Energy-preserving Finite Element Scheme for the Dissipative Elastic Beam

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1 The Dissipative Elastic Beam

The equations governing small elastic deformations of an elastic beam, fixed on one side can be written as

$$
\begin{aligned}
\partial_t q &= p & q, p \in \Omega \\
\partial_t p + p - \nabla \cdot \sigma &= f, \\
\sigma &= \lambda \mathrm{tr}(\epsilon)I + 2\mu\epsilon, \\
\epsilon &= \frac{1}{2}\left(\nabla q + (\nabla q)^T\right), \\
u &= 0, & u \in \partial\Omega,
\end{aligned}
\tag{1}
$$

where $sigma$ is the stress tensor, f is the gravitational force per unit volume, λ and μ are Lamé's elasticity parameters for the material in Ω, I is the identity tensor, tr is the trace operator on a tensor, ϵ is the symmetric strain-rate tensor (symmetric gradient), and q and p are the displacement and velocity vector fields, respectively. We have here assumed isotropic elastic conditions.

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2 Variational Formulation

Here we assume that the solution to (1) belongs to a Hilbert space

$$V = \{u \in L^2 : \int_\Omega u^2 \ dx < \infty, \int_\Omega \|\nabla u\|^2 \ dx < \infty, u = 0 \text{ on } \partial\Omega\}/. \quad (2)$$

Further we assume that $(\cdot, \cdot)$ is the standard inner production on H. Formulating the weak form consist of computing the inner product of (1) with a test function $v \in H$. We can write

$$(\partial_t q, v) = (p, v),$$
$$(\partial_t p, v) + (p, v) - (\nabla \cdot \sigma, v) = (f, v). \quad (3)$$

By introducing the notion of tensorial inner product $\cdot : \cdot$, we can rewrite the latter as

$$(\partial_t q, v) = (p, v),$$
$$(\partial_t p, v) + (p, v) + \int_\Omega \sigma : \nabla v \ dx - \int_{\partial\Omega} (\sigma \cdot n) \cdot v \ dx = (f, v). \quad (4)$$

A finite element discretization of the latter can be obtained using standard methods.

3 Energy preserving extension

A time dissipative and dispersive formulation of equation 1 can be written as

$$\partial_t q = f \quad f, q, p \in \Omega$$
$$\partial_t p - \nabla \cdot \sigma = f,$$
$$f(t, x) + \int_0^t f(\tau, x) \ d\tau = p. \quad (5)$$

Following the footsteps of Figotin et al. 2007, the latter system can be reformulated as a conservative quadratic Hamiltonian system as

$$\partial_t q = f$$
$$\partial_t p - \nabla \cdot \sigma = f,$$
$$\partial_t \phi(t, x, s) = \theta(t, x, s),$$
$$\partial_t \theta(t, x, s) = \partial_s^2 \phi(t, x, s) + \sqrt{2}\delta_0(s)f(t, x), \quad (6)$$

together with the transfer function

$$f(t, x) + \int_0^t f(\tau, x) \, d\tau = p(t, x). \tag{7}$$

Where $\theta, \phi \in V \times \mathcal{H}$, and $\mathcal{H}$ is some appropriate Hilbert space. We equip the space $V \times \mathcal{H}$ with the inner product

$$[u, v] = \int_\Omega \int_{-\infty}^\infty uv \, dx \, d\xi, \qquad u, v \in V \times \mathcal{H}. \tag{8}$$

Note that the added equations correspond to a vibrating string that carries the dissipated energy of the original system in the direction of the added pseudo-space $\mathcal{H}$. This allows us to solve these equations exactly in terms of f

$$\phi(t, x, s) = \frac{\sqrt{2}}{2} \int_0^{t-|s|} f(\tau, x) \, d\tau, \quad \theta(t, x, s) = \frac{\sqrt{2}}{2} \int_0^{t-|s|} f(t - |s|). \tag{9}$$

Then the energy associated to the extended system (6) is

$$H(q, p, \phi, \theta) = \frac{1}{2} \left\{ \int_\Omega \sigma : \nabla q \, dx + \big(p - \phi(t, x, 0), p - \phi(t, x, 0)\big) + [\theta, \theta] + [\partial_s \phi, \partial_s \phi] \right\}. \tag{10}$$

4 Weak Formulation for the Extended System

We may form the weak formulation of the extended system by computing the inner product with a test function $v \in V$.

$$\begin{aligned}
(\partial_t q, v) &= (p, v), \\
(\partial_t p, v) + (p, v) - (\nabla \cdot \sigma, v) &= (f, v), \\
(\partial_t \phi, v) &= (\theta, v) \\
(\partial_t \theta, v) &= (\partial_s^2 \phi, v) + \sqrt{2} \delta_0(s)(f, v)
\end{aligned} \tag{11}$$

together with the transfer function

$$(f, v) + \int_0^t (f, v) \, d\tau = (p, v). \tag{12}$$

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Note that since the integration is only in the direction of the pseudo-space, then the integral can commute with the inner-product. We may use semi a discretization in the space $\mathcal{H}$ to compute the energy as

$$
\begin{aligned}
H(q, p, \phi, \theta) &= \frac{1}{2} \left\{ (\nabla q, \nabla q)_: + (p - \phi(0), p - \phi(0)) + [\partial_s \phi, \partial_s \phi] + [\theta, \theta] \right\} \\
&= \frac{1}{2} \Big\{ (\nabla q, \nabla q)_: + (p - \phi(0), p - \phi(0)) \\
&\quad + \sum_{i=1}^{N} \Delta s_i \int_{\Omega} (\partial_s \phi(t, x, s_i))^2 + (\theta(t, x, s_i)) \ dx \Big\} \\
&= \frac{1}{2} \Big\{ (\nabla q, \nabla q)_: + (p - \phi(0), p - \phi(0)) \\
&\quad + \sum_{i=1}^{N} \Delta s_i (\partial_s \phi, \partial_s \phi) \big|_{s=s_i} + \Delta s_i (\theta, \theta) \big|_{s=s_i} \Big\}
\end{aligned}
$$

$$(13)$$