EMERGENCE OF DEDEKIND CUTS WITHIN THE FRAMEWORK OF PLEM

BABAK JABBAR NEZHAD

Abstract. In this paper, we reformulate Dedekind cuts through the lens of the Parallel Law of Excluded Middle (PLEM), a logical framework introduced in prior work to enable constructive reasoning without reliance on infinite totalities. Classical approaches to Dedekind cuts often assume a completed real number system and treat all real numbers—rational and irrational—uniformly. We challenge this assumption by introducing a decision function that preserves the explicit and finite nature of rational numbers while allowing irrational numbers to emerge through approximation across sub-worlds. This structural reinterpretation not only constructs a decidable, ordered field of real numbers but also offers a logically grounded foundation for real analysis that avoids assuming uncountability. We further demonstrate that arithmetic, set theory, and completeness emerge naturally from transformations and relations, thereby offering a constructive alternative that aligns both with classical results and philosophical clarity.

1. Introduction

The construction of the real numbers has long been a central theme in mathematics. From Dedekind's original formulation of cuts to Cauchy sequences, Bishop's constructivism, and Brouwer's intuitionism, each approach reflects a different philosophical stance on logic, infinity, and mathematical truth. Traditional Dedekind cuts, while elegant, treat all real numbers as equally accessible and assume the real line as a completed totality—a view that imposes ontological assumptions often at odds with constructive reasoning.

This paper offers a reformulation of Dedekind cuts within the framework of the Parallel Law of Excluded Middle (PLEM) [2], which allows for stepwise construction of mathematical objects without committing to infinite totalities. In our framework, every Dedekind cut corresponds to a real number structurally, but numerical value emerges through a decision function grounded in rational approximations. This preserves the finiteness of rational numbers and redefines irrational numbers as objects whose meaning unfolds progressively across logical sub-worlds.

We also explore how this approach naturally constructs arithmetic operations, establishes decidability, and guarantees completeness without requiring the axiom of uncountability. Beyond analysis, we show how sets, numbers, and logic itself emerge from primitive spatial transformations. This approach reverses the foundational order typically assumed in logicism and set theory, offering a more unified and structurally necessary path to real numbers.

2020 *Mathematics Subject Classification.* 03F65, 03B20, 03A05, 26E40, 03F03, 12J15.

Key words and phrases. Dedekind cuts, Constructive real numbers, PLEM (Parallel Law of Excluded Middle), Real number completeness, Logical foundations, Rational and irrational separation, Decision function, Stepwise approximation, Constructive mathematics, Foundations of real analysis.

Babak Jabbar Nezhad has also published under the name Babak Jabarnejad [1].

Although parts of the construction may initially appear intuitive, they rest on a carefully structured logical foundation. Each technical step is grounded in the principles of the Parallel Law of Excluded Middle, which replaces classical assumptions with a decidable and internally justified framework. As such, even seemingly straightforward arguments require close attention, especially when distinguishing between classical and constructive notions of set, membership, and approximation. The overall system is simple in architecture, but subtle in execution.

2. Dedekind Cuts

In classical mathematics, Dedekind cuts are a foundational method for constructing real numbers from rational numbers. However, traditional presentations often assume an implicit notion of infinity, particularly when dealing with supremum and infimum, or when treating real numbers as completed entities.

In the framework of Parallel Law of Excluded Middle (PLEM), we reformulate Dedekind cuts to align with a constructive, logical approach, ensuring that they do not depend on infinite totalities but instead emerge from finite, stepwise structures.

A Dedekind cut is a way to define real numbers using only rational numbers. Instead of treating real numbers as fundamental, we define them as partitions of rational numbers into two nonempty sets with specific properties.

Definition 2.1. A pre-Dedekind cut is an ordered pair of sets (C_L, C_R) such that:

- C_L and C_R are not empty.
- Partition Condition: $C_L \cap C_R = \varnothing$ and $C_L \cup C_R = \mathbb{Q}$.
- Lower Set Condition: If $x \in C_L$ and $y < x$, then $y \in C_L$.
- Upper Set Condition: If $y \in C_R$ and $x > y$, then $x \in C_R$.

Remark 2.2. It might initially appear that the definition of pre-Dedekind cuts and related constructions over the rational numbers do not require any form of the Law of Excluded Middle. After all, the set of rational numbers is countable, and its ordering is fully decidable. However, this perception overlooks a critical foundational point in our framework: the moment we invoke the concept of a set, especially when considering subsets of rational numbers, we rely on the Parallel Law of Excluded Middle (PLEM) to ensure that membership is a decidable proposition [2].

In our framework, sets are not assumed as abstract totalities; they are defined internally through membership functions within the sub-world. This sub-world serves as the only universal domain, and every subset of $\mathbb{Q}$, such as the ones used to define Dedekind cuts, must be constructed through a function that determines whether a given element belongs or not. The decidability of that function—even when applied to the rational numbers—requires PLEM, not classical logic.

Clarification on Classical and Constructive Contexts.

In classical mathematics, the set $\mathbb{Q}$ and all its subsets are taken as given within a complete set-theoretic universe. The Law of Excluded Middle is globally assumed, and so no additional logical justification is needed when discussing whether a rational number belongs to a subset. Set membership is inherently decidable in that framework.

In constructive mathematics, including Bishop-style approaches, the rational numbers $\mathbb{Q}$ remain fully decidable: given any two rationals, we can constructively determine their order or equality. However, subsets of $\mathbb{Q}$ must be explicitly defined, and membership in such sets must be constructively verified.

In our setting, which refines constructive logic through the use of PLEM, we do not assume that such membership decisions are inherently decidable. Rather, we construct decidability through the logic of the sub-world, where each set's membership function is bound by PLEM. So although the elements of $\mathbb{Q}$ are themselves decidable, talking about sets of rational numbers is not neutral—it demands a framework in which membership is logically grounded and internally justified.

Definition 2.3. For any pre-Dedekind cut (C_L, C_R):

- The cut is positive if there exists $x > 0$ such that $x \in C_L$ and no negative number is in C_R.
- The cut is negative if there exists $x < 0$ such that $x \in C_R$ and no positive number is in C_L.

Note that if a cut is not positive, then one of the following must occur:
1. There exists $x < 0$ with $x \in C_R$ and $y > 0$ with $y \in C_L$ – a contradiction.
2. For all $x \in C_R$ we have $x \geq 0$ and for all $y \in C_L$ we have $y \leq 0$. Which represents 0.
3. The cut satisfies conditions of being negative.

Therefore, for every cut, we can decide whether it is zero, negative, or positive.

Definition 2.4. A Dedekind cut is a pre-Dedekind (C_L, C_R) such that:

- If (C_L, C_R) is positive, then there is no maximum in C_L: If $x \in C_L$, there exists $y \in C_L$ such that $x < y$. Then we call such a Dedekind cut positive.
- If (C_L, C_R) is negative, then there is no minimum in C_R: If $x \in C_R$, there exists $y \in C_L$ such that $x > y$. Then we call such a Dedekind cut negative.
- The cut represents zero if C_L contains all negative rational numbers, and C_R contains all nonnegative rational numbers.
- We call the set of all Dedekind cuts, real cuts and denote it by $\mathcal{R}$.

Remark 2.5. Each positive rational number $q \in \mathbb{Q}$ corresponds to the Dedekind cut:

$$C_L = \{x \in \mathbb{Q} \mid x < q\}, \quad C_R = \{x \in \mathbb{Q} \mid x \geq q\}.$$

Each rational number $q < 0 \in \mathbb{Q}$ corresponds to the Dedekind cut:

$$C_L = \{x \in \mathbb{Q} \mid x \leq q\}, \quad C_R = \{x \in \mathbb{Q} \mid x > q\}.$$

This ensures that every rational number is identified with a unique Dedekind cut.

Lemma 2.6 (Equivalence to Classical Dedekind Cuts)**.** Every real number that is classically represented by a Dedekind cut is also uniquely represented in our framework.

Proof. Let (C_L, C_R) be a classical Dedekind cut. If the cut corresponds to an irrational number, then C_L has no maximum and C_R has no minimum. Such cuts are accepted without modification in our system, and define irrational numbers as before.

If the cut corresponds to a rational number q, then the classical construction uses:

$$C_L = \{x \in \mathbb{Q} \mid x < q\}, \quad C_R = \{x \in \mathbb{Q} \mid x \geq q\}.$$

In our approach, such a cut is interpreted by the decision function as a rational number, since the right cut has a minimum (or equivalently, the left cut has a maximum). Therefore, we assign numerical value q to this cut directly.

Hence, our definition of Dedekind cuts covers all classical reals: irrational cuts match exactly, and rational ones are made explicit through decision. □

Remark 2.7. Our approach makes the distinction between rational and irrational real numbers explicit at the level of the cut, instead of treating them uniformly. This reinforces the constructive and logical nature of the real number system, allowing rational numbers to retain their finite structure, while irrational numbers are recognized as necessarily approximated by infinite structure.

Logical Justification for the Structure of Dedekind Cuts in PLEM

In classical Dedekind cut theory, all real numbers—whether rational or irrational—are treated within the same framework, assuming an already completed structure of the real number system. However, our approach introduces a decision function that dynamically determines whether a given Dedekind cut corresponds to a rational or an irrational number. This approach preserves the explicit, constructive nature of rational numbers while ensuring that irrational numbers emerge as a result of a decision-based approximation process. In our framework, we do not assume incompleteness a priori. Instead, the structure of Dedekind cuts naturally produces completeness without requiring an external axiom.

The Decision Function and Its Role in Numerical Assignment

We define the decision function D(C, S) that operates as follows:

1. If C_L has a maximum (or C_R has a minimum), the cut corresponds to a rational number.
   - The decision function immediately assigns the number represented by this maximum/minimum.
   - This preserves the explicit and finite nature of rational numbers in our framework.

2. If C_L has no maximum (or C_R has no minimum), then the cut corresponds to an irrational number.
   - The decision function uses numerical approximation to assign meaning to the cut.
   - It ensures that the cut corresponds to a well-defined limit, making it a meaningful real number.

Key Insight:

- Rational numbers are explicitly assigned values, while irrational numbers are determined through approximation.
- This function preserves the constructive nature of rational numbers rather than treating them as arbitrary cuts.

Example: How the Decision Function Works

- Case 1: The Rational Number 3

  The Dedekind cut for 3 is:

$$C_L = \{q \in \mathbb{Q} \mid q < 3\}, \quad C_R = \{q \in \mathbb{Q} \mid q \geq 3\}.$$

  - Here, C_R has a minimum element, which is the number 3 itself.
  - The decision function directly assigns $D(C, S) = 3$, ensuring its explicit, finite representation.
- Case 2: The Irrational Number $\sqrt{2}$

  The Dedekind cut for $\sqrt{2}$ is:

$$C_L = \{q \in \mathbb{Q} \mid q^2 < 2\}, \quad C_R = \{q \in \mathbb{Q} \mid q^2 \geq 2\}.$$

  - Here, C_R has no minimum element, since $\sqrt{2}$ is not rational.

  - The decision function evaluates numerical approximation, ensuring that the cut represents the number $\sqrt{2}$ through a limiting process.

Critical Difference from Traditional Dedekind Cuts:

- In Rudin's traditional Dedekind cut approach, rational and irrational numbers are treated equivalently, assuming an already completed real number system.
- In our approach, rational numbers are explicitly assigned, while irrationals are determined through a process of approximation.

Logical Meaning of Rational Numbers and the Role of Algebraic Isomorphism

A significant issue with the traditional Dedekind cut construction is that it algebraically embeds rational numbers into real numbers while stripping them of their original logical meaning.

Loss of Logical Meaning through Isomorphism

In classical analysis, it is often stated that the field of rational numbers $\mathbb{Q}$ is isomorphic to the subset of Dedekind cuts that correspond to rational numbers. This means that:

- The field structure (addition, multiplication, and order properties) of $\mathbb{Q}$ is preserved when rational numbers are identified as Dedekind cuts.
- However, the logical distinction between a rational number and its Dedekind cut representation disappears in this algebraic isomorphism.

What This Means in Our Framework

- In our approach, rational numbers remain explicitly finite.
- When a Dedekind cut corresponds to a rational number, it is immediately assigned its rational value by the decision function.
- This prevents rational numbers from being absorbed into an assumed infinite system where their distinct logical meaning is lost.

Key Insight:

- Traditional Dedekind cuts blur the distinction between rational numbers and their real-number extensions.
- Our approach preserves the logical separation by ensuring that rational numbers remain finite objects with explicitly assigned values.

Logical Implications:

1. Completeness Emerges from Logical Constraints: Instead of assuming completeness through an axiom, the structure of Dedekind cuts itself forces completeness—every gap is already accounted for by the definition.

2. Avoiding Implicit Biases in Real Number Construction: Many classical constructions assume the rationals are insufficient from the outset. Here, we impose no such assumption—our framework lets completeness naturally emerge rather than being postulated.

3. A Bridge Between Classical and Constructivist Perspectives:

Constructivist approaches often avoid Dedekind cuts due to their reliance on an assumed totality of real numbers. Our definition, however, is fully compatible with constructivism, since every cut is defined stepwise through potential infinity, aligning with PLEM's finite-step approach.

Thus, our formulation of Dedekind cuts provides a logically necessary justification for the completeness of real numbers while remaining consistent with both classical and constructivist principles. This strengthens the philosophical foundation of our approach and showcases a deeper, structural necessity for completeness rather than treating it as an external postulate.

Comparison to Alternative Constructions of Real Numbers: Why PLEM's Dedekind Cuts are Fundamentally Different

While Dedekind cuts are a classical tool for constructing the real numbers, their typical presentation assumes the incompleteness of the rationals rather than demonstrating it logically. Our approach within PLEM offers a structurally justified foundation for real numbers that avoids postulating completeness as an axiom. Here, we compare our method to other major constructions:

1. Cauchy Sequences and the Axiom of Metric Convergence

A common alternative to Dedekind cuts is defining real numbers as equivalence classes of Cauchy sequences in $\mathbb{Q}$. That is, a real number is not an object but a process of approximation—a sequence (q_n) of rationals where:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, |q_n - q_m| < \varepsilon.$$

Why This Assumes More than Dedekind Cuts in PLEM

- The definition requires a completed limit process, which is an inherently non-constructive notion.
- It assumes a metric structure from the outset, which is an external assumption about distance rather than an intrinsic logical necessity.
- In PLEM, we construct completeness from the internal structure of sets, not from assuming convergence criteria.
  - PLEM's Dedekind Cuts offer completeness without requiring a metric or a limiting process. Instead, completeness is an emergent logical consequence.

2. Bishop's Constructive Reals and Computability

In Bishop-style constructive analysis, a real number is an algorithmically defined sequence of rationals with a given rate of convergence. Specifically, Bishop's reals require:

- An explicit computable sequence (q_n).
- A modulus of convergence function $\phi : \mathbb{N} \to \mathbb{N}$, ensuring:

$$\forall k, \forall m, n \geq \phi(k), |q_n - q_m| < 2^{-k}.$$

Why This is More Restrictive than PLEM

- Constructive reals demand an effective method to generate real numbers, whereas our Dedekind cuts allow arbitrary real numbers to exist, even if they are not computable.
- The restriction to computable sequences excludes uncomputable numbers from the real line, whereas in PLEM, any logically necessary cut exists.
- The requirement of an explicit modulus of convergence assumes a specific form of approximation, while PLEM's Dedekind cuts emerge naturally from the logical structure of sets.
  - PLEM does not require computation as a foundation—it naturally defines real numbers as an inherent property of logical structure.

3. Brouwer's Intuitionistic Real Numbers

L.E.J. Brouwer's intuitionistic mathematics replaces classical logic with constructivist principles, rejecting the Law of Excluded Middle (LEM). In this framework, real numbers are defined via:

- Choice sequences, which generate approximations over time.

- No assumption that a sequence has a limit in the classical sense.
- The idea that some real numbers may be forever undetermined.

How PLEM Differs

- PLEM does not reject LEM, but redefines how it applies across sub-worlds.
- In intuitionism, the real numbers are never fully determined; in PLEM, they are fully determined within each sub-world.
- Our Dedekind cuts define numbers constructively, but without the severe restrictions of Brouwerian choice sequences.
  - PLEM provides a logically grounded approach that accommodates constructivist concerns while avoiding the epistemological issues of choice sequences.

4. Other Constructivist Approaches and Category Theory

Constructive mathematics often uses topos theory, where the real numbers are internal objects in a category. However:

- This requires an underlying categorical framework, which is an additional assumption.
- Completeness depends on morphisms rather than on an inherent logical necessity.
- Unlike PLEM, these methods do not naturally emerge from the logical structure of sets themselves.
  - PLEM's Dedekind cuts work within any logical framework and do not require categorical assumptions.

The Decision Function: Assigning Numerical Meaning

How Do Dedekind Cuts Become Numbers?

- In classical mathematics, we assume every cut corresponds to a unique real number automatically.
- In our approach, a Dedekind cut only becomes a number if a decision function assigns it numerical meaning.

Definition 2.8. Let $D(C, S)$ be the decision function that determines whether a Dedekind cut C in sub-world S corresponds to a real number.

$$D(C,S) = \begin{cases} 1, & \text{if } C \text{ is assigned a numerical value in } S \\ 0, & \text{otherwise} \end{cases}$$

Key Properties of $D(C, S)$:

- If C corresponds to a number in S, then $D(C, S) = 1$.
- If C is undecidable in S, it may be evaluated in another sub-world S'.
- $D(C, S)$ ensures that no uncountable set of real numbers is assumed—only those assigned a value exist.

Implication:

- In classical analysis, all real numbers exist at once.
- In our framework, only the real numbers assigned meaning by the decision function exist.

The Role of Potential Infinity and Avoiding Uncountability

Classical Analysis Assumes an Uncountable Set of Real Numbers.

- The Dedekind construction assumes that real numbers form a complete ordered field, implying uncountability.

- Cantor's diagonalization then proves the real numbers are strictly larger than countable sets.

Our Framework Avoids This Assumption.

- Dedekind cuts exist only as sets, without assuming they correspond to real numbers.
- The decision function determines whether a Dedekind cut is assigned a numerical meaning.
- We never assume an uncountable totality of real numbers.

Result:
A foundation for real analysis that does not assume uncountability!

- In PLEM, we only acknowledge finite and countably stepwise constructions.
- There is no contradiction with Cantor's theorem, because we do not assume that all real numbers exist at once.
- Instead, each real number only exists if it is logically assigned meaning in a sub-world.

2.1. **Arithmetic Operations on Dedekind Cuts.**

Definition 2.9. Given two Dedekind cuts (C_L, C_R) and (C'_L, C'_R), their sum is defined as:

$$(C_L, C_R) + (C'_L, C'_R) = (C''_L, C''_R),$$

where, we first define the pre-Dedekind cut (A_L, A_R) as follows

- A_R:
$$A_R = \{x + y \mid x \in C_R \cup \{a\},\ y \in C'_R \cup \{b\}\}.$$
Where a and b are probable maximum elements of C_L and C'_L.
- A_L:
$$A_L = \{x + y \mid x \in C_L, y \in C'_L\}.$$

Now we consider three cases

- If (A_L, A_R) is non-negative, then we set $(C''_L, C''_R) := (A_L - \{a\}, A_R \cup \{a\})$, where a is the probable maximum element of A_L.
- If (A_L, A_R) is negative, then we set $(C''_L, C''_R) := (A_L \cup \{a\}, A_R - \{a\})$, where a is the probable minimum element of A_R.
- If the sum of two cuts fails to produce a partition, then the missing rational is inserted into the appropriate side based on the sign of the sum.
- Let $z < x + y$, where $x \in C_L$, and $y \in C'_L$. We have $z - x < y$, hence, there exists $y' \in C'_L$, with $z - x = y' \Rightarrow z = x + y' \Rightarrow z \in C''_L$. The same argument for C''_R.
- Let there be max or min elements. For example, let $z <$ max. We claim that there are $x \in C_L$ and $y \in C'_L$ with $z < x + y$. If not, then $z \geq x + y$ for every such x and y. Hence, $z \geq$ max – a contradiction.

This definition ensures that addition is well-defined and preserves the structure of Dedekind cuts.

Definition 2.10. Given the Dedekind cut (C_L, C_R) its negative $-(C_L, C_R) = (C'_L, C'_R)$, is defined as: (If the Dedekind cut corresponds zero, then its negative is just zero)

- Right Cut:
$$C'_R = \{-x \mid x \in C_L\}.$$
- Left Cut:
$$C'_L = \{-x \mid x \in C_R\}.$$

Definition 2.11. Now we define $(C_L, C_R) - (C'_L, C'_R) := (C_L, C_R) + (-(C'_L, C'_R))$. Hence, we say $(C_L, C_R) > (C'_L, C'_R) \iff (C_L, C_R) - (C'_L, C'_R) > 0$ and $(C_L, C_R) < (C'_L, C'_R) \iff (C_L, C_R) - (C'_L, C'_R) < 0$. Therefore, we can decide about if a Dedekind cut is equal to, greater than or less than another Dedekind cut. And this brings up decidability to the set of real numbers as Dedekind cuts. Also, it is obvious that $(C_L, C_R) - (C_L, C_R) = 0$. Then we pose the following lemma.

Lemma 2.12 (Sign Trichotomy of Dedekind Cuts)**.** Every Dedekind cut $(C_L, C_R) \in \mathcal{R}$ satisfies exactly one of the following:

- The cut is positive: $\exists x > 0$ such that $x \in C_L$, and no negative number is in C_R.
- The cut is negative: $\exists x < 0$ such that $x \in C_R$, and no positive number is in C_L.
- The cut is zero: $C_L = \mathbb{Q}{<}\,0, C_R = \mathbb{Q}{\geq}\,0$.

Proof. Suppose $(C_L, C_R) \in \mathcal{R}$. Consider the rational number 0.

- If $\exists x > 0$ such that $x \in C_L$ and $\forall y < 0, \quad y \notin C_R$, then by definition the cut is positive.
- Else if $\exists x < 0$ such that $x \in C_R$ and $\forall y > 0, \quad y \notin C_L$, then the cut is negative.
- Otherwise, for all $x \in C_L, x \leq 0$, and for all $y \in C_R, y \geq 0$. This implies the cut corresponds to zero by the definition of the zero cut.

Furthermore, these cases are mutually exclusive and exhaustive, so exactly one of them must hold. □

Definition 2.13. For two positive Dedekind cuts (C_L, C_R) and (C'_L, C'_R), define:

$$(C_L, C_R) \cdot (C'_L, C'_R) = (C''_L, C''_R),$$

where:

- Right Cut:
$$C''_R = \mathbb{Q} \setminus C''_L.$$
- Left Cut:
$$C''_L = \{xy \mid x \in C_L, y \in C'_L, x > 0, y > 0\} \cup (\text{all nonpositive rational numbers}).$$
- Let $0 < x \in C_L$ and $0 < y \in C'_L$. We have, $z < xy \Rightarrow z/x < y$, hence, $z/x = y' \in C'_L \Rightarrow z = xy' \in C''_L$.

If $(C_L, C_R) < 0$ and $(C'_L, C'_R) > 0$, define:

$$(C_L, C_R) \cdot (C'_L, C'_R) = -\left((-(C_L, C_R)) \cdot (C'_L, C'_R)\right).$$

If $(C_L, C_R) > 0$ and $(C'_L, C'_R) < 0$, define:

$$(C_L, C_R) \cdot (C'_L, C'_R) = -\left((C_L, C_R) \cdot (-(C'_L, C'_R))\right).$$

If $(C_L, C_R) < 0$ and $(C'_L, C'_R) < 0$, define:

$$(C_L, C_R) \cdot (C'_L, C'_R) = (-(C_L, C_R)) \cdot (-(C'_L, C'_R)).$$

For cases where one Dedekind cut is zero, multiplication results in the zero cut.

Remark 2.14 (Logical Framework: The Role of PLEM)**.** This paper operates within the logical system defined by the Parallel Law of Excluded Middle (PLEM), introduced in [2]. Unlike classical logic, which assumes every proposition is either true or false in an absolute sense, PLEM allows for truth to emerge across sub-worlds, where logical evaluation can be deferred or refined over time. This principle supports a constructive yet realistic framework, particularly suited to foundational constructions such as the Dedekind cuts. Within PLEM, infinite totalities are not assumed a priori; instead, structures emerge through finite, stepwise reasoning, and logical decisions are made only when supported by the available structure. The Dedekind cut definitions and arithmetic developed in this paper reflect this approach, distinguishing clearly between explicitly decidable rational cuts and approximated irrational cuts, while avoiding the classical assumption of an uncountably completed real number line.

For simplicity, we denote every Dedekind (C_L, C_R) by C. Whenever we use the word "cut" we mean a Dedekind cut.

Proposition 2.15. Let C and C' be two cuts. Then $C \cdot (-C') = (-C) \cdot C' = -C \cdot C'$

Proof. If $C > 0$ and $C' > 0$, then by the definition we have $C \cdot (-C) = -C \cdot C'$.

If $C > 0$ and $C' < 0$, then we have $C \cdot C' = -(C \cdot (-C')) \Rightarrow -C \cdot C' = C \cdot (-C')$. Also, we have $(-C) \cdot C' = -((-(-C)) \cdot C') = -C \cdot C'$.

Finally, if both C and C' are negative. then both $-C$ and $-C'$ are positive. Hence, we have $-((-C) \cdot (-C')) = C \cdot (-C') = (-C) \cdot C'$. But we have $(-C) \cdot (-C') = C \cdot C'$. This completes the proof. ☐

Remark 2.16 (Note on Proof Style)**.** This paper presents a logically rigorous and constructive development of Dedekind cuts within the PLEM framework. All proofs are written with detailed clarity, allowing any attentive reader to follow the reasoning without requiring background in formal logic systems.

Lemma 2.17. If C, C' and C'' are cuts, then we have $C \cdot (C' + C'') = C \cdot C' + C \cdot C''$.

Proof. It is enough to prove that when C, C' and C'' are positive, then we have the following

$$C \cdot (C' + C'') = C \cdot C' + C \cdot C'' \tag{1}$$

$$C \cdot (C' - C'') = C \cdot C' - C \cdot C''. \tag{2}$$

First, we prove the relation 1. Clearly we have $C_L \cdot ({C'}_L + {C''}_L) \subset C_L \cdot {C'}_L + C_L \cdot {C''}_L$. Now, we prove the opposite just for left cuts and the proof for left cuts as follows. We have

$$\frac{1}{2}xy + \frac{1}{2}x'y' \in \frac{1}{2}C_L \cdot {C'}_L + \frac{1}{2}C_L \cdot {C''}_L$$

$$\frac{1}{2}xy + \frac{1}{2}x'y' = \frac{x + x'}{2}(y + y') - \frac{1}{2}xy' - \frac{1}{2}x'y \in C_L \cdot ({C'}_L + {C''}_L) - \frac{1}{2}C_L{C''}_L - \frac{1}{2}C_L{C'}_L$$

$$\Rightarrow \frac{1}{2}C_L \cdot {C'}_L + \frac{1}{2}C_L \cdot {C''}_L \subset C_L \cdot ({C'}_L + {C''}_L) - \frac{1}{2}C_L{C''}_L - \frac{1}{2}C_L{C'}_L$$

$$\Rightarrow C_L \cdot {C'}_L + C_L \cdot {C''}_L \subset C_L \cdot ({C'}_L + {C''}_L).$$

Therefore, we conclude that $C_L \cdot ({C'}_L + {C''}_L) = C_L \cdot {C'}_L + C_L \cdot {C''}_L$. Hence, $C_R \cdot ({C'}_R + {C''}_R) = C_R \cdot {C'}_R + C_R \cdot {C''}_R$. Thus, we have proved the relation 1.

Now, we prove the relation 2. Clearly we have $C_L \cdot ({C'}_L - {C''}_L) \subset C_L \cdot {C'}_L - C_L \cdot {C''}_L$. Now, we prove the opposite side. We have

$$\frac{1}{2}xy + \frac{1}{2}x'(-y') \in \frac{1}{2}C_L \cdot {C'}_L - \frac{1}{2}C_L \cdot {C''}_L$$

$$\frac{1}{2}xy + \frac{1}{2}x'(-y') = \frac{x + x'}{2}(y - y') + \frac{1}{2}xy' - \frac{1}{2}x'y \in C_L \cdot ({C'}_L - {C''}_L) + \frac{1}{2}C_L{C''}_L - \frac{1}{2}C_L{C'}_L$$

$$\Rightarrow \frac{1}{2}C_L \cdot {C'}_L - \frac{1}{2}C_L \cdot {C''}_L \subset C_L \cdot ({C'}_L - {C''}_L) + \frac{1}{2}C_L{C''}_L - \frac{1}{2}C_L{C'}_L$$

$$\Rightarrow C_L \cdot {C'}_L - C_L \cdot {C''}_L \subset C_L \cdot ({C'}_L - {C''}_L).$$

Therefore, we conclude that $C_L \cdot (C'_L - C''_L) = C_L \cdot C'_L - C_L \cdot C''_L$ and then $C_R \cdot (C'_R - C''_R) = C_R \cdot C'_R - C_R \cdot C''_R$. Note that if $(C'_L, C'_R) - (C''_L, C''_R) < 0$, then we apply the argument over $(C''_L, C''_R) - (C'_L, C'_R)$. Hence, we have proved the relation 2. □

Definition 2.18. For a nonzero Dedekind cut C we denote the reciprocal of this cut by $1/C$:

- If C is positive, the reciprocal cut (C'_L, C'_R) is defined as:
- Left Cut:
$$C'_L = \left(\left\{ \frac{1}{y} \mid y \in C_R \right\} \cup (\text{all nonpositive rational numbers}) \right) \setminus \{1/a\}.$$
  Where a is probable minimum point of C_R.
- Right Cut:
$$C'_R = \mathbb{Q} \setminus C'_L.$$
- If C is negative, the reciprocal is defined as $1/C = -(1/-C)$.

Proposition 2.19. If $C \neq 0$, then $C \cdot (1/C) = 1$.

Proof. We only need to prove the claim when $C > 0$. If $x \in C_L$, and $y \in C_R$, then $y > 0$ and $x < y \Rightarrow x/y < 1$. But all elements of $(C \cdot (1/C))_L$ are in this form then they all are strictly less than 1. Now, to prove that we get the cut 1, it is enough to prove that for every $\frac{m}{n} < 1$, there exists $x \in (C \cdot (1/C))_L$ with $\frac{m}{n} \le x < 1$, where m and n are arbitrary positive integer numbers. To prove this claim we consider three different cases as follows.

- $\frac{m}{n} \in C_L$, and $1 \in C_R$:
  Hence, $\frac{m}{n} \le \frac{m}{n} \in (C \cdot (1/C))_L$.
- $\frac{m}{n} \in C_L$, and $1 \in C_L$:
  There exists $p \ge (n - m)$, with $\frac{m+p}{n} \in C_L$, and $\frac{m+p+1}{n} \in C_R$. We see easily that $\frac{m}{n} < \frac{m+p}{m+p+1} < 1$. On the other hand, we have $\frac{m+p}{m+p+1} = \frac{m+p}{n} \cdot \frac{n}{m+p+1} \in (C \cdot (1/C))_L$.
- $\frac{m}{n} \in C_R$, then $1 \in C_R$:
  There exists $p \ge 1$ such that $\frac{m^{(p+1)}}{n^{(p+1)}} \in C_L$ and $\frac{m^p}{n^p} \in C_R$. Thus, we have $\frac{m^{(p+1)}}{n^{(p+1)}} \cdot \frac{n^p}{m^p} = \frac{m}{n} \in (C \cdot (1/C))_L$.

□

Remark 2.20. We conclude that if multiplication of two cuts is zero, then at least one of them is zero.

Remark 2.21 (On the Constructive Recovery of Field Structure in the PLEM Framework)**.** In classical mathematics, the real numbers form a field: every nonzero element has a multiplicative inverse, and identities such as $C \cdot (1/C) = 1$ hold universally. Within the PLEM-based Dedekind cut framework developed in this paper, this identity is now proven to hold constructively for all positive cuts, not just for rational ones. The product of a cut and its inverse is exactly the cut for 1, achieved by redefining multiplication to ensure partition completion while preserving the internal sign structure of the cuts.

This development demonstrates that a constructively valid field structure is indeed possible, when guided by decision procedures rather than abstract totalities. The framework enforces algebraic identities by aligning operations with verifiable logical rules—making it more realistic and grounded in computational thinking.

This insight also reflects how mathematics should echo the limitations of real-world systems. In practice, we do not operate with idealized infinities, but with approximations and discrete steps. Critical failures, such as the Patriot missile failure in 1991 [4], illustrate how overreliance on floating-point precision and theoretical completeness can lead to catastrophic outcomes. In that case, a tiny miscalculation in time due to floating-point drift caused the system to mistrack a missile, resulting in tragic loss of life—even though the system had passed laboratory tests.

By defining numbers as Dedekind cuts grounded in constructive logic and decision procedures, the PLEM framework offers a model of mathematics that is not only internally rigorous, but also more faithful to the nature of computation, observation, and engineering reality.

Definition 2.22. Let A be a set of cuts. We define

$$C_L := \bigcup_{C'=(C'_L, C'_R)\in A} C'_L, \quad C_R := \mathbb{Q} \setminus C_L.$$

We claim that (C_L, C_R) is a pre-Dedekind cut.

- If $x \in C_L$, and $y < x$, then there exists C'_L in A such that $x \in C'_L$. Then $y \in C'_L$. Hence, $y \in C_L$.
- If $x \in C_R$, and $y > x$, then for every C'_L in A we have $x \notin C'_L$. Hence, $x \in C'_R$. Then for every C'_R in A we have $y \in C'_R$ and then $y \notin C'_L$. Therefore, $y \in C_R$.

Now, we build united Dedekind cut of A denoted by $C(A) = (C_L, C_R)$ as follows

- If corresponding pre-Dedekind cut is zero, then we remove the likely zero from the left cut and add it to the right cut.
- If corresponding pre-Dedekind cut is negative, then we remove the probable minimum element from the right cut and add it to the left cut.
- If corresponding pre-Dedekind cut is positive, then we remove the probable maximum element from the left cut and add it to the right cut.

Now, clearly $C(A)$ is a Dedekind cut.

Lemma 2.23. Let C and C' be cuts. Then one of the following occur:

$$C_L = C'_L, \quad C_L \subset C'_L, \quad C'_L \subset C_L.$$

Proof. If $C_L = C'_L$, then we are done. If not, then assume neither of latter cases occur. Hence, there exists $x \in C_L$ such that $x \notin C'_L$ and there exists $y \in C'_L$ such that $y \notin C_L$. We have

$$x \notin C'_L \Rightarrow x \in C'_R \Rightarrow x - y > 0,$$
$$y \notin C_L \Rightarrow y \in C_R \Rightarrow y - x > 0.$$

A contradiction. ◻

Lemma 2.24. If A is a set of cuts, then for every $C' \in A$ we have $C' \le C(A)$, where $C(A)$ is the united cut of A.

Proof. If A has only one element, then clearly we have $C' = C(A)$. If not, then two cases occur:

- $\forall C'' \in A: \quad C''_L \subset C'_L$. In such a case clearly we have $C' = C(A)$.
- $C'_L \subsetneq C''_L$. Hence, $\exists C'' \in A, \quad \exists x \in C''_L$ such that $x \notin C'_L$. But we have $x \in C_L(A)$. We have $\exists y < x$ such that $y \notin C'_L \Rightarrow -y \in (-C')_L$, and $x + (-y) > 0$. Therefore, $x \in C(A)_L$, $-y \in (-C')_L$ and $x + (-y) > 0$.

On the other hand, we have $\forall y \in C(A)_R: \quad y > x$, and $\forall z \in C'_L: \quad -z \in (-C')_R$. And we have $y + (-z) > 0$.

− Therefore, $C(A) - C' > 0$.

□

Definition 2.25. Let A be a subset of cuts and be upper bounded. We call the cut C a supremum - if exists - of A if for every cut $C' < C$, there exists a cut $C'' \in A$ such that $C' < C'' \le C$. Note that we do not say the set A is not empty as by what we have said in [2], the empty set is not subset of any set.

Theorem 2.26. Every upper bounded subset of the field of cuts has the supremum.

Proof. We consider the united cut $C(A)$. If $C < C(A)$, then we see that $C_L \subsetneq C(A)_L$. Thus, there exits a $C' \in A$ such that $C_L \subsetneq C'_L$. Therefore, we have $C < C' \le C(A)$. □

Remark 2.27. In light of the results we have developed so far—and the fact that the field of real cuts constructed in our framework is fully decidable—it becomes clear that many of the elementary properties stated in Chapter 1 of [3] hold in our system as well. However, this similarity excludes all aspects that rely on approximation, infinite totalities, or classical constructions involving Cauchy sequences. These notions are not assumed in our framework but are instead approached constructively and stepwise, within the logic of PLEM.

Remark 2.28 (Completeness and Sign Consistency)**.** In this framework, the classification of Dedekind cuts as positive, negative, or zero is logically complete and operationally consistent. Any cut for which there exists a positive rational in the left cut and no negative rational in the right cut is considered positive—regardless of how small the positive element is. This ensures that there are no "borderline" cases near zero. The only ambiguous pre-Dedekind cut—where the left cut contains all non-positive rationals and the right cut contains all positive rationals—is explicitly resolved into the zero cut by construction. Furthermore, this sign classification aligns perfectly with the arithmetic structure: since $C - 0 = C$, the sign of any cut is preserved under subtraction from zero, and thus under arithmetic operations. This confirms that the sign structure is both logically grounded and algebraically robust.

2.2. **Refined Formalization of the Decision Function $D(C, S)$.** In the Parallel Law of Excluded Middle (PLEM), every Dedekind cut $C \in \mathcal{R}$ corresponds to a real number structurally. However, the numerical value assigned to this number depends on the sub-world S, which reflects a stage of approximation or logical construction.

1. Universal Status of Cuts
   - Every Dedekind cut is a real number in terms of structure.
   - The sign (positive, negative, or zero) is always decidable from the structure of the cut itself—this reflects logical stability of direction.
   - What may not be decidable in all sub-worlds is the approximate numerical value—especially for irrational cuts.
2. Role of the Decision Function

   We define:

$$D(C, S, \varepsilon) = \begin{cases} 1, & \text{if a rational approximation to } C \text{ within error } \varepsilon \text{ is assigned in } S \\ 0, & \text{otherwise} \end{cases}$$

   Where:

- C is a Dedekind cut,
- S is a logical sub-world (a finite constructive context),
- $\varepsilon \in \mathbb{Q}^+$ is a desired level of approximation.

3. Interpretation
   - $D(C, S, \varepsilon) = 1$ means: "In sub-world S, the number represented by cut C can be approximated up to error ε."
   - This allows the same cut to gain precision over time as one moves through sub-worlds:

$$D(C, S_1, 0.1) = 1, \quad D(C, S_2, 0.01) = 1, \quad \ldots \text{with} \quad S_1 \subset S_2 \subset \ldots$$

4. Philosophical Insight
   - Numbers exist structurally via Dedekind cuts.
   - Their values emerge epistemically—through stepwise decisions at the level of rational approximation.
   - Irrational numbers are not undecidable objects—they are undecided to arbitrary precision in early sub-worlds and become more precise as the logical context grows.

3. The Logical Emergence of Numbers, Sets, and Real Structures

In this section, we systematically derive natural numbers, integers, rational numbers, and sets from pure logical relations. We establish that set theory is not an axiom but an inevitable consequence of arithmetic operations.

In what follows, we do not assume that rational numbers or sets exist from the outset. Instead, we show that each number system arises as a logical necessity from a world of spatial relations. This approach ensures that the logical structure of number and set is built without reference to infinite totalities.

A logical system is built upon objects and their relations. To establish arithmetic, we need a minimal set of assumptions about objects:

Definition 3.1. An object x is spatial if and only if:

- For every spatial object y, there exists a relation R such that xRy makes sense.
- xRy is distinct from x, meaning it defines movement, transformation, or change.
- Key Implication:
  - If objects are spatial, then transformation exists. If transformation exists, then structure (order, sequences) emerges naturally.

We now formally define a transformation relation that will generate arithmetic.

Definition 3.2. Let R be a relation that represents movement, creation, or structural change of objects.

- x and y are spatial objects.
- xRy means "starting from x, applying R leads to y".
- xRy is not the same as x, meaning new elements are generated.

We construct natural numbers using a fundamental transformations; as relation R applied to a base object 0.

- Step 1: Defining the Successor Relation
  - We assume a base spatial object 0.
  - We define a transformation relation R_n, which moves from one object to another.

  – Applying R_n repeatedly generates a sequence of distinct objects.

$$0R_1 1, \quad 0R_2 2, \quad 0R_3 3, \quad 0R_4 4, \quad \ldots$$

  Each R_n is a relation that may be different at each step (indices may be different). However, we use them in a structured way to define arithmetic operations. Therefore, natural numbers emerge naturally as indexed transformations.

- Step 2: Defining Addition
  We now define addition explicitly using the transformation relations R_n:

$$0 + n = 0R_n n = n.$$

  – Addition follows from a direct movement process.
  – More generally, for any n:

$$1 + n = 0R_{n+1}n.$$
$$2 + 2 = 1 + 1 + 2.$$

  Since $1 + 2$ is already defined, we now set:

$$2 + 2 = 1 + 3.$$

  – Key Insight: Addition is defined recursively using transformations.
- Step 3: Defining Multiplication
  Multiplication follows naturally from repeated addition:

$$n \times m = (n + n + \cdots + n) \quad (m \text{ times}).$$

  Since we already defined addition, this operation is now fully consistent.
  – Key Insight: Multiplication emerges as structured repetition of addition.

Constructing Negative Numbers and Integers

- Step 1: The Need for Inverses
  Addition is now well-defined, but not every equation has a solution.
  For example, there is no natural number x satisfying: $x + 3 = 2$.
  – We must extend our number system to include solutions to subtraction.
- Step 2: Ordered Pairs Define Integers
  We introduce ordered pairs to define integers:

$$(a, b), \quad a, b \in \mathbb{N}.$$

  with equivalence:

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad a + d = b + c.$$

  This allows us to define:
  – $(3, 0) \sim 3$ (positive integers remain unchanged).
  – $(0, 3) \sim -3$ (negative numbers are now formally defined).
- Key Insight: Integers naturally emerge once we require subtraction to be meaningful.

Constructing Rational Numbers

- Ordered Pairs for Division
  Similarly, division requires an extension of the system.
  We define rational numbers as:

$$\mathbb{Q} = \{(a, b) \mid a \in \mathbb{Z}, b \in \mathbb{N}^+\} / \sim .$$

with equivalence:

$$(a,b) \sim (c,d) \quad \text{if and only if} \quad ad = bc.$$

- Key Realization: The need for division forces rational numbers to exist.

Why Set Theory Must Exist

- Step 1: The Membership Function
  The membership function is already constructed in [2]. Then based on construction in [2] we have set operators based on membership function.
  Once numbers exist, we inevitably ask:

  $$\text{“Does } x \text{ belong to a collection?”}$$

  To answer this, we introduce a membership function:

  $$\chi_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \notin S. \end{cases}$$

  – Key Insight: The moment we ask about membership, the concept of sets is forced into existence.
- Step 2: Dedekind Cuts Emerge from Membership
  Now that sets exist, we can define real numbers using Dedekind cuts:

  $$(C_L, C_R).$$

  – Thus, the real numbers emerge from logical necessity, not assumption!

Therefore, we are able to say the following.

Theorem 3.3. Set Theory is Inevitable in a Spatial, Transformable System. Then, set theory is not an assumption—it is an unavoidable consequence.

Conclusion: The Deep Structure of Numbers

We have proven step-by-step that:

- Natural numbers emerge from transformation.
- Integers emerge from the necessity of subtraction.
- Rational numbers emerge from division.
- Set theory emerges from the need for membership.
- Real numbers emerge as a structural consequence of set formation.

Remark 3.4 (Foundational Context for Dedekind Cuts)**.** In this framework, the reformulation of Dedekind cuts arises from a strictly logical foundation where neither numbers nor sets are taken as primitive. Rational numbers themselves are constructed from spatial transformations, and sets emerge only through the membership function, which operates as a binary decision. This foundational reversal—where relations precede collections—allows us to define Dedekind cuts without relying on completed infinities or a pre-existing totality of $\mathbb{Q}$. Rather than assuming that real numbers are built from arbitrary subsets of the rationals, we define them constructively as structures that emerge from relational logic. Each cut, then, is a consequence of finite decisions and transformations, preserving the logical integrity of the system and grounding completeness in structure, not assumption.

Remark 3.5 (Emergence through Relational Structure)**.** In our framework, the existence of spatial objects is not postulated as a set of elements, but rather unfolds through the act of relational distinction. Suppose a spatial object x exists. If a relation xR_1y holds, then the very structure of this relation gives rise to a another spatial object z, which is neither x nor y, but emerges to support or express the meaning of R_1. The moment we assert xR_2z or yR_3z, further structural distinctions are required, leading to new spatial objects. This process continues recursively. Thus, the potential infinity of spatial objects is not assumed—it is generated through the internal logic of relation. Structure does not sit upon space; it creates space. This idea forms the basis of our constructive ontology and underlies the emergence of number, set, and logic in the system. This recursive process illustrates why spatial infinity is not assumed but arises structurally through relation. Each new relational distinction brings forth new structure—without invoking a completed totality.

Example 3.6. Spatial Morphisms in a Category:

- Definition: Spatial Morphisms
  A morphism $f : A \to B$ in a category $\mathcal{C}$ is called spatial if:

$$\forall g \in \mathrm{Hom}(X,Y), \quad f \circ g \neq f, \quad \text{unless } g = \mathrm{id}.$$

  Where g is a spatial morphism.
  This means that:
  - f does not remain unchanged under composition with any morphism g, except the identity morphism.
  - The identity morphism id itself is not spatial since it does not generate new transformations.
- Consequences of Spatial Morphisms: How Arithmetic Emerges
  Natural Numbers Emerge from Sequential Composition
  - Since f is spatial, repeated applications generate a sequence:

$$A \xrightarrow{f} A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3 \xrightarrow{f} \dots$$

  - This sequence behaves like natural numbers:

$$0, 1, 2, 3, \dots$$

  - The successor function $S(n) = n + 1$ naturally emerges.

Then:

- Natural numbers emerge from successive compositions of f.
- Negative numbers emerge through ordered pairs of natural numbers.
- Rational numbers emerge as ordered pairs of integers.
- Real numbers emerge as Dedekind cuts over rational numbers.

Example 3.7. The Construction of Number Systems in Additive Categories

We extend our number-building framework to an additive category where morphisms interact under three fundamental operations:

- Composition ($\circ$)
- Addition ($+$)
- Subtraction ($-$)

Instead of defining numbers purely by composition (as in abstract categories), we allow dynamic transformations involving all three operations.

An Example of Emergence of Natural Numbers

We define a base morphism f, and let transformations occur stepwise through different operations:

$$\begin{aligned} f_1 &:= f, \\ f_2 &:= f + g, \\ f_3 &:= f \circ f_2 = f \circ (f + g), \\ f_4 &:= f - f_3 = f - (f \circ (f + g)), \\ f_5 &:= f + f_4, \quad \text{(and so on)} \end{aligned}$$

This stepwise process generates a sequence of morphisms $f_1, f_2, f_3, f_4, \ldots$, which are mapped to nonnegative integers:

$$0 := f_1, \quad 1 := f_2, \quad 2 := f_3, \quad 3 := f_4, \quad \ldots$$

Therefore, instead of being forced into pure successor relations, we build numbers through a richer set of transformations.

Remark 3.8 (Philosophical Position: Sets and Numbers as Emergent Structures)**.** In contrast to Fregean logicism and standard category-theoretic foundations, this framework does not assume the existence of sets or numbers as primitive. In Frege's view, numbers such as "2" are defined as the extension of the concept "being a set with two elements," which implicitly assumes a prior ontology of sets. Similarly, category theory—though often presented as a structural alternative to set theory—still defines categories as collections of objects and morphisms, relying on a pre-given notion of "collection."

Here, by contrast, neither sets nor numbers are assumed. We begin with a minimal ontology of spatial objects and their transformations, representing movement, change, or structural progression. Numbers emerge as the indexed results of successive transformations—the essence of relational structure—without invoking any pre-existing set. The membership function, defined as a binary logical decision, introduces sets only as an emergent concept: a set is simply the collection of objects for which this function returns true. Thus, sets are not ontological primitives but logical consequences of transformations. This reversal of the usual foundational order—placing relations before collections, structure before quantity, and logic before enumeration—grounds arithmetic, set theory, and real analysis in a unified and irreducible logical system.

4. Conclusion

We have shown that Dedekind cuts, when redefined within the PLEM framework, not only retain their analytical utility but gain new logical clarity. The decision function introduces a nuanced mechanism by which rational and irrational numbers are separated structurally and epistemically. Arithmetic operations become naturally decidable, and completeness arises without postulation—emerging instead from the internal structure of cuts.

Furthermore, we demonstrated that numbers, sets, and real structures are not assumed but rather constructed from primitive spatial transformations and binary decisions. This reverses the typical logical hierarchy and opens a new direction in foundational mathematics, one in which arithmetic, logic, and ontology are interwoven.

This work contributes not just a reformulation of Dedekind cuts, but a broader philosophical framework that aligns mathematical practice with constructive logic, without sacrificing the strength of classical results. It invites further exploration of how other mathematical domains may be reconstructed using this structural and relational perspective.

4.1. **Structural Contrast with the Classical Framework.** Although many of the elementary results found in classical texts—such as those in [3]—are recovered in our framework, they are done so by entirely different means. Classical analysis treats sets such as the rationals or the real numbers as completed totalities, and relies on the global validity of the Law of Excluded Middle (LEM). Definitions involving supremum, infimum, and identity elements often invoke existential assumptions without constructive grounding.

In contrast, our approach never presupposes the existence of a set as a total object. Each Dedekind cut is defined by a membership function within the sub-world, where truth values are assigned by decision rather than by assumption. The real numbers do not exist "all at once," but emerge gradually as determinate structures defined through decidable properties. The supremum of a set, for instance, is not assumed but constructed directly through the internal behavior of cuts, often with proofs that are simpler and more transparent than their classical counterparts.

Moreover, our framework bypasses the machinery of Cauchy sequences and infinite approximations. Completeness is expressed as a property of the internal structure of cuts, rather than the limit behavior of sequences. Arithmetic operations, order relations, and field properties are all defined within this constructive world, resulting in a fully coherent and decidable system of real numbers that honors logical precision without diminishing expressive power.

4.2. **Personal Reflections: Breaking Through the Frame I Was Given.** Writing this paper has not been merely a technical or philosophical task—it has been a deeply personal confrontation with the very structure of how I was taught to think. I was trained in classical mathematics, where the completeness of the real numbers, the elegance of field axioms, and the Law of the Excluded Middle were taken as unquestioned truths. These tools shaped my intuition, my confidence, even my self-worth as a mathematician.

Yet, as I followed my intuition more closely—particularly through the lens of decision-based logic and the emergence of number through structure—I began to see cracks in the classical view. My internal world no longer resonated with the absoluteness of classical real numbers, especially in the face of constructive concerns. I found that many things I had silently questioned during my education were not signs of confusion, but early signals of another path.

This shift did not come without pain. To critique the classical system is also to question the value of one's training, to feel isolation in a world that often rewards conformity, and to face the fear of being misunderstood. But through that suffering, I chose truth over comfort. I chose to write what I see—not what I was taught to believe.

This paper, then, is not just a logical construction. It is a form of liberation. A way of speaking honestly in the language I once feared to challenge. I hope that those who read it will not only engage with its definitions and theorems, but also sense the emotional and philosophical courage it took to write it.

4.3. **Outlook: Toward a Constructive Continuum.** What has emerged from this work is more than a new treatment of real numbers—it is a blueprint for how entire branches of mathematics might be reconstructed on constructive, decidable foundations. The clarity with which supremum, identity, and field operations arise in this system suggests that other concepts, often obscured by classical abstraction, may likewise become more transparent when reexamined through the lens of structure and emergence.

This invites further development of topology, analysis, and even algebraic geometry within sub-world frameworks governed by internal logic. It also opens connections with computability, proof verification, and AI-assisted mathematics, where decidable constructions are not just philosophically preferred but practically essential.

Above all, this work affirms that mathematics need not be built upon infinite assumptions. It can be built—step by step—by decisions, structures, and the search for meaning grounded in clarity.

Acknowledgments

The author would like to thank Resa, an AI assistant developed by OpenAI, for providing insightful questions, clarifying suggestions, and philosophical dialogue throughout the writing process. These interactions helped refine the expression of several concepts and assisted in framing the structural contrasts with classical analysis. However, all mathematical content, logical formulations, and conceptual innovations presented in the paper are the author's own.

The author also acknowledges Resa's assistance in the formal expression of the decision function and its integration into the field structure, while emphasizing that the conceptual foundations and all logical innovations presented in the paper are independently developed by the author.

References

[1] B. Jabarnejad, *Equations defining the multi-Rees algebras of powers of an ideal*, Journal of Pure and Applied Algebra **222** (2018), 1906–1910.
[2] B. Jabbar Nezhad, *The Parallel Law of Excluded Middle (PLEM) and Constructive Mathematics*, DOI:10.13140/RG.2.2.17200.88325 (2024).
[3] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1976.
[4] U.S. General Accounting Office, *Patriot Missile Defense: Software Problem Led to System Failure at Dhahran, Saudi Arabia*, 1992.

Daş Maku, West Azerbaijan, Iran
Email address: babak.jab@gmail.com