ON THE IMPLICIT EQUIVALENCE OF THE LAW OF EXCLUDED MIDDLE AND THE COUNTABLE AXIOM OF CHOICE

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"DEDICATED TO MY BELOVED MOTHER SIMZAR HOSSEINZADEH WHO HAS BEEN A SOURCE OF LOVE, SACRIFICE, WISDOM AND CARE TO ME."

1. Abstract

The Axiom of Choice (AC) is well-known to imply the Law of Excluded Middle (LEM) in classical mathematics. However, the relationship between LEM and the Countable Axiom of Choice (CAC) has remained less explored. This paper proves that LEM implies CAC, particularly in the context of constructive mathematics. Furthermore, we demonstrate constructively that CAC implies the Limited Principle of Omniscience (LPO), showing that the existence of a choice function under constructivist principles necessitates a finite routine. This insight challenges the constructivist interpretation that choice functions are inherently constructive, showing instead that they might only simulate constructiveness through an illusion of algorithmic selection. Moreover, since we have established that LEM implies CAC, this relationship introduces a deeper philosophical challenge. If real-world logic is governed by finite routines, then CAC, when applied to practical contexts, effectively implies LEM. This insight suggests that the constructivist acceptance of CAC may inadvertently rely on principles akin to LEM, exposing a subtle inconsistency within the constructivist framework. However, a full exploration of this implication is beyond the scope of this paper and will be addressed in future research, particularly in the context of the Parallel Law of Excluded Middle. Note that in proofs we follow Brouwer's intuitionism and not Bishop's school.

2. Introduction

The Axiom of Choice (AC) and the Law of Excluded Middle (LEM) are foundational principles in classical mathematics. While AC allows for the selection of elements from arbitrary families of non-empty sets, LEM states that for any proposition, either the proposition or its negation is true. It is well-established that AC implies LEM, but whether LEM can imply even a weaker form of AC, such as the Countable Axiom of Choice (CAC), has not been extensively studied, particularly within the realm of constructive mathematics.

In constructive mathematics, and particularly in Bishop's school of constructivism, LEM is rejected while CAC and the Axiom of Dependent Choice (ADC) are accepted. This approach enables constructive mathematicians to develop fields of real and complex numbers while maintaining a constructivist stance. However, this paper challenges the consistency of this approach by presenting rigorous proofs that not only demonstrate LEM implies CAC but also introduce a novel theorem proving CAC implies the Limited Principle of Omniscience (LPO). These observations have been in my mind since I went deep enough to the foundations of mathematics. These additional results significantly strengthens the argument that Bishop's

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constructivism may harbor an implicit reliance on classical logic, even when it explicitly rejects LEM.

This discovery arose from a deep examination of the foundational aspects of mathematics, leading to the insight that under countable sets, CAC implies LEM. This key observation was exposed in my mind through the following observation: In Bishop's constructive analysis, whenever CAC is applied, the base sets are countable, even if the target space is not. Additionally, the binary nature of CAC applications—where concepts are treated as either true or false—aligns closely with LEM, challenging the constructivist claim of avoiding classical principles. Through this analysis, this paper reveals that the constructivist construction of the completeness of real numbers may not be as purely constructive as previously believed.

This work not only fills a critical gap in mathematical logic but also raises important questions about the foundational integrity of Bishop's framework. By demonstrating that CAC implies LPO, we further highlight potential inconsistencies within constructive analysis and algebra, suggesting that the rejection of LEM might not be as complete as intended.

This paper proceeds as follows. First, we prove that LEM implies CAC, then we introduce the constructive notion of the Countable Axiom of Choice (CAC) and present our main theorem demonstrating the equivalence of LEM and CAC under countable sets. We then establish that CAC implies LPO. Then we give two discover universal equivalence of CAC and LEM. In the next section, we provide new insights into the implicit classical assumptions within constructive mathematics. Finally, we discuss the broader impact of these findings on the discourse surrounding mathematical foundations and logical consistency, particularly within Bishop's constructivism.

All proofs in this paper are constructive, not within the framework of Bishop's constructivism, but rather in the framework of Brouwer's intuitionism. This approach adheres to the strict principles of intuitionistic logic, emphasizing finite routines, constructible choices, and the avoidance of the Law of Excluded Middle unless explicitly justified by construction.

3. The Proof of LEM implies CAC

Before we go over the proof we describe constructive notion of the Countable Axiom of Choice as follows.

Countable Axiom of Choice - CAC. *If S is a subset of $\mathbb{N} \times B$, and for each x in $\mathbb{N}$ there exists y in B such that $(x, y) \in S$, then there is a function f from $\mathbb{N}$ to B such that $(x, f(x)) \in S$ for each x in $\mathbb{N}$.*

Note that if we replace $\mathbb{N}$ in above by a finite set, then the axiom is obvious, moreover, we have $y = f(x)$.

Now, let $X = \{A_n : n \in \mathbb{N}\}$ is a countable family of non-empty sets. A choice function for X is a function $f : \mathbb{N} \to \bigcup X$ such that $f(n) \in A_n$ for all $n \in \mathbb{N}$.

On the other hand, existence of the choice function means the existence of a sequence $(P_n)_{n\in\mathbb{N}}$, where $P \in A_n$. By the above, for every $N \in \mathbb{N}$, the sequence $(P_n)_{1\leq n\leq N}$ exists. Note that, when a set is not empty, then by the logic defined in constructive mathematics its elements could be constructed. Hence, there exits a P_n. And to do this we do not need LEM.

Now, we assume that CAC is not an axiom in the direction of our desires when LEM holds. If there is a sequence $(P_n)_{n\in\mathbb{N}}$, then we have a choice function, proving CAC directly. Otherwise, for each $N \in \mathbb{N}$ there exists a breakdown in constructing a sequence. And this is a contradiction since there exists always the finite choice function. Note that here we use LEM to identify denial of not existing a choice function and we don't make any use of CAC.

4. Under Countable Sets CAC is Equivalent to LEM

Theorem 4.1. *If all sets in the world - by the world we mean category of sets - would be at most countable, then CAC would have implied LEM.*

Proof. Our proof is exactly the same as is argued in [3]; just with base change as all setting is considered to be countable. Let $A \coloneqq \{(s,t)\}$, where s and t vary over elements of all elements in the world. Then A is countable. Now, we consider the proposition P and we identify the following. We set $s = t$ iff P. Let $B \coloneqq \{0,1\}$ and let $S \coloneqq \{(s,0),(t,1)\} \subset A \times B$. If $f : A \to B$ is a choice function for S, then either

$$(i)\ f(s) = 1 \text{ or } f(t) = 0,\ \text{ so that } s = t, \text{and therefore } P \text{ holds, or else}$$
$$(ii)\ f(s) = 0 \text{ and } f(t) = 1,\ \text{ so that } s \neq t$$
$$\Rightarrow \text{ Assuming } P \text{ leads to a contradiction, and thus } \neg P \text{ holds.}$$

This completes the proof. □

Corollary 4.2. *If all sets in the world would be at most countable, then CAC would have been equivalent to LEM.*

Observation 4.3. *Now; for instance; we observe the proof of completeness of real numbers in Bishop's school from the book* [3]. *At the page 19 of this book under the chapter of constructive analysis, authors emphasize that only places that CAC is involved is when we construct the sequence* (N_k) *and* $(x_k{}^\infty)$ *by treating the sequence* (x_n)*. If we notice that* N_k*'s are natural numbers and* x_k*'s are rational numbers, then based on Corollary 4.2, we see that Errett Bishop actually uses LEM.*

5. CAC implies LPO

In the following we first state and prove the following foundational theorem. I should mention that there was a dynamic collaboration of human-AI in the exploration of this theorem. As the insight was provided by me and impressively the solid proof was made by the ChatGPT. This rises this concept in capacity of AI in discovering mathematical concepts and arguments in a dynamic and challenging procedure with the human being. Before I go over the theorem let me state the LPO.

A binary sequence is a finite routine that assigns to each positive integer an element $\{0,1\}$. Note that a binary sequence is infinitely processing and free choice. Also, note that a binary sequence is not a function from $\mathbb{N}$ into the set $\{0,1\}$.

The Limited Principle of Omniscience - LPO. *If* (a_n) *is a binary sequence, then either there exists* n *such that* $a_n = 1$*, or else* $a_n = 0$ *for each* n*.*

Now, I state the theorem with the elegant proof of ChatGPT. I say elegant since this proof is so simple but conceptually deep.

Theorem 5.1. *CAC implies LPO.*

Proof. Consider a binary sequence (a_n) where each $a_n \in \{0,1\}$. We construct a countable family of sets $\{A_n :\ n \in \mathbb{N}\}$ as follows:

- If $a_n = 0$, define $A_n = \{0\}$.
- If $a_n = 1$, define $A_n = \{0, 1\}$.

By the Countable Axiom of Choice (CAC), there exists a choice function $f : \mathbb{N} \to \bigcup A_n$ such that $f(n) \in A_n$ for each $n \in \mathbb{N}$.

Now, we analyze the behavior of f:

- If $f(n) = 1$ for some n, this implies that $A_n = \{0, 1\}$, and thus $a_n = 1$.
- If $f(n) = 0$ for all n, this implies that $A_n = \{0\}$, meaning $a_n = 0$ for all n.

Therefore, the choice function f provides a definitive answer to whether there exists an index n for which $a_n = 1$ or if $a_n = 0$ for all n. This is precisely the assertion of the Limited Principle of Omniscience (LPO). □

Note that the significance of the proof above is it makes a finite routine a function. Then we are able to talk about values of this function. And this deep concept was observed by AI.

6. The Philosophical Nature of the Equivalence

We make two different approaches as follows. In two following sections we prove the universal equivalence of CAC and LEM. Note that in the Brouwer's intuitionism framework, denial or negation of a proposition is not merely the absence of truth but rather the presence of a contradiction when is assumed to hold. In the Section 6.1, our proof is based on negation of a proposition in Brouwer's intuitionism framework. But in the Section 6.2, our proof is based on this concept that negation of a proposition is the absence of truth. The dual interpretation of negation - either as contradiction in Brouwer's intuitionism or as the absence of the truth - highlights the nuanced logical landscape in which CAC and LEM interact, the broader philosophical implications of this study.

6.1. **Universal Countability of External Factors.** Now, if we go deep enough on the nature of our proof in Theorem 4.1, then we will see that LEM and CAC are universally equivalent.

We demonstrate that CAC implies LEM beyond countable sets. This universal extension occurs by providing both a philosophical perspective and a formal logical model.

The proof of Theorem 4.1 is based on ordered pairs (s, t), where s and t vary over countable sets. The key insight is that the decision on a proposition P arises from whether s equals t or not, representing a binary choice. When all sets are countable, this method naturally leads to the equivalence of CAC and LEM.

Philosophically, this argument hinges on the principle that all external factors influencing decisions in the real world are fundamentally countable. Whether we are considering physical phenomena, cognitive processes, or logical propositions, the parameters that govern decision-making can be reduced to discrete, observable events. This countability emerges because:

1. Physical Phenomena: Any physical observation is made through finite measurements, sensors, or perceptual events, all of which produce countable data points.
2. Cognitive Processes: Human cognition operates through discrete thoughts, choices, and logical steps, suggesting a countable model of decision-making.
3. Logical Propositions: Propositions in mathematics and logic are typically evaluated in binary terms (true/false), further reinforcing countability.

Therefore, even in seemingly uncountable settings, practical decision-making reduces to a countable sequence of binary choices. This reduction is not merely a mathematical convenience but a reflection of the inherent structure of reality. It follows that CAC, which governs countable choices, is universally sufficient to derive LEM, even in broader, uncountable contexts.

Formal Logical Model:

To formalize this philosophical arguments, consider a scenario where S and T are sets that may be uncountable. In decision-making, we evaluate a proposition P through the relationship between $s \in S$ and $t \in T$. If S and T are uncountable, traditional set theory might suggest that decisions require handeling uncountably many possibilities.

However, the decision-making process inherently involves external factors that are countable. Let F be the set of all external factors influencing the decision on P. We argue that F is countable due to the following reasoning:

1. Decisions in mathematical and physical contexts are based on observable phenomena, which are countable.

2. Any proposition P reduces to evaluating whether specific conditions hold, represented as binary choices (true/false, 0/1).

Since every decision involves comparing a countable set of external factors, the effective domain of (s, t) in any practical decision is countable. Thus, even if S and T are theoretically uncountable, the decision on P depends only on a countable subset of $S \times T$.

This logical model bridges the gap between countable and uncountable scenarios, reinforcing the universal equivalence of CAC and LEM.

6.2. **Philosophical Proof: The Concept of Infinity and CAC Implies LEM.** Building on Theorem 5.1, this section presents an alternative philosophical argument demonstrating that the Countable Axiom of Choice (CAC) implies the Law of Excluded Middle (LEM).

This proof hinges on the nature of infinity and the reduction of decisions to countable binary choices.

The Core Argument

If we accept that there is no more than countable infinity, then CAC would imply LEM, as all decisions could be reduced to countable sequences of binary choices. This perspective arises from the idea that every decision, regardless of complexity, ultimately involves choosing between discrete options—true or false, 0 or 1. Under this framework, the choice function constructed by CAC operates within a countable domain, inherently aligning with the binary simplicity of LEM.

Conversely, if there is more than countable infinity, LEM becomes necessary to handle propositions in such broader contexts. When the concept of infinity expands—particularly in theoretical frameworks that involve uncountable sets—the straightforward reduction to binary choices may no longer apply. This is because uncountable contexts often require reasoning that transcends finite or countably infinite decision-making processes, revealing the full strength and necessity of LEM.

Real-World Decisions and Countability

In practical terms, all decisions in the real world are based on countable steps. Whether considering physical processes, logical deductions, or cognitive choices, the external factors influencing these decisions remain countable. This reinforces the idea that in reality, CAC and LEM are not just equivalent but inseparable, as the decision-making process cannot exceed countable complexity.

Applying Theorem 5.1, the choice function here is merely choosing between 0 or 1—between true and false. However, as the concept of infinity expands (not in the real world but in theoretical constructs), this binary simplicity is lost, and the robustness of LEM is fully revealed. The philosophical insight offered by this proof is that the validity of LEM depends significantly on the concept of infinity being applied. Where the world is countable, CAC is sufficient to establish LEM. Where theoretical infinity dominates, LEM becomes a necessary tool to manage the expanded logical space.

7. An Overview

Now, we explore the broader implications of the equivalence between the Countable Axiom of Choice (CAC) and the Law of Excluded Middle (LEM), particularly within the framework of Bishop's constructive mathematics.

Note that in Brouwer's intuitionism we have the following

- LEM implies LPO.
- LPO implies LLPO (Lesser Limited Principle of Omniscience).

And we proved that

- LEM implies CAC.
- CAC implies LPO.

While Bishop's approach explicitly rejects both LPO and LEM, their constructivism is based on rejecting LPO and not LEM. When they accept and make an extensive use of CAC. Then the proofs that LEM implies CAC and CAC implies LPO challenge the foundational consistency of constructive analysis and algebra as developed by Bishop's school.

The equivalence between CAC and LEM has far-reaching implications beyond constructive analysis. It challenges the foundational integrity of constructivist frameworks by suggesting that classical principles may be embedded implicitly in constructivist methodologies. Moreover, this discovery opens new avenues for exploring the boundaries between constructivist and classical mathematics, potentially leading to more unified logical systems that accommodate both philosophies.

Logical Principles and Their Interconnections

Bishop's constructive mathematics is traditionally grounded in rejecting LPO while maintaining the beauty and constructiveness of explicit constructions. However, if CAC implies LPO, and LPO implies LLPO, then even the acceptance of CAC could introduce LEM implicitly into Bishop's framework. This discovery raises questions about whether constructive methods in analysis and algebra, such as the construction of real numbers, might rely on classical principles that are not immediately apparent.

The Elegant Proof and Its Implications

The elegant proof presented in this paper offers a simple yet profound demonstration that CAC implies LPO. The proof leverages the binary nature of decisions in constructing choice functions, showing that even under the constraints of constructive mathematics, accepting CAC introduces a hidden reliance on LPO, and by extension, LEM.

In one side the proof challenges raised concepts such as undecidablity of real numbers in Bishop's school. On the other sided, the proof reveals that the constructivist acceptance of CAC might inadvertently validate LEM. And these challenge the very foundation of constructive analysis as practiced by Bishop's school.

Constructive Analysis and the Real Numbers

One of the most striking implications of this work is its impact on the constructive construction of the real numbers. Bishop's school claims to avoid non-constructive principles such as LEM while using CAC to ensure the completeness of real numbers. However, if CAC implies LEM universally, then the constructivist approach to real numbers might not be as constructively pure as previously thought.

This insight suggests that the first step of Bishop's constructivism—the construction of the real numbers — may already involve implicit use of LEM. Such a revelation calls for a deeper examination of constructive methods, not only in analysis but also in algebra and other domains.

Finite Routine vs. Choice Function

The philosophical argument presented in the Section 6.2, builds on the distinction between finite routines and choice functions. While Beeson and Bridges [2] introduce countability within constructive mathematics through choice functions, this interpretation may create an illusion of constructiveness. In reality, a choice function merely guarantees the existence of selections from countable sets without providing a finite routine to achieve these selections. In contrast, Brouwer's intuitionistic approach emphasizes that true constructiveness requires a finite routine, where each choice is determined algorithmically through a step-by-step process. This distinction is crucial: our proof that CAC implies LPO demonstrates that under constructivist logic, a valid choice function is essentially equivalent to a finite routine.

Moreover, since LEM implies CAC, this relationship introduces a deeper philosophical challenge. If real-world logic is governed by finite routines, then CAC, when applied to practical contexts, effectively implies LEM. This insight suggests that the constructivist acceptance of CAC may inadvertently rely on principles akin to LEM, exposing a subtle inconsistency within the constructivist framework [1–4, 6]

The equivalence between CAC and LEM challenges the foundational integrity of constructivist frameworks by suggesting that classical principles may be embedded implicitly in constructivist methodologies [1–3]. This discovery also opens new avenues for exploring the boundaries between constructivist and classical mathematics, potentially leading to more unified logical systems that accommodate both philosophies.

8. Future Exploration: The Parallel Law of Excluded Middle

While this paper establishes a foundational equivalence between the Countable Axiom of Choice (CAC) and the Law of Excluded Middle (LEM), future research may explore scenarios where LEM fails in more complex or undecidable contexts. This line of inquiry could involve examining the concept of infinity and its implications for logical frameworks, particularly in situations where classical notions of truth and decidability encounter real-world limitations. The development of an alternative form of LEM, potentially adapted to a more nuanced understanding of reality, remains a compelling area for further exploration.

9. Conclusion

This paper presents a precise and evidence-based argument that under countable sets, LEM and CAC are equivalent. By revealing the implicit use of LEM in Bishop's constructivism, particularly in the construction of real numbers, this work challenges the constructivist claim to a purely constructive approach. Furthermore, the introduction of the theorem proving CAC implies the Limited Principle of Omniscience (LPO) adds a deeper layer to this critique, showing that even Bishop's acceptance of CAC might introduce classical principles into a constructivist framework. Moreover, the findings open the door for future research that could lead to a more logically consistent and transparent mathematical framework.

While this paper focuses specifically on the construction of the real numbers, the broader implication is clear. If LEM is necessary for this fundamental construct, it raises legitimate questions about whether other aspects of Bishop's constructivism might also rely on non-constructive principles.

The disappointment with Bishop's framework is understandable. The constructivist promise of avoiding non-constructive principles such as LEM appears, at least in this instance, unfulfilled. Nevertheless, this work should not be seen as a dismissal of constructivism as a whole but rather as a call for greater scrutiny and a deeper understanding of where and how classical principles might be subtly influencing constructivist methods. This research invites constructive mathematicians to re-examine their assumptions and strive for a more rigorous delineation between constructivist and classical approaches.

Further research is needed to explore whether this equivalence holds in broader constructive settings and what it might mean for the foundations of mathematics. Such an inquiry could lead to a more refined constructivist approach that is both logically consistent and transparent about its foundational assumptions. The concept of decidability, the limits of LEM in complex cases, and the philosophical implications of infinity present compelling avenues for future study.

In conclusion, this paper not only demonstrates the equivalence of LEM and CAC under countable sets but also establishes that CAC implies LPO, revealing an implicit use of LEM within Bishop's school of constructivism. In one side, Bishop's approach to the completeness of real numbers utilizes LEM directly, whether acknowledged or not. On the other side, Bishop's utilization of and undecidablity of real numbers deepens to the question. The significance of this finding lies not only in challenging a specific constructivist methodology but also in contributing to the ongoing discourse about the foundations of mathematics and the role of logical principles in mathematical frameworks. By questioning the foundational integrity of constructivist approaches, this work aims to foster a more transparent and philosophically sound understanding of mathematical logic.

This study not only challenges established constructivist frameworks but also opens new avenues for exploring the logical foundations of mathematics, suggesting that even in constructivism, the shadows of classical principles may subtly persist.

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