A CONSTRUCTIVE FOUNDATION FOR TOPOLOGY: FROM PRE-STRUCTURES TO LIMIT-COMPLEMENT SPACES, WITH A CRITIQUE OF NEIGHBORHOOD SYSTEMS

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Abstract. This paper proposes a new foundation for topology that reconstructs the classical structure without relying on the axiom of arbitrary unions. Beginning with a minimal pre-topological space—where only the empty set, the whole space, and finite unions of open sets are assumed—we define closed sets via limit points or as complements of open sets. Open sets in the full topology are then constructed as complements of these closed sets. This limit-complement approach avoids infinite assumptions while recovering all the classical properties of topology, including closure under arbitrary unions, as a derived property rather than a postulate. The resulting framework aligns exactly with the standard topology in metric spaces but is grounded in a philosophically constructive and predicative structure.

1. Introduction

Topology, as traditionally defined, arises from a collection of subsets called open sets that are closed under arbitrary unions and finite intersections. While powerful, this structure assumes access to infinite unions and unbounded totalities—assumptions that raise deep foundational concerns in constructivist and finitist mathematics.

This paper offers a new foundation for topology, built from minimal principles and grounded in constructive reasoning. We begin with a pre-topological space in which only the empty set, the whole space, and finite unions of open sets are postulated. From this sparse structure, we define closed sets via limit points or as complements of open sets, and we generate the full topology—what we call the limit-complement topology—by taking complements of these closed sets.

This approach reconstructs the essential elements of topology:

- Openness, defined through closure and complementation;
- Arbitrary union, recovered as a derived property rather than an axiom;
- Compactness, redefined constructively through finite coverings and limit behavior.

The resulting framework aligns with classical topology in metric spaces and recovers familiar notions of openness, closedness, and compactness in that context. However, it does so without relying on impredicative reasoning or infinite assumptions. In more abstract topological settings, the definition of compactness diverges from the classical version, offering instead a constructively grounded alternative that emphasizes finite verifiability and limit behavior. This approach provides a philosophically disciplined and verifiably constructive account of what it means for a set to be open, closed, or compact.

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2. Pre-Topological Space

Let X be a nonempty set. A pre-topological space on X consists of a collection $\mathcal{O}$, where $\mathcal{O} = \{\varnothing\} \cup \mathcal{S}$ with $\mathcal{S} \subseteq \mathcal{P}(X)$ (the power set of X) satisfying:
1. $\varnothing \in \mathcal{O}$ and $X \in \mathcal{O}$,
2. If $U, V \in \mathcal{O}$, then $U \cup V \in \mathcal{O}$.

(Finite unions of open sets are open.)
This minimal condition allows us to begin defining essential topological ideas while remaining free of infinite operations.

3. Limit Points and Closed Sets in the Pre-Structure

Let $A \subseteq X$. A point $x \in X$ is called a limit point of A if every open set $U \in \mathcal{O}$ containing x also contains at least one point of A distinct from x. Using this, we define closed sets in the pre-topological space in two ways:

- (a) A set is closed if it contains all its limit points.
- (b) A set is closed if it is the complement of an open set in $\mathcal{O}$.

These two criteria are accepted jointly. Some sets may satisfy one condition and not the other, but both routes are valid for building the collection of closed subsets. This dual perspective is essential for the next stage of construction.

4. Constructing the Limit-Complement Topology

Given a pre-topological space $(X, \mathcal{O})$, we construct a topological space not by assuming closure under arbitrary unions, but by generating open sets constructively as complements of closed sets. These closed sets are defined either as those that contain all their limit points or as complements of basic open sets in $\mathcal{O}$.

Definition 4.1 (Limit-Complement Topology)**.** Let $\mathcal{C}$ be the collection of subsets of X that are considered closed in the pre-topological sense. We define the open sets of the topology as:

$$\mathcal{T} := \{U \subseteq X \mid X \setminus U \in \mathcal{C}\}.$$

This yields a topological space $(X, \mathcal{T})$, where openness emerges naturally from limit-based closure, without assuming any infinite operations.

Corollary 4.2 (Arbitrary unions of open sets are open)**.** Let $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$. Then:

$$X \setminus \left(\bigcup_{\alpha \in A} U_\alpha \right) = \bigcap_{\alpha \in A} (X \setminus U_\alpha)$$

Since each $X \setminus U_\alpha \in \mathcal{C}$ (i.e., is closed), and the intersection of closed sets contains all its limit points, the result is again closed. Thus, the complement of this intersection—i.e., the union—is open.

Conclusion: The topology $\mathcal{T}$ is closed under arbitrary unions as a derived property, not as an axiom. This completes the reconstruction of classical topological behavior through purely constructive means.

5. Compactness in Limit-Complement Topology

We now define compactness using only finite open covers and limit points.

Definition 5.1 (Compact Set)**.** A set $K \subseteq X$ is compact if:

1. There exists an open set $U \in \mathcal{O}$ such that $K \subseteq U$,
2. For every finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$ such that $K \subseteq \bigcup U_i$, all limit points of K are contained in $\bigcup U_i$.

This condition ensures that compactness is not determined by total coverings but by observable limit behavior and finite verifiability.

Remark 5.2. Although the arbitrary union of open sets is not assumed as part of the foundational structure, we proved that it still holds within the limit-complement topology. This result is not meant to reintroduce an infinite axiom through the back door, but rather to demonstrate that the classical behavior of topologies can still emerge—not by postulate, but by consequence. Our constructive stance avoids the concept of infinite totalities precisely by refusing to define topology through arbitrary unions. Instead, openness is reconstructed through the complement of limit-point-closed sets. Similarly, the definition of compactness is designed to bypass the classical notion of open covers ranging over infinite families. Compactness, in this system, is verified only through finite coverings and limit containment, aligning with constructive philosophy. The fact that many classical theorems reappear in this framework is not a retreat to classical logic, but a demonstration of the power and sufficiency of the constructive approach.

6. Agreement with Metric Spaces

While our framework stands on its own philosophical foundation, it is instructive—and reassuring—to observe that in familiar contexts, it fully recovers the classical topology.

Let (X, d) be a metric space. Define a pre-topology by taking as open sets all open balls with rational radii. These form a countable, finitary structure suitable for constructive reasoning. The open sets of the pre-topology are then finite unions of such balls.

From this base, closed sets are defined via limit points or complements of opens, and the full limit-complement topology is constructed.

The following theorem shows that the resulting topology aligns with the standard constructive understanding of openness in metric spaces.

Theorem 6.1 (Constructive Characterization of Openness)**.** Let $U \subseteq X$. Then $U \in \mathcal{T}$ (i.e., U is open in the limit-complement topology) if and only if for every $x \in U$, there exists a rational number $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Proof. ($\Rightarrow$) Suppose $U \in \mathcal{T}$, so its complement $X \setminus U$ is closed, meaning it contains all its limit points.

Let $x \in U$. Then $x \notin X \setminus U$. Since $X \setminus U$ contains all its limit points, and $x \notin X \setminus U$, it follows that:

x is not a limit point of $X \setminus U$.

Therefore, there exists a rational $\varepsilon > 0$ such that the ball $B_\varepsilon(x) \cap (X \setminus U) = \varnothing$, hence $B_\varepsilon(x) \subseteq U$.

($\Leftarrow$) Conversely, suppose for every $x \in U$, there exists a rational $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Let x be a limit point of $X \setminus U$. Then every rational ball around x intersects $X \setminus U$ in a point other than x.

This implies that $x \notin U$. Note that here we use the concept of membership introduced in [2] and the concept of denial of a proposition in constructive mathematics. Hence, $X \setminus U$ contains all its limit points and is therefore closed. Thus, $U \in \mathcal{T}$. □

Theorem 6.2 (Compactness Equals Closed and Bounded in Metric Spaces)**.** Let (X, d) be a metric space. A subset $K \subseteq X$ is compact under the limit-complement definition if and only if it is closed and bounded in the classical sense.

Sketch. If K is classically compact, it is closed and bounded. Boundedness implies it lies within a ball in the pre-topology, and closedness implies it contains all its limit points.

Conversely, if K is compact under our definition, it is bounded and must be closed; otherwise, there would exist a limit point outside K, violating the containment condition. □

7. A Critique of the Neighborhood System

While the neighborhood system is often presented as a local and intuitive way to define a topology, it is fundamentally non-constructive.

Classical Definition:

For each $x \in X$, a collection $\mathcal{N}(x)$ of subsets of X is called a neighborhood system if it satisfies:

1. $x \in N$ for all $N \in \mathcal{N}(x)$,
2. If $N \in \mathcal{N}(x)$ and $N \subseteq M$, then $M \in \mathcal{N}(x)$,
3. If $N_1, N_2 \in \mathcal{N}(x)$, then $N_1 \cap N_2 \in \mathcal{N}(x)$,
4. For every $N \in \mathcal{N}(x)$, there exists $M \in \mathcal{N}(x)$ such that for all $y \in M$, there exists $N_y \in \mathcal{N}(y)$ with $N_y \subseteq N$.

Constructive Critique:

Despite its localized appearance, this structure involves:

- Ambiguous existence: The existential quantifiers over M and the family $\{N_y\}$ lack constructive meaning unless we can explicitly generate them.
- Uncontrolled infinity: The system presumes reasoning over uncountable collections without any bounded or algorithmic structure.
- No generative mechanism: The neighborhood system doesn't arise from a process—it is simply assumed to exist.

Hence, while classically elegant, the neighborhood system is unsuitable for a constructive foundation of topology.

8. Philosophical Motivation and Constructive Discipline

The key motivation behind this framework is to avoid impredicative and infinite assumptions often present in abstract topology. The standard axioms require arbitrary unions of open sets—a move that assumes the totality of potentially uncountable families of sets. Similarly, classical definitions of compactness rely on open covers drawn from such totalities, or on sequence-based arguments that invoke excluded middle and non-constructive convergence.

In contrast, the limit-complement approach:

- Begins with finite, explicit openness in a pre-structure,
- Defines closure through limit points,
- Recovers arbitrary union and classical behavior as emergent, not assumed,
- Introduces a definition of compactness grounded in finite coverings and limit behavior.

This is not just a technical improvement, but a philosophically precise reconstruction of topology. It recasts the core ideas of openness, closure, and compactness into a constructive language, where every notion is defined in terms of verifiable relations, finite families, and observable limit structure. In doing so, it reframes the foundation of topology in a way that aligns with mathematical discipline, logical clarity, and conceptual honesty.

This framework rests on the Parallel Law of Excluded Middle (PLEM), first introduced by the author in [2].

9. Conclusion

We have introduced a constructively grounded definition of topology, built from a pre-topological structure without assuming arbitrary unions of open sets. This approach redefines openness through the lens of closure, and closure through the concept of limit points. In doing so, we recover all classical topological behavior in metric spaces, while offering a more principled and cautious foundation for general topology.

In addition to the core structure, we proposed a new definition of compactness based entirely on finite open covers and limit point containment. This condition—requiring that finite unions of opens covering a set also include all its limit points—aligns with classical compactness in metric spaces, where compactness is equivalent to being closed and bounded. Unlike Bishop's constructive definition, which requires total boundedness and often relies on sequential approximations, the limit-complement framework offers a logically weaker but fully constructive and operationally simpler approach to compactness. It recovers the essential topological behavior of metric spaces while maintaining a cautious, finite, and verifiable foundation.

This paper offers what may be the first general, fully constructive definition of compactness in topology—recovering classical compactness in metric spaces while avoiding the logical and technical burdens found in both classical and constructivist approaches.

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References

[1] B. Jabarnejad, *Equations defining the multi-Rees algebras of powers of an ideal*, Journal of Pure and Applied Algebra **222** (2018), 1906–1910.

[2] B. Jabbar Nezhad, *The Parallel Law of Excluded Middle (PLEM) and Constructive Mathematics*, DOI:10.13140/RG.2.2.17200.88325 (2024).

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