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Three conjectures on extended twin primes and the existence of isoboolean and singular primes inspired by relativistic quantum computing

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Abstract

Starting from a recursive construction of the natural numbers, a Cartesian sum of prime numbers with the elements of the set of powers of 2 is described. Boolean and logical matrices generated through recurring Cartesian sums permit the definition of extended twin primes. That is, primes separated by a gap of a power 2^N, for a natural number N are studied. Besides the set of extended twin primes, through computational techniques that were inspired by relativistic quantum two-component spinor computations, we demonstrate the existence of two other characteristic types of primes which we call iso-Boolean and singular primes, respectively. The iso-Boolean primes possess rows in the Boolean matrix composed of the same elements of the canonical basis of a Boolean Hypercube. The singular primes exhibit a zero Boolean matrix row, and it is computationally established that there are numerous such singular and iso-Boolean primes. All of these novel findings are formulated as three new conjectures on extended twin primes.

Keywords Recursive construction of natural number set · Cartesian sum of primes with powers of 2 · Boolean matrices of a hypercube · Isoboolean and singular primes · Extended twin primes conjecture · Relativistic quantum spinor computations

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1 Introduction

Ever since antiquity, intriguing pursuits for the study of a simple set of numbers, N, the set of natural numbers, and in particular, the subset of prime numbers have been the cynosure of great mathematicians such as Riemann, Hardy, and Ramanujan [1, 2]. They are ubiquitous in that they not only appear in everyday counting but sophisticated research within the realm of natural sciences. Several obvious chemical examples are the sequence of atomic charges in the periodic table of the elements and the sequences constituted by the number of isomers of alkanes [3] and several other sequences from the chemistry literature [4]. Sloane [4] refers to the work of Trinajstić et al. [5] on computer generation of certain classes of molecules as one of the chemically most interesting sequences. Combinatorial enumeration of sequences pertinent to polycyclic aromatics by Trinajstić et al. [5] was inspired by a paper published in 1980 by Balasubramanian et al. [6] on the enumeration of polyhexes, cell-growth problem, and their relevance to the carcinogenicity of benzenoid hydrocarbons [6]. Following this work, Trinajstić et al. [7, 8] have published a series of papers on the enumeration of polycyclic aromatics that culminated in their classic book [5] referred to by Sloane [4] as one of the chemically interesting sequences. Besides, many experimental and computed numerical discrete sets with an appropriate shift of origin and scaling can be transformed into a set of natural numbers.

Among natural numbers, the most interesting subset is formed by the prime numbers, as their distribution and properties have intrigued mathematicians over centuries [1, 2]. Consequently, the present paper deals with some interesting characteristics of the structure of prime numbers not known hitherto, focusing on a novel construction of a new class of primes within the Boolean hypercube representation. That is, from the natural sequence of primes generated through a recursive summation involving powers of two, we seek to construct a new class of extended twin primes.

Relativistic quantum techniques that combine electron correlation effects with relativistic effects within the 2-component spinor representations make extensive use of Boolean logic and quantum bit (q-bit) manipulation techniques. For example, a relativistic two-component spinor representation of a relativistic electronic state is a superposition of two Boolean vectors that correspond to spin up and spin down electronic configurations. Consequently, efficient packing of relativistic spinor configurations is accomplished through Boolean hypercube representations in relativistic configuration interaction techniques. Thus, the computational goal of seeking a new class of extended primes in Boolean hypercube representations is motivated by the computational techniques of relativistic quantum spinors thereby demonstrating the discovery of a new class of extended primes in mathematics inspired by chemical techniques employed in relativistic quantum computations. With this goal in mind, we computationally pursue the set of primes related to such summations that result in the set of extended prime twins. During the process of our computations, we have discovered two classes of primes which we call *iso-Boolean* and *singular* primes both of which are not known today to the best of our knowledge. Three conjectures are proposed from the results of this computational venture.

2 Recursive construction of the set of natural numbers

The natural number set $\mathbb{N}$ can be constructed through a recursive algorithm, as already described in previous works [9–12]. Let $\mathbb{S}_{\mathbb{N}}$ a finite subset of natural numbers, up to some Mersenne number: $N \in \mathbb{N} : M(N) = 2^N - 1$; that is: $\mathbb{S}_{\mathbb{N}} = \{0, 1, 2, ..., M(N)\}$. The subset: $\mathbb{S}_{N+1}$ can be easily constructed by adding the power: $2^{\,N}$ to every element of the set $\mathbb{S}_{\mathbb{N}}$. An operation involving all the elements of $\mathbb{S}_{\mathbb{N}}$, which can be symbolically written as: $\mathbb{T}_N = 2^N \oplus \mathbb{S}_N$, can be used to construct a next natural subset: $\mathbb{S}_{N+1} = \mathbb{S}_N \cup \mathbb{T}_N$. Further details can be found in Appendix 1. The cardinalities of the two sets are readily obtained as $Card\left(\mathbb{S}_N\right) = 2^N$ and $Card\left(\mathbb{T}_N\right) = 2^{(N+1)}$.

2.1 Recursive construction of odd and even natural numbers

The natural number set contains even $\mathbb{E}$ and odd $\mathbb{O}$ numbers subsets, or $\mathbb{N} = \mathbb{E} \cup \mathbb{O} \wedge \mathbb{E} \cap \mathbb{O} = \emptyset$. The recursive generation algorithm can be also applied to the even and odd subsets, because both initial $\mathbb{S}_{\mathbb{N}}$ and generated $\mathbb{S}_{N+1}$ subsets can be decomposed in the same way. That is, for instance: $\mathbb{S}_{\mathbb{N}} = \mathbb{E}_{\mathbb{N}} \cup \mathbb{O}_{\mathbb{N}}$ The recursive algorithm similarly applies to odd sets limited by the Mersenne numbers, so one can write for an odd number subset, defined as: $\mathbb{O}_N = \{1, 3, 5, ..., M(N)\} \wedge Card\left(\mathbb{O}_N\right) = 2^{N-1}$, and also the recursion can be written in this case: $\mathbb{O}_{N+1} = \mathbb{O}_N \cup \left(2^N \oplus \mathbb{O}_N\right)$, so the new cardinality is:$Card\left(\mathbb{O}_{N+1}\right) = 2^N$.

2.2 Prime numbers in the recursive algorithm

The odd number subsets $\mathbb{O}_N$ contain prime numbers subsets $\mathbb{P}_N \subset \mathbb{O}_N$. In the recursive algorithm explained above, the next prime number extended subset: can be constructed from the previous prime subset, $\mathbb{P}_{N+1} = \mathbb{P}_N \cup \mathbb{Q}_N$. However, the extension set: $\mathbb{Q}_N$ can only be written as: $\mathbb{Q}_N \subset \left(2^N \oplus \mathbb{O}_N\right)$, because the extension of prime numbers might come either from a subset of the previous prime subset $\mathbb{P}_N$ or after a subset of the composite odd natural numbers $\mathbb{O}_N$. It is interesting to note that some Mersenne numbers are prime, see, for example, the list on the web page [12]. Furthermore, the largest Mersenne number is also the largest prime number computed so far, although it is under scrutiny. Mersenne numbers can be obtained recursively using appropriate powers of 2: $M(N + 1) = 2^{N+1} - 1 = 2 \times 2^N - 1 = 2^N + 2^N - 1 = 2^N + M(N)$.

In a recent work [11] it has been shown that the cardinalities of the subsets $\mathbb{P}_N$ obtained recursively from the sets $\mathbb{S}_N$ are highly correlated with the associated Mersenne numbers $M(N)$, and can be shown there is a bijection between the two sets. Moreover, prime subset cardinalities grow with a factor of approximately 1.9, instead of factor 2 of the whole natural set. When using the recursive generation of natural numbers as explained above, we obtain two kinds of primes when the prime extension $\mathbb{Q}_N$ of a natural number set is constructed. New prime numbers are such that they are either a) constructed from a subset of the previous primes $\mathbb{P}_N$ or b) they are obtained from a subset of some of the initial odd composite numbers within $\mathbb{O}_N$.

3 Role of the set of powers of 2 in front of prime numbers

The role of the powers $2^{\mathbb{N}}$ in the recursive construction of natural numbers inspires the possibility to use them in the construction of extended twins of prime numbers. Such prime characteristics can be relevant and can transform into a scaffold to find out new properties, new highlights, and new lines of research on the prime subset. In this section, a simple set of theoretical tools for such an exhaustive construction is described.

3.1 The set of powers of 2 and the Cartesian sum of primes

Among the characteristics of prime numbers, one must be also aware of the role played by the elements of the set of powers of 2: $2^{\mathbb{N}} = \left\{2^1, 2^2, ...2^N, ...\right\}$ when added to the elements of the set of primes: $\mathbb{P} \subset \mathbb{N}$. One can construct the elements of a natural superset, denoted by: $\mathbb{A} = 2^{\mathbb{N}} \oplus \mathbb{P}$, using the Cartesian sum, of every power of two with each element of the whole set of prime numbers. Thus, set $\mathbb{A}$ is constructed as a matrix: $A = \left\{A_{pN} = p + 2^N | \forall p \in \mathbb{P} \wedge \forall N \in \mathbb{N}\right\}$, where the elements of each row correspond to the sum of a given prime number p with all the elements of the set $2^{\mathbb{N}}$, and the columns are the result of summing a given power 2^N the whole set of primes. In practice though, one shall use an upper maximal power $2^{N_{\max}}$ and a finite cardinality subset of prime numbers.

3.2 Boolean and logical matrices connecting primes and $2^{\mathbf{N}}$ powers

One can envisage the construction of a Boolean matrix **B** or a logical matrix **L,** of the same dimensions as the matrix **A**, where each element is: 0 or 1, F or T, depending on whether the sum of the corresponding primes with the 2^N powers yield a prime number or not, that is:

$$\mathbf{B} = \left\{B_{pN} = \delta\left[A_{pN} \in \mathbb{P}\right]\right\} \wedge \mathbf{L} = \left\{L_{pN} = \lambda\left[A_{pN} \in \mathbb{P}\right]\right\};$$

where δ[*Expression*] is a Boolean Kronecker's delta yielding 0 if the *Expression* is false or 1 if it is true, and λ[*Expression*] is a logical Kronecker's delta yielding F or T, for the same events. In this manner, choosing a prime number, p in the corresponding row of the Boolean or logical matrices, a powerful tool is constructed to discover the existence of *larger* primes than p, generated via the structure of the matrix **A** elements. The information contained in the rows of matrices **A**, **B**, and **L**, provides information on *larger primes from a given prime*. However, the matrices do not tell if the chosen prime which generates a row of these matrices could itself be the result of summing a power 2^N to the precedent primes. To obtain such information then, a different procedure must be followed. We consider that this might be the subject of further research.

4 Computations

4.1 Relativistic quantum computing

Relativistic quantum computing techniques that seek to find optimized correlated many-electron two-component spinor wave functions employ products of many-electron spinor wave functions where individual electronic states are represented by a superposition of relativistic two-component spinor states. For example, the $(1/2)_{g,u}$ spinors originating from the seventh row 7p-dimers are shown below:

$$\left(\frac{1}{2}\right)_g^{\frac{1}{2}} = \begin{pmatrix} \sigma_g \\ \pi_g^+ \end{pmatrix}, \left(\frac{1}{2}\right)_g^{-\frac{1}{2}} = \begin{pmatrix} \pi_g^- \\ -\sigma_g \end{pmatrix}, \left(\frac{1}{2}\right)_u^{\frac{1}{2}} = \begin{pmatrix} \sigma_u \\ \pi_u^+ \end{pmatrix}, \left(\frac{1}{2}\right)_u^{-\frac{1}{2}} = \begin{pmatrix} \pi_u^- \\ -\sigma_u \end{pmatrix},$$

where: $\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

As the number of such possible spinors combinatorially explodes into billions of two-component spinor configurations in relativistic configuration interaction computations of molecules containing very heavy atoms, each multi-electron spinor configuration is represented by a Boolean string where the symbol α corresponds to spin up and β spin down. Consequently, in relativistic quantum computations, a pack/unpack algorithm is employed for efficient storage and processing of these Boolean strings of spinors and the computations of various relativistic configuration interaction Hamiltonian matrix elements [16, 17]. Spin–orbit electronic configurations are packed into Boolean strings, thus allowing the processing of billions of relativistic electronic spinor configurations [17]. The same technique is used here, where each Boolean string of length 30 for a prime is packed into an integer of 4 bytes by mapping T(α) to 1 and F(β) to 0. Thus, the required storage is dramatically reduced in our algorithm, as we store a packed integer of 4 bytes that represents the Boolean string for each prime number. Such integral labels are written out to a sequential binary file for further processing in the subsequent steps of our algorithm. A similar algorithm was employed previously [18] for the combinatorial construction and manipulation of Hadamard matrices which are orthogonal matrices with elements constructed by the set: $\{1, -1\}$. Consequently, our search for a new class of extended twin primes is stimulated by relativistic quantum computations which in this case have resulted in the discovery of two new classes of extended twin primes, namely iso-Boolean and singular primes.

Any computation involving comparing two primes and comparisons of Boolean hypercube representations of all primes is constrained by the available dynamical access RAM than the CPU power. We have employed efficient pack/unpack algorithms and sequential read/write techniques to overcome any such DRAM constraints. Consequently, for an exhaustive generation, we have chosen: $N_{max} = 30$, and the number of primes was chosen around the maximal prime number contained in the $M(31)$ interval, although an exhaustive listing of all such primes in any form of printed output takes too much space to accommodate in a single tractable printed form. Still, such primes can be printed in chunks. Moreover, in a 32-bit OS, there are limits to the file sizes which are determined by the Mersenne number: $M(31)=2147483647$. However, this is readily overcome in a 64-bit OS. Some of the above-mentioned constraints can be overcome with a 64-bit FORTRAN or other programming languages, like Python, and a 64-bit OS.

4.2 Computational details

All of the computations described herein were carried out using FORTRAN'95 codes. The generation of all prime numbers contained in the $M(31)$ interval is accomplished through an efficient traditional Sieve formula of Eratosthenes. However, to optimize the memory usage at the prime generation stage, we do not store the integral primes, as they would need: $4\times M(31)$ bytes of memory, making further memory-demanding computations intractable. Instead, we store them within Boolean arrays defined as Logical*1 in FORTRAN'95. Thus, reducing the needed memory by a factor of 4. Moreover, the entire logical array is written in a sequential binary file for further read and processing, to generate the Boolean strings within the interval 2^N. Also, for comparisons of various Boolean strings through bit manipulation techniques, and the generation of extended twin primes and singular primes. Each prime is thus stored in a Logical*1 array wherein the truth value T represents the number is a prime and F represents that it is not a prime. Thus, a logical array of $M(31)$ bytes would contain all primes less than $M(31)$. The next section of our algorithm deals with the generation of a Boolean string of length N (for up to $N=30$), where for a given prime number: p, in case that $q=p+2^N$ is also prime, then the Boolean value of T is assigned to the N^{th} entry; otherwise, it is assigned the value of F. Thus, for each prime p in the array of all prime numbers within the Mersenne interval $M(31)$, we generate a Boolean array of length 30. However, explicit storage of the entire Boolean string of length 30 for each prime number in the array of integers in the $M(31)$ interval would take up approximately $2^{31}\times 2^{30}$ bytes, that is 2^{61} bytes of memory. This does not include the memory taken up by the binary array of numbers up to 2^{31}, which would require additional 4×2^{31} bytes of DRAM. Hence, explicit storage of each binary string of length 30 for each prime number in the array of length 2^{31}, makes it computationally nearly impossible, if other numerical comparisons in nested loops have to be carried out.

The next step of our computations involves comparisons of the packed integers generated for each prime, which contains nested loops, and thus for N_p prime numbers, the algorithm includes a set of $\binom{N_p}{2}$ comparisons. Two primes with the same packed nonzero binary integer are then identified as *extended twin primes*. Any prime with a zero-packed binary integer is identified by our code as a *singular prime*. The code then writes out in sequential binary files both extended twin primes and singular primes.

4.3 Conjecture concerning extended twin primes

One must notice the so-called twin primes, that is, prime numbers obtained by summing the first power of 2 to a prime. The first column of matrix **A** will hold this kind of well-studied primes [13, 14]. The first column of matrices **B** or **L** will tell if a given prime is accompanied by a twin or not. Yet, the remnant columns of the corresponding matrices convey if some prime number p has an *extended twin* of the form $p+2^N$. Results show that there is a large number of such extended twins among the list of primes that we tested. Thus, one can formulate:

Conjecture 1 *There exists an infinite number of extended twin primes, including the twin primes themselves.*

The dimension of each row might produce a variable number of extended twins. To describe the results of our computations, one can refer to the row dimension as the precision of calculations. Our computational precision has been thus of $N=30$ when arrays of all primes and their Boolean strings are explicitly considered. However, when a specific prime number needs to be computationally tested, either for a twin prime or a singular prime, our computational arithmetic precision can be extended to a quadruple precision which permits computations up to $N=112$ for a given prime p.

4.4 Further remarkable results

Apart from the possible existence of infinite extended twin primes, there are two other remarkable results found deserving further attention, when perusing the matrices: **A**, **B**, and **L**.

4.4.1 Existence of non-trivial Isoboolean extended twin primes and the supporting computational results

First, in the process of our computations, we discovered the appearance of pairs of primes, which can be named *non-trivial iso-Boolean twins*. They correspond to equal Boolean rows in matrix **B**, with an added characteristic: all the elements of the row are nil except in a coincident position that bears 1. These rows could be associated with vertices of a Boolean hypercube of the appropriate dimension, H, describing the so-called canonical basis: $\left\{\langle\mathbf{e}_I\right| = \left\{\delta_{IJ}\right\}|I,J=1,H\right\}$, see for example [15]. Table 1 shows the results of our computations of some extended iso-Boolean twin pairs together with their binary strings. As can be seen from Table 1, the pairs of primes: {(2411,2423), (49993,49999), (58573,58579), (78691,78697), … (593851,593863), … (50019061,50019091), … (536872811,536872823)} are such non-trivial extended iso-Boolean prime twins. A character has been found concerning the above-mentioned twins, consisting in that the gap between the non-trivial extended twin pairs is a multiple of 6 (see Table 1). The observation that the gap between any non-trivial extended twin pair is expressible as $6\times m$, for a natural number m follows from the fact that any prime larger than 3 can be expressed as $6n\pm 1$ for some natural number n. Note that the converse is not true as a large

Table 1 Enumeration of extended isoboolean twin primes with identical nontrivial binary strings

P\N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
2411	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2423	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
49993	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
49999	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
58573	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
58579	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
78691	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
78697	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
106543	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
106591	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
110753	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
110771	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
593851	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
593863	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
50019061	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
50019091	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
60004817	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
60004859	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
80004527	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
80004581	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
100007119	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
100007143	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
536872811	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
536872823	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

subset of natural numbers that are not primes can be written as $6n \pm 1$ for some natural number n; for example, the sequence: 115, 175, 11999995, … 1199999995, … Furthermore, any given twin pair is either ($6p+1$;$6q+1$) or ($6p$-1;$6q$-1) and never ($6p$-1;$6q+1$), for natural numbers p and q, with $p \neq q$. Hence, it directly follows that the gap between any nontrivial extended twin pair is $6|p\text{-}q|$, which is always a multiple of 6. Consequently, for such non-trivial twins it can be written:

$$\exists p, q \in \mathbb{P} \wedge \exists I \in \mathbb{N} \wedge \left\langle B_p \right| = \left\langle B_q \right| = \langle e_I | \rightarrow |p - q| = m \times 6 \wedge m \in \mathbb{N}.$$

To the best of the authors' knowledge, this kind of non-trivial iso-Boolean primes has not been discovered previously in the literature.

4.4.2 Existence of equivalence classes of non-trivial isoboolean extended twin primes

As can be seen from Table 1, each pair of the non-trivial extended twin primes contains only a singular 1 and all other entries are zeroes in the corresponding row of the Boolean matrix. Consequently, this suggests the possibility of equivalence classes of twin primes of this kind, wherein all members of the class have the same Boolean string. Consider the last two rows of Table 1, which correspond to the twin pair: λ = (536872811, 536872823). These two twin primes have the same Boolean string: β = (000000000000000000100000000000). In the process of exhaustive computational search for such twins, we found yet another twin pair, namely: μ = (400011461, 400011473), which also exhibits the *same Boolean string* β as the non-trivial extended twin pair λ, suggesting that all four primes are equivalent under the computed hypercube Boolean string representation put forward in the present paper. Consequently, this feature exhibited by this kind of twin primes suggests the existence of equivalence classes of primes, such that all members in any given equivalence class have the same Boolean string. In our opinion, such a discovery of the existence of equivalence classes of iso-Boolean primes can result in the possibility of further research on the applications of group theory to the theory of primes through the hypercube Boolean representations that we have considered here. Moreover, all of the twin primes that we have searched for, exhibited a singular 1 in their Boolean string, confirming the existence of equivalence classes of primes under the hypercube Boolean representations.

4.4.3 Singular primes

It is also noteworthy that at the given computational precision, there are primes with rows completely made of zeroes. Some tests to obtain better precision in this special case have augmented it up to $N_{max}=112$ and the zero-row property persisted in quadruple precision. There are prime numbers, via the sums of 2^N, that do not match larger extended twin primes within a great precision, analogous to the row associated with the unique even prime 2, as a trivial example. One can call these primes behaving in this way *singular*. Thus, singular primes have their Boolean rows in the following way: $\exists s \in \mathbb{P} : \langle B_s| = \langle 0|$. More sophisticated zero row primes were computed and are shown below in the set:

{2, 773, 1307, 1433, 1553, 1627, 1811, 1913, … 5348237, 5348333, 5348381, 5348423, … 18906827, 18906961, 18907103, 18907201, …}

where are written the first and some of the following singular primes holding this characteristic, obtained in this computation, showing just some limited intermediate elements. A longer list of such singular primes is shown in Table 2. The present computations show, for example, there are 428924475 of such singular primes lying within the natural number interval [1, 453000000], suggesting almost every prime is singular (around 95%), a result that was not known up to now. The zero Boolean string continues beyond $N=30$, for example, up to $N_{max}=112$. As demonstrated for the singular prime $p=1307$ in Table 3, for which all the generated numbers: $q=p+2^N$ up to $N_{max}=112$ are not primes, thus yielding a zero Boolean string in the corresponding row of matrix **B**. It is interesting to discuss the possibility that, besides the prime number 2, there are other singular primes, which have not been previously generated by summing a power of two to a lesser prime. If such a prime

Table 2 List of some of the members of the *singular* prime set (primes with all-zero Boolean string)

2 773 1307 1433 1553 1627 1811 1913 2131 2297 2371 2417 2477 2719 2777 2837 2897 3023 3181 3347 3797 3863
3889 4283 4591 5011 5101 5297 5507 5821 5897 6379 6421 6781 6803 6971 7013 7159 7253 7411 7643 7793 7937
7951 8161 8447 8543 8839 9049 9103 9323 9649 9721 9833 10181 10613 10711 10973 11213 11251 11437 11491
12301 12379 12619 13103 13577 13591 13999 14033 14537 14551 14657 14717 14957 15173 15289 15683 15797
15889 16127 16193 16519 16547 16741 16831… 9020797 9021137 9021151 9021241 9021269 9021379
9021601…18889433 18889681 18890149 18890177 18890203 18890273 18890467 18890987 18891017 18891113
18891317 18891473 18891581 18891827 18891841 18892217 18892411 18892579 18893003 18893101 18893179
18893299 18893593 18893731 18893843 18894049 18894143 18894301 18894377 18894677 18894713 18894833
18894923 18895117 18895193 18895463 18895511 18895703 18896237 18896327 18896399 18896417 18896587
18896623 18896897 18896993 18897031 18897173 18897227 18897317 18897497 18897737 18897833 18897871
18897899 18897919 18897947 18898193 18898351 18898487 18898591 18898963 18899299 18899371 18899449
18899761 18899899 18900061 18900347 18900361 18900619 18900857 18900881 18900983 18901013 18901291
18901667 18901741 18901781 18901877 18901901 18901907 18902041 18902357 18902593 18902641 18902717
18902791 18903023 18903037 18903163 18903197 18903323 18903413 18903553 18903601 18903637 18903743
18903919 18903947 18904037 18904681 18904819 18904871 18904987 18904997 18905191 18905573 18905743
18905923 18906061 18906721… 18906827 18906961 18907103 18907201…

Table 3 Computed numbers $q = p + 2^N$ for the prime, $p = 1307$, resulting in the 112-dimensional binary string (0, 0, …0, 0)

N	q	N	q	N	q	N	q
1	1309	30	1073743131	59	576460752303424795	88	309485009821345068724782363
2	1311	31	2147484955	60	1152921504606848283	89	618970019642690137449563419
3	1315	32	4294968603	61	2305843009213695259	90	1237940039285380274899125531
4	1323	33	8589935899	62	4611686018427389211	91	2475880078570760549798249755
5	1339	34	17179870491	63	9223372036854777115	92	4951760157141521099596498203
6	1371	35	34359739675	64	18446744073709552923	93	9903520314283042199192995099
7	1435	36	68719478043	65	36893488147419104539	94	19807040628566084398385988891
8	1563	37	137438954779	66	73786976294838207771	95	39614081257132168796771976475
9	1819	38	274877908251	67	147573952589676414235	96	79228162514264337593543951643
10	2331	39	549755815195	68	295147905179352827163	97	158456325028528675187087901979
11	3355	40	1099511629083	69	590295810358705653019	98	316912650057057350374175802651
12	5403	41	2199023256859	70	1180591620717411304731	99	633825300114114700748351603995
13	9499	42	4398046512411	71	2361183241434822608155	100	1267650600228229401496703206683
14	17691	43	8796093023515	72	4722366482869645215003	101	2535301200456458802993406412059
15	34075	44	17592186045723	73	9444732965739290428699	102	5070602400912917605986812822811
16	66843	45	35184372090139	74	18889465931478580856091	103	10141204801825835211973625644315
17	132379	46	70368744178971	75	37778931862957161710875	104	20282409603651670423947251287323
18	263451	47	140737488356635	76	75557863725914323420443	105	40564819207303340847894502573339
19	525595	48	281474976711963	77	151115727451828646839579	106	81129638414606681695789005145371
20	1049883	49	562949953422619	78	302231454903657293677851	107	162259276829213363391578010289435
21	2098459	50	1125899906843931	79	604462909807314587354395	108	324518553658426726783156020577563
22	4195611	51	2251799813686555	80	1208925819614629174707483	109	649037107316853453566312041153819
23	8389915	52	4503599627371803	81	2417851639229258349413659	110	1298074214633706907132624082306331
24	16778523	53	9007199254742299	82	4835703278458516698826011	111	2596148429267413814265248164611355
25	33555739	54	18014398509483291	83	9671406556917033397650715	112	5192296858534827628530496329221403
26	67110171	55	36028797018965275	84	19342813113834066795300123		
27	134219035	56	72057594037929243	85	38685626227668133590598939		
28	268436763	57	144115188075857179	86	77371252455336267181196571		
29	536872219	58	288230376151713051	87	154742504910672534362391835		

appears to exist, then one can call it *isolated*. Isolated primes must be necessarily obtained summing some powers 2^N to composite odd numbers.

From the computational results thus obtained we have displayed below a pair of extra conjectures regarding iso-Boolean and singular primes:

Conjecture 2 *There exists an infinite number of equivalence classes of iso-boolean extended twin primes such as the ones displayed in* Table 1 *starting with {5,191…}, {17,137,5477,29387,53147,76367, 79397,….518057,549257,577937}, {19,229,9739,10099, 32359, …}, {23,2843,8963, 13163,15263,29753,…}, {31, 1201,4441,5281,8431,9661,10651,12253,17011,21751,24181,28591,30661, 32491, 52951,54331,78691,78697…}, {61,1951,3823,6571,7351,7621,7681,11551,13711,16231,17551,18541,18691,19471, 21481,23203, 25741, …}, {2411,2423,4463,4493,4973, 6323, …}, {49993,49999,52183, 52543…}, {6709,8329,10639, 11059, …58573, 58579,59809, 60169, …}, {3061, 4231, 6451, 8311,… 50221,51481,…106543, 106591…} and so forth. Furthermore, the separation between any two isoboolean extended twin primes is always 6q, for some natural number q.*

Conjecture 3 *There exists an infinite number of extended singular primes, such as the ones displayed in* Table 2 *starting with* {*2, 773, 1307, 1433, 1553, 1627, 1811, 1913, 2131, 2297, 2371, 2417, 2477,2719, 2777, 2837, 2897, 3023, 3181,3347, 3797, 3863, 3889, 4283, 4591, 5011, 5101, 5297, 5507, 5821, 5897,6397, 6781,… 17293,….32003,…}.*

There are chemical parallels between isoboolean primes and the related hypercube representations. First and foremost the discovery of isoboolean as well as singular primes were stimulated by relativistic quantum computing in that the spinor representations of relativistic quantum states are Boolean in nature, and thus any relativistic electronic configuration is readily represented in Boolean space. As the vertices of an n-dimensional hypercube provide representations of Boolean strings, they also represent the possible spinor products for n-dimensional relativistic quantum states. The n-dimensional hypercubes also provide representations of water clusters, $(H_2O)_n$, taking into account of their fluxional nature. Thus the related n-dimensional hyperoctahedral groups of the Boolean hypercubes, which are wreath products, find applications in the representations of the rovibronic levels of nonrigid water clusters [19, 20]. In mathematical sciences, graph theoretical representations and the interconnectivity of such prime numbers could provide significant insights into extremely difficult problems of primes such as the Chowla conjecture concerning prime numbers [21–23]. It is hoped that the present study would stimulate such interdisciplinary explorations into unravelling connections between graph theory, number theory and topological indices. Finally isoboolean twin primes are reminiscent of molecular isomers in that the isoboolean twin primes are different primes that have the same Boolean representation. The analog of molecular chirality is yet to be discovered in the context of isoboolean twin primes as we are yet to find two's complement of a Boolean representation of a prime that would also correspond to a Boolean string of another prime. Such topics could be the subject matter of future studies.

5 Conclusions

Inspired by relativistic quantum computing, we have formulated a recursive technique for representing prime numbers in the Boolean hypercube space through Boolean and logical matrices. We have also developed computational techniques that employed elegant pack/unpack and sequential read/write algorithms for an exhaustive generation of such Boolean matrices for prime numbers. During our computational generation, we have discovered a new set of extended twin primes which exhibit the same Boolean string in the hypercube Boolean representation of powers of 2. Furthermore, we have demonstrated the possibility of the existence of equivalence classes of such extended twin primes, where all members in any equivalence class contain the same computed Boolean string. Owing to these results it is plausible to formulate a conjecture about the existence of an infinite number of extended twin primes. We have also shown the existence of singular primes which exhibit a completely zero Boolean string for even large $N_{max}=112$. The existence of both equivalence classes of twin primes and singular primes suggests the possibility of future group theoretical applications that may make use of hyperoctahedral groups to these special classes of primes, for partitioning them into equivalence classes. We have established that about 95% of the primes are singular under the Boolean hypercube classification.

Appendix 1: the Cartesian sum of two number sets

Suppose two numerical sets are known, in our case natural number sets: $\mathbb{U} = \left\{ u_I | I = 1, U \right\}$ and $\mathbb{V} = \left\{ v_J | J = 1, V \right\}$. The Cartesian sum $\mathbb{X} = \mathbb{U} \oplus \mathbb{V}$ of such sets defines a new set X, whose elements are simply constructed by all the possible sums of the sets entering the Cartesian sum: $\mathbb{X} = \left\{ x_{IJ} = u_I + v_J | I = 1, U; J = 1, V \right\}$. The recursive algorithm to construct natural numbers, as defined earlier [9–12], and in this paper, can be seen as a particular case of the Cartesian sum of two sets, where one of the involved sets has cardinality one.

The Cartesian sum of two sets is commutative: $\mathbb{X} = \mathbb{U} \oplus \mathbb{V} = \mathbb{V} \oplus \mathbb{U}$. The Cartesian sum of two sets can be extended to any number of sets.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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