THE PARALLEL LAW OF EXCLUDED MIDDLE (PLEM) AND CONSTRUCTIVE MATHEMATICS

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ABSTRACT. The Parallel Law of Excluded Middle (PLEM) introduces a novel logical framework where mathematical truth is distributed across structured sub-worlds rather than assumed to be absolute. Unlike classical logic, which enforces a single truth evaluation, or intuitionistic logic, which restricts non-constructive proofs, PLEM ensures that truth is systematically determined via a decision procedure function that operates between sub-worlds. This function governs how mathematical statements are resolved across different logical contexts, ensuring consistency while allowing for local variation in truth values.

This paper develops the formalism of PLEM by defining set operations, power sets, and membership functions in a way that preserves logical consistency while avoiding paradoxes such as Russell's paradox and the Grothendieck hierarchy. Unlike classical set theory, where universal sets lead to contradictions, PLEM allows each sub-world to have its own well-defined universal set, whose complement is the empty set. Furthermore, the power set operation, a foundational concept in set theory, is well-defined for proper sets but vanishes when applied to the full universe, preventing infinite regress.

By introducing a framework where only potential infinity exists, PLEM aligns with constructive mathematics while preserving a form of the Law of Excluded Middle through its structured logical distribution. The result is a philosophically coherent and mathematically rigorous alternative to classical set theory, providing a new foundation for mathematical truth.

1. Introduction

Mathematical logic has long been divided between classical set theory, which assumes an absolute notion of truth, and constructivist approaches, which require explicit verification of mathematical statements [9]. While classical logic relies on the Law of Excluded Middle (LEM) to assert that every statement is either true or false, constructivism rejects this principle unless a proof is explicitly given. This leads to fundamental tensions in set theory, particularly in how infinity, universal sets, and power sets are treated.

This paper introduces the Parallel Law of Excluded Middle (PLEM), a framework that distributes logical truth across interconnected sub-worlds. Instead of assuming a single universal truth evaluation, PLEM introduces a decision procedure function that determines whether a proposition is true in a given sub-world or whether its resolution must be deferred to another sub-world. This structured approach allows for logical consistency while avoiding paradoxes associated with absolute universality [8].

PLEM also provides a new perspective on infinity and set operations. Classical set theory assumes that infinite sets exist as completed entities, leading to paradoxes such as the power set hierarchy [6] and debates over the existence of uncountable sets [4]. In contrast, PLEM

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ensures that infinity is only potential, meaning that infinite collections must emerge through finite or stepwise countable processes. Unlike classical constructivist approaches, PLEM deos not assume the existence of infinite sequences as cmpleted entities, nor deos it rely on choice principles such as the Countable Axiom of Choice [2]. This ensures that mathematical structures remain explicitly generable rather than assumed as pre-existing totalities.

Main Contributions of This Paper

1. A New Logical Foundation
   - PLEM provides a novel logical framework that reconciles aspects of constructivism and classical logic while avoiding their respective limitations.
   - Unlike intuitionistic logic, which restricts the Law of Excluded Middle (LEM), PLEM preserves a modified form of LEM by distributing truth across structured sub-worlds.
2. Introduction of the Decision Procedure Function
   - A core feature of PLEM is a formal decision procedure function that determines whether a mathematical statement is resolved within a given sub-world or whether it requires resolution in another sub-world.
   - This function ensures logical consistency across sub-worlds, providing a structured alternative to absolute truth assumptions.
3. Set-Theoretic Consistency
   - The framework defines set operations, membership, and complements in a way that eliminates classical paradoxes.
   - Unlike classical set theory, PLEM ensures that every set is constructed in a logically controlled manner, preventing contradictions that arise from unrestricted comprehension.
4. Resolution of Classical Paradoxes
   - PLEM naturally eliminates Russell's paradox [8] by ensuring that universal sets only exist within their respective sub-worlds, preventing contradictions.
   - The framework avoids the problems of absolute universality that arise in classical set theory, providing an internally consistent structure for mathematical truth.
5. Redefinition of the Power Set
   - Unlike classical set theory, where power sets are assumed to exist for every set, PLEM ensures that the power set operation is meaningful only for proper sets.
   - When applied to the full universe, the power set vanishes, preventing infinite regress and paradoxes such as the Grothendieck hierarchy [6].
6. Constructive Treatment of Infinity
   - PLEM rejects completed infinity, ensuring that all infinite structures are generated through explicit stepwise processes [2].
   - This aligns with intuitionistic and constructive mathematics, preserving mathematical rigor while avoiding assumptions about uncountable sets [4].

By formalizing PLEM as a mathematically rigorous alternative to classical set theory, this paper lays the foundation for new approaches to logic, proof theory, and the philosophy of mathematics.

2. The Parallel Law of Excluded Middle (PLEM)

To introduce the Parallel Law of Excluded Middle, we first clarify several key concepts: sub-worlds, finite routines, and decision procedures. These notions will help us formalize PLEM and distinguish its local vs. global logical scope.

Sub-worlds. We use the term *sub-world* to mean an individual logical universe or context within our framework. Each sub-world can be thought of as a self-contained domain in which mathematical statements have truth values according to that world's information and rules. In other words, a sub-world is an internally "consistent world" (or logical frame) with its own set of true propositions. Different sub-worlds may correspond to different sets of assumptions, perspectives, or stages of knowledge. We denote by $\mathcal{W}$ the collection of all sub-worlds under consideration. Importantly, each sub-world is *local*: it may not represent the whole of mathematical reality, but rather a portion of it (a particular viewpoint or scenario). This setup allows truth to be a relative notion - what is true in one sub-world might be false in another, and some statements might remain undecided (neither proven nor refuted) within a given sub-world. The union of all sub-worlds, in contrast, provides a kind of global perspective encompassing multiple possibilities in parallel.

Finite routines and decision procedures. Within any single sub-world, the only way to establish truth or falsehood of statements is via finite processes. By a *finite routine*, we mean a procedure or algorithm consisting of a finite sequence of steps that can be carried out to obtain information or construct objects. In particular, any infinite sequence arising in a sub-world (for example, a sequence of bits $(a_n)_{n\in\mathbb{N}}$ with $a_n \in 0, 1$ for each n) is not viewed as a completed infinite object from the outset, but rather as an *infinitely proceeding* process generated by a finite rule at each step. This aligns with the constructive viewpoint: one can always perform a finite initial segment of the routine, but there is no assumption that an actual infinite totality is given all at once. A *decision procedure* is then any finite routine that yields a definite yes/no answer to a given question or proposition. In a sub-world, a statement P is considered *decidable* if there exists some finite routine (a proof, calculation, or other effective method) that acts as a decision procedure to determine P's truth value within that world. If such a procedure exists, the sub-world can affirm either P or $\neg P$ as true after finitely many steps. However, if no decision procedure is available in that sub-world, the statement may remain undetermined (neither true nor false locally). This captures the idea that each sub-world may be logically incomplete from a global standpoint, since some propositions could be independent of the information or axioms present in that world.

With these definitions in place, we now formulate the central principle:

Parallel Law of Excluded Middle - PLEM. *Either a statement or its negation is true, but possibly in different sub-worlds.* More formally, we adopt the following axiom: for every proposition P,

$$(\exists,\ W \in \mathcal{W} :\ W \vDash P)\ \vee\ (\exists\, W' \in \mathcal{W} :\ W' \vDash \neg P).$$

Here $W \vDash P$ denotes that P is true in sub-world W. In words, PLEM asserts that for any statement P, *either P holds in at least one sub-world or $\neg P$ holds in at least one sub-world*. This is a global logical principle spanning all sub-worlds in $\mathcal{W}$. It generalizes the classic Law of Excluded Middle (LEM) to a scenario of multiple parallel contexts: there is no absolute proposition left without a truth value somewhere in the collection of worlds. Crucially, PLEM does *not* require that P or $\neg P$ be decided within a single given world; it only requires that one of them is realized in *some* world. In fact, it allows (and in practice, expects) that different worlds may give different truth values to the same statement P. For instance, one sub-world W_1 might verify P (making P true in W_1), while another sub-world W_2 might refute P (making $\neg P$ true in W_2). PLEM guarantees that no statement is *globally* undecidable: the dichotomy P vs. $\neg P$ will be resolved by at least one world, even if it is not resolved in every world.

This framework distinguishes the **local** validity of LEM from its **global** validity. *Locally* (i.e. within a single sub-world), the usual Law of Excluded Middle need not hold universally. In a constructive sub-world, for example, one does not assert $P \vee \neg P$ unless one has a decision procedure to confirm either P or $\neg P$. Such a world might treat certain propositions as neither established nor refuted (so P is not known to be true and also not known to be false in that world). In a *classical* sub-world, on the other hand, one could postulate that every statement is either true or false in that world (the traditional LEM holds locally in that world). The PLEM does not force each individual world to obey LEM; instead, it requires that across the collection of all worlds, the dichotomy is covered. In other words, *global truth* is achieved in a *distributed* way: if a particular world fails to decide P, then some parallel world will decide it (one way or the other). No proposition is absolutely excluded from truth or falsehood; one of the two outcomes must manifest in the "multiverse" of sub-worlds. This parallel satisfaction of LEM ensures logical completeness at the global level without demanding that any single world be omniscient. Each sub-world can maintain internal consistency and even a form of logical restraint (as in constructivism), while PLEM as a global axiom guarantees that the collection $\mathcal{W}$ as a whole does not leave P indeterminate.

Mathematical truth under PLEM. Under PLEM, the truth of a mathematical statement becomes a two-tier notion. First, one can speak of truth *within* a specific sub-world: a statement can be true in W_1 and false in W_2, for example, if W_1 has adopted or proven that statement and W_2 has refuted or counterexampled it. Second, one can consider the *global status* of the statement across all sub-worlds: PLEM ensures that globally, the statement is not in a limbo of undefined truth-either it finds support in some world or its negation finds support in some world (often both will occur, in different worlds). This means that mathematical truth is no longer a single absolute boolean value, but rather a spectrum of possibilities distributed across sub-worlds, governed by the PLEM axiom. A consequence is that if we cannot construct a proof of P in our current world W, we do not conclude $\neg P$ outright (as a strict classical logician might via LEM); instead, we acknowledge that perhaps $\neg P$ holds in another world, or conversely P might hold in some alternative world even if W cannot see it. PLEM thus guides us to consider parallel scenarios to understand the truth of P. In effect, it *governs mathematical truth by extending the law of excluded middle to the ensemble of sub-worlds*: the truth-value of each proposition is decided, but the decision may live in a world beyond the one we are in. This philosophy aligns with the idea of exploring multiple models or interpretations of a theory to find where a statement can be settled. It provides a logical justification for the existence of these parallel models (hence the term *Parallel* LEM): if a statement is not resolved here, there must exist a resolution elsewhere. Indeed, one might view the PLEM axiom as *postulating the existence of parallel worlds* whenever needed to realize the two sides of every proposition. In particular, the acceptance of PLEM implies that there cannot be a proposition that is unprovable and also irrefutable in *all* contexts; if nothing in our current framework can decide it, we extend the framework with an additional sub-world where a decision is made. This is reminiscent of how, in model theory or set theory, independent statements (like the Continuum Hypothesis) lead us to consider multiple models: one where the statement holds and one where it fails. PLEM elevates this practice to an axiom, insisting that such a bifurcation (or more generally, a covering of all logical possibilities) always exists.

The Role of Infinity and the Empty Set in PLEM

PLEM introduces a fundamental shift in how mathematics treats infinity and the empty set. Unlike classical set theory, which often assumes completed infinite collections, PLEM operates strictly within finite or stepwise countable constructions. In this framework, actual infinity is not an entity—instead, what is often called "infinity" is treated as a process that is never fully realized but can be indefinitely extended.

This potential infinity approach ensures that all mathematical constructions remain explicitly generative, meaning that every set must emerge through finite or countably sequential steps rather than being assumed as an already existing infinite totality.

Similarly, the empty set is not treated as a fundamental primitive but is instead constructed through explicit membership conditions. In PLEM, the empty set is defined as the unique set whose membership function assigns 0 to every possible element in a given sub-world. This means:

$$A(a) = 0 \quad \forall a \text{ in a given sub-world.}$$

Thus, in PLEM, the empty set is not an absolute, unique object but instead context-dependent, varying across different sub-worlds. Each sub-world has its own empty set based on its own element structure. This distinction is crucial: rather than signifying absolute "nothingness," the empty set in PLEM serves as the neutral base for further constructions, allowing the systematic formation of larger sets.

Within any single sub-world, only finite or countably stepwise processes exist, ensuring that no absolute infinite set can arise. However, when considering the totality of all sub-worlds, a new form of infinity emerges—not as a singular, completed object, but as the unbounded extensibility of logical structures.

In PLEM, the collection of all sub-worlds, denoted as $\mathcal{W}$, is not an actualized infinite set but an indefinitely extensible system. This ensures that:

1. Each sub-world is finite or stepwise countable.
2. The network of sub-worlds allows for an infinite expansion of mathematical contexts, ensuring that every well-formed proposition is resolved somewhere.
3. This structured approach to infinity prevents paradoxes that arise from treating infinity as an absolute totality.

Thus, PLEM retains the usefulness of infinite extensions while avoiding the pitfalls of completed infinity. This aligns with the philosophical insight that "in parallel logics, there might be a true infinite," but only in the sense of an open-ended logical landscape, not an all-encompassing infinite entity.

The Constructive Role of the Empty Set in PLEM

PLEM also fundamentally changes how the empty set is used in mathematical proofs.

In classical logic, existence is often established through contradiction: one assumes an object does not exist, derives a contradiction, and concludes that the object must exist. This method implicitly assumes a starting state of "nothingness" from which existence emerges.

PLEM rejects this approach.

Instead, existence in PLEM must be established through explicit construction. The empty set is not a mere absence but a structured mathematical object that serves as the starting configuration for set formation. This ensures that all mathematical existence claims are explicitly built, rather than inferred from contradictions.

The Empty Complement Dilemma and the Universal Set in PLEM

A crucial consequence of PLEM's framework is the treatment of universal sets and complements. In classical set theory, the concept of a universal set is often avoided due to paradoxes, but in PLEM, each sub-world itself serves as a universal set within its domain.

Universal Set in Each Sub-World

- In each sub-world, there exists a unique universal set, which is simply the sub-world itself.
- This universal set contains all elements of the sub-world and is well-defined.
- The complement of the universal set is the empty set, as expected:

$$S^c = \varnothing.$$

Complements Are Well-Defined in PLEM

- Unlike classical set theory, where complements are sometimes defined relative to a larger "universal" discourse, PLEM ensures that:
- Every set has a well-defined complement inside its sub-world.
- There is no need to define complements relative to an external larger set.
- Given a large set E, the complement $E \setminus A$ is naturally defined and does not require extra assumptions.

Thus, PLEM allows complements to exist within sub-worlds in a logically consistent way without introducing paradoxes.

- The classical paradox of the complement of the universal set does not arise because the sub-world itself is a valid universal set, and its complement is simply the empty set.
- Unlike classical set theory, where absolute universality causes contradictions, PLEM ensures that universal sets exist in a well-defined, local manner without leading to inconsistencies.

PLEM as a Balance Between Constructive and Classical Truth

The core principle of PLEM is that mathematical truth is not localized to a single world but distributed across a structured plurality of sub-worlds.

This creates a balanced framework between:

1. Constructivist caution (truth must be explicitly generated within each sub-world).
2. Classical completeness (globally, every mathematical proposition is ultimately resolved somewhere).

Key Consequences:

- Within each sub-world, truth must be established by finite or stepwise processes—absolute totalities are not assumed.
- Globally, PLEM ensures that every mathematical proposition is decided somewhere (avoiding complete undecidability).

Thus, PLEM retains the spirit of the Law of Excluded Middle while avoiding paradoxes from assuming an all-knowing, universal logical structure.

The Final Takeaways:

1. The empty set is constructed, not assumed as "nothing."

2. Infinity is not contained within a single world but emerges through the open-ended network of sub-worlds.

3. Pathological contradictions (such as the universal set paradox) are avoided, not by restriction, but by logical design.

This provides a philosophically coherent and mathematically rigorous foundation, where truth is established through finite means within each world and guaranteed globally through the interplay of parallel logical systems.

PLEM rejects absolute totalities, ensuring that no contradictions arise from set-theoretic universals or infinite assumptions. However, it still preserves the classical intuition that mathematical statements are ultimately decidable—not necessarily within a single world, but across the full structure of sub-worlds.

This leads to a new landscape for mathematical reasoning, where:

- Every mathematical object is built step by step.
- Infinity is never assumed, only extended.
- Logical consistency is preserved by design, not just by restrictions.

Thus, PLEM bridges the gap between constructivist and classical approaches, offering a structured, logically sound, and philosophically rich foundation for mathematics.

3. Set Operations, the Power Set, and the Resolution of Classical Paradoxes in PLEM

Non-Existence of an Absolute Universal Set:

The assumption of a universal set in classical set theory is well known to lead to contradictions. Russell's paradox, first formulated by Russell [8], arises from the assumption that unrestricted set comprehension allows the construction of the set of all sets that do not contain themselves, leading to an inconsistency in membership conditions. ZF set theory resolves this by restricting comprehension through axioms like Regularity and Separation [3].

Additionally, Grothendieck universes, introduced to handle large mathematical structures, attempt to provide a hierarchy where power sets can be taken indefinitely without leading to paradoxes [6]. However, this relies on axiomatic assumptions that permit infinite set extensions, which PLEM explicitly rejects.

One line of reasoning is that U would have to contain itself as an element (since U is a set and U contains all sets, so $U \in U$). However, Zermelo-Fraenkel axioms like Regularity forbid any set from containing itself; indeed, if U existed, it must contain itself, which contradicts the axiom that no set is an element of itself.

Another argument uses the unrestricted comprehension principle: if U exists, one can attempt to form the set

$$R = \{x \in U \mid x \notin x\}$$

of all sets in U that do not contain themselves. By construction, R would be a subset of U and thus a set. Yet Russell's paradox shows R can neither contain itself nor avoid containing itself consistently.

How PLEM Resolves This Issue

- PLEM does not allow an absolute universal set, but it does allow a universal set within each sub-world, which is simply the sub-world itself.
- Since each sub-world operates independently, the contradictions of Russell's paradox do not arise.

- The complement of the universal set in a sub-world is just the empty set $S^c = \varnothing$, avoiding paradoxes.
- Unlike ZF set theory, where the "universe" is often treated as a proper class, in PLEM the universe is always a well-defined set inside each sub-world.

In everyday mathematics, when we talk about the complement of a set, we implicitly refer to a complement relative to a fixed universe of discourse, rather than an all-encompassing universal set. PLEM ensures that such universal sets exist locally within each sub-world while avoiding the contradictions of absolute universality.

Now, before we start to go over set operations let us note that such binary 0 and 1 could be defined in each sub-world satisfying the following: $1 + 0 = 0 + 1 = 1 + 1 = 1$, $1 \cdot 1 = 1$ and $0 \cdot 0 = 1 \cdot 0 = 0 \cdot 1 = 0$.

Let A be a set in a sub-world as well a be an element in this sub-world. If $a \in A$ in this sub-world, then we are done. Otherwise, $a \notin A$ in this sub-world or in another sub-world. If $a \notin A$ in this sub-world, then we are done and if $a \notin A$ in another sub-world, then a and A are common among both sub-worlds. Hence, $a \notin A$ in the original sub-world as well. Therefore, in PLEM, sets are defined through their membership functions, which provide a binary indicator of whether an element belongs to a set. This allows us to define fundamental set operations in a consistent and constructive way.

Each set A is associated with a membership function:

$$A(a) := \begin{cases} 1, & \text{if } a \in A, \\ 0, & \text{if } a \notin A. \end{cases}$$

Using the membership function, the fundamental set operations are defined as follows:

- Union: $(A \cup B)(a) = A(a) + B(a)$.
- Intersection: $(A \cap B)(a) = A(a) \cdot B(a)$.
- Complement: $A^c(a) = 1 - A(a)$.
- Set Difference (Subtraction of Sets): $(A - B)(a) = A(a) \cdot (1 - B(a))$.
  - This ensures that an element is in A - B only if it is in A but not in B.

These definitions are finite and constructive, ensuring they align with PLEM's foundational principles.

Definition of the Empty Set

For every element a, the empty set is represented by $A(a) = 0$ for all a. This definition aligns with a constructive viepoint, where the empty set is not treated as an abstract object rather as the sequence of zeros across all elements, avoiding potential ontological concerns.

How the Empty Set Exists but is Not a Subset

In classical set theory, the empty set $\varnothing$ is defined as the set with no elements, and it is assumed to be a subset of every set. However, in PLEM, the empty set exists, but it is not a subset of anything.

Why the Empty Set is Not a Subset in PLEM

- In classical set theory, it is argued that since $\varnothing$ has no elements, it "does not contradict" any set.
- However, in PLEM, the empty set is literally nothing, and nothing cannot be a part of something.

- Since subset relations are determined by membership, and the empty set has no defined membership in any set, it follows that:

$$\varnothing \notin A \quad \text{for any set } A.$$

- This corrects a fundamental flaw in classical set theory: assuming nothing has some kind of implicit presence in all sets.

Conclusion: The empty set exists, but it is not a subset of anything—it is a defined object but does not participate in set membership relationships.

De Morgan's Laws

With these definitions, De Morgan's Laws naturally hold:

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

These laws follow directly from the definitions of union, intersection, and complement using binary sequences. Specifically, for the first law:

- The complement of the union of two sets is:

$$(A \cup B)^c(a) = 1 - (A(a) + B(a)) = (1 - A(a)) \cdot (1 - B(a)) = A^c(a) \cdot B^c(a),$$

which is exactly the intersection of the complements.

For the second law:

- The complement of the intersection of two sets is:

$$(A \cap B)^c(a) = 1 - (A(a) \cdot B(a)) = (1 - A(a)) + (1 - B(a)) = A^c(a) + B^c(a),$$

which is exactly the union of the complements.

These identities align with the standard De Morgan's Laws in Boolean algebra, making the application of these laws in our context both straightforward and consistent. With these definitions, De Morgan's Laws naturally hold, demonstrating that our binary sequence approach is algebraically consistent with classical set operations. This provides a computationally well-defined perspective on complements.

The Power Set in PLEM

In classical set theory, the power set $\mathcal{P}(A)$ is defined as the set of all subsets of A. However, this definition relies on assumptions that do not hold in PLEM, such as:

- The assumption that the empty set is always a subset.
- The assumption that power sets can be infinitely iterated.
- The assumption that the universe itself has a power set, leading to paradoxes.

The PLEM Definition of Power Sets

In PLEM, we define power set membership using the function:

$$\mathcal{P}(A)(B) := (1 - \chi(S - A)) \cdot \chi(B - A) \cdot \prod_{a \in A} B(a).$$

where:

- S is the sub-world, the universe of discourse in which A exists.
- $\chi(X)$ is the characteristic function:

$$\chi(X) = \begin{cases} 0, & \text{if } X \neq \varnothing \\ 1, & \text{if } X = \varnothing. \end{cases}$$

- $\chi(S - A)$ ensures that the power set is only defined for proper subsets of S.
- $\chi(B - A)$ ensures that B is a subset of A.

- $\prod_{a\in A} B(a)$ ensures that all elements of A are present in B.

Resolution of Russell's Paradox and Grothendieck Universes

Russell's Paradox Does Not Arise in PLEM

- In classical set theory, Russell's paradox occurs when we consider the set of all sets that do not contain themselves:

$$R = \{x \mid x \notin x\}.$$

- This leads to a contradiction: Does R contain itself?
- In PLEM, this paradox does not arise because:
  - Power sets are not assumed to exist for all sets.
  - The power set of the universe itself vanishes (see below).
  - Self-referential sets cannot be formed, because power sets are constrained by the logical structure of sub-worlds.

The Grothendieck Universe Hierarchy Does Not Exist in PLEM

- Grothendieck universes are built by successively taking power sets of power sets.
- This assumes that the universe itself can be expanded infinitely.
- In PLEM, this assumption is false, because the power set of the full universe vanishes.

The Vanishing of the Power Set of the Universe

- When applying the power set formula to the entire universe S, we get:

$$\mathcal{P}(S) = (1 - \chi(S - S)) \cdot \chi(B - S) \cdot \prod_{a\in S} B(a).$$

- Since $S - S = \varnothing$, we get $\chi(S - S) = 1$, meaning: $1 - \chi(S - S) = 0$.
- This forces: $\mathcal{P}(S) = 0$.

Implication:

- The power set of the universe is the empty set.
- This means there is no external expansion of the universe beyond itself.
- This prevents paradoxes and infinite regressions.

Final Statement

In classical set theory, power sets are assumed to exist as absolute objects, leading to paradoxes and unnecessary complexity. In PLEM, we redefine power sets so that they are meaningful for proper sets but vanish when applied to the entire universe. This ensures logical consistency, removes the paradoxes of Russell and Grothendieck, and establishes a structured foundation for set theory without infinite regressions. Furthermore, the empty set is not a subset of anything, as nothing cannot be contained within something. This leads to a new perspective on mathematical and physical existence.

The Non-Existence of Uncountable Sets in PLEM

In standard set theory, uncountability is typically established using Cantor's diagonal argument. This proof assumes that given any countable list of infinite binary sequences, one can systematically construct a sequence that is not in the list by modifying the nth digit of the nth sequence. However, this argument relies on the Axiom of Countable Choice, which is not available in PLEM.

In our framework:

- Every set is explicitly constructed through finite or stepwise countable processes.

- The membership function of a set is a binary sequence, but this sequence must be explicitly defined rather than assumed to exist by an abstract diagonalization argument.
- Since the diagonal construction requires making an infinite number of independent choices, it does not hold in PLEM, where such global choices are not permitted.

Thus, there are no uncountable sets in PLEM because the very process used to prove uncountability depends on an assumption (countable choice) that is absent in our logic. Every set in PLEM must be constructible within a sub-world, and constructible processes do not lead to uncountable totalities.

4. Decision Procedure

The Role of the Decision Procedure in PLEM

In classical logic, a proposition is any well-formed statement that has a definite truth value (true or false). However, in PLEM, we redefine what qualifies as a proposition within a sub-world:

- A proposition in a given sub-world is a statement whose truth value is decidable within that sub-world.
- If the truth value cannot be determined within the sub-world, then it is not a proposition in that sub-world but only a proposition in the broader logical structure of PLEM.
- The decision function governs how such statements are resolved across sub-worlds, ensuring logical consistency.

Thus, unlike classical logic, where every proposition is assumed to have a truth value, in PLEM, a statement only qualifies as a proposition if it can be logically evaluated within a sub-world or transferred to another sub-world for resolution.

When Do We Need the Decision Function?

PLEM introduces an important distinction in how truth values are assigned:

1. Evaluating a Specific Element Within a Set (Decision Function Not Required)

- When we talk about a specific element a and a specific set A, we can determine membership directly using the membership function:

$$A(a) = \begin{cases} 1, & \text{if } a \in A, \\ 0, & \text{if } a \notin A. \end{cases}$$

- Here, we do not need the decision function because the set A is explicitly defined.

2. Evaluating a Variable (Decision Function Is Required)

- If a is not a specific element but a variable, and A is defined by a logical condition, then the decision function is needed to determine membership.
- For example, if A is defined by a property P(x), then:

$$D(P(a), S) = \begin{cases} 1, & \text{if } P(a) \text{ is verifiable within sub-world } S, \\ 0, & \text{if } P(a) \text{ is false within sub-world } S, \\ D(P(a), S'), & \text{if the truth of } P(a) \text{ must be determined in another sub-world } S'. \end{cases}$$

- This ensures that the truth value of P(a) is determined within a structured logical context, rather than assumed universally.

Formal Definition of the Decision Function
The decision function governs how truth is assigned to statements within PLEM:

$$D(P(a_1,\ldots,a_n),S) = \begin{cases} 1, & \text{if } P(a_1,\ldots,a_n) \text{ is verifiable within sub-world } S, \\ 0, & \text{if } P(a_1,\ldots,a_n) \text{ is false within sub-world } S, \\ D(P(a_1,\ldots,a_n),S'), & \text{if the truth of } P(a_1,\ldots,a_n) \text{ must be determined in another sub-world } S'. \end{cases}$$

where:

- $P(a_1,\ldots,a_n)$ is a mathematical or logical function (e.g., a predicate or set membership condition).
- S is the current sub-world.
- S' is a related sub-world where $P(a_1,\ldots,a_n)$ may be resolved if it is undecidable in S.

The Decision Function and Set Operations
Since set operations correspond to propositional structures in PLEM, the decision function extends naturally to them.

- Union. A proposition involving the union of two sets follows the rule:
$$D(A \cup B, S) = D(A,S) + D(B,S).$$
This means that if an element belongs to either A or B, it belongs to $A \cup B$.
- Intersection. For the intersection of two sets:
$$D(A \cap B, S) = D(A,S) \cdot D(B,S).$$
This ensures that an element is in $A \cap B$ only if it belongs to both A and B.
- Complement. For the complement of a set:
$$D(A^c, S) = 1 - D(A,S).$$
If an element is in A, then it is not in A^c, and vice versa.
- Set Difference (Subtraction). For the difference of two sets (subtraction):
$$D(A - B, S) = D(A,S) \cdot (1 - D(B,S)).$$
An element belongs to A–B if it is in A but not in B.

De Morgan's Laws in PLEM
Since De Morgan's Laws express fundamental relationships between union, intersection, and complement, they must also be formulated using the decision function:

- First De Morgan's Law:
$$D((A \cup B)^c, S) = D(A^c,S) \cdot D(B^c,S).$$
This states that the complement of a union is equal to the intersection of the complements.
- Second De Morgan's Law:
$$D((A \cap B)^c, S) = D(A^c,S) + D(B^c,S).$$
This states that the complement of an intersection is equal to the union of the complements.

Since PLEM constructs logical truth in a structured way, these laws hold consistently within each sub-world.

Connection to Frege's Concept of Function

Frege defined functions as statements with variables, meaning that their truth value depends on input values. In PLEM:

- A function $P(a_1, \ldots, a_n)$ is a mathematical proposition, and its truth is determined within a given sub-world.
- The decision function governs whether $P(a_1, \ldots, a_n)$ can be evaluated directly or must be deferred to another sub-world.
- Set operations are special cases of propositional truth functions, meaning that their evaluation follows the same logical structure as general functions.

The Decision Function for Power Sets in PLEM

In PLEM, the decision function systematically determines truth values across sub-worlds. For power sets, the challenge is ensuring that all evaluations happen within a single sub-world while preserving logical consistency.

We define the decision function for power set membership as:

$$D(\mathcal{P}(A)(B), S) = \big(1 - \chi(F(S) - D(A,S))\big) \cdot \chi(D(B - A, S)) \cdot \prod_{a \in A} D(B(a), S).$$

where:

1. $F(S)$ Determines the Sub-World for Decision-Making

$$F(S) = \begin{cases} S, & \text{if all decisions can be made in } S, \\ S', & \text{if some decisions must be deferred to } S'. \end{cases}$$

   - Ensures all evaluations are made in a single sub-world.
   - Prevents fragmentation of truth across multiple sub-worlds.

2. Explanation of Each Term in the Formula
   - $\chi(F(S) - D(A,S))$: Ensures that the decision process correctly aligns with the logical structure of PLEM.
   - $\chi(D(B - A, S))$: Ensures that the subset B satisfies membership conditions within A.
   - $\prod_{a \in A} D(B(a), S)$: Evaluates whether all elements of B belong to A within sub-world S.
     - If all components can be evaluated in S, the decision remains in S.
     - If some components require evaluation in S', then $F(S) = S'$ ensures that all decisions are made consistently.
     - This prevents contradictions and aligns with PLEM's structured logic.

5. Constructive Definition of Integer Arithmetic (Succ/Pred Approach)

Basic Setup: We take 0 as the base integer, and two unary operations: successor $S(x) = x + 1$ and predecessor $P(x) = x - 1$. We assume the key inverse law P(S(x)) = S(P(x)) = x for all integers x . In this constructive framework (consistent with the Parallel Law of Excluded Middle), integers are generated by iterating S or P starting from 0, rather than by set-theoretic encodings. All definitions and proofs proceed by explicit construction (using these functions) without invoking non-constructive axioms or set abstractions like the empty set.

1. Addition via Successor Composition

Definition (Addition): For any integers a and b, define $a + b$ by applying the successor or predecessor function repeatedly, according to the structure of b. Formally:

- Base case: $0 + 0 := 0$.
- If $a > 0$, then $a + 0 = 0 + a := S(P(a))$.
- If $a < 0$, then $a + 0 = 0 + a := P(S(a))$.
- Successor step: If $b = S(c)$ (i.e. $b = c + 1$), then define $a + b := S(a + c)$. Equivalently, $a + (c + 1) = (a + c) + 1$, meaning we add 1 to $a + c$.
- Predecessor step: If $b = P(c)$ (i.e. $b = c-1$), then define $a+b := P(a+c)$. Equivalently, $a + (c - 1) = (a + c)–1$, meaning we subtract 1 from $a + c$.

This definition ensures addition is built by composition of S and P. Intuitively, to compute $a + b$, we start with a and perform b "steps" : if b is positive, we successively apply S (add 1) for each unit in b; if b is negative, we successively apply P (subtract 1) for each unit in $|b|$. In other words, $a + n$ (for natural n) means apply S n times to a , and $a + (-n)$ means apply P n times to a. This is exactly the Peano-style iteration of the successor function for addition, extended to negative integers by allowing iteration of the predecessor function.

Basic properties: From the definition, we immediately get the familiar properties of addition, now proven constructively: Identity: a + 0 = a. Successor Rule: $a + S(c) = S(a + c)$ by definition . For example, $5 + 3 = 5 + S(2) = S(5 + 2) = S(7) = 8$. Predecessor Rule: $a + P(c) = P(a + c)$. In particular, adding a predecessor corresponds to subtracting one: e.g. $5 + (-3)$ can be seen as $5 + P(P(P(0))) = P(P(P(5 + 0))) = P(P(P(5))) = 2$. Formally, one can show for all integers x, y:

$$x + P(y) = P(x + y),$$

which is the right-inverse analog of the successor rule . Dually, $P(x) + y = P(x + y)$ as well (left inverse law) . These equations confirm that subtraction is built into addition via the predecessor function.

Crucially, we did not introduce subtraction as a separate primitive operation. Instead, subtraction is integrated as the inverse of addition: the presence of $P(x) = x - 1$ allows us to handle "negative addition" inherently. In practice, to subtract b from a, we add the predecessor of b repeatedly. For instance, $a–b$ can be understood as $a + (P(P(\dots P(0)\dots)))$ with b applications of P if b is positive. This approach yields the usual outcome since each predecessor step undoes a successor step of addition. Thus all cases of addition (adding positive, zero, or negative) are covered by the single recursive definition above.

2. Multiplication as Repeated Addition (Positive and Negative)

Definition (Multiplication): Given integers a and b, define the product ab by iterating addition (this is a second-order iteration, since addition itself was defined by iterating successor/predecessor). Formally:

- Let $a > 0$. Then $1 \times a = a \times 1 = S(P(a))$.
- Let $a, b > 0$. If $b = S(c)$ (i.e. $b = c + 1$ and $c \geq 1$), then $ab := (ac) + a$. This means $a(c + 1) = (ac) + a$, reflecting that multiplying by a positive number is adding a repeatedly (e.g. $a \times 3 = a + a + a$).
- Let $a, b < 0$. Then in this case we simply go with the above case as $ab := (-a)(-b)$.
- Let $a > 0$, $b < 0$. Then in this case we simply go with the first case as $ab := -(a)(-b)$.
- Let $a < 0$, $b > 0$. Then in this case we simply go with the first case as $ab := -(-a)(b)$.

All these properties hold constructively since the definitions are given by explicit recursion without case distinctions beyond the structural form of the second argument.

3. Multiplication by Zero via Distributivity

To rigorously establish the rule $0 \times n = 0$, $n \times 0 = 0$, and $0 \times 0 = 0$ in this constructive setting, we rely on the distributive law and the existence of additive inverses (which we have, thanks to predecessor). The proof is straightforward:

- $0 \times n = (1 - 1) \times n = n - n = 0$.
- $n \times 0 = n \times (1 - 1) = n - n = 0$.
- $0 \times 0 = (1 - 1) \times (1 - 1) = 0$.

This uses that $x(y+z) = xy+xz$, when $x, y, z \neq 0$. The key step was using distributivity which is not an axiom and could be proven. The rest is purely logical manipulation allowed in our constructive setting: we didn't need to invoke any case distinction or excluded middle, just the algebraic properties of addition and multiplication that we've constructively established (existence of additive inverses and the validity of distributive law for our definitions). The result $0 \times n = 0$ is thus derived constructively and holds for all integers n.

4. Concluding Remarks (Constructivity and PLEM)

In this refined approach, every operation is defined by a concrete rule (successor/predecessor iterations), and every property is proved using those rules. We avoided any set-theoretic construction of numbers (such as defining 0 as $\varnothing$, 1 as $\varnothing$, etc.) - instead, we work directly with the operations (0, S, P) that generate the integers. This keeps the development explicitly constructive: an integer is not an abstract set but something we can build by a finite sequence of $+1$ or -1 steps, and operations like addition and multiplication are given by algorithms following those steps.

Finally, this approach is compatible with the Parallel Law of Excluded Middle (PLEM) framework in the sense that we haven't assumed the ordinary law of excluded middle or non-constructive existence principles at any point. All proofs (such as $0 \times n = 0$) are done by direct algebraic manipulation and induction on the structure of numbers, which are methods allowed in constructive mathematics. The PLEM framework likely provides a context where one can in parallel consider classical reasoning, but here we did not need to invoke it - the development stays within a constructivist-friendly paradigm. Thus, the integer arithmetic operations and their fundamental properties are established with full clarity and rigor, grounded in the simple successor/predecessor mechanics.

6. Conclusion

This paper develops the Parallel Law of Excluded Middle (PLEM) as a structured alternative to classical and intuitionistic logic, ensuring that mathematical truth is systematically distributed across interconnected sub-worlds. By introducing the decision procedure function, PLEM establishes a formal mechanism for determining whether a statement is true within a sub-world or whether its truth value depends on a broader logical structure. This resolves key paradoxes, particularly those involving universal sets and power sets, while preserving logical expressibility.

A major contribution of PLEM is its constructive approach to infinity. Unlike classical set theory, which assumes the existence of infinite sets outright, PLEM only allows potential infinity, meaning that infinite collections emerge through explicit generative processes. This ensures that set-theoretic operations remain well-defined without leading to contradictions.

Key Takeaways :

- The decision procedure function ensures logical consistency across sub-worlds.
- Set operations, membership, and complements are formally defined, eliminating Russell's paradox and preventing infinite regress.
- The power set is well-defined for proper sets but does not extend to the full universe, avoiding paradoxes such as the Grothendieck hierarchy.
- Infinity is always potential, preventing contradictions related to actual infinity.

7. Future Directions

PLEM introduces a new foundation for mathematical logic, opening multiple avenues for further research:

1. Proof Theory and Computability – Investigating how the decision procedure function aligns with formal proof systems and computational logics.

2. Category-Theoretic Extensions – Exploring how PLEM could be formalized using topos theory and higher-order structures [1]

3. Mathematical Logic and Alternative Foundations – Comparing PLEM with paraconsistent logics [7] and constructive type theories.

By restructuring the foundations of set theory, PLEM provides a powerful framework that balances constructivist caution with classical expressibility. Future work will explore its applications to mathematical proof theory, formal systems, and higher-order logic.

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