Smooth Priors and Random Fields

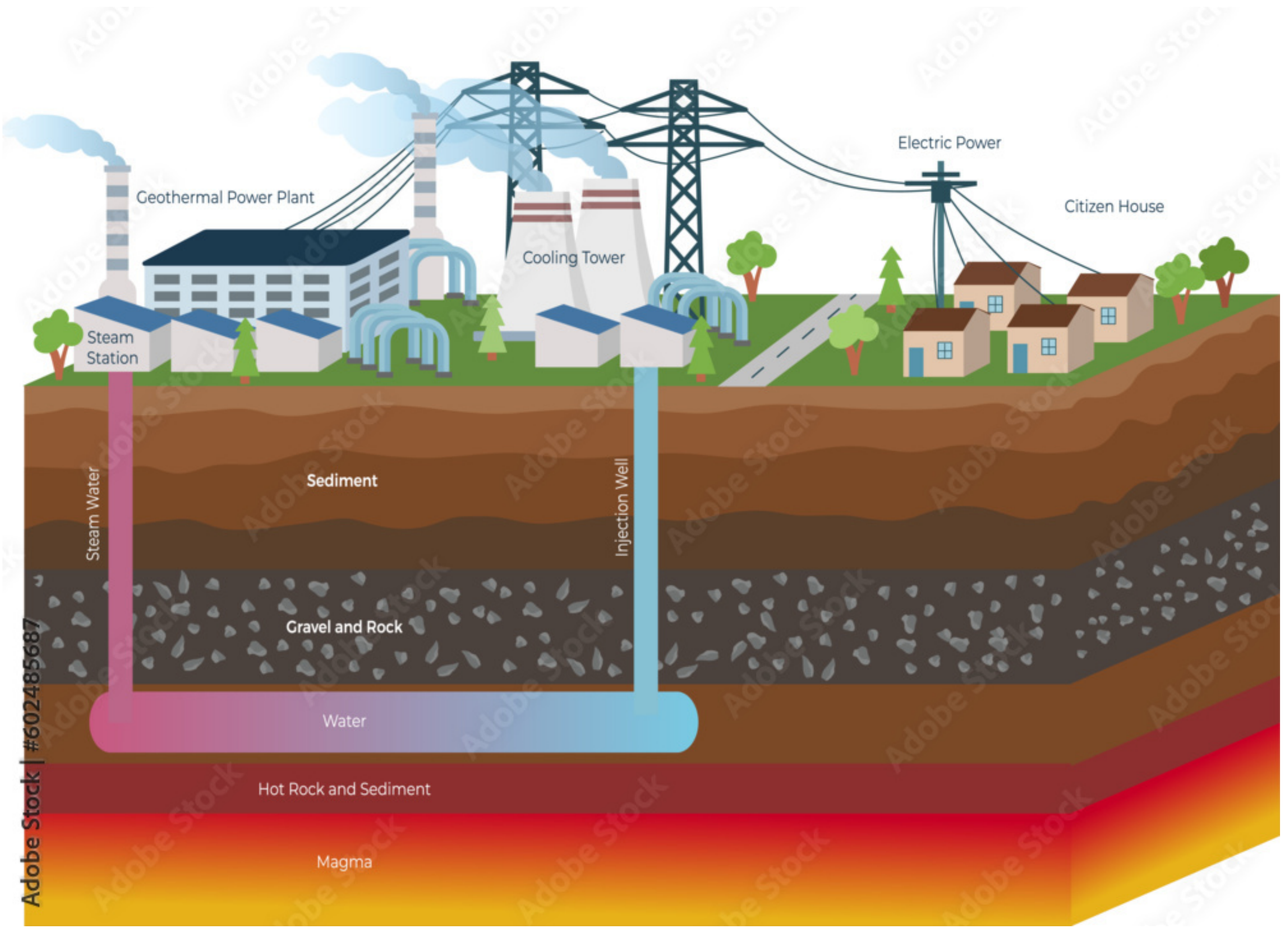
Gaussian and Exponential priors

Babak Maboudi - day 5 - Jyväskylä summer school

Geo-thermal Power Stations

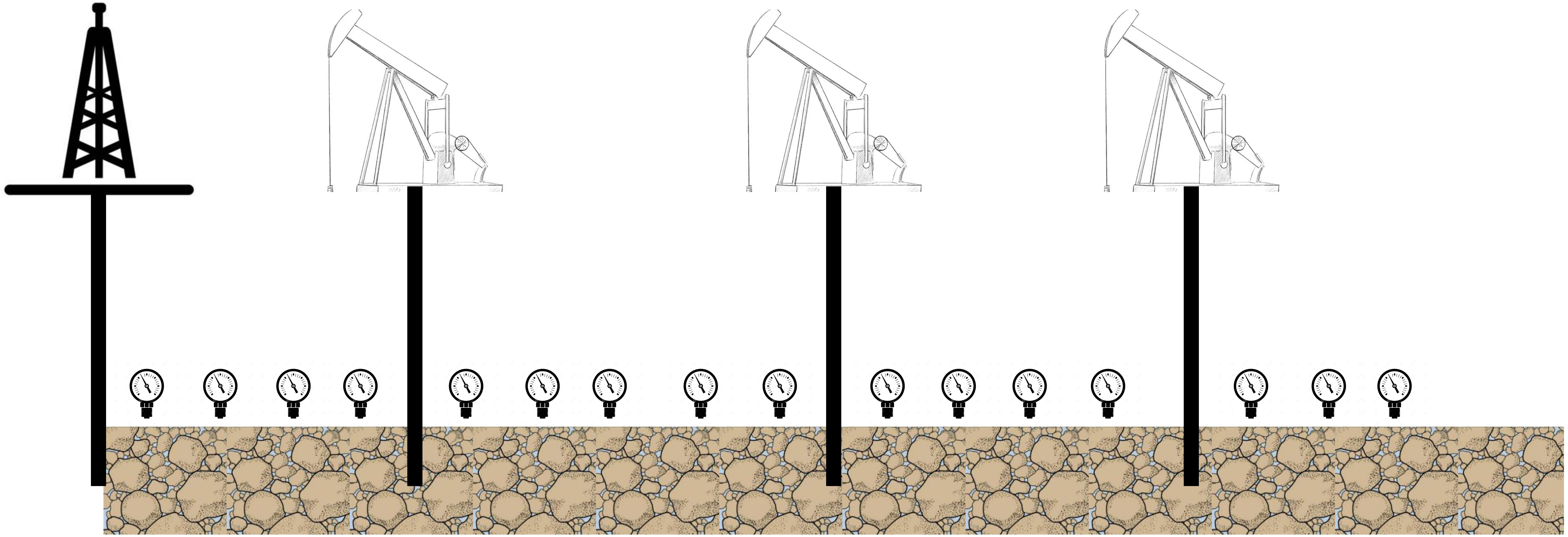
Motivation for today’s project

Krafla Power Station, Iceland



Porosity Estimation

1D flow in porous medium



Porosity Estimation

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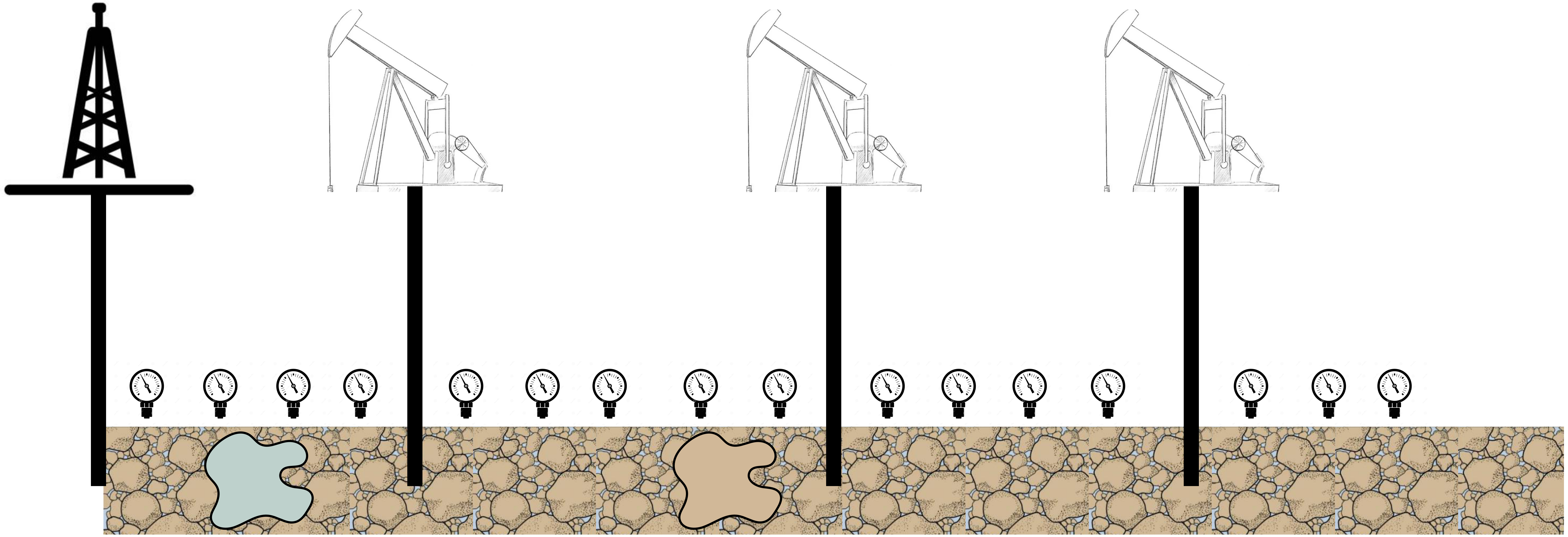


Smooth Priors for Bayesian Inverse Problems

The covariance function method

- We want to create a random function $X(s)$, with $s \in [0,1)$.
- We want realizations of X be *continuous* and *differentiable*.

Smooth Priors for Bayesian Inverse Problems

The covariance function method

- Let us first discretize $x(s)$, by choosing step-size $\Delta s = 1/N$.

$$x(s) \approx \mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N), \quad \mathbf{x}_i = x(i\Delta s)$$

- We can now specify how $\mathbf{x}_i$ are correlated using a covariance function:
  - Gaussian covariance function (infinitely many times differentiable):

$$f(\mathbf{x}_i, \mathbf{x}_j) = c_{ij} := \exp(-\frac{|\mathbf{x}_i - \mathbf{x}_j|^2}{2\ell^2})$$

  - Exponential covariance function (1-time differentiable):

$$f(\mathbf{x}_i, \mathbf{x}_j) = c_{ij} := \exp(-\frac{|\mathbf{x}_i - \mathbf{x}_j|}{\ell})$$

- ℓ is called the *length scale* and controls how far apart points have strong correlation.

Smooth Priors for Bayesian Inverse Problems

The covariance function method

- Once we have the correlations, we can create a covariance matrix:

$$C_X = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1,N} \\ c_{21} & c_{22} & \cdots & c_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{N,N} \end{pmatrix}$$

- We can then create a Gaussian prior for X following

$$X \sim \mathcal{N}(0, C_X), \quad \text{with} \qquad \pi_X(\mathbf{x}) \propto \exp(-\frac{1}{2}\mathbf{x}^T C_X^{-1}\mathbf{x})$$

Independent Components of X

- Since we enforced spatial correlation for components of $X(s)$ then, by definition, it's components, i.e., $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ are correlated. This makes inference really challenging!
- Theorem: If the covariance matrix C of a multi-variate random variable X is symmetric and positive-definite, then the random variable
$$Z := C^{-1/2}X$$
has independent and standard normal components, where $C^{-1/2}$ is the precision matrix, or the inverse or the Cholesky factor $C^{1/2}$
- The formulation of $X = C^{1/2}Z$ for a random function is referred to as the *Karhunen-Loève expansion of X.*

Exercise 1

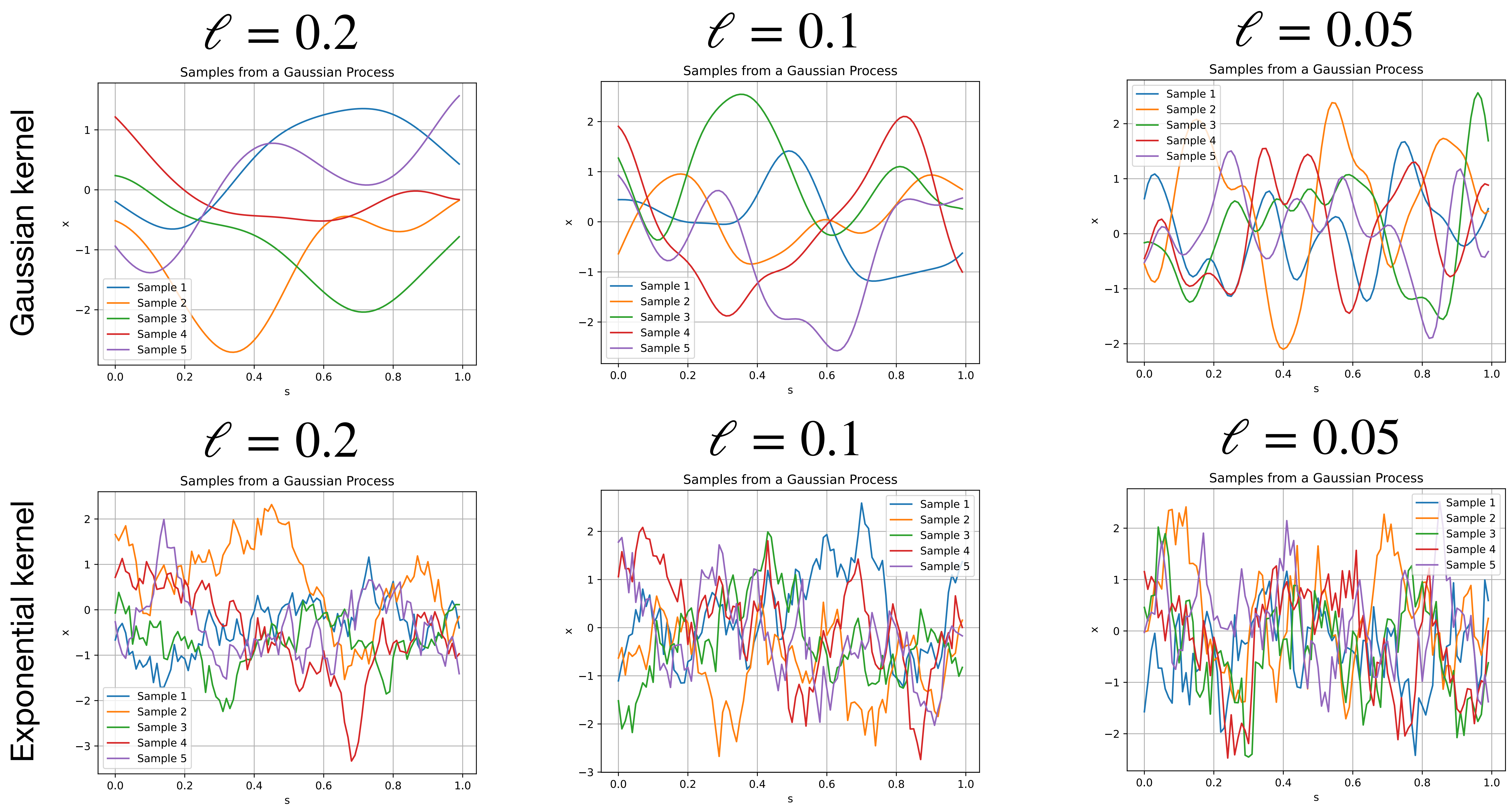
- Create a covariance matrix following once the Gaussian cov. function, and once following the exponential cov. function.

- Sample random functions from the distribution $\mathcal{N}(0, C_X)$. For different length scales 0.05, 0.1, and 0.2. Explain what you see.

- Compute the Cholesky factor $C_X^{1/2}$.

- For the Gaussian covariance kernel, set the length scale to 0.1 and plot 5 samples from $C^{1/2}Z$, where $Z \sim \mathcal{N}(0, I_N)$.

Samples from a Gaussian Random Field



Hydraulic Imaging

Problem formulation

- The hydraulic imaging can be formulated in terms of the PDE problem:

$$\frac{\partial}{\partial s}\left(\exp(x(s))\frac{\partial p_i(s)}{\partial s}\right) = q_i\delta(s_i^{\text{well}}), \qquad i = 1,\ldots,N_{\text{well}}$$

  - $p_i(s)$ is the pressure profile when the ith injector is active
  - q_i is the injection rate at the location of ith well
  - $\exp(x)$ is the porosity profile.
  - x is a continuous and possibly differentiable function.

Hydraulic Imaging

Notation

- We refer to $\mathbf{x}$ to be the unknown in the inverse problem.
- We refer to $\mathbf{p}_i = F_i(\mathbf{x})$ to refer to the experiment of injecting in the ith well and recording the pressure profile $\mathbf{p}_i$ (solving 1 PDE).
- We can write the inverse problem as

$$\mathbf{y}_i = F_i(\mathbf{x}) + \varepsilon_i, \quad \text{for } i = 1,\ldots,5$$

Hydraulic Imaging

Run the hydraulic problem

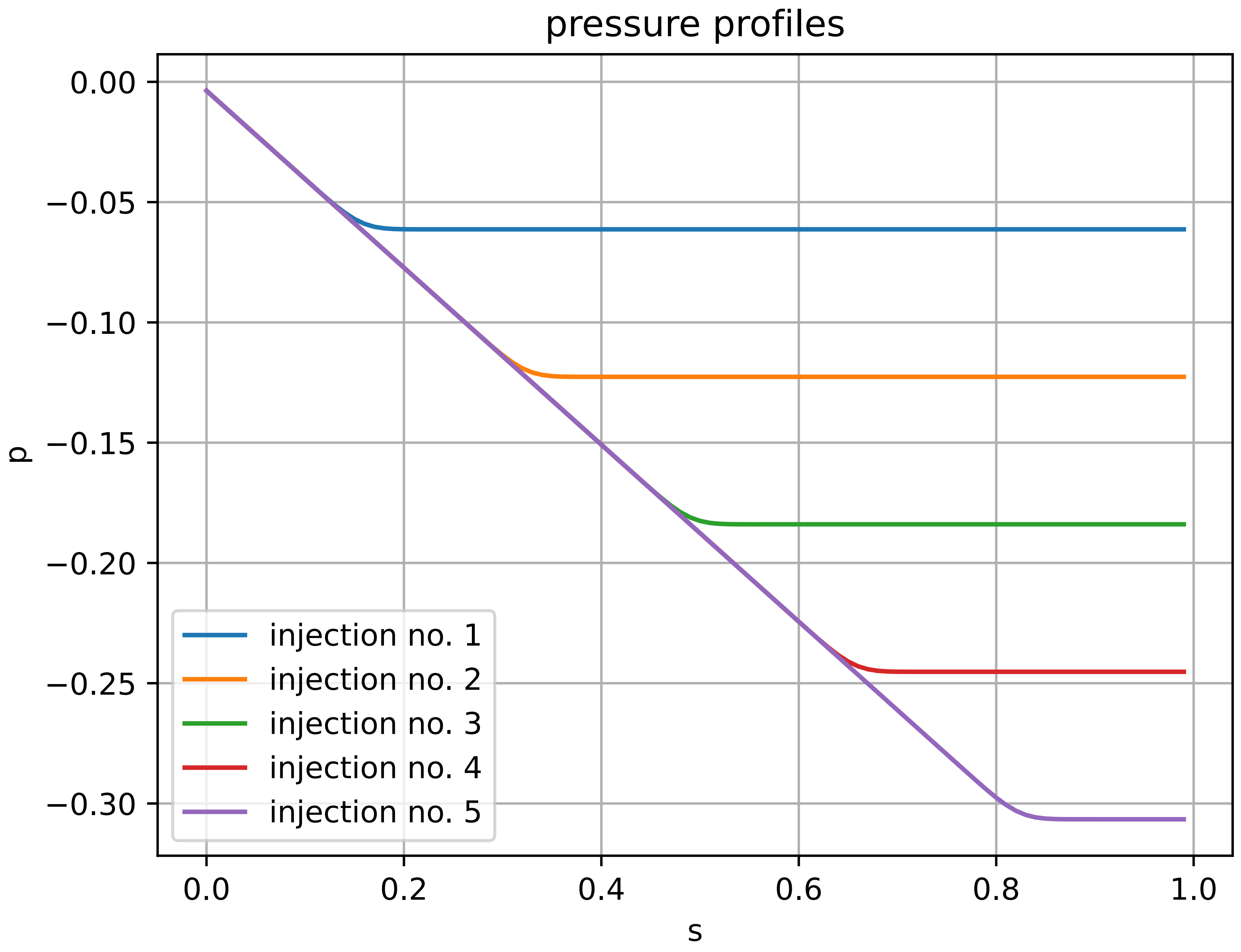
```
import numpy as np
import matplotlib.pyplot as plt

from hydraulic import hydraulic_class

N_points = 128

hydraulic = hydraulic_class(N_points) # initiate hydraulic problem with N discretization points

X = np.ones(N_points) # constant porosity profile
p = hydraulic.forward(X) # returns the pressure profile for 5 injections
```



Hydraulic Imaging

Bayesian Formulation

- First we introduce random variables:

$$\mathbf{Y}_i = F_i(\mathbf{X}) + E_i$$

- Y_i, $i = 1,\ldots,5$, are random variables of pressure profile for ith injection.

- E_i, $i = 1,\ldots,5$, are random variables of noise for ith injection.

- X is the random variable for log-porosity.

Hydraulic Imaging

Bayesian Formulation

- We now formulate the Bayes' rule

$$\pi_{X|Y_1,\ldots,Y_5} \propto \pi_{Y_1,\ldots,Y_5|X}\pi_X$$

- Under independence of measurements we can further break down the posterior

$$\pi_{X|Y_1,\ldots,Y_5} \propto \pi_{Y_1|X}\cdots\pi_{Y_5|X}\pi_X$$

- Now we need to specify each distribution on the right hand side.

Hydraulic Imaging

Bayesian Formulation

- Let us break down the terms in Bayes' rule:

$$\pi_{X|Y_1,\ldots,Y_5} \propto \pi_{Y_1|X} \cdots \pi_{Y_5|X} \pi_X$$

  - $\pi_{Y_i|X}$ is the likelihood density for the ith injection/measurement:

$$\pi_{Y_i|X}(\mathbf{y}_i^{\text{obs}}) \propto \exp\left(-\frac{\|F_i(\mathbf{x}) - \mathbf{y}_i^{\text{obs}}\|_2^2}{2\sigma_i^2}\right),$$

    where σ_i is the noise standard deviation for the ith experiment.

  - $\pi_X(\mathbf{x})$ is the prior density for a smooth function

$$\pi_X(\mathbf{x}) \propto \exp\left(-\frac{\mathbf{x} C_X^{-1} \mathbf{x}}{2}\right)$$

Hydraulic Imaging

Bayesian Formulation with Reparameterization

- Let us break down the terms in Bayes' rule:

$$\pi_{Z|Y_1,\ldots,Y_5} \propto \pi_{Y_1|Z} \cdots \pi_{Y_5|Z} \pi_Z$$

  - $\pi_{Y_i|X}$ is the likelihood density for the ith injection/measurement:

$$\pi_{Y_i|Z}(\mathbf{y}_i^{\mathrm{obs}}) \propto \exp(-\frac{\|F_i(C_X^{1/2}\mathbf{z}) - \mathbf{y}_i^{\mathrm{obs}}\|_2^2}{2\sigma_i^2}),$$

    where σ_i is the noise standard deviation for the ith experiment.

  - $\pi_Z(\mathbf{z})$ is the prior density for a smooth function

$$\pi_Z(\mathbf{z}) \propto \exp\left(-\frac{\|\mathbf{z}\|_2^2}{2}\right)$$

Exercise 2

- Write a Python function that computes the log-prior density corresponding to a Gaussian covariance function with correlation length $\ell = 0.1$.
- Write a Python function that computes the log-likelihood density function which combines the 5 experiments for the hydraulic imaging problem.
- Combine the two to write a Python function that computes the log-posterior of the hydraulic imaging problem.
- Use the random-walk Metropolis-Hasting algorithm to sample from the posterior distribution.
- Plot the posterior mean and also point-wise variance.