SYMPLECTIC MODEL-REDUCTION WITH A WEIGHTED INNER PRODUCT*

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Abstract. In the recent years, ~~a~~ considerable attention has been ~~geared~~ PAID toward~~s~~ preserving structures and invariants in reduced basis methods, IN ORDER ~~to~~ enhance the stability and ~~the~~ robustness of the reduced system. In the context of Hamiltonian systems, symplectic model reduction ~~is an approach that tends~~ SEEKS to construct a reduced order system that preserves the symplectic symmetry of Hamiltonian systems. However, the symplectic methods are based on the standard Euclidean inner products and are not suitable for problems that are equipped with a more general inner product. In this paper we generalize the symplectic model reduction method ~~such that it can adapt to the norms and the~~ TO ALLOW FOR THE inner products most appropriate to the problem while preserving the symplectic symmetry of Hamiltonian systems. To construct a reduced basis and accelerate evaluation of nonlinear terms, a greedy generation of a symplectic basis is proposed. Furthermore, it is shown that the greedy approach yields a norm bounded reduced basis. The accuracy and the stability of this model reduction technique is illustrated through the ~~simulations of~~ DEVELOPMENT OF REDUCED MODELS FOR a vibrating elastic beam and the sine-Gordon equation.

Key words. Structure Preserving, Weighted MOR, Hamiltonian Systems, Greedy Reduced Basis, Symplectic DEIM

AMS subject classifications. 78M34, 34C20, 35B30, 37K05, 65P10, 37J25

1. Introduction.

Reduced order models have emerged as a powerful approach to cope with ~~the evermore~~ INCREASINGLY complex new applications in engineering and science. These methods~~, specially the reduced basis (RB) methods~~, substantially reduce the dimensionality of the problem by constructing a reduced ~~order~~ configuration space. Exploration of the reduced space is then possible with significant acceleration [22, 19].

Over the past decade, ~~RB~~ REDUCED BASIS methods ~~had a great~~ HAVE DEMONSTRATED GREAT success in lowering the computational costs of solving elliptic and parabolic differential equations [23, 24]. However, model reduction of hyperbolic problems remains ~~one of the today's challenges~~ A CHALLENGE. ~~These~~ SUCH problems often arise from a set of conservation laws and invariants. ~~Such~~ THESE intrinsic structures are lost ~~over the course of~~ DURING model reduction which result in a qualitatively wrong, and sometimes unstable, reduced system [3].

Moreover, THE simulation of reduced models incurs solution error and THE estimation of this error is essential in ~~the~~ applications of model reduction [20, 36, 16]. Finding tight error bounds for ~~the~~ GENERAL reduced system has shown to be computationally expensive and often impractical. Therefore, when one is interested in a cheap surrogate for the error ~~incurred~~ or when the conserved quantity is an output of the system, it becomes imperative to preserve system structures ~~through model order reduction~~ IN THE REDUCED MODEL.

Recently, THE construction of RB methods that ~~can~~ conserve intrinsic structures has attracted ~~a vast~~ GREAT attention [2, 1, 25, 15, 7, 10, 6, 33]. Structure preservation in model reduction not only constructs a physically meaningful reduced system, but ~~it~~ can

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also enhance the robustness and stability of the reduced system. In system theory, conservation of passivity can be found in the work of [34, 18]. Energy preserving and inf-sup stable methods for finite element methods (FEM) are developed in [15, 4]. Also, a conservative model reduction technique for finite-volume methods ~~is seen~~ PROPOSED in [9].

In the context of Lagrangian and Hamiltonian systems, recent works provide a promising approach to THE constructing ION OF a robust and stable reduced system. Carlberg, Tuminaro, and Boggs [11] suggest that a reduced order model of a Lagrangian system ~~should~~ be identified by an approximate Lagrangian on a reduced order configuration space. This allows the reduced system to inherit the geometric structure of the original system. Similar approach has been adopted in the work of Maboudi Afkham and Hesthaven [2] and ~~also~~ in the work of Peng and Mohseni [33] for Hamiltonian systems. They construct a low-order symplectic linear vector space, i.e. a vector space equipped with a symplectic 2-form, as the reduced space. Once the symplectic reduced space is generated, a symplectic projection result in a physically meaningful reduced system. A proper time-stepping scheme can then preserve the Hamiltonian structure of the reduced system. It is shown in [2, 33] that this approach preserves the overall dynamics of the original system and ~~helps with~~ EMPHASIZES the stability of the reduced system. Despite the success of these method in model reduction of Hamiltonian systems, these ~~problems~~ TECHNIQUES are only compatible with the ~~standard Euclidean or the~~ L^2 inner product. Therefore, the computational structures that arise from a natural inner product of a problem will be lost ~~over~~ DURING model reduction.

CHANGE THE ORDER - [33] WAS FIRST

Weak formulations and inner-products, defined on a Hilbert space, are at the core of the error analysis of many numerical methods for solving partial differential equations. Therefore, it is natural to seek ~~for~~ model reduction methods that consider such features. At THE discrete level, these features often require a Euclidean vector space to be equipped with a generalized inner product associated with a weight matrix X. Many works ~~have been conducted to make~~ EVADES conventional model reduction techniques compatible with such inner products [37]. However, a model reduction method that simultaneously preserves the symplectic symmetry of Hamiltonian systems ~~is still~~ REMAINS unknown.

In this paper, we ~~attempt~~ SEEK to combine the classical model reduction methods with respect to a weight matrix with the symplectic model reduction. The reduced system constructed by the new method is a generalized Hamiltonian system and the low order configuration space associated with this system is a symplectic linear vector space with a non-standard symplectic 2-form. It is demonstrated that the new method can be viewed as the natural extension to [2], AND therefore, ~~it carries~~ RETAINS the structure preserving features, e.g. symplecticITY ~~ness~~ and stability. We also present a greedy approach for THE construct ION OF a generalized symplectic basis for the reduced system. ~~It is well known that~~ HOWEVER, structured matrices are in general not norm bounded [26]. We show that the condition number of the basis generated by the greedy method is bounded by the condition number of the weight matrix X. Finally, to accelerate the evaluation of THE nonlinear terms in the reduced system, we present a variation of the discrete empirical interpolation method (DEIM) that preserves the symplectic structure of the reduced system.

What remains of this paper is organized as follows. In section 2 we cover the required background on the Hamiltonian and the generalized Hamiltonian systems. Section 3 summarizes the classical model reduction routine with respect to a weighted norm and the symplectic model reduction method with respect to the standard Euclidean inner product. We introduce the symplectic model reduction method with respect to a weighted inner product in section 4. Section 5 illustrates the performance of the new method through ~~numerical simulation of the equations governing~~ a vibrating beam and the sine-Gordon equation. We offer conclusive remarks in section 6.

A ~~FEW~~

2. Hamiltonian systems. In this section we discuss the basic concepts ~~around~~ of the geometry of symplectic linear vector spaces and introduce Hamiltonian and Generalized Hamiltonian systems.

2.1. Generalized Hamiltonian systems. Let $(\mathbb{R}^{2n}, \Omega)$ be a symplectic linear vector space, with $\mathbb{R}^{2n}$ as the configuration space and $\Omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ a closed, skew-symmetric and non-degenerate 2-form on $\mathbb{R}^{2n}$. Given a smooth Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$, the *generalized Hamiltonian equation* of evolution reads

$$\begin{cases} \dot{z} = J_{2n} \nabla_z H(z), \\ z(0) = z_0. \end{cases} \tag{1}$$

J_{2n} DEPENDS ON z ?

Here $z \in \mathbb{R}^{2n}$ is the configuration coordinates and J_{2n} is a full-rank and skew-symmetric $2n \times 2n$ *structure* matrix such that $\Omega(x, y) = x^T J_{2n} y$, for all state vectors $x, y \in \mathbb{R}^{2n}$ [30]. Note that there always exists a coordinate transformation $\tilde{z} = T^{-1} z$, with $T \in \mathbb{R}^{2n \times 2n}$, such that J_{2n} takes the form of the *standard* symplectic structure matrix

T DEPENDS ON z – COMPUTABLE?

$$\mathbb{J}_{2n} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \tag{2}$$

in the new coordinate system [13]. Here 0_n and I_n are the zero matrix and the identity matrix of size $n \times n$, respectively. A central feature of Hamiltonian systems is conservation of the Hamiltonian ~~which we summarize in the following theorem.~~

THEOREM 2.1. *[30] Consider the flow $\phi_t : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ of the Hamiltonian system (1). Then $H \circ \phi_t = H$.*

Under a general coordinate transformation, the equations of evolution of a Hamiltonian system might not take the form (1). ~~It turns out that~~ INDEED only transformations which preserve the symplectic form, *symplectic transformations*, ~~also~~ preserve the form of a Hamiltonian system [21]. Suppose that $(\mathbb{R}^{2n}, \Omega)$ and $(\mathbb{R}^{2k}, \Lambda)$ are two symplectic linear vector spaces. A transformation $\alpha : \mathbb{R}^{2n} \to \mathbb{R}^{2k}$ is a symplectic transformation if

$$\Omega(x, y) = \Lambda(\alpha(x), \alpha(y)), \quad \text{for all } x, y \in \mathbb{R}^{2n}. \tag{3}$$

In matrix notation, i.e. when we ~~agree upon~~ CONSIDER a set of basis vectors for $\mathbb{R}^{2n}$ and $\mathbb{R}^{2k}$, a linear symplectic transformation is a matrix $A \in \mathbb{R}^{2n \times 2k}$ that satisfies

$$A^T J_{2n} A = J_{2k}. \tag{4}$$

We are interested in a class of symplectic transformations that transform a symplectic structure J_{2n} into the standard symplectic structure $\mathbb{J}_{2k}$.

DEFINITION 2.2. *Let $J_{2n} \in \mathbb{R}^{2n \times 2n}$ be a full-rank skew-symmetric structure matrix. A matrix $A \in \mathbb{R}^{2n \times 2k}$ is J_{2n}-symplectic if*

$$A^T J_{2n} A = \mathbb{J}_{2k}. \tag{5}$$

Note that in the literature [30, 21], symplectic transformations ~~are only~~ refer~~red~~ to $\mathbb{J}_{2n}$-symplectic matrices, in contrast to Definition 2.2.

It is natural to expect a numerical integrator that solves (1) to also satisfy the conservation law expressed in Theorem 2.1. Conventional numerical time integrators, e.g. the Runge-Kutta methods, do not generally PRE ~~conserve~~ the symplectic symmetry of Hamiltonian systems WHICH ~~and~~ often result in a wrong behavior of the solution over long time-integration. The class of time-integrators for (1) that preserve the Hamiltonian are called *Poisson integrators* [21]. To construct a general Poisson integrator, ~~first~~ we SEEK ~~find~~ a coordinate transformation $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $\tilde{z} = T^{-1}z$, such that $J_{2n} = T\mathbb{J}_{2n}T^T$. Then, a *symplectic integrator* ~~can~~ preserves the symplectic structure of the transformed system. The *Störmer-Verlet* scheme is an example of a second order symplectic time-integrator ~~given as~~

$$
(6)\qquad
\begin{aligned}
q_{m+1/2} &= q_m + \frac{\Delta t}{2} \cdot \nabla_p \tilde{H}(p_m, q_{m+1/2}),\\
p_{m+1} &= p_m - \frac{\Delta t}{2} \cdot \left(\nabla_q \tilde{H}(p_m, q_{m+1/2}) + \nabla_q \tilde{H}(p_{m+1}, q_{m+1/2}) \right),\\
q_{m+1} &= q_{m+1/2} + \frac{\Delta t}{2} \cdot \nabla_p \tilde{H}(p_{m+1}, q_{m+1/2}).
\end{aligned}
$$

Here, $\tilde{z} = (q^T, p^T)^T$, $\tilde{H}(\tilde{z}) = H(T^{-1}z)$ and the subscript m denotes ~~the~~ time-stepping. ~~index~~. Note that it is important to use a backward stable method to compute the transformation T. In this paper we use the symplectic Gaussian elimination method with ~~a~~ complete pivoting to compute the decomposition $J_{2n} = T\mathbb{J}_{2n}T^T$. However, one may use a more computationally efficient method, e.g., a Cholesky-like factorization proposed in [8] or the isotropic Arnoldi/Lanczos methods [31]. There are a few known numerical integrators that can preserve the symplectic symmetry of a generalized Hamiltonian system without requiring the computation of the transformation matrix T [21]. The implicit midpoint rule

$$
(7)\qquad z_{m+1} = z_m + \Delta t \cdot J_{2n} \nabla_z H \left(\frac{z_{m+1} + z_m}{2} \right),
$$

for (1) is an example of such integrators. For more on the construction and the applications of Poisson/symplectic integrators, we refer the reader to [21].

3. Model order reduction. In this section we summarize the fundamentals of model order reduction~~. We~~ AND discuss the conventional approach to model order reduction with a weighted inner product. We then RECALL ~~outline~~ the main results presented in [2] regarding symplectic model reduction. In section 4 we shall combine the two concepts to introduce the symplectic model reduction of Hamiltonian systems with respect to a weighted inner product.

3.1. Model-reduction with a weighted inner product. Consider a dynamical system of the form

$$
(8)\qquad
\begin{cases}
\dot{x}(t) = f(t, x),\\
x(0) = x_0.
\end{cases}
$$

where $x \in \mathbb{R}^m$ and $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is some continuous function. In this paper we ~~may~~ assume that time t is the only parameter ON WHICH ~~that~~ the solution vector x depends~~ on~~. Nevertheless, it is straight forward to generalize the findings of this paper to the case where x also depends on a LARGER set of ~~physical or geometrical~~ parameters that belong to a closed and bounded subset ~~of a Euclidean space.~~ IS WELL APPROXIMATED BY

Suppose that x ~~lies on or very close to~~ a low dimensional linear subspace with the basis $V = [v_1|\dots|v_k]$, $v_i \in \mathbb{R}^m$ for $i = 1, \dots, k$. The approximated solution to (8) in this basis reads

$$x = Vy, \tag{9}$$

where y ARE ~~is~~ the expansion coefficients of x in the basis V. Note that the ~~method of projecting~~ PROJECTION OF x onto the ~~span~~ SPAN(V) space of V depends on the inner product and the norm defined on (8). We define the weighted inner product

$$[x, y]_X = x^T X y, \quad \text{for all } x, y \in \mathbb{R}^m, \tag{10}$$

for some symmetric and positive-definite matrix $X \in \mathbb{R}^{m\times m}$. AND ~~We also~~ refer to $\|\cdot\|_X$ as the X-norm associated to this inner product. If we choose V to be an orthonormal basis with respect to the X-norm ($V^T X V = I_k$), then the operator

$$P_{X,V}(x) = VV^T Xx, \quad \text{for all } x \in \mathbb{R}^m \tag{11}$$

$VV^TXVy = VIy = Vy = x$

becomes idempotent, i.e. ~~This means that~~ $P_{X,V}$ is a projection operator onto the span space of V.

Now suppose that $\{x(t_i)\}_{i=1}^N$ is a collection of N ~~samples of the~~ solutions to (8) at instances $t_1, \dots, t_N$. We SEEK ~~would like to find~~ V such that it minimizes the collective projection error of the samples onto the span space of V. This corresponds to the minimization problem

$$\begin{aligned} &\underset{V\in\mathbb{R}^{m\times k}}{\text{minimize}} \quad \sum_{i=1}^{N} \|\overbrace{x_i - P_{X,V}(x_i)}^{x(t_i) - P_{X,V}x(t_i)}\|_X, \\ &\text{subject to} \quad V^T X V = I_k. \end{aligned} \tag{12}$$

Following ~~the derivations in~~ [35] the above minimization is equivalent to

$$\begin{aligned} &\underset{V\in\mathbb{R}^{m\times k}}{\text{minimize}} \quad \|\tilde{S} - \tilde{V}\tilde{V}^T\tilde{S}\|_2, \\ &\text{subject to} \quad \tilde{V}^T\tilde{V} = I_k. \end{aligned} \tag{13}$$

where $\tilde{V} = X^{1/2}V$, $\tilde{S} = X^{1/2}S$, S is the matrix of ~~that contains~~ samples $\{x(t_i)\}_{i=1}^N$ in its columns, referred to as the *snapshot matrix*, and $X^{1/2}$ is the matrix square root of X. According to Schmidt-Mirsky-Eckart-Young theorem [29] the solution $\tilde{V}$ to the minimization (13) is the truncated singular value decomposition (SVD) of $\tilde{S}$. The basis V can then be computed from $V = X^{-1/2}\tilde{V}$. The reduced ~~system~~ MODEL of (8) ~~to (8)~~ USING ~~corresponding~~ to the basis V and the projection $P_{X,V}$, ~~then~~ is

$$\begin{cases} \dot{y}(t) = V^T X f(t, Vy), \\ y(0) = V^T X x_0. \end{cases} \tag{14}$$

If k can be chosen such that $k \ll m$, then the reduced system (14) can potentially be evaluated significantly faster than the full order system (8). We refer the reader to [22, 35] for further information regarding the development and the efficiency of reduced order models.

It is worthwhile NOTING ~~to note~~ that the proper orthogonal decomposition (POD) method [22] corresponds to taking $X = I_m$. In this case, the projection $P_{I_m,V}$, constructed from the solution to (12), becomes the POD-Galerkin projection [22].

3.2. Symplectic model-reduction. Conventional model reduction methods, e.g. those introduced in subsection 3.1, do no generally preserve the conservation law expressed in Theorem 2.1. This often result in the lack of robustness in the reduced system over long time-integration. In this section we summarize the main findings of [2] regarding symplectic model order reduction of Hamiltonian systems with respect to the standard Euclidean inner product. Symplectic model reduction aims to construct a reduced system that conserves the geometric symmetry expressed in Theorem 2.1 which helps with the stability of the reduced system. Consider a Hamiltonian system of the form

$$\begin{cases} \dot{z}(t) = \mathbb{J}_{2n} L z(t) + \mathbb{J}_{2n} \nabla_z f(z), \\ z(0) = z_0. \end{cases} \tag{15}$$

Here $z \in \mathbb{R}^{2n}$ is the state vector, $L \in \mathbb{R}^{2n \times 2n}$ is a symmetric and positive-definite matrix and $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is some smooth enough function. Note that the Hamiltonian for system (15) is given by $H(z) = z^T L z + f(z)$. Suppose that the solution to (15) lies on a low dimensional symplectic subspace. Let $A \in \mathbb{R}^{2n \times 2k}$ be a $\mathbb{J}_{2n}$-symplectic basis containing the basis vectors $A = [e_1|\dots|e_k|f_1|\dots|f_k]$, such that $z = Ay$ with y the expansion coefficients of z in this basis. Using the symplectic inverse $A^+ = \mathbb{J}_{2k}^T A^T \mathbb{J}_{2n}$ we can construct the reduced system

$$\dot{y} = A^+ \mathbb{J}_{2n} (A^+)^T A^T L A y + A^+ \mathbb{J}_{2n} (A^+)^T \nabla_y f(Ay). \tag{16}$$

We refer the reader to [2] for the details of derivation. It is shown in [33] that $(A^+)^T$ is also $\mathbb{J}_{2n}$-symplectic, therefore $A^+ \mathbb{J}_{2n} (A^+)^T = \mathbb{J}_{2k}$ and (16) reduces to

$$\dot{y}(t) = \mathbb{J}_{2k} A^T L A y + \mathbb{J}_{2k} \nabla_y f(Ay). \tag{17}$$

This system is a Hamiltonian system with the Hamiltonian $\tilde{H}(y) = y^T A^T L A y + f(Ay)$. To reduced the complexity of evaluating the nonlinear term in (17), we may apply the discrete empirical interpolation method (DEIM) [5, 12]. Assuming that $\nabla_z f(z)$ lies on a low dimensional subspace with a basis U the DEIM approximation reads

$$\nabla_z f(z) \approx U (\mathcal{P}^T U)^{-1} \mathcal{P}^T \nabla_z f(z). \tag{18}$$

Here $\mathcal{P}$ is the interpolating index matrix [12]. For a general choice of U the approximation in (18) destroys the Hamiltonian structure in (17). It is shown in [2] that by taking $U = (A^+)^T$ we can recover the Hamiltonian structure in (17). Therefore, the reduced system to (15) becomes

$$\begin{cases} \dot{y}(t) = \mathbb{J}_{2k} A^T L A y + \mathbb{J}_{2k} (\mathcal{P}^T (A^+)^T)^{-1} \mathcal{P}^T \nabla_z f(Ay), \\ y(0) = A^+ z_0. \end{cases} \tag{19}$$

Note that the Hamiltonian formulation of (19) allows us to integrate it using a symplectic integrator. This conserves the symmetry expressed in Theorem 2.1 at the level of the reduced system. It is also shown in [2, 33] that the stability of the critical points of (15) is preserved in the reduced system and the difference of the Hamiltonians of the two system (15) and (19) is constant. Therefore, the overall behavior (19) is close to the full order Hamiltonian system (15). In the next subsection we discuss methods for generating a $\mathbb{J}_{2n}$-symplectic basis A.

3.3. Greedy generation of a $\mathbb{J}_{2n}$-symplectic basis. Suppose that S is the snapshot matrix containing the time instances $\{z(t_i)\}_{i=1}^N$ of the solution to (15). We ~~would like to find~~ SEEK the $\mathbb{J}_{2n}$-symplectic basis A such that the collective symplectic projection error of samples in S onto the span space of A is minimized.

$$\begin{aligned} &\underset{A\in\mathbb{R}^{n\times k}}{\text{minimize}} && \|S - P_{I,A}^{\text{symp}}(S)\|_2, \\ &\text{subject to} && A^T \mathbb{J}_{2n} A = I_{2k}. \end{aligned} \tag{20}$$

Here $P_{I,A}^{\text{symp}} = AA^+$ is the symplectic projection operator with respect to the standard Euclidean inner production onto the ~~span~~ space ~~of~~ SPANNED BY A. Note that $P_{I,A}^{\text{symp}} \circ P_{I,A}^{\text{symp}} = P_{I,A}^{\text{symp}}$ [33, 2].

Direct approaches to solve (20) are often inefficient. Some SVD-type solutions to (20) are proposed by [33]. However, these solutions are only suited for the standard Euclidean inner product and cannot be generalized to ~~be compatible with~~ GENERAL a weighted inner product.

The greedy generation of a $\mathbb{J}_{2n}$-symplectic basis aims to find a near optimal solution to (20) in an iterative ~~procedure~~ PROCESS. This method ~~enhances~~ INCREASES the overall accuracy of the basis by adding the best possible basis vectors at each iteration. Suppose that $A_{2k} = [e_1|\dots|e_k|f_1|\dots|f_k]$ is a $\mathbb{J}_{2n}$-symplectic basis. We ~~may~~ assume that A_{2k} is also an ortho-normal basis [2]. The first step of the greedy methods is to find the snapshot $z(t^{k+1})$, that is worst approximated by the basis A_{2k}:

$$t^{k+1} := \underset{t}{\text{argmax}}\ \|z(t) - P_{I,A}^{\text{symp}}(z(t))\|_2. \tag{21}$$

If e_{k+1} is the vector obtained by $\mathbb{J}_{2n}$-orthogonalizing $z(t^{k+1})$ with respect to A_{2k} [2], then the enriched basis A_{2k+2} reads

$$A_{2k+2} = [e_1|\dots|e_k|e_{k+1}|f_1|\dots|f_k|\mathbb{J}_{2n}^T e_{k+1}]. \tag{22}$$

It is easily checked that A_{2k+2} is $\mathbb{J}_{2n}$-symplectic and ortho-normal. We ~~point out~~ NOTE that the choice of THE orthogonalization routine generally depends on the application. In this paper we use the symplectic Gram-Schmidt (GS) process as the orthogonalization routine. However the isotropic Arnoldi method or the isotropic Lanczos method [31] are ~~among~~ backward stable alternatives.

THE Evaluation of the projection error is impractical for parameter-dependent problems. The loss in the Hamiltonian function can be used as a cheap surrogate to the projection error,

$$\omega^{k+1} = \underset{\omega\in\Omega}{\text{argmax}}\ |H(z(\omega)) - H(P_{I,A}^{\text{symp}}(z(\omega)))|. \tag{23}$$

– RELATION BETWEEN (21) AND (23) UNKNOWN

Here $\Omega \subset \mathbb{R}^d$ is a closed and bounded set of parameters ~~that~~ ON WHICH the original Hamiltonian system depends ~~on~~. It is shown in [2] that the loss in the Hamiltonian is constant in time. Therefore, ω^{k+1} can be identified prior to time integration.

We summarized the greedy algorithm for generating a $\mathbb{J}_{2n}$-symplectic basis in Algorithm 1. The first loop constructs a $\mathbb{J}_{2n}$-symplectic for the Hamiltonian system (15), and the second loop adds the nonlinear snapshots to the symplectic inverse of the basis. We refer the reader to [2] for more detail on the generation and properties of ~~a~~ THE symplectic basis. In section 4 we will show how this algorithm can be generalized to support any weighted inner product.

Algorithm 1 The greedy algorithm for generation of a $\mathbb{J}_{2n}$-symplectic basis

Input: Tolerated projection error δ, initial condition z_0

$t^1 \leftarrow t = 0$
$e_1 \leftarrow z_0$
$A \leftarrow [e_1 | \mathbb{J}_{2n}^T e_1]$
$k \leftarrow 1$
while $\|z(t) - P_{I,A}^{\text{symp}}(z(t))\|_2 > \delta$ for all $t \in [0,T]$
    $t^{k+1} := \underset{t\in[0,T]}{\text{argmax}}\ \|z(t) - P_{I,A}^{\text{symp}}(z(t))\|_2$
    $\mathbb{J}_{2n}$-orthogonalize $z(t^{k+1})$ to obtain e_{k+1}
    $A \leftarrow [e_1|\dots|e_{k+1}|\mathbb{J}_{2n}^T e_1|\dots,\mathbb{J}_{2n}^T e_{k+1}]$
    $k \leftarrow k+1$
end while
compute $(A^+)^T = [e_1'|\dots|e_k'|\mathbb{J}_{2n}^T e_1'|\dots|\mathbb{J}_{2n}^T e_k']$
while $\|\nabla f(z(t)) - P_{I,(A^+)^T}^{\text{symp}}(\nabla f(z(t)))\|_2 > \delta$ for all $t \in [0,T]$
    $t^{k+1} := \underset{t\in[0,T]}{\text{argmax}}\ \|\nabla f(z(t)) - P_{I,(A^+)^T}^{\text{symp}}(\nabla f(z(t)))\|_2$
    $\mathbb{J}_{2n}$-orthogonalize $\nabla f(z(t^{k+1}))$ to obtain e_{k+1}'
    $(A^+)^T \leftarrow [e_1'|\dots|e_{k+1}'|\mathbb{J}_{2n}^T e_1'|\dots|\mathbb{J}_{2n}^T e_{k+1}']$
    $k \leftarrow k+1$
end while
$A \leftarrow \left(\left((A^+)^T\right)^+\right)^T$

Output: $\mathbb{J}_{2n}$-symplectic basis A.

4. Symplectic model-reduction with a weighted inner product.

In this section we will combine the concept of model reduction with a weighted inner product, DISCUSSED in subsection 3.1 with the symplectic model reduction discussed in subsection 3.2. We will ~~discuss How~~ ARGUE THAT the new method can be viewed as a natural extension ~~to~~ of the original symplectic method. Finally we generalize the greedy method for the symplectic basis generation, and the symplectic model reduction of nonlinear terms to be compatible with any weighted inner product.

4.1. Generalization of the symplectic projection.

As discussed in subsection 3.1, ~~proper~~ THE error analysis of methods for solving partial differential equations often require ~~using~~ THE USE OF a weighted inner product. This is particularly important when dealing with Hamiltonian systems, where the system energy can induce a norm that is fundamental to the dynamics of the system.

Consider a Hamiltonian system of the form (15) together with the weighted inner product defined in (10) with $m = 2n$. Also suppose that the solution z to (15) ~~lies on~~ IS WELL APPROXIMATED BY a $2k$ dimensional symplectic subspace with the basis A. We ~~would like~~ SEEK to construct a projection operator that minimizes the projection error with respect to the X-norm while preserving the symplectic dynamics of (15) in the projected space. Consider the operator $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined as

$$P = A\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X. \tag{24}$$

It is easy to show that P is idempotent if and only if

$$\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X A = I_{2k}, \tag{25}$$

IN WHICH CASE ~~This means that~~ P is a projection operator onto the span space of A. Suppose that S is the snapshot matrix containing the time samples $\{z(t_i)\}_{i=1}^N$ of the solution to (15). We SEEK ~~like to find~~ the basis A that minimizes the projection error of the samples in S with respect to P~~,~~

$$\begin{aligned} &\underset{A\in\mathbb{R}^{2n\times 2k}}{\text{minimize}} && \|S - P(S)\|_X, \\ &\text{subject to} && \mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X A = I_{2k}. \end{aligned} \tag{26}$$

~~By~~ (24) we have

$$\begin{aligned} \|S - P(S)\|_X &= \|S - A\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X S\|_X \\ &= \|X^{1/2}S - X^{1/2}A\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X S\|_2 \\ &= \|\tilde{S} - \tilde{A}\tilde{A}^+\tilde{S}\|_2. \end{aligned} \tag{27}$$

Here $\tilde{S} = X^{1/2}S$, $\tilde{A} = X^{1/2}A$ and $\tilde{A}^+ = \mathbb{J}_{2k}^T\tilde{A}^T J_{2n}$ is the sympletic inverse of $\tilde{A}$ with respect to the skew-symmetric matrix $J_{2n} = X^{1/2}\mathbb{J}_{2n}X^{1/2}$. Note that the symplectic inverse in (27) is a generalization of the symplectic inverse introduced in subsection 3.2. Therefore, we may use the same notation (the superscript $+$) for both. We summarized the properties of this generalization in Theorem 4.1. With THIS ~~the introduced~~ notation~~s,~~ the condition (25) turns into $\tilde{A}^+\tilde{A} = I$ which is equivalent to $\tilde{A}^T J_{2n}\tilde{A} = \mathbb{J}_{2k}$. In other words, this condition implies that $\tilde{A}$ has to be a J_{2n}-symplectic matrix. Finally we can rewrite the minimization (26) as

$$\begin{aligned} &\underset{A\in\mathbb{R}^{2n\times 2k}}{\text{minimize}} && \|\tilde{S} - P_{X,\tilde{A}}^{\text{symp}}(\tilde{S})\|_2, \\ &\text{subject to} && \tilde{A}^T J_{2n}\tilde{A} = \mathbb{J}_{2k}. \end{aligned} \tag{28}$$

where $P_{X,\tilde{A}}^{\text{symp}} = \tilde{A}\tilde{A}^+$ is the symplectic projection with respect to the X-norm onto the span of $\tilde{A}$. At ~~the~~ first glance, the minimization (28) might look similar to (20). However, since $\tilde{A}$ is J_{2n}-symplectic, and the projection operator depends on X, we need to seek ~~for~~ AN alternative approach~~es~~ to find a near optimal solution to (28). AS ~~Similar to~~ (20), direct approaches to solving (28) are impractical. Furthermore, there are no SVD-type methods known to the authors, that solves (28). However, the greedy generation of the symplectic basis can be generalized to generate a near optimal basis $\tilde{A}$. The generalized greedy method is discussed in subsection 4.2.

Now suppose that a basis A that solves (28) is AVAILABLE ~~in hand~~ such that $z = Ay$ with $y \in \mathbb{R}^{2k}$~~,~~ BEING the expansion coefficients of z in the basis of A. Using (25) we may write the reduced system to (15) as

$$\dot{y} = \mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X \mathbb{J}_{2n} L A y + \mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X \mathbb{J}_{2n} \nabla_z f(z). \tag{29}$$

Since $\nabla_z H(z) = Lz + \nabla_z f(z)$, we may use the chain rule to write

$$\nabla_z H(z) = (\mathbb{J}_{2k}^T A^T X \mathbb{J}_{2n} X)^T \nabla_y H(Ay). \tag{30}$$

Finally the reduced system (29) ~~simplifies to~~ BECOMES

$$\begin{cases} \dot{y}(t) = J_{2k}A^T L A y + J_{2k}\nabla_y f(Ay), \\ y(0) = \tilde{A}^+ X^{1/2} z_0. \end{cases} \tag{31}$$

where $J_{2k} = \tilde{A}^{+} J_{2n} (\tilde{A}^{+})^T$ is a skew-symmetric matrix. The system (31) is a generalized Hamiltonian system with the Hamiltonian defined as $\tilde{H}(y) = y^T A^T L A y + f(Ay)$. Therefore, a Poisson integrator ~~can~~ preserves the symplectic symmetry associated with (31).

We close this section by summarizing the properties of the sympelctic inverse in ~~the form of~~ the following theorem.

THEOREM 4.1. *Let $A \in \mathbb{R}^{2n\times 2k}$ be a J_{2n}-symplectic basis where $J_{2n} \in \mathbb{R}^{2n}$ is a full rank and skew-symmetric matrix. Furthermore, suppose that $A^+ = \mathbb{J}_{2k}^T A^T J_{2n}$ is the symplectic psudo-inverse. Then the following holds:*

1. $A^+ A = I_{2k}$.
2. $(A^+)^T$ *is* J_{2n}^{-1}*-symplectic.*
3. $\left(\left((A^+)^T \right)^+ \right)^T = A.$
4. *Let $J_{2n} = X^{1/2} \mathbb{J}_{2n} X^{1/2}$. Then A is ortho-normal with respect to the X-norm, if and only if $(A^+)^T$ is ortho-normal with respect to the X^{-1}-norm.*

Proof. It is straight forward to show all statements using the definition of a symplectic basis. ◻

4.2. Greedy generation of a J_{2n}-symplectic basis.

In this section we modify the greedy algorithm introduced in subsection 3.3 to construct a J_{2n}-symplectic basis. Ortho-normalization is an essential step in ~~most~~ greedy approaches to basis generation [22, 35]. Here, we summarize a variation of the Gram-Schmidt orthogonalization process, known as the *symplectic Gram-Schmidt* process.

Suppose that Ω is a symplectic form defined on $\mathbb{R}^{2n}$ such that $\Omega(x,y) = x^T J_{2n} y$, for all $x, y \in \mathbb{R}^{2n}$ and some full rank and skew-symmetric matrix $J_{2n} = X^{1/2} \mathbb{J}_{2n} X^{1/2}$. We would like to build a basis of size $2k+2$ in an iterative manner~~.~~ AND ~~We~~ start with some initial vector, e.g. $e_1 = z_0$. It is known that a symplectic basis is even dimensional [30]. We may take Te_1, where $T = X^{-1/2} \mathbb{J}_{2n}^T X^{1/2}$ as a candidate for the second basis vector. It is easily checked that $\tilde{A}_2 = [e_1 | Te_1]$ is J_{2n}-symplectic and consequently, $\tilde{A}_2$ is the first basis generated by the greedy approach. Next, suppose that $\tilde{A}_{2k} = [e_1|\dots|e_k|Te_1|\dots|Te_k]$ is generated in THE k-th step of the greedy method and $z \notin \text{span}\left(\tilde{A}_{2k}\right)$ is provided. We aim to J_{2n}-orthogonalize z with respect to the basis $\tilde{A}_{2k}$. This means we ~~should find~~ SEEK coefficients $\alpha_i, \beta_i \in \mathbb{R}$, for $i = 1, \dots, k$ such that

$$\Omega\left(z + \sum_{i=1}^{k} \alpha_i e_i + \sum_{i=1}^{k} \beta_i T e_i, \sum_{i=1}^{k} \bar{\alpha}_i e_i + \sum_{i=1}^{k} \bar{\beta}_i T e_i \right) = 0, \tag{32}$$

for all possible $\bar{\alpha}_i, \bar{\beta}_i \in \mathbb{R}$, $i = 1, \dots, k$. It is easily checked that this problem has the unique solution

$$\alpha_i = -\Omega(z, Te_i), \quad \beta = \Omega(z, e_i). \tag{33}$$

If we take $\tilde{z} = z - \sum_{i=1}^{k} \Omega(z, Te_i) e_i + \sum_{i=1}^{k} \Omega(z, e_i) Te_i$, then the next candidate pair of basis vectors are $e_{k+1} = \tilde{z} / \|\tilde{z}\|_X$ and Te_{k+1}. Finally, the basis generated at the $(k+1)$th step of the greedy method is ~~given by~~

$$\tilde{A}_{2k+2} = [e_1|\dots|e_k|e_{k+1}|Te_1|\dots|Te_k|Te_{k+1}]. \tag{34}$$

It is checked easily that $\tilde{A}_{2k+2}$ is J_{2n}-symplectic. We ~~point out~~ NOTE that the symplectic Gram-Schmidt orthogonalization process is chosen due to its simplicity. However, in problems where there is a need for a large basis, this process might be impractical. In such cases, one may use a backward stable routine, e.g. the isotropic Arnoldi method or the isotropic Lanczos method [31].

It is well known that symplectic ~~bases~~, in general, ~~are~~ IS not norm bounded [27]. The following theorem guarantees that the greedy method for generating a J_{2n}-symplectic basis yields a bounded basis.

THEOREM 4.2. *The basis generated by the greedy method for constructing a J_{2n}-symplectic basis is ortho/normal with respect to the X-norm.*

Proof. Let $\tilde{A}_{2k} = [e_1|\dots,e_k|Te_1|\dots|Te_k]$ be the J_{2n}-symplectic basis generated at the kth step of the greedy method. Using the fact that $\tilde{A}_{2k}$ is J_{2n}-symplectic, one can check that

$$[e_i,e_j]_X = [Te_i,Te_j]_X = \Omega(e_i,Te_j) = \delta_{i,j}, \quad i,j=1,\dots,k, \tag{35}$$

and

$$[e_i,Te_j]_X = \Omega(e_i,e_j) = 0 \quad i,j=1,\dots,k, \tag{36}$$

where $\delta_{i,j}$ is the Kronecker delta function. This ~~shows~~ ENSURES that $\tilde{A}_{2k}^T X \tilde{A}_{2k} = I_{2k}$, i.e., $\tilde{A}_{2k}$ is an ortho/normal basis with respect to the X-norm. □

We ~~point out~~ NOTE that if we take $X = I_{2n}$, then the greedy process generates a $\mathbb{J}_{2n}$-symplectic basis. ~~As the matter of fact,~~ With this choice, the greedy method discussed above becomes identical to the greedy process discussed in subsection 3.3, since $T = X^{-1/2}\mathbb{J}_{2n}^T X^{1/2} = \mathbb{J}_{2n}^T$.

For identifying the best vectors to be added to a set of basis vectors, we may use similar error functions to those introduced in subsection 3.3. The projection error can be used to identify the snapshot that is worst approximated by a given basis $\tilde{A}_{2k}$:

$$t^{k+1} := \underset{t}{\text{argmax}}\ \|X^{1/2}z(t) - P^{\text{symp}}_{X,\tilde{A}_{2k}}(X^{1/2}z(t))\|_2. \tag{37}$$

~~Or a~~ Alternatively we can use the loss in the Hamiltonian function in (23) for parameter dependent problems. We summarized the greedy method for generating a J_{2n}-symplectic matrix in Algorithm 2.

It is shown in [2] that under natural assumptions on the solution manifold of (15), the original greedy method for symplectic basis generation converges exponentially fast. We expect similar convergence rate for the generalized greedy method, since the X-norm is topologically equivalent to the standard Euclidean norm [17], for a full rank matrix X.

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4.3. Efficient evaluation of nonlinear terms. THE ~~Evaluating~~ the nonlinear term CUT in (31) still ~~contain~~ RETAINS a computational complexity proportional to the size of the full order system (15). To overcome this inefficiency, we ~~may~~ take AN ~~a similar~~ approach SIMILAR to subsection 3.2. The DEIM approximation of the nonlinear term in (31) yields

$$\dot{y} = J_{2k}A^T LAy + \tilde{A}^+ X^{1/2}\mathbb{J}_{2n}U(\mathcal{P}^T U)^{-1}\mathcal{P}^T \nabla_z f(z). \tag{38}$$

Here U is a basis for the nonlinear snapshots $\{\nabla_z f(z(t_i))\}_{i=1}^N$, and $\mathcal{P}$ is the interpolating index matrix [12]. As discussed in subsection 3.2, for a general choice of U, the reduced system (31) does not ~~take~~ RETAIN a Hamiltonian form. Applying the chain rule on (38) gives

$$\dot{y} = J_{2k}A^T LAy + \tilde{A}^+ X^{1/2}\mathbb{J}_{2n}U(\mathcal{P}^T U)^{-1}\mathcal{P}^T X^{1/2}(\tilde{A}^+)^T \nabla_y f(Ay). \tag{39}$$

Algorithm 2 The greedy algorithm for generation of a J_{2n}-symplectic basis

Input: Tolerated projection error δ, initial condition z_0, the snapshots $\{\tilde{z}(t_i)\}_{i=1}^N = \{X^{1/2}z(t_i)\}_{i=1}^N$, full rank matrix $X = X^T > 0$

$T \leftarrow X^{-1/2}\mathbb{J}_{2n}^T X^{1/2}$
$t^1 \leftarrow t = 0$
$e_1 \leftarrow X^{1/2} z_0$
$\tilde{A} \leftarrow [e_1 | T e_1]$
$k \leftarrow 1$
while $\|\tilde{z}(t) - P^{\text{symp}}_{X,\tilde{A}}(\tilde{z}(t))\|_2 > \delta$ for all $t \in [0, T]$
  $t^{k+1} := \underset{t\in[0,T]}{\text{argmax}} \|\tilde{z}(t) - P^{\text{symp}}_{X,\tilde{A}}(\tilde{z}(t))\|_2$
  J_{2n}-orthogonalize $\tilde{z}(t^{k+1})$ to obtain e_{k+1}
  $\tilde{A} \leftarrow [e_1|\dots|e_{k+1}|Te_1|\dots|Te_{k+1}]$
  $k \leftarrow k + 1$
end while
$A \leftarrow X^{-1/2}\tilde{A}$

Output: J_{2n}-symplectic basis $\tilde{A}$ and the reduced basis A

~~Now~~ If we require $U = X^{1/2}(\tilde{A}^+)^T$ then the complex expression in (39) reduces to

$$\dot{y} = J_{2k}A^T L A y + J_{2k}\nabla_y f(Ay), \tag{40}$$

and hence we recover the Hamiltonian structure. This ~~gives rise to~~ YIELDS the reduced system

$$\begin{cases} \dot{y}(t) = J_{2k}A^T L A y + J_{2k}(\mathcal{P}^T X^{1/2}(\tilde{A}^+)^T)^{-1}\mathcal{P}^T\nabla_z f(z), \\ y(0) = \tilde{A}^+ X^{1/2} z_0. \end{cases} \tag{41}$$

We now discuss how to ensure THAT $X^{1/2}(\tilde{A}^+)^T$ ~~to be~~ IS a basis for the nonlinear snapshots. Note that if $z \in \operatorname{span}\left(X^{1/2}(\tilde{A}^+)^T\right)$ then $X^{-1/2}z \in \operatorname{span}\left((\tilde{A}^+)^T\right)$. Therefore, it is sufficient to require $(\tilde{A}^+)^T$ to be a basis for $\{X^{-1/2}\nabla_z f(z(t_i))\}_{i=1}^N$. Theorem 4.1 SUGGESTS ~~indicates~~ that $(\tilde{A}^+)^T$ is a J_{2n}^{-1}-symplectic basis and that the transformation between $\tilde{A}$ and $(\tilde{A}^+)^T$ does not affect the symplectic feature of the bases. Consequently, from A we may compute $(\tilde{A}^+)^T$ and enrich it with snapshots $\{X^{-1/2}\nabla_z f(z(t_i))\}_{i=1}^N$. Once $(\tilde{A}^+)^T$ represents the nonlinear term with the desired accuracy, we may compute $\tilde{A} = \left(\left((\tilde{A}^+)^T\right)^+\right)^T$ to obtain the reduced basis for (41). Note that Theorem 4.1 implies that $(\tilde{A}^+)^T$ is orthonormal with respect to the X^{-1}-norm. This affects the ortho-normalization process. We summarize ~~the~~ process of generating a basis for the nonlinear terms in Algorithm 3.

- ACCURACY WHEN $U = X^{1/2}(A^+)^T$?

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4.4. Offline/online decomposition. Model order reduction becomes particularly useful for parameter dependent problems or in multi-query settings. For the purpose of most efficient computation, it is important to delineate high dimensional $(\mathcal{O}(n))$ OFFLINE computations from low dimensional ONLINE $(\mathcal{O}(k))$ ones ~~in the implementation phase~~. Time intensive high dimensional quantities are computed only once for a given problem in the offline phase and THE cheaper low dimensional computations can be performed in the online phase. This segregation or compartmentalization of quantities, according to their computational cost, is referred to as the offline/online decomposition.

More precisely, one can decompose the computations into THE following stages:

Algorithm 3 Generation of a basis for nonlinear terms

Input: Tolerated projection error δ, J_{2n}-symplectic basis $\tilde{A}$ of size $2k$, the snapshots $\{\tilde{z}(t_i)\}_{i=1}^N = \{X^{-1/2}\nabla_z f(z(t_i))\}_{i=1}^N$, full rank matrix $X = X^T > 0$

1. $T \leftarrow X^{1/2}\mathbb{J}_{2n}^T X^{-1/2}$
2. compute $(\tilde{A}^+)^T$
3. **while** $\|\tilde{z}(t) - P^{\text{symp}}_{X^{-1},(\tilde{A}^+)^T}(\tilde{z}(t))\|_2 > \delta$ for all $t \in [0,T]$
4. $\quad t^{k+1} := \underset{t\in[0,T]}{\text{argmax}}\ \|\tilde{z}(t) - P^{\text{symp}}_{X^{-1},(\tilde{A}^+)^T}(\tilde{z}(t))\|_2$
5. $\quad J_{2n}^{-1}$-orthogonalize $\tilde{z}(t^{k+1})$ to obtain e_{k+1}
6. $\quad (\tilde{A}^+)^T \leftarrow [e_1|\dots|e_{k+1}|Te_1|\dots|Te_{k+1}]$
7. $\quad k \leftarrow k+1$
8. **end while**
9. $\tilde{A} \leftarrow \left(\left((\tilde{A}^+)^T\right)^+\right)^T$

Output: J_{2n}-symplectic basis $\tilde{A}$

Offline stage: Quantities in this stage are computed only once and then used in the online stage.

1. Generate the weighted snapshots $\{X^{1/2}z(t_i)\}_{i=1}^N$ and the weighted snapshots of the nonlinear term $\{X^{-1/2}.\nabla_z f(z(t_i))\}_{i=1}^N$
2. Generate a J_{2n}-symplectic basis for the solution snapshots and the snapshots of the nonlinear terms FOLLOWING ~~according~~ Algorithms 2 and 3, respectively.
3. Assemble the reduced order model (19).

THE *Online stage:* Reduced model (19) is SOLVED ~~simulated~~ for multiple parameter sets and THE desired output is extracted ~~in this stage~~.

~~We use this decomposition to perform experiments in the next section.~~

5. Numerical results.

LET US NOW ~~In this section we~~ discuss the performance of the symplectic model reduction with a weighted inner product. In subsection 5.1 we apply this model reduction ~~method~~ to the equations of a vibrating elastic beam. AND WE ~~We~~ examine the evaluation of nonlinear terms in the model reduction of the sine-Gordon equation, in section subsection 5.1.

5.1. The elastic beam equation.

Consider the equations governing small deformations of a clamped elastic body $\Gamma \subset \mathbb{R}^3$ as

$$
(42)\qquad \begin{cases}
u_{tt}(t,x) = \nabla\cdot\sigma + f, & x\in\Gamma,\\
u(0,x) = \vec{0}, & x\in\Gamma,\\
\sigma\cdot n = T, & x\in\partial\Gamma_T,\\
u(t,x) = 0, & x\in\partial\Gamma\backslash\partial\Gamma_T,
\end{cases}
$$

PERHAPS A DIFFERENT LETTER AS T IS USED IN TRANSFORMATION?

and

$$
(43)\qquad \sigma = \lambda(\nabla\cdot u)I + \mu(\nabla u + (\nabla u)^T).
$$

Here u is the displacement vector field defined on Γ, subscript t denotes derivative with respect to time, σ is the stress tensor, f is the body force per unit volume, λ and μ are Lamé's elasticity parameters for THE material in Γ, I is the identity tensor, n

is the outward unit normal vector at the boundary and $T : \partial\Gamma_T \to \mathbb{R}^3$ is the traction at the boundary $\partial\Gamma_T$ [28].

AND

We define a vector valued function space ~~where we~~ seek ~~for~~ the solution to (42) as $V = \{u \in L^2(\Gamma) : \|\nabla u\|_2 \in L^2 \text{ and } u = \vec{0} \text{ on } \partial\Gamma_T\}$, equipped with the standard L^2 inner product $(\cdot,\cdot) : V \times V \to \mathbb{R}$. To formulate the weak formulation of (42), we multiply it with the vector valued test function $v \in V$ and integrate over Γ

$$\int_\Gamma u_{tt} \cdot v \; dx = \int_\Gamma (\nabla \cdot \sigma) \cdot v \; dx + \int_\Gamma f \cdot v \; dx. \tag{44}$$

We may integrate the first term on the right hand side by parts to obtain

$$\int_\Gamma u_{tt} \cdot v \; dx = -\int_\Gamma \sigma : \nabla v \; dx + \int_{\partial\Gamma_T} (\sigma \cdot n) \cdot v \; ds + \int_\Gamma f \cdot v \; dx, \tag{45}$$

where $\sigma : \nabla v$ is the tensor inner product. Note that the skew-symmetric part of ∇v vanishes over the product $\sigma : \nabla v$, since σ is symmetric. By prescribing the boundary conditions to (45) we recover

$$\int_\Gamma u_{tt} \cdot v \; dx = -\int_\Gamma \sigma : \mathrm{Sym}(\nabla v) \; dx + \int_{\partial\Gamma_T} T \cdot v \; ds + \int_\Gamma f \cdot v \; dx, \tag{46}$$

IS

with $\mathrm{Sym}(\nabla v) = (\nabla v + (\nabla v)^T)/2$. The variational form associated to (42) ~~then reads~~

$$(u_{tt}, v) = -a(u,v) + L(v), \quad u, v \in V, \tag{47}$$

where

$$\begin{aligned} a(u,v) &= \int_\Gamma \sigma : \mathrm{Sym}(\nabla v) \; dx, \\ L(v) &= \int_{\partial\Gamma_T} T \cdot v \; ds + \int_\Gamma f \cdot v \; dx. \end{aligned} \tag{48}$$

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To obtain the FEM discretization of (47), we ~~discretize~~ the domain Γ ~~into a triangulated mesh, which discretizes the domain into a set of disjoint tetrahedrons. Furthermore,~~ we define vector valued piece-wise linear basis functions $\{\phi_i\}_{i=1}^{N_h}$, referred to as the *hat functions*, ~~and~~ THE define the FEM space V_h as an approximation of V over these set of basis functions. Projecting (47) onto V_h ~~gives us~~ the discretized weak form

YIELDS

$$((u_h)_{tt}, v_h) = -a(u_h, v_h) + L(v_h), \quad u_h, v_h \in V_h. \tag{49}$$

Any particular function u_h can be expressed as $u_h = \sum_{i=1}^{N_h} q_i \phi_i$, where q_i, $i = 1, \dots, N_h$, are the expansion coefficients. Therefore, by choosing test functions $v_h = \phi_i$, $i = 1, \dots, N_h$, we obtain the ordinary differential equation

$$M\ddot{q} = -Kq + g_q. \tag{50}$$

where $q = (q_1, \dots, q_{N_h})^T$, $M \in \mathbb{R}^{N_h \times N_h}$ is given as $M_{i,j} = (\phi_i, \phi_j)$, $K \in \mathbb{R}^{N_h \times N_h}$ is given as $K_{i,j} = a(\phi_i, \phi_j)$ and $g_q = (L(v_1), \dots, L(v_{N_h}))^T$. ~~We~~ now introduce the canonical coordinate $p = M\dot{q}$ to recover the Hamiltonian system

$$\dot{z} = \mathbb{J}_{2N} Lz + g_{qp}, \tag{51}$$

TOO MANY L

where

$$z = \begin{pmatrix} q \\ p \end{pmatrix}, \quad L = \begin{pmatrix} K & 0 \\ 0 & M^{-1} \end{pmatrix}, \quad g_{qp} = \begin{pmatrix} 0 \\ g_p \end{pmatrix}, \tag{52}$$

together with the Hamiltonian function $H(z) = z^T L z + z^T \mathbb{J}_{2N_h}^T g_{qp}$. An appropriate FEM setup leads to a symmetric and positive-definite matrix L. ~~Therefore~~, HENCE it ~~seams~~ natural to take $X = L$, the energy matrix associated to (51). System parameters are summarized in the table below. For further information regarding the problem ~~set up~~, we refer ~~the reader~~ to [28].

Domain shape	box: $l_x = 1,\ l_y = 0.2,\ l_z = 0.2$
No. mesh points	$n_x = 10,\ n_y = 4,\ n_z = 4$
Time discretization size	$\Delta t = 0.01$
Gravitational force	$g = (0, 0, -0.4)^T$
Traction	$T = \vec{0}$
Lamé parameters	$\lambda = 1.25,\ \mu = 1.0$
Degree of freedom	$2N_h = 1650$

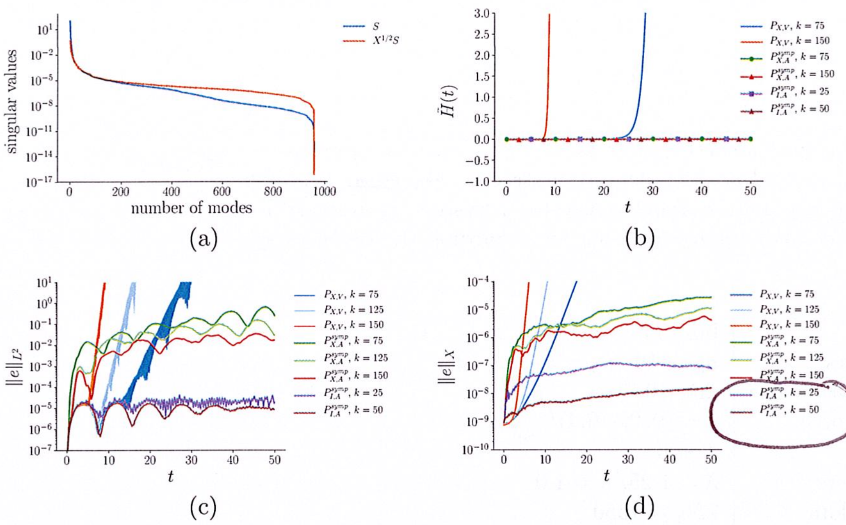
We used the Störmer-Verlet scheme (6) to integrate (51) in time and generate the TEMPORAL ~~time~~ snapshots. Projection operators $P_{X,V}$, $P_{I,A}^{\text{symp}}$ and $P_{X,\tilde{A}}^{\text{symp}}$ are ~~then~~ constructed FOLLOWING ~~according to~~ subsections 3.1 to 3.3, respectively. After computing the transformation $J_{2k} = T \mathbb{J}_{2k} T^T$, the reduced systems CONSTRUCTED ~~resulted~~ from $P_{I,A}^{\text{symp}}$ and $P_{X,\tilde{A}}^{\text{symp}}$, are integrated RECOVERED in time using the Störmer-Verlet scheme. The reduced system ~~yielded~~ by $P_{X,V}$ is integrated using a second order implicit Runge-Kutta method. Note that the Störmer-Verlet scheme is not used since the canonical form of a Hamiltonian system is destroyed when ~~applying~~ $P_{X,V}$, IS APPLIED.

Figure 1(a) shows the decay of the singular values of the TEMPORAL ~~time~~ snapshots S and $X^{1/2}S$, respectively. We notice that the decay of the two matrices are different and ~~as~~ the ~~matter of fact the~~ decay saturates for $X^{1/2}S$.

Figure 1(b) shows the conservation of the Hamiltonian for all the methods ~~mentioned~~ DISCUSSED above. ~~It is observed~~ CONFIRMING that the symplectic methods ~~do~~ preserve the Hamiltonian and THE system energy. However, the Hamiltonian blows up for the reduced system constructed by the projection $P_{X,V}$. It is also noticeable that as the basis V gets larger, the Hamiltonian blows up faster.

Figure 1(c) shows the L^2-error between the projected systems and the full order system. We notice that the reduced system obtained by the non-symplectic method is unstable. ~~It is known that classical reduce basis methods generally become exceedingly unstable as the reduced basis is enriched. This can also be observed here where~~ AND THAT the reduced system, constructed by $P_{X,V}$, is more unstable as k increases. On the other hand, THE symplectic methods yield a stable reduced system. ~~As the matter of fact, even as~~ ALTHOUGH the system, constructed by the projection $P_{X,\tilde{A}}^{\text{symp}}$, is not based on the L^2 projection, the error remains bounded with respect to the L^2 norm.

Figure 1(d) also indicates that the classical model reduction method based on the projection $P_{X,V}$ does not ~~necessarily~~ yield a stable reduced system. However, the symplectic methods provide a stable reduced system ~~under long time integration~~. We observe that the original symplectic APPROACH ~~also~~ provides an accurate solution with respect to the X-norm. Nevertheless, the relation between the two norms ~~in general~~ depends on the problem set up and the initial and the boundary conditions [14].



BETTER? WHY?

FIG. 1. *Numerical results related to the beam equation. (a) the decay of the singular values. (b) conservation of the Hamiltonian. (c) error with respect to the L^2 norm. (d) error with respect to the X-norm.*

5.2. The sine-Gordon equation. The sine-Gordon equation arise S in differential geometry and quantum physics [32], ~~This equation is~~ a nonlinear generalization of the linear wave equation, ~~given as~~ AS ON THE FORM

$$\begin{cases} u_t(t,x) = v, \quad x \in \Gamma, \\ v_t(t,x) = u_{xx} - \sin(u), \\ u(t,0) = 0, \\ u(t,L) = 2\pi. \end{cases} \tag{53}$$

Here, $x \in \Gamma$ is an SEGMENT ~~one-dimensional box~~ of length L, $u, v : \Gamma \to \mathbb{R}$ are scalar functions. The Hamiltonian associated ~~to~~ WITH (53) is

$$H(q,p) = \int_\Gamma \frac{1}{2}v^2 + \frac{1}{2}u_x^2 + 1 - \cos(u)\, dx. \tag{54}$$

One can VERIFY ~~check~~ that $u_t = \delta H/\delta v$ and $v_t = -\delta H/\delta u$. The sine-Gordon equation admits the soliton solution

$$u(t,x) = 4\text{arctan}\left(\exp\left(\pm\frac{x - x_0 - ct}{\sqrt{1-c^2}}\right)\right), \tag{55}$$

where the plus and minus signs are known as the *kink* and the *anti-kink* solutions, respectively. Here $|c| < 1$ is the wave speed. We discretize the LINE ~~box~~ into n equi-distant grid point $x_i = i\Delta x$, $i = 1, \dots, n$. Furthermore, we use standard finite-differences schemes to discretize (53) ~~to~~ AND obtains

$$\dot{z} = \mathbb{J}_{2n} L z + \mathbb{J}_{2n} g(z) + \mathbb{J}_{2n} c_b. \tag{56}$$

Here $z = (q^T, p^T)^T$, $q(t) = (u(t, x_1), \dots, u(t, x_N))^T$, $p(t) = (v(t, x_1), \dots, v(t, x_N))^T$, c_b is the term corresponding to the boundary conditions and

$$L = \begin{pmatrix} D_x^T D_x & 0_N \\ 0_N & I_n \end{pmatrix}, \quad g(z) = \begin{pmatrix} \sin(q) \\ \vec{0} \end{pmatrix}, \tag{57}$$

where D_x is the standard matrix differentiation operator. We may take $X = L$ as the weight matrix associated to (56). The discrete Hamiltonian, ~~then~~ takes the form

$$H_{\Delta x} = \Delta x \cdot \frac{1}{2}\|p\|^2 + \Delta x \cdot \|D_x q\|^2 + \sum_{i=1}^{n} \Delta x \cdot (1 - \cos(q_i)). \tag{58}$$

The system parameters ~~can be found in the table below~~. ARE GIVEN AS

Domain legnth	box: $L = 50$
No. grid points	$n = 500$
Time discretization size	$\Delta t = 0.01$
Wave speed	$c = 0.2$

The midpoint scheme (7) is used to integrate (53) in time and generate the snapshot matrix S. Similar to the previous subsection, projection operators $P_{X,V}$, $P_{I,A}^{\text{symp}}$ and $P_{X,\tilde{A}}^{\text{symp}}$ are used to construct a reduced system. To accelerate the evaluation of the nonlinear term, the symplectic methods discussed in subsections 3.1 and 3.2 are coupled with the projection operators $P_{I,A}^{\text{symp}}$ and $P_{X,A}^{\text{symp}}$, respectively. Furthermore, the DEIM approximation is used for THE efficient evaluation of the reduced system, obtained by the projection $P_{X,V}$. The midpoint rule is also used to integrate the reduced systems in time. Figure 2 shows the numerical results corresponding to THE reduced model, without treating the nonlinear terms, while the results corresponding to the accelerated evaluation of the nonlinear term are presented in Figure 3. APPROPRIATELY

Figure 2(a) shows the decay of the singular values of matrices S and $X^{1/2}S$. AS IN ~~Similar to~~ the previous section, we observe a saturation in the decay of the singular values of $X^{1/2}S$ compared to the singular values of S. This indicates that THE reduced ~~bases~~ BASIS, based on a weighted inner product, should be chosen to be larger to provide AN ~~similar~~ accuracy ~~as those which are~~ SIMILAR TO THOSE based on the L^2-inner product.

Figure 2(b) displays the error in the Hamiltonian. It is observed that the symplectic approaches conserve the Hamiltonian. However, the classical approaches do not necessarily conserve the Hamiltonian. We point out THAT using the projection operator $P_{X,V}$ ~~helped with~~ ENSURES the boundedness of the Hamiltonian. The contrary is observed when we apply the POD with respect to the L^2 inner-product, i.e. applying the projection operator $P_{I,V}$. This can be seen in the results presented in [33], where the unboundedness of the Hamiltonian is observed when $P_{I,V}$ is applied to the sine-Gordon equation. Nevertheless, only the symplectic model reduction ~~routines~~ consistently preserveS the Hamiltonian.

Figure 2(c) ~~exhibits~~ SHOWS the L^2 error between the solution of the projected systems and the original system. The behavior of the solution is investigated for $k = 100$, $k = 125$ and $k = 150$. We observe that all systems ~~that~~ WHICH are projected with respect to the X-norm are bounded. As the results in [33] suggest, the Euclidean inner-product does not necessarily yield a bounded reduced system. Moreover, we notice that the symplectic projection $P_{X,\tilde{A}}^{\text{symp}}$ result in a substantially more accurate reduced system compared to the reduced system yielded from $P_{X,V}$. This is because the

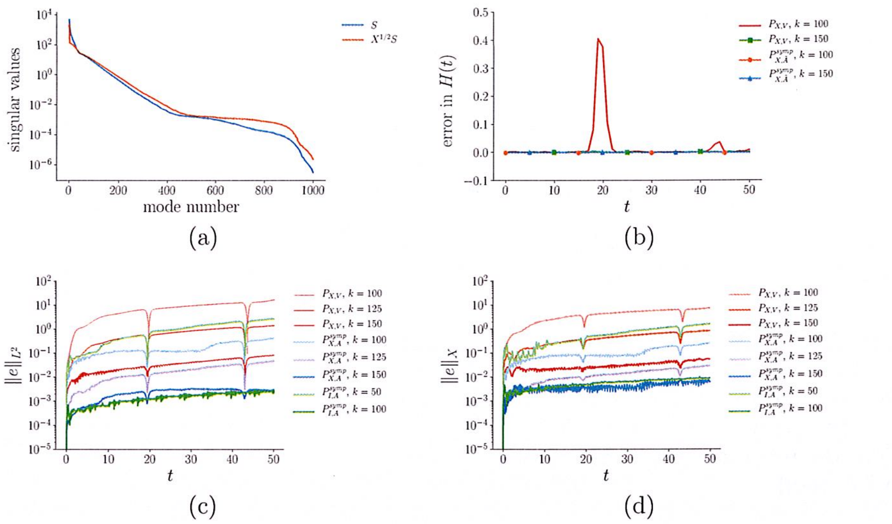


FIG. 2. *Numerical results related to the sine-Gordon equation. (a) the decay of the singular values. (b) error in the Hamiltonian. (c) error with respect to the L^2 norm. (d) error with respect to the X-norm.*

overall behavior of the original system ~~due to~~ As the symplelctic structure is translated correctly to the reduced system, constructed with the symplectic projection. ~~Whereas for systems constructed by the classical method, the Hamiltonian structure is lost.~~

The error with respect to the X-norm between the solution of the original system and the projected systems are presented in Figure 2(d). We see that the behavior of the X-norm error is similar to the L^2 norm, however the growth of the error is slower for methods based on a weighted inner product. Note that the connection between the error in the L^2-norm and the X-norm is problem dependent. ~~It is~~ We also observed that symplectic methods are substantially more accurate.

Figure 3 shows the performance of the different model reduction methods, when an efficient method is adopted in evaluating the nonlinear term in (56). This figure compares the symplectic approaches against non-symplectic methods. For all simulations, the size of the reduced basis for (56) is kept as $k = 100$. The size of the basis of the nonlinear term is then chosen as $k_n = 75$ and $k_n = 100$. For symplectic methods, a basis for the nonlinear term is constructed according to Algorithm 3, whereas for non-symplectic methods, the DEIM is applied. Note that for symplectic methods, the basis for the nonlinear term is added to the symplectic basis A. This means that the size of the reduced system is larger compared to the classical approach.

Figure 3(a) and Figure 3(b) show the error with respect to the L^2-norm and the X-norm between the solution of the projected systems compared to the solution of the original system, respectively. We observe that all solutions are bounded and the behavior of the error in the L^2-norm and the X-norm is similar. We observe that enriching the DEIM basis, does not increase the overall accuracy of the system projected using $P_{X,V}$ ~~(orange lines)~~. Furthermore, applying the DEIM to a symplectic reduced system ~~(green lines)~~ also destroys the symplectic nature of the reduced system, as suggested in subsection 4.3. Therefore, it is essential to adopt a symplectic approach to reduce the complexity of evaluati~~ng~~on of the nonlinear terms. We observe that

The

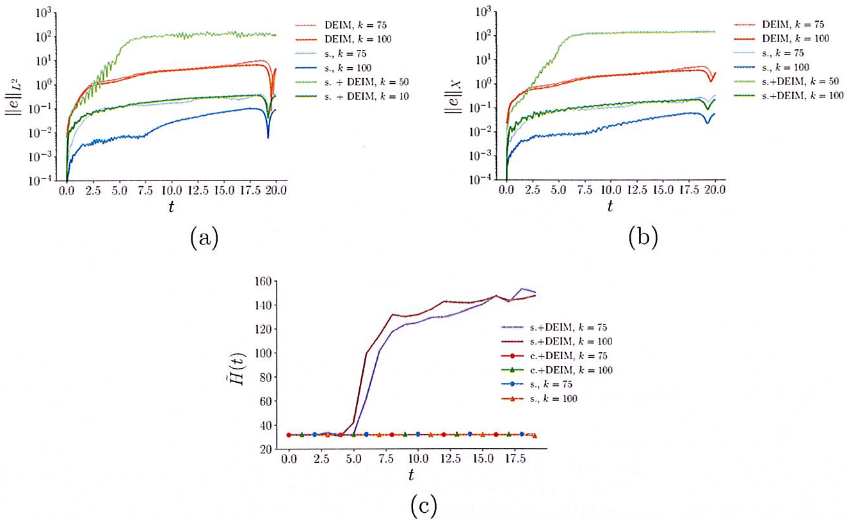


FIG. 3. *Numerical results related to the sine-Gordon equation with efficient evaluation of the nonlinear terms. Here, "c.+DEIM" indicates classical model reduction with the DEIM, "s.+DEIM" indicates symplectic model reduction with the DEIM and "s." indicates symplectic model reduction with symplectic treatment of the nonlinear term. (a) error with respect to the* L^2 *norm. (b) error with respect to the* X*-norm. (c) error in the Hamiltonian.*

the symplectic method presented in subsection 4.3 provides not only an accurate approximation of the nonlinear term, but also preserves the symplectic structure of the reduce system. Moreover, enriching such a basis consistently increases the accuracy of the solution, as suggested in Figure 3(a) and Figure 3(b).

Figure 3(b) shows the conservation of the Hamiltonian for different methods. It is again observed ~~seen~~ that applying the DEIM to a symplectic reduced system destroys the Hamiltonian structure, therefore the Hamiltonian is not preserved.

6. Conclusion.

~~In this paper~~ We present a model reduction routine that combines the classic model reduction method, develops with respect to a weighted inner product, with ~~the~~ symplectic model reduction. This allows the reduced system to be defined with respect to the norms and inner-products, that are natural to the problem. Furthermore, the symplectic nature of the reduced system ~~will~~ preserves the Hamiltonian structure of the original system, which result in robustness and enhanced stability in the reduced system.

We demonstrate that including the weighted inner-product in the sypplectic model reduction can be viewed as a natural extension of the symplectic method. Therefore, the stability preserving properties of the symplectic method generalize naturally to the new method.

Numerical results suggest the classic model reduction methods with respect to an weighted inner product can help with the boundedness of the system. However, only the symplectic treatment ~~can~~ consistently increases the accuracy of the reduced system. This is consistent with the fact the symplectic methods preserve the Hamiltonian structure.

We ~~also exhibit~~ show that to accelerate the evaluation of the nonlinear terms, adopting a symplastic approach is essential. This allows an accurate reduced ~~system~~ model that is consistently enhanced when the basis for the nonlinear term is enriched.

Hence, the symplectic model-reduction with respect to a weigther inner product can provide an accurate and robust reduced system that ~~carry~~ the norms and inner products most appropriate to the problem. ALLOWS THE USE OF

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