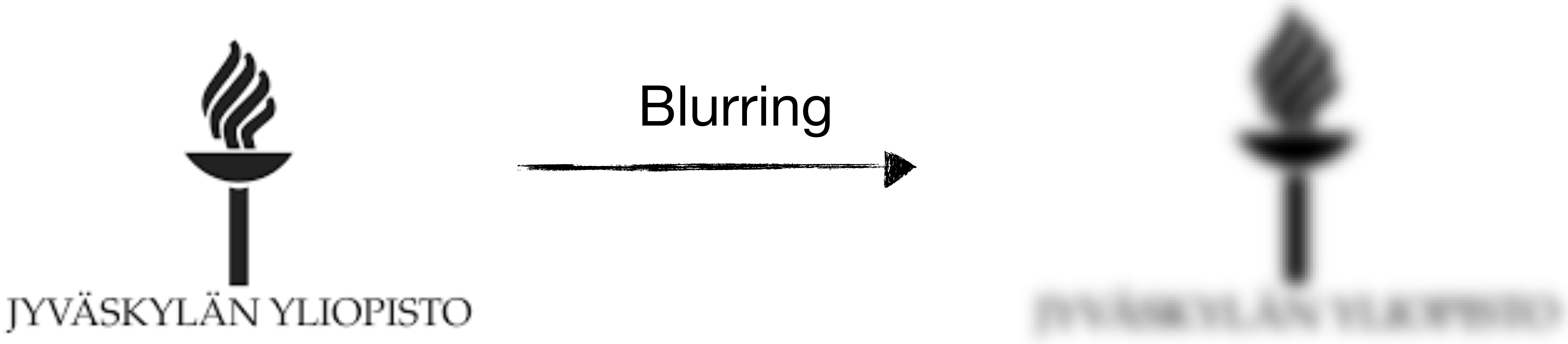
Prior Engineering

How to choose the right prior for your Bayesian inverse problems

Babak Maboudi - day 4 - Jyväskylä summer school 2025

The De-blurring Problem



De-blurring inverse problem in 1D

- Let $x(s)$ represent (continuous) pixel intensities or an image in 1D, with $s \in [0,1)$.

- We can express blurring using a convolution operation:

$$y(s) = (g * x)(s) := \int_0^1 g(t - s)x(s)\ dt$$

g is called a kernel and a typical kernel is a Gaussian kernel:

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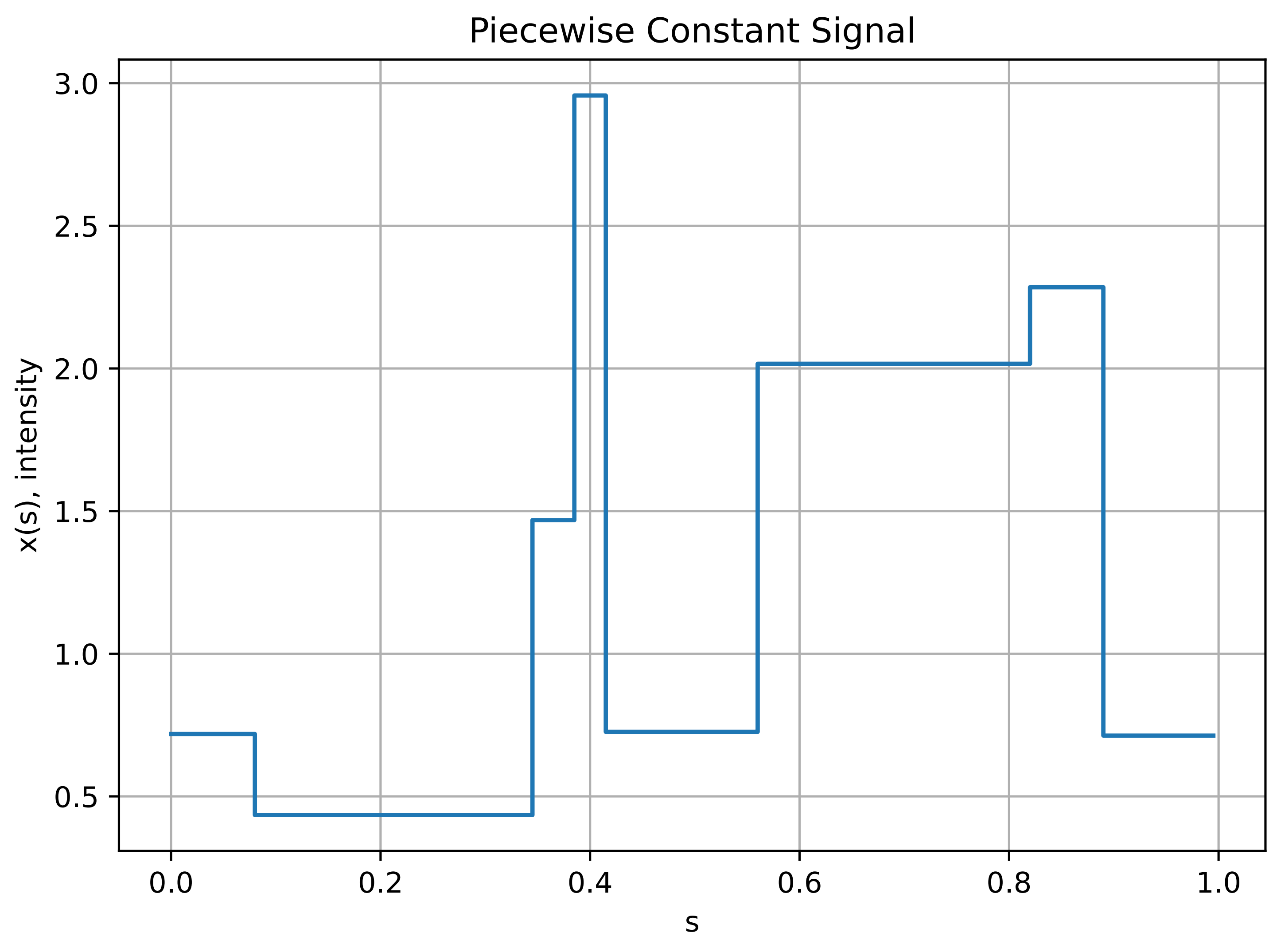
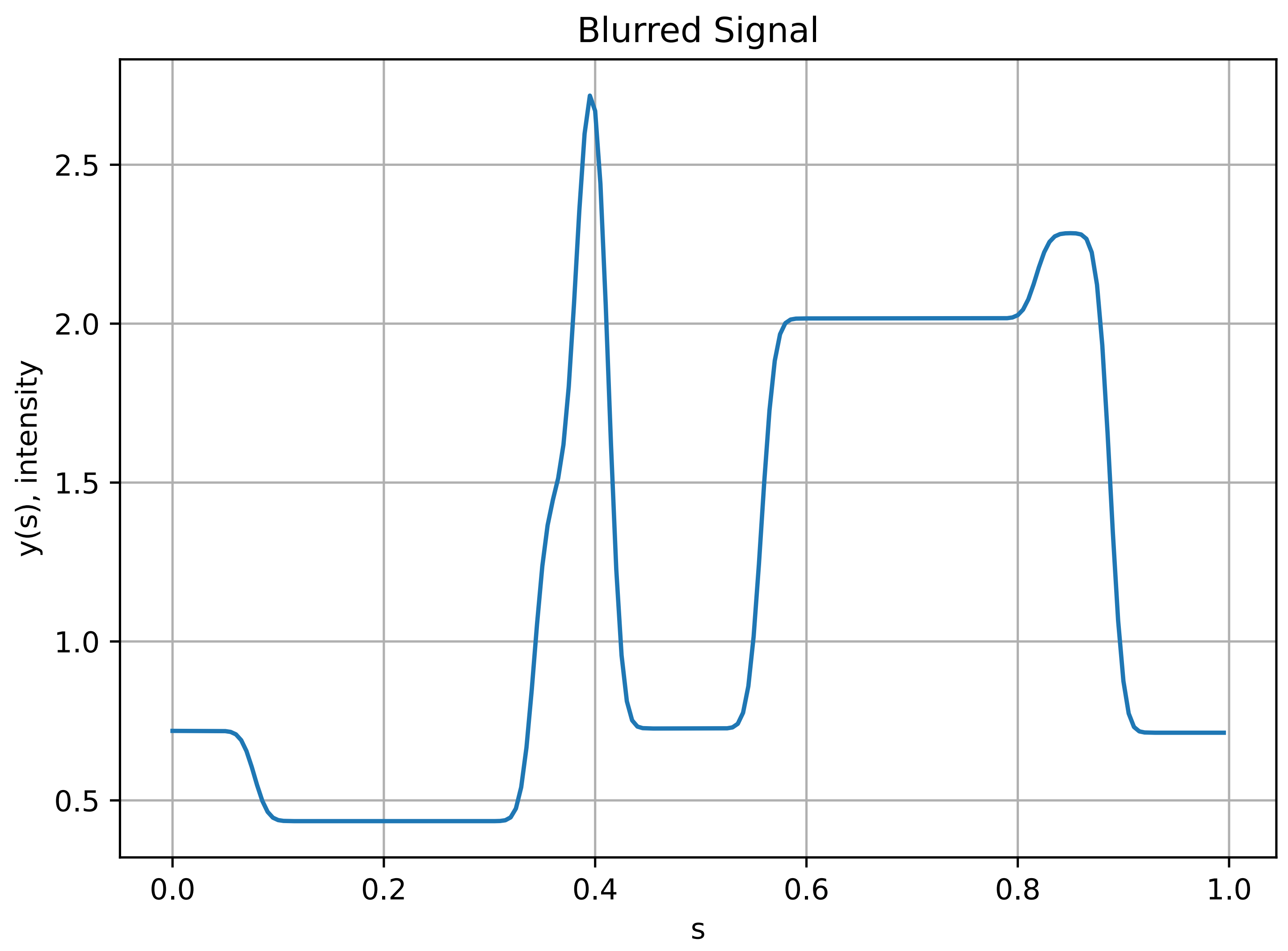


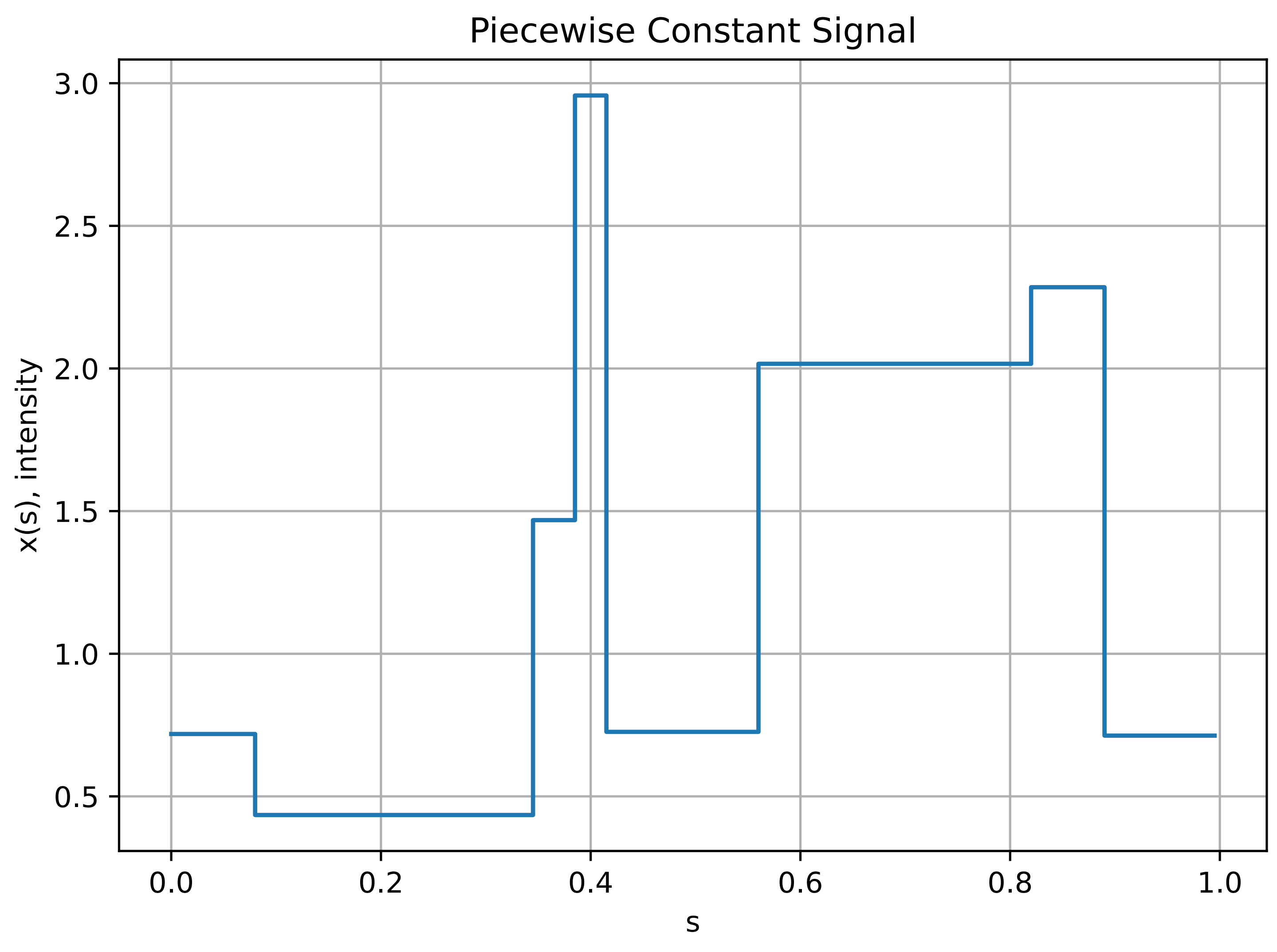


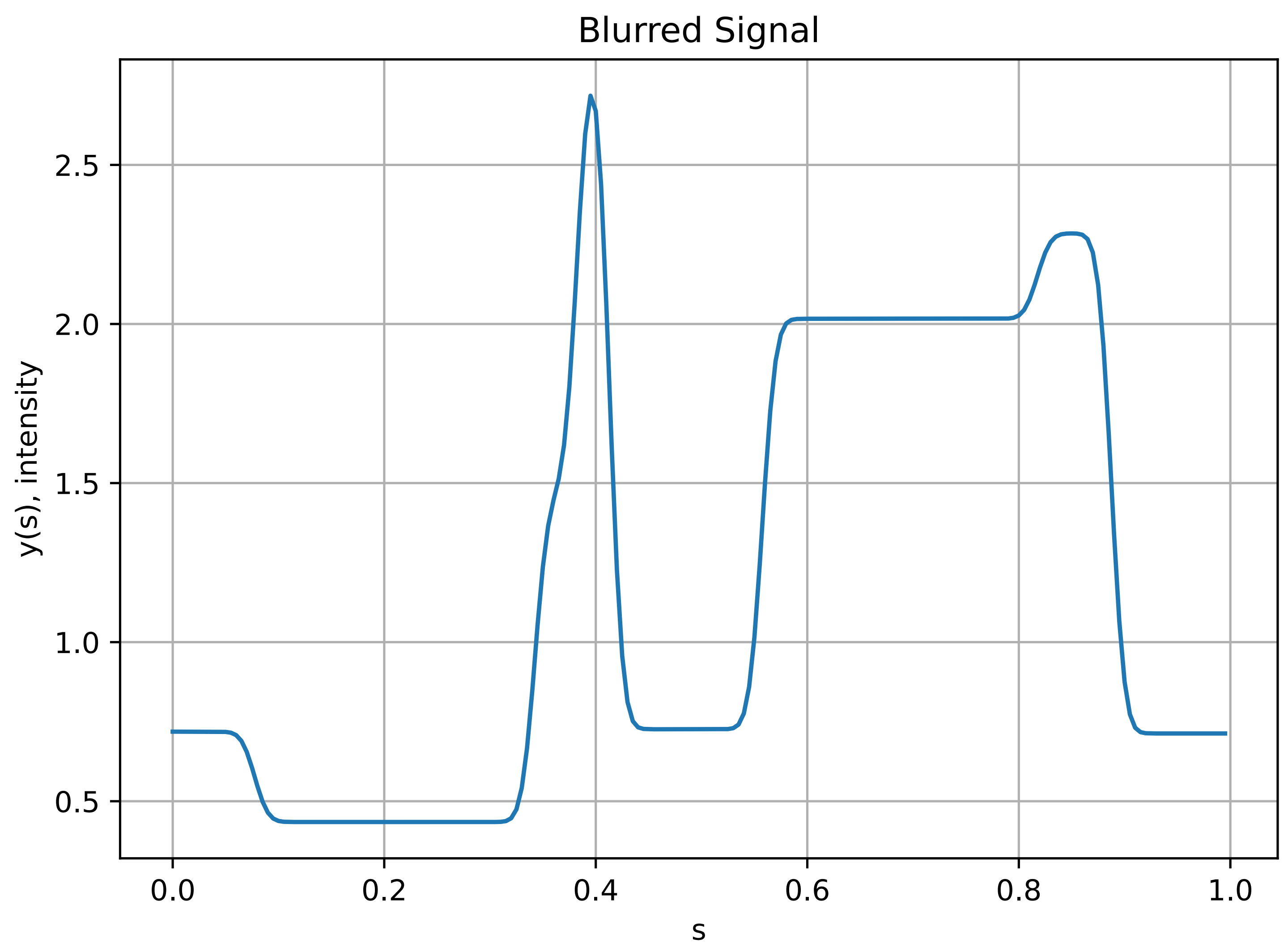
Discrete De-blurring problem

- We discretize $x(s)$ with a vector $\mathbf{x} \in \mathbb{R}^N$ with $x_i = x(i\Delta s)$, and $\Delta s = 1/N$.
- We can express discrete blurring operation as an (implicit) matrix vector multiplication:

$$\mathbf{y} = A\mathbf{x}$$

$$A \in \mathbb{R}^{N \times N}, \mathbf{y} \in \mathbb{R}^N.$$





Statistical De-Blurring

Exercise 1 (HW4)

- Complete the code in `exercise_1.py` in day 4 folder
- Discretize the unit interval with a vector s that contains the elements $i\Delta s$ with $i = 1,\ldots,100$ and $\Delta s = 1/100$. This is already in the code!
- Create a Numpy vector `x` that approximates the step signal

$$x(s) = \begin{cases} 1, & 0.2 \leq s \leq 3.5, \\ 2, & 0.5 \leq s \leq 0.7, \\ 0 & \text{otherwise.} \end{cases}$$

- Use the Python function `A` to add blurr to `x` and create blurred measurement `y`. Use blurring standard deviation δ (in the code `delta`) to be 1. Furthermore, use the suggested code to add noise to the blurred signal to create a noisy measurement `y_obs`.
- Plot the original signal `x` and the blurred measurement `y` and noisy measurements `y_obs`.

Formulating the de-blurring problem as an inverse problem

Exercise 2

- Formulate the problem as an inverse problem:

$$\mathbf{y} = \mathscr{A}\mathbf{x} + \mathbf{e}$$

- Here $\mathbf{e} \sim \mathscr{N}(0, \sigma^2 I_N)$, i.e., noise is independent and standard normal for each component of $\mathbf{y}$.

Bayesian formulation of the De-blurring problem

Posing a Bayesian inverse problem

- Define parameters as random variables:

$$Y = \mathscr{A}\mathbf{X} + E$$

  - $E \in \mathbb{R}^N$ is noise random variable.
  - $Y \in \mathbb{R}^N$ is measurement random variable.
  - $X \in \mathbb{R}^N$ is the unknown random variable.

Posing a Bayesian inverse problem

- Now we need to define the distribution for each component $E, Y|X,$, and X to use the Bayes' formula:

$$\pi_{X|Y=\mathbf{y}} \propto \pi_{Y|X}(\mathbf{y})\pi_X(\mathbf{x}).$$

  - E is a Gaussian noise with, therefore,

$$E \sim \mathcal{N}(0, \sigma^2 I_N), \qquad \pi_E(e) \propto \exp(-\frac{\|e\|_2^2}{2\sigma^2})$$

  - $Y|X$ is translation of E with AX, i.e.,

$$Y|X = \mathcal{N}(\mathcal{A}X, \sigma^2 I_N), \quad \pi_{Y|X=x}(y_{\text{obs}}) \propto \exp(-\frac{\|y_{\text{obs}} - \mathcal{A}x\|_2^2}{2\sigma^2})$$

  - The art of Bayesian inversion is to choose π_X relevant to the problem.

Gaussian i.i.d. priors

- The most basic prior we can choose is that each pixel is independent from the other pixels and follows a standard normal distribution.
- Therefore,

$$X \sim \mathcal{N}(0, I_N), \qquad \pi_X(\mathbf{x}) \propto \exp(-\frac{\|\mathbf{x}\|_2^2}{2})$$

Exercise 3

- Complete `exercise_2.py` from day4:
- Create a Python function that computes the log of (un-normalized) prior density.
- Create a Python function that computes the log of (un-normalized) likelihood density.
- Combine the two to create a Python function that computes the log of (un-normalized) posterior density.
- Use the random-walk Metropolis-Hastings algorithm to compute 10000 posterior samples. Choose the step-size c such that you have acceptance rate of 23%. [The Metropolis-Hastings must be adjusted to log]
- Plot the mean.
- For each pixel plot the variance of the samples, i.e., restrict all samples to a specific pixel and then estimate its variance.
- Plot the pixel-wise variance as a measure for uncertainty. Explain what you see.

Prior on Increments

JYVÄSKYLÄN YLIOPISTO

Prior on Increments

- Instead of specifying a prior on pixel intensities, we want to specify their relation to their neighbors, i.e,
$$\mathbf{x}_{i+1} - \mathbf{x}_i \sim \mathcal{N}(0,\alpha^2), \quad i = 1,\ldots,N-1.$$
- There are 2 approaches to implement this:

1. We can define the differences matrix:
$$D = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 & 1 \end{pmatrix} \text{ and } \pi_X(\mathbf{x}) \propto \exp(-\frac{\|D\mathbf{x}\|_2^2}{2\sigma_{\text{prior}}^2})$$
Remember boundary condition in D and prior parameters in σ_{prior}.

2. Or we can define increment variables $\mathbf{z}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$ and insert an integration matrix T in the likelihood:
$$T = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}, \ \pi_Z(z) \propto \exp\left(-\frac{\|\mathbf{z}\|_2^2}{2\sigma_{\text{prior}}^2}\right), \ \pi_{Y|Z}(\mathbf{y}_{\text{obs}}) \propto \exp\left(-\frac{\|AT\mathbf{z} - \mathbf{y}_{\text{obs}}\|}{2\sigma^2}\right),$$
Remember boundary condition in T and prior parameters in σ_{prior}.

Prior on Increments

Exercise (HW4)

- Fill-in the exercise in `exercise_3.py` for day 4.
- Implement the Gaussian prior on increments and repeat the sampling and uncertainty quantification like the previous exercise. What difference do you observe and why?
- Where are you the most confident in reconstruction?

Prior on Increments

Sparsity in Jump

- If we know that there are only a few changes in intensity we can modify the prior on the jumps:
  - Gaussian distribution:

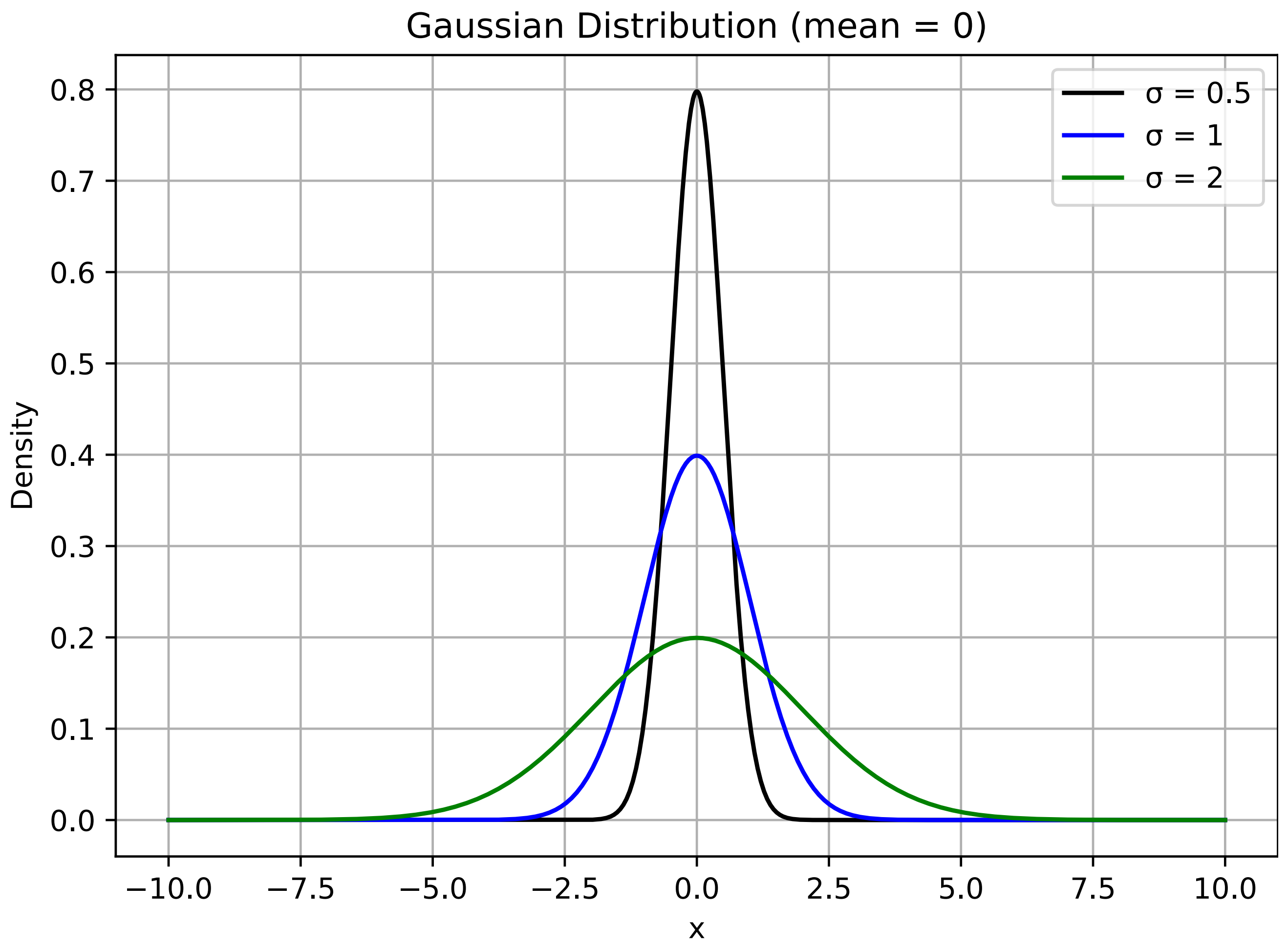
$$\pi_Z(\mathbf{z}) \propto \exp\left(-\frac{\|\mathbf{z}\|_2^2}{2\sigma^2_{prior}}\right)$$

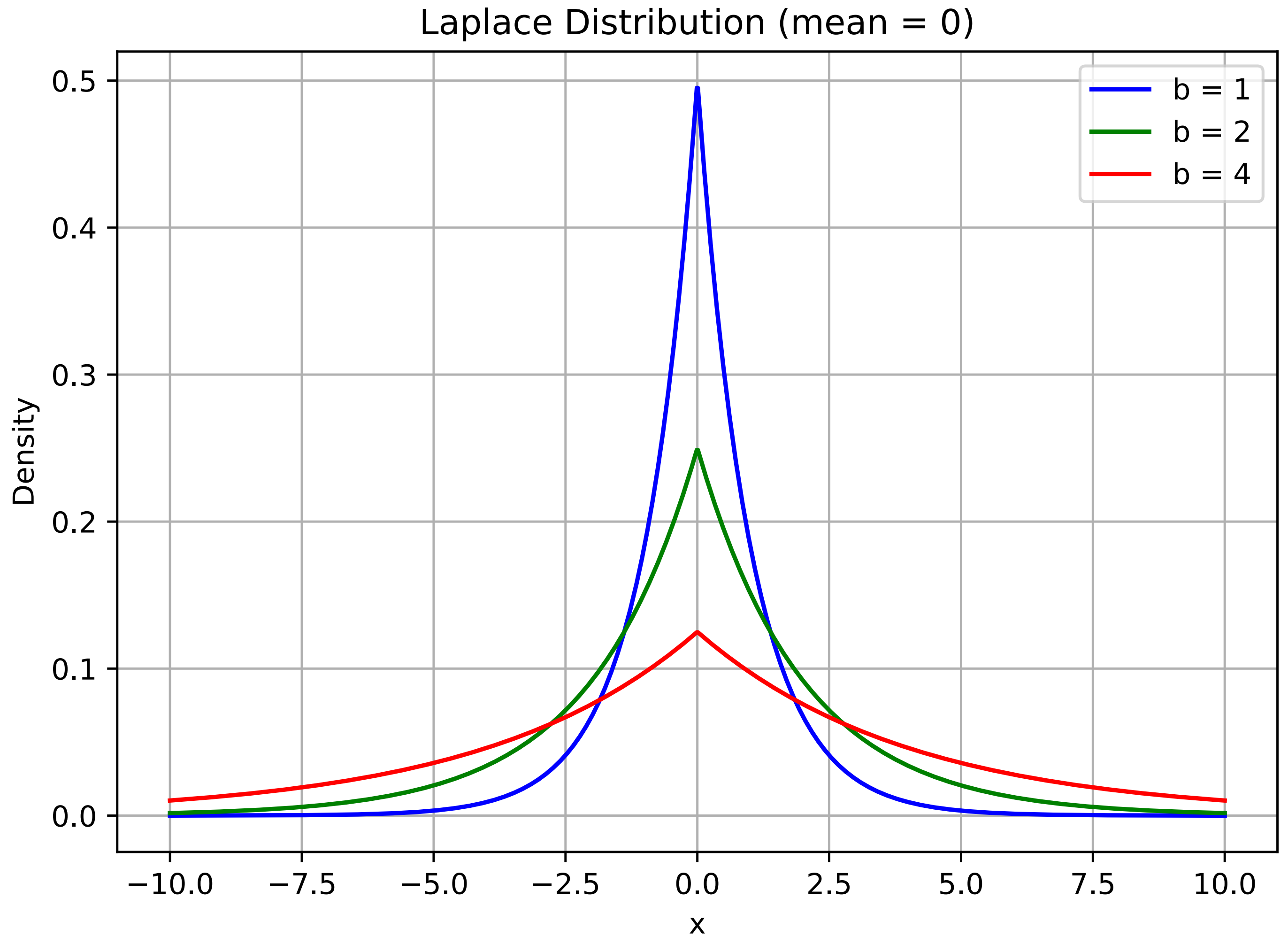
  - Laplace distribution:

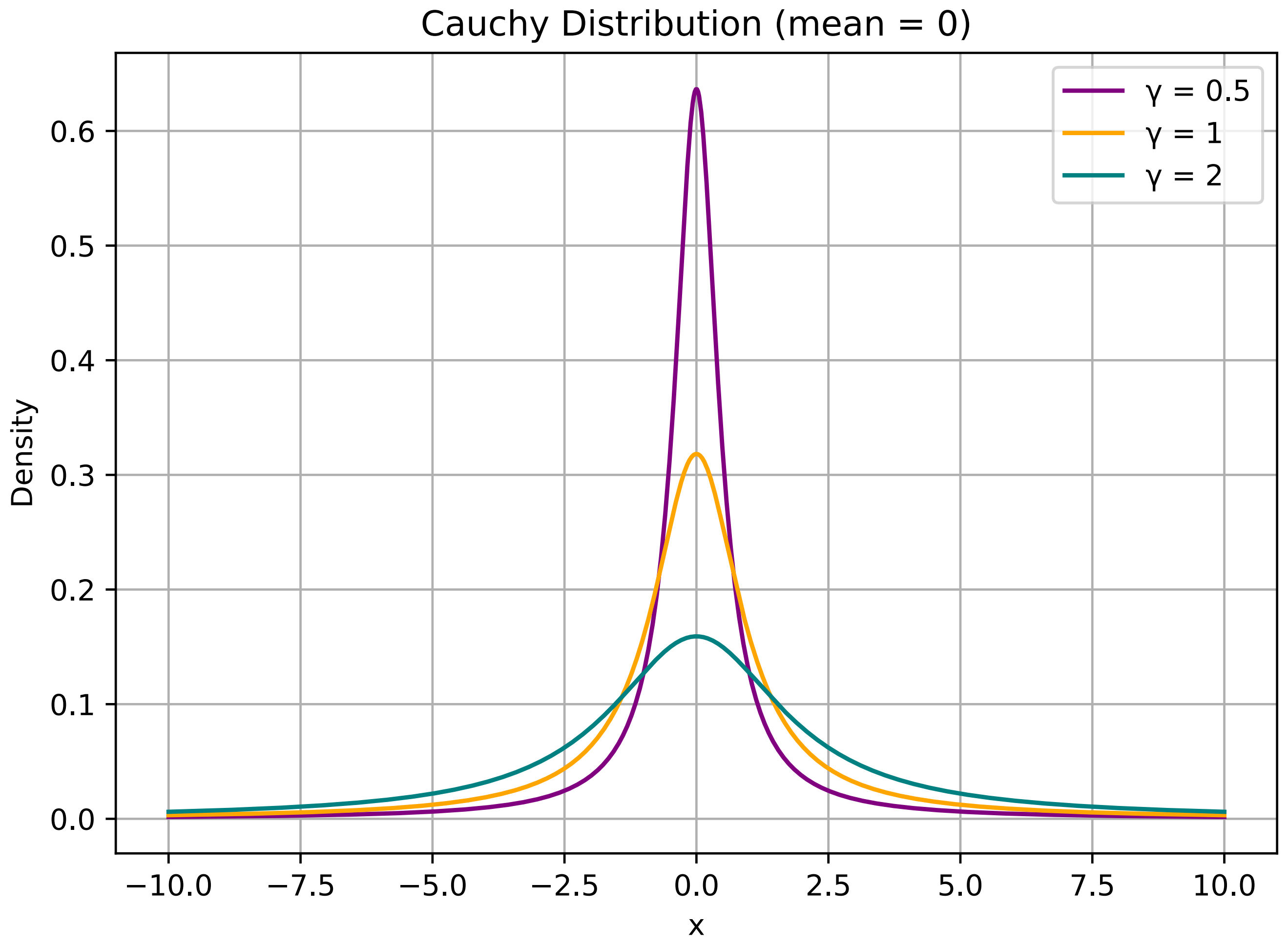
$$\pi_Z(\mathbf{z}) \propto \exp\left(-\frac{\|\mathbf{z}\|_1}{b}\right)$$

  - Cauchy distribution:

$$\pi_Z(\mathbf{z}) \propto \prod_i \frac{1}{1+\frac{\mathbf{z}_i^2}{\gamma^2}}$$







Prior on Increments

Exercise (HW4)

- Repeat the experiments with sparsity promoting Laplace and Cauchy priors. Tune the prior parameters to obtain desired characteristics in the reconstruction.

- Compare with the previous results.