

# Model Reduction of Finite Element Hamiltonian Systems With Respect to The Energy Norm

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## Abstract

Here we summarize the basic concepts on how we generalize the model reduction with respect to energy norm to Hamiltonian systems.

## 1 Preliminaries

### 1.1 Finite Element Formulation

Consider the wave equation:

$$\begin{aligned}\partial_t q - p &= 0, \\ \partial_t p - \Delta q &= f, \\ q(x, 0) &= q_0(x), \\ p(x, 0) &= p_0(x).\end{aligned}\tag{1}$$

defined on the domain  $\Omega \times [0, T] \subset \mathbb{R}^n \times \mathbb{R}$ . We assume that the solution  $(q, p)$  to the system of differential equation (1) belongs to  $H_{\text{per}}^1 \times H_{\text{per}}^1$  where

$$H_{\text{per}}^1 = \{u \in L^2 : \|\nabla u\| \in L^2 \text{ and } u \text{ is periodic on } \Omega\}.\tag{2}$$

We denote  $(\cdot, \cdot)$  to be  $L^2$  inner product. The solution to (1) also satisfies the weak form of finding  $(q, p)$  such that

$$\begin{aligned}(\partial_t q, u) - (p, u) &= 0, \\ (\partial_t p, v) + (\nabla q, \nabla v) &= 0.\end{aligned}\tag{3}$$

The semi-discrete mixed formulation of (4) is to find  $(q_h, p_h) : [0, T] \times [0, T] \rightarrow U_h \times V_h$  such that

$$\begin{aligned}(\partial_t q_h, u_h) - (p_h, u_h) &= 0, \\ (\partial_t p_h, v_h) + (\nabla q_h, \nabla v_h) &= 0.\end{aligned}\tag{4}$$

where  $U_h$  and  $V_h$  are finite dimensional linear subspaces of  $H_{\text{per}}^1$ . Let  $\{\phi_i\}_1^{\dim(U_h)}$  and  $\{\psi_i\}_1^{\dim(V_h)}$  be the basis functions for  $U_h$  and  $V_h$  respectively. We define the mass matrices

$$\begin{aligned}M_{i,j}^q &= (\phi_j, \phi_i), \\ M_{i,j}^p &= (\psi_j, \psi_i).\end{aligned}\tag{5}$$

Further we define the stiffness matrices

$$\begin{aligned}K_{i,j}^q &= (\phi_j, \psi_i), \\ K_{i,j}^p &= (\nabla \psi_j, \nabla \phi_i).\end{aligned}\tag{6}$$

The semi-discrete form (4) also satisfies the system of ordinary differential equations

$$\begin{aligned}M^q q_t - K^q p &= 0, \\ M^p p_t + K^p q &= M^p f.\end{aligned}\tag{7}$$

The energy corresponding to the Hamiltonian system (1) is defined by

$$H(q, p) = \frac{1}{2}(p, p) + \frac{1}{2}(\nabla q, \nabla q).\tag{8}$$

The Hamiltonian defines an inner product on  $(\cdot, \cdot)_H : H_{\text{per}}^1 \times H_{\text{per}}^1 \rightarrow \mathbb{R}$  denoted by

$$((q_1, p_1), (q_2, p_2))_H = \frac{1}{2}(p_1, p_2) + \frac{1}{2}(\nabla q_1, \nabla q_2),\tag{9}$$

and the corresponding energy norm  $\|(q, p)\|_H = \sqrt{H(q, p)}$ .

An essential feature of Hamiltonian systems is the conservation of the energy and how it evolves under numerical time-integration. For a solution

$(q, p)$  to the Hamiltonian equation (1) we have:

$$\begin{aligned}
\frac{d}{dt} \|(q, p)\|_H^2 &= \frac{1}{2} \left( \frac{d}{dt} (p, p) + \frac{d}{dt} (\nabla q, \nabla q) \right) \\
&= (\partial_t p, p) + (\nabla \partial_t q, \nabla q) \\
&= (\Delta q + f, p) + (\nabla q, \nabla q) \\
&= (f, p) - (\nabla q, \nabla q) + (\nabla q, \nabla q) \\
&= (f, p)
\end{aligned} \tag{10}$$

By taking the integral over  $[0, T]$  we obtain

$$H(T) = H(0) + \int_0^T (f, p) \, dt. \tag{11}$$

Now applying the Cauchy-Schwartz inequality yields,

$$\begin{aligned}
H(T) &\leq H(0) + \int_0^T \|f(\cdot, t)\|_{L^2} \cdot \|p(\cdot, t)\|_{L^2} \, dt \\
&\leq H(0) + \sup_{0 \leq t < T} (H) \cdot \int_0^T \|f(\cdot, t)\|_{L^2} \, dt
\end{aligned} \tag{12}$$

where the last inequality is due to the fact that the energy inner product defines a norm on  $H_{\text{per}}^1$ .

## 1.2 Energy Preservation in Semi-Discrete mixed Formulation

Let  $U_h \in U$  and  $V_h \in V$  be finite dimensional proper subspaces of  $U$  and  $V$  respectively. Furthermore, suppose that  $\pi_u : U \rightarrow U_h$  and  $p_v : V \rightarrow V_h$  be the  $L^2$  projection operators. By adding and subtracting  $(\pi_U q, u_h)$  and  $(\pi_V p, v_h)$  to the semi-discrete mixed formulation (4) we obtain

$$\begin{aligned}
(\dot{q}, u_h) + (\pi_U \dot{q}, u_h) - (\pi_U \dot{q}, u_h) - (p, u_h) + (\pi_V p, u_h) - (\pi_V p, u_h) &= 0 \\
(\dot{p}, v_h) + (\pi_V \dot{p}, v_h) - (\pi_V \dot{p}, v_h) + (\nabla q, \nabla v_h) + (\nabla \pi_U q, \nabla v_h) - (\nabla \pi_U q, \nabla v_h) &= (f, v_h)
\end{aligned} \tag{13}$$

Having in mind that  $q - \pi_U q$  and  $p - \pi_V p$  are orthogonal to  $U_h$  and  $V_h$ , respectively, we can omit many terms from above to obtain

$$\begin{aligned}
(\pi_U \dot{q}, u_h) - (\pi_V p, u_h) &= (\pi_U \dot{q} - \dot{q}, u_h) \\
(\pi_V \dot{p}, v_h) + (\nabla \pi_U q, \nabla v_h) &= (\pi_V \dot{p} - \dot{p}, v_h) + (f, v_h)
\end{aligned} \tag{14}$$

Now if we add the original weak form (4) to the above we retrieve

$$\begin{aligned} (\pi_U \dot{q} - \dot{q}_h, u_h) - (\pi_V p - p_h, u_h) &= (\pi_U \dot{q} - \dot{q}, u_h) \\ (\pi_V \dot{p} - \dot{p}_h, v_h) + (\nabla \pi_U q - \nabla q_h, \nabla v_h) &= (\pi_V \dot{p} - \dot{p}, v_h) \end{aligned} \quad (15)$$

We define new variables  $\theta = \pi_U q - q_h \in U_h$ ,  $\rho = \pi_V p - p_h \in V_h$ ,  $\mu = \pi_U q - q$  and  $\xi = \pi_V p - p$ . Then equation (15) turns into

$$\begin{aligned} (\dot{\theta}, u_h) - (\rho, u_h) &= (\dot{\mu}, u_h) \\ (\dot{\rho}, v_h) + (\nabla \theta, \nabla v_h) &= (\dot{\xi}, v_h) \end{aligned} \quad (16)$$

To bound this we use the energy norm

$$\begin{aligned} \frac{d}{dt} H(\theta, \rho)^2 &= \frac{d}{dt} \|(\theta, \rho)\|^2 = (\dot{\rho}, \rho) + (\nabla \dot{\theta}, \nabla \theta) \\ &= (\dot{\xi}, \rho) - (\nabla \theta, \nabla \rho) - (\rho, \Delta \theta) - (\dot{\mu}, \Delta \theta) \\ &= (\dot{\xi}, \rho) + (\nabla \dot{\mu}, \nabla \theta) \\ &= (\dot{\xi}, \rho) + (\dot{\mu}, \theta)_\nabla \end{aligned} \quad (17)$$

where  $(\cdot, \cdot)_\nabla = (\nabla \cdot, \nabla \cdot)$ . Finally by applying the Cauchy inequality we obtain

$$\begin{aligned} H^2(\theta, \rho)^2(T) &\leq \int_0^T \|\dot{\xi}\| \|\rho\| + \|\dot{\mu}\|_\nabla \|\theta\|_\nabla dt \\ &\leq \sqrt{2} \int_0^T \underbrace{(\|\rho\| + \|\theta\|_\nabla)}_{H(\theta, \rho)} (\|\dot{\xi}\| + \|\dot{\mu}\|_\nabla) dt \\ &\leq \sqrt{2} \cdot \sup_{0 \leq t \leq T} H(\theta, \rho) \int_0^T \|\dot{\xi}\| + \|\dot{\mu}\|_\nabla dt \end{aligned} \quad (18)$$

Here we used the fact that  $H(\theta, \rho)(0) = 0$ . And now by a theorem in [Symplectic-mixed finite element approximation of linear acoustic wave equations, Robert C. Kirby Thinh Tri Kieu] we get

$$H^2(\theta, \rho)(T) \leq \sqrt{2} \int_0^T \|\dot{\xi}\| + \|\dot{\mu}\|_\nabla dt \quad (19)$$

By assuming regularity on  $\dot{\xi}$  and  $\dot{\mu}$  and with theory of interpolation we can achieve several error estimations on the Finite Element solution.

### 1.3 Error Estimation in the Energy Norm

We define the error in the energy/Hamiltonian norm as

$$\epsilon(t) = \|(q - q_h, p - p_h)\|_H \quad (20)$$

Using the triangle inequality, we obtain

$$\sup \epsilon(t) \leq \underbrace{\sup \|\mu, \xi\|_H}_I + \underbrace{\sup \|\theta, \rho\|_H}_{II} \quad (21)$$

For part  $I$  we can directly apply error estimation in the theory of interpolation, under regularity assumptions on  $q$  and  $p$ . Also for part  $II$  we can apply the error estimate obtain in (19).

## 2 Model Reduction With Respect to the Energy Inner Product

The energy norm appears naturally in the error analysis of the finite element methods. Suppose that  $a(\cdot, \cdot)$  is the bilinear form corresponding to the variational formulation

$$a(u, v) = L(v), \quad u, v \in V \quad (22)$$

Where  $V$  is some appropriate Hilbert space. The finite element discretization of equation (22) is

$$a(u_h, v_h) = L(v_h), \quad u_h, v_h \in V_h \subset V \quad (23)$$

The energy inner product associated to (24) is defined as

$$(u_h, v_h)_a = a(u_h, v_h), \quad (24)$$

which implies the energy norm  $\|\cdot\|_a$ . In vector notation we have

$$(u_h, v_h)_a = \bar{u}^T X \bar{v} = u, \quad (25)$$

where  $\bar{u}$  and  $\bar{v}$  are expansion coefficients of  $u_h$  and  $v_h$  in the finite element basis, and  $X$  is a positive definite matrix, usually taken to be the stiffness

matrix. Note that we can rewrite an energy norm in terms of the 2-norm as  $\|\bar{u}\|_a = \|X^{1/2}u\|_2$ .

The energy inner product induces a projection. Energy projection of function  $u_h$  onto  $e$  reads

$$(u_h, e)_a \cdot e = \bar{u}_h^T X \bar{e} \cdot \bar{e} = \bar{e} \bar{e}^T X u_h. \quad (26)$$

Therefore the matrix  $\bar{e} \bar{e}^T X$  is the energy projection operator in the matrix notation. Now suppose that  $W = [w_1, \dots, w_k]$  is an expansion coefficients of a basis with respect to some finite element basis. Then the energy projection of a function  $s$  onto the span space of  $W$  would be  $WW^T X s$ . It is often desirable to find a basis  $W$  that minimizes the energy projection error of a set of functions  $\{s_1, \dots, s_N\}$ . We have

$$\begin{aligned} \min \sum_{i=1}^N \|s_i - WW^T X s_i\|_a &= \min \sum_{i=1}^N \|X^{1/2} s_i - X^{1/2} WW^T X s_i\|_2 \\ &= \min \sum_{i=1}^N \|\tilde{s}_i - \tilde{W} \tilde{W}^T \tilde{s}_i\|_2 \\ &= \min \|\tilde{S} - \tilde{W} \tilde{W}^T \tilde{S}\|_2. \end{aligned} \quad (27)$$

Where  $\tilde{W} = X^{1/2}W$  and  $\tilde{s}_i = X^{1/2}s_i$  and  $\tilde{S}$  is the matrix containing  $\tilde{s}_i$ . The Smidth-Mirsky theorem implies that the solution to the above minimization is the truncated singular value decomposition of the matrix  $\tilde{S}$ .

## 2.1 Symplectic-Model Reduction with respect to an Energy Inner Product

The solution of the minimization problem (27), the basis  $W$ , is not necessarily compatible with the time integration of choice, e.g., a symplectic time integrator. Note that the basis for the snapshots  $s_i$  is  $X^{-1/2}\tilde{W}$ . Therefore we require the matrix  $A = X^{-1/2}\tilde{W}$  to be a symplectic basis. This means

$$A^T J A = \tilde{W}^T X^{-1/2} J X^{-1/2} W = J \quad (28)$$

where  $J$  is the standard symplectic matrix. Now if we define the skew-symmetric matrix  $\tilde{J} = X^{-1/2} J X^{-1/2}$  then the minimization (27) turns into the constrained minimization

$$\begin{aligned} &\min \|\tilde{S} - \tilde{W} \tilde{W}^T \tilde{S}\|_2, \\ &\text{subject to } \tilde{W}^T \tilde{J} \tilde{W} = J. \end{aligned} \quad (29)$$