Geometric MOR of Mechanical Systems

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1 Mechanical systems

1.1 Continuous system

Consider the linear mechanical system

$$M(\mu)\ddot{q} + D(\mu)\dot{q} + K(\mu)q = 0 \tag{1}$$

where M, D, K are parameter μ dependent matrices and q is the unknown. We can write this as a first order system by introducing momentum p as below

$$\begin{aligned} \dot{q} &= M^{-1}p, \\ \dot{p} &= -DM^{-1}p - Kq. \end{aligned} \tag{2}$$

Due to the presence of damping in the system, it is not Hamiltonian but can be turned into one by enlarging the phase space. This attains the following objectives:

1. We can use geometric MOR techniques on the enlarged system.
2. The Hamiltonian can work as a Lyapunov function and hence can be used to derive and implement new error estimators for this mechanical system.

Keeping this in mind, let us define an enlarged phase space

$$U = \begin{bmatrix} u(t) \\ \theta(s,t) \\ \phi(s,t) \end{bmatrix} \tag{3}$$

where $u = [q^T, p^T]^T$. Over this phase space, we define Hamiltonian $\mathcal{H}$ as below

$$\mathcal{H}(U) = \frac{1}{2}\|\mathcal{K}U\|^2 \tag{4}$$
$$= \frac{1}{2}\|Su - T\phi\|^2 + \frac{1}{2}q^T Kq + \frac{1}{2}\int_{-\infty}^{\infty}\{\|\theta(s,t)\|^2 + \|\partial_s\phi(s,t)\|^2\}ds \tag{5}$$

where

$$\mathcal{K} = \begin{bmatrix} S & 0 & -T \\ 0 & 1 & 0 \\ 0 & 0 & \partial_s \end{bmatrix}, S = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad T\phi = \int_{-\infty}^{\infty}\zeta(s)\phi(s)ds,$$
$$\zeta(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\omega s)\sqrt{2\omega Im\hat{\chi}(\omega)}d\omega, \quad \chi(\tau) = \begin{bmatrix} DM^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{6}$$

The equations of motion corresponding to this Hamiltonian in the enlarged space read

$$\partial_t U(s,t) = \mathcal{J}\nabla_U\mathcal{H} = \mathcal{J}\mathcal{K}^T\mathcal{K}U(s,t) \tag{7}$$

Here $\nabla_U\mathcal{H}$ denotes the vector of variational derivatives and

$$\mathcal{J} = \begin{bmatrix} J & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{8}$$

1.2 Discrete system

(Symplectic) implicit midpoint rule is:

$$\frac{U^{n+1,i+1/2} - U^{n,i+1/2}}{\Delta t} = \mathcal{J}\nabla_U\mathcal{H}(U^{n+1/2,i+1/2}). \tag{9}$$

Here n and i are the time and spatial indices, respectively, and $n + 1/2$ or $i + 1/2$ denote corresponding midpoints.

Hamiltonian-preserving discrete gradient method reads

$$\frac{U^{n+1,i+1/2} - U^{n,i+1/2}}{\Delta t} = \mathcal{J}\bar{\nabla}_U\mathcal{H}(U^{n+1/2,i+1/2}). \tag{10}$$

where $\bar{\nabla}$ is a discrete gradient.

1.3 Lyapunov function

Since $\mathcal{H}$ satisfies

$$\mathcal{H}(0) = 0, \mathcal{H}(U) > 0 \text{ for } U \neq 0, \text{ and } \frac{d}{dt}\mathcal{H} = 0,$$

it is a good candidate for the Lyapunov function of the extended system.