j=1}^{N-1} v_j^2 l_j\right)^{1/2}. \end{aligned} \tag{4.14}$$

Using (4.10) and the Cauchy–Schwarz inequality we estimate the last term in (4.13) by

$$
\begin{aligned}
R_2 &\leqslant Ch\left(||\rho'(u)||_\infty + ||b'||_\infty ||u||_\infty\right) \sum_{j=0}^{N-1} x_{j+\frac{1}{2}} |v_{j+1} - v_j| \\
&\leqslant Ch\left(||\rho'(u)||_\infty + ||b'||_\infty ||u||_\infty\right) \\
&\quad \times \left(x_1 |v_1| + \sum_{j=1}^{N-1} x_{j+\frac{1}{2}}^{1/2} \left(\frac{x_{j+1}^{\alpha_j} - x_j^{\alpha_j}}{b_{j+\frac{1}{2}} (x_{j+1}^{\alpha_j} + x_j^{\alpha_j})} \right)^{1/2} \right. \\
&\quad \left. \times \frac{x_{j+\frac{1}{2}}^{1/2} b_{j+\frac{1}{2}}^{1/2} (x_{j+1}^{\alpha_j} + x_j^{\alpha_j})^{1/2}}{(x_{j+1}^{\alpha_j} - x_j^{\alpha_j})^{1/2}} |v_{j+1} - v_j| \right) \\
&\leqslant Ch\left(||\rho'(u)||_\infty + ||b'||_\infty ||u||_\infty\right) \\
&\quad \times \left[x_1 |v_1| + \left(\sum_{j=1}^{N-1} x_{j+1/2} \frac{x_{j+1}^{\alpha_j} - x_j^{\alpha_j}}{b_{j+\frac{1}{2}} (x_{j+1}^{\alpha_j} + x_j^{\alpha_j})} \right)^{\frac{1}{2}} \right. \\
&\quad \left. \times \left(\sum_{j=1}^{N-1} b_{j+\frac{1}{2}} x_{j+\frac{1}{2}} \frac{x_{j+1}^{\alpha_j} + x_j^{\alpha_j}}{x_{j+1}^{\alpha_j} - x_j^{\alpha_j}} (v_{j+1} - v_j)^2 \right)^{\frac{1}{2}} \right] \\
&\leqslant Ch\left(||\rho'(u)||_\infty + ||b'||_\infty ||u||_\infty\right) \\
&\quad \times \left[x_1 |v_1| + \left(\sum_{j=1}^{N-1} \frac{x_{j+\frac{1}{2}}}{b_{j+\frac{1}{2}}} \frac{x_{j+1}^{\alpha_j} - x_j^{\alpha_j}}{x_{j+1}^{\alpha_j} + x_j^{\alpha_j}} \right)^{1/2} ||v_h||_{1,h} \right].
\end{aligned}
\tag{4.15}
$$

Note that $h_j/(2x_{j+1/2}) = h_j/(2x_j + h_j) < 1$ for $j \geqslant 1$. Using Taylor's expansion we have

$$
\begin{aligned}
\frac{x_{j+1}^{\alpha_j} - x_j^{\alpha_j}}{x_{j+1}^{\alpha_j} + x_j^{\alpha_j}} &= \frac{\left(1 + \frac{h_j}{2x_{j+1/2}}\right)^{\alpha_j} - \left(1 - \frac{h_j}{2x_{j+1/2}}\right)^{\alpha_j}}{\left(1 + \frac{h_j}{2x_{j+1/2}}\right)^{\alpha_j} + \left(1 - \frac{h_j}{2x_{j+1/2}}\right)^{\alpha_j}} \\
&= \frac{\left(1 + \alpha_j \mathcal{O}\left(\frac{h_j}{2x_{j+1/2}}\right)\right) - \left(1 - \alpha_j \mathcal{O}\left(\frac{h_j}{2x_{j+1/2}}\right)\right)}{\left(1 + \alpha_j \mathcal{O}\left(\frac{h_j}{2x_{j+1/2}}\right)\right) + \left(1 - \alpha_j \mathcal{O}\left(\frac{h_j}{2x_{j+1/2}}\right)\right)} \\
&\leqslant C\alpha_j \frac{h_j}{x_{j+1/2}}.
\end{aligned}
$$

From this we see that the last sum in (4.15) can be estimated as

$$
\sum_{j=1}^{N-1} \frac{x_{j+1/2}}{b_{j+1/2}} \frac{x_{j+1}^{\alpha_j} - x_j^{\alpha_j}}{x_{j+1}^{\alpha_j} + x_j^{\alpha_j}} \leqslant C \sum_{j=1}^{N-1} h_j \frac{\alpha_j}{b_{j+1/2}} \leqslant CX,
\tag{4.16}
$$

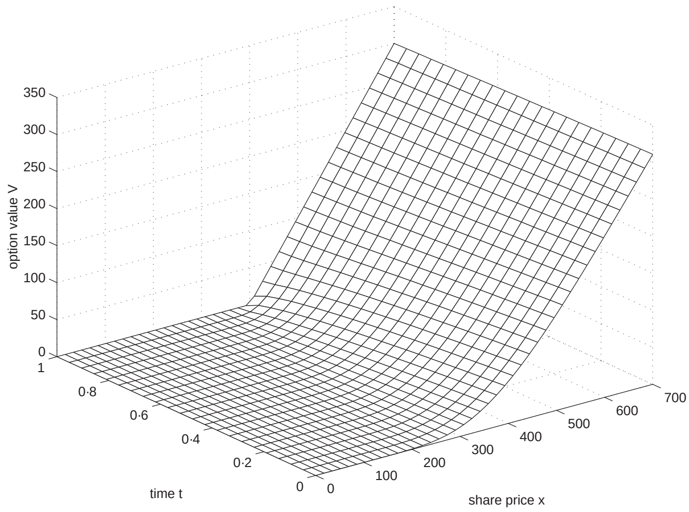


FIG. 2. Computed option value for Test 1.

because $\alpha_j = b_{j+1/2}/a$. Substituting the estimate (4.16) into (4.15) we have

$$R_2 \leqslant Ch \left(||\rho'(u)||_\infty + ||b'||_\infty ||u||_\infty\right) \left(h|v_1| + ||v_h||_{1,h}\right).$$

Now, replacing R_1 and R_2 in (4.13) by (4.14) and the above estimate, respectively, we obtain

$$|B(u_h - u_I, v_h)| \leqslant Ch \left(||\rho'(u)||_\infty + ||u'||_\infty + ||b'||_\infty + ||c'||_\infty + ||f'||_\infty\right) ||v_h||_h.$$

On taking $v_h = u_h - u_I$ in the above estimate and using Theorem 4.2 we obtain (4.12).

Finally, we note that since the norm $||\cdot||_h$ depends only on the nodal values of a given function at the mesh nodes, Theorem 4.4 still holds if we assume that u_I is the interpolant of u in the conventional piecewise linear finite element space constructed on T_h.

5. Numerical experiments

To show the efficiency and usefulness of the discretization method, various European option test problems, with different initial and boundary conditions and different choices of parameters, were solved. The splitting parameter θ is equal to 0.5 in each of the test problems so that the time stepping scheme has second-order accuracy. All the numerical results were computed in double precision on a Pentium PC under Linux.

Test 1. Call option with the final condition (2.4) and boundary conditions (2.7a)–(2.7b). Parameters: $X = 700$, $T = 1{\cdot}0$, $r = 0{\cdot}1$, $\sigma = 0{\cdot}3$, $d = 0{\cdot}04$ and $E = 400$.

To solve this problem, we chose to divide $(0, X)$ and $(0, T)$ uniformly into 41 and 21 sub-intervals, respectively. The numerical result is depicted in Fig. 2.

TABLE 1 *Computed errors in the two discrete norms*

norm\mesh	11 × 5	21 × 9	41 × 17	81 × 33	161 × 65
$\|\| \cdot \|\|_h$	2·178	1·070	0·511	0·240	0·104
$\|\| \cdot \|\|_{h,\infty}$	1·013	0·551	0·267	0·128	0·055

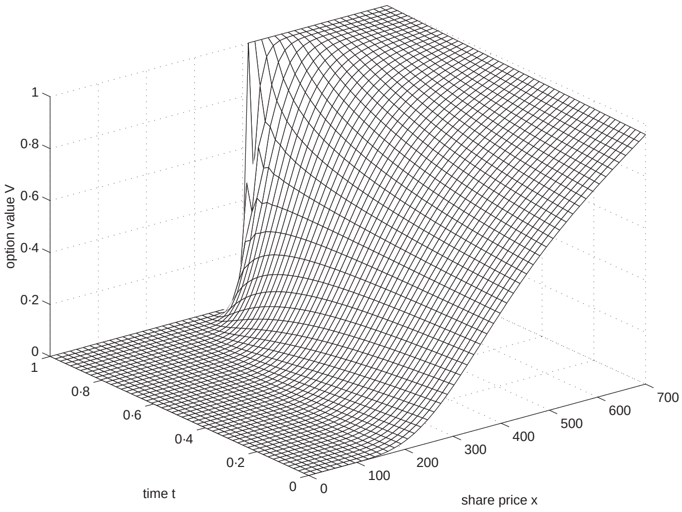


FIG. 3. Computed option value for Test 2.

To demonstrate the theoretical rates of convergence numerically, we chose an initial uniform mesh with 11×5 mesh nodes, and then refined it uniformly four times. We used the solution on the uniform mesh with 641×257 nodes as the 'exact' solution. The computed errors in $|| \cdot ||_h$ at $t = 0$ and the discrete maximum norm, $|| \cdot ||_{h,\infty}$, defined by

$$||u - u_h||_{h,\infty} := \max_{1\leqslant i\leqslant N-1;1\leqslant k\leqslant K} |u_i^k - u(x_i^k)|$$

are listed in Table 1. Clearly the numbers in the table show that the rates of convergence are approximately equal to 1.

Test 2. Call option with the final condition (2.5) and boundary conditions (2.7a)–(2.7b). Parameters: $X = 700$, $T = 1$, $r = 0{\cdot}1$, $\sigma = 0{\cdot}4$, $d = 0{\cdot}04$, $B = 1$ and $E = 400$.

A 51×51 uniform mesh is chosen for the solution of this problem, and the numerical solution on this mesh is plotted in Fig. 3. From the figure it is seen that the numerical solution from our method is non-oscillatory.

Test 3. Call option with the final condition (2.4) and boundary conditions (2.7a)–(2.7b). Parameters: $X = 700$, $T = 1$, $r = 0{\cdot}1 + 0{\cdot}02\sin(10Tt)$, $\sigma = 0{\cdot}4$, $d = 0{\cdot}06x/X$ and $E = 400$.

This problem has non-constant coefficients r and d. To solve this problem, a 41×21 uniform mesh is chosen. The numerical solution on this mesh is plotted in Fig. 4. Again,

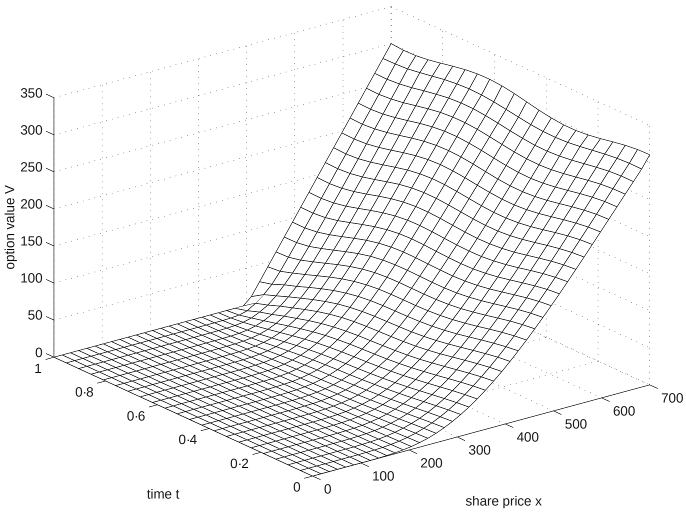


FIG. 4. Computed option value for Test 3.

from the figure it is obvious that the numerical solution from our method is smooth for all $t \in [0, 1)$.

Test 4. Portfolio of options. Combinations of different options have step final conditions such as the 'bullish vertical spread' payoff defined in (2.6). In this example, we assume that the final condition is a 'butterfly spread' delta function given by

$$u(x, T) = \begin{cases} 1, & x \in (X_1, X_2), \\ -1, & x \in (X_2, X_3), \\ 0, & \text{otherwise}, \end{cases}$$

and the boundary conditions are assumed to be homogeneous. This model problem arises from a portfolio of three types of options with different exercise prices (see Wilmott *et al.*, 1993, Section 13.2). The other parameters are: $X = 100$, $T = 1$, $X_1 = 40$, $X_2 = 50$, $X_3 = 60$, $r = 0{\cdot}1 + 0{\cdot}02\sin(10Tt)$, $\sigma = 0{\cdot}4$, $d = 0{\cdot}06x/X$ and $E = 400$.

To solve this problem a uniform mesh with 61×61 mesh nodes is used. The numerical solution is shown in Fig. 5.

6. Conclusion

In this paper we proposed and analysed a novel fitted finite volume method for the Black–Scholes equation governing option pricing. The method can also be formulated as a finite element method. Stability of the discretization has been proved and an $\mathcal{O}(h)$ error bound for the spatial discretization has been established. It has also been shown that the system matrix of the discretization is an M-matrix so that the discrete maximum principle is satisfied by the discretization. Numerical experiments, performed to demonstrate the

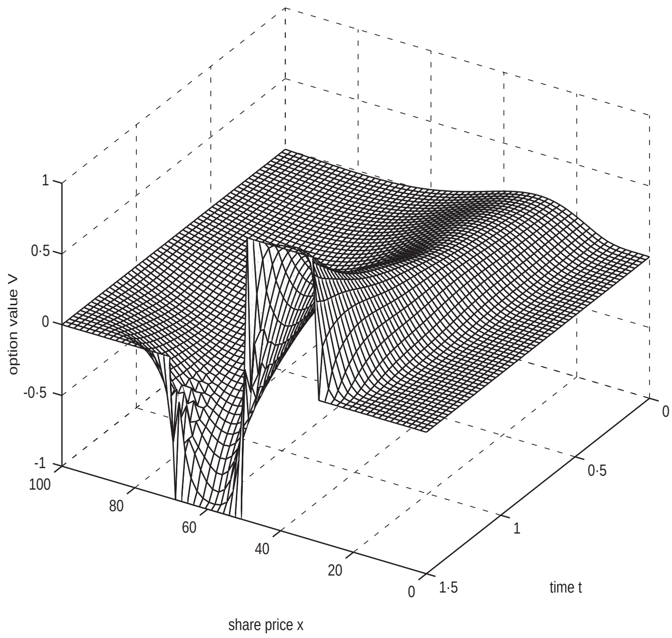


FIG. 5. Computed option value for Test 4.

effectiveness of the method, showed that the approach is stable and accurate even when layers are present.

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