Problem for Day 4. *In this problem, we consider the statistical inverse problem of 1D de-blurring, formulated as*

$$Y = \mathcal{A}X + E, \tag{1}$$

where X, Y and E are all random variables in $\mathbb{R}^n$, and $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a blurring matrix. Complete the Python code `exercise_1.py` *in day 4 folder following these instructions.*

1. *the vector* `s` *contains a discretization of the interval $[0, 1]$ with step size $\Delta s = 1/100$. Use* `s` *to create a vector* `x` *that approximates the signal*

$$x(s) = \begin{cases} 1, & 0.2 \leq s \leq 3.5, \\ 2, & 0.5 \leq s \leq 0.7, \\ 0 & \text{otherwise.} \end{cases}$$

2. *Use the Python function* `A` *to add blurr to* `x` *and create blurred measurement* `y` *and add noise to create noisy measurement* `y_obs`*, using blurring standard deviation $\delta = 1$ (in the code this is defined by* `delta`*). Plot the original signal* `x` *and the blurred measurements* `y` *and noisy measurements* `y_obs`*.*

Now we use the Bayes' theorem to construct the posterior as

$$\pi_{X|Y=\mathbf{y}}(\mathbf{x}) \propto \pi_{Y|X}(\mathbf{y})\pi_X(\mathbf{x})$$

and investigate different choices for the prior $\pi_X(\mathbf{x})$.

Uncorrelated prior: *the first exercise is to choose the prior $\pi_X(\mathbf{x})$ to be*

$$X \sim \mathcal{N}(0, I_n),$$

where I_n is the n-dimensional identity matrix. Continue completing the code following these instructions.

3. *Write the density function of the posterior distribution up to the proportionality constant when $E \sim \mathcal{N}(0, \sigma^2 I_n)$, i.e.,*

$$\pi_{X|Y=\mathbf{y}}(\mathbf{x}) \propto \cdots$$

4. *Complete the code in* `exercise_2.py` *using the steps below in order to enable sampling from the posterior distribution of this inverse problem:*

   (a) *write a Python function* `prior` *that computes the log (un-normalized) prior density, i.e.,*

$$\log \pi_X(\mathbf{x}) = -\frac{1}{2}\|x\|_2^2$$

(b) *write a Python function* **`likelihood`** *that computes the log (un-normalized) likelihood density, i.e.,*

$$\log \pi_{Y|X=\mathbf{x}}(\mathbf{y}) = -\frac{1}{2\sigma^2}\|\mathbf{y} - \mathcal{A}\mathbf{x}\|_2^2$$

(c) *write a Python function* **`posterior`** *that computes the log (un-normalized) posterior deinsty.*

(d) *using the random-walk Metropolis-Hasting algorithm, draw 10000 samples. Choose the step size c that gives you about 23% acceptance rate.*

5. *plot the posterior mean.*

6. *plot the point-wise variance as a measure of uncertainty, i.e., limit your samples to a point, e.g., $s = i\Delta s$ for a particular i, now your samples are real-valued samples. You can compute the variance of these samples. Then repeat for all points in* **`s`**.

7. *Where are the regions that has high(er) uncertainty.*

Increment (correlated) prior: *In this exercise we will set the prior on increments instead of pixel values. Let us first introduce the random variable for the increments:*

$$\mathbf{z}_i = \mathbf{x}_{i+1} - \mathbf{x}_i, \qquad i = 1, \dots, n-1,$$

and let $\mathbf{z}$ *be the vector that contains* $\mathbf{z}_i$. *Define the Matrix* T *that transforms* $\mathbf{z}$ *into* $\mathbf{x}$, *i.e.,* $\mathbf{x} = T\mathbf{z}$

8. *Write how the matrix* T *look like.*

Now let Z *be the random variable associate with the vector* $\mathbf{z}$, *with the prior density function* $\pi_Z(\mathbf{z})$ *(the exact density will be discussed below).*

8. *Reformulate the statistical de-blurring inverse problem using the change of variable* $X = TZ$ *and* $\mathbf{x} = T\mathbf{z}$. *(you need to reformulate the prior to be* π_Z *and the likelihood to be* $\pi_{Y|Z}$*)*

Now we will investigate different types of priors for Z. *We will use Gaussian prior, Laplace prior and Cauchy priors. Laplace and Cauchy priors will promote sparsity in jumps, i.e., we, a priori, expect fewer jumps in the signal.*

8. *In the Python code* **`exercise_3.py`** *for day 4, write 3 functions that can compute the log density function for the prior, once for the Gaussian* $\mathcal{N}(0, \sigma_{prior}^2 I_n)$, $\sigma_{prior} = 0.1$,

$$\pi_Z(\mathbf{z}) \propto \exp(-\frac{\|\mathbf{z}\|_2^2}{2\sigma_{prior}^2}),$$

once for the Laplace prior $Laplace(0, b)$, *zero mean and scaling* $b = 0.05$

$$\pi_Z(\mathbf{z}) \propto \exp(-\frac{\|\mathbf{z}\|_1}{b})$$

and Once for the Cauchy Cauchy$(0, \gamma)$*, zero mean and scaling* $\gamma = 0.005$.

$$\pi_Z(\mathbf{z}) \propto \prod_{i=1}^{n-1} \frac{1}{1 + (\frac{\mathbf{z}_i^2}{\gamma^2})}$$

Don't forget to take the log of the prior!!!

9. *Repeat the sampling as with the uncorrelated case. For each of the choices of prior make sure that you set the step size in the Metropolis-Hastings method such that you achieve acceptance rate of about 23%. Plot the posterior mean and point-wise variance. Explain the differences with respect to the uncorrelated prior. Which of these priors are the most suitable for the problem and why?*