PARADOX ON THE COUNTABLE AXIOM OF CHOICE

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"DEDICATED TO L.E.J. BROUWER"

Abstract. Bishop's constructive mathematics school rejects the Law of Excluded Middle, but instead vastly makes use of weaker versions of the Choice. In this paper we pioneer an example, which shows that this road is not consistent, as our example provides a paradox. Therefore, rejecting the Law of Excluded Middle, and as an alternative using the Countable Axiom of Choice and the Axiom of Dependent Choice, still does not create a consistent structure. Actually, constructively; the Countable Axiom of Choice is an implication of the Axiom of Dependent Choice.

1. Introduction

This paper is written in the framework of Bishop's constructive mathematics. Throughout the entirety of this paper, by constructive mathematics, we mean Bishop's constructive mathematics.

Both the Law of Excluded Middle and the Axiom of Choice are accepted in classical mathematics but are rejected in constructive mathematics. The Zermelo's Axiom of Choice revolutionized mathematics in the last century. Even Bishop's school of constructive mathematicians could not be completely released from this axiom and broadly made use of weaker versions of it, namely, the Countable Axiom of Choice (CAC) and the Axiom of Dependent Choice (ADC).

In Bishop's constructive mathematics, one rejects the Law of Excluded Middle but practices CAC and ADC extensively. This approach allows constructive mathematicians to build fields of real and complex numbers and to develop analysis on these fields. However, as this paper demonstrates, this reliance on CAC and ADC leads to an inherent paradox within the constructive framework.

A crucial observation is that to construct countable infinity within this framework, one must inevitably rely on either the Law of Excluded Middle or the Axiom of Choice. This dependency contradicts the essence of Brouwer's constructivism, which demands explicit construction rather than assumptions. Therefore, the term "constructive" in Bishop's school might be a misnomer, as it reintroduces the same assumption-based reasoning that constructivism originally intended to avoid. In true constructivism, as envisioned by Brouwer, infinity should be constructible step-by-step, but Bishop's approach indirectly relies on the acceptance of infinite sets as a given, undermining the constructivist philosophy.

Furthermore, this paradoxical pattern is not limited to Bishop's constructive mathematics alone. For example, in fuzzy theory, while the Law of Excluded Middle (LEM) is rejected, the theory still employs the continuity and completeness of the real numbers. However, constructing the real number system and ensuring its completeness fundamentally rely on LEM

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in classical mathematics. By rejecting LEM, fuzzy theory indirectly depends on Bishop's constructive framework, where completeness is achieved using the Countable Axiom of Choice (CAC) instead. This dependency highlights a hidden reliance on classical or semi-classical principles, despite the overt rejection of binary logic. This reveals a hidden inconsistency within fuzzy theory, where logical ambiguities arise from trying to avoid the Law of Excluded Middle while still utilizing classical analytic tools. This observation indicates that the problem of inconsistency might extend beyond constructive mathematics to other frameworks that attempt to balance between classical and constructive principles. At the end, in the fuzzy theory they reject the Law of Excluded Middle then they reject the following

$$A \cap \overline{A} = \varnothing, \ A \cup \overline{A} = X,$$

where A is a set, X is the universal set and the set $\overline{A}$ is the complement of A. But they accept the following

$$\overline{\overline{A}} = A, \ \overline{A \cup B} = \overline{A} \cap \overline{B}, \ \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

This is just ambiguity, as latter identities are implications of first identities.

What we manifest in the present paper is very short and clear by introducing a straightforward example. The example is similar to the one that we have presented in [4], but this time in the framework of Bishop's constructive mathematics; with totally different arguments and different functions in used. The example takes place over the field of complex numbers but at the same time makes use of constructive properties of algebraic numbers. Roughly speaking, we introduce a nonzero differentiable function on a simply connected open set whose zeros are not isolated; certainly we introduce this example under constructive interpretation.

2. A glimpse of foundation

In this section we list some results in Bishop's constructive mathematics. Actually, in entirety of this paper by constructive mathematics we mean Bishop's constructive mathematics.

We say a set is discrete if the equality is decidable.

The Law of Excluded Middle is rejected in constructive mathematics, which asserts that $P \vee \neg P$ holds for any statement P, where $\neg P$ is the denial of P.

Axiom of Choice. If S is a subset of $A \times B$, and for each x in A there exists y in B such that $(x, y) \in S$, then there is a function f from A to B such that $(x, f(x)) \in S$ for each x in A.

Countable Axiom of Choice - CAC. This is the Axiom of Choice with A being the set of positive integers.

Axiom of Dependent Choice - ADC. Let A be a nonempty set and R be a subset of $A \times A$ such that for each a in A there is an element a' in A with $(a, a') \in R$. Then there is a sequence $a_0, a_1, \dots$ of elements of A such that $(a_i, a_{i+1}) \in R$ for each i.

The Axiom of Choice is not accepted in Bishop's constructive mathematics, as it implies the Law of Excluded Middle. CAC and ADC are accepted and widely are used in this school. Actually, the ADC implies the CAC.

Remark 2.1. Algebra is the manipulation of symbols without (necessarily) regard for their meaning. And fields in general are not purely algebraic notion. Although, a discrete field is an algebraic notion, in the sense that we do algebra in it, but a field that is not discrete is not an algebraic notion. Such as fields of real and complex numbers which are not discrete - from viewpoint of constructive mathematics -.

Notation 2.2. We denote the field of complex numbers by $\mathbb{C}$. We also denote the field of algebraic numbers by $\mathbb{C}^{\alpha}$. By the filed of algberaic numbers we mean complex numbers that are algebraic over the field of rational numbers. Finally, we denote the field of real numbers by $\mathbb{R}$. We also denote the field of real algebraic numbers by $\mathbb{R}^{\alpha}$.

The open sphere of radius $r > 0$ about a point x in a metric space X is the subset

$$S(x,r) := \{y \in X; \rho(x,y) < r\}$$

of X.

The complement of a set S in X is the set $X - S := \{x \in X; \forall s \in S : x \neq s\}$. Note that the complement of an open set is a closed set.

Let X be a metric space. We define the closed sphere for $r \geq 0$ to be

$$Sc(x,r) := \{y \in X; \rho(x,y) \leq r\}.$$

Note that the closed sphere in $\mathbb{R}$ and $\mathbb{C}$ is compact.

An open set $U \subset \mathbb{C}$ is connected if any two points of U can be joined by a path in U. We say that the open set U is simply connected if it is connected and every closed path in U is null-homotopic.

We don't go in detail over definitions of located set, compact set and et cetera in Bishop's constructive mathematics, as they are found in [1]. We just mention the following

For each located set $K \subset \mathbb{C}$ and each $r > 0$, we write

$$K_r := \{z \in \mathbb{C}; \rho(z,K) \leq r\}.$$

A totally bounded set $K \subset \mathbb{C}$ is well contained in an open set $U \subset \mathbb{C}$ if $K_r \subset U$ for some $r > 0$. We then write $K \subset\subset U$. Note that if $K \subset \mathbb{C}$ is totally bounded, then K_r is compact for each $r > 0$.

Lemma 2.3. Let $U \subset \mathbb{C}$ be open, and $f : U \to \mathbb{C}$ be uniformly continuous on each closed sphere well contained in U. Then f is continuous on U.

Now, we state the following theorem from [2]

Theorem 2.4. Let f be analytic and not identically zero on the connected open set U. Let K be a compact set well contained in U, and $\epsilon > 0$. Then either $\inf\{|f(z)|; z \in K\} > 0$ or there exist finitely many points $z_1, \dots, z_n$ of U and an analytic function g on U such that

$$f(z) = (z - z_1)(z - z_2)\dots(z - z_n)g(z), \ (z \in U),$$

such that $\inf\{|g(z)|; z \in K\} > 0$, and $\rho(z_n, K) < \epsilon$ for each K.

Note that on an open set analytic condition is weaker than differentiable condition in constructive complex analysis.

We say the function $f : \mathbb{C} \to \mathbb{C}$ is an entire function if it is differentiable on $\mathbb{C}$.

Functions of

$$\begin{aligned} f : \mathbb{C} \to \mathbb{C}, &\quad f(z) = \cos(z), \\ g : \mathbb{C} \to \mathbb{C}, &\quad g(z) = \sin(z), \\ h : \mathbb{C} \to \mathbb{C}, &\quad h(z) = \mathbf{exp}(z), \end{aligned}$$

are entire functions, where for $z \in \mathbb{C}$ one defines

$$\mathbf{exp}(x+\mathbf{i}y) = \exp(x)(\cos(y)+\mathbf{i}\sin(y)), \qquad \text{where, } x,y \in \mathbb{R},$$
$$\cos(z) = \frac{\mathbf{exp}(\mathbf{i}z)+\mathbf{exp}(-\mathbf{i}z)}{2}, \qquad \sin(z) = \frac{\mathbf{exp}(\mathbf{i}z)-\mathbf{exp}(-\mathbf{i}z)}{2\mathbf{i}}.$$

A complex number w is a logarithm of a complex number z if $\mathbf{exp}(w) = z$. Every complex number $z \neq 0$ has at least one logarithm w, and numbers

$$w + 2\pi\mathbf{i}k \ (k \in \mathbb{Z})$$

form the totality of logarithms of z. If U is simply connected open subset $\mathbb{C} - \{0\}$ (this is the metric complement which the same as constructive complement mentioned above), z_0 is any point of U, and a is any logarithm of z_0, then there exists a unique differentiable function g on U, with $g'(z) = z^{-1}$ and $g(z_0) = a$. Then we have $\mathbf{exp}(g(z)) = z$, where $z \in U$. The function g is called a branch of the logarithmic function on U.

If f is a differentiable function from an open subset U of $\mathbb{C}$ into an open subset V of $\mathbb{C}$, and g is differentiable on V, then $g \circ f$ is differentiable on U if $f(K) \subset\subset V$ for each compact set $K \subset\subset U$.

Clearly, if g is an entire function and f is differentiable on the open set U, then mentioned conditions above are automatically satisfied; so that $g \circ f$ is differentiable on U.

Theorem 2.5. Let f be a continuous function on an open set $U \subset \mathbb{C}$, with partial derivatives f_x and f_y on U satisfying the Cauchy-Riemann equations

$$f_y = \mathbf{i}f_x.$$

Then f is differentiable on U, and $f' = f_x$.

3. Some facts in constructive mathematics

In the following we only use the Countable Axiom of Choice and the Axiom of Dependent Choice. And there is no use of the Law of Excluded Middle or any implications of it. As, all proofs are made in the framework of Bishop's constructive mathematics. We constructively prove some new results in which we did not find in the literature although we have used results in the literature of constructive mathematics to prove our results.

Theorem 3.1. Let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial with $a_i \in \mathbb{C}^\alpha$. Then we have $p(x) = a_n(x - x_1)\ldots(x - x_n)$, where $x_i \in \mathbb{C}^\alpha$.

By discussed arguments in [1, Chapter 2, Section 7], we have the following

$$\begin{array}{cc}
(1)\ln(xy) = \ln(x) + \ln(y), & x,y \in \mathbb{R}, x>0, y>0,\\
(2)\ln(x/y) = \ln(x) - \ln(y), & x,y \in \mathbb{R}, x>0, y>0,\\
(3)\exp(x+y) = \exp(x)\exp(y), & x,y \in \mathbb{R}.\\
(4)\exp(x) > 0, & x \in \mathbb{R},\\
(5)\exp(-x) = 1/\exp(x), & x \in \mathbb{R},\\
(6)\exp(x-y) = \exp(x)/\exp(y), & x,y \in \mathbb{R}.\\
(7)\exp(\ln(x)) = x, & x \in \mathbb{R}, x>0.
\end{array}$$

Also, for every $a \in \mathbb{R}$, $a > 0$, they define

$$a^x := \exp(x\ln(a)), \ x \in \mathbb{R}. \tag{1}$$

So that if $0 < a \in \mathbb{R}$, then $\sqrt{a} > 0$.

If $a+b=1$, and $a,b \geq 0$, then by [1, Propsition 2.6], $(a+b)-b = a+(b-b) = a+0 = a = 1-b \geq 0$. Hence by [1, Propsition 2.11], $b \leq 1$. Similarly we have $a \leq 1$. Hence we conclude that $0 \leq a, b \leq 1$.

If $a, b \in \mathbb{R}^\alpha$, and $ab < 0$, then $ab \neq 0$. Since $\mathbb{R}^\alpha$ is a discrete field, we have $a \neq 0$ and $b \neq 0$. Now, since $\mathbb{R}^\alpha$ is discrete we have $a > 0$ or $a < 0$. If $a > 0$, then by [1, Propsition 2.11], $a^{-1} > 0$. Hence, by[1, Propsition 2.11], we have $aa^{-1}b < 0 \Rightarrow b < 0$. If $a < 0$, then by [1, Propsition 2.6, Proposition 2.11], we have $0 = a + (-a) < 0 + (-a) \Rightarrow -a > 0$. Thus $(-a)^{-1} > 0$. On the other hand, we have $ab < 0 \Rightarrow (-a)(-b) < 0 \Rightarrow -b < 0 \Rightarrow b > 0$. Therefore, we proved the following. If $a, b \in \mathbb{R}^\alpha$ and $ab < 0$, then either ($a > 0$ and $b < 0$) or ($a < 0$ and $b > 0$).

Suppose that $a, b \in \mathbb{R}^\alpha$, and $a^2 + b^2 = 1$. Then by what we have said above we have $0 \leq a^2 \leq 1 \Rightarrow a^2 - 1 \leq 0 \Rightarrow (a-1)(a+1) \leq 0$. Then we conclude that $(a-1)(a+1) = 0$ or $(a-1)(a+1) < 0$. If $(a-1)(a+1) = 0$, then $a = 1$ or $a = -1$. If $(a-1)(a+1) < 0$, then by what we have said we have $-1 < a < 1$. Therefore, we have proved that if $a, b \in \mathbb{R}^\alpha$ and $a^2 + b^2 = 1$, then $-1 \leq a, b \leq 1$.

Now if we observe [1, Chapter 2, Section 7, page 58-61], then we see that on these pages authors define functions sine and cosine. They also construct the number π as twice the first positive zero of the cosine function. They also prove identities $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$, $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\cos(x)^2 + \sin(x)^2 = 1$, which lead us to obtain all elementary trigonometric identities related to sine, cosine and the number π. The function $f(x) = \sin x$ is a strictly increasing function from $(-\pi/2, \pi/2)$ onto $(-1, 1)$. Since $\sin 0 = 0$, for every $a \in \mathbb{R}^\alpha$, $-1 < a < 0$, there is a unique $-\pi/2 < \theta_1 < 0$, such that $\sin\theta_1 = a$, and for every $a \in \mathbb{R}^\alpha$, $0 < a < 1$, there is a unique $0 < \theta_1 < \pi/2$, such that $\sin\theta_1 = a$. Moreover, for every $a \in \mathbb{R}^\alpha$, $-1 < a < 0$, there is a unique $-\pi < \theta_1 < -\pi/2$, such that $\sin\theta_1 = a$, and for every $a \in \mathbb{R}^\alpha$, $0 < a < 1$, there is a unique $\pi/2 < \theta_1 < \pi$, such that $\sin\theta_1 = a$. We have a similar argument for the function $f(x) = \cos x$. On the other hand, we have $\cos(-\pi/2) = \cos(\pi/2) = 0$, $\sin 0 = \sin(\pi) = 0$, $\sin(-\pi/2) = -1$, $\sin(\pi/2) = 1$, $\cos(\pi) = -1$, $\cos 0 = 1$. Therefore, for every $a \in \mathbb{R}^\alpha$, $-1 < a < 0$ (resp. $0 < a < 1$), there are exactly two $-\pi < \theta_1, \theta_2 \leq \pi$, such that $\sin\theta_1 = \sin\theta_2 = a$, and there are exactly two $-\pi < \gamma_1, \gamma_2 \leq \pi$, such that $\cos\gamma_1 = \cos\gamma_2 = a$. Now, suppose that $a, b \in \mathbb{R}^\alpha$, and $a^2 + b^2 = 1$. Hence $-1 \leq a \leq 1$. We consider four different cases. Case 1, $a = 0$. We have $\sin(\pi) = \sin 0 = a = 0$. Thus $b = \pm 1$. If $b = 1$, then we take $\theta = 0$, and we have $\sin(\theta) = a = 0$, and $\cos(\theta) = b = 1$. If $b = -1$, then we take $\theta = \pi$, and we have $\sin(\theta) = a = 0$, and $\cos(\theta) = b = -1$. Case 2, $a = 1$. Then $b = 0$. We take $\theta = \pi/2$, and we have $\sin(\theta) = a = 1$, $\cos(\theta) = b = 0$. Case 3, $a = -1$. Then $b = 0$. We take $\theta = -\pi/2$, and we have $\sin(\theta) = a = -1$, $\cos(\theta) = b = 0$. Case 4, $a \neq 0, 1, -1$. Hence $-1 < a < 0$ or $0 < a < 1$. If $-1 < a < 0$, then there are exactly two $-\pi < \theta_1 < -\pi/2, \quad -\pi/2 < \theta_2 < 0$, such that $\sin(\theta_1) = \sin(\theta_2) = a$. But we have the identity $\sin^2(x) + \cos^2(x) = 1 \Rightarrow \cos^2(\theta_1) + a^2 = 1 = \cos^2(\theta_2) + a^2 \Rightarrow b^2 = \cos^2(\theta_1) = \cos^2(\theta_2)$. On the other hand, $-1 < b < 0$ or $0 < b < 1$. If $-1 < b < 0$, then we pick $-\pi < \theta_1 < -\pi/2$, and we know that $-1 < \cos(\theta_1) < 0$. So that $b = \cos(\theta_1)$. If $0 < b < 1$, then we pick $-\pi/2 < \theta_2 < 0$, and we know that $0 < \cos(\theta_2) < 1$. So that $b = \cos(\theta_2)$. On the other hand, $b^2 = A^2$ has only two values for b. Then in either cases θ_1 or θ_2 is unique. If $0 < a < 1$, we have a similar argument.

Therefore, we have the following. For every $c + \mathbf{i}d$, where $c, d \in \mathbb{R}^\alpha$ and $c^2 + d^2 = 1$, there is a unique $\theta \in (-\pi, \pi]$ such that $c + \mathbf{i}d = \cos\theta + \mathbf{i}\sin\theta$.

Now if $c, d \in \mathbb{R}^\alpha$, and they are arbitrary and at least one of them is nonzero, then we have $c + \mathbf{i}d = \sqrt{c^2+d^2}\left(\frac{c}{\sqrt{c^2+d^2}} + \mathbf{i}\frac{d}{\sqrt{c^2+d^2}}\right)$, where $-1 \leq \frac{c}{\sqrt{c^2+d^2}}, \frac{d}{\sqrt{c^2+d^2}} \leq 1$. Therefore, for each $c, d \in \mathbb{R}^\alpha$, we have $c + \mathbf{i}d = r(\cos\theta + \mathbf{i}\sin\theta)$, where $\theta \in (-\pi, \pi]$ is unique, and $r \in \mathbb{R}^\alpha$, $r \geq 0$.

For $c+\mathbf{i}d = r(\cos\theta + \mathbf{i}\sin\theta)$, $r>0$, $-\pi<\theta\le\pi$, we denote θ as the principal argument of the algebraic number, and we write $\mathrm{Arg}(c+\mathbf{i}d)=\theta$.

For $c+\mathbf{i}d = r(\cos\theta + \mathbf{i}\sin\theta)$, $-\pi<\theta\le\pi$, $r>0$, we call r the absolute value of $c+\mathbf{i}d$ and we denote it by $|c+\mathbf{i}d|$.

We state the *complex conjugate root theorem* in our desired context which is valid in constructive mathematics.

Theorem 3.2. If $p(x)$ is a polynomial over the field of rational numbers, and $a+\mathbf{i}b$ is a root of $p(x)$, where $a,b\in\mathbb{R}$, then its conjugate $a-\mathbf{i}b$ is also a root of $p(x)$.

Therefore, from *complex conjugate root theorem* we conclude that the absolute value of an algebraic number is also an algebraic number. Actually, we get more which says if $z=r(\cos\theta+\mathbf{i}\sin\theta)$ is an algebraic number, then $\cos\theta$ and $\sin\theta$ are also algebraic numbers; certainly beside r.

By [1, Definition 2.8], $\mathbf{exp}(x+\mathbf{i}y)=\exp(x)(\cos y+\mathbf{i}\sin y)$, where $x,y\in\mathbb{R}$. For $z_1,z_2\in\mathbb{C}$, they prove that $\mathbf{exp}(z_1+z_2)=\mathbf{exp}(z_1)\mathbf{exp}(z_2)$. Note that if $z\in\mathbb{R}$, then the exp function defined on $\mathbb{R}$ coincides with the defined $\mathbf{exp}$ function on $\mathbb{C}$. Therefore,, for the algebraic number $c+\mathbf{i}d=r(\cos\theta+\mathbf{i}\sin\theta)$; we are able to use the notation as $c+\mathbf{i}d=r\mathbf{exp}(\mathbf{i}\theta)$, where $-\pi<\theta\le\pi$. So that for two algebraic numbers $\mathbf{exp}(\mathbf{i}\theta_1)$, $\mathbf{exp}(\mathbf{i}\theta_2)$ we have

$$\mathbf{exp}(\mathbf{i}\theta_1)\mathbf{exp}(\mathbf{i}\theta_2)=\mathbf{exp}(\mathbf{i}(\theta_1+\theta_2)),\ \mathbf{exp}(-\mathbf{i}\theta_1)=\frac{1}{\mathbf{exp}(\mathbf{i}\theta_1)}.$$

Now, we define the natural complex logarithmic function Ln_1 on $A:=\{x+\mathbf{i}y;\ x>0\}$ as

$$Ln_1(z):=\ln(\sqrt{x^2+y^2})+\mathbf{i}\arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right).$$

If we apply Lemma 2.3, and using Cauchy-Riemann equations, then we conclude that the function Ln_1 is differentiable on A. And we see that for the algebraic number $z=r\mathbf{exp}(\mathbf{i}\theta)$ we have $Ln_1(r\mathbf{exp}(\mathbf{i}\theta))=\ln(r)+\mathbf{i}\theta$, where $-\pi<\theta\le\pi$, where $-\pi<\theta\le\pi$ is the principal argument. And using the principal argument for two algebraic numbers z_1 and z_2 we have the following

$$Ln_1(z_1z_2)=Ln_1(z_1)+Ln_1(z_2)+2\mathbf{i}k\pi,\quad k=0,1,-1,$$
$$Ln_1(z_1/z_2)=Ln_1(z_1)-Ln_1(z_2)+2\mathbf{i}k\pi,\quad k=0,1,-1.$$

Note that above identities remain well-defined when we work with the original function of

$$Ln_1(z)=\ln(\sqrt{x^2+y^2})+\mathbf{i}\arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right).$$

Now, for a complex number $x+\mathbf{i}y$, we define the natural complex logarithmic function Ln_2 on $B:=\{x+\mathbf{i}y;\ x<0\}$ as

$$Ln_2(z):=\ln(\sqrt{x^2+y^2})+\mathbf{i}\arccos\left(\frac{y}{\sqrt{x^2+y^2}}\right).$$

If we apply Lemma 2.3, and using Cauchy-Riemann equations, then we conclude that the function Ln_2 is differentiable on B. And we see that for the algebraic number $z=r\mathbf{exp}(\mathbf{i}\theta)$ we have $Ln_2(r\mathbf{exp}(\mathbf{i}\theta))=\ln(r)-\mathbf{i}\theta+\mathbf{i}\pi/2$, where $-\pi<\theta\le\pi$ is the principal argument.

Now, we apply Theorem 2.4, to prove the following version related to algebraic numbers.

Theorem 3.3. Let f be analytic and not identically zero on the connected open set U. Let K be a compact set well contained in U and f is not identically zero on K. Then either $\inf\{|f(z)|; z \in K\} > 0$, or for only finitely many algebraic numbers $x_i \in K$ we have $f(x_i) = 0$.

Proof. Applying Theorem 2.4, we have $f(z) = (z - z_1)(z - z_2)\dots(z - z_n)g(z)$, where z is every arbitrary element in K. If $x_1 \in \mathbb{C}^\alpha$, and $x_1 \in K$, then $g(x_1) \neq 0$. If $f(x_1) = 0$, then $(x_1-z_1)(x_1-z_2)\dots(x_1-z_n) = 0$. We let $h(z) := (z-z_1)(z-z_2)\dots(z-z_n) = z^n+a_{n-1}z^{n-1}+\dots+a_0$. Now, if $f(x_1) = 0$, then $h(x_1) = 0$. On the other hand, we have $h(z) = (z - x_1)q(z) + p$, where

$$\begin{aligned} q(z) = z^{n-1} + z^{n-2}(x_1 + a_{n-1}) + z^{n-3}({x_1}^2 + x_1 a_{n-1} + a_{n-2}) \\ + \dots + z({x_1}^{n-2} + {x_1}^{n-3}a_{n-1} + \dots + a_2) \\ +({x_1}^{n-1} + {x_1}^{n-2}a_{n-1} + \dots + x_1 a_2 + a_1), \\ p = {x_1}^n + {x_1}^{n-1}a_{n-1} + \dots + x_1 a_1 + a_0. \end{aligned}$$

Therefore, $h(z) = (z-x_1)q(z)$. Now, if there exists another similar $x_2 \neq x_1$, then $q(x_2) = 0$. If this procedure stops we are done. If not, then $(z-z_1)(z-z_2)\dots(z-z_n) = (z-x_1)(z-x_2)\dots(z-x_n)$. If there is another x_{n+1}, then $(z - x_1)(z - x_2)\dots(z - x_n) = (z - x_1)(z - x_2)\dots(z - x_{n+1})$; this is because $\mathbb{C}^\alpha[z]$ is a UFD. Hence, $x_n = x_{n+1}$. This completes the proof. ☐

Remark 3.4. Note that the proof of Theorem 3.3, is constructive as we have $\neg(x \neq 0) \equiv (x = 0)$. So that the complement of $\mathbb{C}-\{0\}$ is the set $\{0\}$. Therefore, we may consider f a function from $\mathbb{C}^\alpha$ onto $\mathbb{C}-\{0\}\cup\{0\}$. If the range of f is $\mathbb{C}-\{0\}$, then we are done; and this function is constructive by the Countable Axiom of Choice. Otherwise, we get a contradiction; therefore, its negation is true (that is how it works in constructive mathematics) which means there is an algebraic number which goes to zero. And we proceed.

4. The example

The example provided in this paper is straightforward yet revealing. It introduces a nonzero differentiable function on a simply connected open set whose zeros are not isolated, demonstrating a paradox under constructive interpretation.

The example utilizes the field of complex numbers along with the constructive properties of algebraic numbers. By analyzing the behavior of this function, we show that accepting the Countable Axiom of Choice within Bishop's constructive mathematics inevitably leads to a paradox

First, we fix in the entirety of this section the simply connected open set $\mathcal{D} := \{x + \mathbf{i}y; -2 < x < -\epsilon, -1 + \epsilon < y < 1 - \epsilon\}$, where ϵ is a very small and positive algebraic number. Now, we consider following functions which are defined on $\mathcal{D}$, $F_1(z) := Ln_1\left(\frac{-z}{\mathbf{i}z+1}\right)$, $F_2(z) := Ln_1\left(\frac{\mathbf{i}z}{\mathbf{i}z+1}\right)$, and $F_3(z) := Ln_1\left(1 - \sin(-\mathbf{i}\,\mathrm{Ln}(z))\right)$, where the function Ln is the branch of the logarithmic function in which is differentiable on the simply connected open set of $\mathcal{D}$.

Now, we discuss differentiablity of F_1, F_2, F_3, on $\mathcal{D}$.

(1) F_1.

We have $\frac{-x-\mathbf{i}y}{\mathbf{i}(x+\mathbf{i}y)+1} = \frac{-x}{x^2+y^2-2y+1} + \mathbf{i}\frac{x^2-y+y^2}{x^2+y^2-2y+1}$. But $x^2+y^2-2y+1 < 4+(1-\epsilon)^2+2(1-\epsilon)+1 = 8-4\epsilon+\epsilon^2 \Rightarrow \frac{-x}{x^2+y^2-2y+1} > \frac{\epsilon}{8-4\epsilon+\epsilon^2}$. Hence, $\frac{-z}{\mathbf{i}z+1}$ is well contained in the open set of $\mathcal{A} := \{x + \mathbf{i}y;\ x > 0\}$. Therefore, F_1 is differentiable on $\mathcal{D}$.

(1) F_2.

We have $\frac{\mathbf{i}(x+\mathbf{i}y)}{\mathbf{i}(x+\mathbf{i}y)+1} = \frac{x^2-y+y^2}{x^2+y^2-2y+1} + \mathbf{i}\frac{x}{x^2+y^2-2y+1}$. But $\frac{x^2-y+y^2}{x^2+y^2-2y+1} > \frac{\epsilon^2+\epsilon-1}{8-4\epsilon+\epsilon^2}$. Hence for $\epsilon = \frac{4}{5}$, $\frac{\mathbf{i}z}{\mathbf{i}z+1}$ is well contained in the open set of $\mathcal{A} := \{x + \mathbf{i}y;\ x > 0\}$. Therefore, F_2 is differentiable on $\mathcal{D}$.

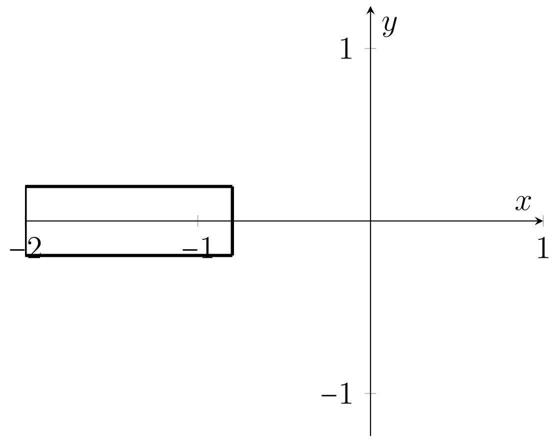

FIGURE 1. The open simply connected set of $\mathcal{D}$.

(1) F_3.

We have $1-\sin(-\mathbf{i}\,\mathrm{Ln}(z)) = \frac{1}{2}\frac{2x^2+2y^2+y(-x^2-y^2-1)}{x^2+y^2} + \mathbf{i}\frac{1}{2}\frac{x^3+xy^2-x}{x^2+y^2}$. But we have $2\epsilon^2 < 2x^2+2y^2$, and $-(1-\epsilon)^2-5 < -x^2-y^2-1 < -\epsilon^2-1$, then $(-(1-\epsilon)^2-5)(1-\epsilon) < y(-x^2-y^2-1)$. Therefore, $-6+8\epsilon-\epsilon^2+\epsilon^3 < 2x^2+2y^2+y(-x^2-y^2-1)$. On the other hand, we get $x^2+y^2 < 4+(1-\epsilon)^2$. So that if we choose $\epsilon := \frac{4}{5}$, we see that $1-\sin(-\mathbf{i}\,\mathrm{Ln}(z))$ is well contained in the open set of $\mathcal{A} := \{x+\mathbf{i}y;\ x>0\}$. Therefore, F_3 is differentiable on $\mathcal{D}$.

So that the simply connected open set of $\mathcal{D} := \{x+\mathbf{i}y;\ -2 < x < \frac{-4}{5},\ \frac{-1}{5} < y < \frac{1}{5}\}$ is shown in the Figure 1, where it is the region inside thick segments.

On the other hand, if the algebraic number $z := \mathbf{exp}(\mathbf{i}\theta)$, then we have

$$\begin{aligned}\frac{-z}{\mathbf{i}z+1} &= \frac{-\mathbf{exp}(\mathbf{i}\theta)}{\mathbf{exp}(\mathbf{i}\pi/2+\mathbf{i}\theta)+1}\\ &= \frac{-\mathbf{exp}(\mathbf{i}\theta)}{2\cos(\theta/2+\pi/4)\mathbf{exp}(\mathbf{i}\theta/2+\mathbf{i}\pi/4)}\\ &= \frac{-\mathbf{exp}(-\mathbf{i}\pi/4)\mathbf{exp}(\mathbf{i}\theta/2)}{2\cos(\theta/2+\pi/4)}.\end{aligned}$$

And we have

$$\begin{aligned}\frac{\mathbf{i}z}{\mathbf{i}z+1} &= \frac{z}{z-\mathbf{i}}\\ &= \frac{\mathbf{exp}(\mathbf{i}\theta)}{\cos(\theta)+\mathbf{i}(\sin(\theta)-1)}\\ &= \frac{\mathbf{exp}(\mathbf{i}\theta)}{\sin(\pi/2+\theta)+2\mathbf{i}(\sin(\theta/2-\pi/4)\cos(\theta/2+\pi/4))}\\ &= \frac{\mathbf{exp}(\mathbf{i}\theta)}{2\sin(\pi/4+\theta/2)\cos(\pi/4+\theta/2)+2\mathbf{i}(\sin(\theta/2-\pi/4)\cos(\theta/2+\pi/4))}\\ &= \frac{\mathbf{exp}(\mathbf{i}\theta)}{2\cos(\theta/2+\pi/4)(\cos(\theta/2-\pi/4)+\mathbf{i}\sin(\theta/2-\pi/4))}\\ &= \frac{\mathbf{exp}(\mathbf{i}\pi/4)\mathbf{exp}(\mathbf{i}\theta/2)}{2\cos(\theta/2+\pi/4)}.\end{aligned}$$

And we have

$$1 - \sin(-\mathbf{i}\,\mathrm{Ln}(z)) = 1 - \sin(\theta) = 1 + \cos(\theta + \pi/2) = 2\cos^2(\theta/2 + \pi/4).$$

Now, we are ready to proceed with the example.

We consider the function

$$f : \mathcal{D} \to \mathbb{C}, \quad f(z) := Ln_1(F_1(z)) + Ln_1(F_2(z)) + Ln_1(F_3(z)) + Ln_2(z) + \mathrm{Ln}(2) + \mathbf{i}\pi - \mathbf{i}\pi/2.$$

By the argument above the function f is differentiable on $\mathcal{D}$. Where $\mathcal{D}$ is an open simply connected set.

Now, we evaluate f in two set of points in the simply connected open set of $\mathcal{D}$. So that we consider two different cases as follow.

Case (I). We consider the algebraic number $z := \mathbf{exp(i}\theta)$, $\pi/2 < \theta < \pi$.

We have

$$\begin{aligned} f(z) &= Ln_1\Big(\frac{-\mathbf{exp(-i}\pi/4)\mathbf{exp(i}\theta/2)}{2\cos(\theta/2+\pi/4)}\Big) + Ln_1\Big(\frac{\mathbf{exp(i}\pi/4)\mathbf{exp(i}\theta/2)}{2\cos(\theta/2+\pi/4)}\Big) \\ &+ Ln_1(2\cos^2(\theta/2+\pi/4)) + Ln_2(\mathbf{exp(i}\theta)) + Ln_1(2) + \mathbf{i}\pi - \mathbf{i}\pi/2 \\ &= \mathbf{i}\theta - 2Ln_1(2) - 2Ln_1(-\cos(\theta/2+\pi/4)) + 2Ln_1(-\cos(\theta/2+\pi/4)) - \mathbf{i}\theta + 2Ln_1(2) = 0. \end{aligned}$$

Case (II). We consider the algebraic number $z := \mathbf{exp(i}\theta)$, $-\pi < \theta < -\pi/2$.

We have

$$\begin{aligned} f(z) &= Ln_1\Big(\frac{-\mathbf{exp(-i}\pi/4)\mathbf{exp(i}\theta/2)}{2\cos(\theta/2+\pi/4)}\Big) + Ln_1\Big(\frac{\mathbf{exp(i}\pi/4)\mathbf{exp(i}\theta/2)}{2\cos(\theta/2+\pi/4)}\Big) \\ &+ Ln_1(2\cos^2(\theta/2+\pi/4)) + Ln_2(\mathbf{exp(i}\theta)) + Ln_1(2) + \mathbf{i}\pi - \mathbf{i}\pi/2 \\ &= \mathbf{i}\theta - 2Ln_1(2) - 2Ln_1(\cos(\theta/2+\pi/4)) + 2Ln_1(\cos(\theta/2+\pi/4)) - \mathbf{i}\theta + 2Ln_1(2) + 2\mathbf{i}\pi \\ &= 2\mathbf{i}\pi \neq 0. \end{aligned}$$

Note that in the first case $\pi/4 < \theta/2 < \pi/2$, then $\pi/2 < \theta/2 + \pi/4 < 3\pi/4$. Hence, we have $-\cos(\theta/2 + \pi/4) > 0$. And in the second case $-\pi/2 < \theta/2 < -\pi/4$, then $-\pi/4 < \theta/2 + \pi/4 < 0$. Hence, we have $\cos(\theta/2 + \pi/4) > 0$.

Now, we consider $Sc(-1, \frac{1}{5} - \delta) \subset\subset \mathcal{D}$, where $\delta > 0$ is a very small algebraic number. Clearly there are infinitely many algebraic numbers with the absolute value of 1 inside $Sc(-1, \frac{1}{5} - \delta)$; above and under the x-axis; examples are $\mathbf{exp}(-\mathbf{i}\pi + \frac{\mathbf{i}\pi}{n})$ and $\mathbf{exp}(\mathbf{i}\pi - \frac{\mathbf{i}\pi}{n})$ for big enough n's. And on infinitely many of these algebraic numbers f is zero and on infinitely many of these algebraic numbers f is nonzero. Therefore, regarding Theorem 3.3, entire the above procedure leads us to a paradox. As the theorem says that since f is analytic on the simply connected open set $\mathcal{D}$. The $Sc(-1, \frac{1}{5} - \delta)$ is a compact set well contained in $\mathcal{D}$, and f is not identically zero on $Sc(-1, \frac{1}{5} - \delta)$. And $\inf\{|f(z)|; z \in Sc(-1, \frac{1}{5} - \delta)\} = 0$, then there are finitely many algebraic numbers $x \in Sc(-1, \frac{1}{5} - \delta)$ such that $f(x) = 0$. This is the paradox.

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