Table of Contents

Numerical Integration

• Basic Introduction

Numerical Integration is the process of computing the value of a definite integral from a set of numerical values of the integrand. It is necessary that the integrand has no singularities the domain under consideration. In the present days of computers, knowledge of numerical integration is necessary because computers do not go through the analytic process of integration.

The term "Numerical Integration" first appears in 1915 in the publication *A Course in Interpolation and Numeric Integration for the Mathematical Laboratory* by **David Gibb**.

The problem of numerical integration is solved by representing the integrand by an interpolation formula and then integrating this formula between desired limits. This process is called mechanical quadrature. When it is applied to the integration of a function of single variable. Numerical integration is used to evaluate a definite integral when there is no closed-form expression for the integral or when the explicit function is not known, and the data is available in tabular form only.

Numerical integration methods can generally be described as combining evaluations of the integrand to get an approximation to the integral. An important part of the analysis of any numerical integration method is to study the behavior of the approximation error as a function of the number of integrand evaluations.

• General Quadrature Formula for Equidistant ordinates

The general quadrature formula for equidistant ordinates is given below :

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^6}{6} - 2n^5 + \frac{35n^4}{4} - \frac{50n^3}{3} + \frac{12n^2}{1}\right)\frac{\Delta^5 y_0}{5!} + \ldots\ldots\right]$$

• Trapezoidal Rule

The trapezoidal rule for numerical integration is given below :

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[\frac{1}{2}(y_0 + y_n) + (y_1 + y_2 + \ldots\ldots + y_{n-1})\right]$$

• Simpson's One-Third Rule

The Simpson's one-third rule for numerical integration is given below:

$\int_{x_0}^{x_0+nh} y\,dx = \frac{1}{3}h[(y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2})]$

• Simpson's Three-Eight Rule

The Simpson's three-eight rule for numerical integration is given below :

$\int_{x_0}^{x_n} y\,dx = \frac{3}{8}h[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots\ldots + y_{n-3})]$

• Weddle's Rule

The Weddle's rule for numerical integration is given below :

$\int_{x_0}^{x_n} y\,dx = \frac{3}{10}h[(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots\ldots)]$

• i - Problem & Solution based on Trapezoidal Rule

Problem: Evaluate $\int_0^6 \frac{1}{(1+x)^2}\,dx$ by using Trapezoidal rule.

Solution: Let $y = \frac{1}{(1+x)^2}$. Here a = 0, b = 6, we shall divide the interval into six equal parts. Hence $h = \frac{b-a}{n} \Rightarrow \frac{6-0}{6} = 1$.

Now we find the value of $y = \frac{1}{(1+x)^2}$ for each point of sub-division are given below:

x	$y = \frac{1}{(1+x)^2}$
$x_0 = 0$	$y_0 = 1$
$x_1 = x_0 + h = 1$	$y_1 = 0.2500$
$x_2 = x_0 + 2h = 2$	$y_2 = 0.1111$
$x_3 = x_0 + 3h = 3$	$y_3 = 0.0625$
$x_4 = x_0 + 4h = 4$	$y_4 = 0.0400$
$x_5 = x_0 + 5h = 5$	$y_5 = 0.0277$
$x_6 = x_0 + 6h = 6$	$y_6 = 0.0204$

By Trapezoidal rule, we get $\int_0^6 \frac{1}{(1+x)^2}\,dx = \int_{x_0}^{x_0+6h} y\,dx$

$= h\left[\frac{1}{2}(y_0 + y_6) + (y_1 + y_2 + y_3 + y_4 + y_5)\right]$

$= 1[(\frac{1}{2} \times 1.0204) + \ 0.4913] = \ 1.002$ (**Approximately**)

• ii - Problem & Solution based on Simpson's One-Third Rule

Problem: Find $\int_0^1 \frac{1}{1+x^2}\,dx$ by using Simpson's one-third rule. Hence obtain the approximate value of π in each case.

Solution: Let $y = \frac{1}{1+x^2}$. Here a = 0, b = 1, we shall divide the interval into six equal parts. Hence $h = \frac{b-a}{n} \Rightarrow \frac{1-0}{6} = \frac{1}{6}$.

Now we find the value of $y = \frac{1}{1+x^2}$ for each point of sub-division are given below:

x	$y = \frac{1}{1+x^2}$
$x_0 = 0$	$y_0 = 1$
$x_1 = x_0 + h = \frac{1}{6}$	$y_1 = 0.9729$
$x_2 = x_0 + 2h = \frac{2}{6}$	$y_2 = 0.9000$
$x_3 = x_0 + 3h = \frac{3}{6}$	$y_3 = 0.8000$
$x_4 = x_0 + 4h = \frac{4}{6}$	$y_4 = 0.6923$
$x_5 = x_0 + 5h = \frac{6}{6}$	$y_5 = 0.5901$
$x_6 = x_0 + 6h = \frac{6}{6} = 1$	$y_6 = 0.5000$

By Simpson's $\frac{1}{3}$ rule, we get $\int_0^1 \frac{1}{1+x^2}\,dx = \int_{x_0}^{x_6} \frac{1}{1+x^2}\,dx$

$= \frac{1}{3}h[(y_0 + y_{n6}) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$= \frac{1}{18}[1.5000 + 9.4525 + 3.1846] \ = \ 0.7853$

By actual integration $\int_0^1 \frac{1}{1+x^2}\,dx = [tan^{-1} x]_0^1 = tan^{-1} 1 - \ tan^{-1} 0 = \frac{\pi}{4}$

$\therefore \frac{\pi}{4} = 0.7853 \quad \Rightarrow \pi = 0.7853 \times 4 = 3.1416$ (**Approximately**)

• iii - Problem & Solution based on Simpson's Three-Eight Rule

Problem: Compute the value of integral $\int_0^{\frac{\pi}{2}} \sqrt{1-\frac{1}{4}sin^2\,t}\;dt$ up to four decimal place by using Simpson's three-eight rule.

Solution: Let $y = \sqrt{1-\frac{1}{4}sin^2\,t}$. Here a = 0, b = $\frac{\pi}{2}$, we shall divide the interval into six equal parts. Hence $h = \frac{b-a}{n} \Rightarrow \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12}$.

Now we find the value of $y = \sqrt{1-\frac{1}{4}sin^2\,t}$ for each point of sub-division are given below:

t	$y = \sqrt{1-\frac{1}{4}sin^2\,t}$
$t_0 = 0$	$y_0 = 1$
$t_1 = t_0 + h = \frac{\pi}{12}$	$y_1 = 0.9916$
$t_2 = t_0 + 2h = \frac{2\pi}{12}$	$y_2 = 0.9682$
$t_3 = t_0 + 3h = \frac{3\pi}{12}$	$y_3 = 0.9354$
$t_4 = t_0 + 4h = \frac{4\pi}{12}$	$y_4 = 0.9014$
$t_5 = t_0 + 5h = \frac{5\pi}{12}$	$y_5 = 0.8756$
$t_6 = t_0 + 6h = \frac{6\pi}{12}$	$y_6 = 0.8660$

By Simpson's $\frac{3}{8}$ rule, we get $\int_0^{\frac{\pi}{2}} \sqrt{1-\frac{1}{4}sin^2\,t}\;dt = \int_{t_0}^{t_6} y\,dt$

$$= \frac{3}{8}h[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8} \times \frac{\pi}{12}[1.8660 + 3.7368 + 1.8708]$$

$$= \frac{\pi}{32}[7.4736] = 0.7337$$ (**Approximately**)

• iv - Problem & Solution based on Weddle's Rule

Problem: Evaluate $\int_0^6 \frac{1}{(1+x)^2}\,dx$ by using Weddle's rule.

Solution: Let $y = \frac{1}{(1+x)^2}$. Here a = 0, b = 6, we shall divide the interval into six equal parts. Hence $h = \frac{b-a}{n} \Rightarrow \frac{6-0}{6} = 1$.

Now we find the value of $y = \frac{1}{(1+x)^2}$ for each point of sub-division are given below:

x	$y = \frac{1}{(1+x)^2}$
$x_0 = 0$	$y_0 = 1$
$x_1 = x_0 + h = 1$	$y_1 = 0.2500$
$x_2 = x_1 + 2h = 2$	$y_2 = 0.1111$
$x_3 = x_2 + 3h = 3$	$y_3 = 0.0625$
$x_4 = x_3 + 4h = 4$	$y_4 = 0.0400$
$x_5 = x_4 + 5h = 5$	$y_5 = 0.0277$
$x_6 = x_5 + 6h = 6$	$y_6 = 0.0204$

By Weddle's rule, we get $\int_0^6 \frac{1}{(1+x)^2}\, dx = \int_{x_0}^{x_6} y\, dx$

$= \frac{3}{10} h[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6]$

$= \frac{3\times1}{10}[1 + (5 \times 0.2500) + 0.1111 + (6 \times 0.0625) + 0.0400 + (5 \times 0.0277) + (2 \times 0.0204]$

$= \frac{3}{10}[2.9554] = 0.8866$ (**Approximately**)

• v - Problem & Solution based on Simpson's Three-Eight Rule

Problem: A river is 80 meters wide. The depth 'd' in meters at a distance x meters from one back is given by the following table. Calculate the area of cross-section of the river using Simpson's one-third rule.

x :	0	10	20	30	40	50	60	70	80
d :	0	4	7	9	12	15	14	8	3

Solution: Represent the distance x along x-axis and depth d along y-axis, then $y_0, y_1, y_2, \dots, y_8$ are the values of d in the table.

Here a = 0, b = 80 and h = 10.

∴ Area of cross-section is = $\int_0^{80} d\, dx$

By Simpson's one-third rule, we have

A = $\int_0^{80} d\, dx = \frac{10}{3}[(0+3)] + 4(4+9+15+8) + 2(7+12+14)]$

= 710 sq. meters.

• C++ Program for Simpson's 1/3 Rule with Output •

```
Evaluate logx dx within limit 4 to 5.2.

First we will divide interval into six equal parts as number of interval should be even.
```

x	:	4	4.2	4.4	4.6	4.8	5.0	5.2
logx	:	1.38	1.43	1.48	1.52	1.56	1.60	1.64

```
Now we can calculate approximate value of integral using above formula:
     = h/3[( 1.38 + 1.64) + 4 * (1.43 + 1.52 + 1.60 ) +2 *(1.48 + 1.56)]
     = 1.82785

Hence the approximation of above integral is 1.82785 using Simpson's 1/3 rule.
```

Program.cpp

```cpp
#include <bits/stdc++.h>
using namespace std;

// Function to calculate f(x)
float func (float x)
{
    return log (x);
}

// Function for approximate integral
float simpsons_ (float ll, float ul, int n)
{
// Calculating the value of h
    float h = (ul - ll) / n;

// Array for storing value of x and f(x)
    float x[10], fx[10];

// Calculating values of x and f(x)
    for (int i = 0; i <= n; i++) {
        x[i] = ll + i * h;
        fx[i] = func (x[i]);
    }

// Calculating result
    float res = 0;
    for (int i = 0; i <= n; i++) {
        if (i == 0 || i == n)
            res += fx[i];
        else if (i % 2 != 0)
            res += 4 * fx[i];
        else
            res += 2 * fx[i];
    }
    res = res * (h / 3);
    return res;
}

// Driver program
int main()
{
    float lower_limit = 4; // Lower limit
    float upper_limit = 5.2; // Upper limit
    int n = 6; // Number of interval
    cout << simpsons_ (lower_limit, upper_limit, n);
    return 0;
}
```

Output: 1.82785

[Program's Source Code is uploaded into ideone, link: https://ideone.com/o4hkkR]