
Problem for Day 1. *Consider the statistical inverse problem*

$$Y = f(X) + E, \tag{1}$$

where $X \in \mathcal{N}(0, I_n), E \sim \mathcal{N}(0, \sigma^2 I_m)$ where I_n and I_m are an $n \times n$ and $m \times m$ identity matrices, $\sigma > 0$ is the standard deviation of noise, and $f : \mathbb{R}^n \to \mathbb{R}^m$ is the forward operator. Furthermore, we can write the probability density function of X and E as

$$\pi_X(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n}} \exp\left(-\frac{1}{2}\|\mathbf{x}\|_2^2\right),$$
$$\pi_E(\mathbf{e}) = \frac{1}{\sqrt{(2\pi)^m \sigma^2}} \exp\left(-\frac{\|\mathbf{e}\|^2}{2\sigma^2}\right).$$

1. *Use Bayes' theorem to write the density function of the posterior distribution $\pi_{X|Y=\mathbf{y}}(\mathbf{x})$ up to a proportionality constant:*

   $$\pi_{X|Y=\mathbf{y}} \propto \cdots$$

   Assume that f is smooth but not necessarily linear. Only express it in terms of f, $\mathbf{x}$, $\mathbf{y}$, and σ.

2. *Show that the maximum a posteriori (MAP) estimate*

   $$\mathbf{x}_{MAP} := \arg\max_{\mathbf{x}} \ \pi_{X|Y=\mathbf{y}}(\mathbf{x})$$

   is equivalent to the solution of a Tikhonov regularized optimization problem:

   $$\mathbf{x}_{Tik} := \arg\min_{\mathbf{x}} \ \|f(\mathbf{x}) - \mathbf{y}\|_2^2 + \lambda\|\mathbf{x}\|^2,$$

   for some $\lambda > 0$. Express λ in terms of σ.