A PARADOX ON THE LAW OF EXCLUDED MIDDLE IN THE FRAMEWORK OF CATEGORY OF SET

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"DEDICATED TO MY BELOVED MOTHER SIMZAR HOSSEINZADEH WHO HAS BEEN A SOURCE OF LOVE, WISDOM AND CARE TO ME, TO MY FATHER FIROUZ WHO WAS MY FIRST MATH TEACHER, TO MY BROTHER MASOOD FOR HIS SUPPORT, EMOTIONALLY, MENTALLY AND SCIENTIFICALLY, AND TO MY SISTER NEDA FOR HER LOVE AND SUPPORT."

ABSTRACT. In this paper we pioneer an example, which shows that if we accept the Law of Excluded Middle, then we come into a paradox.

1. Introduction

Before we proceed, we should take into account that the knowledge of humanity is based on what our ancestors have made, discovered and created. This is promised with respect to the logic that they have pictured from the world. This is a model whose source is their own and personal picture of the world, different from other pictures. Therefore, there is no the Absolute. Rather, there are perspectives.

The Law of Excluded Middle is the most discussed axiom of mathematics; besides Euclid's Axiom of Parallels and the Axiom of Choice. As we know the Law of Excluded Middle is accepted in classical mathematics but is rejected in constructive mathematics. The Law of Excluded Middle has an ancient history which backtracks to Aristotle. This law is strongly tied with the knowledge of humanity including science and philosophy. Notice that the Axiom of Choice itself implies the Law of Excluded Middle. Therefore, the Axiom of Choice is rejected in constructive mathematics, but not weaker versions of this axiom.

What we manifest in the present paper is very short and clear by introducing a straight-forward example; although to build such an example one needs to make a hard work. And it is clear that by ideas that we are presenting in our example one could build similar examples which are not in our interest. As doing this just would be practicing useless generalizations which sometimes continues in math community. The example occurs over the field of complex numbers. We introduce a nonzero analytic function on a connected open set whose zeros are not isolated. We know that if we accept the Law of Excluded Middle, then we accept: 1. The field of complex numbers is a discrete field- a field is discrete if it is decidable-. 2. We accept some facts in analysis and algebra. So that; using these facts; the Law of Excluded Middle guides us to the paradox.

This reminds me the famous word of the great mathematician Leopold Kronecker "God created the integers, all the rest is the work of Man". And what Erret Bishop adds "Mathematics belongs to man, not God". "If God has mathematics of his own that needs to be done, let him do it himself". Regardless to believing or not believing in God, one would say that mathematics is the most compelling miracle made by the man which penetrates its route in a genius creation through the summit of knowledge; although, the summit is not so clear to see.

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Notice that there are counterexamples for the Law of Excluded Middle; these examples belong to two families. One family of these examples are set in the framework of constructive mathematics that are based on the built logic in constructive mathematics which do not internally deny the Law of Excluded Middle in classical mathematics. The other family of these examples are set in the regular language; in one side the regular language is not exact as the connection of this language with the true and firm thought is intermediate; on the other side these examples made by words do not fit in the category of set which has the strongest logic behind itself and it has the most sensible and exact connection with the true thought.

At the end, let us say a little more to appreciate philosophy as we believe that the motivation of mathematics is based on the philosophy.

As we said the Law of Excluded Middle is an implication of the Axiom of Choice. On the other hand, there is a strong connection between analytic philosophy and symbolic logic as the first one makes use of the second one. And the symbolic logic; first formalized by Gottlob Frege specifically in [1]; is based on the Law of Excluded Middle, as Frege says one observes whether a statement is either true or false in an abstract scheme called formal logic. So that any uncertainty on the Axiom of Choice and the Law of Excluded Middle derives ambiguity in the analytic philosophy. And also vagueness in the philosophy of mind and language; developed by the prominent philosopher Ludwig Wittgenstein. Where, in [3], Wittgenstein says: "1.12 For the totality of facts determines both what is the case, and also all that is not the case.", "2.11 The picture presents the facts in logical space, the existence and non-existence of atomic facts.". However, any sort of uncertainty on the Axiom of Choice and the Law of Excluded Middle does not influence some of other traditions of philosophy such as existentialism. As the philosophy of existentialism is already the philosophy of uncertainty in the meaningless world around us; as Franz Kafka stares at lightness from darkness under nonsensical circumstances to create sensible concepts.

In ages, the Law of Excluded Middle led humanity to believe that estimations of facts in the world are exact and accurate. We don't mean numerical estimations but we mean making exact and firm statements based on; in reality; non-exact facts. Yes, we live in the world of uncertainties and our interpretation of the world is based on estimations. Making exact statements has continued to the extent that the man gathers data and makes firm decisions based of these information. For example, one clear and ongoing fact occurs in the math education, where bunch of articles; based on data science; are replaced by previous analytical manuscripts in this area of research. An area which is strongly connected to psychology, philosophy and logic. So that I assume the following whispered words in my ears: in psychology one cannot discover exact concepts; instead they try to interpret traces made by the patient. And the interpretation will go wrong if the one looks at the trace from his own perspective and not this human's perspective.

2. A LITTLE OF BACKGROUND

First of all, one should be familiar with following axioms.

The most famous principle is the Law of Excluded Middle, which asserts that $P \vee \neg P$ holds for any statement P, where $\neg P$ is the denial of P.

Axiom of Choice. If S is a subset of $A \times B$, and for each x in A there exists y in B such that $(x,y) \in S$, then there is a function f from A to B such that $(x,f(x)) \in S$ for each x in A.

Countable Axiom of Choice - CAC. This is the axiom of choice with A being the set of positive integers.

Axiom of Dependent Choice - ADC. Let A be a nonempty set and R be a subset of $A \times A$ such that for each a in A there is an element a' in A with $(a, a') \in R$. Then there is a sequence a_0, a_1, \ldots of elements of A such that $(a_i, a_{i+1}) \in R$ for each i.

I leave this to the reader to see that to prove following true facts in the framework of classical mathematics we only need the Law of Excluded Middle, and not any kind of the Choice; including the Countable Axiom of Choice (CAC) and the Axiom of Dependent Choice (ADC).

We denote the field of real numbers (resp. the field of complex numbers) by \mathbb{R} , (resp. \mathbb{C}). We say a function $f := D \subset \mathbb{C} \to \mathbb{C}$, is holomorphic if it is differentiable in each $z \in D$. The reader should be familiar with the concept of analytic functions in complex variables. If $f := D \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic function and D is an open set, then f is analytic. We also know that if $f : D_1 \to \mathbb{C}$, and $g : D_2 \to \mathbb{C}$ are analytic functions and $g(D_2) \subset D_1$, then $f \circ g : D_2 \to \mathbb{C}$ is an analytic function. We know that we represent every complex number in the polar coordinate system under this condition that the principal argument of each complex number is in the interval of $(-\pi, \pi]$.

The logarithmic function denoted by Ln is analytic on $\mathbb{C} - \{x + \mathbf{i}y; y = 0, x \le 0\}$. As we define $\operatorname{Ln}(z) = \operatorname{Ln}(r) + \mathbf{i}\theta$, where $z = re^{\mathbf{i}\theta}$, r > 0, and $-\pi < \theta \le \pi$.

We say the function $f: \mathbb{C} \to \mathbb{C}$ is an entire function if it is analytic on \mathbb{C} . Functions of

$$f: \mathbb{C} \to \mathbb{C}, \quad f(z) = \cos(z),$$

 $g: \mathbb{C} \to \mathbb{C}, \quad g(z) = \sin(z),$
 $h: \mathbb{C} \to \mathbb{C}, \quad h(z) = e^z,$

are entire functions, where

$$\cos(z) = \frac{e^{\mathbf{i}z} + e^{-\mathbf{i}z}}{2}, \quad \sin(z) = \frac{e^{\mathbf{i}z} - e^{-\mathbf{i}z}}{2\mathbf{i}}.$$

Suppose $f := D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function, and D is an open connected subset of the field of complex numbers. If f is not identically zero, then points of the set $S := \{z \in D, f(z) = 0\}$ are isolated.

3. The example

Before we introduce the example we need to do some discussions as follow. Also, note that for simplicity for the exponential function we use the symbol exp.

Let $\mathcal{A} := \{x + \mathbf{i}y; \ y = 0, x \le 0\} = \{r; \ r \le 0\}$. We want to see when $F_1(z) := \frac{z}{\mathbf{i}z+1} \in \mathcal{A}$, $F_2(z) := \frac{\mathbf{i}z}{\mathbf{i}z+1} \in \mathcal{A}$, and $F_3(z) := \mathbf{i}\exp(\mathbf{i}c)(1-\sin(-\mathbf{i}\operatorname{Ln}(z))) \in \mathcal{A}$, where c is a fixed number such that $-\pi < c < 0$, and $\cos(c) \ne 0$; we mean $c \ne -\pi/2$.

We have $\frac{z}{\mathbf{i}z+1} \in \mathcal{A}$ iff $\frac{z}{\mathbf{i}z+1} + r = 0$, where $r \geq 0$. If $z := x + \mathbf{i}y$, where $x, y \in \mathbb{R}$, then we conclude that $x = \frac{-r}{r^2+1}$, and $y = \frac{r^2}{r+1}$. Hence, we have $x^2 + y^2 - y = 0 \Rightarrow |z - \frac{\mathbf{i}}{2}| = \frac{1}{2}$, where, $x \leq 0$, and $y \geq 0$, and |z| is the absolute value of the associated complex number. So that we have the semicircle as shown in the Figure 1, with the red color.

Also, we get $\frac{\mathbf{i}z}{\mathbf{i}z+1} \in \mathcal{A} \Rightarrow z = \mathbf{i}\frac{r}{r+1}$, where $r \ge 0$. So that we have the segment as shown in the Figure 2, with the red color.

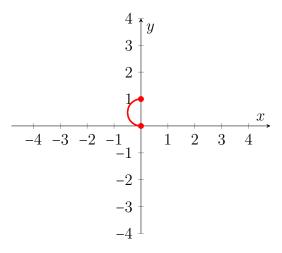


FIGURE 1. The semicircle of $|z - \frac{\mathbf{i}}{2}| = \frac{1}{2}, \ x \le 0, \ y \ge 0.$

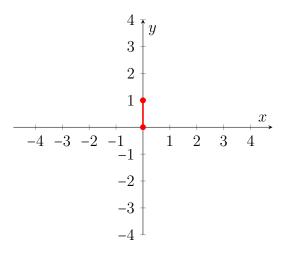


FIGURE 2. The segment $z = \mathbf{i} \frac{r}{r+1}, r \ge 0.$

Now, we want to see when $F_3(z) = \mathbf{i} \exp(\mathbf{i}c)(1 - \sin(-\mathbf{i} \operatorname{Ln}(z))) \in \mathcal{A}$. For simplicity we set $z := s \exp(\mathbf{i}\theta)$, where s > 0, and $-\pi < \theta \le \pi$; we exclude the origin. We have

$$\mathbf{i} \exp(\mathbf{i}c)(1 - \sin(-\mathbf{i}\operatorname{Ln}(z))) \in \mathcal{A}$$

$$\Rightarrow \mathbf{i} \exp(\mathbf{i}c)(1 - \sin(-\mathbf{i}\operatorname{Ln}(z))) = -r, \text{ where, } r \ge 0,$$

$$\Rightarrow \mathbf{i} \exp(\mathbf{i}c)(1 - \frac{1}{2\mathbf{i}}(s\exp(\mathbf{i}\theta) - s^{-1}\exp(-\mathbf{i}\theta))) = -r$$

$$\Rightarrow (2\mathbf{i}s\exp(\mathbf{i}c) - (s^{2}\exp(\mathbf{i}\theta + \mathbf{i}c) - \exp(-\mathbf{i}\theta + \mathbf{i}c))) = -2rs$$

$$\Rightarrow 2\mathbf{i}s\cos(c) - 2s\sin(c) - s^{2}\cos(\theta + c) - s^{2}\mathbf{i}\sin(\theta + c) + \cos(-\theta + c) + \mathbf{i}\sin(-\theta + c) = -2rs$$

Therefore, we conclude that

(1)
$$2s\cos(c) - s^2\sin(\theta)\cos(c) - s^2\cos(\theta)\sin(c) + \cos(\theta)\sin(c) - \sin(\theta)\cos(c) = 0,$$

$$(2) \qquad -2s\sin(c) - s^2\cos(\theta)\cos(c) + s^2\sin(\theta)\sin(c) + \cos(\theta)\cos(c) + \sin(\theta)\sin(c) = -2rs.$$

From the relation (1), we get $(2s - s^2 \sin(\theta) - \sin(\theta))\cos(c) = (s^2 - 1)\cos(\theta)\sin(c)$. Then $2s - s^2 \sin(\theta) - \sin(\theta) = 0$ iff $(s^2 - 1)\cos(\theta) = 0$. If s = 1, then we get $2 - 2\sin(\theta) = 0 \Rightarrow \theta = \pi/2$, and in the case that $\cos(\theta) = 0$, we get $\theta = \pm \pi/2$, and then we get $2s \pm s^2 \pm 1 = 0 \Rightarrow s = 1$; note that s is a non-negative number. Therefore, we exclude points of $\pm i$.

Now, we consider the case that always $2s - s^2 \sin(\theta) - \sin(\theta) \neq 0$, $(s^2 - 1)\cos(\theta) \neq 0$. But the discriminant of the polynomial $2s - s^2 \sin(\theta) - \sin(\theta)$ in terms of s is equal to $1 - \sin(\theta)^2 \geq 0$. Hence, $\sin(\theta) = 0$. Therefore, we have $\cos(c) = \cos(\theta) \frac{(s^2 - 1)\sin(c)}{2s}$. If in the relation 2, we replace $\cos(c)$ by this value, then we get

$$-2s\sin(c) - \cos(\theta)^{2} \frac{(s^{2} - 1)\sin(c)}{2s} s^{2} + \cos(\theta)^{2} \frac{(s^{2} - 1)\sin(c)}{2s} = -2rs$$

$$\Rightarrow -2s\sin(c) - \frac{(s^{2} - 1)\sin(c)}{2s} s^{2} + \frac{(s^{2} - 1)\sin(c)}{2s} = -2rs$$

$$\Rightarrow \frac{(1 + s^{2})^{2}}{2s}\sin(c) = 2rs.$$

Since $\sin(c) < 0$, this case is out of order. Note that we have already excluded the origin.

Remark 3.1. Notice that in the discussion above clearly we used the Law of Excluded Middle as follows.

Let s > 0, $-\pi < \theta \le \pi$, and,

$$\mathscr{P}: (\exists (s,\theta); 2s - s^2 \sin(\theta) - \sin(\theta) = 0).$$

Then

$$\neg \mathscr{P}: (\forall (s,\theta); 2s - s^2 \sin(\theta) - \sin(\theta) \neq 0).$$

Moreover, $\mathscr{P} \vee \neg \mathscr{P}$ holds for the statement \mathscr{P} .

On the other hand, if $z = \exp(i\theta)$, then we have

$$F_1(z) = \frac{\exp(i\theta)}{\exp(i\pi/2 + i\theta) + 1}$$

$$= \frac{\exp(i\theta)}{2\cos(\theta/2 + \pi/4)\exp(i\theta/2 + i\pi/4)}$$

$$= \frac{\exp(-i\pi/4)\exp(i\theta/2)}{2\cos(\theta/2 + \pi/4)}.$$

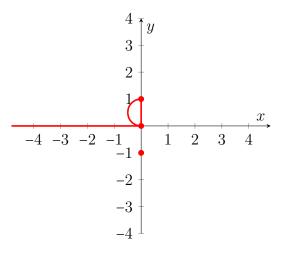


FIGURE 3. The region \mathcal{D} .

And we have

$$F_{2}(z) = \frac{\mathbf{i}z}{\mathbf{i}z+1} = \frac{z}{z-\mathbf{i}}$$

$$= \frac{\exp(\mathbf{i}\theta)}{\cos(\theta) + \mathbf{i}(\sin(\theta) - 1)}$$

$$= \frac{\exp(\mathbf{i}\theta)}{\sin(\pi/2 + \theta) + 2\mathbf{i}(\sin(\theta/2 - \pi/4)\cos(\theta/2 + \pi/4))}$$

$$= \frac{\exp(\mathbf{i}\theta)}{2\sin(\pi/4 + \theta/2)\cos(\pi/4 + \theta/2) + 2\mathbf{i}(\sin(\theta/2 - \pi/4)\cos(\theta/2 + \pi/4))}$$

$$= \frac{\exp(\mathbf{i}\theta)}{2\cos(\theta/2 + \pi/4)(\cos(\theta/2 - \pi/4) + \mathbf{i}\sin(\theta/2 - \pi/4))}$$

$$= \frac{\exp(\mathbf{i}\pi/4)\exp(\mathbf{i}\theta/2)}{2\cos(\theta/2 + \pi/4)}.$$

And we have

$$F_3(z) := \mathbf{i} \exp(\mathbf{i}c)(1 - \sin(\theta)) = \mathbf{i} \exp(\mathbf{i}c)(1 + \cos(\theta + \pi/2))$$
$$= 2\mathbf{i} \exp(\mathbf{i}c)\cos^2(\theta/2 + \pi/4).$$

We fix the open connected set of \mathcal{D} as shown in the Figure 3, where red lines, the red point and inside of the semicircle are excluded.

Now, we are ready to proceed with the example.

We consider the function

$$f: \mathcal{D} \to \mathbb{C}, \quad f(z) := \operatorname{Ln}(F_1(z)) + \operatorname{Ln}(F_2(z)) + \operatorname{Ln}(F_3(z)) - \operatorname{Ln}(z) + \operatorname{Ln}(2) - \frac{i\pi}{2} - ic.$$

By the argument above the function f is analytic on \mathcal{D} . Where \mathcal{D} is an open connected set. Now, we evaluate f in two set of points in the connected open set of \mathcal{D} . So that we consider two different cases as follow.

Case (I).
$$0 < \theta < \pi/2$$
.

We have

$$f(z) = \operatorname{Ln}\left(\frac{\exp(-i\pi/4)\exp(i\theta/2)}{2\cos(\theta/2 + \pi/4)}\right) + \operatorname{Ln}\left(\frac{\exp(i\pi/4)\exp(i\theta/2)}{2\cos(\theta/2 + \pi/4)}\right)$$
$$+ \operatorname{Ln}\left(2i\exp(ic)\cos^{2}(\theta/2 + \pi/4)\right) - \operatorname{Ln}\left(\exp(i\theta)\right) + \operatorname{Ln}\left(2\right) - \frac{i\pi}{2} - ic$$
$$= i\theta - 2\operatorname{Ln}\left(2\right) - 2\operatorname{Ln}\left(\cos(\theta/2 + \pi/4)\right) - i\theta + \operatorname{Ln}\left(2\right) + \frac{i\pi}{2} + ic$$
$$+ 2\operatorname{Ln}\left(\cos(\theta/2 + \pi/4)\right) + \operatorname{Ln}\left(2\right) - \frac{i\pi}{2} - ic = 0.$$

Case (II). $\pi/2 < \theta < \pi$. We have

$$f(z) = \operatorname{Ln}(\frac{\exp(-i\pi/4)\exp(i\theta/2)}{2\cos(\theta/2 + \pi/4)}) + \operatorname{Ln}(\frac{\exp(i\pi/4)\exp(i\theta/2)}{2\cos(\theta/2 + \pi/4)})$$

$$+ \operatorname{Ln}(2i\exp(ic)\cos^{2}(\theta/2 + \pi/4)) - \operatorname{Ln}(\exp(i\theta)) + \operatorname{Ln}(2) - \frac{i\pi}{2} - ic$$

$$= \operatorname{Ln}(\frac{\exp(-i\pi/4 + i\theta/2 - i\pi)}{-2\cos(\theta/2 + \pi/4)}) + \operatorname{Ln}(\frac{\exp(i\pi/4 + i\theta/2 - i\pi)}{-2\cos(\theta/2 + \pi/4)})$$

$$+ \operatorname{Ln}(2i\exp(ic)\cos^{2}(\theta/2 + \pi/4)) - \operatorname{Ln}(\exp(i\theta)) + \operatorname{Ln}(2) - \frac{i\pi}{2} - ic$$

$$= i\theta - 2\operatorname{Ln}(2) - 2\operatorname{Ln}(-\cos(\theta/2 + \pi/4)) - 2i\pi - i\theta + \operatorname{Ln}(2) + \frac{i\pi}{2} + ic$$

$$+ \operatorname{Ln}((-\cos(\theta/2 + \pi/4))^{2}) + \operatorname{Ln}(2) - \frac{i\pi}{2} - ic$$

$$= -2\operatorname{Ln}(-\cos(\theta/2 + \pi/4)) - 2i\pi + 2\operatorname{Ln}(-\cos(\theta/2 + \pi/4)) = -2i\pi \neq 0.$$

Note that in this case $\pi/4 < \theta/2 < \pi/2$, then $\pi/2 < \theta/2 + \pi/4 < 3\pi/4$, and $0 < \theta/2 - \pi/4 < \pi/4$. Hence, we have $-\cos(\theta/2 + \pi/4) > 0$, and $-\exp(-i\pi/4 + i\theta/2) = \exp(-i\pi/4 + i\theta/2 - i\pi)$, and $-\exp(i\pi/4 + i\theta/2) = \exp(i\pi/4 + i\theta/2 - i\pi)$. Where $-\pi < -\pi/4 + \theta/2 - \pi < -3\pi/4$, and $-\pi/2 < \pi/4 + \theta/2 - \pi < -\pi/4$ so they are principal arguments.

So that in the connected open set \mathcal{D} , the function f is analytic, not identically zero, and some of its zeros are not isolated. And this is a paradox.

Remark 3.2. If the reader is concerned with the open set \mathcal{D} to be simply connected, then they may consider this region to be the upper plane where the mentioned semicircle in the example and in its inside are excluded.

Remark 3.3. With very small modification one may switch the logarithmic function to the radical function and so gives similar examples; only with a different view.

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