

# STATS 235: Modern Data Analysis

## Classification and Regression Trees

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# Tree Models

- Decision trees are models that recursively partition the input space into regions and define a local map between each region and the response variable.

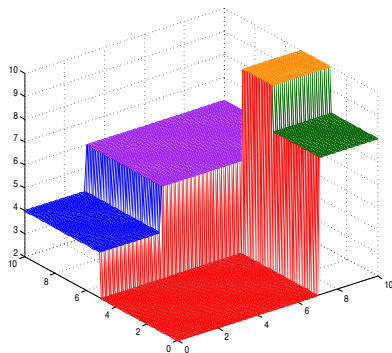
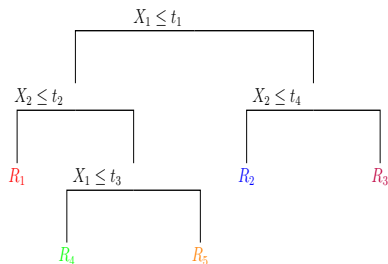


Figure 16.1 in Murphy (2012).

- Starting from the root, when a rule is satisfied, we move to the left branch; otherwise, we move to the right branch.
- When we reach a leaf, we use the subset of data, which fall in the corresponding region, to estimate the response variable.
- The corresponding model can be presented as follows:

$$\hat{y}_i = \sum_{m=1}^M \hat{y}_m I(x_i \in R_m)$$

- For regression model,  $\hat{y}_m$  can be simply the mean response in region  $R_m$ , or the regression estimate using the subset in  $R_m$ .
- For classification models, we use the sample proportions,  $p_{mc}$ , within region  $R_m$  as the estimate for the probability of class  $c$ .
- We usually assign a case to the class with the highest probability.

- The most commonly used decision tree method is Classification and Regression Tree (CART).
- To build a CART model, we first *grow* a tree using recursive *binary* splits.
- We usually stop the procedure when some stopping criterion is met. For example: the leaves must have at least  $m$  observations.
- The resulting model is typically too complex.
- Next, we *prune* the tree to obtain a simpler model and to avoid overfitting.

# Growing a Tree

- To grow a tree,
  - ▶ we choose a *cost* function, which typically reflects “impurity”
  - ▶ at each node (starting with the whole data at the root), we find the best input variable (feature),  $j^*$ , to split the data,
  - ▶ and find the best cutoff,  $t^*$  for the split.

- For numerical variables, the best  $(j^*, t^*)$  is defined as follows:

$$(j^*, t^*) = \arg \min_{j \in \{1, \dots, p\}} \min_t \{ \text{cost}(x_{ij}, y_i : x_{ij} \leq t) + \text{cost}(x_{ij}, y_i : x_{ij} > t) \}$$

- When  $x_j$  is categorical with  $K$  categories, the splits are usually based on one group versus all other groups:  $x_{ij} = k$  vs.  $x_{ij} \neq k$ .

- For regression trees, the square error cost function is commonly used,

$$\text{cost}(\mathcal{D}) = \sum_{i \in \mathcal{D}} (y_i - \hat{y}_{\mathcal{D}})^2$$

- $\hat{y}_{\mathcal{D}}$  is the estimate of the response variable (e.g., mean or regression estimate) using the observations in  $\mathcal{D}$ .

- For classification trees, commonly used cost functions are
  - ▶ misclassification rate

$$\text{cost}(\mathcal{D}) = \frac{1}{|\mathcal{D}|} \sum_{i \in \mathcal{D}} (y_i \neq \hat{y}_i)$$

- ▶ Entropy

$$\text{cost}(\mathcal{D}) = \sum_{c=1}^C p_c \log \frac{1}{p_c}$$

- ▶ Gini index

$$\text{cost}(\mathcal{D}) = \sum_{c=1}^C p_c (1 - p_c)$$

- $p_c$  is the sample proportion of class  $c$  in the subset  $\mathcal{D}$ .



# Cost Functions

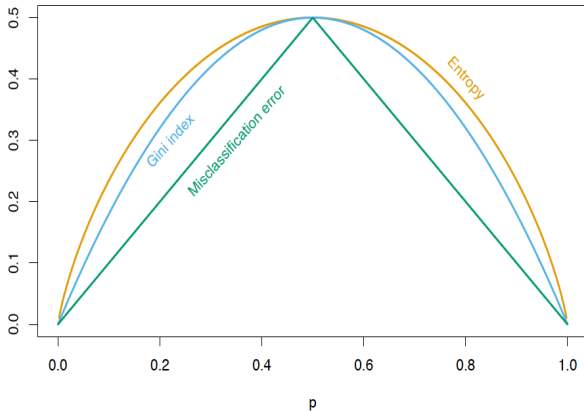


Figure 9.3 in Hastie et al. (2010) for binary classification.

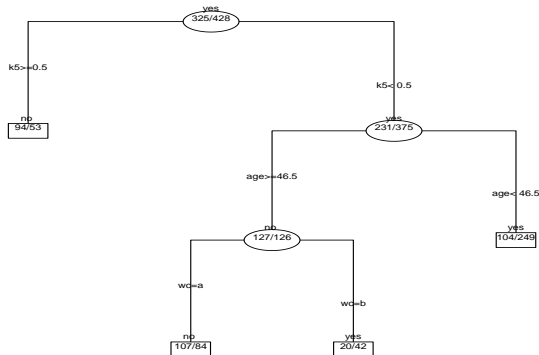
- After growing a full tree, we prune it back to avoid overfitting.
- Pruning involves collapsing some internal nodes to find a subtree without increasing the overall cost substantially.
- We can find the cross-validation costs for all possible subtrees,  $T$ , and choose the subtree with the lowest *cost-complexity* value defined as

$$C_{\alpha}(T) = \sum_{m=1}^{|T|} n_m C_m + \alpha |T|$$

where  $|T|$  is the number of terminal nodes for the subtree;  $n_m$  and  $C_m$  are the number of observations and cost for the  $m$ th terminal node;  $\alpha > 0$  is a tuning parameter.

# US Women's labor-force participation

- The dataset Mroz in the package car includes the work status of 753 women along with 7 other variables
- We obtain the following model for predicting US women's work status after growing and pruning a tree model



# Pros and Cons of Trees

- Pros

- ▶ Easy to interpret
- ▶ Perform automatic variable selection
- ▶ Automatically captures interactions and nonlinear relationships
- ▶ Robust to outliers

- Cons

- ▶ Low predictive power
- ▶ High variability

- To reduce the variance of prediction,  $\hat{y} = \hat{f}(x)$ , we could take a collection of bootstrap samples, build a separate model based on each sample, and aggregate predictions from all models into a single prediction with a similar bias but lower variance
- Given a training,  $z = \{(x_1, y_1), \dots, (x_n, y_n)\}$ , we generate a collection of  $B$  bootstrap samples,

$$z^{(b)} = \{(x_1^*, y_1^*), \dots, (x_n^*, y_n^*)\}, \quad b = 1, \dots, B$$

- The overall prediction for the input  $x$  is the average over all bootstrap samples,

$$\hat{f}_B(x) = \frac{1}{B} \sum_{b=1}^B \hat{f}^{(b)}(x)$$

where  $\hat{f}^{(b)}(x)$  is the prediction based on the model built on the  $b^{\text{th}}$  bootstrap sample

- This approach is called *bootstrap aggregation* or *bagging*
- Bagging could improve models with low bias and high variance, such as trees
- For regression trees, we simply average over bootstrap trees
- for classification trees, we can either average over class probabilities from bootstrap trees, or if we are just interested in prediction, we can predict the class with the majority “vote”
- The latter resembles the “Wisdom of Crowds” idea (Surowiecki, 2004): making decisions based on the consensus of *independent* and *weakly-informed* voters (here, individual trees)
- Bagged trees, however, are not independent; *random forests* models are designed to address this issue

- *Random forests* models (Breiman, 2001) attempt to reduce variance and improve predictive power by using *bagged* and *de-correlated* (decoupled) trees
- A random forests model is an ensemble of trees, each developed based on a random subset of input variables (features) and a bootstrap sample of observations.
- For regression, we average over the estimates from individual trees.
- For classification, each tree casts a vote, and we choose the class with the highest vote.

**for**  $b = 1$  to  $B$  **do**

Bootstrap: Sample  $n$  cases with replacement from the data.

Decouple: Use the bootstrap sample to grow a random-forest tree  $T_b$ , but before each split, sample  $m < p$  features randomly (typically  $m = \sqrt{p}$ ) and then find the best feature/cutoff among the  $m$  variables.

**end for**

Output an ensemble of trees  $\{T_1, \dots, T_B\}$ .

For regression, average the estimates from individual trees.

For classification, use the majority rule.



# Out-of-bag samples

- An advantage of random forests models is their use of out-of-bag (OOB) samples to provide a fast assessment of model performance
- To this end, the response value for the  $i^{th}$  case in the training set is estimated by averaging over all bootstrap trees that did not include the  $i^{th}$  case in their bootstrap sample
- The resulting prediction accuracy is very similar to that of leave-one-out procedure (without fitting the model  $n$  times)

- The idea behind bagging and random forests is that an ensemble of *weak learners* can collectively perform well
- *Boosting* extends this idea by sequentially applying a base learner (e.g., a classification tree) to weighted versions of data
- The base learners are *adaptive basis-functions models*, such as GAM, trees, and random forests, with the following general form

$$f(x) = \sum_{m=1}^M \beta_m \phi_m(x)$$

where the basis functions,  $\phi(x)$ , are learned from the data

- Tree models, for example, can be written as follows:

$$f(x) = \sum_{m=1}^M w_m I(x \in R_m)$$

where  $w_m$  is the mean of the response variable over the region  $R_m$

- For classification trees, we start by applying the weak learner (a tree model performing slightly better than random guessing) to the training set, giving all observations equal weights:  $1/n$
- Next, we reapply the weak learner to the data, but this time we give higher weights to cases that were misclassified previously
- We continue this procedure for  $M$  iterations, where at each iteration we increase the weights for cases that were misclassified previously and decrease weights for those that were correctly classified

- This process results in a sequence of  $M$  classifiers:  $G_1(x), \dots, G_M(x)$
- For a binary classification problem with the response variable  $y \in \{-1, 1\}$ , we combine the individual votes using a weighted majority vote as follows:

$$G(x) = \text{sign}\left(\sum_{m=1}^M \alpha_m G_m(x)\right)$$

- The weights  $\alpha_1, \dots, \alpha_M$ , are chosen such that more accurate classifiers have higher weights
- The following plot illustrates this approach

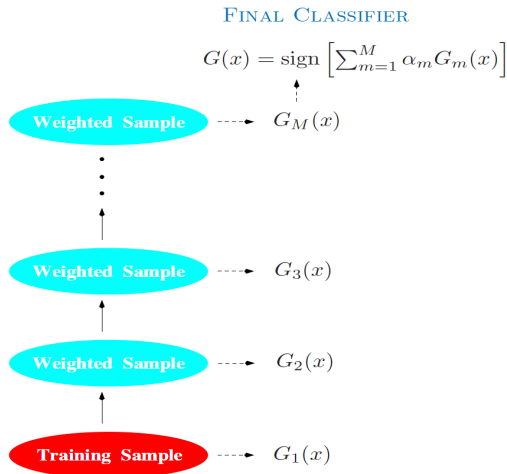


Figure 10.1 in Hastie et al. (2010).

- Boosting algorithms attempt to minimize the loss function

$$L(y, f(x))$$

- For binary classification, we could use the 0-1 loss function,  $L = I(\text{sign}(f) \neq y)$ ; however, this is not differentiable
- Alternatively, we could use exponential loss,  $L = \exp(-yf)$ , or logloss,  $L = \log(1 + \exp(-yf))$

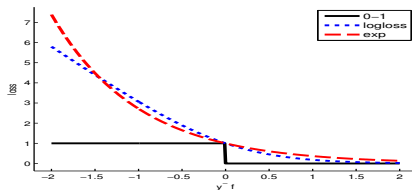


Figure 16.9 in Murphy (2012).

- Using exponential loss leads to one of the most common boosting algorithms called AdaBoost (see Hastie et al. (2010) for more details)

Initialize the observations weights,  $w_i = 1/n$ ,  $i = 1, \dots, n$

**for**  $m = 1$  to  $M$  **do**

Fit a base learner  $G_m$  (e.g., tree) to the weighted training data

Find the overall error

$$err_m = \frac{\sum_{i=1}^n w_i I(y_i \neq G_m(x_i))}{\sum_{i=1}^n w_i}$$

Calculate the model weights:  $\alpha_m = \log((1 - err_m)/err_m)$

Update the observations weights:  $w_i \leftarrow w_i \exp(\alpha_m I(y_i \neq G_m(x_i)))$

**end for**

Output  $G(x) = \text{sign}(\sum_{m=1}^M \alpha_m G_m(x))$

# Forward stagewise additive modeling (optional)

- To understand the above algorithm, we note that boosting can be regarded as an additive basis expansion method
- For classification models, each  $G_m(x) \in \{-1, 1\}$  is a basis
- In general, we can write them as

$$\begin{aligned} f(x) &= \sum_{m=1}^M \beta_m \phi_m(x) \\ &= \sum_{m=1}^M \beta_m \phi(x, \gamma_m) \end{aligned}$$

- We estimate  $f(x)$  by minimizing a loss function

$$\min \sum_{i=1}^n L(y_i, \sum_{m=1}^M \beta_m \phi(x_i, \gamma_m))$$



# Forward stagewise additive modeling

- In general, this optimization is computationally intensive; therefore, it is common to find an approximate solution by sequentially adding a basis function, while keeping the previous ones (and their corresponding parameters) fixed
- This strategy is known as *forward stagewise additive modeling*:
  - 1 Initialize  $f_0(x)$
  - 2 For  $m = 1, \dots, M$ 
    - a) Find  $(\beta_m, \gamma_m) = \operatorname{argmin} \sum_{i=1}^n L(y_i, f_{m-1}(x_i) + \beta\phi(x_i, \gamma))$
    - b) Set  $f_m(x) = f_{m-1}(x) + \beta_m\phi(x_i, \gamma_m)$

# Forward stagewise additive modeling

- For AdaBoost, we have

$$L(y, f(x)) = \exp(-yf(x))$$

and

$$\phi_m(x) = G_m(x) \in \{-1, 1\}$$

Therefore, optimization is in terms of  $\beta$  and  $G_m(x)$  with respect to the exponential loss function

$$\begin{aligned}(\beta_m, G_m) &= \operatorname{argmin} \sum_{i=1}^n \exp[-y_i(f_{m-1}(x_i) + \beta G(x_i))] \\ &= \operatorname{argmin} \sum_{i=1}^n w_i^{(m)} \exp[-\beta y_i G(x_i)]\end{aligned}$$

where  $w_i^{(m)} = \exp(y_i f_{m-1}(x_i))$ , i.e., observation weight, which only depends on the previous model, not the optimization parameters

# Forward stagewise additive modeling

- We can write the objective function as

$$e^{-\beta} \sum_{y_i=G(x_i)} w_i^{(m)} + e^{\beta} \sum_{y_i \neq G(x_i)} w_i^{(m)} =$$

$$(e^{\beta} - e^{-\beta}) \sum_{i=1}^n w_i^{(m)} I(y_i \neq G(x_i)) + e^{\beta} \sum_{i=1}^n w_i^{(m)}$$

Therefore, for any value of  $\beta > 0$ , optimization with respect to  $G_m(x)$  involves finding a classifier that minimizes the weighted error

$$G_m = \operatorname{argmin} \sum_{i=1}^n w_i^{(m)} I(y_i \neq G(x_i))$$

# Forward stagewise additive modeling

- Given  $G_m$ , we solve the optimization problem with respect to  $\beta$ ,

$$\beta_m = \frac{1}{2} \log \frac{1 - \text{err}_m}{\text{err}_m}$$

where

$$\text{err}_m = \frac{\sum_{i=1}^n w_i^{(m)} I(y_i \neq G_m(x_i))}{\sum_{i=1}^n w_i^{(m)}}$$

Using  $(\beta_m, G_m)$ , we set

$$f_m(x) = f_{m-1}(x) + \beta_m G_m(x)$$

# Forward stagewise additive modeling

- Using  $(\beta_m, G_m)$ , we set

$$f_m(x) = f_{m-1}(x) + \beta_m G_m(x)$$

Therefore, the next set of observation weights are

$$w_i^{(m+1)} = w_i^{(m)} \exp[-\beta_m y_i G_m(x_i)]$$

Because  $-y_i G_m(x_i) = 2I(y_i \neq G_m(x_i)) - 1$ , we can rewrite the weights as

$$w_i^{(m+1)} = w_i^{(m)} \exp[\alpha_m I(y_i \neq G_m(x_i))]$$

where

$$\alpha_m = 2\beta_m = \log \frac{1 - \text{err}_m}{\text{err}_m}$$