

# Calculus Made Easy Problem Sets

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# Calc Made Easy

1998 Edition

✓ = verified

✓ = done but  
not verified

- ✓ 4. Simplest Cases  
Ex. 1, p. 58
- ✓ 5. Next Stage: What to Do with Constants  
Ex. 2, p. 64
- ✓ 6. Sums, Differences, Products, and Quotients  
Ex. 3, p. 76
- ✓ 7. Successive Differentiation  
Ex. 4, p. 82
- ✓ 8. When Time Varies  
Ex. 5, p. 91
- ✓ 9. The Chain Rule (Introducing a Useful Dodge)  
Ex. 6, p. 100  
Ex. 7, p. 101
- ✓ 10. Geometric Meaning of Differentiation  
Ex. 8, p. 113
- ✓ 11. Maxima and Minima  
Ex. 9, p. 130
- ✓ 12. Curvature of Curves  
Ex. 10, p. 137
- ✓ 13. Partial Fractions and Inverse Functions  
Ex. 11, p. 147
- ✓ 14. Compound Interest and Organic Growth  
Ex. 12, p. 166  
Ex. 13, p. 173
- ✓ 15. How to Deal with Sines and Cosines  
Ex. 14, p. 183
- ✓ 16. Partial Differentiation  
Ex. 15, p. 189
- ✓ 17. Integration  
Ex. 16, p. 197
- ✓ 18. Integrating as the Reverse of Differentiating  
Ex. 17, p. 209
- ✓ 19. On Finding Areas by Integrating  
Ex. 18, p. 225
- ✓ 20. Dodges, Pitfalls, and Triumphs  
Ex. 19, p. 233
- ✓ 21. Solving Differential Equations  
Ex. 20, p. 248
- ✓ 22. A Little More about Curvature of Curves  
Ex. 21, p. 260
- ✓ 23. How to Find the Length of an Arc on a Curve  
Ex. 22, p. 274

Bonus Exercises ✓

# Simplest Derivatives

## Calc Made Easy, Exercises 1

$$1) \quad y = x^3; \quad y' = 3x^2$$

$$2) \quad y = x^{-\frac{3}{2}}; \quad y' = -\frac{3}{2}x^{-\frac{5}{2}}$$

$$3) \quad y = x^{2a}; \quad y' = 2a x^{(2a-1)}$$

$$4) \quad v = t^{2.4}; \quad v' = 2.4t^{1.4}$$

$$5) \quad z = \sqrt[3]{v}; \quad z' = \frac{1}{3}v^{-\frac{2}{3}}$$

$$6) \quad y = \sqrt[3]{x^{-5}}; \quad y' = -\frac{5}{3}x^{-\frac{8}{3}}$$

$$7) \quad u = x^{-\frac{8}{5}}; \quad u' = -\frac{8}{5}x^{-\frac{13}{5}}$$

$$8) \quad y = 2x^a; \quad y' = 2ax^{(a-1)}$$

$$9) \quad y = x^{\frac{3}{q}}; \quad y' = \frac{3}{q}x^{\frac{3-q}{q}}$$

$$10) \quad y = x^{-\frac{m}{n}}; \quad y' = -\frac{m}{n}x^{-\frac{(m+n)}{n}}$$

## Calc Made Easy, Exercises 2, p. 64

$$1) \quad y = ax^3 + b; \quad y' = 3ax^2$$

$$2) \quad y = 13x^{\frac{3}{2}} - c; \quad y' = \frac{39}{2}x^{\frac{1}{2}}$$

$$3) \quad y = 12x^{\frac{1}{2}} + c^{\frac{1}{2}}; \quad y' = 6x^{-\frac{1}{2}}$$

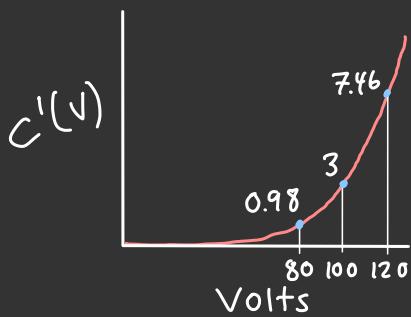
$$4) \quad y = c^{\frac{1}{2}}x^{\frac{1}{2}}; \quad y' = \frac{c^{\frac{1}{2}}}{2}x^{-\frac{1}{2}}$$

$$5) \quad v = \frac{az^n - 1}{c}; \quad v' = \frac{na}{c}z^{n-1}$$

$$6) \quad y = 1.18t^2 + 22.4; \quad y' = 2.36t$$

$$7) \quad l(t) = a(1 + bt) \\ l'(t) = ab, \text{ where } b = 0.000012$$

$$8) C(V) = aV^b; C'(V) = abV^{b-1}$$



$$a = 0.5 \times 10^{-10}$$

$$b = 6$$

$$9) n = (DL)^{-1} \sqrt{\frac{gT}{\pi\sigma}} \quad \frac{\partial n}{\partial D} = -D^{-2} L^{-1} \sqrt{\frac{gT}{\pi\sigma}}$$

$n$  - frequency

$D$  - diameter

$L$  - length

$\sigma$  - specific gravity

$T$  - force

$$\frac{\partial n}{\partial L} = -D^{-1} L^{-2} \sqrt{\frac{gT}{\pi\sigma}}$$

$$\frac{\partial n}{\partial T} = (2DL)^{-1} \sqrt{\frac{g}{\pi\sigma}} T^{-\frac{1}{2}}$$

$$\frac{\partial n}{\partial \sigma} = -(2DL)^{-1} \sqrt{\frac{gT}{\pi}} \sigma^{-\frac{3}{2}}$$

$$10) P = \left( \frac{2E}{1-\sigma^2} \right) \frac{t^3}{D^3} \quad dP/dt = 3 \left( \frac{2E}{1-\sigma^2} \right) \frac{t^2}{D^3}$$

$$4t \ll D \quad dP/dD = -3 \left( \frac{2E}{1-\sigma^2} \right) \frac{t^3}{D^4}$$

This implies  $dP/dD = -D/t$ .

$$11) \text{ circle: } C = 2\pi r; C' = 2\pi$$

$$A = \pi r^2; A' = 2\pi r$$



# Differentiation Practice

from Calc Made Easy, Ch. 6, p. 71

$$0) \quad y = \frac{bx^5 + c}{x^2 + a}$$

$$y' = \frac{5bx^4(x^2 + a) - (bx^5 + c)2x}{(x^2 + a)^2}$$

$$= \frac{5bx^6 + 5abx^4 - 2bx^6 - 2cx}{(x^2 + a)^2}$$

$$= \frac{3bx^6 + 5abx^4 - 2cx}{(x^2 + a)^2}$$

$$3) \quad z = a\theta^{-2/3} - b\theta^{-1/5} - c$$

$$z' = -\frac{2a}{3}\theta^{-5/3} + \frac{b}{5}\theta^{-6/5}$$

$$z'' = \frac{10a}{9}\theta^{-8/3} - \frac{6b}{25}\theta^{-11/5}$$

$$z''' = -\frac{80}{27}a\theta^{-11/3} + \frac{6b}{125}b\theta^{-16/5}$$

$$5) \quad y = (2x-3)(x+1)^2$$

$$= (2x-3)(x^2+2x+1)$$

$$y' = 2(x^2+2x+1) + (2x-3)(2x+2)$$

$$= 2[x^2+2x+1 + 2x^2 + 2x - 3x - 3]$$

$$= 2(3x^2+x-2)$$

$$= 2(3x-2)(x+1)$$

$$y'' = 2(6x+1) \quad y''' = 12$$

$$7) \quad w = \left(\theta + \frac{1}{\theta}\right)\left(\sqrt{\theta} + \frac{1}{\sqrt{\theta}}\right)$$

$$= (\theta + \theta^{-1})(\theta^{1/2} + \theta^{-1/2})$$

$$= (\theta^{3/2} + \theta^{1/2} + \theta^{-1/2} + \theta^{-3/2})$$

$$w' = \frac{3}{2}\theta^{1/2} + \frac{1}{2}\theta^{-1/2} - \frac{1}{2}\theta^{-3/2} - \frac{3}{2}\theta^{-5/2}$$

$$w'' = \frac{3}{4}\theta^{-1/2} - \frac{1}{4}\theta^{-3/2} + \frac{3}{4}\theta^{-5/2} + \frac{15}{4}\theta^{-7/2}$$

$$w''' = -\frac{3}{8}\theta^{-3/2} + \frac{3}{8}\theta^{-5/2} - \frac{15}{8}\theta^{-7/2} - \frac{105}{8}\theta^{-9/2}$$

$$1) \quad y = \frac{a}{b^2}x^3 - \frac{a^2}{b}x + \frac{a^2}{b^2}$$

$$y' = \frac{3a}{b^2}x^2 - \frac{2a}{b}$$

$$y'' = \frac{6a}{b^2}x \quad y''' = \frac{6a}{b^2}$$

$$2) \quad y = 2a\sqrt{bx^3} - \frac{3b\sqrt[3]{a}}{x} - 2\sqrt{ab}$$

$$y' = 3a\sqrt{bx} + \frac{3b\sqrt[3]{a}}{x^2}$$

$$y'' = \frac{3}{2}a\sqrt{\frac{b}{x}} - \frac{6b\sqrt[3]{a}}{x^3}$$

$$y''' = -\frac{3}{4}a\frac{\sqrt{b}}{\sqrt{x^3}} + \frac{18b\sqrt[3]{a}}{x^4}$$

$$4) \quad v = (at^2 - bt + c)^3$$

$$v' = 3(at^2 - bt + c)^2 \cdot (2at - b)$$

$$v'' = b(at^2 - bt + c)(2at - b)^2 +$$

$$6a(at^2 - bt + c)^2$$

$$6) \quad y = ax^3(x-b) = ax^4 - abx^3$$

$$y' = 4ax^3 - 3abx^2 \quad \begin{matrix} a = 0.5 \\ b = 3 \end{matrix}$$

$$y'' = 12ax^2 - 6abx = 6x^2 - 9x$$

$$y''' = 24ax - 6ab = 12x - 9$$

$$8) \quad y = a(1 + a\sqrt{x} + a^2x)^{-1}$$

$$y' = -a(1 + a\sqrt{x} + a^2x)^{-2} \left(\frac{1}{2}a\sqrt{x} + a^2\right)$$

$$9) y = \frac{x^2}{x^2+1}$$

$$\begin{aligned}y' &= 2x(x^2+1) + x^2(2x) \\&= 2x^3 + 2x + 2x^3 \\&= 4x^3 + 2x = 2x(2x^2 + 1)\end{aligned}$$

$$\begin{aligned}10) y &= \frac{a+\sqrt{x}}{a-\sqrt{x}} \\y' &= \frac{\frac{1}{2}\bar{x}^{-\frac{1}{2}}(a-x^{\frac{1}{2}}) - (a+x^{\frac{1}{2}})(-\frac{1}{2}\bar{x}^{-\frac{1}{2}})}{(a-x^{\frac{1}{2}})^2} \\&= \frac{a}{x^{\frac{1}{2}}(a-x^{\frac{1}{2}})^2}\end{aligned}$$

$$11) \theta = \frac{1-a\sqrt[3]{t^2}}{1+a\sqrt[2]{t^3}}$$

$$\dot{\theta} = \frac{-(\frac{2}{3})a t^{-\frac{1}{3}} \left(1+a\sqrt[3]{t^2}\right) - \left(1-a\sqrt[2]{t^3}\right)^{\frac{3}{2}} a t^{\frac{1}{2}}}{\left(1+a\sqrt[2]{t^3}\right)^2}$$

12) A reservoir shaped like a  $45^\circ$  pyramid frustum is being filled at a constant rate. Our goal is to express the rate at which the height changes.

$$V = \frac{h}{3}(A + a + \sqrt{Aa})$$

$$= \frac{h}{3}((2h+p)^2 + p^2 + (2h+p)p)$$

$$= \frac{h}{3}(4h^2 + 4hp + p^2 + p^2 + 2hp + p^2)$$

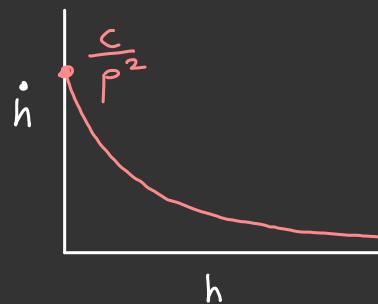
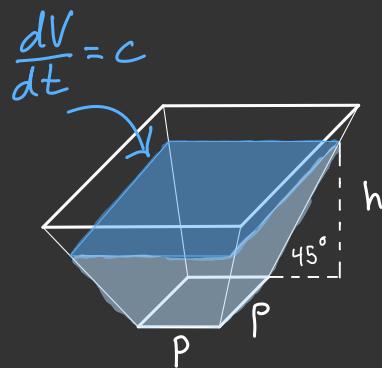
$$= \frac{4}{3}h^3 + 2h^2p + hp^2$$

$$\dot{V} = 4h^2\dot{h} + 4h\dot{h}p + p^2\dot{h}$$

$$= \dot{h}(4h^2 + 4hp + p^2) = c$$

$$\dot{h} = \frac{c}{(2h+p)^2}$$

$$\dot{h} = \frac{c}{(2h+p)^2}$$



$$13) P = \left(\frac{a+t}{b}\right)^5 ; P' = 5\left(\frac{a+t}{b^5}\right)^4$$

$$t=100, a=40, b=140, P' = 128$$

# Sums, Differences, Products, Quotients

from Calc. Made Easy, Exercises 3, p. 76

$$1) v = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$v' = 1 + \cancel{\frac{2x}{2!}} + \cancel{\frac{3x^2}{3 \cdot 2!}} + \cancel{\frac{4x^3}{4 \cdot 3 \cdot 2!}} + \dots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$v' = v = v'' = v'''$$

(naturally, since  $\sum_{i=0}^{\infty} x^i / i! = \exp(x)$ )

$$y = (a+x)^2$$

$$y' = 2(a+x) \quad y'' = 2$$

$$y = ax^2 + bx + c$$

$$y' = 2ax + b \quad y'' = 2a$$

$$y = (x+a)^3$$

$$y' = 3(x+a)^2 \quad y'' = 6x + 6a$$

Amazingly, Thompson makes us do this one before teaching the chain rule...

$$2) w = at - \frac{1}{2}bt^2$$

$$\dot{w} = a - bt \quad \ddot{w} = -b$$

$$\dddot{w} = 0$$

$$3) y = (x + \sqrt{-1})(x - \sqrt{-1})$$

$$= x^2 - i^2$$

$$y' = 2x \quad y'' = 2 \quad y''' = 0$$

$$4) y = (ax + bx^2)(c + dx + gx^3)$$

$$y''' = (a + 2bx)(c + dx + gx^3) + (ax + bx^2)(d + 3gx^2)$$

$$y''' = 2b(c + dx + gx^3) + (a + 2bx)(d + 3gx^2) + (a + 2bx)(d + 3gx^2) + (ax + bx^2)6gx$$

$$(a = 197 \quad b = -34 \quad c = 7 \quad d = 22 \quad g = -83)$$

$$5) x = (y+3)(y+5)$$

$$x' = y+5 + y+3$$

$$= 2y + 8 = 2(y+4)$$

$$x'' = 2 \quad x''' = 0$$

$$6) y = ax(b + cx^2)$$

$$y' = ab + 3acx^2$$

$$= a(b + 3cx^2)$$

$$y'' = bacx \quad y''' = bac$$

$$(a = 1.3709 \quad b = 112.6 \quad c = 45.202)$$

$$7) \quad y = \frac{2x+3}{3x+2}$$

$$y' = \frac{2(3x+2) - (2x+3)3}{(3x+2)^2} = -5(3x+2)^{-2}$$

$$y'' = 30(3x+2)^{-3}$$

$$y''' = -270(3x+2)^{-4}$$

$$8) \quad y = \frac{1+x+2x^2+3x^3}{1+x+2x^2}$$

$$\begin{aligned} y' &= \frac{(1+4x+9x^2)(1+x+2x^2) - (1+x+2x^2+3x^3)(1+4x)}{(1+x+2x^2)^2} \\ &= \frac{(1+x+2x^2+4x+4x^2+8x^3+9x^2+9x^3+18x^4) - }{(1+x+2x^2)^2} \\ &\quad \left( (1+x+2x^2+3x^3+4x+4x^2+8x^3+12x^4) \right) / (1+x+2x^2)^2 \\ &= 9x^2+6x^3+6x^4 / (1+x+2x^2)^2 \\ &= 3x^2(3+2x+2x^2) / (1+x+2x^2)^2 \end{aligned}$$

$$9) \quad y = \frac{ax+b}{cx+d}$$

$$y' = \frac{a(cx+d) - (ax+b)c}{(cx+d)^2}$$

$$= \cancel{acx+ad} - \cancel{acx-cb}$$

$$= \frac{ad-cb}{(cx+d)^2}$$

$$10) \quad y = \frac{x^n + a}{x^{-n} + b}$$

$$\begin{aligned}y' &= \frac{nx^{n-1}(x^{-n} + b) + (x^n + a)n(-x^{-(n+1)})}{(x^{-n} + b)^2} \\&= \frac{nx^{-1} + bnx^{n-1} + nx^{-1} + anx^{-(n+1)}}{(x^{-n} + b)^2} \\&= \frac{n(2x^{-1} + bx^{n-1} + ax^{-(n+1)})}{(x^{-n} + b)^2}\end{aligned}$$

$$11) \quad C = a + bt + ct^2$$

$$C' = b + 2ct$$

$$12) \quad R = R_0(1 + at + bt^2)$$

$$R' = R_0(a + 2bt)$$

$$R = R_0(1 + at + b\sqrt{t})$$

$$R' = R_0(a + \frac{1}{2}bt^{-\frac{1}{2}})$$

$$R = R_0(1 + at + bt^2)^{-1}$$

$$R' = -R_0(a + 2bt)(1 + at + bt^2)^{-2}$$

$$13) \quad E = a(1 - b(t - c) + d(t - c)^2)$$

$$E' = a(2d(t - c) - b)$$

$$14) \quad E = a + bl + \frac{c + kl}{i}$$

$$\frac{\partial E}{\partial l} = b + k_i$$

$$\frac{\partial E}{\partial i} = \frac{-(c + kl)}{i^2}$$

# SUCCESSIONAL DIFFERENTIATION PRACTICE

CALC MADE EASY EXERCISES 4, p. 82

$$1) \quad y = 17x + 12x^2$$

$$y' = 17 + 24x$$

$$y'' = 24$$

$$2) \quad y = \frac{x^2 + a}{x + a}$$

$$y' = \frac{2x(x+a) - (x^2 + a)}{(x+a)^2}$$

$$= \frac{2x^2 + 2xa - x^2 - a}{(x+a)^2} = \frac{x^2 + 2xa - a}{(x+a)^2}$$

$$y'' = \frac{(2x+2a)(x+a)^2 - (x^2 + 2xa - a)2(x+a)}{(x+a)^4}$$

$$= 2(x+a)^{-3} \left( (x+a)^2 - (x^2 + 2xa - a) \right)$$

$$= 2(x+a)^{-3} \left( \cancel{x^2} + \cancel{2xa} + a^2 - \cancel{x^2} - \cancel{2xa} + a \right)$$

$$= \frac{2a(a+1)}{(x+a)^3}$$

$$3) \quad y = 1 + x + \frac{x^2}{2} + \frac{x^3}{3 \cdot 2 \cdot 1} + \frac{x^4}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$y' = 1 + x + \frac{x^2}{2} + \frac{x^3}{3 \cdot 2 \cdot 1}$$

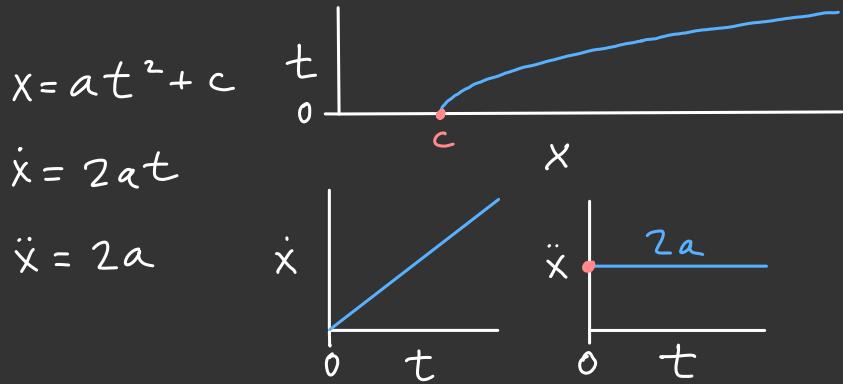
$$y'' = 1 + x + \frac{x^2}{2}$$

Added second derivatives of Ex. 3, 1-7  
 & Chapter 6, 1-7

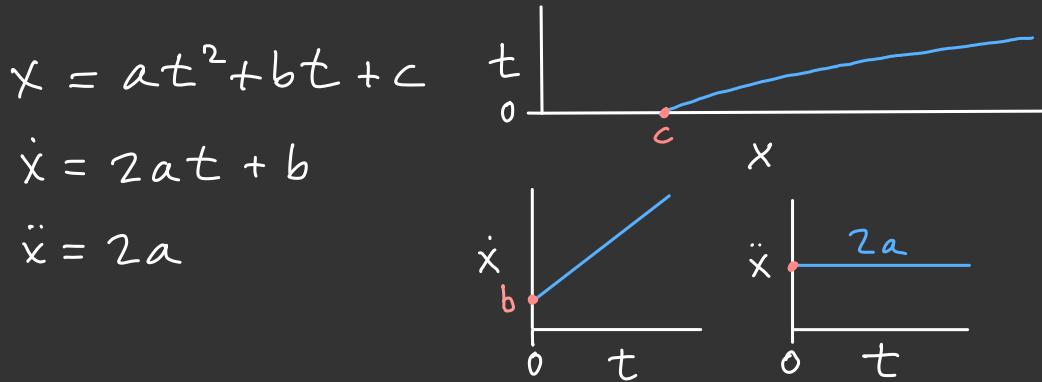
# Rates of Change

from Calc. Made Easy Ch. 8

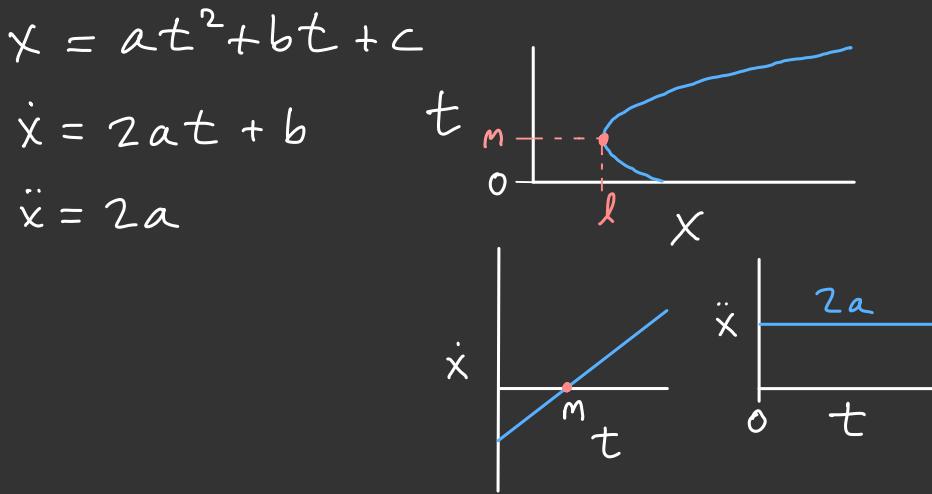
1) Constant Acceleration, Zero Initial Velocity



2) Constant Acceleration, Positive Initial Velocity



3) Constant Acceleration, Negative Initial Velocity

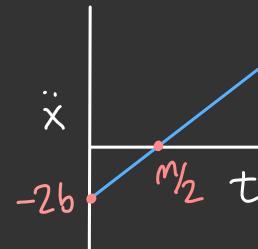
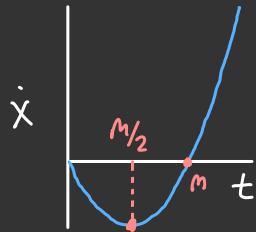
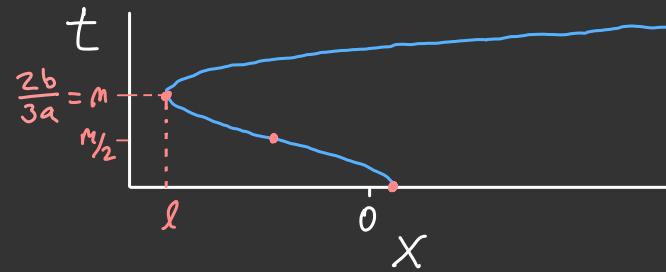


## 4&5) Variable Acceleration

$$x = at^3 - bt^2 + c$$

$$\dot{x} = 3at^2 - 2bt$$

$$\ddot{x} = 6at - 2b$$

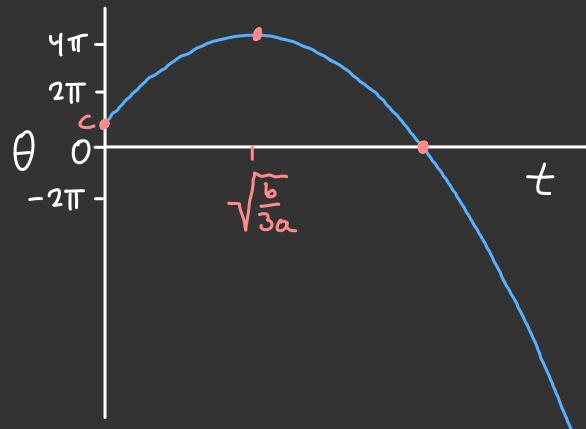


## 6) Rotational Acceleration

$$\theta = c + bt - at^3$$

$$\dot{\theta} = b - 3at^2$$

$$\ddot{\theta} = -6at$$



From Calc. Made Easy, Exercises 5, p. 91

$$1) \quad y = a + bt^2 + ct^4$$

$$\dot{y} = 2bt + 4ct^3$$

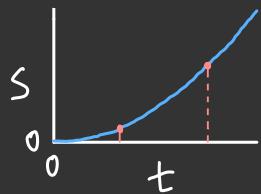
$$\ddot{y} = 2b + 12ct^2$$

$$6) \quad \theta = 2.1 - 3.2t + 4.8t^2$$

$$\dot{\theta} = -3.2 + 9.6t$$

$$\ddot{\theta} = 9.6$$

(2)  $s = 16t^2$



$$7) \quad s = 6.8t^3 - 10.8t$$

$$\dot{s} = 20.4t^2 - 10.8$$

$$\ddot{s} = 40.8t$$

$$3) \quad x = at - \frac{1}{2}gt^2$$

$$\dot{x} = a - gt$$

$$\ddot{x} = -g$$

$$8) \quad h = 0.5 + \frac{1}{10}\sqrt[3]{t-125}$$

$$\dot{h} = \frac{1}{30} (t-125)^{-\frac{2}{3}}$$

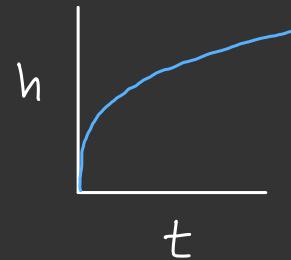
$$\ddot{h} = -\frac{1}{45} (t-125)^{-\frac{5}{3}}$$

$$4 \& 5) \quad s = 12 - 4.5t + 6.2t^2$$

$$\dot{s} = -4.5 + 12.4t$$

$$@ t=4, \dot{s}=45.1$$

$$\ddot{s} = 12.4$$



$$\begin{aligned}
 9) \quad P &= \frac{4}{4+t^2} + 0.8t - 1 \\
 &= 4(4+t^2)^{-1} + 0.8t - 1 \\
 \dot{P} &= -4(4+t^2)^{-2} 2t + 0.8 \\
 &= -8t(t^2+4)^{-2} + 0.8 \\
 \ddot{P} &= \frac{-8(t^2+4)^2 + 8t \cdot 2(t^2+4) \cdot 2t}{(t^2+4)^4} \\
 &= \frac{8(t^2+4)(-t^2-4+4t^2)}{(t^2+4)^3} = \frac{(3t^2-4)}{(t^2+4)^3}
 \end{aligned}$$

10)  $s = t^n \quad \dot{s} = nt^{(n-1)} \quad \ddot{s} = n(n-1)t^{(n-2)}$

(i) Find  $n$  so that  $n \cdot b^{(n-1)} = 2 \cdot a^{(n-1)}$   
for some  $a \neq b$ .

$$\log 2 + (n-1) \log a = (n-1) \log b$$

$$n = 1 + \frac{\log 2}{\log(b/a)}$$

In the problem,  $b=10$  &  $a=5$ , so  $n=2$ .

(ii)  $nd^{(n-1)} = n(n-1)d^{(n-2)}$   
for some value  $d$ .

$$\text{so } d = n-1 \dots$$

$$\text{if } d=0, n=1$$

# Chain Rule Practice

from Calc Made Easy, Ch. 9 (added numbers)

$$1) \quad y = (x^2 + a^2)^{\frac{3}{2}}$$

$$y' = \frac{3}{2}(x^2 + a^2)^{\frac{1}{2}} \cancel{x}$$

$$2) \quad Y = \left( m - n x^{\frac{2}{3}} + \frac{P}{x^{\frac{1}{3}}} \right)^\alpha$$

$$Y' = \alpha \left( m - n x^{\frac{2}{3}} + \frac{P}{x^{\frac{1}{3}}} \right)^{\alpha-1} \left( -\frac{2}{3} n x^{-\frac{1}{3}} - \frac{1}{3} P x^{-\frac{4}{3}} \right)$$

$$3) \quad y = \sqrt{a+x} \quad 4) \quad y = (a+x^2)^{-\frac{1}{2}}$$

$$y' = \frac{1}{2}(a+x)^{-\frac{1}{2}} \quad y' = -\frac{1}{2}(a+x^2)^{-\frac{3}{2}} \cancel{x}$$

$$5) \quad y = (x^3 - a^2)^{-\frac{1}{2}}$$

$$y' = -\frac{1}{2}(x^3 - a^2)^{-\frac{3}{2}} \cdot 3x^2$$

$$6) \quad y = \sqrt{\frac{(1-x)}{(1+x)}} = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$$

$$\begin{aligned} \frac{dy}{dx} &= f'g + fg' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)(1+x)^{-\frac{1}{2}} + (1-x)^{\frac{1}{2}}(-\frac{1}{2})(1+x)^{-\frac{3}{2}} \cdot 1 \\ &= (-\frac{1}{2}) \left[ (1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}} + (1-x)'(1-x)^{-\frac{1}{2}}(1+x)^{-1}(1+x)^{-\frac{1}{2}} \right] \\ &= (-\frac{1}{2})(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}} \left[ 1 + (1-x)(1+x)^{-1} \right] \\ &= (-\frac{1}{2})(1-x^2)^{-\frac{1}{2}} \left( (1+x)(1+x)^{-1} + (1-x)(1+x)^{-1} \right) \\ &= (-\frac{1}{2})(1-x^2)^{-\frac{1}{2}} \left( 1 + \cancel{(1+x + 1-x)} \right) \\ &= - (1-x^2)^{-\frac{1}{2}}(1+x)^{-1} = \frac{-1}{(1+x)\sqrt{1-x^2}} \end{aligned}$$

$$7) \quad y = \frac{x^{\frac{3}{2}}}{(1+x^2)^{\frac{1}{2}}} = x^{\frac{3}{2}}(1+x^2)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} + x^{\frac{3}{2}}(-\frac{1}{2})(1+x^2)^{-\frac{3}{2}} \cancel{x}$$

$$= (\frac{3}{2})(1+x^2)^{-\frac{3}{2}}x^{\frac{1}{2}}(3(1+x^2) - 2x^2)$$

$$= (\frac{1}{2})x^{\frac{1}{2}}(1+x^2)^{-\frac{3}{2}}(3+x^2)$$

$$= \frac{(3+x^2)x^{\frac{1}{2}}}{2(1+x^2)^{\frac{3}{2}}}$$

$$8) \quad y = x^3 + (x^2 + x + a)^{\frac{3}{2}}$$

$$\frac{dy}{dx} = 3x^2 + (\frac{3}{2})(x^2 + x + a)^{\frac{1}{2}}(2x+1)$$

$$= (\frac{3}{2}) \left( 2x^2 + (2x+1)(x^2 + x + a)^{\frac{1}{2}} \right)$$

$$9) \quad y = \left( \frac{a^2 + x^2}{a^2 - x^2} \right)^{\frac{1}{2}} \left( \frac{a^2 - x^2}{a^2 + x^2} \right)^{\frac{1}{3}} = \left( \frac{a^2 + x^2}{a^2 - x^2} \right)^{\frac{1}{6}} = (a^2 + x^2)^{\frac{1}{6}} (a^2 - x^2)^{-\frac{1}{6}}$$

$$\frac{dy}{dx} = \frac{1}{6} (a^2 + x^2)^{-\frac{5}{6}} \cdot \cancel{2x} (a^2 - x^2)^{-\frac{1}{6}} + (a^2 + x^2)^{\frac{1}{6}} (+\cancel{\frac{1}{3}}) (a^2 - x^2)^{-\frac{7}{6}} (+\cancel{2x})$$

$$= \left(\frac{1}{3}\right) \times (a^2 + x^2)^{\frac{1}{6}} (a^2 - x^2)^{-\frac{1}{6}} \left( (a^2 + x^2)^{-1} + (a^2 - x^2)^{-1} \right)$$

$$= \left(\frac{1}{3}\right) \times (a^2 + x^2)^{\frac{1}{6}} (a^2 - x^2)^{-\frac{1}{6}} \left( \frac{a^2 - x^2 + a^2 + x^2}{a^4 - x^4} \right)$$

$$= \left(\frac{2}{3}\right) \times (a^2 + x^2)^{\frac{1}{6}} (a^2 - x^2)^{-\frac{1}{6}} \left( \frac{a^2}{a^4 - x^4} \right)$$

$$= \frac{2x}{3} \left( \frac{a^2 + x^2}{a^2 - x^2} \right)^{\frac{1}{6}} \left( \frac{a^2}{a^4 - x^4} \right)$$

10) For some function  $y^n$ , find its derivative with respect to  $y$ .

$$\frac{dy^n}{dy^5} \quad \text{let } x = y^5, \text{ so } y^n = x^{\frac{n}{5}} \quad \frac{dx^{\frac{n}{5}}}{dx} = \frac{n}{5} x^{\frac{n-5}{5}}$$

$$\frac{dy^n}{dy^5} = \frac{n}{5} y^{n-5}$$

$$11) \quad y = \frac{x^{\frac{3}{2}}}{b} (a-x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = b^{-1} \left( \frac{3}{2} x^{\frac{1}{2}} (a-x)^{\frac{1}{2}} + x^{\frac{3}{2}} \left(\frac{1}{2}\right) (a-x)^{-\frac{1}{2}} (-1) \right)$$

$$= b^{-1} 2^{-1} x^{\frac{1}{2}} \left( 3(a-x)^{\frac{1}{2}} - x (a-x)^{-\frac{1}{2}} \right)$$

$$= b^{-1} 2^{-1} x^{\frac{1}{2}} (a-x)^{-\frac{1}{2}} \left( 3(a-x) - x \right)$$

$$= (2b)^{-1} (a-x)^{-\frac{1}{2}} (3ax^{\frac{1}{2}} - 4x^{\frac{3}{2}})$$

$$= \frac{x^{\frac{1}{2}} (3a - 4x)}{2b \sqrt{a-x}}$$

Agony!

$$\frac{d^2y}{dx^2} = (2b)^{-1} \left( \left(+\frac{1}{2}\right) (a-x)^{-\frac{3}{2}} \cancel{(3ax^{\frac{1}{2}} - 4x^{\frac{3}{2}})} + (a-x)^{-\frac{1}{2}} \left( \frac{3}{2} ax^{-\frac{1}{2}} - 6x^{\frac{1}{2}} \right) \right)$$

$$= (2b)^{-1} (a-x)^{-\frac{3}{2}} \left( \left(\frac{1}{2}\right) (3ax^{\frac{1}{2}} - 4x^{\frac{3}{2}}) + (a-x) \left( \frac{3}{2} ax^{-\frac{1}{2}} - 6x^{\frac{1}{2}} \right) \right)$$

## Additional Examples with multiple steps.

$$1) \quad z = 3x^4 \quad \frac{dy}{dv} = \frac{1}{2}(1+v)^{-\frac{1}{2}} \quad \frac{dz}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dz} \cdot \frac{dz}{dx}$$

$$v = 7z^{-2} \quad \frac{dv}{dz} = -14z^{-3} \quad = \frac{1}{2}(1+v)^{-\frac{1}{2}} \cdot (-14z^{-3}) \cdot (12x^3)$$

$$y = (1+v)^{\frac{1}{2}} \quad \frac{dz}{dx} = 12x^3$$

$$2) \quad t = \frac{1}{5\sqrt{\theta}} \quad \frac{dt}{d\theta} = -\frac{1}{10}\theta^{-\frac{3}{2}}$$

$$x = t^3 + \frac{t}{2} \quad \frac{dx}{dt} = 3t^2 + \frac{1}{2}$$

$$v = 7x^2(x-1)^{-\frac{1}{3}} \quad \frac{dv}{dx} = 14x(x-1)^{-\frac{1}{3}} + 7x^2(-\frac{1}{3})(x-1)^{-\frac{4}{3}}$$

$$\frac{dv}{d\theta} = \left(\frac{7}{3}\right)x(x-1)^{-\frac{4}{3}}(5x-6)$$

$$\frac{dv}{d\theta} = \left(\frac{7}{3}\right)x(x-1)^{-\frac{4}{3}}(5x-6)(3t^2 + \frac{1}{2})(-\frac{1}{10}\theta^{-\frac{3}{2}})$$

$$3) \quad \theta = 3\alpha^2 x^{-\frac{3}{2}} = 3\alpha^2 x^{-\frac{1}{2}} \quad \frac{d\theta}{dx} = -\frac{3}{2}\alpha^2 x^{-\frac{3}{2}}$$

$$\omega = (1-\theta^2)^{\frac{1}{2}}(1+\theta) \quad \frac{d\omega}{d\theta} = \frac{1}{2}(1-\theta^2)^{-\frac{1}{2}}(-2\theta)(1+\theta) + (1-\theta^2)^{\frac{1}{2}}$$

$$\phi = \sqrt{3} - (\omega\sqrt{2})^{-1} \quad \frac{d\phi}{d\omega} = (\omega\sqrt{2})^{-\frac{2}{2}} \cdot \cancel{\sqrt{2}}$$

# MORE CHAIN RULE PRACTICE

Calc Made Easy, Exercises 6, p.100

$$1) \quad y = (x^2 + 1)^{\frac{1}{2}}$$

$$y' = \cancel{\frac{1}{2}}(x^2 + 1)^{-\frac{1}{2}} \cancel{x}$$

$$2) \quad y = (x^2 + a^2)^{\frac{1}{2}}$$

$$y' = \cancel{\frac{1}{2}}(x^2 + a^2)^{-\frac{1}{2}} \cancel{x}$$

$$3) \quad y = (a+x)^{-\frac{1}{2}}$$

$$y' = (-\frac{1}{2})(a+x)^{-\frac{3}{2}}$$

$$= \frac{-1}{2\sqrt[2]{(a+x)^3}}$$

$$4) \quad y = a(a-x^2)^{-\frac{1}{2}}$$

$$y' = \cancel{(-\frac{1}{2})}a(a-x^2)^{-\frac{3}{2}} \cancel{(-2x)}$$

$$= a \cdot x (a-x^2)^{-\frac{3}{2}}$$

$$5) \quad y = \bar{x}^2(x^2 - a^2)^{\frac{1}{2}}$$

$$y' = (-2)\bar{x}^3(x^2 - a^2)^{\frac{1}{2}} + \cancel{\bar{x}^2}(\frac{1}{2})(x^2 - a^2)^{-\frac{1}{2}} \cancel{2}$$

$$= \bar{x}^3(x^2 - a^2)^{-\frac{1}{2}} (-2(x^2 - a^2) + x^2)$$

$$= \bar{x}^3(x^2 - a^2)^{-\frac{1}{2}} (2a^2 - x^2)$$

$$6) \quad y = \frac{(x^4 + a)^{\frac{1}{3}}}{(x^3 + a)^{\frac{1}{2}}} = (x^4 + a)^{\frac{1}{3}}(x^3 + a)^{-\frac{1}{2}}$$

$$y' = \frac{1}{3}(x^4 + a)^{-\frac{2}{3}} 4x^3(x^3 + a)^{-\frac{1}{2}} + (x^4 + a)^{\frac{1}{3}} (-\frac{1}{2})(x^3 + a)^{-\frac{3}{2}} 3x^2$$

$$= x^2(x^3 + a)^{-\frac{3}{2}} (x^4 + a)^{\frac{1}{3}} \left( 4x^3(x^3 + a) (x^4 + a)^{-1} - (\frac{3}{2}) \right)$$

$$7) \quad y = \frac{a^2 + x^2}{(a+x)^2}$$

$$y' = \frac{2x(a+x)^2 - (a^2 + x^2) 2(a+x)}{(a+x)^4} = \frac{2(a+x)(x(a+x) - (a^2 + x^2))}{(a+x)^4}$$

$$= 2(a+x)^{-3} (ax + x^2 - a^2 - x^2) = 2a(x-a)(x+a)^{-3}$$

$$8) \frac{dy^5}{dy^2} \quad \text{let } y^2 = x, \text{ so } y^5 = x^{5/2}$$

$$\frac{dx^{5/2}}{dx} = \frac{5}{2}x^{3/2} = \frac{5}{2}y^3 = \frac{dy^5}{dy^2}$$

$$9) y = \frac{(1-\theta^2)^{1/2}}{1-\theta}$$

$$y' = \frac{\cancel{2}(1-\theta^2)^{-1/2}(-2\theta)(1-\theta) + (1-\theta^2)^{1/2}(-1)}{(1-\theta)^2}$$

$$= (1-\theta)^2 \left[ (1-\theta^2)^{1/2} - \theta(1-\theta)(1-\theta^2)^{-1/2} \right]$$

$$= (1-\theta)^{-1} (1-\theta^2)^{-1/2}$$

Calc Made Easy, Exercises 7, p. 101

$$1) v = \frac{1}{2}x^3 \quad \frac{dv}{dx} = \frac{3}{2}x^2$$

$$v = 3(v+u^2) \quad \frac{dv}{du} = 3 + 6u$$

$$w = 1/v^2 \quad \frac{dw}{dv} = -2v^{-3}$$

$$\frac{dw}{dx} = -2\cancel{v}^{-3}(3+6u)^{3/2}x^2$$

$$2) y = 3x^2 + \sqrt{2} \quad \frac{dy}{dx} = 6x$$

$$z = \sqrt{1+y} \quad \frac{dz}{dy} = \frac{1}{2}(1+y)^{-1/2}$$

$$v = (\sqrt{3} + 4z)^{-1} \quad \frac{dv}{dz} = -(\sqrt{3} + 4z)^{-2} \cdot 4$$

$$\frac{dv}{dx} = -(\sqrt{3} + 4z)^{-2} \cdot 4 \cdot \cancel{\frac{1}{2}}(1+y)^{-1/2} \cdot 6x$$

$$3) y = x^3 z^{-1/2} \quad \frac{dy}{dx} = 3^{1/2} x^2$$

$$z = (1+y)^2 \quad \frac{dz}{dy} = 2(1+y)$$

$$v = (1+z)^{-1/2} \quad \frac{dv}{dz} = -\frac{1}{2}(1+z)^{-3/2}$$

$$\frac{dv}{dx} = -\cancel{\frac{1}{2}}(1+\cancel{z})^{-3/2} \cancel{\frac{1}{2}}(1+y) 3^{1/2} x^2$$

$$4) \quad v = a + x; \quad v' = 1$$

$$y = 2a^3 \ln v - v(5a^2 - 2av + \frac{1}{3}v^2)$$

$$= 2a^3 \ln v - 5a^2 v + 2av^2 - \frac{1}{3}v^3$$

$$y' = 2a^3 v^{-1} - 5a^2 + 4av - v^2$$

$$= v^{-1}(2a^3 - 5a^2 v + 4av^2 - v^3)$$

$$= (x+a)^{-1} \left[ 2a^3 - 5a^2 x - 5a^3 + 4a(x^2 + 2xa) + a^2 \right]$$

$$= (x+a)^{-1} \left[ a^3 + 4ax^2 + 3a^2 x - (x+a)^3 \right]$$

$$= \frac{x^2(a-x)}{x+a}$$

$$5) \quad x = a(\theta - \sin \theta) \quad \frac{dx}{d\theta} = a(1 - \cos \theta)$$

$$y = a(1 - \cos \theta) \quad \frac{dy}{d\theta} = a \sin \theta$$

$$\text{so } \left(\frac{dy}{d\theta}\right) / \left(\frac{dx}{d\theta}\right) = \frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta} = \cot \frac{1}{2}\theta$$

$$6) \quad x = a \cos^3 \theta \quad \frac{dx}{d\theta} = 3a \cos^2 \theta (-\sin \theta)$$

$$y = a \sin^3 \theta \quad \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$$

$$\frac{dy}{dx} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$$

$$7) \quad y = \ln(\sin(x^2 - a^2))$$

$$\frac{dy}{dx} = (\sin(x^2 - a^2))^{-1} \cdot \cos(x^2 - a^2) \cdot 2x = 2x \cot(x^2 - a^2)$$

$$8) v = x + y \Rightarrow y = v - x \quad \text{find } \frac{dy}{dx}$$

$$4x = 2v - \ln(2v-1) \quad \frac{dy}{dv} = 1 - \frac{dx}{dv}$$

$$x = \frac{1}{2}v - \frac{\ln(2v-1)}{4} = 1 - \left(\frac{v-1}{2v-1}\right)$$

$$\frac{dx}{dv} = \frac{1}{2} - \frac{1}{2v(2v-1)} \cdot 2 = \frac{2v-1-v+1}{2v-1} = \frac{v}{2v-1}$$

$$= \frac{1}{2} \left(1 - \frac{1}{2v-1}\right)$$

$$= \frac{1}{2} \left(\frac{2v-1-1}{2v-1}\right) = \frac{1}{2} \left(\frac{2(v-1)}{2v-1}\right) \quad \frac{dy}{dv} = \frac{v}{2v-1}$$

$$\frac{dy}{dx} = \frac{v}{2v-1} \cdot \frac{2v-1}{v-1} = \frac{v}{v-1} = \frac{x+y}{x+y-1}$$

# Geometric Meaning of Differentiation

from Calc Made Easy, Exercises 8, p. 112

$$1) \quad y = \frac{3}{4}x^2 - 5$$

$$y' = \frac{3}{2}x$$

$$2) \quad y = 0.12x^3 - 2$$

$$y' = 0.36x^2$$

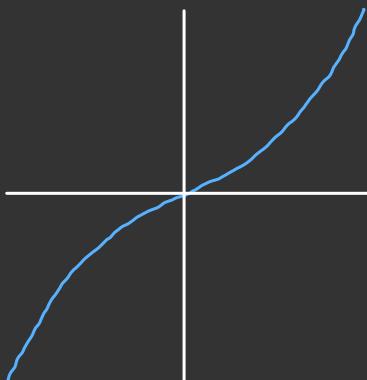
$$3) \quad y = (x-a)(x-b)$$

$$\begin{aligned} y' &= 2x - (a+b) \\ &= 0 @ x = (a+b)/2 \end{aligned}$$

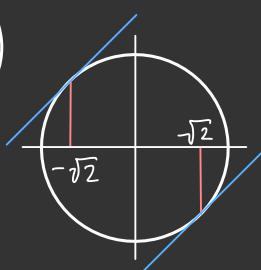
$$4) \quad y = x^3 + 3x = x(x^2 + 3)$$

$$\begin{aligned} y' &= (x^2 + 3) + x \cdot 2x \\ &= 3x^2 + 3 = 3(x^2 + 1) \end{aligned}$$

$$y'' = 6x$$



5)



$$x^2 + y^2 = 4$$

$$y^2 = 4 - x^2$$

$$y = \sqrt{4-x^2} \quad (\text{only positive root})$$

$$y' = \frac{1}{2}(4-x^2)^{-1/2}(-2x) = 1$$

$$-x = (4-x^2)^{1/2}$$

$$x^2 = 4 - x^2$$

$$2x^2 = 4$$

$$x^2 = 2$$

$$x = \pm \sqrt{2}$$

$$6) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right)$$

$$y = \pm b \left(1 - \frac{x^2}{a^2}\right)^{1/2}$$

$$y' = \pm b \left(1 - \frac{x^2}{a^2}\right)^{-1/2} \cdot -\frac{2x}{a^2} = \pm \frac{b}{a^2} x \left(1 - \frac{x^2}{a^2}\right)^{-1/2}$$

$$\text{if } x = 1, \quad y' = \pm \frac{b}{a} (a^2 - 1)^{-1/2}$$

$$7) \quad y = 5 - 2x + 0.5x^3$$

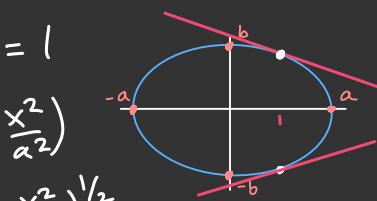
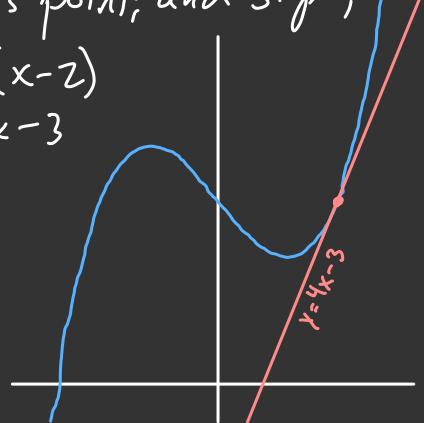
$$y' = -2 + 1.5x^2$$

$$@ x = 2, \quad y = 5, \quad y' = 4$$

given this point, and slope,

$$(y-5) = 4(x-2)$$

$$y = 4x - 3$$

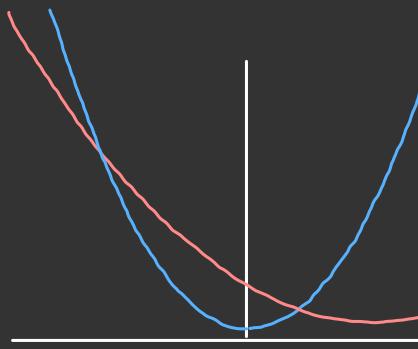


8) Given two functions

$$f(x) = 3.5x^2 + 2$$

$$g(x) = x^2 - 5x + 9.5$$

We can see they intersect at  $\hat{x} = \{-3, 1\}$ .



When  $x=1$ , the first line has a slope of 7, and the second slope is -3, with  $\hat{y}=5.5$ .

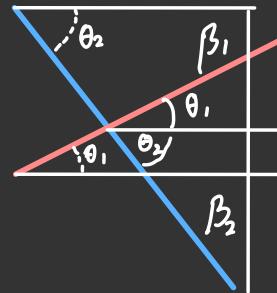
Since the arctan of each slope gives its angle on the unit circle, the diff of the arctans gives the angle between the two lines, at the point of intersection. Thus,

$$\tan^{-1}(7) - \tan^{-1}(-3) = 2.68 \text{ rad or } 153.4^\circ$$

When  $x = -3$ ,  $\beta_1 = -21$ ,  $\beta_2 = -11$ , so

$$\tan^{-1}(-11) - \tan^{-1}(-21) \approx 0.04 \text{ radians.}$$

or  $2.29^\circ$



9)

$$y = \sqrt{25 - x^2} \quad y' = \frac{1}{2}(25-x^2)^{-1/2}(-2x)$$

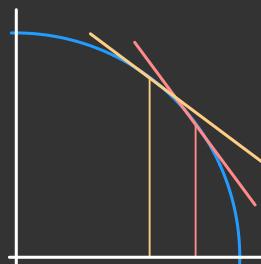
$$@ x=3, y=4, y' = -3/4$$

$$@ x=4, y=3, y' = -4/3$$

so the tangents are

$$y = -\frac{3}{4}x + \frac{25}{4}$$

$$y = -\frac{4}{3}x + \frac{25}{3}$$



which intersect at  $x = 25/7$ . The angle between the tangents at this intersection is

$$\tan^{-1}(-3/4) - \tan^{-1}(-4/3) \approx 0.28 \text{ radians}$$

or  $16.04^\circ$

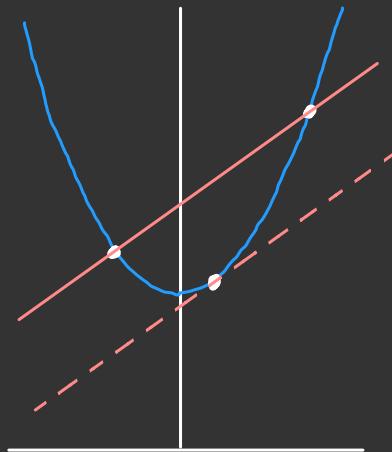
$$10) \quad y = 3x^2 + 2$$

$$y = 2x - b$$

Setting these equal to each other,  
the values of  $x$  at which  
the two lines touch are given by

$$x = \frac{1}{3} \pm \sqrt{-\left(\frac{5}{9} + \frac{b}{3}\right)}$$

The only real solution is if  $b = -\frac{5}{3}$



# Practice Finding Maxima & Minima

from Calc. Made Easy, Ch. 11

$$y = x^2 - 4x + 7$$

$$= (x-2)^2 + 3$$

$$\frac{dy}{dx} = 2x - 4$$

$$= 0 \text{ at } \hat{x} = 2$$

$$Y = 3x - x^2$$

$$= -(x - 3/2)^2 + 9/4$$

$$\frac{dy}{dx} = 3 - 2x \text{ so } \hat{x} = 3/2$$

$$y = (x-a)^2 - d^2$$

$$y = x^2 - 2ax + a^2 - d^2$$

$$\hat{x} = \{ a \pm d \}$$

$$y = 4x + 1/x$$

$$\frac{dy}{dx} = 4 - 1/x^2$$

$$\text{so } \hat{x} = \pm 1/2$$

$$Y = \frac{x^2 - x}{(x - 1/2)^2 + 1/4}$$

$$\frac{dy}{dx} = 2x - 1 \text{ so } \hat{x} = 1/2$$

$$y = ax^3 + bx + c$$

$$\frac{dy}{dx} = 3ax^2 + b$$

$$= 0 \text{ if } \hat{x} = \pm \sqrt{-b/3a}$$

$$y = \frac{1}{3}x^3 - 2x^2 + 3x + 1$$

$$\frac{dy}{dx} = x^2 - 4x + 3$$

$$= (x-2)^2 - 1$$

$$= 0 \text{ when } \hat{x} = \{ 1, 3 \}$$

$$(y-b)^2 + (x-a)^2 = r^2$$

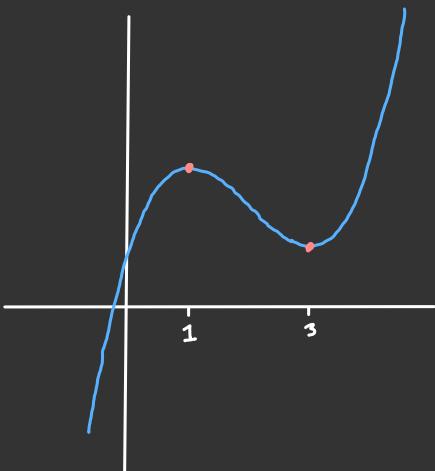
$$y = \pm \sqrt{r^2 - (x-a)^2} + b$$



$$\frac{dy}{dx} = \frac{1}{2} \left( r^2 - (x-a)^2 \right)^{-1/2} (-2(x-a))$$

$$= \frac{-(x-a)}{\left( r^2 - (x-a)^2 \right)^{1/2}}$$

$$= 0 \text{ when } x = a$$



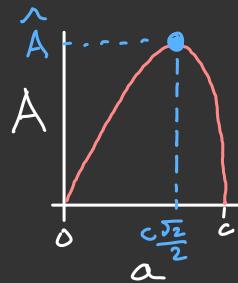
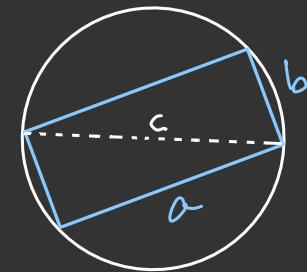
1) For a given circle, what are the dimensions of the inscribed rectangle with the largest possible area?

constraint: for a given  $c$ ,

$$a^2 + b^2 = c^2 \\ b = \sqrt{c^2 - a^2} \quad (b \text{ must be positive})$$

goal: maximize  $A = ab = a\sqrt{c^2 - a^2}$

$$\frac{dA}{da} = \sqrt{c^2 - a^2} + a \frac{1}{2}(c^2 - a^2)^{-\frac{1}{2}}(-2a) \\ = \frac{(c^2 - a^2) - a^2}{(c^2 - a^2)^{\frac{1}{2}}} \\ = 0 \text{ when } c^2 = 2a^2 \text{ or } a = c \frac{\sqrt{2}}{2} \approx 0.7c$$



This implies  $\hat{a} = \hat{b}$ , so the area-maximizing shape is a square!

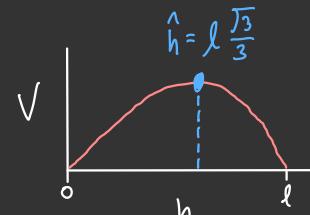
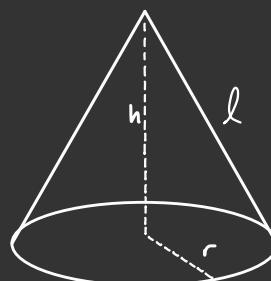
In these terms, the area of the circle is  $\pi c^2 / 4$ , so the ratio of the areas of the circle & square is  $\pi/2 \approx 1.57$ .

2) For a cone with a fixed side length, what is the volume-maximizing radius?

$$l^2 = h^2 + r^2 \Rightarrow l^2 - h^2 = r^2$$

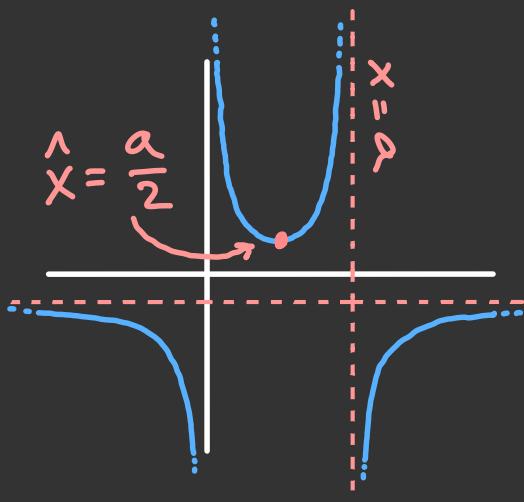
$$V = \frac{\pi r^2 h}{3} = \frac{\pi (l^2 - h^2) h}{3}$$

$$\frac{dV}{dh} = \frac{\pi}{3} \left( (l^2 - h^2) + h(-2h) \right) \\ = \frac{\pi}{3} (l^2 - 3h^2) \\ = 0 \text{ when } \hat{h} = \frac{l\sqrt{3}}{3} \approx 0.58l$$

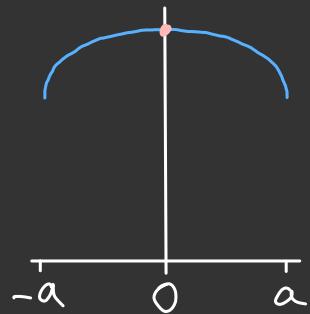


This implies that  $\hat{r} = \hat{h}\sqrt{2} = l \frac{\sqrt{6}}{3} \approx 0.82l$ .

$$\begin{aligned}
 3) \quad & y = \frac{x}{a-x} + \frac{a-x}{x} \\
 &= \frac{x^2}{(a-x)x} + \frac{(a-x)^2}{(a-x)x} \\
 \lim_{x \rightarrow \pm\infty} &= (-2) \left( \frac{x^2 - ax + \frac{a^2}{2}}{x^2 - ax} \right) \\
 &= \frac{(a-x)+x}{(a-x)^2} + \frac{(-1)x - (a-x)}{x^2} \\
 &= a \left( \frac{x^2 - (a-x)^2}{(a-x)^2 x^2} \right) \\
 &= \frac{2a^2(x - a/2)}{x^2(a-x)^2} \\
 &= 0 \text{ when } \hat{x} = a/2
 \end{aligned}$$



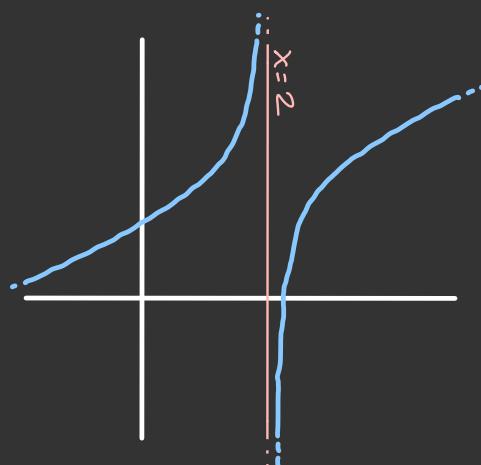
$$\begin{aligned}
 4) \quad & y = \sqrt{a-x} + \sqrt{a+x} \\
 \frac{dy}{dx} &= -\frac{1}{2}(a-x)^{-1/2} + \frac{1}{2}(a+x)^{-1/2} \\
 &= 0 \text{ when } (a+\hat{x})^{-1/2} = (a-\hat{x})^{-1/2} \\
 &\text{or } \hat{x} = 0
 \end{aligned}$$



$$5) \quad y = \frac{x^2 - 5}{2x - 4}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x(2x-4) - (x^2-5)2}{(2x-4)^2} \\ &= \frac{x^2 - 4x + 5}{4(x^2 - 4x + 4)} = \frac{\frac{1}{4}(x-2)^2 + 1}{(x-2)^2} \end{aligned}$$

The solutions for  $dy/dx = 0$  are  $\{2 \pm i\}$ , so this function has no max or min!



$$6) \quad y = x^2 \pm x^{5/2}$$

$$\begin{aligned} \frac{dy}{dx} &= 2x - \frac{5}{2}x^{3/2} \\ &= 0 \text{ if } \hat{x} = \frac{16}{25} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 2x + \frac{5}{2}x^{3/2} \\ &= 0 \text{ if } \hat{x}^{1/2} = -\frac{4}{5}, \\ &\text{which is impossible, so} \\ &\text{no extrema exist.} \end{aligned}$$



7) A cylinder with a height twice the radius is growing such that its surface area  $A$  changes at rate  $dA/dt = \gamma \text{ cm}^2/\text{sec}$ .

What is the rate the volume changes?

$$A = 2\pi r(t)^2 + 2\pi r(t)h = 6\pi r(t)^2$$

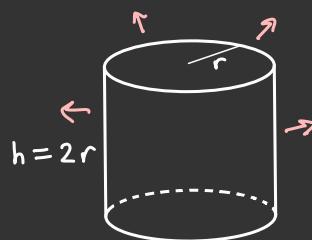
$$V = \pi r(t)^2 h = 2\pi r(t)^3$$

$$\frac{dA}{dt} = 12\pi r(t) \cdot \frac{dr}{dt} = \gamma$$

$$\frac{dr}{dt} = \frac{\gamma}{12\pi r(t)}$$

$$\frac{dV}{dt} = 6\pi r(t)^2 \frac{dr}{dt} = \frac{\gamma r(t)}{2}$$

In Calc Made Easy,  $\gamma = 20$ .



# Calc. Made Easy, Exercises 9, p. 130

1)  $y = \frac{x^2}{x+a}$

$$\frac{dy}{dx} = \frac{x(x+2a)}{(x+a)^2}$$

$$= 0 \text{ if } \hat{x} = \{0, -2a\}$$

$$\text{if } a=1, \hat{x} = \{0, -2\}$$

Plots show 0 is a max,  
 $-2a$  is a min.

2)  $y = \frac{x}{a^2+x^2}$

$$\frac{dy}{dx} = \frac{a^2-x^2}{(a^2+x^2)^2}$$

$$= 0 \text{ if } \hat{x} = \pm a$$

3) Maximize the area of  
 a rectangle with a  
 fixed perimeter.



$$P = 2(b+h)$$

$$\frac{P-2b}{2} = h$$

$$A = b h$$

$$= \frac{b(P-2b)}{2}$$

$$\frac{dA}{db} = \frac{P}{2} - 2b$$

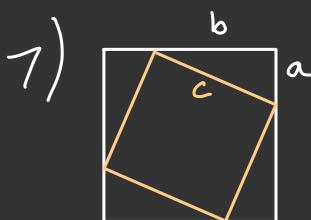
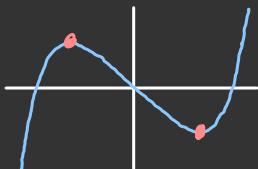
$$= 0 \text{ if } b = P/4$$

4) For a triangle with a fixed perimeter, the maximum possible area is when all three sides are equal. Proof: fix any one side; the max area is thus maximizing height, which is when two remaining sides are equal.  
 By symmetry, all sides must be equal!

6)  $y = x^5 - bx$

$$\frac{dy}{dx} = 5x^4 - b$$

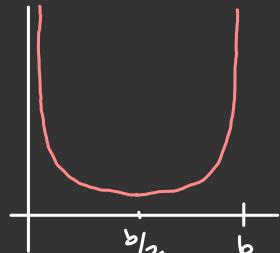
$$= 0 \text{ if } \hat{x} = \pm (b/5)^{1/4}$$



5)  $y = \frac{a}{x} + \frac{a}{b-x} = \frac{ab}{x(b-x)}$

$$\frac{dy}{dx} = -\frac{ab}{x(b-x)} \cdot \frac{(b-2x)}{x(b-x)}$$

$$= 0 \text{ if } \hat{x} = b/2$$



$$4(a+b) = P$$

$$b = P/4 - a$$

$$a^2 + b^2 = c^2$$

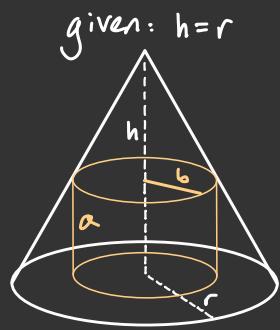
$$a^2 + (P/4 - a)^2 = c^2$$

$$a^2 + (P/4)^2 - 2a/4 + a^2 = c^2$$

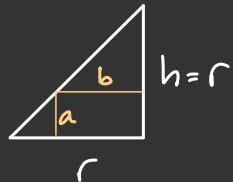
$$\frac{d(c^2)}{da} = 4a - \cancel{\frac{1}{2}} =$$

$$= 0 \text{ if } a = P/8, \text{ so } a = b$$

8)

given:  $h=r$ 

Note in cross section:

where  $a+b=r$ .

To maximize the cylinder's volume

$$V = \pi b^2 \cdot a = \pi b^2 \cdot (r-b) = \pi b^2 r - \pi b^3$$

$$\frac{dV}{db} = 2\pi b r - 3\pi b^2 = 0 \quad \text{if} \quad 2\cancel{\pi} b r = 3\cancel{\pi} b^2$$

$$2r = 3b$$

$$\hat{b} = \frac{2}{3}r \quad \hat{a} = \frac{1}{3}r$$

To maximize the lateral area:

$$A = 2\pi b \cdot a = 2\pi(r-a) \cdot a$$

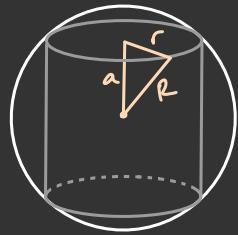
$$\frac{dA}{da} = 2\pi(r-2a), \quad \text{so } \hat{a} = \frac{1}{2}r, \quad \hat{b} = \frac{1}{2}r$$

To maximize total area:

$$A = 2\pi b \cdot a + 2\pi b^2 = 2\pi b r$$

The area is maximized as  $b$  goes to  $r$ .

9)



$R$  is fixed, so  $a$  is a function of  $r$

$$a^2 = R^2 - r^2 \quad a = \sqrt{R^2 - r^2}$$

$$\begin{aligned} V &= r^2 \pi \cdot 2a & L &= 2\pi r \cdot 2a & A &= 2\pi r^2 + 2\pi r \cdot 2a \\ & & & & &= 2\pi r(r + 2a) \end{aligned}$$

What inscribed cylinder has the largest volume?

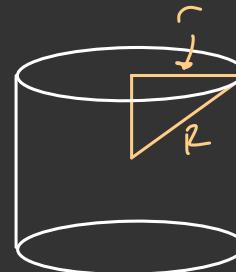
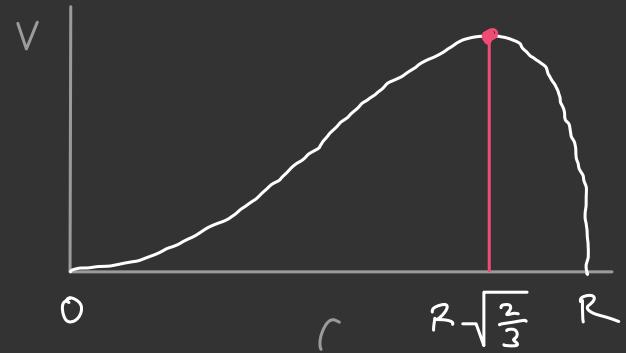
$$V = r^2 \pi \cdot 2 \sqrt{R^2 - r^2}$$

$$\frac{dV}{dr} = 2\pi \left[ 2r \sqrt{R^2 - r^2} + r^2 \cancel{\frac{1}{2}} (R^2 - r^2) (-\cancel{2/r}) \right]$$

$$= 2\pi r \sqrt{R^2 - r^2} \left[ 2 - r^2 \cancel{\frac{1}{R^2 - r^2}} \right]$$

$$= \frac{2\pi r \sqrt{R^2 - r^2} (2R^2 - 3r^2)}{R^2 - r^2} = 0$$

$$r = R \sqrt{\frac{2}{3}} \quad a = R \sqrt{\frac{1}{3}}$$



1.41:1

What inscribed cylinder has the largest lateral surface area?

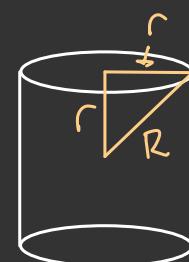
$$L = 2\pi r \cdot 2 \sqrt{R^2 - r^2}$$

$$\frac{dL}{dr} = 4\pi \left[ -\cancel{R^2 - r^2} + r \cancel{\frac{1}{2}} \sqrt{R^2 - r^2} (-\cancel{2/r}) \right]$$

$$= 4\pi \sqrt{R^2 - r^2} \left( 1 - r^2 \cancel{\frac{1}{R^2 - r^2}} \right)$$

$$= 4\pi \sqrt{R^2 - r^2} \frac{(R^2 - 2r^2)}{R^2 - r^2}$$

$$r = R/\sqrt{2}$$



1:1

What inscribed cylinder has the largest surface area?

$$A = 2\pi r(r + 2\sqrt{R^2 - r^2})$$

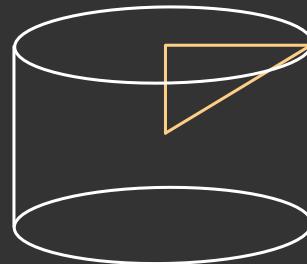
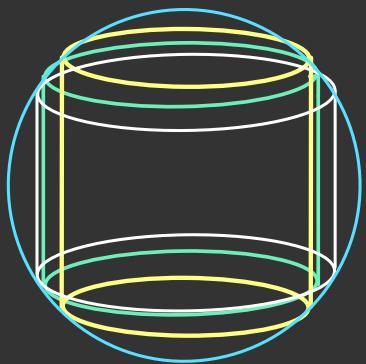
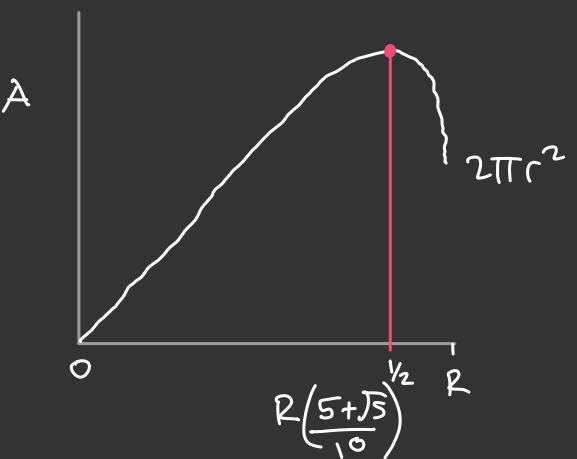
$$= 2\pi r^2 + 4\pi r\sqrt{R^2 - r^2}$$

$$\frac{dA}{dr} = 4\pi r + 4\pi \sqrt{R^2 - r^2} \frac{(R^2 - 2r^2)}{R^2 - r^2}$$

$$r(R^2 - r^2)^{\frac{1}{2}} + R^2 - 2r^2 = 0$$

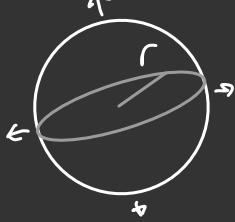
$$r = R - \sqrt{\frac{5 + \sqrt{5}}{10}}$$

Solved w/  
mathematica!



1.62 : 1

10)



If a sphere's volume is expanding at a constant rate, what's happening to its surface area?

$$V = \frac{4}{3}\pi r^3 \quad A = 4\pi r^2$$

$$\frac{dV}{dt} = a$$

$$\text{so } \frac{dr}{dt} = \frac{a}{4\pi r^2}$$

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$a = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$= 8\pi r \cdot \frac{dr}{dt}$$

$$= 8\pi r \left( \frac{a}{4\pi r^2} \right)$$

$$\frac{dA}{dt} = \frac{2a}{r}$$

11)



Which cone inscribed in a sphere has the largest volume?

Notation:

$$V = \frac{\pi r^2 (R+a)}{3} \text{ where } r^2 + a^2 = R^2.$$

$$\text{So: } V = \frac{\pi}{3} \cdot (R^2 - a^2)(R+a)$$

$$= \frac{\pi}{3} (R-a)(R+a)^2$$

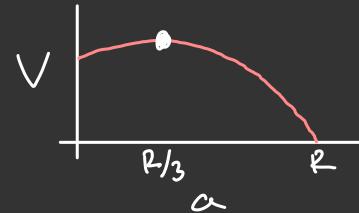
$$= \frac{\pi}{3} (R^3 + R^2 a - Ra^2 - a^3)$$

$$\frac{dV}{da} = \frac{\pi}{3} (R^2 - 2Ra - 3a^2) = 0$$

$$\text{if } a = \frac{2R \pm \sqrt{4R^2 - 4 \cdot (-3)R^2}}{2 \cdot (-3)}$$

$$= \frac{2R \pm \sqrt{16R^2}}{-6} = \frac{2R \pm 4R}{-6}$$

$$= \frac{R \pm 2R}{-3} \text{ or } \{-R, \frac{R}{3}\}$$



which implies  $a = \frac{2\sqrt{2}}{3} R \approx 0.94R$

# Calc Made Easy, Ch. 12

$$y = 4x^2 - 9x - 6$$

$$= 4(x - \frac{9}{8})^2 - \frac{177}{16}$$

$$y' = 8x - 9$$

$$= 0 @ \hat{x} = \frac{9}{8}$$

$y'' = 8$ , so this is a min

$$y = -4x^2 + 9x + 6$$

$$= -4(x - \frac{9}{8})^2 + \frac{177}{16}$$

$$y' = -8x + 9$$

$$= 0 @ \hat{x} = \frac{9}{8}$$

$y'' = -8$ , so this is a max

$$y = x^3 - 3x + 16$$

$$y' = 3x^2 - 3$$

$$= 0 @ \hat{x} = \pm 1$$

$$y'' = 6x, \text{ so}$$

$\hat{x} = 1$  is a min,

$\hat{x} = -1$  is a max

$$y = \frac{x-1}{x^2+2}$$

$$y' = \frac{(x^2+2) - (x-1)2x}{(x^2+2)^2}$$

$$= \frac{x^2+2 - 2x^2+2x}{(x^2+2)^2}$$

$$= \frac{-x^2+2x+2}{(x^2+2)^2}$$

$$= 0 @ \hat{x} = \{1 \pm \sqrt{3}\}$$

Graphing reveals  $1+\sqrt{3}$  is a max &  $1-\sqrt{3}$  is a min, but you can use  $y''$  too...

$$C = aP + \frac{b}{c+P} + d \quad (a, b, c, d \text{ all positive})$$

$$C' = a - b(c+P)^{-2} = 0$$

when

$$a = \frac{b}{(c+P)^2}$$

$$P = \{-c \pm \sqrt{b/a}\}$$

By inspection,  $-c + \sqrt{b/a}$  is a minimum!

# Lighting a Building

Calc Made Easy, p. 136

Our goal is to model the hourly cost of lighting a building, in cents. Each lamp has an expected lifespan of  $t$  hours, and replacing one costs  $r$  cents. The rate of replacement is  $t^{-1}$ , so for  $n$  lamps in use, the hourly average cost of replacement is  $nr/t$ .

We also must consider the energy costs of each lamp. Parameter  $x$  measures the lamp's efficiency, in watts per candle, and  $l$  measures the candles of light per lamp. The cost of electricity, in cents per kWh, is  $p$ . Thus, the cost of running  $n$  lamps for an hour must be  $n \cdot l \cdot x \cdot p / 1000$ .

Thus, the overall operations cost is

$$C = n \left( \frac{r}{t} + \frac{l \cdot x \cdot p}{1000} \right)$$

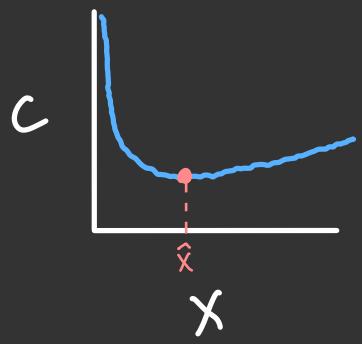
which increases linearly with  $n, r, l, a, p$  &  $\frac{1}{t}$  (the rate of replacement).

We connect the two parts together by assuming that the lifespan of a lamp is a power law of its efficiency:  $t = mx^a$ . That is, the more watts required per candle, the longer the lifespan (?)

Given  $t$  hrs, we can find the cost-minimizing efficiency to be

$$\hat{x} = \left( \frac{nr \cdot 1000}{ml} \right)^{\left(\frac{1}{n+1}\right)}$$

Increasing  $x$  from  $\hat{x}$  and each candle costs more. Decreasing  $x$  from  $\hat{x}$ , and the higher efficiency of each lamp causes it to expire faster, driving up replacement costs.



# MORE GRAPHICAL OPTIMA

Calc Made Easy, Exercises 10, p. 137

$$1) \quad y = x^3 + x^2 - 10x + 8$$

$$y' = 3x^2 + 2x - 10$$

$$\hat{x} = \frac{-2 \pm \sqrt{4 - 4 \cdot 3 \cdot (-10)}}{6} = \frac{-2 \pm \sqrt{124}}{6}$$

$$= \frac{-2 \pm 2\sqrt{31}}{6} = \frac{-1 \pm \sqrt{31}}{3}$$

$$2) \quad y = \frac{b}{a}x - cx^2 = -c(x^2 - \frac{b}{ac}x + 0)$$

$$y' = \frac{b}{a} - 2cx$$

$$= 0 \text{ when } x = \frac{b}{2ac}$$

$$y'' = -2c, \text{ so a max}$$

$$3) \quad y = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$y' = -\cancel{\frac{2}{2}}x + \cancel{\frac{4x^3}{24}} = \frac{x^3}{6} - x$$

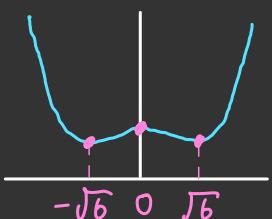
$$= \frac{1}{6}(x^2 - 6) = \frac{1}{6}(x - \sqrt{6})(x + \sqrt{6})$$

So optima exist at  $\{\pm\sqrt{6}, 0\}$

$$y'' = \frac{8x^2}{6} - 1.$$

If  $x=0$ , negative, max.

If  $x=\pm\sqrt{6}$ , positive, min.



$$y = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

$$y' = -\cancel{\frac{2}{2}}x + \cancel{\frac{4x^3}{24}} - \cancel{\frac{6x^5}{720}} = -x + \frac{x^3}{6} - \frac{x^5}{120}$$

$$= -\frac{x}{120} \left( 120 + \frac{x^3}{20} - x^5 \right)$$

This has the same shape & number of critical points as the simpler version... (but why?)

$$4) \quad y = 2x + 1 + \frac{5}{x^2}$$

$$y' = 2 - 2\left(\frac{5}{x^3}\right)$$

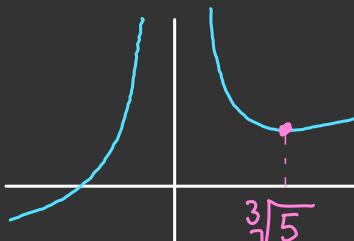
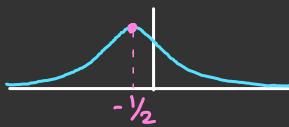
$$= 0 \text{ when } \hat{x}^3 = 5, \text{ so } \hat{x} = \sqrt[3]{5}!$$

$$y'' = 40x^{-5}, \text{ which is } + \text{ at } \hat{x}, \text{ min!}$$

$$5) \quad y = 3(x^2 + x + 1)^{-1}$$

$$y' = -3(x^2 + x + 1)^{-2}(2x + 1)$$

$$= 0 \text{ when } \hat{x} = -\frac{1}{2}$$



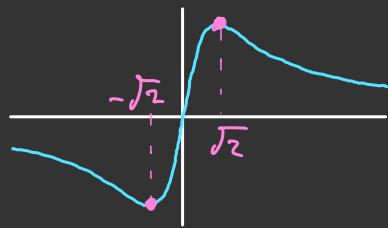
$$6) \quad y = \frac{5x}{2+x^2} = 5x(2+x^2)^{-1}$$

$$y' = 5(2+x^2)^{-1} + 5x(-1)(2+x^2)^{-2} \cdot 2x$$

$$= 5(2+x^2)^{-2}(2+x^2 - 2x^2)$$

$$= 5(2+x^2)^{-2}(2-x^2)$$

$$= 0 \text{ when } x = \pm\sqrt{2}$$



$$7) \quad y = \frac{3x}{x^2-3} + \frac{x}{2} + 5$$

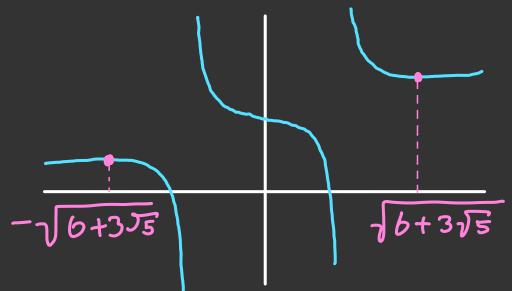
$$y' = \frac{3(x^2-3) - 3x \cdot 2x}{(x^2-3)^2} + \frac{1}{2}$$

$$= \frac{x^4 - 12x^2 - 9}{2(x^2-3)^2} = \frac{(x^2-6)^2 - 45}{2(x^2-3)^2}$$

$$= 0 \text{ if } x^2 - 6 = \pm 3\sqrt{5} \quad (\text{only } + \text{ for real solutions})$$

$$x^2 = 6 + 3\sqrt{5}$$

$$x = \pm\sqrt{6+3\sqrt{5}} \approx \pm 3.56$$



$$8) \quad y = ax^2 + b(1-x)^2$$

$$y' = 2ax - 2b(1-x)$$

$$= 0 \text{ if } x = b/(a+b)$$

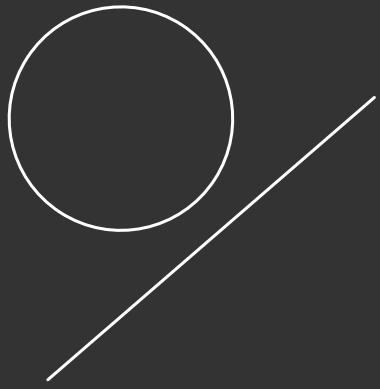
In Calc Made Easy,

$$a = 3 \text{ and } b = 2$$

$$9) v = x(a+bx+cx^2)^{-1}$$

$$\begin{aligned}
 v' &= \frac{(a+bx+cx^2) - x(b+2cx)}{(a+bx+cx^2)^2} \\
 &= \frac{a+bx+cx^2 - bx - 2cx^2}{(a+bx+cx^2)^2} \\
 &= \frac{(a-cx^2)(a+bx+cx^2)}{(a+bx+cx^2)^2} \\
 &= 0 \text{ if } x = \pm\sqrt{\frac{a}{c}} \quad \begin{matrix} \text{MISTAKE in} \\ \text{book can be} \\ + \text{ or } - \end{matrix}
 \end{aligned}$$

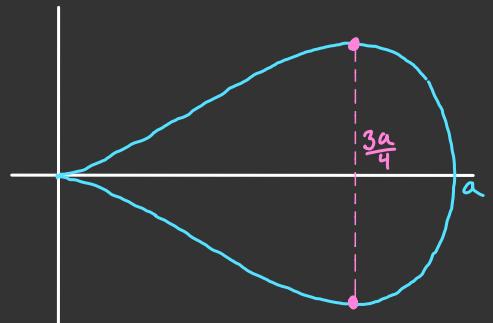
10) See next page!



$$11) y = \pm \frac{x}{b} \sqrt{x(a-x)}$$

Real values of  $y$  only exist between 0 &  $a$ .  
Taking the positive side,  $y = b^{-1}x(xa-x^2)^{1/2}$

$$\begin{aligned}
 y' &= b^{-1}(xa-x^2)^{1/2} + x b^{-1} \frac{1}{2}(xa-x^2)^{-1/2} (a-2x) \\
 &= b^{-1}(xa-x^2)^{-1/2} (xa-x^2 + \frac{1}{2}(a-2x)) \\
 &= b^{-1}(xa-x^2)^{-1/2} (xa-x^2 + \frac{ax}{2} - x^2) \\
 &= b^{-1}(xa-x^2)^{-1/2} (3\frac{a}{2}x - 2x^2) \\
 &= 0 \text{ when } x = \{0, \frac{3a}{4}\}
 \end{aligned}$$



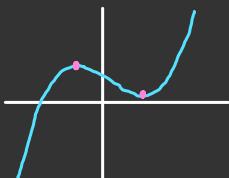
The graph shows  $\frac{3a}{4}$  is max for positive  $y$ .

By symmetry,  $\frac{3a}{4}$  is also the min of the negative side.

$$12) y = 4x^3 - x^2 - 2x + 1$$

$$y' = 12x^2 - 2x - 2$$

$$= 0 \text{ if } x = \{-\frac{1}{3}, \frac{1}{2}\}$$



# Optimal Ship Speed

Calc Made Easy, Exercises 10, Problem 10

The costs to run our ship as we voyage can be broken up into two categories: fuel + running costs. Fuel use increases with the ship's speed, while the total running costs (wages, stores, etc.) increase with the voyage's duration. This sets up a trade-off between the two kinds of cost, implying the existence of a cost-optimizing speed.

Define the running costs as a fixed rate of energy per unit time,  $b$ , so the total running costs of a trip of length  $t$  is  $b \cdot t$ . Define  $x$  as the rate of fuel consumed. The total energy cost of the trip is therefore

$$c = bt + xt = (b+x)t$$

We assume the average speed  $v$  is proportional to the cube-root of the rate of energy consumed in a given unit of time via

$$v = g x^{1/3}$$

Both  $t$  &  $x$  can be written in terms of  $v$ :

$$t = d/v$$

$$x = \left(\frac{v}{g}\right)^3$$

Hence

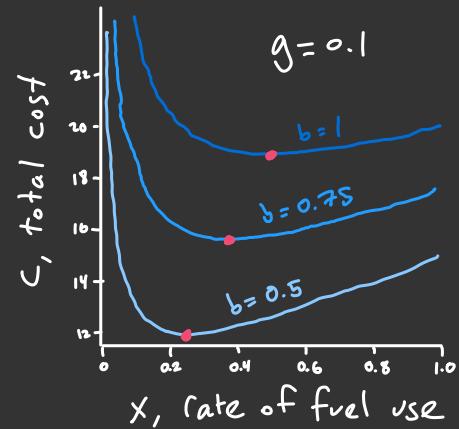
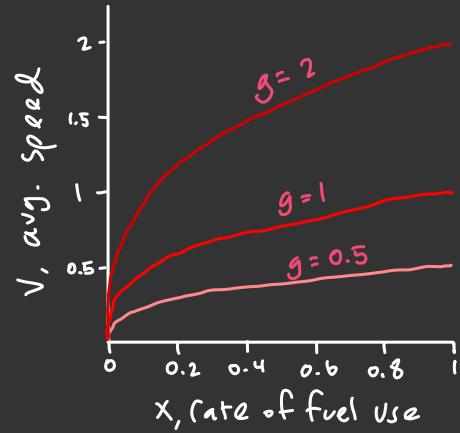
$$c = \left(b + \frac{v^3}{g^3}\right) \frac{d}{v} = d \left(\frac{b}{v} + \frac{v^2}{g^3}\right)$$

Differentiating with respect to  $v$  shows a min at

$$v^* = g(b/2)^{1/3}$$

This implies a very simple rule: the optimal rate of fuel consumption should always be half of the ship's running costs:

$$x^* = b/2$$



# PARTIAL FRACTIONS REVIEW

It all depends on the type of denominator.

## CASE I: DISTINCT LINEAR FACTORS

$$\frac{3x+1}{x^2-1} = \frac{3x+1}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

$$\frac{3x+1}{x^2-1} = \frac{2}{x-1} + \frac{1}{x+1}$$

$$(3x+1) = A_1(x+1) + A_2(x-1)$$

$$= (A_1 + A_2)x + (A_1 - A_2)$$

$$A_1 + A_2 = 3$$

$$A_1 - A_2 = 1$$

$$A_1 = 2$$

$$A_2 = 1$$

$$\frac{4x^2+2x-14}{x^3+3x^2-x-3} = \frac{A_1}{x+3} + \frac{A_2}{x-1} + \frac{A_3}{x+1}$$

$$= \frac{2}{x+3} - \frac{1}{x-1} + \frac{3}{x+1}$$

$$4x^2+2x-14 = A_1(x+1)(x+3) + A_2(x+3)(x-1) + A_3(x+1)(x-1)$$

$$= (A_1 + A_2 + A_3)x^2 + (4A_2 + 2A_3)x - (A_1 - 3A_2 + 3A_3)$$

$$4 = A_1 + A_2 + A_3$$

$$2 = 4A_2 + 2A_3$$

$$14 = A_1 - 3A_2 + 3A_3$$

$$A_1 = 2$$

$$A_2 = -1$$

$$A_3 = 3$$

## CASE II: DISTINCT QUADRATIC FACTORS

$$\frac{-x^2-3}{(x^2+1)(x+1)} = \frac{A_1x+B_1}{x^2+1} + \frac{B_2}{x+1} = \frac{x-1}{x^2+1} - \frac{2}{x+1}$$

$$-x^2-3 = (x+1)(A_1x+B_1) + B_2(x^2+1)$$

$$= (A_1+B_2)x^2 + (A_1+B_1)x + B_1 + B_2$$

$$-1 = A_1 + B_2$$

$$0 = A_1 + B_1$$

$$-3 = B_1 + B_2$$

$$A_1 = 1$$

$$B_2 = -2$$

$$B_1 = -1$$

$$\frac{x^3-2}{(x^2+1)(x^2+2)} = \frac{A_1x+B_1}{x^2+1} + \frac{A_2x+B_2}{x^2+2} = \frac{-x-2}{x^2+1} + \frac{2x+2}{x^2+2}$$

$$x^3-2 = (x^2+2)(A_1x+B_1) + (x^2+1)(A_2x+B_2)$$

$$= (A_1+A_2)x^3 + (B_1+B_2)x^2 + (2A_1+A_2)x + (2B_1+B_2)$$

$$1 = A_1 + A_2$$

$$0 = B_1 + B_2$$

$$0 = 2A_1 + A_2$$

$$-2 = 2B_1 + B_2$$

$$A_1 = -1$$

$$A_2 = 2$$

$$B_1 = -2$$

$$B_2 = 2$$

CASE III: REPEATING LINEAR FACTORS

$$\frac{3x^2 - 2x + 1}{(x+1)^2(x-2)} = \frac{A_1}{(x+1)^2} + \frac{A_2}{x+1} + \frac{A_3}{x-2} = \frac{-2}{(x+1)^2} + \frac{2}{x+1} + \frac{1}{x-2}$$

$$3x^2 - 2x + 1 = (x-2)A_1 + (x+1)(x-2)A_2 + (x+1)^2 A_3$$

$$\begin{aligned} &= A_1 x - 2A_1 + (x^2 - x - 2)A_2 + (x^2 + 2x + 1)A_3 \\ &= \underline{A_1 x} - 2A_1 + \underline{A_2 x^2} - \underline{A_2 x} - 2A_2 + \underline{A_3 x^2} + \underline{2A_3 x} + A_3 \\ &= (A_2 + A_3)x^2 + (A_1 - A_2 + 2A_3)x - 2A_1 - 2A_2 + A_3 \end{aligned}$$

$$\begin{array}{ll} \text{so : } & A_2 + A_3 = 3 \\ & A_1 - A_2 + 2A_3 = -2 \\ & -2A_1 - 2A_2 + A_3 = 1 \end{array} \quad \begin{array}{l} A_1 = -2 \\ A_2 = 2 \\ A_3 = 1 \end{array}$$

$$\frac{3x - 1}{(2x^2 - 1)^2(x+1)} = \frac{A_1 x + A_2}{(2x^2 - 1)^2} + \frac{A_3 x + A_4}{2x^2 - 1} + \frac{A_5}{x+1} = \frac{8x - 5}{(2x^2 - 1)^2} + \frac{8x - 8}{2x^2 - 1} - \frac{4}{x+1}$$

$$(A_1 x + A_2)(x+1) + (A_3 x + A_4)(2x^2 - 1)(x+1) + A_5 (2x^2 - 1)^2$$

$$= A_1 x^2 + A_1 x + A_2 x + A_2 + (A_3 x + A_4)(2x^3 + 2x^2 - x - 1) + A_5 (4x^4 - 4x^2 + 1)$$

$$\begin{aligned} &= A_1 x^2 + A_1 x + A_2 x + A_2 + 2A_3 x^4 + 2A_3 x^3 - A_3 x^2 - A_3 x \\ &\quad + 2A_4 x^3 + 2A_4 x^2 - A_4 x - A_4 + 4A_5 x^4 - 4A_5 x^2 + A_5 \end{aligned}$$

$$\begin{aligned} &= (4A_5 + 2A_3)x^4 + (2A_3 + 2A_4)x^3 + (A_1 - A_3 + 2A_4 - 4A_5)x^2 \\ &\quad + (A_1 + A_2 - A_3 - A_4)x + (A_2 - A_4 + A_5) \end{aligned}$$

$$\begin{array}{ll} \text{so : } & 4A_5 + 2A_3 = 0 \rightarrow A_5 = -\frac{1}{2}A_3 \\ & 2A_3 + 2A_4 = 0 \rightarrow A_4 = -A_3 \\ & A_1 - A_3 + 2A_4 - 4A_5 = 0 \rightarrow A_1 - A_3 - 2A_3 + 2A_3 = 0 \end{array}$$

$$A_1 + A_2 - A_3 - A_4 = 3 \quad \text{so } A_1 = A_3$$

$$A_2 - A_4 + A_5 = -1 \quad \begin{array}{l} A_3 + A_2 = 3 \\ A_2 + \frac{1}{2}A_3 = -1 \end{array}$$

$$\begin{array}{l} (3 - A_3) + \frac{1}{2}A_3 = -1 \\ 3 - \frac{1}{2}A_3 = -1 \\ 4 = \frac{1}{2}A_3 \end{array}$$

$$\begin{array}{l} A_3 = 8 \quad A_1 = 8 \\ A_2 = -5 \quad A_4 = -8 \quad A_5 = -4 \end{array}$$

## Additional Examples

$$\frac{4x+1}{(x+1)^3} = \frac{4(z-1)+1}{z^3} = \frac{4}{z^2} - \frac{3}{z^3}$$

$$y = \frac{5-4x}{6x^2+7x-3} = \frac{5-4x}{(2x+3)(3x-1)} = \frac{A_1}{2x+3} + \frac{A_2}{3x-1} = \frac{-2}{2x+3} + \frac{1}{3x-1}$$

$$A_1(3x-1) + A_2(2x+3) = 5-4x$$

$$\begin{aligned} 3A_1x - A_1 + 2A_2x + 3A_2 &= 5-4x \\ (3A_1 + 2A_2)x - A_1 + 3A_2 & \end{aligned}$$

When calculating  $y'$ , it's  
much easier to decompose  
 $y$  first.

$$\begin{aligned} 3A_1 + 2A_2 &= -4 \\ -A_1 + 3A_2 &= 5 \end{aligned}$$

$$A_1 = 3A_2 - 5$$

$$\begin{aligned} 3(3A_2 - 5) + 2A_2 &= -4 \\ 9A_2 - 15 + 2A_2 &= -4 \end{aligned}$$

$$(11A_2 - 15 = -4)$$

$$\begin{aligned} A_2 &= 1 \\ A_1 &= -2 \end{aligned}$$

# PARTIAL FRACTIONS PRACTICE

Exercises 11, page 147

$$1) \frac{3x+5}{(x-3)(x+4)} = \frac{1}{x+4} + \frac{2}{x-3}$$

$$\begin{aligned} 3x+5 &= A_1(x-3) + A_2(x+4) \\ &= A_1x - 3A_1 + A_2x + 4A_2 \\ &= (A_1 + A_2)x + 4A_2 - 3A_1 \end{aligned}$$

$$\begin{aligned} 3 &= A_1 + A_2 \\ 5 &= 4A_2 - 3A_1 \end{aligned}$$

$$4) \frac{x+1}{x^2-7x+12} = \frac{-4}{x-3} + \frac{5}{x-4}$$

$$\begin{aligned} x+1 &= (x-4)A_1 + (x-3)A_2 \\ &= A_1x - 4A_1 + A_2x - 3A_2 \\ &= (A_1 + A_2)x - 4A_1 - 3A_2 \end{aligned}$$

$$\begin{aligned} 1 &= A_1 + A_2 \\ 1 &= -4A_1 - 3A_2 \end{aligned}$$

$$\begin{aligned} A_1 &= -4 \\ A_2 &= 5 \end{aligned}$$

$$2) \frac{3x-4}{(x-1)(x-2)} = \frac{2}{x-2} + \frac{1}{x-1}$$

$$\begin{aligned} 3x-4 &= (x-1)A_1 + (x-2)A_2 \\ &= (A_1 + A_2)x - A_1 - 2A_2 \end{aligned}$$

$$\begin{aligned} 3 &= A_1 + A_2 \\ -4 &= -A_1 - 2A_2 \end{aligned}$$

$$5) \frac{x-8}{(2x+3)(3x-2)} = \frac{1}{13} \left( \frac{19}{2x+3} - \frac{22}{3x-2} \right)$$

$$\begin{aligned} x-8 &= (2x+3)A_1 + (3x-2)A_2 \\ &= 2A_1x + 3A_1 + 3A_2x - 2A_2 \\ &= (2A_1 + 3A_2)x + 3A_1 - 2A_2 \end{aligned}$$

$$\begin{aligned} 1 &= 2A_1 + 3A_2 \\ -8 &= 3A_1 - 2A_2 \end{aligned}$$

$$\begin{aligned} A_1 &= -22/13 \\ A_2 &= 19/13 \end{aligned}$$

$$3) \frac{3x+5}{x^2+x-12} = \frac{2}{x-3} + \frac{1}{x+4}$$

$$\begin{aligned} 3x+5 &= (x+4)A_1 + (x-3)A_2 \\ &= A_1x + 4A_1 + A_2x - 3A_2 \\ &= (A_1 + A_2)x + 4A_1 - 3A_2 \end{aligned}$$

$$\begin{aligned} 3 &= A_1 + A_2 \\ 5 &= 4A_1 - 3A_2 \end{aligned}$$

$$6) \frac{x^2-13x+26}{(x-2)(x-3)(x-4)} = \frac{2}{x-2} + \frac{4}{x-3} - \frac{5}{x-4}$$

$$\begin{aligned} x^2-13x+26 &= A_1(x-3)(x-4) + A_2(x-2)(x-4) + A_3(x-3)(x-2) \\ &= (A_1 + A_2 + A_3)x^2 - (7A_1 + 6A_2 + 5A_3)x \\ &\quad + 12A_1 + 8A_2 + 6A_3 \end{aligned}$$

$$\begin{aligned} 1 &= A_1 + A_2 + A_3 \rightarrow A_1 = 2 \\ 13 &= 7A_1 + 6A_2 + 5A_3 \quad A_2 = 4 \\ 26 &= 12A_1 + 8A_2 + 6A_3 \quad A_3 = -5 \end{aligned}$$

$$7) \frac{x^2 - 3x + 1}{(x-1)(x+2)(x-3)} = \frac{1}{6(x-1)} + \frac{11}{15(x+2)} + \frac{1}{10(x-3)}$$

$$\begin{aligned} x^2 - 3x + 1 &= A_1(x+2)(x-3) + A_2(x-1)(x-3) + A_3(x+2)(x-1) \\ &= (A_1 + A_2 + A_3)x^2 - (A_1 + 4A_2 - A_3)x - 6A_1 + 3A_2 - 2A_3 \end{aligned}$$

$$\begin{aligned} 1 &= A_1 + A_2 + A_3 & A_1 &= \frac{1}{6} \\ 3 &= A_1 + 4A_2 - A_3 & A_2 &= \frac{11}{15} \\ 1 &= -6A_1 + 3A_2 - 2A_3 & A_3 &= \frac{1}{10} \end{aligned}$$

$$8) \frac{5x^2 + 7x + 1}{(2x+1)(3x-2)(3x+1)} = \frac{-5/7}{2x+1} + \frac{7/63}{3x-2} + \frac{7/9}{3x+1}$$

$$\begin{aligned} 5x^2 + 7x + 1 &= A_1(3x-2)(3x+1) + A_2(2x+1)(3x+1) + A_3(2x+1)(3x-2) \\ &= A_1(9x^2 + 3x - 6x - 2) + A_2(6x^2 + 2x + 3x + 1) + A_3(6x^2 - 4x + 3x - 2) \\ &= x^2(9A_1 + 6A_2 + 6A_3) + x(-3A_1 + 5A_2 - A_3) - 2A_1 + A_2 - 2A_3 \end{aligned}$$

$$\begin{aligned} 5 &= 9A_1 + 6A_2 + 6A_3 & A_1 &= -\frac{5}{7} \\ 7 &= -3A_1 + 5A_2 - A_3 & A_2 &= \frac{7}{63} \\ 1 &= -2A_1 + A_2 - 2A_3 & A_3 &= \frac{7}{9} \end{aligned}$$

$$9) \frac{x^2}{x^3 - 1} = \frac{x^2}{(x-1)(x^2+x+1)} = \frac{A_1}{x-1} + \frac{A_2x+B_1}{x^2+x+1} = \frac{1}{3} \left( \frac{1}{(x-1)} + \frac{2x+1}{(x^2+x+1)} \right)$$

$$\begin{aligned} x^2 &= (x^2 + x + 1)A_1 + (x-1)(A_2x + B_1) \\ &= \underline{A_1}x^2 + \underline{A_1}x + A_1 + \underline{A_2}x^2 + \underline{B_1}x - \underline{A_2}x - B_1 \\ &= (\underline{A_1} + A_2)x^2 + (A_1 + B_1 - A_2)x + A_1 - B_1 \end{aligned}$$

$$\begin{aligned} A_1 + A_2 &= 1 & A_1 + A_2 &= 1 & A_1 &= \frac{1}{3} \\ A_1 + B_1 - A_2 &= 0 \Rightarrow 2A_1 = A_2 & A_2 &= \frac{2}{3} & A_2 &= \frac{2}{3} \\ A_1 - B_1 &= 0 & B_1 &= \frac{1}{3} & B_1 &= \frac{1}{3} \end{aligned}$$

$$10) \frac{x^4 + 1}{x^3 + 1} = x + \frac{1-x}{x^3 + 1} = x + \frac{1-x}{(x+1)(x^2-x+1)} = x + \frac{1}{3} \left( \frac{1-2x}{x^2-x+1} + \frac{2}{x+1} \right)$$

gotta simplify  
with polynomial  
long division:

$$\begin{array}{r} x \\ x^3 + 1 \overline{) x^4 + 1} \\ \underline{- (x^4 + x)} \\ \hline -x \end{array}$$

Case II:  
distinct  
quadratic  
factors

$$\begin{aligned} 1-x &= (x+1)(Ax + B) + C(x^2 - x + 1) \\ &= (A+C)x^2 + (A+B-C)x + (B+C) \end{aligned}$$

$$\frac{Ax+B}{x^2-x+1} + \frac{C}{x+1}$$

$$\text{implies } A = -\frac{2}{3}, B = \frac{1}{3}, C = \frac{2}{3}$$

$$11) \frac{5x^2 + 6x + 4}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} = \frac{3}{x+1} + \frac{2x+1}{x^2+x+1}$$

$$\begin{aligned} 5x^2 + 6x + 4 &= (x^2 + x + 1)A + (x+1)(Bx+C) \\ &= \underline{A}x^2 + \underline{A}x + A + \underline{B}x^2 + \underline{B}x + \underline{C}x + C \\ &= (A+B)x^2 + (A+B+C)x + (A+C) \end{aligned}$$

$$\begin{aligned} A+B &= 5 \\ A+B+C &= 6 \quad C=1 \quad A=3 \quad B=2 \\ A+C &= 4 \end{aligned}$$

$$12) \frac{x}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} = \frac{1}{x-1} - \frac{1}{x-2} + \frac{2}{(x-2)^2}$$

$$\begin{aligned} x &= A(x-2)^2 + B(x-2)(x-1) + C(x-1) \\ &= A(x^2 - 4x + 4) + B(x^2 - 3x + 2) + C(x-1) \\ &= (A+B)x^2 - (4A+3B-C)x + 4A+2B-C \end{aligned}$$

$$\begin{aligned} 0 &= A+B \\ 1 &= -(4A+3B-C) \quad -(4A-3A-2A)=1 \\ 0 &= 4A+2B-C \quad A=1 \\ C &= 2A \quad B=-1 \\ & \quad C=2 \end{aligned}$$

$$13) \frac{x}{(x^2-1)(x+1)} = \frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{1}{2(x+1)^2}$$

$$\begin{aligned} x &= A(x+1)^2 + B(x-1)(x+1) + C(x-1) \\ &= A(x^2 + 2x + 1) + B(x^2 - 1) + C(x-1) \\ &= (A+B)x^2 + (2A+C)x + (A-B-C) \end{aligned}$$

$$\begin{aligned} A &= -B & A &= \frac{1}{4} \\ 2A+C &= 1 & B &= -\frac{1}{4} \\ A-B-C &= 0 & C &= \frac{1}{2} \\ 2A &= C & & \end{aligned} \quad \begin{aligned} -3A-2(A-1)-4(-\frac{1}{4}) &= 9 \\ -3A-6(A-1) &= 9 \\ -3A-6A+6 &= 9 \\ -9A &= 3 \\ A &= -\frac{1}{3} \end{aligned}$$

$$14) \frac{x+3}{(x+2)^2(x-1)} = \frac{A}{(x+2)^2} + \frac{B}{x+2} + \frac{C}{x-1} = \frac{-1}{3(x+2)^2} - \frac{4}{9(x+2)} + \frac{4}{9(x-1)}$$

$$\begin{aligned} x+3 &= A(x-1) + B(x+2)(x-1) + C(x+2)^2 \\ &= A(x-1) + B(x^2 + x - 2) + C(x^2 + 4x + 4) \\ &= (B+C)x^2 + (A+B+4C)x - A - 2B + 4C \end{aligned}$$

$$\begin{aligned} C &= \frac{4}{9} \\ B &= -\frac{4}{9} \\ -3(\frac{4}{9}) - \frac{4}{9} + 4\frac{4}{9} &= \frac{4}{9} \\ -\frac{12}{9} - \frac{4}{9} + \frac{16}{9} &= \frac{4}{9} \\ \frac{4}{9} &= \frac{4}{9} \end{aligned}$$

$$\begin{aligned} \frac{1}{3} + 2 \cdot \frac{4}{9} + 4 \cdot \frac{4}{9} &= \frac{1}{3} + \frac{8}{9} + \frac{16}{9} \\ \frac{3}{9} + \frac{8}{9} + \frac{16}{9} &= \frac{27}{9} = 3 \checkmark \\ -A - 2B + 4C &= 3 \quad -A + B + 4C = 1 \Rightarrow A + 3C = 1 \quad A = 1 - 3C \\ -A - 2(\frac{1}{3}) + 4(\frac{1}{3}) &= 3 \quad = 1 + 3B \quad \frac{A-1}{-3} = C \\ \frac{A-1}{3} &= B \end{aligned}$$

$$15) \frac{3x^2 + 2x + 1}{(x+2)(x^2+x+1)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

so we work with:  $A(x^2+x+1)^2 + (Bx+C)(x+2)(x^2+x+1) + (Dx+E)(x+2)$

$$\begin{aligned} & A(x^4 + x^3 + x^2 + x^3 + x^2 + x + x^2 + x + 1) \\ & + (Bx+C)(x^3 + x^2 + x + 2x^2 + 2x + 2) \\ & + (Dx+E)(x+2) \\ & = A(x^4 + 2x^3 + 3x^2 + 2x + 1) \\ & + (Bx^4 + 3Bx^3 + 3Bx^2 + 2Bx \\ & + Cx^3 + 3Cx^2 + 3Cx + 2C \\ & + Dx^2 + 2Dx + Ex + 2E) \\ & = (A+B)x^4 + (2A+3B+C)x^3 \\ & + (3A+3B+3C+D)x^2 + (2A+2B+3C+2D+E)x \\ & + A+2C+2E \end{aligned}$$

which implies

$$\begin{array}{lll} 0 = A+B & A = -B \\ 0 = 2A+3B+C & \Rightarrow C = A \\ 3 = 3A+3B+3C+D & 3A - 3A + 3A + D = 3 \\ 2 = 2A+2B+3C+2D+E & 2A - 2A + 3A + 2D + E = 2 \\ 1 = A+2C+2E & A + 2A + 2E = 1 \\ & \Downarrow & D = 3(1-A) \\ & 3A + D = 3 & \\ & 3A + 2D + E = 2 & \Rightarrow 3A + 6(1-A) + E = 2 \\ & A + 2A + 2E = 1 & 3A + 2E = 1 \end{array}$$

$$\begin{array}{l} 3A + 6 - 6A + E = 2 \\ 3A + 2E = 1 \end{array}$$

$$\frac{3x^2 + 2x + 1}{(x+2)(x^2+x+1)^2} = \frac{1}{x+2} + \frac{1-x}{x^2+x+1} - \frac{1}{(x^2+x+1)^2}$$

$$\begin{array}{r} -3A + E = -4 \\ 3A + 2E = 1 \\ \hline 3E = -3 \end{array}$$

$$\begin{array}{ll} B = -1 & E = -1 \\ C = 1 & A = 1 \\ D = 0 & \end{array}$$

$$16) \frac{5x^2+8x-12}{(x+4)^3} = \frac{A}{(x+4)^2} + \frac{B}{(x+4)} + \frac{C}{x+4} = \frac{36}{(x+4)^3} - \frac{32}{(x+4)^2} + \frac{5}{x+4}$$

$$\begin{aligned} 5x^2+8x-12 &= A + B(x+4) + C(x+4)^2 \\ &= A + Bx + 4B + C(x^2+8x+16) \\ &= (A+4B+16C) + (B+8C)x + Cx^2 \end{aligned}$$

$$C = 5$$

$$B+8C = 8 \quad B = -32$$

$$(A+4B+16C) = -12$$

$$A - 128 + 80 = -12$$

$$A - 48 = -12$$

$$A = 36$$

$$17) \frac{7x^2+9x-1}{(3x-2)^4} = \frac{A}{(3x-2)^4} + \frac{B}{(3x-2)^3} + \frac{C}{(3x-2)^2} + \frac{D}{3x-2} = \boxed{\frac{73}{9(3x-2)^4} + \frac{55}{9(3x-2)^3} + \frac{7}{9(3x-2)}}$$

$$\begin{aligned} 7x^2+9x-1 &= A + B(3x-2) + C(3x-2)^2 + D(3x-2)^3 \\ &= A + 3Bx - 2B + C(9x^2 - 12x + 4) \\ &\quad + D(3x-2)(3x-2)(3x-2) \\ &= A + 3Bx - 2B + 9Cx^2 - 12Cx + 4C \\ &\quad + D(3x-2)(9x^2 - 12x + 4) \\ &= A + 3Bx - 2B + 9Cx^2 - 12Cx + 4C \\ &\quad + D(27x^3 - 36x^2 + 12x - 18x^2 + 24x - 8) \\ &= 27Dx^3 + 9Cx^2 - 36Dx^2 - 18Dx^2 \\ &\quad + 3Bx - 12Cx + 12Dx + 24Dx \\ &\quad + A - 2B + 4C - 8D \\ &= 27Dx^3 + (9C - 54D)x^2 \\ &\quad + (3B - 12C + 36D)x \\ &\quad + A - 2B + 4C - 8D \end{aligned}$$

which implies

$$0 = 27D$$

$$7 = 9C - 54D$$

$$9 = 3B - 12C + 36D$$

$$-1 = A - 2B + 4C - 8D$$

$$D = 0$$

$$C = \frac{7}{9}$$

$$\begin{aligned} 9 &= 3B - 12\left(\frac{7}{9}\right) \\ -1 &= A - 2B + 4\left(\frac{7}{9}\right) \end{aligned}$$

↓

$$-1 = A - 2\left(\frac{55}{9}\right) + 4\left(\frac{7}{9}\right)$$

$$-9 = 9A - 110 + 28$$

$$-9 = 9A - 82$$

$$73 = 9A$$

$$A = \frac{73}{9}$$

$$81 = 27B - 12 \cdot 7$$

$$81 = 27B - 84$$

$$+ 84$$

$$+ 84$$

$$165 = 27B$$

$$B = \frac{165}{27} = \frac{5 \cdot 33}{27} = \frac{5 \cdot 11}{3 \cdot 9} = \frac{55}{9}$$

$$B = \frac{55}{9}$$

$$18) \frac{x^2}{(x^3-8)(x-2)} = \frac{x^2}{(x-2)^2(x^2+2x+4)} = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{Cx+D}{x^2+2x+4}$$

$$= \frac{1}{3(x-2)^2} + \frac{1}{6(x-2)} - \frac{1}{6(x^2+2x+4)}$$

$$\begin{aligned} x^2 &= A(x^2+2x+4) + B(x-2)(x^2+2x+4) + (Cx+D)(x^2-4x+4) \\ &= Ax^2 + 2Ax + 4A + B(x^3 + 2x^2 + 4x - 2x^2 - 4x^2 + 8) \\ &\quad + (Cx+D)(x^2-4x+4) \\ &= Ax^2 + 2Ax + 4A + Bx^3 - 8B \\ &\quad + Cx^3 - 4Cx^2 + 4Cx + Dx^2 - 4Dx + 4D \end{aligned}$$

$$\begin{aligned} x^2 &= (B+C)x^3 \\ &\quad + (A-4C+D)x^2 \\ &\quad + (2A+4C-4D)x \\ &\quad + 4A+4D-8B \end{aligned}$$

$$\begin{aligned} A-2B-2D &= 0 \\ -(A-2B+D=0) \\ -3D &= 0 \\ D &= 0 \end{aligned}$$

$$\begin{aligned} B+C &= 0 \quad \text{so } B = -C \\ A-4C+D &= 1 \\ 2A+4C-4D &= 0 \\ 4A+4D-8B &= 0 \end{aligned} \Rightarrow \begin{aligned} A+4B+D &= 1 \\ A-2B-2D &= 0 \Rightarrow -(A-2B=0) \\ A+D-2B &= 0 \\ A-2 \cdot \frac{1}{6} &= 0 \\ A &= \frac{1}{3} \end{aligned} \begin{aligned} A+4B &= 1 \\ 4B &= 1 \\ B &= \frac{1}{4} \\ C &= -\frac{1}{4} \end{aligned}$$

# INVERSE DIFFERENTIATION

OCT 26  
2022

CHAPTER 13, p. 147

$$y = 3x \Rightarrow x = \frac{y}{3}$$

$$\frac{dy}{dx} = 3 \quad \frac{dx}{dy} = \frac{1}{3}$$

$$y = 4x^2 \Rightarrow x = \sqrt{\frac{y}{2}}$$

$$\frac{dy}{dx} = 8x \quad \frac{dx}{dy} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4\sqrt{y}} = \frac{1}{8x}$$

$$y = \sqrt{\frac{3}{x} - 1} \Rightarrow x = \frac{3}{1+y^2} = 3(1+y^2)^{-1}$$

$$= \left(\frac{3}{x} - 1\right)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{3}{x} - 1\right)^{-\frac{1}{2}} \cdot -3x^{-2} \quad \frac{dx}{dy} = -3(1+y^2)^{-2} \cdot 2y = -\frac{2}{3} \left(\frac{3}{x} - 1\right)^{\frac{1}{2}} x^{-2}$$

$$y = (\theta+5)^{\frac{1}{3}} \Rightarrow y^3 = \theta + 5$$

$$\frac{dy}{d\theta} = \frac{1}{3}(\theta+5)^{-\frac{2}{3}} \quad \frac{d\theta}{dy} = 3y^2 = 3(\theta+5)^{\frac{2}{3}}$$

Exer. I 5, 6, 7

$$5) z = \sqrt[3]{v} \quad \frac{dz}{dv} = \frac{1}{3} v^{-\frac{2}{3}}$$

$$\begin{aligned} z^3 &= v \\ \frac{dy}{dz} &= 3z^2 \\ &= 3v^{\frac{2}{3}} \\ \frac{dz}{dv} &= (3v^{\frac{2}{3}})^{-1} \end{aligned}$$

$$6) y = \sqrt[3]{x^{-5}} = x^{-\frac{5}{3}}$$

$$\frac{dy}{dx} = -\frac{5}{3} x^{-\frac{8}{3}}$$

$$\begin{aligned} y^3 &= x^{-5} = \frac{1}{x^5} \\ x^5 &= y^3 \\ x &= y^{-\frac{3}{5}} \\ \frac{dx}{dy} &= -\frac{3}{5} y^{\frac{8}{5}} \end{aligned}$$

$$7) u = x^{-\frac{8}{5}} \quad \frac{du}{dx} = -\frac{8}{5} x^{-\frac{13}{5}}$$

$$\begin{aligned} u^{-\frac{5}{8}} &= x \quad \frac{dx}{du} = -\frac{5}{8} u^{-\frac{13}{8}} \\ &= -\frac{5}{8} (x^{-\frac{8}{5}})^{-\frac{13}{8}} \\ &= -\frac{5}{8} x^{\frac{13}{5}} \end{aligned}$$

Ch. 9, Ex. 1, 2, 4

$$1) y = \sqrt{a+x} \quad \frac{dy}{dx} = \frac{1}{2}(a+x)^{-\frac{1}{2}}$$

$$y^2 - a = x$$

$$\frac{dx}{dy} = 2y = 2(a+x)^{\frac{1}{2}}$$

$$2) y = (a+x^2)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{2}(a+x^2)^{-\frac{3}{2}} \cdot 2x$$

$$\begin{aligned} y^{-2} &= a+x^2 \\ y^{-2}-a &= x^2 \\ (y^{-2}-a)^{\frac{1}{2}} &= x \\ \cancel{1/2} (y^{-2}-a)^{-\frac{1}{2}} \cdot (-2)y^{-3} &= x \\ ((a+x^2)^{-\frac{1}{2}} - a)^{-\frac{1}{2}} \cdot (a+x^2)^{-\frac{1}{2}-3} &= x \\ &= x^{-1} \cdot (a+x^2)^{\frac{3}{2}} \end{aligned}$$

$$4) y = (x^3 - a^2)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{3}{2}(x^3 - a^2)^{-\frac{3}{2}} \cdot x^2$$

$$\begin{aligned} y^{-2} &= x^3 - a^2 \\ y^{-2} + a^2 &= x^3 \\ x &= (y^{-2} + a^2)^{\frac{1}{3}} \\ \frac{dx}{dy} &= \frac{1}{3} (y^{-2} + a^2)^{-\frac{2}{3}} \cdot (-2)y^{-3} \\ &= (-\frac{2}{3}) x^2 (x^3 - a^2)^{-\frac{3}{2}} \end{aligned}$$

Exer. 6, 1, 2, 3, 4

$$1) y = (x^2 + 1)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \cancel{x} (x^2 + 1)^{-\frac{1}{2}} x$$
$$\begin{aligned} y^2 &= x^2 + 1 \\ y^2 - 1 &= x^2 \\ \sqrt{y^2 - 1} &= x \\ \frac{dx}{dy} &= \frac{1}{2} (y^2 - 1)^{-\frac{1}{2}} \cancel{y} = x^{-1} (x^2 + 1)^{\frac{1}{2}} \end{aligned}$$

$$2) y = \sqrt{x^2 + a^2}$$
$$\frac{dy}{dx} = \frac{1}{2} (x^2 + a^2)^{-\frac{1}{2}} \cancel{x}$$
$$\begin{aligned} y^2 &= x^2 + a^2 \\ y^2 - a^2 &= x^2 \\ \frac{dx}{dy} &= \cancel{2}(y^2 - a^2)^{-\frac{1}{2}} \cdot \cancel{y} = x^{-1} (x^2 + a^2)^{\frac{1}{2}} \end{aligned}$$

$$3) y = (a+x)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2} (a+x)^{-\frac{1}{2}}$$
$$\begin{aligned} y^2 &= a+x \\ y^2 - a &= x \\ \frac{dx}{dy} &= 2y = 2(a+x)^{\frac{1}{2}} \end{aligned}$$

$$4) y = a(a-x^2)^{\frac{1}{2}} \Rightarrow (a-x^2)^{\frac{1}{2}} = \frac{a}{y}$$
$$\begin{aligned} \frac{dy}{dx} &= +\frac{a}{2}(a-x^2)^{-\frac{1}{2}} \cdot (-2x) \\ &= a x (a-x^2)^{-\frac{3}{2}} \end{aligned}$$
$$\begin{aligned} a-x^2 &= a^2/y^2 \\ \frac{a-a^2/y^2}{a^2/y^2} &= \frac{x}{x^2} \\ \frac{\sqrt{a-a^2/y^2}}{a^2/y^2} &= x \\ \frac{(a-a^2/y^2)^{-\frac{1}{2}}}{a^{-1}x^{-1}(a-x^2)^{\frac{3}{2}}} &= \frac{a^2/y^2}{a^2/y^2} = \frac{dx}{dy} \end{aligned}$$

# The Calculus of Growth

Calc Made Easy, Ch. 14

Examples, p. 173

$$1) y = e^{\alpha x} (-\alpha x)$$

$$y' = -\alpha e^{\alpha x} (-\alpha x)$$

$$\ln y = -\alpha x$$

$$y' = -\alpha y$$

$$2) y = e^{\alpha x} (x^2/3)$$

$$y' = e^{\alpha x} (x^2/3) \cdot 2/3 x$$

or

$$\ln y = x^2/3$$

$$y' = y \cdot 2x/3$$

$$3) y = e^{\alpha x} (2x/(x+1)}$$

$$y' = 2 \frac{e^{\alpha x} (2x/(x+1))}{(x+1)^2}$$

$$\ln y = 2x/(x+1)$$

$$y' = y \left[ \frac{2(x+1) - 2x}{(x+1)^2} \right] = \frac{y \cdot 2}{(x+1)^2}$$

$$4) y = e^{\alpha x} (\sqrt{x^2+a})$$

$$y' = e^{\alpha x} (\sqrt{x^2+a}) \cdot \frac{1}{2} (x^2+a)^{-1/2} \cdot 2x$$

$$\ln y = \sqrt{x^2+a}$$

$$y' = y \cdot \frac{1}{2} (x^2+a)^{-1/2} \cdot 2x$$

$$5) y = \ln(a+x^3)$$

$$y' = \frac{3x^2}{a+x^3}$$

$$6) y = \ln(3x^2 + \sqrt{a+x^2})$$

$$y' = (3x^2 + \sqrt{a+x^2})^{-1} (6x + \cancel{\frac{1}{2}} (a+x^2)^{-1/2} \cdot 2x)$$

$$7) y = (x+3)^2 \sqrt{x-2}$$

$$y' = 2(x+3) \sqrt{x-2} + (x+3)^2 \frac{1}{2} (x-2)^{-1/2}$$

$$\frac{4(x+3)(x-2) + (x+3)^2}{(x-2)^{1/2}} = \frac{5(x+3)(x-1)}{2(x-2)^{1/2}}$$

$$\ln y = 2 \ln(x+3) + \frac{1}{2} \ln(x-2)$$

$$y' = y \left[ \frac{2}{x+3} + \frac{1}{2(x-2)} \right]$$

$$= y \left[ \frac{4x-8+x+3}{2(x-2)(x+3)} \right] = \frac{5y}{2} \frac{(x-1)}{(x-2)(x+3)}$$

$$8) y = (x^2+3)^3 (x^3-2)^{2/3}$$

$$y' = 3(x^2+3)^2 \cancel{2x} (x^3-2)^{2/3} + (x^2+3)^3 \cancel{2/3} (x^3-2)^{-1/3} \cancel{3x^2}$$

$$= 2x(x^2+3)^2 (x^3-2)^{-1/3} \left[ 3(x^3-2) + (x^2+3) \cdot x \right]$$

$$2x(x^2+3)^2 (x^3-2)^{1/3} \left[ \begin{matrix} 3x^3-6+x^3+3x \\ (4x^3+3x-6) \end{matrix} \right]$$

or

$$\ln y = 3 \ln(x^2+3) + 2/3 \ln(x^3-2)$$

$$y' = y \left[ \frac{3}{x^2+3} \cdot 2x + \frac{2}{3(x^3-2)} \cdot 3x^2 \right]$$

$$= 2xy \left[ \frac{3(x^3-2) + x(x^2+3)}{(x^2+3)(x^3-2)} \right]$$

$$= 2xy \frac{(4x^3+3x-6)}{(x^2+3)(x^3-2)}$$

$$9) \quad y = \frac{(x^2+a)^{1/2}}{(x^3-a)^{1/3}}$$

$$\ln y = \frac{1}{2} \ln(x^2+a) - \frac{1}{3} \ln(x^3-a)$$

$$y' = y \left[ \frac{1}{2(x^2+a)} \cdot 2x - \frac{1}{3(x^3-a)} \cdot 3x^2 \right]$$

$$= y \left[ \frac{x(x^3-a) - x^2(x^2+a)}{(x^2+a)(x^3-a)} \right]$$

$$= y \left[ \frac{-xa - x^2a}{(x^2+a)(x^3-a)} \right]$$

$$= \frac{-x\alpha y(1-x)}{(x^2+a)^{1/2}(x^3-a)^{1/3}}$$

$$= \frac{-\alpha x(1-x)}{(x^2+a)^{1/2}(x^3-a)^{1/3}}$$

Further Examples, p. 171

$$1) \quad C = \frac{E}{R} \left( 1 - e^{-Rt/L} \right)$$

at  $t = \frac{L}{R}$ ,  $C$  is  $(1 - \frac{1}{e})$  times  
the original amount...

$$3) \quad Q = Q_0 e^{-\lambda t}$$

$$\frac{1}{2} = e^{-\lambda t_{1/2}}$$

$$\ln(\frac{1}{2}) = -\lambda t_{1/2}$$

$$t_{1/2} = \frac{\ln(\frac{1}{2})}{-\lambda}$$

$$10) \quad y = \frac{1}{\ln x}$$

sometimes written  
 $\ln^2 x$

$$y' = \frac{-1}{(\ln x)^2} \cdot \frac{1}{x}$$

$$11) \quad y = (\ln x)^{1/3}$$

$$y' = \frac{1}{3} (\ln x)^{-2/3} \frac{1}{x}$$

$$= \frac{1}{3x(\ln x)^{2/3}}$$

$$12) \quad y = a^{-\alpha x^2}$$

$$\ln y = -\alpha x^2 \ln a$$

$$y' = y \cdot -2\alpha \ln a x$$

$$y' = -2a^{1-\alpha x^2} \ln a x$$

$$2) \quad I = I_0 e^{-Kt}$$

$$2 = e^{-Kt_{1/2}}$$

$$t_{1/2} = \frac{\ln 2}{-K}$$

# Exponential Derivatives Practice

CALC MADE EASY, Exercises 12, p. 166

$$1) y = b(e^{ax} - e^{-ax})$$

$$\begin{aligned}\frac{dy}{dx} &= abe^{ax} + abe^{-ax} \\ &= abe^{ax}(e^{2ax} + 1)\end{aligned}$$

$$8) y = (3x^2 + 1)e^{-5x}$$

$$\frac{dy}{dx} = 6x \cdot e^{-5x} - 5e^{-5x}(3x^2 + 1)$$

$$9) y = \ln(x^a + a)$$

$$\frac{dy}{dx} = \frac{1}{x^a + a} \cdot a \cdot x^{a-1}$$

$$10) y = (3x^2 - 1)(\sqrt{x} + 1)$$

$$= 3x^2\sqrt{x} + 3x^2 - \sqrt{x} - 1$$

$$\frac{dy}{dx} = 7.5x^{1.5} + 6x - \frac{1}{2}x^{-0.5}$$

$$11) y = \frac{\ln(x+3)}{x+3}$$

$$\frac{dy}{dx} = \frac{\frac{x+3}{x+3} - \ln(x+3)}{(x+3)^2} = \frac{1 - \ln(x+3)}{(x+3)^2}$$

$$12) y = a^x x^a$$

$$= a^x \ln a x^a + a^x a x^{a-1}$$

$$= a^x x^{a-1} [x \ln a + a]$$

$$5) W = Pv^n$$

$$\frac{dw}{dv} = n p v^{n-1}$$

$$13) s = ay^2 \ln\left(\frac{1}{y}\right)$$

$$\begin{aligned}\frac{ds}{dy} &= a \left[ 2y \ln\left(\frac{1}{y}\right) - \cancel{y^2} \cdot \cancel{y} \cdot \cancel{y^2} \right] \\ &= a y [2 \ln\left(\frac{1}{y}\right) - 1] = 0 \\ \ln\left(\frac{1}{y}\right) &= 1/2\end{aligned}$$

$$7) y = 3e^{-\frac{x}{x-1}}$$

$$\frac{dy}{dx} = 3e^{-\frac{x}{x-1}} \cdot \frac{-(x-1)+x}{(x-1)^2} = \frac{3}{(x-1)^2} e^{-\frac{x}{x-1}}$$

$$\begin{cases} y = \exp(-1/2) \\ y = e^{-1/2} \end{cases}$$

$$14) \quad y = x^3 - \ln x$$

$$\frac{dy}{dx} = 3x^2 - \frac{1}{x} = 0$$

$$3x^3 - 1 = 0$$

$$\text{so min is } \hat{x} = \left(\frac{1}{3}\right)^{\frac{1}{3}} \approx 0.693$$

$$15) \quad y = \ln(axe^x)$$

$$= \ln(a) + \ln(x) + \ln(e^x)$$

$$= \ln(a) + \ln(x) + x$$

$$\frac{dy}{dx} = \frac{1}{x} + 1$$

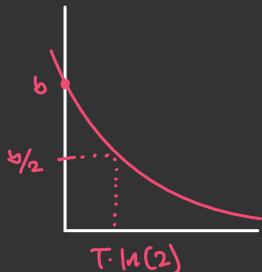
$$16) \quad y = (\ln(ax))^3$$

$$\frac{dy}{dx} = \frac{3\ln(ax)^2}{ax} \cancel{a} = \frac{3}{x} \ln(ax)^2$$

## More Growth Exercises

Calc Made Easy Ex. 13, p. 173

$$1) \quad y = b e^{-t/\tau} \quad \tau = b \ln(2)$$



$$2) \quad t_{1/2} = 24 \quad \Rightarrow \quad y = b e^{-t \ln(2)/24}$$

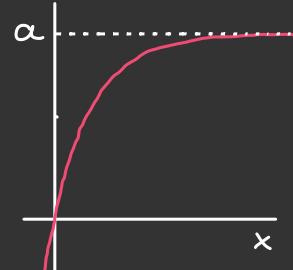
$$y_{100} = b e^{-t_{100} \ln(2)/24}$$

$$\ln(10^2) = -\frac{t_{100} \ln(2)}{24}$$

$$t_{100} = 2 \cdot 24 \log_2(10) \approx 159.45$$

$$3) \quad y = a(1 - e^{-2t}) \quad = a(1 - (\frac{1}{2})^n)$$

$$n = \log_2(\exp(bx))$$



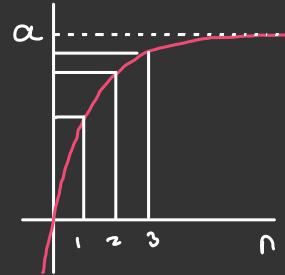
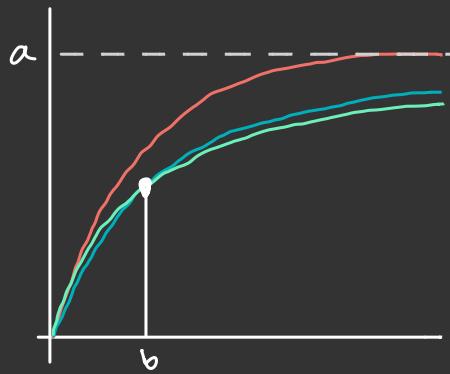
$$4) \quad y = \frac{2a}{\pi} \arctan(x/b)$$

$$y = a(1 - e^{-x/b})$$

$$= a(1 - (\frac{1}{2})^n); x = \ln((\frac{1}{2})^n)b$$

$$y = \frac{ax}{x+b} = \frac{an}{n+1}; n = \frac{x}{b}$$

Also see Ex. 14, #11



$$5) \quad y = x^x \quad y' = x^x \cdot (\ln(x) + 1)$$

$$= e^{x \cdot \ln(x)}$$

$$y = (e^x)^x \quad y' = e^{x^2} \cdot 2x$$

$$y = e^{x^x} = e^v$$

$$y' = e^v \cdot v' = e^{x^x} \cdot x^x (\ln(x) + 1)$$

$$6) \quad Q = Q_0 e^{-\lambda t}$$

$$\frac{1}{2} Q_0 = Q_0 e^{-\lambda t}$$

$$\ln(\frac{1}{2}) = -\lambda t_{1/2}$$

$$t_{1/2} = \ln(\frac{1}{2}) / -\lambda$$

$$\text{if } \lambda = 5, \quad t_{1/2} \approx 0.14 \text{ sec}$$

$$7) V = V_0 \exp(-t/kR)$$

$$\begin{aligned}V_0 &= 20 \\k &= 4 \times 10^{-6} \\R &= 10^4\end{aligned}$$

$$V(0.1) = 1.64$$

$$V(0.01) = 15.58$$

$$8) Q = Q_0 \exp(-\mu t)$$

$t$	$Q$	$16 = 20 \exp(-\mu \cdot 10)$	$\mu = 0.00037?$
0	20	$\ln(\frac{16}{20}) = -10\mu$	
10	16		answer key is wrong!
31.1	10	$\mu = \ln(\frac{1}{5})/-10 \approx 0.022$	

$$\frac{1}{2} Q = Q_0 \exp(-\mu t_{1/2}) \quad t_{1/2} = \frac{\ln(1/2) \cdot 10}{\ln(1/5)}$$

$$\ln(1/2) = -\mu t_{1/2}$$

$$9) i = i_0 \exp(-\beta l) \quad \beta = 0.0114$$

$$l \text{ if } \frac{i}{i_0} = 0.08$$

$$\alpha = \frac{i}{i_0} \text{ if } l = 40$$

$$0.08 \cancel{i_0} = \cancel{i_0} \exp(-\beta l)$$

$$-\frac{\ln(0.08)}{0.0114} = l$$

$$\alpha \cancel{i_0} = \cancel{i_0} \exp(\beta \cdot l)$$

$$\alpha = \exp(-40 \cdot 0.0114)$$

$$\approx 0.63$$

$$221.6 \approx l$$

$$10) p = p_0 \exp(-\kappa h)$$

$h$	$p$	$\kappa$
0	760	
10	199.2	0.133
20	42.4	0.144
50	0.32	0.155

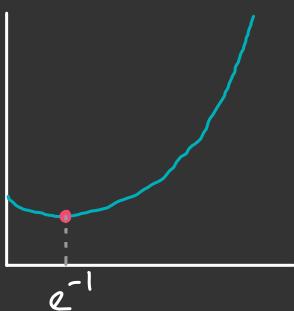
$\frac{\ln\left(\frac{p}{760}\right)}{-h} = \kappa$

$\kappa = 0.144$

$$11) y = x^x$$

$$y' = x^x (\ln x + 1) = 0$$

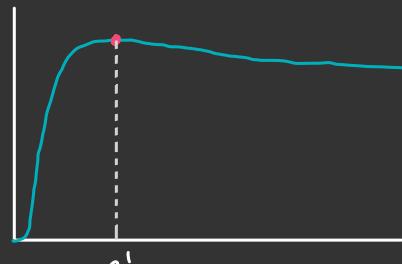
$$\hat{x} = \exp(-1)$$



$$12) y = x^{1/x} = \exp\left(\frac{1}{x} \ln x\right)$$

$$y' = \exp\left(\frac{1}{x} \ln x\right) \cdot \left(-\frac{1}{x^2} \ln x + \frac{1}{x^2}\right)$$

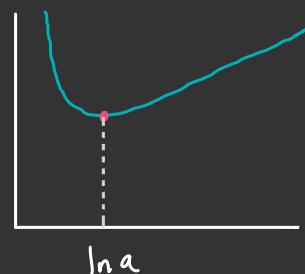
$$\hat{x} = \exp(1)$$



$$13) y = x^{\alpha/x}, \quad \alpha > 1$$

$$\begin{aligned}y' &= \alpha^{\alpha/x} - x^{\alpha/x} \ln \alpha \hat{x}^2 \\&= \alpha^{\alpha/x} (1 - \hat{x}^2 \ln \alpha)\end{aligned}$$

$$\hat{x} = \ln \alpha$$



# THE CALCULUS OF TRIGONOMETRY

Calc Made Easy Ch. 15, p. 175

$$1) y = \arcsin x$$

$$\begin{aligned} x &= \sin y \\ \frac{dx}{dy} &= \cos y = \sqrt{1-x^2} \\ \frac{dy}{dx} &= (1-x^2)^{-\frac{1}{2}} \end{aligned}$$

$$2) y = \cos^3 \theta$$

$$\frac{dy}{dx} = 3 \cos^2 \theta \cdot (-\sin \theta)$$

$$3) y = \sin(x+a)$$

$$\frac{dy}{dx} = \cos(x+a)$$

?

$$4) y = \ln \sin \theta$$

$$y' = \frac{1}{\sin \theta} \cdot \cos \theta = \cot \theta$$

$$5) y = \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$y' = \frac{-\sin^2 \theta - \cos^2 \theta}{\sin^2 \theta} = -(\frac{1 + \cot^2 \theta}{\sin^2 \theta}) = -\csc^2 \theta$$

$$6) y = \tan 3\theta$$

$$y' = \sec^2 3\theta \cdot 3$$

$$7) y = \sqrt{1+3\tan^2 \theta} - \frac{1}{2}$$

$$y' = \frac{1}{2}(1+3\tan^2 \theta) \cdot 6\tan \theta \cdot \sec \theta$$

$$8) y = \sin x \cos x$$

$$\begin{aligned} y' &= \cos x \sin x - \sin^2 x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

$$\sin M - \sin N = 2 \cos\left(\frac{M+N}{2}\right) \sin\left(\frac{M-N}{2}\right)$$

$$\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \sinh x / \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\begin{aligned} \frac{d \sin x}{dx} &= \cos x & \sec x &= \frac{1}{\cos x} \\ \frac{d \cos x}{dx} &= -\sin x & \csc x &= \frac{1}{\sin x} \\ \frac{d \tan x}{dx} &= \sec^2 x & \cot x &= \frac{1}{\tan x} \end{aligned}$$

$$\arcsin x = \sin^{-1} x$$

$$\arccos x = \cos^{-1} x$$

$$\operatorname{arctan} x = \tan^{-1} x$$

$$\text{if } \theta = \arccos x, \quad \sin \theta = \sqrt{1-x^2} \quad \text{for } x \in [-1, 1]$$

$$\text{if } \theta = \arcsin x, \quad \cos \theta = \sqrt{1-x^2} \quad \text{for } x \in [-1, 1]$$

$$\sec^2 x = 1 + \tan^2 x$$

if  $\theta = \arctan x$ , then

$$\sin \theta = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \cos \theta = \frac{1}{\sqrt{1+x^2}}$$

# Trig Derivative Exercises

CALC MADE EASY Ex. 14, p. 183

$$1) \quad y = A \sin(\theta - \frac{\pi}{2})$$

$$y' = A \cos(\theta - \frac{\pi}{2})$$

$$y = \sin^2 \theta$$

$$\begin{aligned} y' &= 2 \cos \theta \sin \theta \\ &= \sin 2\theta \end{aligned}$$

$$y = \sin 2\theta$$

$$y' = 2 \cos 2\theta$$

$$y = \sin^3 \theta$$

$$= 3 \sin^2 \theta \cdot \cos \theta$$

$$y = \sin 3\theta$$

$$= 3 \cos 3\theta$$

$$2) \quad y = \sin \theta \cos \theta$$

$$y' = \cos^2 \theta - \sin^2 \theta = 0 \text{ if } \sin \theta = \cos \theta$$

$$\tan \theta = 1$$

$$\hat{\theta} = \tan^{-1}(1) = \frac{\pi}{4} + 2\pi N$$

$$3) \quad y = \frac{\cos(2\pi n t)}{2\pi}$$

$$y' = \underline{-\sin(2\pi n t) \frac{2\pi n}{2\pi}}$$

$$4) \quad y = \sin(a^x)$$

$$y' = \cos(a^x) a^x \ln a$$

$$5) \quad y = \ln(\cos(x))$$

$$y' = -\frac{\sin(x)}{\cos(x)} = -\tan(x)$$

$$6) \quad y = a \sin(x+b)$$

$$y' = a \cos(x+b)$$

$$7) \quad y = a \sin(x-b)$$

$$y' = a \cos(x-b) = 0 \text{ if } x = \cos^{-1}(0) + b = \frac{\pi}{2} + b$$

$$y'' = -a \sin(x-b) = 0 \text{ if } x = \sin^{-1}(0) + b = b$$

$$y'(b) = a$$

$$a = 100 \quad b = 15^\circ$$

$$x = 75^\circ$$

$$y' = 100 \cos(75^\circ - 15^\circ) = 50$$

$$8) \quad y = \sin \theta \sin 2\theta$$

$$y' = \cos \theta \sin 2\theta + \sin \theta \cos 2\theta \cdot 2$$

$$9) \quad y = a \tan^n(\theta)$$

$$y' = a n \tan^{n-1}(\theta) \sec^2(\theta) \cdot n \theta^{n-1}$$

$$10) \quad y = e^x \sin^2 x$$

$$y' = e^x \sin^2 x + e^x 2 \sin x \cos x$$

$$= e^x \sin x (\sin x + 2 \cos x)$$

$$= e^x (\sin^2 x + \sin 2x)$$

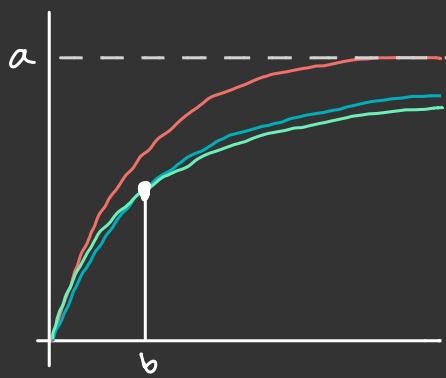
$\sin 2x = 2 \sin x \cos x$

$$(1) \quad y = \frac{2a}{\pi} \arctan(x/b)$$

$$y = a(1 - \exp(-x/b))$$

$$= a(1 - (\frac{1}{2})^n); x = \ln(\frac{1}{2})b$$

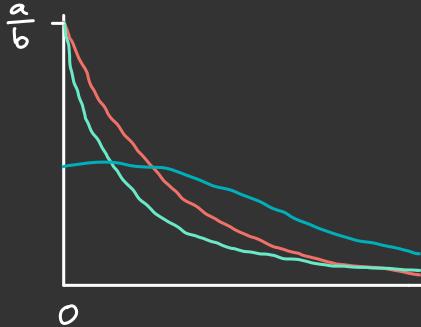
$$y = \frac{ax}{x+b} = \frac{an}{n+1}; n = \frac{x}{b}$$



$$y' = \frac{2a}{\pi b (1+x^2)}$$

$$y' = a/b \exp(-x/b)$$

$$y' = \frac{ab}{(x+b)^2}$$



$$(2) \quad y = \sec x = \frac{1}{\cos x}$$

$$y' = \frac{\sin x}{\cos^2 x}$$

$$= \tan x \sec x$$

$$y = \arccos x$$

$$x = \cos y$$

$$\frac{dx}{dy} = -\sin y$$

$$y' = -\frac{1}{\sin y} = -\frac{1}{\sin(\arccos(x))}$$

$$= -(1-x^2)^{-1/2}$$

$$y = \operatorname{arcsec} x$$

$$x = \sec y$$

$$\frac{dx}{dy} = \tan y \cdot x$$

$$y' = \frac{1}{x \tan(\operatorname{arcsec} x)}$$

$$= \frac{1}{x \sqrt{x^2-1}}$$

$$y = \arctan x$$

$$x = \tan y$$

$$dx/dy = \sec^2 y$$

$$y = \tan x \sqrt{3 \sec x}$$

$$y' = \sec^2 x \cdot \sec'^{1/2} x \cdot \sqrt{3}$$

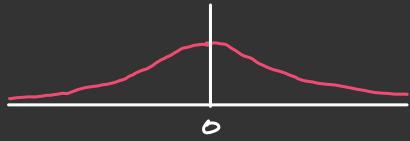
$$+ \tan x \frac{1}{2}(3 \sec x)^{-1/2} \cdot 3 \tan x \sec x$$

$$= \sqrt{3} \cdot \sec^{5/2} x + \frac{3 \tan^2 x \sec x}{\sqrt{3} \cdot 2 \cdot \sec^{1/2} x}$$

$$= \sqrt{3} \sec^{1/2} x \left( \sec^2 x + \frac{\tan^2 x}{2} \right)$$

$$= \frac{\sqrt{3} \sec x (3 \sec^2 x - 1)}{2}$$

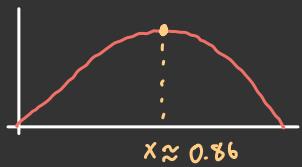
using  
 $\sec^2 x = \frac{1}{1 + \tan^2 x}$



$$13) y = \sin(a\theta + b)$$
$$y' = C \sin(a\theta + b) \cdot C - C \cos(a\theta + b) a$$

$$14) y = \theta^3 + 3 \sin(\theta + 3) - 3^{\sin \theta} - 3^\theta$$
$$y' = 3\theta^2 + 3 \cos(\theta + 3) - 3^{\sin \theta} \ln 3 \cdot \cos \theta - 3^\theta \ln 3$$

$$15) y = \theta \cos \theta$$
$$y' = \cos \theta - \theta \sin \theta = 0 \text{ if } \theta = \cot \theta$$
$$\theta \approx \pm 0.86$$



# PARTIAL DIFFERENTIATION

$$1) w = 2ax^2 + 3bxy + 4cy^3$$

$$\frac{\partial w}{\partial x} = 4ax + 3by$$

$$\frac{\partial w}{\partial y} = 3bx + 12cy^2$$

HENCE:

$$dw = (4ax + 3by)dx + (3bx + 12cy^2)dy$$

$$2) z = x^y \quad \frac{\partial z}{\partial x} = yx^{y-1} \quad \frac{\partial z}{\partial y} = x^y \ln x$$

so

$$dz = yx^{y-1} \cdot dx + x^y \ln x \cdot dy$$

$$= x^{y-1} (ydx + x \ln x dy)$$

$$3) V = \frac{1}{3}\pi r^2 h$$

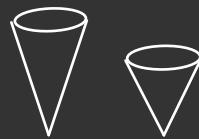
If radius varies for a fixed height:

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h$$



If height varies for a fixed radius:

$$\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$



If both height & radius change:

$$dV = \frac{\partial V}{\partial r} \cdot dr + \frac{\partial V}{\partial h} dh$$

$$= \frac{2}{3}\pi rh dr + \frac{1}{3}\pi r^2 dh$$

$$= \frac{1}{3}\pi r (2h dr + r dh)$$

$$\frac{dV}{dr} = \frac{1}{3}\pi r (2h + r \frac{dh}{dr})$$

$$\frac{dV}{dh} = \frac{1}{3}\pi r (2h \frac{dr}{dh} + r)$$

4)  $y = f(w) + f(v)$

*w = x + at*

*v = x - at*

*generic, differentiable functions*

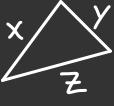
$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial w} + \frac{\partial f}{\partial v} = f'(w) + f'(v)$$

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial w} \cdot a - \frac{\partial f}{\partial v} a$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial w^2} + \frac{\partial^2 f}{\partial v^2}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 f}{\partial w^2} \cdot a^2 + \frac{\partial^2 f}{\partial v^2} a^2$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 \cdot \frac{\partial^2 f}{\partial x^2}$$

5)   $x+y+z=2s$ , for some constants  $s$

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

$$= \sqrt{s(s-x)(s-y)(s-(2s-x)-y)}$$

$$= \sqrt{s(s-x)(s-y)(-s+x+y)}$$

$$P = (s-x)(s-y)(x+y-s)$$

$$\frac{\partial P}{\partial x} = - (s-y)(x+y-s) + (s-x)(s-y)$$

$$= (s-y)(2(s-x)-y) = 0$$

$$\frac{\partial P}{\partial y} = (s-x)(2(s-y)-x) = 0$$

$\begin{cases} s=y \text{ or } y=2(s-x) \\ s=x \text{ or } y=s-\frac{x}{2} \end{cases}$

$$\begin{cases} x = 2s/3 \\ y = 2s/3 \\ z = 2s/3 \end{cases}$$

$$s = \frac{3x}{2}$$

6) For a fixed  $V$ , what are the dimensions of a lidless box that minimize its surface area?

By symmetry, we can assume the base must be a square.

$$V = s^2 h \quad \text{iso-volumes: } h = V/s^2$$

$$A = 4sh + s^2 \quad \text{iso-areas: } h = \frac{A - s^2}{4s}$$

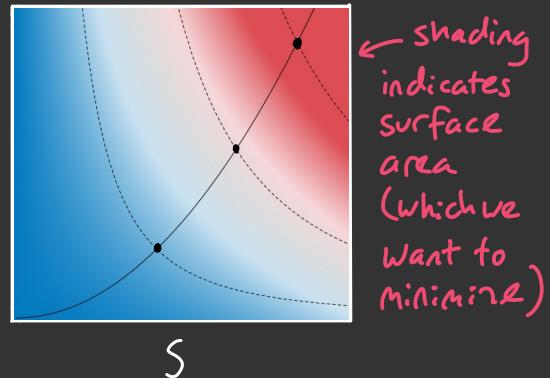
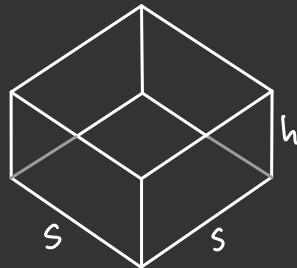
$$A = 4 \frac{V}{s^2} \cdot \frac{s}{2} + s^2$$

$$\frac{\partial A}{\partial s} = -\frac{4V}{s^2} + s^2 = 0 \quad \text{if } s^2 = 2\sqrt{V}$$

which implies

$$h = \frac{V}{s^2} = \frac{V}{2\sqrt{V}} = \frac{\sqrt{V}}{2}$$

$$h = \left(\frac{s}{2}\right)^2$$



Dotted lines show iso-volumes.  
For any given volume, the min. Surface area is at  $h = (s/2)^2$ .

# PARTIAL DIFFERENTIATION PRACTICE

CALC MADE EASY EXERCISES 15, PAGE 189

$$1) z = \frac{x^3}{3} - 2x^3y - 2y^2x + \frac{y^3}{3}$$

$$\frac{\partial z}{\partial x} = x^2 - 6x^2y - 2y^2$$

$$\frac{\partial z}{\partial y} = -2x^3 - 4yx + \frac{y^2}{3}$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= (x^2 - 6x^2y - 2y^2)dx + (-2x^3 - 4yx + \frac{y^2}{3})dy$$

$$2) A = x^2yz + xy^2z + xyz^2 + x^2y^2z^2$$

$$\frac{\partial A}{\partial x} = 2xyz + y^2z + yz^2 + 2xy^2z^2$$

$$\frac{\partial A}{\partial y} = x^2z + 2xyz + xz^2 + 2x^2yz^2$$

$$\frac{\partial A}{\partial z} = x^2y + xy^2 + 2xyz + 2x^2y^2z$$

$$dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz$$

4) Find the total derivative of  $y = v^v$ .

$$3) r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{r}(r^2)^{-\frac{1}{2}} \cdot 1(x-a) = \frac{(x-a)}{r}$$

so by symmetry, the gradient is

$$\nabla r = \left( \frac{x-a}{r}, \frac{y-b}{r}, \frac{z-c}{r} \right); r \neq 0 \quad \text{gradients are vector fields}$$

Hence the requested sum is

$$\frac{\partial r}{\partial x} + \frac{\partial r}{\partial y} + \frac{\partial r}{\partial z} = r^{-1}((x-a) + (y-b) + (z-c))$$

Interpretation: the directional derivative along the vector  $(1, 1, 1)$

The second partial derivatives:

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - (x-a)\frac{\partial r}{\partial x}}{r^2} = \frac{1}{r} - \frac{(x-a)^2}{r^3}$$

so

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{3}{r} - \frac{1}{r^3}((x-a)^2 + (y-b)^2 + (z-c)^2)$$

$$= \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

This expression is the Laplacian of  $r$ , aka.  $\nabla^2 r$  or  $\Delta r$

$$y = v^v$$

$$\frac{\partial y}{\partial v} = v v^{v-1} \quad \frac{\partial y}{\partial v} = v^v \ln v$$

$$dy = v v^{v-1} dv + v^v \ln v dv ;$$

$$\text{hence } \frac{dy}{dv} = v v^{v-1} + v^v \ln v \frac{dv}{dv}$$

$$\frac{dy}{dv} = v v^{v-1} \frac{du}{dv} + v^v \ln v$$

Note: for some function  $f(x_1, x_2, \dots, x_n)$  where each  $x_i$  is a function of  $t$ , the total derivative is

$$\frac{df}{dt} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \frac{dx_i}{dt}$$

aka. the multivariate chain rule

conceptually: total derivative = gradient  $\times$  velocity vector

not exactly the same as a directional derivative!

there's also a "total differential" form:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

$$\nabla = \text{"nabla"} \text{ or "del"}; \quad \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \dots \right)$$

$$5) \quad y = u^3 \sin v \quad \frac{\partial y}{\partial u} = 3u^2 \sin v \quad \frac{\partial y}{\partial v} = u^3 \cos v$$

$$y = (\sin x)^u \quad \frac{\partial y}{\partial u} = (\sin x)^u \ln(\sin x) \quad \frac{\partial y}{\partial x} = u(\sin x)^{u-1} \cos x$$

$$y = \frac{\ln u}{v} \quad \frac{\partial y}{\partial u} = \frac{1}{uv} \quad \frac{\partial y}{\partial v} = -\frac{\ln u}{v^2}$$

$$6) \quad x \cdot y \cdot z = k \quad \frac{\partial n}{\partial x} = 1 \quad \frac{\partial n}{\partial y} = 1 \quad \frac{\partial n}{\partial z} = 1$$

$$x + y + z = m \quad dM = dx + dy + dz; \quad \frac{dn}{dx} = 1 + \frac{dy}{dx} + \frac{dz}{dx} = 1 - \frac{k}{zx^2} - \frac{k}{yx^2}$$

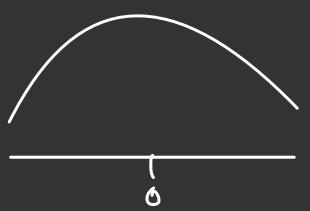
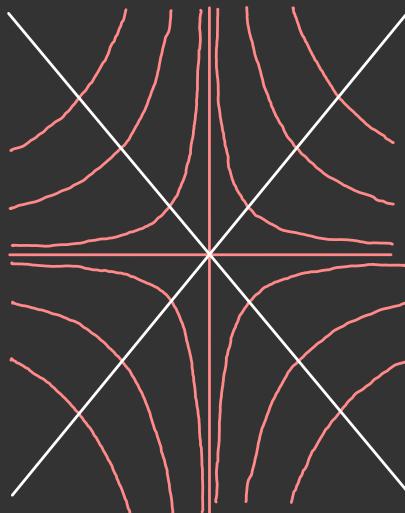
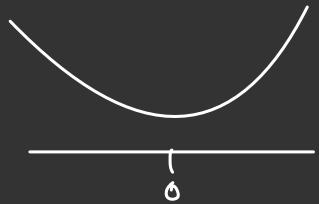
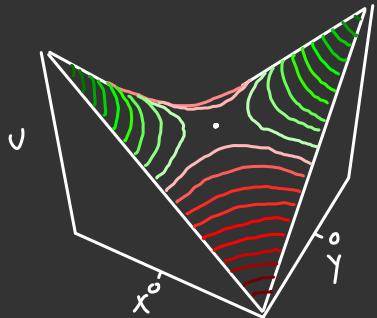
$$7) \quad u = -ax + xy - by \quad \text{isoclines at:}$$

$$\frac{\partial u}{\partial x} = -a + y = 0 \quad y = a$$

$$\frac{\partial u}{\partial y} = -b + x = 0 \quad x = b$$

$y = \frac{u + ax}{x - b}$

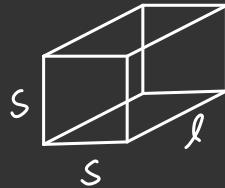
point  $(a, b)$  is a saddle



8) Maximize  $V = s^2 l$  for a given  $D = l + 4s$ .

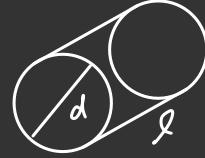
$$V = s^2 l = s^2 (D - 4s)$$

$$6s = l + 4s \quad l = 2s$$



$$\frac{\partial V}{\partial s} = 2sD - 12s^2 = 0 \quad \text{so} \quad 6s = D \quad l = 2s$$

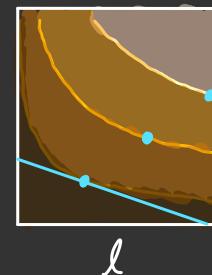
Maximize  $V = \pi(d/2)^2 l$  for a given  $S = \pi d + l$ .



$$V = \frac{\pi d^2}{4} (S - \pi d) = \frac{\pi d^2 S}{4} - \frac{\pi^2 d^3}{4}$$

$$\frac{\partial V}{\partial d} = \frac{\pi d S}{2} - \frac{3\pi^2 d^2}{4} = 0 \quad \text{if} \quad d = \frac{2S}{3\pi}$$

For a given  $S$ , we find the maximum  $V$  along  $d = 2l/\pi$



$$\text{So } S = 3l \quad \& \quad l = \frac{\pi d}{2} \quad \text{or} \quad l = \pi r$$

$$9) \quad \begin{aligned} \pi &= a+b+c \\ S &= \sin(a) \sin(b) \sin(c) \end{aligned}$$

$$\frac{\partial S}{\partial a} = \sin(a) \sin(2b+a) = 0 \quad \text{if} \quad a = \{0, \pi\} \quad \text{or} \quad 2b+a = \pi \quad a = \pi - 2b$$

$$\frac{\partial S}{\partial b} = \sin(b) \sin(2a+b) = 0 \quad \text{if} \quad b = \{0, \pi\} \quad \text{or} \quad 2a+b = \pi \quad a = \frac{\pi - b}{2} \quad b = \pi - 2a$$

$$\pi - 2b = \frac{\pi - b}{2} \quad -\frac{2\pi}{2} + 4b = \frac{\pi - b}{2} - \pi + b \quad \pi = 3b \quad b = \pi/3 \quad \text{so by symmetry, } a = b = c = \pi/3$$

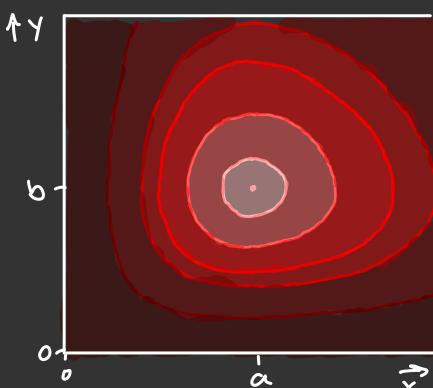
$$\pi - 3b = 0$$

$$10) \quad u = \frac{e^{x/a+y/b}}{xy}$$

$$\frac{\partial u}{\partial x} = \frac{(x-a)e^{x/a+y/b}}{y x^2} = \frac{(x-a)}{x} \cdot u$$

$$\frac{\partial u}{\partial y} = \frac{(y-b)e^{x/a+y/b}}{x y^2} = \frac{(y-b)}{y} \cdot u$$

The critical point is  $(a, b)$ , which by inspection is a minimum.

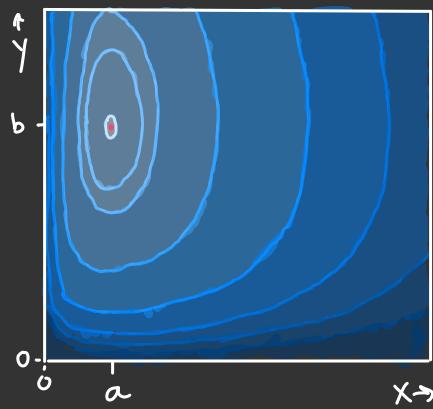


$$(1) \quad v = y + \frac{x}{a} - b \ln y - \ln x$$

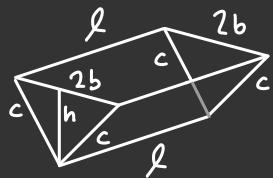
$$\frac{\partial v}{\partial x} = \frac{1}{a} - \frac{1}{x} = 0 \text{ if } x = a$$

$$\frac{\partial v}{\partial y} = 1 - \frac{b}{y} = 0 \text{ if } y = b$$

The point  $(a, b)$  is a minimum of  $v$ .



(2)



$$S = 2lc + 2bh$$

$$V = bhl$$

Our goal is to find the values of  $b, h$ , &  $l$  which minimize  $S$ , for a given  $V$ .

First, let's express  $l$  &  $c$  in terms of  $V, b+h$ :

$$c = \sqrt{b^2 + h^2} \quad l = \sqrt[3]{V/bh}$$

So then:

$$S = \frac{2V}{bh} \sqrt{b^2 + h^2} + 2bh$$

Because  $b+h$  are exchangeable in  $S$ , by symmetry,  
 $\hat{b} = \hat{h}$  at the joint min. Thus,

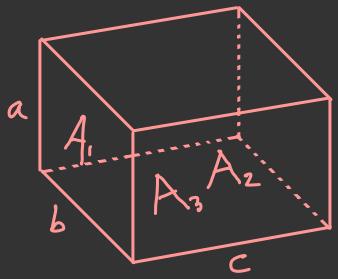
$$\begin{aligned} S &= \frac{2V}{h^2} \sqrt{2h^2} + 2h^2 \\ &= \frac{2\sqrt{2}V}{h} + 2h^2 \end{aligned}$$

$$\frac{dS}{dh} = -\frac{2\sqrt{2}V}{h^2} + 4h = 0 \text{ if } h = \left(\frac{\sqrt{2}}{2}V\right)^{1/3} = \hat{h}.$$

$$\text{This implies } \hat{l} = \sqrt[3]{2V}$$

# Partial Differentiation Practice

Calc Made Easy, Ch 16



$V = abc$  For any given  $V$ ,  
 $A_1 = ab$  we want to find the  
 $A_2 = ac$  values of  $a, b,$  &  $c$  that  
 $A_3 = bc$  minimize  $2A_1 + 2A_2 + A_3.$

$$w = 2A_1 + 2A_2 + A_3 \quad c = \frac{V}{ab}$$

$$= 2ab + 2ac + bc$$

$$= 2ab + 2a\frac{V}{ab} + \frac{bV}{ab}$$

$$\frac{\partial w}{\partial a} = 2b - Vb^{-2} \rightarrow 2\hat{b} = V\hat{b}^{-2}$$

$$2\hat{b}^3 = V$$

$$\hat{b} = \sqrt[3]{V/2}$$

$$\frac{\partial w}{\partial b} = 2a - Va^{-2} \rightarrow 2\hat{a} = V\hat{a}^{-2}$$

$$2\hat{a}^3 = V$$

$$\hat{a} = \sqrt[3]{V/2}$$

$$V = \sqrt[3]{V/2} \sqrt[3]{V/2} \hat{c}$$

$$\frac{V}{\sqrt[3]{3}} = \frac{(V/2)^{2/3} \hat{c}}{\sqrt[3]{3}}$$

$$\sqrt[3]{3} = \hat{c}/2^{2/3}$$

$$\hat{c} = 2^{2/3} \sqrt[3]{3} \frac{2^{2/3}}{2^{2/3}} = 2\sqrt[3]{V/2}$$

So, the area-minimizing dimensions are

$$\left( \sqrt[3]{V/2}, \sqrt[3]{V/2}, 2\sqrt[3]{V/2} \right)$$

# Integration Practice

Calc. Made Easy, Ch. 18

$$\frac{dy}{dx} = ax^b \quad y = \int \frac{dy}{dx} dx = \int ax^b dx = a \int x^b dx = \frac{a}{b+1} x^{b+1} + C$$

$$\frac{dy}{dx} = 24x^{11} \Rightarrow y = \int 24x^{11} dx = 2x^{12} + C$$

$$\int (a+b)(x+1) dx = \left(\frac{a+b}{2}\right)x^2 + (a+b)x + C$$

$$\int gt^{1/2} dt = \frac{2g}{3} t^{3/2} + C$$

$$\int (x^3 - x^2 + x) dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

$$\int ax^b dx = \left(\frac{a}{b+1}\right)x^{b+1} + C \Rightarrow \text{if } a=9.75, b=2.25, y = 3x^{3.75}$$

$$\int ax^{-1} dx = a \ln x + C \quad \text{if } x > 0$$

$$\int (x+1)(x+2) dx = \int (x^2 + 3x + 2) dx = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C$$

Since  $y = x(\ln x - 1) \Rightarrow \frac{dy}{dx} = \ln x + \cancel{\frac{x}{x}} - 1$   
it follows that  $\int \ln x dx = x(\ln x - 1)$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

# Some Easy Series

Calc Made Easy, Ex. 1b, p. 197

1.  $S = \frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \dots$

$$= \frac{2}{3} \left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right) = \frac{2}{3} A$$

$$= \frac{2}{3} \left( 1 + \frac{1}{2} \underbrace{\left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right)}_A \right)$$

$$A = 1 + \frac{1}{2} A$$

$$A = 2 \text{ so } S = \frac{4}{3}$$

2.  $S = -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots =$

$$= \sum_{i=1}^{\infty} (-1)^i \cdot \frac{1}{i}$$

This series is convergent, b/c  $|a_{n+1}| < |a_n|$ ,  
and  $\lim_{n \rightarrow \infty} a_n = 0$ .

In general,  $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \ln(1+x)$

so here  $x = 1 \& S = \ln(2) \approx 0.69$

3.  $\ln(1.3) = 0.3 - \frac{0.3^2}{2} + \frac{0.3^3}{3} - \frac{0.3^4}{4} + \dots$

$\approx 0.2625$   
(actually  $\approx 0.2624$ )

4.  $y' = \frac{1}{4}x$

$$\text{so } y = \frac{1}{8}x^2 + C$$

5.  $y' = 2x + 3$   
 $y = x^2 + 3x + C$

# Integration by Power Rule

Calc Made Easy, Exercises 17, p. 209

$$1. \int \sqrt{4ax} dx = \frac{4}{3}\sqrt{a}x^{3/2} + C$$

$$2. \int \frac{3}{x^4} dx = -\frac{3}{x^3} + C$$

$$3. \int a^{-1}x^3 dx = \frac{1}{4a}x^4 + C$$

$$4. \int (x^2 + a) dx = \frac{1}{3}x^3 + ax + C$$

$$5. \int 5x^{-7/2} dx = -2x^{-5/2} + C$$

$$6. \int (4x^3 + 3x^2 + 2x + 1) dx = x^4 + x^3 + x^2 + x + C$$

$$7. \int \left( \frac{ax}{2} + \frac{bx^2}{3} + \frac{cx^3}{4} \right) dx = \frac{a}{4}x^2 + \frac{b}{9}x^3 + \frac{c}{16}x^4 + C$$

$$8. \int \left( \frac{x^2 + a}{x + a} \right) dx = \int \left( \frac{a^2 + a}{x + a} - a + x \right) dx$$

$$= \ln|x+a| (a^2 + a) - ax + \frac{1}{2}x^2 + C$$

$$\begin{aligned} & x-a \\ & x+a \overline{)x^2 + a} \\ & \quad - (x^2 + xa) \\ & \quad \quad (a - xa) \\ & \quad - (-xa - a^2) \\ & \quad \quad \quad a + a^2 \end{aligned}$$

remainder

$$\frac{x^2 + a}{x + a} = x - a + \frac{a + a^2}{x + a}$$

using algebraic  
long division

$$9. \int (x+3)^3 dx = \int (x^2 + 6x + 9)(x+3) dx = \int (x^3 + 9x^2 + 27x + 27) dx \\ = \frac{1}{4}x^4 + 3x^3 + \frac{27}{2}x^2 + 27x + C$$

$$10. \int (x+2)(x-a) dx = \int (x^2 + (2-a)x - 2a) dx = \frac{1}{3}x^3 + \frac{(2-a)}{2}x^2 - 2ax + C$$

$$11. \int (\sqrt{x} + \sqrt[3]{x}) 3a^2 dx = 3a^2 \left(\frac{3}{2}\right)x^{3/2} + 3a^2 \left(\frac{4}{3}\right)x^{4/3} + C$$

$$12. \int (\sin \theta - \frac{1}{2}) \frac{d\theta}{3} = \frac{1}{3} \int \sin \theta d\theta - \frac{1}{6} \int d\theta = -\frac{1}{3} \cos \theta - \frac{1}{6}\theta + C$$

$$13. \int \cos^2 a\theta d\theta = \frac{1}{2} \int (1 + \cos(2a\theta)) = \frac{\theta}{2} + \frac{\sin(2a\theta)}{4a} + C$$

$$14. \int \sin^2 \theta d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C$$

$$15. \int \sin^2 a\theta d\theta = \frac{1}{2} \int (1 + \sin(2a\theta)) = \frac{\theta}{2} - \frac{\sin(2a\theta)}{4a} + C$$

$$16. \int e^{3x} dx = \frac{1}{3}e^{3x} + C$$

$$17. \int \frac{dx}{1+x} = \ln|1+x| + C$$

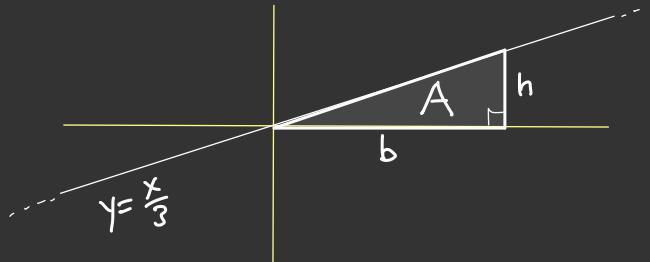
$$18. \int \frac{dx}{1-x} = -\ln|1-x| + C$$

$$\begin{cases} (\cos(x))^2 = \frac{1}{2}(\cos(2x) + 1) \\ (\sin(x))^2 = \frac{1}{2}(1 - \cos(2x)) \end{cases}$$

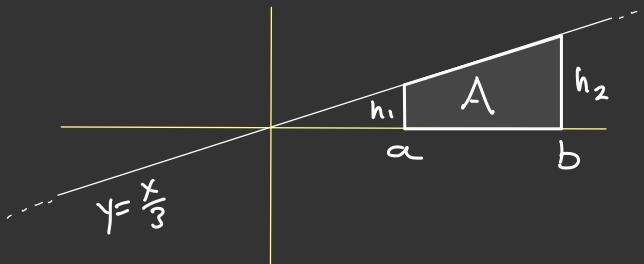
# Area Integration Practice

Calculus Made Easy, Ch. 19

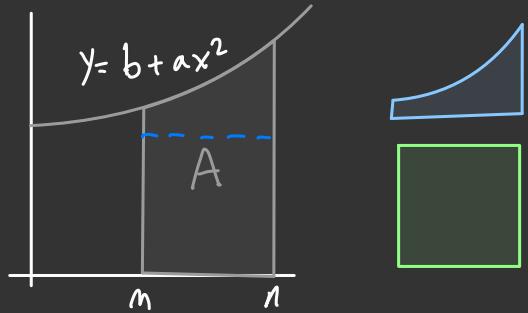
$$\begin{aligned}
 A &= \int_0^b y dx = \int_0^b \frac{x}{3} dx = \frac{1}{3} \int_0^b x dx \\
 &= \frac{1}{3} \left[ \frac{1}{2} x^2 + C \right]_0^b = \frac{1}{6} \cdot b^2 = \frac{1}{2} \cdot b \left( \frac{b}{3} \right) \\
 &= \frac{1}{2} b \cdot h
 \end{aligned}$$



$$\begin{aligned}
 A &= \int_a^b y dx = \int_a^b \frac{x}{3} dx = \frac{1}{3} \left[ \frac{1}{2} x^2 + C \right]_a^b \\
 &= \frac{1}{3} \left( \frac{1}{2} b^2 - \frac{1}{2} a^2 \right) = \frac{1}{2} \left( b \cdot \frac{b}{3} - a \cdot \frac{a}{3} \right) \\
 &= \frac{1}{2} (b \cdot h_2 - a \cdot h_1)
 \end{aligned}$$

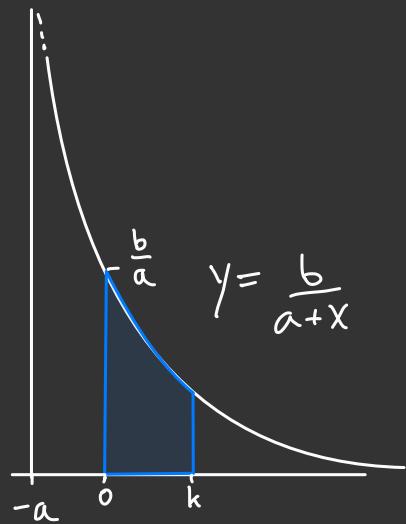


$$\begin{aligned}
 A &= \int_m^n y dx = \int_m^n (b + ax^2) dx \\
 &= b \int_m^n dx + a \int_m^n x^2 dx \\
 &= b [x + C]_m^n + a \left[ \frac{1}{3} x^3 + C \right]_m^n \\
 &= b(n-m) + a \left( \frac{1}{3} n^3 - \frac{1}{3} m^3 \right)
 \end{aligned}$$



$$A = \int_0^k \frac{b}{a+x} dx = b \left[ \ln(x+a) + C \right]_0^k$$

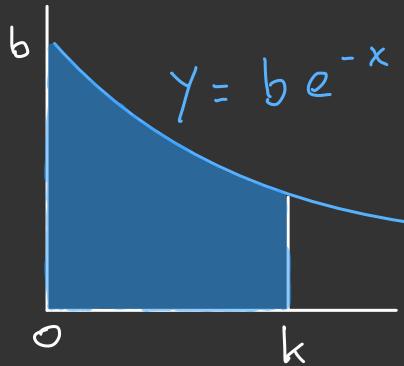
$$= b \ln \left( \frac{k+a}{a} \right)$$



$$A = \int_0^k b e^{-x} dx$$

$$= b \left[ -e^{-x} + C \right]_0^k$$

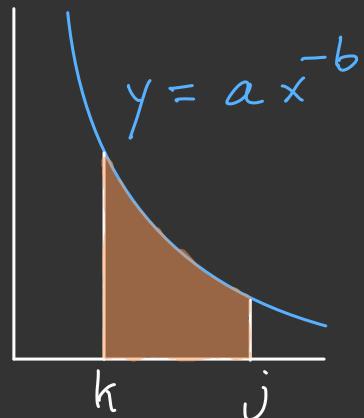
$$= b (1 - e^{-k})$$



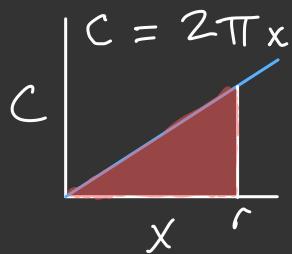
$$A = \int_k^j a x^{-b} dx$$

$$= a \left[ \left( \frac{1}{-b+1} \right) \cdot x^{-b+1} + C \right]_k^j$$

$$= \left( \frac{a}{-b+1} \right) \left[ j^{(-b+1)} - k^{(-b+1)} \right]$$

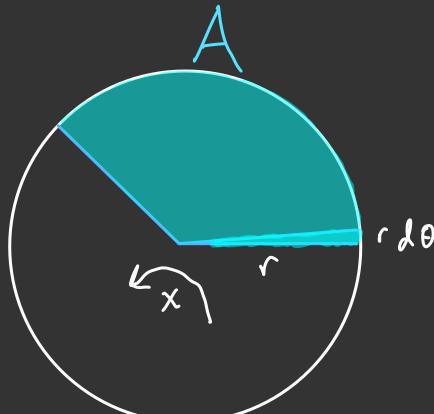


$$\begin{aligned}
 A &= \int_0^r 2\pi x \, dx \\
 &= 2\pi \left[ \frac{1}{2}x^2 + C \right]_0^r \\
 &= \pi r^2
 \end{aligned}$$



$$\begin{aligned}
 A &= \int_0^x \frac{r \cdot d\theta}{2} \\
 &= \frac{r^2}{2} [\theta + C]_0^x \\
 &= \frac{\pi x^2}{2}
 \end{aligned}$$

If  $x = 2\pi$ ,  $A = \frac{2\pi \cdot r^2}{2} = \pi r^2$

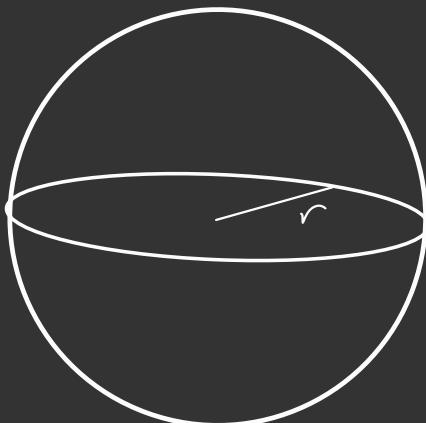


The surface area of a sphere with radius  $r$  is given by

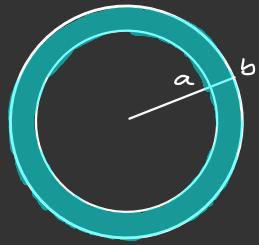
$$A = 4\pi r^2$$

so the volume is

$$\begin{aligned}
 V &= \int_0^a 4\pi r^2 dr = \\
 &= \left[ \frac{4}{3}\pi r^3 + C \right]_0^a \\
 &= \frac{4}{3}\pi a^3
 \end{aligned}$$



$$\begin{aligned}
 A &= \int_a^b 2\pi r \, dr \\
 &= [\pi r^2 + C]_a^b \\
 &= \pi b^2 - \pi a^2
 \end{aligned}$$

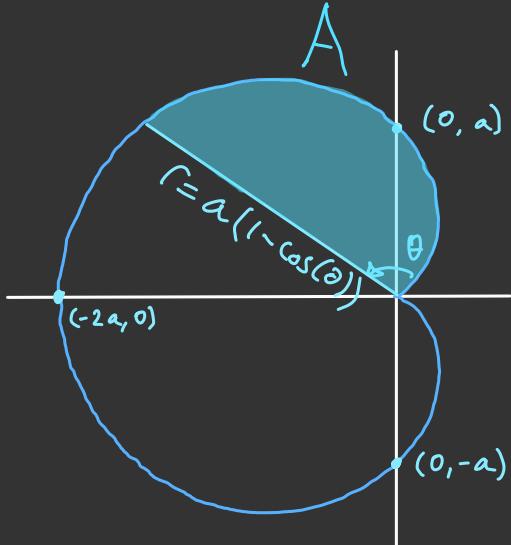


$$y = x - x^2 = -((x - \frac{1}{2})^2 - \frac{1}{4})$$

$$\begin{aligned}
 A &= \int_0^t (x - x^2) dx = [\frac{1}{2}x^2 - \frac{1}{3}x^3 + C]_0^t \\
 &= \frac{1}{2}t^2 - \frac{1}{3}t^3
 \end{aligned}$$



# Area of a <sup>half of</sup> Cartioid



A cardioid (Greek for "heart") is defined by the equation

$$r(\theta) = a(1 - \cos \theta)$$

for angle  $\theta$  & distance  $r$ .



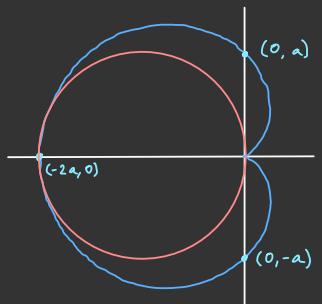
$$0 \leq \theta \leq \pi \text{ (half the heart)}$$

$$\begin{aligned} A &= \int_0^\theta \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_0^\theta a^2 (1 - \cos(\theta))^2 d\theta = \frac{a^2}{2} \int_0^\theta (1 - \cos \theta)(1 - \cos \theta) d\theta \\ &= \frac{a^2}{2} \int_0^\theta (1 - 2\cos \theta + \cos^2 \theta) d\theta = \frac{a^2}{2} \left( \int_0^\theta 1 d\theta - 2 \int_0^\theta \cos \theta d\theta + \int_0^\theta \cos^2 \theta d\theta \right) \\ &= \frac{a^2}{2} \left( \left[ x + c \right]_0^\theta - 2 \left[ \sin x + c \right]_0^\theta + \left[ \frac{1}{2} [x + \sin x \cos x] \right]_0^\theta \right) = \frac{a^2}{2} \left( \theta - 2 \sin \theta + \frac{1}{2} (\theta + \sin \theta \cos \theta) \right) \\ &= \frac{a^2}{2} \left( \frac{3}{2} \theta - 2 \sin \theta + \frac{\sin \theta \cos \theta}{2} \right) \end{aligned}$$

$$\text{if } \theta = \frac{\pi}{2}, A = \frac{a^2}{2} \left( \frac{3\pi}{4} - 2 \right) = a^2 \left( \frac{3\pi}{8} - 1 \right)$$

$$\text{if } \theta = \pi, A = a^2 \cdot (3\pi/4)$$

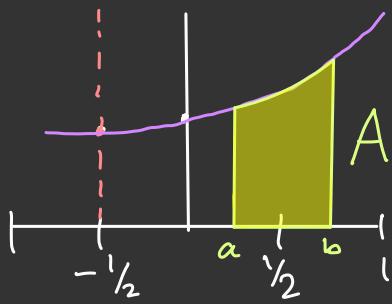
 The area of the whole cardioid is thus  $A = \frac{3}{2}\pi a^2$ , or 1.5 times the area of a circle with radius  $a$ .



# More Fun with Area Integration

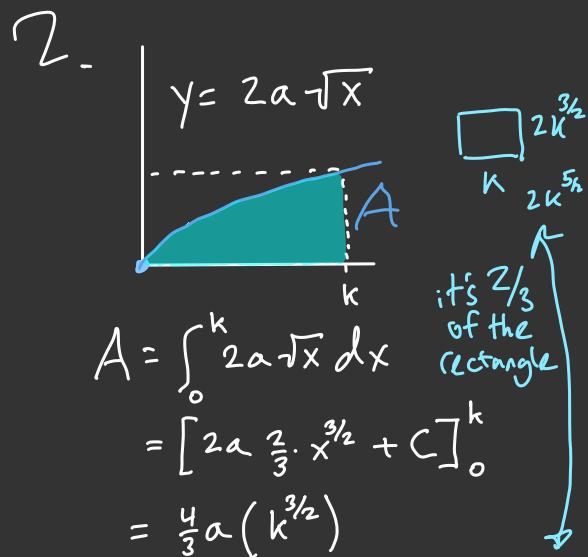
Calc. Made Easy, Exercises 18, p. 225

$$1. \quad y = x^2 + x + 5 \\ = (x + \frac{1}{2})^2 + 4\frac{3}{4}$$



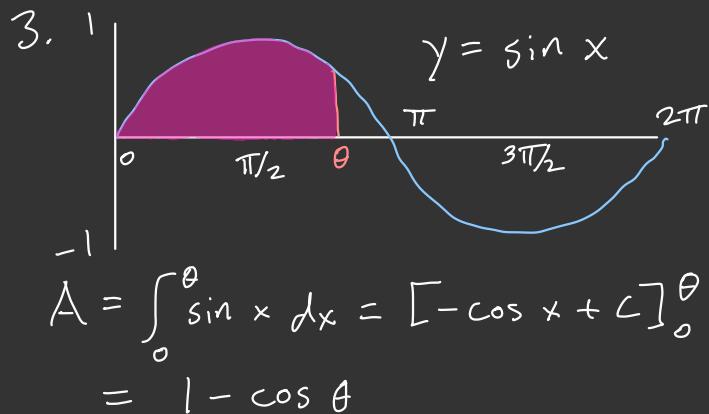
$$A = \int_a^b (x^2 + x + 5) dx \\ = \left[ \frac{1}{3}x^3 + \frac{1}{2}x^2 + 5x + C \right]_a^b \\ = \frac{1}{3}b^3 + \frac{1}{2}b^2 + 5b + C \\ - \frac{1}{3}a^3 - \frac{1}{2}a^2 - 5a - C$$

$$\text{If } a=0, b=6, \quad A = \frac{1}{3}6^3 + \frac{1}{2}6^2 + 6 \cdot 5 \\ = 6 \cdot 6 \cdot 2 + 6 \cdot 3 + 6 \cdot 5 \\ = 6(6 \cdot 2 + 3 + 5) \\ = 6(12 + 3 + 5) \\ = 120$$

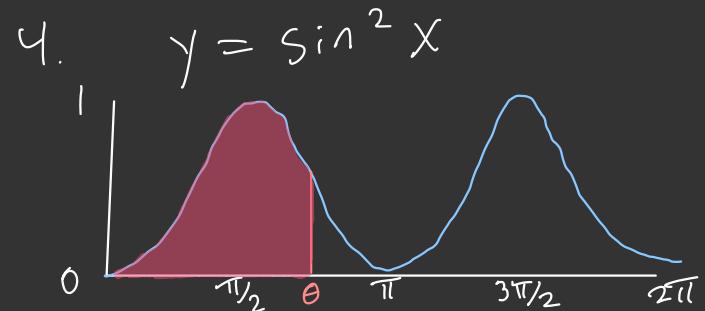


$$A = \int_0^k 2x - \sqrt{x} dx \\ = \left[ 2x \cdot \frac{2}{3}x^{3/2} + C \right]_0^k \\ = \frac{4}{3}k \left( k^{3/2} \right)$$

$$\text{if } k=a, \quad A = \frac{4}{3}a^{5/2} = \frac{2}{3}2k^{5/2}$$

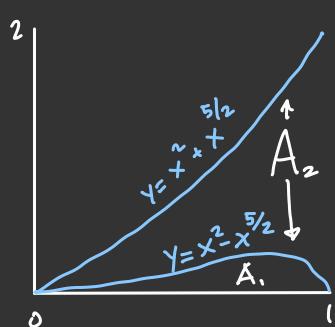


$$A = \int_0^\theta \sin x dx = [-\cos x + C]_0^\theta \\ = 1 - \cos \theta$$



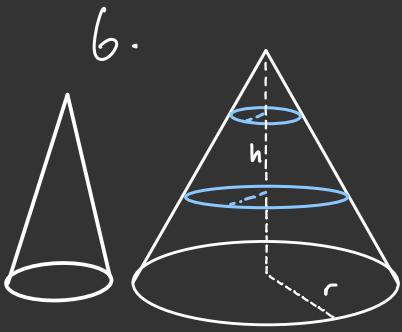
$$A = \int_0^\theta \sin^2 x dx = \left[ \frac{x}{2} - \frac{\sin 2x}{4} + C \right]_0^\theta \\ = \frac{\theta}{2} - \frac{\sin 2\theta}{4}$$

5.

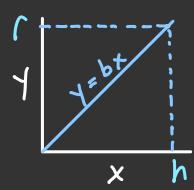


$$A_1 = \int_0^1 (x^2 - x^{5/2}) dx \\ = \left[ \frac{1}{3}x^3 - \frac{2}{7}x^{7/2} + C \right]_0^k \\ = \frac{1}{3}k^3 - \frac{2}{7}k^{7/2} \\ \text{if } k=1, \quad A_1 = \frac{1}{21}, \quad A_2 = \frac{4}{7}$$

$$A_2 = \int_0^k (x^2 + x^{5/2}) dx - A_1 \\ = \left[ \frac{1}{3}x^3 + \frac{2}{7}x^{7/2} \right]_0^k - A_1 \\ = \frac{1}{3}k^3 + \frac{2}{7}k^{7/2} - \frac{1}{3}k^3 + \frac{2}{7}k^{7/2} \\ = \frac{4}{7}k^{7/2}$$



$$V = \int_0^r \pi y^2 dx = b^2 \int_0^k \pi x^2 dx \\ = b^2 \pi \left[ \frac{1}{3} x^3 \right]_0^k = \frac{b^2 \pi k^3}{3} = \frac{r^2 \pi h}{3}$$



7.

$$y = x^3 - \ln x$$

$$A = \int_0^k (x^3 - \ln x) dx \\ = \left[ \frac{1}{4} x^4 - x(\ln x - 1) + C \right]_0^k \\ = \frac{1}{4} k^4 - k(\ln k - 1) \\ \text{if } k=1, A = \frac{5}{4}$$

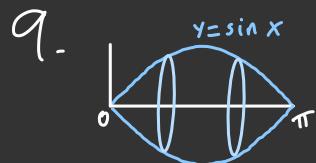
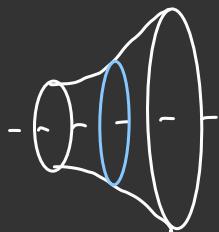
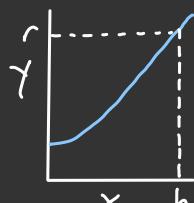
8.

$$y = \sqrt{1+x^2}$$

$$V = \int_0^k \pi y^2 dx \\ = \int_0^h \pi (1+x^2) dx \\ = \pi \left[ x + \frac{1}{3} x^3 + C \right]_0^h$$

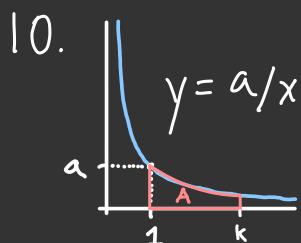
$$= \pi \left( h + \frac{1}{3} h^3 \right) = \frac{h}{3} \pi (2+r^2) = \frac{2\pi h}{3} + \frac{\pi r^2 h}{3}$$

If  $h=4$ ,  $V = 4\pi + \frac{64}{3}\pi = \frac{76}{3}\pi$



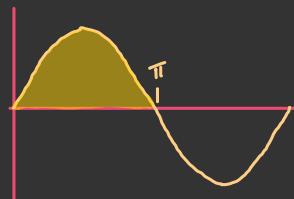
$$V = \int_0^\theta \pi \sin^2 x dx \\ = \pi \left[ \frac{x}{2} - \frac{\sin 2x}{4} + C \right]_0^\theta \\ = \frac{\pi}{2} \left( \theta - \frac{\sin 2\theta}{2} \right) \\ = \frac{\pi}{2} (\theta - \sin \theta \cos \theta)$$

If  $\theta=\pi$ ,  $V=\pi^2/2$ .



$$A = \int_1^k a/x dx \\ = a \left[ \ln x + C \right]_1^k \\ = a \ln k$$

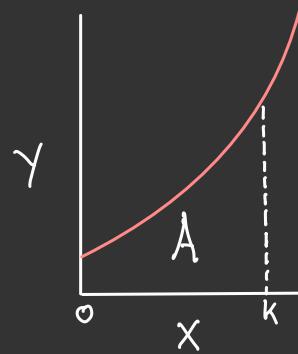
II)  $\int_0^\pi \sin x dx = \left[ -\cos x \right]_0^\pi = -(\cos(\pi) - \cos(0)) = 2$



$$12. \quad y = x^2 + 3x + 2 = (x + \frac{3}{2})^2 - \frac{1}{4}$$

$$\begin{aligned} A &= \int_0^k (x^2 + 3x + 2) dx \\ &= \left[ \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x \right]_0^k \\ &= \frac{1}{3}k^3 + \frac{3}{2}k^2 + 2k \end{aligned}$$

$$\bar{y} = \frac{1}{3}k^2 + \frac{3}{2}k + 2$$



$$\bar{y} = \sqrt{\frac{1}{k} \int_0^k y^2 dx} ?$$

$$\begin{aligned} y^2 &= (x^2 + 3x + 2)^2 \\ &= x^4 + 3x^3 + 2x^2 + 3x^3 + 9x^2 + 6x + 2x^2 + 6x + 4 \\ &= x^4 + 6x^3 + 11x^2 + 14x + 4 \end{aligned}$$

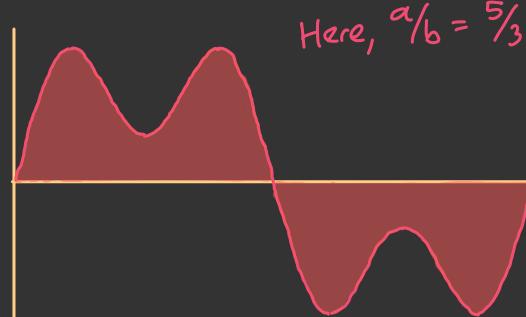
$$\frac{1}{k} \int_0^k y^2 dx = k^{-1} \left[ \frac{1}{5}x^5 + \frac{6}{4}x^4 + \frac{11}{3}x^3 + \frac{14}{2}x^2 + 4x \right]_0^k$$

$$\text{so } \bar{y} = \left( \frac{1}{5}k^5 + \frac{3}{2}k^4 + \frac{11}{3}k^3 + 7k + 4 \right)^{1/2}$$

$$13. \quad y = a \sin x + b \sin 3x$$

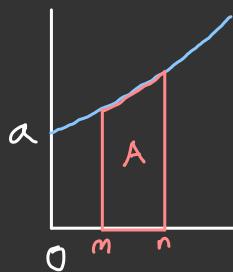
$$\begin{aligned} A &= a \int_0^{2\pi} \sin x dx + b \int_0^{2\pi} \sin 3x dx \\ &= a \left[ -\cos x \right]_0^{2\pi} + b \left[ -\frac{\cos 3x}{3} \right]_0^{2\pi} \\ &= -a[1-1] - \frac{b}{3}[1-1] = 0 \end{aligned}$$

also true for any integers multiplied by  $x!$



$$14. \quad y = a e^{bx}$$

$$\begin{aligned} A &= \int_m^n a e^{bx} dx \\ &= a \left[ \frac{1}{b} e^{bx} + C \right]_m^n \\ &= a \left( \frac{1}{b} e^{bn} - \frac{1}{b} e^{bm} \right) \\ &= \frac{a}{b} e^{bn} - \frac{a}{b} e^{bm} \end{aligned}$$



$$a = 3.42 \quad b = 0.21$$

$$m = 2 \quad n = 8$$

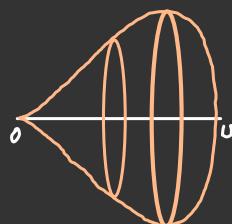
$$\bar{y} \approx 10.43$$

$$A \approx 62.60$$

15. See "Area of a Cartioid"

$$16. \quad y = \pm a \times \sqrt{x(u-x)}$$

$$\begin{aligned} V &= \int_0^k \pi a^2 x^2 (x(u-x)) dx \\ &= a^2 \pi \int_0^k x^3 u - x^4 dx \\ &= a^2 \pi \left[ \frac{1}{4} x^4 u - \frac{1}{5} x^5 \right]_0^k \\ &= a^2 \pi \left( \frac{1}{4} k^4 u - \frac{1}{5} k^5 \right) \\ &= a^2 \pi k^4 \left( \frac{1}{4} u - \frac{1}{5} k \right) \end{aligned}$$



$$\text{When } k=u, \quad V = \frac{a^2 \pi}{20} u^5$$

# INTEGRATION BY PARTS

CALC MADE EASY, CHAPTER 20, P. 227

$$1) \quad y = \int w \sin w \, dw$$

$$= \int f g' = f g - \int f' g + C$$

$$\begin{aligned} f &= w & g' &= \sin w \\ f' &= 1 & g &= -\cos w \end{aligned}$$

$$= w(-\cos w + C) - \int -\cos w \, dw + C$$

$$= -w \cos w + \sin w + C$$

$$2) \quad \int x e^x \, dx$$

$$\begin{aligned} f &= x & g' &= e^x \\ f' &= 1 & g &= e^x \end{aligned}$$

$$= x e^x - \int 1 \cdot e^x \, dx$$

$$= x e^x - e^x = e^x(x-1) + C$$

$$\begin{aligned} f &= x^2 & g' &= \sin x \\ f' &= 2x & g &= -\cos x \end{aligned}$$

$$4) \quad \int x^2 \sin x \, dx$$

$$= -x^2 \cos x + 2 \int x \cos x \, dx \xrightarrow{\text{REPEAT!}}$$

$$\begin{aligned} f &= x & g' &= \cos x \\ f' &= 1 & g &= \sin x \end{aligned}$$

$$= -x^2 \cos x + 2 \left( x \sin x - \int 1 \cdot \sin x \, dx \right)$$

$$= -x^2 \cos x + 2x \sin x - 2 \cdot (-\cos x) + C$$

$$= -x^2 \cos x + 2(x \sin x + \cos x) + C$$

$$5) \quad \int \sqrt{1-x^2} \, dx = \int \sqrt{1-x} \sqrt{1+x} \, dx$$

$$\begin{aligned} f &= \sqrt{1+x} & g' &= \sqrt{1-x} \\ f' &= \frac{1}{2\sqrt{1+x}} & g &= \frac{-1}{2\sqrt{1-x}} \end{aligned}$$

$$- \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \, dx$$

$$- \frac{\sqrt{1+x}}{2\sqrt{1-x}} - \frac{1}{2} \sin^{-1}(x) + C$$

INTEGRATION BY PARTS GOES BACKWARDS FROM THE PRODUCT RULE:

$$\frac{d(fg)}{dx} = f \frac{dg}{dx} + \frac{df}{dx} \cdot g$$

$$fg = \int f \frac{dg}{dx} \cdot dx + \int \frac{df}{dx} g \, dx$$

$$\int f \frac{dg}{dx} \cdot dx = fg - \int \frac{df}{dx} g \, dx$$

SO, FOR AN INTEGRATION OF A PRODUCT, WE PICK ONE TERM TO BE  $f$  AND ONE TO BE  $\frac{dg}{dx}$

# INTEGRATION BY SUBSTITUTION

CALC MADE EASY, CHAPTER 20, P. 230

$$1) \int \sqrt{3+x} dx \quad \begin{matrix} u = 3+x \\ \frac{du}{dx} = 1 \end{matrix}$$

$$= \int \sqrt{u} du$$

$$= \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \frac{2}{3} (3+x)^{\frac{3}{2}} + C$$

$$2) \int \frac{dx}{e^x + e^{-x}} \quad \begin{matrix} u = e^x \\ \frac{du}{dx} = e^x \end{matrix}$$

$$= \int \frac{du}{u \cdot (u+1)} = \int \frac{du}{u^2+1} = \tan^{-1} u + C$$

$$= \tan^{-1} e^x + C$$

$$3) \int \frac{dx}{x^2+2x+3} = \int \frac{dx}{(x+1)^2+(\sqrt{2})^2} \quad \begin{matrix} u = x+1 \\ \frac{du}{dx} = 1 \end{matrix}$$

$$= \int \frac{du}{u^2+\sqrt{2}^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{x+1}{\sqrt{2}} \right) + C$$

Knowing that:

$$\left( \begin{array}{l} y = \frac{1}{\alpha} \tan^{-1} \left( \frac{x}{\alpha} \right) \\ \frac{dy}{dx} = \frac{1}{x^2+\alpha^2} \end{array} \right)$$

# PARTIAL FRACTIONS IN INTEGRATION

CALC MADE EASY, CHAPTER 20, p. 231

$$\int \frac{dx}{x^2+2x-3} = \frac{1}{4} \int \left( \frac{1}{x-1} - \frac{1}{x+3} \right) dx = \frac{1}{4} (\ln(x-1) - \ln(x+3)) + C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \int \left( \frac{1}{a-x} + \frac{1}{a+x} \right) dx = \frac{1}{2a} (\ln(a+x) - \ln(a-x)) + C$$

see Boole's Differential Equations

Bonus problem from Reddit:

$$\begin{aligned} \int \frac{5x^2}{x^2+4x+3} dx &= \int \frac{5x^2}{(x+3)(x+1)} dx = \int \frac{(-45/2)}{x+3} dx + \int \frac{(5/2)}{x+1} dx + \int 5 dx \\ &= -\frac{45}{2} \log(x+3) + \left(\frac{5}{2}\right) \log(x+1) + 5x + C \end{aligned}$$

# Practice With Differential Equations

CALC MADE EASY, EX. 20, p. 248

$$1) \frac{dT}{d\theta} = \mu T$$

$$\int \frac{dT}{T} = \mu \int d\theta$$

$$\ln T = \mu \theta$$

$$T = \exp(\mu \theta) \cdot T_0$$

$$2) \frac{d^2 s}{dt^2} = a \quad \frac{ds}{dt} = v \quad \text{if } t=0, s=0$$

$$\frac{ds}{dt} = at + v$$

$$s = \frac{at^2}{2} + vt$$

$$3) \frac{di}{dt} + 2i = \sin 3t \quad i=0 \text{ when } t=0$$

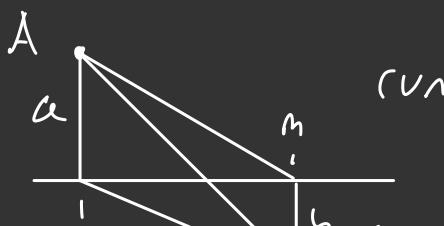
$$\frac{du}{dt} = e^{2t} \frac{di}{dt} + 2e^{2t} \cdot i = e^{2t} \sin 3t$$

$$u = e^{2t} \cdot i$$

$$u = \int e^{2t} \sin 3t \, dt = \frac{\exp(2t)}{2^2 + 3^2} (2 \sin 3t - 3 \cos 3t) + C$$

$$i = \frac{1}{13} \left( 2 \sin 3t - 3 \cos 3t \right) + \frac{3}{13}$$

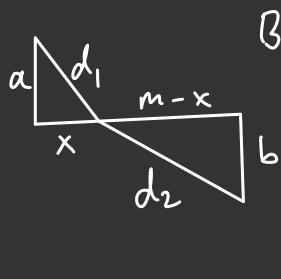
# Path Minimization



$$r = \alpha \cdot s$$

$$\alpha > 1$$

$$P = \frac{t_1}{t_1 + t_2}$$



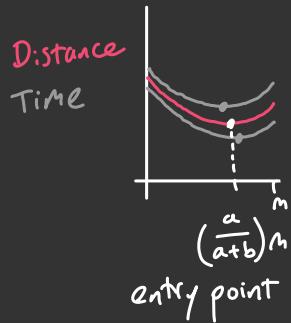
$$D = \sqrt{d_1^2 + d_2^2} = \sqrt{a^2 + x^2} + \sqrt{b^2 + (m-x)^2}$$

$$\hat{x} = \frac{a}{a+b} m$$

$$T = t_1 \cdot r + t_2 \cdot s$$

$$= t_1 \alpha s + t_2 s$$

$$T = s (t_1 \alpha + t_2)$$



Minimun has no analytic solution?

$$d_1 = r \cdot t_1 = \alpha s t_1 = \sqrt{a^2 + x^2}$$

$$\text{so } t_1 = \frac{\sqrt{a^2 + x^2}}{\alpha s}$$

$$d_2 = s t_2 = \sqrt{b^2 + (m-x)^2}$$

$$\text{so } t_2 = \frac{\sqrt{b^2 + (m-x)^2}}{s}$$

$$T = s \left( \frac{\sqrt{a^2 + x^2}}{\alpha s} + \frac{\sqrt{b^2 + (m-x)^2}}{s} \right)$$

$$= \alpha^{-1} \sqrt{a^2 + x^2} + \sqrt{b^2 + (m-x)^2}$$

# SHORTEST PATHS IN A RACE FROM CALC MADE EASY PAGE 300

WHEN RACING AROUND A SHARP CORNER, WE HAVE THE OPTION TO CUT ACROSS ROUGH TERRAIN, AT A REDUCED SPEED. THE CORNER IS  $1$  UNIT AWAY, THE DESTINATION IS  $b$  UNITS FROM THE CORNER. IF WE TAKE A SHORTCUT  $x$  UNITS FROM THE CORNER, WE ONLY HAVE  $h$  UNITS LEFT, BUT AT A FRACTION  $\gamma$  OF OUR NORMAL SPEED. THE TOTAL TIME IS THUS

$$t = \underbrace{(1-x)}_{\text{TIME BEFORE WE CUT}} + \underbrace{\frac{h}{\gamma}}_{\text{SHORTCUT TIME}}$$

DIFFERENTIATING,

$$\frac{dt}{dx} = \frac{x}{\gamma(x^2+b^2)^{1/2}} - 1$$

WHICH IS MINIMIZED AT

$$\hat{x} = b \sqrt{\frac{\gamma^2}{1-\gamma^2}}$$

THE BEST SHORTCUT IS THUS A FUNCTION OF  $b \leq \gamma$ . FOR ANY SPECIFIC  $b$ , WE CAN ADJUST  $\gamma$  TO MAKE  $\hat{x} \rightarrow 0$  AS  $\gamma \rightarrow 0$ , SO NEVER CUT, TO  $\hat{x}=1$  IF  $\gamma \geq (1+b^2)^{-1/2}$ , SO IMMEDIATELY CUT.

IN CALC MADE EASY,  $b=3/4$ ,  $\gamma=3/4$ , SO  $\hat{x}=0.85$ , ABOUT 15% ALONG (MARKED IN FIG. 3)

