Energy-preserving Finite Element Scheme for the Dissipative Elastic Beam

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October 19, 2017

1 The Dissipative Elastic Beam

The equations governing small elastic deformations of an elastic beam, fixed on one side can be written as

$$\begin{aligned}
\partial_t q &= p & q, p \in \Omega \\
\partial_t p + p - \nabla \cdot \sigma &= f, & \\
\sigma &= \lambda \mathrm{tr}(\epsilon) I + 2\mu\epsilon, & \\
\epsilon &= \frac{1}{2}\left(\nabla q + (\nabla q)^T\right), & \\
u &= 0, & u \in \partial\Omega,
\end{aligned} \tag{1}$$

where *sigma* is the stress tensor, f is the gravitational force per unit volume, λ and μ are Lamé's elasticity parameters for the material in Ω, I is the identity tensor, tr is the trace operator on a tensor, ϵ is the symmetric strain-rate tensor (symmetric gradient), and q and p are the displacement and velocity vector fields, respectively. We have here assumed isotropic elastic conditions.

2 Variational Formulation

Here we assume that the solution to (1) belongs to a Hilbert space

$$V = \{u \in L^2 : \int_\Omega u^2 \; dx < \infty, \int_\Omega \|\nabla u\|^2 \; dx < \infty, u = 0 \text{ on } \partial\Omega\}/. \tag{2}$$

Further we assume that $(\cdot, \cdot)$ is the standard inner production on H. Formulating the weak form consist of computing the inner product of (1) with a test function $v \in H$. We can write

$$\begin{aligned} (\partial_t q, v) &= (p, v), \\ (\partial_t p, v) + (p, v) - (\nabla \cdot \sigma, v) &= (f, v). \end{aligned} \tag{3}$$

By introducing the notion of tensorial inner product $\cdot : \cdot$, we can rewrite the latter as

$$\begin{aligned} (\partial_t q, v) &= (p, v), \\ (\partial_t p, v) + (p, v) + \int_\Omega \sigma : \nabla v \; dx - \int_{\partial\Omega} (\sigma \cdot n) \cdot v \; dx &= (f, v). \end{aligned} \tag{4}$$

A finite element discretization of the latter can be obtained using standard methods.

3 Energy preserving extension

A time dissipative and dispersive formulation of equation 1 can be written as

$$\begin{aligned} \partial_t q &= f \quad\; f, q, p \in \Omega \\ \partial_t p - \nabla \cdot \sigma &= f, \\ f(t, x) + \int_0^t f(\tau, x) \; d\tau &= p. \end{aligned} \tag{5}$$

Following the footsteps of Figotin et al. 2007, the latter system can be reformulated as a conservative quadratic Hamiltonian system as

$$\begin{aligned} \partial_t q &= f \\ \partial_t p - \nabla \cdot \sigma &= f, \\ \partial_t \phi(t, x, s) &= \theta(t, x, s), \\ \partial_t \theta(t, x, s) &= \partial_s^2 \phi(t, x, s) + \sqrt{2}\delta_0(s) f(t, x), \end{aligned} \tag{6}$$

together with the transfer function

$$f(t,x) + \int_0^t f(\tau,x)\ d\tau = p(t,x). \tag{7}$$

Where $\theta, \phi \in V \times \mathcal{H}$, and $\mathcal{H}$ is some appropriate Hilbert space. We equip the space $V \times \mathcal{H}$ with the inner product

$$[u,v] = \int_\Omega \int_{-\infty}^{\infty} uv\ dx\ d\xi, \qquad u,v \in V \times \mathcal{H}. \tag{8}$$

Note that the added equations correspond to a vibrating string that carries the dissipated energy of the original system in the direction of the added pseudo-space $\mathcal{H}$. This allows us to solve these equations exactly in terms of f

$$\phi(t,x,s) = \frac{\sqrt{2}}{2}\int_0^{t-|s|} f(\tau,x)\ d\tau, \quad \theta(t,x,s) = \frac{\sqrt{2}}{2}\int_0^{t-|s|} f(t-|s|). \tag{9}$$

Then the energy associated to the extended system (6) is

$$H(q,p,\phi,\theta) = \frac{1}{2}\left\{\int_\Omega \sigma : \nabla q\ dx + \big(p-\phi(t,x,0), p-\phi(t,x,0)\big) + [\theta,\theta] + [\partial_s\phi, \partial_s\phi]\right\}. \tag{10}$$

4 Weak Formulation for the Extended System

We may form the weak formulation of the extended system by computing the inner product with a test function $v \in V$.

$$\begin{aligned}(\partial_t q, v) &= (p,v),\\ (\partial_t p, v) + (p,v) - (\nabla\cdot\sigma, v) &= (f,v),\\ (\partial_t\phi, v) &= (\theta, v)\\ (\partial_t\theta, v) &= (\partial_s^2\phi, v) + \sqrt{2}\delta_0(s)(f,v)\end{aligned} \tag{11}$$

together with the transfer function

$$(f,v) + \int_0^t (f,v)\ d\tau = (p,v). \tag{12}$$

Note that since the integration is only in the direction of the pseudo-space, then the integral can commute with the inner-product. We may use semi a discretization in the space $\mathcal{H}$ to compute the energy as

$$
\begin{aligned}
H(q,p,\phi,\theta) &= \frac{1}{2}\left\{(\nabla q, \nabla q)_{:} + (p-\phi(0), p-\phi(0)) + [\partial_s\phi, \partial_s\phi] + [\theta,\theta]\right\} \\
&= \frac{1}{2}\Big\{(\nabla q, \nabla q)_{:} + (p-\phi(0), p-\phi(0)) \\
&+ \sum_{i=1}^{N} \Delta s_i \int_\Omega (\partial_s\phi(t,x,s_i))^2 + (\theta(t,x,s_i))\ dx\Big\} \\
&= \frac{1}{2}\Big\{(\nabla q, \nabla q)_{:} + (p-\phi(0), p-\phi(0)) \\
&+ \sum_{i=1}^{N} \Delta s_i (\partial_s\phi, \partial_s\phi)\big|_{s=s_i} + \Delta s_i(\theta,\theta)\big|_{s=s_i}\Big\}
\end{aligned}
\tag{13}
$$

4.1 Existence and uniqueness of the solution

Let us rewrite eqs. (11) and (12)

$$
\begin{aligned}
(\partial_t q, u_q) &= (\tilde{f}, u_q), \\
(\partial_t p, u_p) - (\nabla\cdot\sigma, u_p) &= (f, u_p), \\
(\partial_t \phi, u_\phi) &= (\theta, u_\phi) \\
(\partial_t \theta, u_\theta) &= (\partial_s^2\phi, u_\theta) + \sqrt{2}\delta_0(s)(\tilde{f}, u_\theta) \\
(\tilde{f}, u_{\tilde{f}}) + \int_0^t (\tilde{f}, u_{\tilde{f}})\ d\tau &= (p, u_{\tilde{f}}).
\end{aligned}
\tag{14}
$$

Notice that the vector of unknowns and the test functions

$$
U = \begin{bmatrix} q \\ p \\ \phi \\ \theta \\ \tilde{f} \end{bmatrix} V = \begin{bmatrix} u_q \\ u_p \\ u_\phi \\ u_\theta \\ u_{\tilde{f}} \end{bmatrix}
\tag{15}
$$

are of the same dimension as required for well-posedness of a weak formulation. Rewriting Equation (14) we get the following defining equation

$$
a(\partial_t U, V) + b(U,V) = l_f(V)
\tag{16}
$$

where a, b, and l_f are bilinear and linear forms respectively with their obvious definitions. A time-discrete formulation of this equation is

$$c(U^k, V) = \tilde{l_f}(V) \tag{17}$$

where $c = a + \Delta t b$ and $\tilde{l_f} = \Delta t l_f - a$. Therefore, according to Lax Milgram theorem, the weak form (14) has a unique solution *assuming* the bilinear form c is coercive and bounded and the linear form $\tilde{l_f}$ belongs to the dual space V'.

[1]

References

[1] D. Wirtz, D. Sorensen, and B. Haasdonk. A posteriori error estimation for DEIM reduced nonlinear dynamical systems. *SIAM Journal on Scientific Computing*, 36(2):A311–A338, 2014.