Problem 1. *Consider the statistical inverse problem*

$$Y = f(X) + E, \tag{1}$$

where $X \in \mathcal{N}(0, \Sigma), E \sim \mathcal{N}(0, \sigma^2 I_m)$ *where* $\Sigma \in \mathbb{R}^{n\times n}$ *is a symmetric and positive definite covariance matrix,* I_m *is an* $m \times m$ *identity matrix,* $\sigma > 0$ *is the standard deviation of noise, and* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is the forward operator. Furthermore, we can write the probability density function of* X *and* E *as*

$$\pi_X(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} \exp\left(-\frac{1}{2}\mathbf{x}^T \Sigma^{-1} \mathbf{x}\right),$$
$$\pi_E(\mathbf{e}) = \frac{1}{\sqrt{(2\pi)^m \sigma^2}} \exp\left(-\frac{\mathbf{e}^T\mathbf{e}}{2\sigma^2}\right).$$

1. *Use Bayes' theorem to write the density function of the posterior distribution* $\pi_{X|Y=\mathbf{y}}(\mathbf{x})$ *up to a proportionality constant:*

   $$\pi_{X|Y=\mathbf{y}} \propto \cdots$$

   Assume that f *is smooth but not necessarily linear. Only express it in terms of* f, $\mathbf{x}$, $\mathbf{y}$, Σ, *and* σ.

2. *Show that the maximum a posteriori (MAP) estimate*

   $$\mathbf{x}_{MAP} := \arg\max_{\mathbf{x}} \ \pi_{X|Y=\mathbf{y}}(\mathbf{x})$$

   is equivalent to the solution of a Tikhonov regularized optimization problem:

   $$\mathbf{x}_{Tik} := \arg\min_{\mathbf{x}} \ \|f(\mathbf{x}) - \mathbf{y}\|_2^2 + \lambda\|\mathbf{x}\|^2,$$

   for some $\lambda > 0$. *Express* λ *in terms of* Σ *and* σ.