Calculus Made Easy Problem Sets

v0.6, 19 Jan. 2026

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Calc Made Easy

1998 Edition

✓ = verified
✓ = done but not verified

Simplest Derivatives

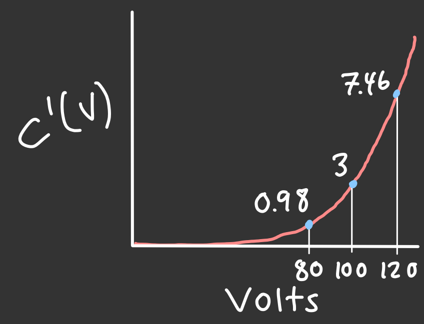
Calc Made Easy, Exercises 1

1) $y = x^{13}$; $y' = 13x^{12}$
2) $y = x^{-3/2}$; $y' = -\frac{3}{2} x^{-5/2}$
3) $y = x^{2a}$; $y' = 2a\,x^{(2a-1)}$
4) $u = t^{2.4}$; $u' = 2.4t^{1.4}$
5) $z = \sqrt[3]{u}$; $z' = \frac{1}{3} u^{-2/3}$
6) $y = \sqrt[3]{x^{-5}}$; $y' = -\frac{5}{3} x^{-8/3}$
7) $u = x^{-8/5}$; $u' = -\frac{8}{5} x^{-13/5}$
8) $y = 2x^{a}$; $y' = 2a\,x^{(a-1)}$
9) $y = x^{3/q}$; $y' = \frac{3}{q} x^{\frac{3-q}{q}}$
10) $y = x^{-m/n}$; $y' = -\frac{m}{n} x^{-(m+n)/n}$

Calc Made Easy, Exercises 2, p. 64

1) $y = ax^3 + 6$; $y' = 3ax^2$
2) $y = 13x^{3/2} - c$; $y' = \frac{39}{2} x^{1/2}$
3) $y = 12x^{1/2} + c^{1/2}$; $y' = 6x^{-1/2}$
4) $y = c^{1/2} x^{1/2}$; $y' = \frac{c^{1/2}}{2} x^{-1/2}$
5) $u = \frac{az^n - 1}{c}$; $u' = \frac{na}{c} z^{n-1}$
6) $y = 1.18t^2 + 22.4$; $y' = 2.36t$
7) $l(t) = a(1 + bt)$
$l'(t) = ab$, where $b = 0.000012$

8) $c(V) = aV^b$; $c'(V) = abV^{b-1}$



$a = 0.5 \times 10^{-10}$
$b = 6$

9) $n = (DL)^{-1}\sqrt{\frac{gT}{\pi\sigma}}$

$$\frac{\partial n}{\partial D} = -D^{-2}L^{-1}\sqrt{\frac{gT}{\pi\sigma}}$$

$$\frac{\partial n}{\partial L} = -D^{-1}L^{-2}\sqrt{\frac{gT}{\pi\sigma}}$$

$$\frac{\partial n}{\partial T} = (2DL)^{-1}\sqrt{\frac{g}{\pi\sigma}}\;T^{-1/2}$$

$$\frac{\partial n}{\partial \sigma} = -(2DL)^{-1}\sqrt{\frac{gT}{\pi}}\;\sigma^{-3/2}$$

n - frequency
D - diameter
L - length
σ - specific gravity
T - force

10) $P = \left(\frac{2E}{1-\sigma^2}\right)\frac{t^3}{D^3}$

$$dP/dt = 3\left(\frac{2E}{1-\sigma^2}\right)\frac{t^2}{D^3}$$

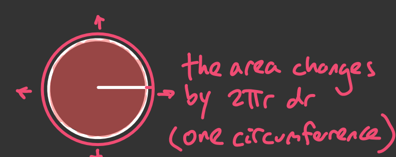
$4t \ll D$

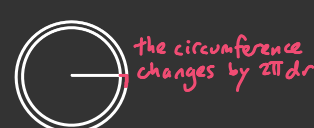
$$dP/dD = -3\left(\frac{2E}{1-\sigma^2}\right)\frac{t^3}{D^4}$$

This implies $dt/dD = -D/t$.

11) circle: $C = 2\pi r$; $C' = 2\pi$

$A = \pi r^2$; $A' = 2\pi r$





Differentiation Practice

from Calc Made Easy, Ch. 6, p. 71

0) $y = \dfrac{bx^5 + c}{x^2 + a}$

$$y' = \frac{5bx^4(x^2+a) - (bx^5+c)2x}{(x^2+a)^2}$$

$$= \frac{5bx^6 + 5abx^4 - 2bx^6 - 2cx}{(x^2+a)^2}$$

$$= \frac{3bx^6 + 5abx^4 - 2cx}{(x^2+a)^2}$$

1) $y = \dfrac{a}{b^2}x^3 - \dfrac{a^2}{b}x + \dfrac{a^2}{b^2}$

$$y' = \frac{3a}{b^2}x^2 - \frac{2a}{b}$$

$$y'' = \frac{6a}{b^2}x \qquad y''' = \frac{6a}{b^2}$$

2) $y = 2a\sqrt{bx^3} - \dfrac{3b\sqrt[3]{a}}{x} - 2\sqrt{ab}$

$$y' = 3a\sqrt{bx} + \frac{3b\sqrt[3]{a}}{x^2}$$

$$y'' = \frac{3}{2}a\sqrt{\frac{b}{x}} - \frac{6b\sqrt[3]{a}}{x^3}$$

$$y''' = -\frac{3}{4}a\frac{\sqrt{b}}{\sqrt{x^3}} + \frac{18b\sqrt[3]{a}}{x^4}$$

3) $z = a\theta^{-2/3} - b\theta^{-1/5} - c$

$$z' = -\frac{2a}{3}\theta^{-5/3} + \frac{b}{5}\theta^{-6/5}$$

$$z'' = \frac{10}{9}a\theta^{-8/3} - \frac{6}{25}b\theta^{-11/5}$$

$$z''' = -\frac{80}{27}a\theta^{-11/3} + \frac{66}{125}b\theta^{-16/5}$$

4) $v = (at^2 - bt + c)^3$

$$v' = 3(at^2 - bt + c)^2 \cdot (2at - b)$$

$$v'' = 6(at^2 - bt + c)(2at - b)^2 + 6a(at^2 - bt + c)^2$$

5) $y = (2x-3)(x+1)^2$

$$= (2x-3)(x^2+2x+1)$$

$$y' = 2(x^2+2x+1) + (2x-3)(2x+2)$$

$$= 2[x^2 + 2x + 1 + 2x^2 + 2x - 3x - 3]$$

$$= 2(3x^2 + x - 2)$$

$$= 2(3x-2)(x+1)$$

$$y'' = 2(6x+1) \qquad y''' = 12$$

6) $y = ax^3(x-b) = ax^4 - abx^3$

$a = 0.5$

$b = 3$

$$y' = 4ax^3 - 3abx^2$$

$$y'' = 12ax^2 - 6abx = 6x^2 - 9x$$

$$y''' = 24ax - 6ab = 12x - 9$$

7) $w = \left(\theta + \dfrac{1}{\theta}\right)\left(\sqrt{\theta} + \dfrac{1}{\sqrt{\theta}}\right)$

$$= (\theta + \theta^{-1})(\theta^{1/2} + \theta^{-1/2})$$

$$= (\theta^{3/2} + \theta^{1/2} + \theta^{-1/2} + \theta^{-3/2})$$

$$w' = \tfrac{3}{2}\theta^{1/2} + \tfrac{1}{2}\theta^{-1/2} - \tfrac{1}{2}\theta^{-3/2} - \tfrac{3}{2}\theta^{-5/2}$$

$$w'' = \tfrac{3}{4}\theta^{-1/2} - \tfrac{1}{4}\theta^{-3/2} + \tfrac{3}{4}\theta^{-5/2} + \tfrac{15}{4}\theta^{-7/2}$$

$$w''' = -\tfrac{3}{8}\theta^{-3/2} + \tfrac{3}{8}\theta^{-5/2} - \tfrac{15}{8}\theta^{-7/2} - \tfrac{105}{8}\theta^{-9/2}$$

8) $y = a(1 + a\sqrt{x} + a^2x)^{-1}$

$$y' = -a(1 + a\sqrt{x} + a^2x)^{-2}\left(\tfrac{1}{2}ax^{-1/2} + a^2\right)$$

9) $y = \frac{x^2}{x^2+1}$

$$y' = 2x(x^2+1) + x^2(2x)$$
$$= 2x^3 + 2x + 2x^3$$
$$= 4x^3 + 2x = 2x(2x^2+1)$$

10) $y = \frac{a+\sqrt{x}}{a-\sqrt{x}}$

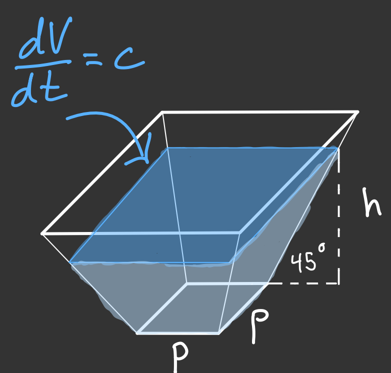
$$y' = \frac{\frac{1}{2}x^{-1/2}(a - x^{1/2}) - (a + x^{1/2})\left(-\frac{1}{2}x^{-1/2}\right)}{(a - x^{1/2})^2}$$
$$= \frac{a}{x^{1/2}(a - x^{1/2})^2}$$

11) $\theta = \frac{1 - a\sqrt[3]{t^2}}{1 + a\sqrt[2]{t^3}}$

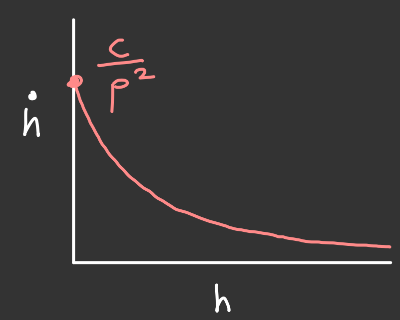
$$\dot{\theta} = \frac{-(2/3)\,a\,t^{-1/3}\left(1 + a\sqrt[3]{t^2}\right) - \left(1 - a\sqrt[2]{t^3}\right)\frac{3}{2}a\,t^{1/2}}{\left(1 + a\sqrt[2]{t^3}\right)^2}$$

12) A reservoir shaped like a 45° pyramid frustrum is being filled at a constant rate. Our goal is to express the rate at which the height changes.

$$V = \frac{h}{3}\left(A + a + \sqrt{Aa}\right)$$
$$= \frac{h}{3}\left((2h+p)^2 + p^2 + (2h+p)p\right)$$
$$4h^2 + 4hp + p^2 + p^2 + 2hp + p^2$$
$$= \frac{h}{3}\left(4h^2 + 6hp + 3p^2\right)$$
$$= \frac{4}{3}h^3 + 2h^2p + hp^2$$



$$\dot{V} = 4h^2\dot{h} + 4h\dot{h}p + p^2\dot{h}$$
$$= \dot{h}(4h^2 + 4hp + p^2) = c$$
$$= \dot{h}(2h+p)^2 = c$$
$$\dot{h} = c/(2h+p)^2$$



$$\dot{h} = \frac{c}{(2h+p)^2}$$

13) $P = \left(\frac{a+t}{b}\right)^5$; $P' = 5\frac{(a+t)^4}{b^5}$

$t = 100$, $a = 40$, $b = 140$, $P' = 1/28$

Sums, Differences, Products, Quotients

from Calc. Made Easy, Exercises 3, p. 76

1) $v = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$v' = 1 + \frac{\cancel{2}x}{\cancel{2}\cdot 1} + \frac{\cancel{3}x^2}{\cancel{3}\cdot 2\cdot 1} + \frac{\cancel{4}x^3}{\cancel{4}\cdot 3\cdot 2\cdot 1} + \ldots$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$v' = v = v'' = v'''$

(naturally, since $\sum_{i=0}^{\infty} x^i/i! = \exp(x)$)

$y = (a+x)^2$

$y' = 2(a+x)$ $\quad y'' = 2$

$y = ax^2 + bx + c$

$y' = 2ax + b$ $\quad y'' = 2a$

$y = (x+a)^3$

$y' = 3(x+a)^2$ $\quad y'' = 6x + 6a$

Amazingly, Thompson makes us do this one before teaching the chain rule...

2) $w = at - \frac{1}{2}bt^2$

$\dot{w} = a - bt$ $\quad \ddot{w} = -b$

$\dddot{w} = 0$

3) $y = (x + \sqrt{-1})(x - \sqrt{-1})$

$= x^2 - i^2$

$y' = 2x$ $\quad y'' = 2$ $\quad y''' = 0$

4) $y = (ax + bx^2)(c + dx + gx^3)$

$y'' = (a + 2bx)(c + dx + gx^3)$
$+ (ax + bx^2)(d + 3gx^2)$

$y''' = 2b(c + dx + gx^3) +$
$(a + 2bx)(d + 3gx^2) +$
$(a + 2bx)(d + 3gx^2) +$
$(ax + bx^2)6gx$

($a = 197$ $b = -34$ $c = 7$ $d = 22$ $g = -83$)

5) $x = (y+3)(y+5)$

$x' = y + 5 + y + 3$

$= 2y + 8 = 2(y+4)$

$x'' = 2$ $\quad x''' = 0$

6) $y = ax(b + cx^2)$

$y' = ab + 3acx^2$

$= a(b + 3cx^2)$

$y'' = 6acx$ $\quad y''' = 6ac$

($a = 1.3709$ $b = 112.6$ $c = 45.202$)

7) $y = \dfrac{2x+3}{3x+2}$

$$y' = \frac{2(3x+2) - (2x+3)3}{(3x+2)^2} = -5(3x+2)^{-2}$$

$$y'' = 30(3x+2)^{-3}$$

$$y''' = -270(3x+2)^{-4}$$

8) $y = \dfrac{1+x+2x^2+3x^3}{1+x+2x^2}$

$$y' = \frac{(1+4x+9x^2)(1+x+2x^2) - (1+x+2x^2+3x^3)(1+4x)}{(1+x+2x^2)^2}$$

$$= (\cancel{1} + \cancel{x} + \cancel{2x^2} + \cancel{4x} + \cancel{4x^2} + \cancel{8x^3} + 9x^2 + \overset{6}{\cancel{9}}x^3 + 1\overset{6}{\cancel{8}}x^4) -$$
$$(\cancel{1} + \cancel{x} + \cancel{2x^2} + \cancel{3x^3} + \cancel{4x} + \cancel{4x^2} + \cancel{8x^3} + \cancel{12x^4}) / (1+x+2x^2)^2$$

$$= 9x^2 + 6x^3 + 6x^4 / (1+x+2x^2)^2$$

$$= 3x^2(3+2x+2x^2) / (1+x+2x^2)^2$$

9) $y = \dfrac{ax+b}{cx+d}$

$$y' = \frac{a(cx+d) - (ax+b)c}{(cx+d)^2}$$

$$= \cancel{acx} + ad - \cancel{acx} - cb$$

$$= \frac{ad-cb}{(cx+d)^2}$$

10) $y = \dfrac{x^n + a}{x^{-n} + b}$

$$y' = \frac{nx^{n-1}(x^{-n}+b) + (x^n+a)nx^{-(n+1)}}{(x^{-n}+b)^2}$$

$$= \frac{nx^{-1} + bnx^{n-1} + nx^{-1} + anx^{-(n+1)}}{(x^{-n}+b)^2}$$

$$= \frac{n\left(2x^{-1} + bx^{n-1} + ax^{-(n+1)}\right)}{(x^{-n}+b)^2}$$

11) $C = a + bt + ct^2$

$C' = b + 2ct$

12) $R = R_0(1 + at + bt^2)$

$R' = R_0(a + 2bt)$

$R = R_0(1 + at + b\sqrt{t})$

$R' = R_0(a + \frac{1}{2}bt^{-1/2})$

$R = R_0(1 + at + bt^2)^{-1}$

$R' = -R_0(a + 2bt)(1 + at + bt^2)^{-2}$

13) $E = a(1 - b(t-c) + d(t-c)^2)$

$E' = a(2d(t-c) - b)$

14) $E = a + bl + \dfrac{c + kl}{i}$

$\partial E/\partial l = b + k/i$

$\partial E/\partial i = \dfrac{-(c+kl)}{i^2}$

SUCCESIVE DIFFERENTIATION PRACTICE

CALC MADE EASY EXERCISES 4, p. 82

1) $y = 17x + 12x^2$

$y' = 17 + 24x$

$y'' = 24$

2) $y = \dfrac{x^2+a}{x+a}$

$y' = \dfrac{2x(x+a) - (x^2+a)}{(x+a)^2}$

$= \dfrac{2x^2 + 2xa - x^2 - a}{(x+a)^2} = \dfrac{x^2 + 2xa - a}{(x+a)^2}$

$y'' = \dfrac{(2x+2a)(x+a)^2 - (x^2+2xa-a)2(x+a)}{(x+a)^4}$

$= 2(x+a)^{-3}\left((x+a)^2 - (x^2 + 2xa - a)\right)$

$= 2(x+a)^{-3}\left(\cancel{x^2} + \cancel{2xa} + a^2 - \cancel{x^2} - \cancel{2xa} + a\right)$

$= \dfrac{2a(a+1)}{(x+a)^3}$

3) $y = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3\cdot2\cdot1} + \dfrac{x^4}{4\cdot3\cdot2\cdot1}$

$y' = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3\cdot2\cdot1}$

$y'' = 1 + x + \dfrac{x^2}{2}$

Added second derivatives of Ex. 3, 1-7
& Chapter 6, 1-7

Rates of Change

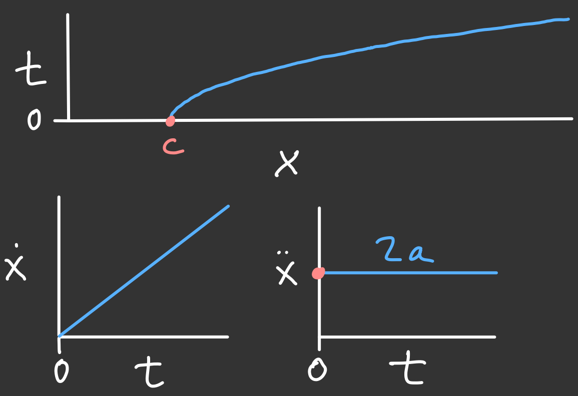
from Calc. Made Easy Ch. 8

1) Constant Acceleration, Zero Initial Velocity

$x = at^2 + c$

$\dot{x} = 2at$

$\ddot{x} = 2a$

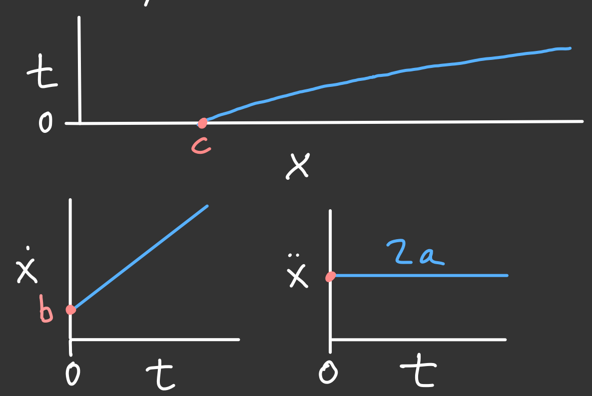


2) Constant Acceleration, Positive Initial Velocity

$x = at^2 + bt + c$

$\dot{x} = 2at + b$

$\ddot{x} = 2a$

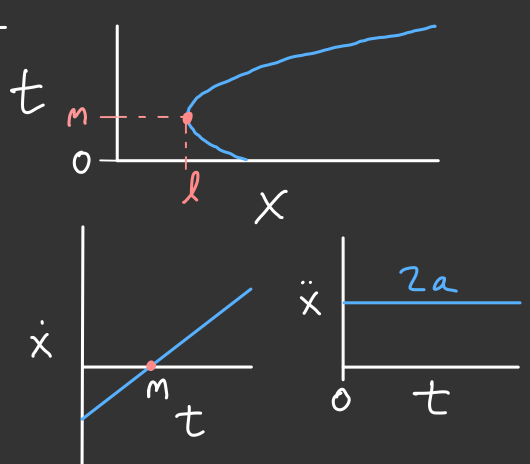


3) Constant Acceleration, Negative Initial Velocity

$x = at^2 + bt + c$

$\dot{x} = 2at + b$

$\ddot{x} = 2a$

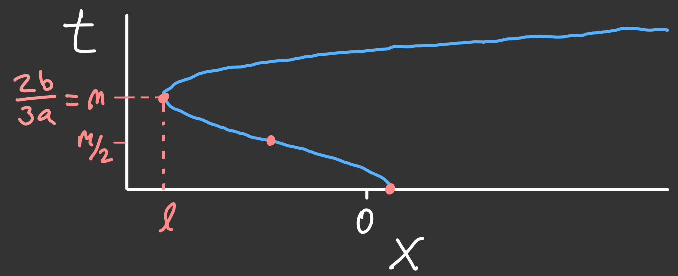


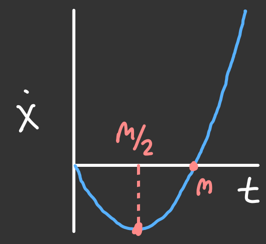
4&5) Variable Acceleration

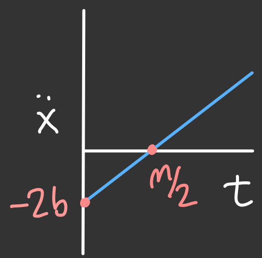
$x = at^3 - bt^2 + c$

$\dot{x} = 3at^2 - 2bt$

$\ddot{x} = 6at - 2b$





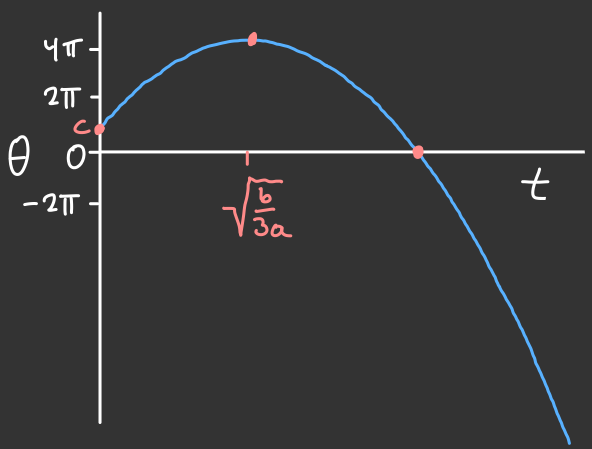


6) Rotational Acceleration

$\theta = c + bt - at^3$

$\dot{\theta} = b - 3at^2$

$\ddot{\theta} = -6at$



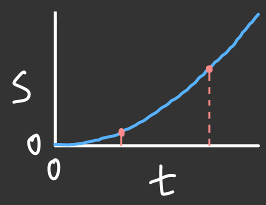
from Calc. Made Easy, Exercises 5, p. 91

1) $y = a + bt^2 + ct^4$

$\dot{y} = 2bt + 4ct^3$

$\ddot{y} = 2b + 12ct^2$

2) $s = 16t^2$



3) $x = at - \frac{1}{2}gt^2$

$\dot{x} = a - gt$

$\ddot{x} = -g$

4 & 5) $s = 12 - 4.5t + 6.2t^2$

$\dot{s} = -4.5 + 12.4t$

@ $t = 4$, $\dot{s} = 45.1$

$\ddot{s} = 12.4$

6) $\theta = 2.1 - 3.2t + 4.8t^2$

$\dot{\theta} = -3.2 + 9.6t$

$\ddot{\theta} = 9.6$

7) $s = 6.8t^3 - 10.8t$

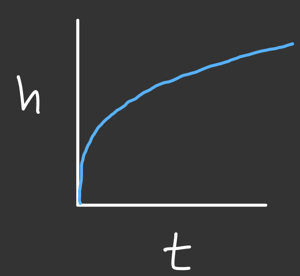
$\dot{s} = 20.4t^2 - 10.8$

$\ddot{s} = 40.8t$

8) $h = 0.5 + \frac{1}{10}\sqrt[3]{t - 125}$

$\dot{h} = \frac{1}{30}(t - 125)^{-2/3}$

$\ddot{h} = -\frac{1}{45}(t - 125)^{-5/3}$



9)

$$P = \frac{4}{4+t^2} + 0.8t - 1$$

$$= 4(4+t^2)^{-1} + 0.8t - 1$$

$$\dot{P} = -4(4+t^2)^{-2}\,2t + 0.8$$

$$= -8t(t^2+4)^{-2} + 0.8$$

$$\ddot{P} = \frac{-8(t^2+4)^2 + 8t \cdot 2(t^2+4) \cdot 2t}{(t^2+4)^4}$$

$$= \frac{8\cancel{(t^2+4)}\left(-t^2-4+4t^2\right)}{(t^2+4)^{\cancel{4}\,3}} = \frac{(3t^2-4)8}{(t^2+4)^3}$$

10) $s = t^n \quad \dot{s} = nt^{(n-1)} \quad \ddot{s} = n(n-1)t^{(n-2)}$

(i) Find n so that $\cancel{n} \cdot b^{(n-1)} = 2\cancel{n}a^{(n-1)}$ for some a & b.

$$\log 2 + (n-1)\log a = (n-1)\log b$$

$$n = 1 + \frac{\log 2}{\log(b/a)}$$

In the problem, $b = 10$ & $a = 5$, so $n = 2$.

(ii) $nd^{(n-1)} = n(n-1)d^{(n-2)}$ for some value d.

so $d = n-1 \ldots$

if $d = 10$, $n = 11$

Chain Rule Practice

from Calc Made Easy, Ch. 9 (added numbers)

1) $y = (x^2+a^2)^{3/2}$

$y' = \frac{3}{\cancel{2}}(x^2+a^2)^{1/2}\,\cancel{2}x$

2) $y = \left(m - nx^{2/3} + \frac{p}{x^{4/3}}\right)^a$

$y' = a\left(m - nx^{2/3} + p/x^{4/3}\right)^{a-1}\left(-\tfrac{2}{3}nx^{-1/3} - \tfrac{4}{3}px^{-7/3}\right)$

3) $y = \sqrt{a+x}$

$y' = \frac{1}{2}(a+x)^{-1/2}$

4) $y = (a+x^2)^{-1/2}$

$y' = -\frac{1}{\cancel{2}}(a+x^2)^{-3/2} \cdot \cancel{2}x$

5) $y = (x^3-a^2)^{-1/2}$

$y' = -\frac{1}{2}(x^3-a^2)^{-3/2} \cdot 3x^2$

6) $y = \sqrt{\frac{(1-x)}{(1+x)}} = (1-x)^{1/2}(1+x)^{-1/2}$

$dy/dx = f'g + fg' = \frac{1}{2}(1-x)^{-1/2}(-1)(1+x)^{-1/2} + (1-x)^{1/2}(-\tfrac{1}{2})(1+x)^{-3/2} \cdot 1$

$= (-\tfrac{1}{2})\left[(1-x)^{-1/2}(1+x)^{-1/2} + (1-x)^{1}(1-x)^{-1/2}(1+x)^{-1}(1+x)^{-1/2}\right]$

$= (-\tfrac{1}{2})(1-x)^{-1/2}(1+x)^{-1/2}\left[1 + (1-x)(1+x)^{-1}\right]$

$= (-\tfrac{1}{2})(1-x^2)^{-1/2}\left((1+x)(1+x)^{-1} + (1-x)(1+x)^{-1}\right)$

$= (-\cancel{\tfrac{1}{2}})(1-x^2)^{-1/2}(1+x)^{-1}\cancel{(1+x+1-x)}$

$= -(1-x^2)^{-1/2}(1+x)^{-1} = \dfrac{-1}{(1+x)\sqrt{1-x^2}}$

7) $y = \dfrac{x^{3/2}}{(1+x^2)^{1/2}} = x^{3/2}(1+x^2)^{-1/2}$

$dy/dx = (\tfrac{3}{2})x^{1/2}(1+x^2)^{-1/2} + x^{3/2}(-\tfrac{1}{\cancel{2}})(1+x^2)^{-3/2}\,\cancel{2}x$

$= (\tfrac{1}{2})(1+x^2)^{-3/2}x^{1/2}\left(3(1+x^2) - 2x^2\right)$

$= (\tfrac{1}{2})x^{1/2}(1+x^2)^{-3/2}(3+x^2)$

$= \dfrac{(3+x^2)\,x^{1/2}}{2\,(1+x^2)^{3/2}}$

8) $y = x^3 + (x^2+x+a)^{3/2}$

$dy/dx = 3x^2 + (\tfrac{3}{2})(x^2+x+a)^{1/2}(2x+1)$

$= (\tfrac{3}{2})\left(2x^2 + (2x+1)(x^2+x+a)^{1/2}\right)$

9) $y = \left(\frac{a^2+x^2}{a^2-x^2}\right)^{1/2}\left(\frac{a^2-x^2}{a^2+x^2}\right)^{1/3} = \left(\frac{a^2+x^2}{a^2-x^2}\right)^{1/6} = (a^2+x^2)^{1/6}(a^2-x^2)^{-1/6}$

$dy/dx = \frac{1}{\cancel{6}_3}(a^2+x^2)^{-5/6}\cdot\cancel{2}x\,(a^2-x^2)^{-1/6} + (a^2+x^2)^{1/6}\left(+\frac{1}{\cancel{6}_3}\right)(a^2-x^2)^{-7/6}(+\cancel{2}x)$

$= (1/3)x\,(a^2+x^2)^{1/6}(a^2-x^2)^{-1/6}\left((a^2+x^2)^{-1} + (a^2-x^2)^{-1}\right)$

$= (1/3)x\,(a^2+x^2)^{1/6}(a^2-x^2)^{-1/6}\left(\frac{a^2 \cancel{-x^2} + a^2 \cancel{+x^2}}{a^4-x^4}\right)$

$= (2/3)x\,(a^2+x^2)^{1/6}(a^2-x^2)^{-1/6}(a^4-x^4)^{-1}a^2$

$= \frac{2x}{3}\left(\frac{a^2+x^2}{a^2-x^2}\right)^{1/6}\left(\frac{a^2}{a^4-x^4}\right)$

10) For some function y^n, find its derivative with respect to y^5.

$\frac{dy^n}{dy^5}$ let $x = y^5$, so $y^n = x^{n/5}$ $\frac{dx^{n/5}}{dx} = \frac{n}{5}x^{\frac{n-5}{5}}$

$\frac{dy^n}{dy^5} = \frac{n}{5}y^{n-5}$

11) $y = \frac{x^{3/2}}{b}(a-x)^{1/2}$

$dy/dx = b^{-1}\left(3/2\,x^{1/2}(a-x)^{1/2} + x^{3/2}(1/2)(a-x)^{-1/2}(-1)\right)$

$= b^{-1}2^{-1}x^{1/2}\left(3(a-x)^{1/2} - x(a-x)^{-1/2}\right)$

$= b^{-1}2^{-1}x^{1/2}(a-x)^{-1/2}\left(3(a-x) - x\right)$

$= (2b)^{-1}(a-x)^{-1/2}(3ax^{1/2} - 4x^{3/2})$

$= \frac{x^{1/2}(3a-4x)}{2b\sqrt{a-x}}$

Agony!

$d^2y/dx^2 = (2b)^{-1}\left((+1/2)(a-x)^{-3/2}\cancel{(-1)}(3ax^{1/2} - 4x^{3/2}) + (a-x)^{-1/2}(3/2\,ax^{-1/2} - 6x^{1/2})\right)$

$= (2b)^{-1}(a-x)^{-3/2}\left((1/2)(3ax^{1/2} - 4x^{3/2}) + (a-x)(3/2\,ax^{-1/2} - 6x^{1/2})\right)$

Additional Examples with multiple steps.

1)

$$z = 3x^4 \qquad \frac{dy}{dv} = \frac{1}{2}(1+v)^{-1/2} \qquad \frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dz}\cdot\frac{dz}{dx}$$

$$v = 7z^{-2} \qquad \frac{dv}{dz} = -14z^{-3} \qquad = \frac{1}{2}(1+v)^{-1/2}\cdot(-14z^{-3})\cdot(12x^3)$$

$$y = (1+v)^{1/2}$$

$$\frac{dz}{dx} = 12x^3$$

2)

$$t = \frac{1}{5\sqrt{\theta}} \qquad dt/d\theta = -\frac{1}{10}\theta^{-3/2}$$

$$x = t^3 + \frac{t}{2} \qquad dx/dt = 3t^2 + \frac{1}{2}$$

$$v = 7x^2(x-1)^{-1/3} \qquad dv/dx = 14x(x-1)^{-1/3} + 7x^2(-1/3)(x-1)^{-4/3}$$

$$= (7/3)\,x\,(x-1)^{-4/3}(5x-6)$$

$$dv/d\theta = (7/3)\,x\,(x-1)^{-4/3}(5x-6)\,(3t^2 + 1/2)\,(-1/10\,\theta^{-3/2})$$

3)

$$\theta = 3a^2 x\, x^{-3/2} = 3a^2 x^{-1/2} \qquad d\theta/dx = -3/2\, a^2 x^{-3/2}$$

$$w = (1-\theta^2)^{1/2}(1+\theta) \qquad dw/d\theta = \cancel{\tfrac{1}{2}}(1-\theta^2)^{-1/2}(-\cancel{2}\theta)(1+\theta) + (1-\theta^2)^{1/2}$$

$$= (1-\theta^2)^{-1/2}(1-\theta-2\theta^2)$$

$$\phi = \sqrt{3} - (w\sqrt{2})^{-1} \qquad d\phi/dw = (w\sqrt{2})^{\cancel{-2}\,-1}\cdot\cancel{\sqrt{2}}$$

More Chain Rule Practice

Calc Made Easy, Exercises 6, p.100

1) $y=(x^2+1)^{1/2}$

$y'=\frac{\cancel{1}}{\cancel{2}}(x^2+1)^{-1/2}\,\cancel{2}x$

2) $y=(x^2+a^2)^{1/2}$

$y'=\frac{\cancel{1}}{\cancel{2}}(x^2+a^2)^{-1/2}\cdot\cancel{2}x$

3) $y=(a+x)^{-1/2}$

$y'=(-\tfrac{1}{2})(a+x)^{-3/2}$

$=\dfrac{-1}{2\sqrt[2]{(a+x)^3}}$

4) $y=a(a-x^2)^{-1/2}$

$y'=(\cancel{-\tfrac{1}{2}})a(a-x^2)^{-3/2}(\cancel{-2}x)$

$=ax(a-x^2)^{-3/2}$

5) $y=x^{-2}(x^2-a^2)^{1/2}$

$y'=(-2)x^{-3}(x^2-a^2)^{1/2}+x^{\cancel{-2}\,-1}(\cancel{\tfrac{1}{2}})(x^2-a^2)^{-1/2}\cdot\cancel{2x}$

$=x^{-3}(x^2-a^2)^{-1/2}\left(-2(x^2-a^2)+x^2\right)$

$=x^{-3}(x^2-a^2)^{-1/2}(2a^2-x^2)$

6) $y=\dfrac{(x^4+a)^{1/3}}{(x^3+a)^{1/2}}=(x^4+a)^{1/3}(x^3+a)^{-1/2}$

$y'=\tfrac{1}{3}(x^4+a)^{-2/3}4x^3(x^3+a)^{-1/2}+(x^4+a)^{1/3}(-\tfrac{1}{2})(x^3+a)^{-3/2}3x^2$

$=x^2(x^3+a)^{-3/2}(x^4+a)^{1/3}\left(\tfrac{4x}{3}(x^3+a)(x^4+3)^{-1}-(\tfrac{3}{2})\right)$

7) $y=\dfrac{a^2+x^2}{(a+x)^2}$

$y'=\dfrac{2x(a+x)^2-(a^2+x^2)2(a+x)}{(a+x)^4}=\dfrac{2\cancel{(a+x)}\left(x(a+x)-(a^2+x^2)\right)}{(a+x)^{\cancel{4}\,3}}$

$=2(a+x)^{-3}(ax\cancel{+x^2}-a^2\cancel{-x^2})=2a(x-a)(x+a)^{-3}$

8) $\frac{dy^5}{dy^2}$ let $y^2 = x$, so $y^5 = x^{5/2}$

$$\frac{dx^{5/2}}{dx} = \tfrac{5}{2}x^{3/2} = \tfrac{5}{2}y^3 = \frac{dy^5}{dy^2}$$

9) $y = \frac{(1-\theta^2)^{1/2}}{1-\theta}$

$$y' = \frac{\cancel{\tfrac{1}{2}}(1-\theta^2)^{-1/2}(-\cancel{2}\theta)(1-\theta) \cancel{+} (1-\theta^2)^{1/2}\cancel{(-1)}}{(1-\theta)^2}$$

$$= (1-\theta)^{-2}\left[(1-\theta^2)^{1/2} - \theta(1-\theta)(1-\theta^2)^{-1/2}\right]$$

$$= (1-\theta)^{-1}(1-\theta^2)^{-1/2}$$

Calc Made Easy, Exercises 7, p. 101

1) $u = \frac{1}{2}x^3$ $\quad du/dx = \frac{3}{2}x^2$

$v = 3(u+u^2)$ $\quad dv/du = 3 + 6u$

$w = 1/v^2$ $\quad dw/dv = -2v^{-3}$

$$dw/dx = -\cancel{2}v^{-3}(3+6u)\,3/\cancel{2}\,x^2$$

2) $y = 3x^2 + \sqrt{2}$ $\quad dy/dx = 6x$

$z = \sqrt{1+y}$ $\quad dz/dy = \frac{1}{2}(1+y)^{-1/2}$

$v = (\sqrt{3}+4z)^{-1}$ $\quad dv/dz = -(\sqrt{3}+4z)^{-2}\cdot 4$

$$dv/dx = -(\sqrt{3}+4z)^{-2}\cdot \overset{2}{\cancel{4}}\cdot \cancel{\tfrac{1}{2}}(1+y)^{-1/2}\cdot 6x$$

3) $y = x^3 3^{-1/2}$ $\quad dy/dx = 3^{1/2}x^2$

$z = (1+y)^2$ $\quad dz/dy = 2(1+y)$

$u = (1+z)^{-1/2}$ $\quad du/dz = -\frac{1}{2}(1+z)^{-3/2}$

$$du/dx = -\cancel{\tfrac{1}{2}}(1+z)^{-3/2}\,\cancel{2}(1+y)\,3^{1/2}x^2$$

4) $u = a + x$; $u' = 1$

$$y = 2a^3 \ln u - u\left(5a^2 - 2au + \tfrac{1}{3}u^2\right)$$
$$= 2a^3 \ln u - 5a^2 u + 2au^2 - \tfrac{1}{3}u^3$$
$$y' = 2a^3 u^{-1} - 5a^2 + 4au - u^2$$
$$= u^{-1}\left(2a^3 - 5a^2 u + 4au^2 - u^3\right)$$
$$= (x+a)^{-1}\left(2a^3 - 5a^2(x+a) + 4a(x+a)^2 - (x+a)^3\right)$$
$$= (x+a)^{-1}\left[2a^3 - 5a^2 x - 5a^3 + 4a\left(x^2 + 2xa + a^2\right) - (x+a)^3\right]$$
$$= (x+a)^{-1}\left[a^3 + 4ax^2 + 3a^2 x - (x+a)^3\right]$$
$$= \frac{x^2(a-x)}{x+a}$$

5) $x = a(\theta - \sin\theta)$ $\quad dx/d\theta = a(1 - \cos\theta)$

$y = a(1 - \cos\theta)$ $\quad dy/d\theta = a\sin\theta$

so $(dy/d\theta)/(dx/d\theta) = dy/dx = \dfrac{\sin\theta}{1-\cos\theta} = \cot\tfrac{1}{2}\theta$

6) $x = a\cos^3\theta$ $\quad dx/d\theta = 3a\cos^2\theta(-\sin\theta)$

$y = a\sin^3\theta$ $\quad dy/d\theta = 3a\sin^2\theta\cos\theta$

$$\frac{dy}{dx} = \frac{\cancel{3a}\sin^{\cancel{2}}\theta\,\cancel{\cos\theta}}{-\cancel{3a}\cos^{\cancel{2}}\theta\,\cancel{\sin\theta}} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

7) $y = \ln\left(\sin(x^2 - a^2)\right)$

$dy/dx = \left(\sin(x^2 - a^2)\right)^{-1} \cdot \cos(x^2 - a^2) \cdot 2x = 2x\cot(x^2 - a^2)$

8) $u = x + y \Rightarrow y = u - x$ find dy/dx

$4x = 2u - \ln(2u-1)$

$x = \frac{1}{2}u - \frac{\ln(2u-1)}{4}$

$dx/du = \frac{1}{2} - \frac{1}{\cancel{4}_{2}(2u-1)} \cdot \cancel{2}$

$= \frac{1}{2}\left(1 - \frac{1}{2u-1}\right)$

$= \frac{1}{2}\left(\frac{2u-1-1}{2u-1}\right) = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}(u-1)}{2u-1}\right)$

$dy/du = 1 - dx/du$

$= 1 - \left(\frac{u-1}{2u-1}\right)$

$= \frac{\cancel{2}u - \cancel{1} \cancel{-}u \cancel{+1}}{2u-1} = \frac{u}{2u-1}$

$dy/du = \frac{u}{2u-1}$

$\frac{dy}{dx} = \frac{u}{\cancel{2u-1}} \cdot \frac{\cancel{2u-1}}{u-1} = \frac{u}{u-1} = \frac{x+y}{x+y-1}$

Geometric Meaning of Differentiation

from Calc Made Easy, Exercises 8, p. 112

1) $y = \frac{3}{4}x^2 - 5$

$y' = \frac{3}{2}x$

2) $y = 0.12x^3 - 2$

$y' = 0.36x^2$

3) $y = (x-a)(x-b)$

$y' = 2x - (a+b)$

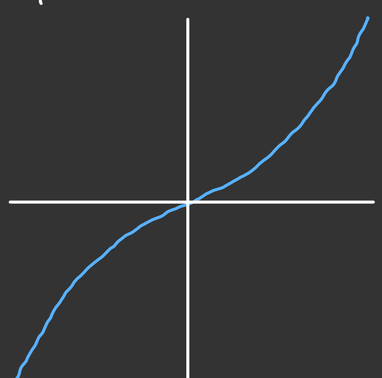
$= 0 \;@\; \hat{x} = (a+b)/2$

4) $y = x^3 + 3x = x(x^2+3)$

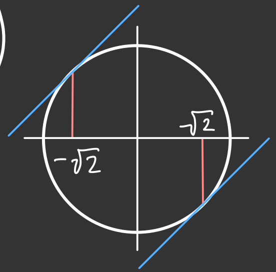
$y' = (x^2+3) + x \cdot 2x$

$= 3x^2 + 3 = 3(x^2+1)$

$y'' = 6x$



5)



$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \sqrt{4-x^2}$

$y' = \cancel{\tfrac{1}{2}}(4-x^2)^{-1/2}(-\cancel{2}x) = 1$

$-x = (4-x^2)^{1/2}$

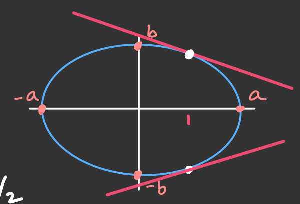
$x^2 = 4 - x^2$

$2x^2 = 4$

$x^2 = 2$

$x = \pm\sqrt{2}$

6) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

$y = \pm b\left(1 - \frac{x^2}{a^2}\right)^{1/2}$

$y' = \frac{1}{\cancel{2}} b\left(1 - \frac{x^2}{a^2}\right)^{-1/2} \cdot \frac{-\cancel{2}x}{a^2} = \pm\frac{xb}{a^2}\left(1 - \frac{x^2}{a^2}\right)^{-1/2}$

if $x = 1$, $y' = \pm\frac{b}{a}(a^2-1)^{-1/2}$

7) $y = 5 - 2x + 0.5x^3$

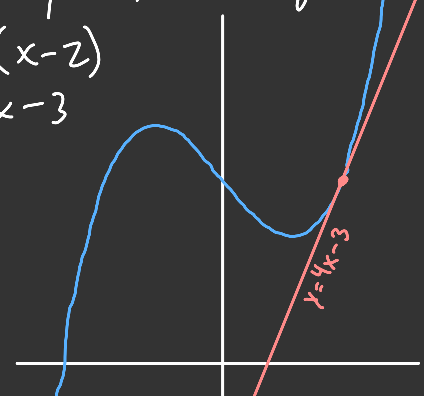
$y' = -2 + 1.5x^2$

$@\; x = 2,\; y = 5,\; y' = 4$

given this point, and slope,

$(y-5) = 4(x-2)$

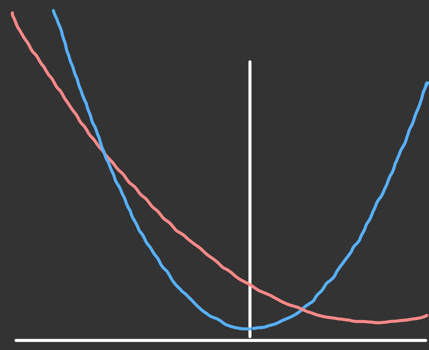
$y = 4x - 3$



8) Given two functions

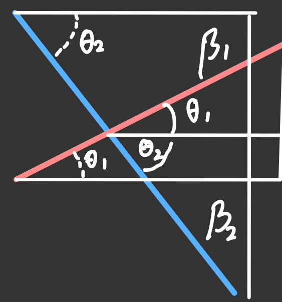
$f(x) = 3.5x^2 + 2$

$g(x) = x^2 - 5x + 9.5$



we can see they intersect at $\hat{x} = \{-3, 1\}$.

When $x = 1$, the first line has a slope of 7, and the second slope is -3, with $\hat{y} = 5.5$.



Since the arctan of each slope gives its angle on the unit circle, the diff of the arctans gives the angle between the two lines, at the point of intersection. Thus,

$$\tan^{-1}(7) - \tan^{-1}(-3) = 2.68 \text{ rad or } 153.4°$$

When $x = -3$, $\beta_1 = -21$, $\beta_2 = -11$, so

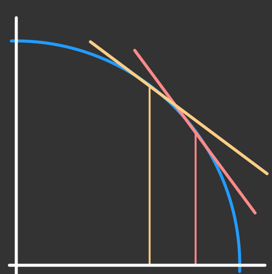
$$\tan^{-1}(-11) - \tan^{-1}(-21) \approx 0.04 \text{ radians.}$$

or $2.29°$

9)

$y = \sqrt{25 - x^2}$ $\quad y' = \cancel{\tfrac{1}{2}}(25 - x^2)^{-1/2}(-\cancel{2}x)$

@ $x = 3$, $y = 4$, $y' = -3/4$
@ $x = 4$, $y = 3$, $y' = -4/3$



so the tangents are

$y = -3/4\, x + 25/4$
$y = -4/3\, x + 25/3$

which intersect at $x = 25/7$. The angle between the tangents at this intersection is

$$\tan^{-1}(-3/4) - \tan^{-1}(-4/3) \approx 0.28 \text{ radians}$$

or $16.04°$

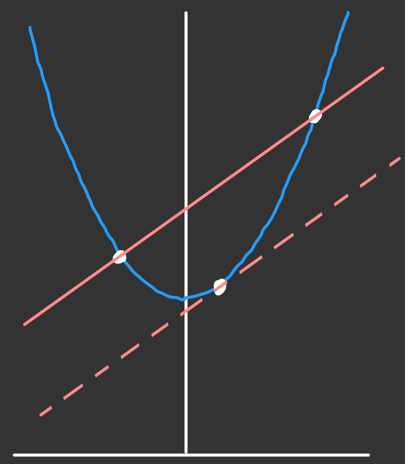
10) $y = 3x^2 + 2$

$y = 2x - b$

Setting these equal to each other, the values of x at which the two lines touch are given by

$$\hat{x} = \frac{1}{3} \pm \sqrt{-\left(\frac{5}{9} + \frac{b}{3}\right)}$$

The only real solution is if $b = -\frac{5}{3}$



Practice Finding Maxima & Minima

from Calc. Made Easy, Ch. 11

$y=(x-m)^2-d^2$
$y=x^2-2mx+m^2-d^2$
$\hat{x}=\{m\pm d\}$

$y=x^2-4x+7$
$=(x-2)^2+3$
$\frac{dy}{dx}=2x-4$
$=0$ at $\hat{x}=2$

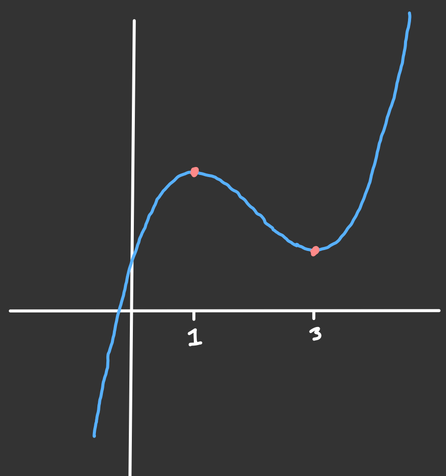
$y=3x-x^2$
$=-(x+3/2)^2+9/4$
$\frac{dy}{dx}=3-2x$ so $\hat{x}=3/2$

$y=ax^3+bx+c$
$\frac{dy}{dx}=3ax^2+b$
$=0$ if $\hat{x}=\pm\sqrt{-b/3a}$

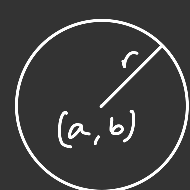
$y=4x+1/x$
$\frac{dy}{dx}=4-1/x^2$
so $\hat{x}=\pm 1/2$

$y=x^2-x$
$=(x-1/2)^2+1/4$
$\frac{dy}{dx}=2x-1$ so $\hat{x}=1/2$

$y=1/3\,x^3-2x^2+3x+1$
$\frac{dy}{dx}=x^2-4x+3$
$=(x-2)^2-1$
$=0$ when $\hat{x}=\{1,3\}$

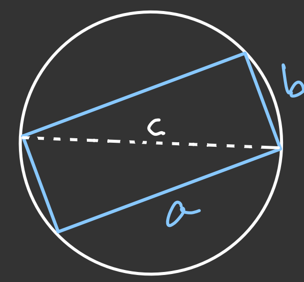


$(y-b)^2+(x-a)^2=r^2$
$y=\pm\sqrt{r^2-(x-a)^2}+b$



$\frac{dy}{dx}=\cancel{\frac{1}{2}}(r^2-(x-a)^2)^{-1/2}(-\cancel{2}(x-a))$
$=\frac{-(x-a)}{(r^2-(x-a)^2)^{1/2}}$
$=0$ when $x=a$

1) For a given circle, what are the dimensions of the inscribed rectangle with the largest possible area?



constraint: for a given c,

$$a^2 + b^2 = c^2$$

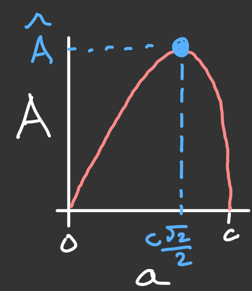
$$b = \sqrt{c^2 - a^2}$$ (b must be positive)

goal: maximize $A = ab = a\sqrt{c^2 - a^2}$

$$\frac{dA}{da} = \sqrt{c^2-a^2} + a\frac{1}{\cancel{2}}(c^2-a^2)^{-1/2}(-\cancel{2}a)$$

$$= \frac{(c^2-a^2) - a^2}{(c^2-a^2)^{1/2}}$$

$$= 0 \text{ when } c^2 = 2\hat{a}^2 \text{ or } \hat{a} = c\frac{\sqrt{2}}{2} \approx 0.7c$$

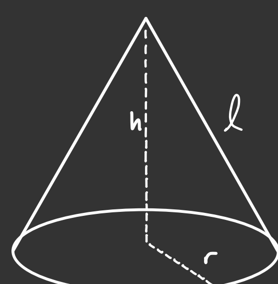


This implies $\hat{a} = \hat{b}$, so the area-maximizing shape is a square!

In these terms, the area of the circle is $\pi c^2/4$, so the ratio of the areas of the circle & square is $\pi/2 \approx 1.57$.

2) For a cone with a fixed side length, what is the volume-maximizing radius?

$$\ell^2 = h^2 + r^2 \Rightarrow \ell^2 - h^2 = r^2$$

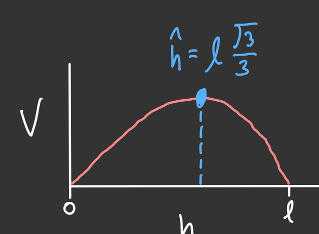


$$V = \frac{\pi r^2 h}{3} = \frac{\pi(\ell^2-h^2)h}{3}$$

$$\frac{dV}{dh} = \frac{\pi}{3}\left((\ell^2-h^2) + h(-2h)\right)$$

$$= \frac{\pi}{3}(\ell^2 - 3h^2)$$

$$= 0 \text{ when } \hat{h} = \frac{\ell\sqrt{3}}{3} \approx 0.58\,\ell$$



This implies that $\hat{r} = \hat{h}\sqrt{2} = \ell\frac{\sqrt{6}}{3} \approx 0.82\,\ell$.

3) $y = \frac{x}{a-x} + \frac{a-x}{x}$

$= \frac{x^2}{(a-x)x} + \frac{(a-x)^2}{(a-x)x}$

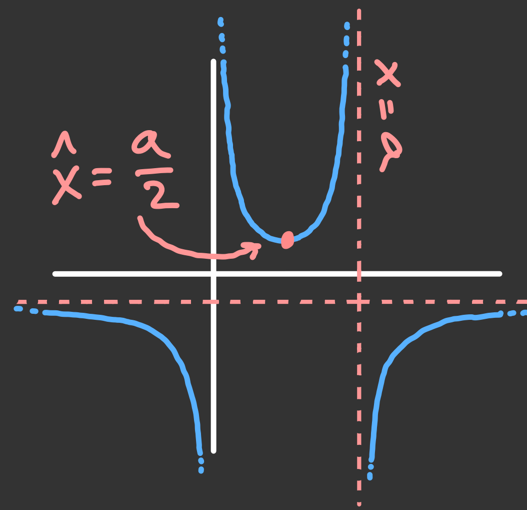
$\lim_{x \to \pm\infty} = (-2)\left(\frac{x^2 - ax + \frac{a^2}{2}}{x^2 - ax}\right)$

$= \frac{(a \cancel{-x}) \cancel{+ x}}{(a-x)^2} + \frac{\cancel{(-1)x} - (a \cancel{-x})}{x^2}$

$= a\left(\frac{x^2 - (a-x)^2}{(a-x)^2 x^2}\right)$

$= \frac{2a^2(x - a/2)}{x^2(a-x)^2}$

$= 0$ when $\hat{x} = a/2$

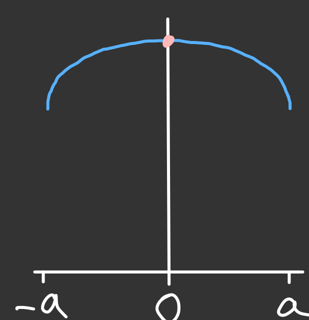


4) $y = \sqrt{a-x} + \sqrt{a+x}$

$\frac{dy}{dx} = -\frac{1}{2}(a-x)^{-1/2} + \frac{1}{2}(a+x)^{-1/2}$

$= 0$ when $(a+\hat{x})^{-1/2} = (a-\hat{x})^{-1/2}$

or $\hat{x} = 0$

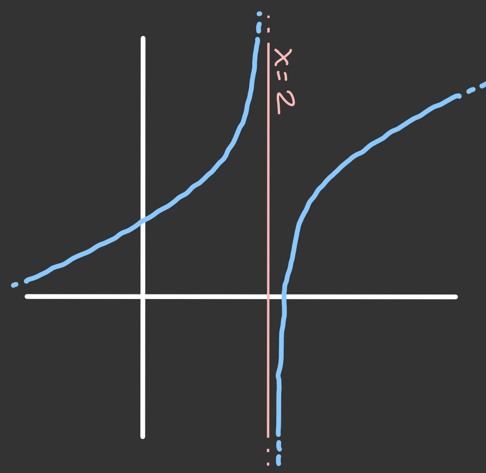


5) $y = \frac{x^2-5}{2x-4}$

$$\frac{dy}{dx} = \frac{2x(2x-4)-(x^2-5)2}{(2x-4)^2}$$

$$= \frac{x^2-4x+5}{4(x^2-4x+4)} = \frac{1}{4}\,\frac{(x-2)^2+1}{(x-2)^2}$$

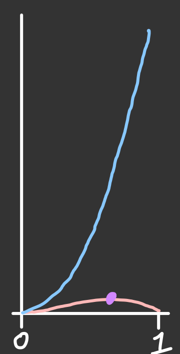
The solutions for $dy/dx = 0$ are $\{2 \pm i\}$, so this function has no max or min!



6) $y = x^2 \pm x^{5/2}$

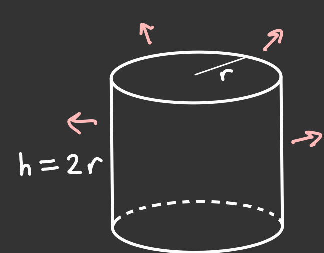
$$\frac{dy}{dx} = 2x - 5/2\, x^{3/2}$$
$$= 0 \text{ if } \hat{x} = 16/25$$

$$\frac{dy}{dx} = 2x + 5/2\, x^{3/2}$$
$= 0$ if $\hat{x}^{1/2} = -4/5$, which is impossible, so no extrema exist.



7) A cylinder with a height twice the radius is growing such that its surface area A changes at rate $dA/dt = \gamma$ cm^2/sec.

What is the rate the volume changes?



$$A = 2\pi r(t)^2 + 2\pi r(t) h = 6\pi r(t)^2$$

$$V = \pi r(t)^2 h = 2\pi r(t)^3$$

$$\frac{dA}{dt} = 12\pi r(t) \cdot \frac{dr}{dt} = \gamma$$

$$\frac{dr}{dt} = \frac{\gamma}{12\pi r(t)}$$

$$\frac{dV}{dt} = 6\pi r(t)^2 \frac{dr}{dt} = \frac{\gamma\, r(t)}{2}$$

In Calc Made Easy, $\gamma = 20$.

Calc. Made Easy, Exercises 9, p.130

1) $y = \frac{x^2}{x+a}$

$dy/dx = \frac{x(x+2a)}{(x+a)^2}$

$= 0$ if $\hat{x} = \{0, -2a\}$

if $a=1$, $\hat{x} = \{0, -2\}$

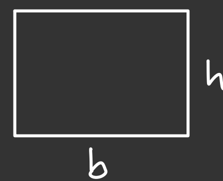
Plots show 0 is a max, $-2a$ is a min.

2) $y = \frac{x}{a^2+x^2}$

$dy/dx = \frac{a^2-x^2}{(a^2+x^2)^2}$

$= 0$ if $\hat{x} = \pm a$

3) Maximize the area of a rectangle with a fixed perimeter.



$p = 2(b+h)$

$\frac{p-2b}{2} = h$

$A = bh$

$= \frac{b(p-2b)}{2}$

$\frac{dA}{db} = \frac{p}{2} - 2b$

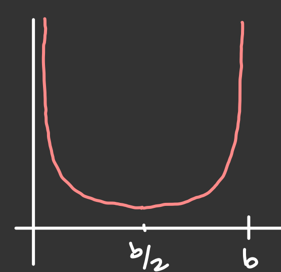
$= 0$ if $b = p/4$

4) For a triangle with a fixed perimeter, the maximum possible area is when all three sides are equal. Proof: fix any one side; the max area is thus maximizing height, which is when two remaining sides are equal. By symmetry, all sides must be equal!

5) $y = \frac{a}{x} + \frac{a}{b-x} = \frac{ab}{x(b-x)}$

$\frac{dy}{dx} = \frac{-ab}{x(b-x)} \frac{(b-2x)}{x(b-x)}$

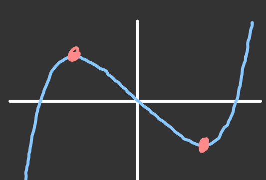
$= 0$ if $\hat{x} = b/2$



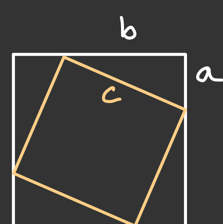
6) $y = x^5 - bx$

$dy/dx = 5x^4 - b$

$= 0$ if $\hat{x} = \pm(b/5)^{1/4}$



7)



$4(a+b) = P$

$b = P/4 - a$

$a^2 + b^2 = c^2$

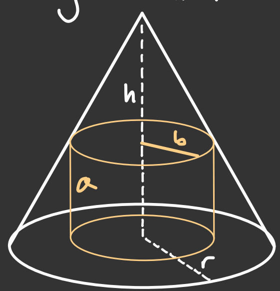
$a^2 + (P/4 - a)^2 = c^2$

$a^2 + (P/4)^2 - 2a/4 + a^2 = c^2$

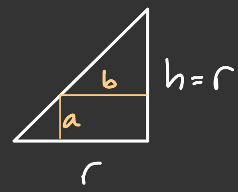
$\frac{d(c^2)}{da} = 4a - \cancel{2}/\cancel{4} =$

$= 0$ if $a = P/8$, so $a = b$

8) given: $h=r$



Note in cross section:



where $a+b=r$.

To maximize the cylinder's volume

$$V=\pi b^2\cdot a=\pi b^2\cdot(r-b)=\pi b^2 r-\pi b^3$$

$$\frac{dV}{db}=2\pi b r-3\pi b^2=0 \text{ if } 2\cancel{\pi}\cancel{b}r=3\cancel{\pi}b^{\cancel{2}}$$

$$2r=3b$$

$$\hat{b}=2/3\, r \quad \hat{a}=1/3\, r$$

To maximize the lateral area:

$$A=2\pi b\cdot a=2\pi(r-a)\cdot a$$

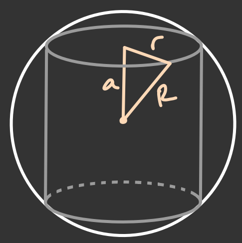
$$\frac{dA}{da}=2\pi(r-2a), \quad \text{so } \hat{a}=1/2\, r,\ \hat{b}=1/2\, r$$

To maximize total area:

$$A=2\pi b\cdot a+2\cdot\pi b^2=2\pi b r$$

The area is maximized as b goes to r.

9)



R is fixed, so a is a function of r

$$a^2 = R^2 - r^2 \qquad a = \sqrt{R^2 - r^2}$$

$$V = r^2 \pi \cdot 2a \qquad L = 2\pi r \cdot 2a \qquad A = 2\pi r^2 + 2\pi r\, 2a = 2\pi r(r + 2a)$$

What inscribed cylinder has the largest volume?

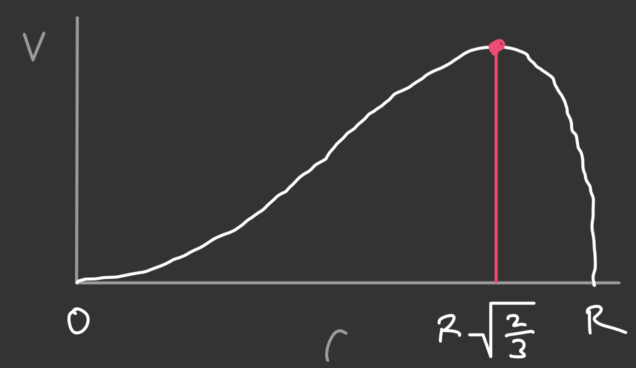
$$V = r^2 \pi\, 2\sqrt{R^2 - r^2}$$

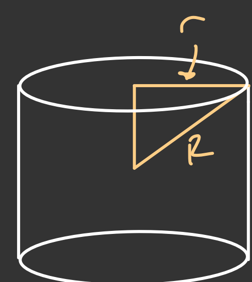
$$\frac{dV}{dr} = 2\pi\left[2r\sqrt{R^2 - r^2} + r^2 \cancel{\tfrac{1}{2}}(R^2 - r^2)^{-1/2}(-\cancel{2}r)\right]$$

$$= 2\pi r\sqrt{R^2 - r^2}\left[2 - r^2/(R^2 - r^2)\right]$$

$$= \frac{2\pi r\sqrt{R^2 - r^2}\,(2R^2 - 3r^2)}{R^2 - r^2} = 0$$

$$\hat{r} = R\sqrt{\tfrac{2}{3}} \qquad \hat{a} = R\sqrt{1/3}$$





1.41 : 1

What inscribed cylinder has the largest lateral surface area?

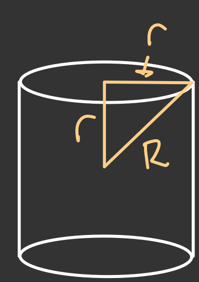
$$L = 2\pi r\, 2\sqrt{R^2 - r^2}$$

$$\frac{dL}{dr} = 4\pi\left[\sqrt{R^2 - r^2} + r\,\cancel{\tfrac{1}{2}}\sqrt{R^2 - r^2}^{\,-1}(-\cancel{2}r)\right]$$

$$= 4\pi\sqrt{R^2 - r^2}\left(1 - r^2/(R^2 - r^2)\right)$$

$$= 4\pi\sqrt{R^2 - r^2}\,\frac{(R^2 - 2r^2)}{R^2 - r^2}$$

$$\hat{r} = R/\sqrt{2}$$



1 : 1

What inscribed cylinder has the largest surface area?

$$A = 2\pi r\left(r + 2\sqrt{R^2 - r^2}\right)$$

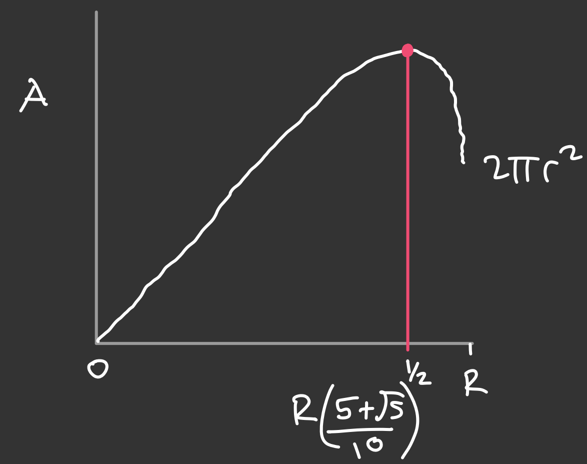
$$= 2\pi r^2 + 4\pi r\sqrt{R^2 - r^2}$$

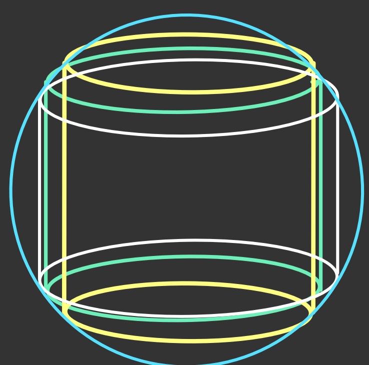
$$\frac{dA}{dr} = 4\pi r + 4\pi\sqrt{R^2 - r^2}\,\frac{\left(R^2 - 2r^2\right)}{R^2 - r^2}$$

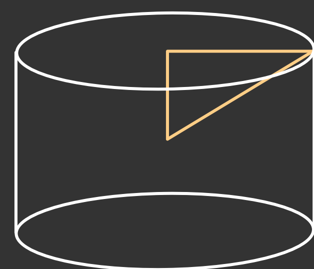
$$r\left(R^2 - r^2\right)^{1/2} + R^2 - 2r^2 = 0$$

$$\hat{r} = R\sqrt{\frac{5 + \sqrt{5}}{10}}$$

solved w/ mathematica!

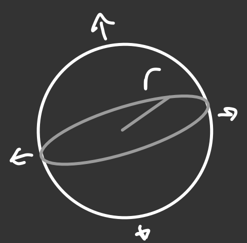






1.62 : 1

10)



If a sphere's volume is expanding at a constant rate, what's happening to its surface area?

$V = \frac{4}{3}\pi r^3 \qquad A = 4\pi r^2$

$\frac{dV}{dt} = a \qquad$ so $\frac{dr}{dt} = \frac{a}{4\pi r^2}$

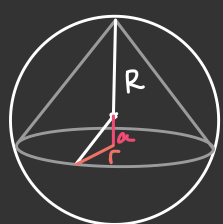
$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$

$a = 4\pi r^2 \cdot \frac{dr}{dt}$

$$\frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt} = 8\pi r\cdot\frac{dr}{dt} = 8\pi r\left(\frac{a}{4\pi r^2}\right)$$

$$\frac{dA}{dt} = \frac{2a}{r}$$

11)



Which cone inscribed in a sphere has the largest volume?

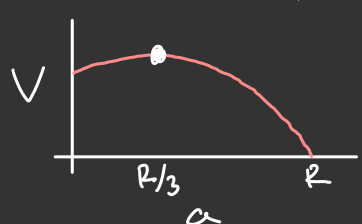
Notation:

$V = \frac{\pi r^2 (R+a)}{3}$ where $r^2 + a^2 = R^2$.

So: $= \frac{\pi}{3}\cdot(R^2 - a^2)(R+a)$

$= \frac{\pi}{3}(R-a)(R+a)^2$

$= \frac{\pi}{3}(R^3 + R^2 a - Ra^2 - a^3)$



$\frac{dV}{da} = \frac{\pi}{3}(R^2 - 2Ra - 3a^2) = 0$

if $\hat{a} = \frac{2R \pm \sqrt{4R^2 - 4\cdot(-3)R^2}}{2\cdot(-3)}$

$= \frac{2R \pm \sqrt{16R^2}}{-6} = \frac{2R \pm 4R}{-6}$

$= \frac{R \pm 2R}{-3}$ or $\{-R, R/3\}$

which implies $\hat{r} = \frac{2\sqrt{2}}{3} R \approx 0.94R$

Calc Made Easy, Ch. 12

$y = 4x^2 - 9x - 6$
$ = 4(x - 9/8)^2 - 177/16$
$y' = 8x - 9$
$ = 0$ @ $\hat{x} = 9/8$,
$y'' = 8$, so this is a min

$y = -4x^2 + 9x + 6$
$ = -4(x - 9/8)^2 + 177/16$
$y' = -8x + 9$
$ = 0$ @ $\hat{x} = 9/8$
$y'' = -8$, so this is a max

$y = x^3 - 3x + 16$
$y' = 3x^2 - 3$
$ = 0$ @ $\hat{x} = \pm 1$
$y'' = 6x$, so
$\hat{x} = 1$ is a min,
$\hat{x} = -1$ is a max

$$y = \frac{x-1}{x^2+2}$$

$$y' = \frac{(x^2+2) - (x-1)2x}{(x^2+2)^2}$$

$$= \frac{x^2 + 2 - 2x^2 + 2x}{(x^2+2)^2}$$

$$= \frac{-x^2 + 2x + 2}{(x^2+2)^2}$$

$= 0$ @ $\hat{x} = \{1 \pm \sqrt{3}\}$

Graphing reveals $1+\sqrt{3}$ is a max & $1-\sqrt{3}$ is a min, but you can use y'' too...

$$C = aP + \frac{b}{c+P} + d \quad (a, b, c, d \text{ all positive})$$

$$C' = a - b(c+P)^{-2} = 0$$

when

$$a = \frac{b}{(c+P)^2}$$

$$P = \{-c \pm \sqrt{b/a}\}$$

By inspection, $-c + \sqrt{b/a}$ is a minimum!

Lighting a Building

Calc Made Easy, p. 136

Our goal is to model the hourly cost of lighting a building, in cents. Each lamp has an expected lifspan of t hours, and replacing one costs r cents. The rate of replacement is t^{-1}, so for n lamps in use, the hourly average cost of replacement is nr/t.

We also must consider the energy costs of each lamp. Parameter x measures the lamp's efficiency, in watts per candle, and l measures the candles of light per lamp. The cost of electricity, in cents per kWh, is p. Thus, the cost of running n lamps for an hour must be $n \cdot l \cdot x \cdot p / 1000$.

Thus, the overall operations cost is

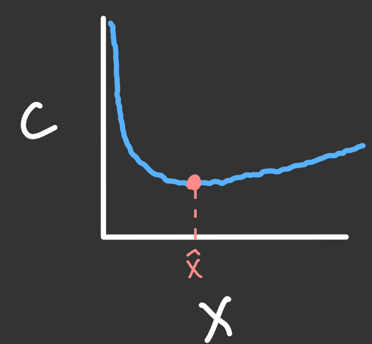
$$C = n\left(\frac{r}{t} + \frac{l \cdot x \cdot p}{1000}\right)$$

which increases linearly with n, r, l, a, p & $1/t$ (the rate of replacement).

We connect the two parts together by assuming that the lifespan of a lamp is a power law of its efficiency: $t = m x^n$. That is, the more watts required per candle, the longer the lifespan (?)

Given this, we can find the cost-minimizing efficiency to be

$$\hat{x} = \left(\frac{nr\,1000}{mal}\right)^{\left(\frac{1}{n+1}\right)}$$



Increasing x from $\hat{x}$ and each candle costs more. Decreasing x from $\hat{x}$, and the higher efficiency of each lamp causes it to expire faster, driving up replacement costs.

More Graphical Optima

Calc Made Easy, Exercises 10, p. 137

1) $y = x^3 + x^2 - 10x + 8$

$y' = 3x^2 + 2x - 10$

$\hat{x} = \dfrac{-2 \pm \sqrt{4 - 4\cdot 3\cdot(-10)}}{6} = \dfrac{-2 \pm \sqrt{124}}{6}$

$= \dfrac{-2 \pm 2\sqrt{31}}{6} = \dfrac{-1 \pm \sqrt{31}}{3}$

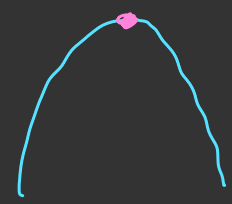
2) $y = \frac{b}{a}x - cx^2 = -c(x^2 - \frac{b}{a}x + 0)$

$-c(x - \frac{b}{2ac})^2 + (\frac{b}{2a})^2$

$y' = \frac{b}{a} - 2cx$

$= 0$ when $x = \frac{b}{2ac}$

$y'' = -2c$, so a max



3) $y = 1 - \frac{x^2}{2} + \frac{x^4}{24}$

$y' = -\frac{\cancel{2}x}{\cancel{2}} + \frac{\cancel{4}x^3}{\cancel{24}\,6} = \frac{x^3}{6} - x$

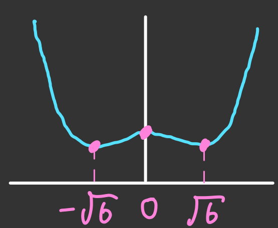
$= \frac{x}{6}(x^2 - 6) = \frac{x}{6}(x - \sqrt{6})(x + \sqrt{6})$

So optima exist at $\{\pm\sqrt{6}, 0\}$

$y'' = \frac{\cancel{3}x^2}{\cancel{6}\,2} - 1.$

If $x = 0$, negative, max.

If $x = \pm\sqrt{6}$, positive, min.



$y = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$

$y' = -\frac{\cancel{2}x}{\cancel{2}} + \frac{\cancel{4}x^3}{\cancel{24}\,6} - \frac{\cancel{6}x^5}{\cancel{720}\,120} = -x + \frac{x^3}{6} - \frac{x^5}{120}$

$= -\frac{x}{120}\left(120 + \frac{x^3}{20} - x^5\right)$

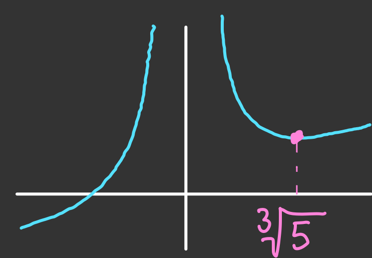
This has the same shape & number of critical points as the simpler version... (but why?)

4) $y = 2x + 1 + \frac{5}{x^2}$

$y' = 2 - 2\left(\frac{5}{x^3}\right)$

$= 0$ when $\hat{x}^3 = 5$, so $\hat{x} = \sqrt[3]{5}$!

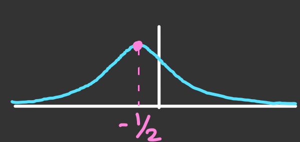
$y'' = 40x^{-5}$, which is + at $\hat{x}$, min!



5) $y = 3(x^2 + x + 1)^{-1}$

$y' = -3(x^2 + x + 1)^{-2}(2x + 1)$

$= 0$ when $\hat{x} = -\frac{1}{2}$



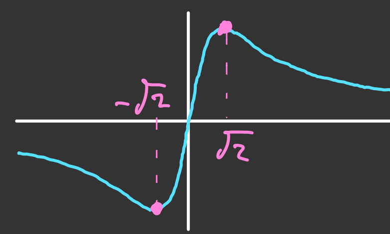
6) $y = \frac{5x}{2+x^2} = 5x(2+x^2)^{-1}$

$y' = 5(2+x^2)^{-1} + 5x(-1)(2+x^2)^{-2}\,2x$

$= 5(2+x^2)^{-2}(2+x^2-2x^2)$

$= 5(2+x^2)^{-2}(2-x^2)$

$= 0$ when $\hat{x} = \pm\sqrt{2}$



7) $y = \frac{3x}{x^2-3} + \frac{x}{2} + 5$

$y' = \frac{3(x^2-3) - 3x\cdot 2x}{(x^2-3)^2} + \frac{1}{2}$

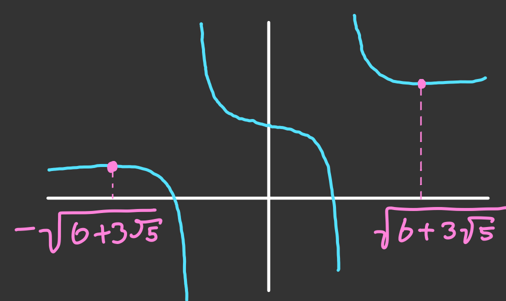
$= \frac{x^4 - 12x^2 - 9}{2(x^2-3)^2} = \frac{(x^2-6)^2 - 45}{2(x^2-3)^2}$

(only + for real solutions)

$= 0$ if $x^2 - 6 = +3\sqrt{5}$

$x^2 = 6 + 3\sqrt{5}$

$x = \pm\sqrt{6+3\sqrt{5}} \approx \pm 3.56$



8) $y = ax^2 + b(1-x)^2$

$y' = 2ax - 2b(1-x)$

$= 0$ if $\hat{x} = b/(a+b)$

In Calc Made Easy,

$a = 3$ & $b = 2$

9) $v = x(a+bx+cx^2)^{-1}$

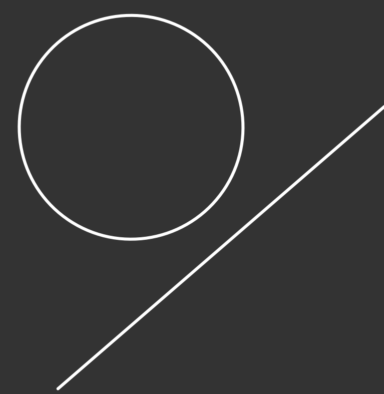
$$v' = \frac{(a+bx+cx^2) - x(b+2cx)}{(a+bx+cx^2)^2}$$

$$= \frac{a+\cancel{bx}+cx^2-\cancel{bx}-2cx^2}{(a+bx+cx^2)^2}$$

$$= (a-cx^2)(a+bx+cx^2)^{-2}$$

$$= 0 \text{ if } x = \pm\sqrt{\tfrac{a}{c}}$$

Mistake in book => can be + or −

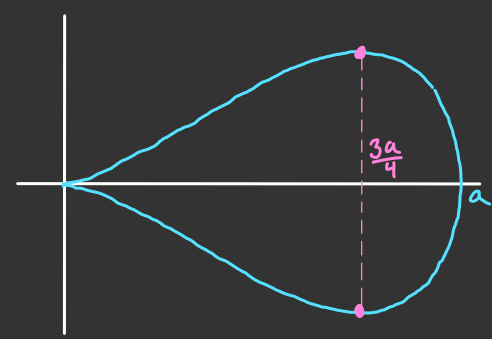


10) See next page!

11) $y = \pm \frac{x}{b}\sqrt{x(a-x)}$

Real values of y only exist between 0 & a.
Taking the positive side, $y = b^{-1}x(xa-x^2)^{1/2}$

$$y' = b^{-1}(xa-x^2)^{1/2} + xb^{-1}\tfrac{1}{2}(xa-x^2)^{-1/2}(a-2x)$$
$$= b^{-1}(xa-x^2)^{-1/2}\left(xa-x^2+\tfrac{x}{2}(a-2x)\right)$$
$$= b^{-1}(xa-x^2)^{-1/2}\left(xa-x^2+\tfrac{xa}{2}-x^2\right)$$
$$= b^{-1}(xa-x^2)^{-1/2}\left(\tfrac{3a}{2}x-2x^2\right)$$
$$= 0 \text{ when } x=\{0, 3a/4\}$$

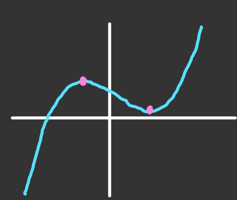


The graph shows $3a/4$ is max for positive y.
By symmetry, $3a/4$ is also the min of the negative side.

12) $y = 4x^3 - x^2 - 2x + 1$

$y' = 12x^2 - 2x - 2$

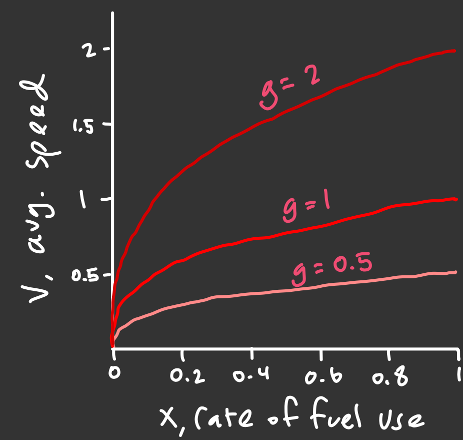
$= 0$ if $x = \{-1/3, 1/2\}$



Optimal Ship Speed

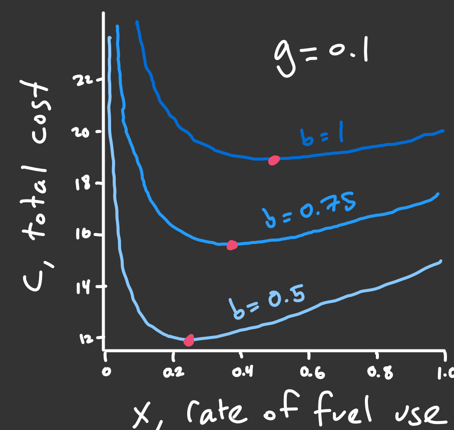
Calc Made Easy, Exercises 10, Problem 10

The costs to run our ship as we voyage can be broken up into two categories: fuel & running costs. Fuel use increases with the ship's speed, while the total running costs (wages, stores, etc.) increase with the voyage's duration. This sets up a trade-off between the two kinds of cost, implying the existence of a cost-optimizing speed.



Define the running costs as a fixed rate of energy per unit time, b, so the total running costs of a trip of length t is $b \cdot t$. Define x as the rate of fuel consumed. The total energy cost of the trip is therefore

$$c = bt + xt = (b+x)t$$



We assume the average speed v is proportional to the cube-root of the rate of energy consumed in a given unit of time via

$$v = g x^{1/3}$$

Both t & x can be written in terms of v:

$$t = d/v$$

$$x = \left(\frac{v}{g}\right)^3$$

Hence

$$c = \left(b + \frac{v^3}{g^3}\right)\frac{d}{v} = d\left(\frac{b}{v} + \frac{v^2}{g^3}\right)$$

Differentiating with respect to v shows a min at

$$v^* = g\left(b/2\right)^{1/3}$$

This implies a very simple rule: the optimal rate of fuel consumption should always be <u>half</u> of the ships running costs:

$$x^* = b/2$$

PARTIAL FRACTIONS REVIEW

It all depends on the type of denominator.

CASE I: DISTINCT LINEAR FACTORS

$$\frac{3x+1}{x^2-1} = \frac{3x+1}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

$$\frac{3x+1}{x^2-1} = \frac{2}{x-1} + \frac{1}{x+1}$$

$$(3x+1) = A_1(x+1) + A_2(x-1)$$
$$= (A_1+A_2)x + (A_1-A_2)$$

$$\to \quad A_1 + A_2 = 3 \qquad A_1 - A_2 = 1 \quad \to \quad A_1 = 2 \qquad A_2 = 1$$

$$\frac{4x^2+2x-14}{x^3+3x^2-x-3} = \frac{A_1}{x+3} + \frac{A_2}{x-1} + \frac{A_3}{x+1}$$

$$= \frac{2}{x+3} - \frac{1}{x-1} + \frac{3}{x+1}$$

$$4x^2+2x-14 = A_1(x-1)(x+1) + A_2(x+3)(x+1) + A_3(x+3)(x-1)$$

$$= (A_1+A_2+A_3)x^2 + (4A_2+2A_3)x - (A_1-3A_2+3A_3)$$

$$4 = A_1 + A_2 + A_3 \qquad 2 = 4A_2 + 2A_3 \qquad 14 = A_1 - 3A_2 + 3A_3$$

$$\to \quad A_1 = 2 \qquad A_2 = -1 \qquad A_3 = 3$$

CASE II: DISTINCT QUADRATIC FACTORS

$$\frac{-x^2-3}{(x^2+1)(x+1)} = \frac{A_1x+B_1}{x^2+1} + \frac{B_2}{x+1} = \frac{x-1}{x^2+1} - \frac{2}{x+1}$$

$$-x^2-3 = (x+1)(A_1x+B_1) + B_2(x^2+1)$$
$$= (A_1+B_2)x^2 + (A_1+B_1)x + B_1 + B_2$$

$$-1 = A_1 + B_2 \qquad 0 = A_1 + B_1 \qquad -3 = B_1 + B_2$$

$$\to \quad A_1 = 1 \qquad B_2 = -2 \qquad B_1 = -1$$

$$\frac{x^3-2}{(x^2+1)(x^2+2)} = \frac{A_1x+B_1}{x^2+1} + \frac{A_2x+B_2}{x^2+2} = \frac{-x-2}{x^2+1} + \frac{2x+2}{x^2+2}$$

$$x^3-2 = (x^2+2)(A_1x+B_1) + (x^2+1)(A_2x+B_2)$$
$$= (A_1+A_2)x^3 + (B_1+B_2)x^2 + (2A_1+A_2)x + (2B_1+B_2)$$

$$1 = A_1 + A_2 \qquad 0 = B_1 + B_2 \qquad 0 = 2A_1 + A_2 \qquad -2 = 2B_1 + B_2$$

$$\to \quad A_1 = -1 \qquad A_2 = 2 \qquad B_1 = -2 \qquad B_2 = 2$$

CASE III: REPEATING LINEAR FACTORS

$$\frac{3x^2-2x+1}{(x+1)^2(x-2)} = \frac{A_1}{(x+1)^2} + \frac{A_2}{x+1} + \frac{A_3}{x-2} = \frac{-2}{(x+1)^2} + \frac{2}{x+1} + \frac{1}{x-2}$$

$$3x^2-2x+1 = (x-2)A_1 + (x+1)(x-2)A_2 + (x+1)^2A_3$$

$$= A_1x - 2A_1 + (x^2-x-2)A_2 + (x^2+2x+1)A_3$$

$$= \underline{A_1x} - 2A_1 \underline{+ A_2x^2} \underline{- A_2x} - 2A_2 \underline{+ A_3x^2} + \underline{2A_3x} + A_3$$

$$= (A_2+A_3)x^2 + (A_1 - A_2 + 2A_3)x - 2A_1 - 2A_2 + A_3$$

so:

$$A_2 + A_3 = 3$$
$$A_1 - A_2 + 2A_3 = -2$$
$$-2A_1 - 2A_2 + A_3 = 1$$

$$A_1 = -2$$
$$A_2 = 2$$
$$A_3 = 1$$

$$\frac{3x-1}{(2x^2-1)^2(x+1)} = \frac{A_1x+A_2}{(2x^2-1)^2} + \frac{A_3x+A_4}{2x^2-1} + \frac{A_5}{x+1} = \frac{8x-5}{(2x^2-1)^2} + \frac{8x-8}{2x^2-1} - \frac{4}{x+1}$$

$$(A_1x+A_2)(x+1) + (A_3x+A_4)(2x^2-1)(x+1) + A_5(2x^2-1)^2$$

$$= A_1x^2 + A_1x + A_2x + A_2 + (A_3x+A_4)(2x^3+2x^2-x-1) + A_5(4x^4-4x^2+1)$$

$$= A_1x^2 + A_1x + A_2x + A_2 + 2A_3x^4 + 2A_3x^3 - A_3x^2 - A_3x$$
$$+ 2A_4x^3 + 2A_4x^2 - A_4x - A_4 + 4A_5x^4 - 4A_5x^2 + A_5$$

$$= (4A_5+2A_3)x^4 + (2A_3+2A_4)x^3 + (A_1 - A_3 + 2A_4 - 4A_5)x^2$$
$$+ (A_1 + A_2 - A_3 - A_4)x + (A_2 - A_4 + A_5)$$

so:

$$4A_5 + 2A_3 = 0 \rightarrow A_5 = -\tfrac{1}{2}A_3$$
$$2A_3 + 2A_4 = 0 \rightarrow A_4 = -A_3$$
$$A_1 - A_3 + 2A_4 - 4A_5 = 0 \rightarrow A_1 - A_3 - 2A_3 + 2A_3 = 0$$

so $A_1 = A_3$

$$A_1 + A_2 - A_3 - A_4 = 3$$
$$A_2 - A_4 + A_5 = -1$$

$$\rightarrow$$

$$A_3 + A_2 = 3$$
$$A_2 + \tfrac{1}{2}A_3 = -1$$

$$(3 - A_3) + \tfrac{1}{2}A_3 = -1$$
$$3 - \tfrac{1}{2}A_3 = -1$$
$$4 = \tfrac{1}{2}A_3$$

$A_3 = 8$ $\quad A_1 = 8$

$A_2 = -5$ $\quad A_4 = -8$ $\quad A_5 = -4$

Additional Examples

$$\frac{4x+1}{(x+1)^3} = \frac{4(z-1)+1}{z^3} = \frac{4}{z^2} - \frac{3}{z^3}$$

$$y = \frac{5-4x}{6x^2+7x-3} = \frac{5-4x}{(2x+3)(3x-1)} = \frac{A_1}{2x+3} + \frac{A_2}{3x-1} = \frac{-2}{2x+3} + \frac{1}{3x-1}$$

$$A_1(3x-1) + A_2(2x+3) = 5-4x$$

$$3A_1x - A_1 + 2A_2x + 3A_2 = 5-4x$$

$$(3A_1+2A_2)x - A_1 + 3A_2$$

when calculating y', its much easier to decompose y first.

$$3A_1 + 2A_2 = -4$$
$$-A_1 + 3A_2 = 5$$

$$A_1 = 3A_2 - 5$$

$$3(3A_2-5) + 2A_2 = -4$$
$$9A_2 - 15 + 2A_2 = -4$$
$$11A_2 - 15 = -4$$
$$A_2 = 1$$
$$A_1 = -2$$

PARTIAL FRACTIONS PRACTICE

Exercises 11, page 147

1) $\frac{3x+5}{(x-3)(x+4)} = \frac{1}{x+4} + \frac{2}{x-3}$

$$3x+5 = A_1(x-3) + A_2(x+4)$$
$$= A_1 x - 3A_1 + A_2 x + 4A_2$$
$$= (A_1 + A_2)x + 4A_2 - 3A_1$$

$3 = A_1 + A_2$ $\quad A_1 = 1$
$5 = 4A_2 - 3A_1$ $\quad A_2 = 2$

2) $\frac{3x-4}{(x-1)(x-2)} = \frac{2}{x-2} + \frac{1}{x-1}$

$$3x-4 = (x-1)A_1 + (x-2)A_2$$
$$= (A_1+A_2)x - A_1 - 2A_2$$

$3 = A_1 + A_2$ $\quad A_1 = 2$
$-4 = -A_1 - 2A_2$ $\quad A_2 = 1$

3) $\frac{3x+5}{x^2+x-12} = \frac{2}{x-3} + \frac{1}{x+4}$

$$3x+5 = (x+4)A_1 + (x-3)A_2$$
$$= A_1 x + 4A_1 + A_2 x - 3A_2$$
$$= (A_1 + A_2)x + 4A_1 - 3A_2$$

$3 = A_1 + A_2$ $\quad A_1 = 2$
$5 = 4A_1 - 3A_2$ $\quad A_2 = 1$

4) $\frac{x+1}{x^2-7x+12} = \frac{-4}{x-3} + \frac{5}{x-4}$

$$x+1 = (x-4)A_1 + (x-3)A_2$$
$$= A_1 x - 4A_1 + A_2 x - 3A_2$$
$$= (A_1 + A_2)x - 4A_1 - 3A_2$$

$1 = A_1 + A_2$ $\quad A_1 = -4$
$1 = -4A_1 - 3A_2$ $\quad A_2 = 5$

5) $\frac{x-8}{(2x+3)(3x-2)} = \frac{1}{13}\left(\frac{19}{2x+3} - \frac{22}{3x-2}\right)$

$$x-8 = (2x+3)A_1 + (3x-2)A_2$$
$$= 2A_1 x + 3A_1 + 3A_2 x - 2A_2$$
$$= (2A_1 + 3A_2)x + 3A_1 - 2A_2$$

$1 = 2A_1 + 3A_2$ $\quad A_1 = -22/13$
$-8 = 3A_1 - 2A_2$ $\quad A_2 = 19/13$

6) $\frac{x^2-13x+26}{(x-2)(x-3)(x-4)} = \frac{2}{x-2} + \frac{4}{x-3} - \frac{5}{x-4}$

$$x^2-13x+26 = A_1(x-3)(x-4) + A_2(x-2)(x-4) + A_3(x-3)(x-2)$$

$$= (A_1+A_2+A_3)x^2 - (7A_1 + 6A_2 + 5A_3)x$$
$$+ 12A_1 + 8A_2 + 6A_3$$

$1 = A_1 + A_2 + A_3$ $\quad \to A_1 = 2$
$13 = 7A_1 + 6A_2 + 5A_3$ $\quad A_2 = 4$
$26 = 12A_1 + 8A_2 + 6A_3$ $\quad A_3 = -5$

7) $\dfrac{x^2-3x+1}{(x-1)(x+2)(x-3)} = \dfrac{1}{6(x-1)} + \dfrac{11}{15(x+2)} + \dfrac{1}{10(x-3)}$

$$\begin{aligned} x^2-3x+1 &= A_1(x+2)(x-3) + A_2(x-1)(x-3) + A_3(x+2)(x-1) \\ &= (A_1+A_2+A_3)x^2 - (A_1+4A_2-A_3)x - 6A_1 + 3A_2 - 2A_3 \end{aligned}$$

$$\begin{aligned} 1 &= A_1 + A_2 + A_3 \qquad & A_1 &= 1/6 \\ 3 &= A_1 + 4A_2 - A_3 & A_2 &= 11/15 \\ 1 &= -6A_1 + 3A_2 - 2A_3 & A_3 &= 1/10 \end{aligned}$$

8) $\dfrac{5x^2+7x+1}{(2x+1)(3x-2)(3x+1)} = \dfrac{-5/7}{2x+1} + \dfrac{71/63}{3x-2} + \dfrac{7/9}{3x+1}$

$$\begin{aligned} 5x^2+7x+1 &= A_1(3x-2)(3x+1) + A_2(2x+1)(3x+1) + A_3(2x+1)(3x-2) \\ &= A_1(9x^2+3x-6x-2) + A_2(6x^2+2x+3x+1) + A_3(6x^2-4x+3x-2) \\ &= x^2(9A_1+6A_2+6A_3) + x(-3A_1+5A_2-A_3) - 2A_1 + A_2 - 2A_3 \end{aligned}$$

$$\begin{aligned} 5 &= 9A_1 + 6A_2 + 6A_3 \qquad & A_1 &= -5/7 \\ 7 &= -3A_1 + 5A_2 - A_3 & A_2 &= 71/63 \\ 1 &= -2A_1 + A_2 - 2A_3 & A_3 &= 7/9 \end{aligned}$$

9) $\dfrac{x^2}{x^3-1} = \dfrac{x^2}{(x-1)(x^2+x+1)} = \dfrac{A_1}{x-1} + \dfrac{A_2x+B_1}{x^2+x+1} = \dfrac{1}{3}\left(\dfrac{1}{(x-1)} + \dfrac{2x+1}{(x^2+x+1)}\right)$

$$\begin{aligned} x^2 &= (x^2+x+1)A_1 + (x-1)(A_2x+B_1) \\ &= A_1x^2 + A_1x + A_1 + A_2x^2 + B_1x - A_2x - B_1 \\ &= (A_1+A_2)x^2 + (A_1+B_1-A_2)x + A_1 - B_1 \end{aligned}$$

$$\begin{aligned} A_1 + A_2 &= 1 & & & & \\ A_1 + B_1 - A_2 &= 0 \quad \Rightarrow & A_1 + A_2 &= 1 \qquad & A_1 &= 1/3 \\ A_1 - B_1 &= 0 & 2A_1 &= A_2 & A_2 &= 2/3 \\ & & & & B_1 &= 1/3 \end{aligned}$$

10) $\dfrac{x^4+1}{x^3+1} = x + \dfrac{1-x}{x^3+1} = x + \dfrac{1-x}{(x+1)(x^2-x+1)} = x + \dfrac{1}{3}\left(\dfrac{1-2x}{x^2-x+1} + \dfrac{2}{x+1}\right)$

gotta simplify with polynomial long division:

$$\begin{array}{r} x \phantom{{}+1} \\ x^3+1 \overline{\big) \, x^4+1} \\ -(x^4+x) \\ \hline 1-x \end{array}$$

Case II: distinct quadratic factors

$$\frac{Ax+B}{x^2-x+1} + \frac{C}{x+1}$$

$$\begin{aligned} 1-x &= (x+1)(Ax+B) + C(x^2-x+1) \\ &= (A+C)x^2 + (A+B-C)x + (B+C) \end{aligned}$$

implies $A = -2/3$, $B = 1/3$, $C = 2/3$

11) $\frac{5x^2+6x+4}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} = \frac{3}{x+1} + \frac{2x+1}{x^2+x+1}$

$$\begin{aligned} 5x^2+6x+4 &= (x^2+x+1)A + (x+1)(Bx+C) \\ &= Ax^2+Ax+A + Bx^2+Bx+Cx+C \\ &= (A+B)x^2 + (A+B+C)x + (A+C) \end{aligned}$$

$A+B=5$
$A+B+C=6$ $\quad C=1 \quad A=3 \quad B=2$
$A+C=4$

12) $\frac{x}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} = \frac{1}{x-1} - \frac{1}{x-2} + \frac{2}{(x-2)^2}$

$$\begin{aligned} x &= A(x-2)^2 + B(x-2)(x-1) + C(x-1) \\ &= A(x^2-4x+4) + B(x^2-3x+2) + C(x-1) \\ &= (A+B)x^2 - (4A+3B-C)x + 4A+2B-C \end{aligned}$$

$0 = A+B$
$1 = -(4A+3B-C)$ $\quad -(4A-3A-2A)=1$
$0 = 4A+2B-C$ $\quad A=1$
$C=2A$ $\quad B=-1$
$C=2$

13) $\frac{x}{(x^2-1)(x+1)} = \frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{1}{2(x+1)^2}$

$$\begin{aligned} x &= A(x+1)^2 + B(x-1)(x+1) + C(x-1) \\ &= A(x^2+2x+1) + B(x^2-1) + C(x-1) \\ &= (A+B)x^2 + (2A+C)x + (A-B-C) \end{aligned}$$

$A=-B$ $\quad A=1/4$
$2A+C=1$ $\quad B=-1/4$
$A-B-C=0$ $\quad C=1/2$
$2A=C$

$-3A-2(A-1)-4(A-1)=9$
$-3A-6(A-1)=9$
$-3A-6A+6=9$
$-9A=3$
$A=-1/3$

14) $\frac{x+3}{(x+2)^2(x-1)} = \frac{A}{(x+2)^2} + \frac{B}{x+2} + \frac{C}{x-1} = \frac{-1}{3(x+2)^2} - \frac{4}{9(x+2)} + \frac{4}{9(x-1)}$

$$\begin{aligned} x+3 &= A(x-1) + B(x+2)(x-1) + C(x+2)^2 \\ &= A(x-1) + B(x^2+x-2) + C(x^2+4x+4) \\ &= (B+C)x^2 + (A+B+4C)x - A - 2B + 4C \end{aligned}$$

$C=4/9$
$B=-4/9$

$-3/9 - 4/9 + 4\cdot 4/9$
$-7/9 + 16/9$
$= 9/9$ ✓

$1/3 + 2\cdot 4/9 + 4\cdot 4/9$
$3/9 + 8/9 + 16/9$
$= 27/9 = 3$ ✓

$B=-C$
$A+B+4C=1 \rightarrow A+3C=1 \quad A=1-3C$
$-A-2B+4C=3 \quad\quad = 1+3B$
$-A-2\left(\frac{A-1}{3}\right)+4\left(\frac{A-1}{-3}\right)=3$

$\frac{A-1}{-3}=C$

$\frac{A-1}{3}=B$

15) $\dfrac{3x^2+2x+1}{(x+2)(x^2+x+1)^2} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+x+1} + \dfrac{Dx+E}{(x^2+x+1)^2}$

so we work with: $A(x^2+x+1)^2 + (Bx+C)(x+2)(x^2+x+1) + (Dx+E)(x+2)$

$$A(x^4+x^3+x^2+x^3+x^2+x+x^2+x+1)$$
$$+(Bx+C)(x^3+x^2+x+2x^2+2x+2)$$
$$+(Dx+E)(x+2)$$

$$= A(x^4+2x^3+3x^2+2x+1)$$
$$+(Bx+C)(x^3+3x^2+3x+2)$$
$$+(Dx+E)(x+2)$$

$$= Ax^4+2Ax^3+3Ax^2+2Ax+A$$
$$+Bx^4+3Bx^3+3Bx^2+2Bx$$
$$+Cx^3+3Cx^2+3Cx+2C$$
$$+Dx^2+2Dx+Ex+2E$$

$$=(A+B)x^4+(2A+3B+C)x^3$$
$$+(3A+3B+3C+D)x^2+(2A+2B+3C+2D+E)x$$
$$+A+2C+2E$$

which implies

$$\begin{aligned} 0&=A+B\\ 0&=2A+3B+C\\ 3&=3A+3B+3C+D\\ 2&=2A+2B+3C+2D+E\\ 1&=A+2C+2E \end{aligned} \implies \begin{aligned} &A=-B\\ &C=A\\ &3A-3A+3A+D=3\\ &2A-2A+3A+2D+E=2\\ &A+2A+2E=1 \end{aligned}$$

$\Downarrow$ $\qquad D=3(1-A)$

$$\begin{aligned} &3A+D=3\\ &3A+2D+E=2\\ &A+2A+2E=1 \end{aligned} \implies \begin{aligned} &3A+6(1-A)+E=2\\ &3A+2E=1 \end{aligned}$$

$$3A+6-6A+E=2$$
$$3A+2E=1$$

$$\begin{aligned} -3A+E&=-4\\ 3A+2E&=1\\ \hline 3E&=-3 \end{aligned}$$

$B=-1 \qquad E=-1$
$C=1 \qquad A=1$
$\qquad\qquad D=0$

$$\frac{3x^2+2x+1}{(x+2)(x^2+x+1)^2} = \frac{1}{x+2} + \frac{1-x}{x^2+x+1} - \frac{1}{(x^2+x+1)^2}$$

16) $\dfrac{5x^2+8x-12}{(x+4)^3} = \dfrac{A}{(x+4)^3} + \dfrac{B}{(x+4)^2} + \dfrac{C}{x+4} = \dfrac{36}{(x+4)^3} - \dfrac{32}{(x+4)^2} + \dfrac{5}{x+4}$

$$\begin{aligned} 5x^2+8x-12 &= A + B(x+4) + C(x+4)^2 \\ &= A + Bx + 4B + C(x^2+8x+16) \\ &= (A+4B+16C) + (B+8C)x + Cx^2 \end{aligned}$$

$C = 5$

$B + 8C = 8 \qquad B = -32$

$(A+4B+16C) = -12$

$A - 128 + 80 = -12$

$A - 48 = -12$

$A = 36$

17) $\dfrac{7x^2+9x-1}{(3x-2)^4} = \dfrac{A}{(3x-2)^4} + \dfrac{B}{(3x-2)^3} + \dfrac{C}{(3x-2)^2} + \dfrac{D}{3x-2} = \boxed{\dfrac{73}{9(3x-2)^4} + \dfrac{55}{9(3x-2)^3} + \dfrac{7}{9(3x-2)}}$

$$\begin{aligned} 7x^2+9x-1 &= A + B(3x-2) + C(3x-2)^2 + D(3x-2)^3 \\ &= A + 3Bx - 2B + C(9x^2-12x+4) \\ &\quad + D(3x-2)(3x-2)(3x-2) \\ &= A + 3Bx - 2B + 9Cx^2 - 12Cx + 4C \\ &\quad + D(3x-2)(9x^2-12x+4) \\ &= A + 3Bx - 2B + 9Cx^2 - 12Cx + 4C \\ &\quad + D(27x^3 - 36x^2 + 12x - 18x^2 + 24x - 8) \\ &= 27Dx^3 + 9Cx^2 - 36Dx^2 - 18Dx^2 \\ &\quad + 3Bx - 12Cx + 12Dx + 24Dx \\ &\quad + A - 2B + 4C - 8D \\ &= 27Dx^3 + (9C - 54D)x^2 \\ &\quad + (3B - 12C + 36D)x \\ &\quad + A - 2B + 4C - 8D \end{aligned}$$

which implies

$0 = 27D$

$7 = 9C - 54D$

$9 = 3B - 12C + 36D$

$-1 = A - 2B + 4C - 8D$

$D = 0$

$C = 7/9$

$\Rightarrow$

$9 = 3B - 12(7/9)$

$-1 = A - 2B + 4(7/9)$

$\Downarrow$

$81 = 27B - 12\cdot 7$

$81 = 27B - 84$

$+84 \qquad +84$

$165 = 27B$

$B = \dfrac{165}{27} = \dfrac{5\cdot 33}{27} = \dfrac{5\cdot \cancel{3}\cdot 11}{\cancel{3}\cdot 9} = \dfrac{55}{9}$

$B = 55/9$

$\Leftarrow$

$-1 = A - 2\left(\dfrac{55}{9}\right) + 4\left(7/9\right)$

$-9 = 9A - 110 + 28$

$-9 = 9A - 82$

$73 = 9A$

$A = 73/9$

18) $$\frac{x^2}{(x^3-8)(x-2)} = \frac{x^2}{(x-2)^2(x^2+2x+4)} = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{Cx+D}{x^2+2x+4}$$

$$\boxed{= \frac{1}{3(x-2)^2} + \frac{1}{6(x-2)} - \frac{1}{6(x^2+2x+4)}}$$

$$\begin{aligned} x^2 &= A(x^2+2x+4) + B(x-2)(x^2+2x+4) + (Cx+D)(x-2)^2 \\ &= Ax^2 + 2Ax + 4A + B(x^3 + \cancel{2x^2} \cancel{+4x} \cancel{-2x^2} \cancel{-4x} - 8) \\ &\quad + (Cx+D)(x^2-4x+4) \\ &= Ax^2 + 2Ax + 4A + Bx^3 - 8B \\ &\quad + Cx^3 - 4Cx^2 + 4Cx + Dx^2 - 4Dx + 4D \end{aligned}$$

$$\begin{aligned} x^2 = &(B+C)x^3 \\ &+ (A-4C+D)x^2 \\ &+ (2A+4C-4D)x \\ &+ 4A + 4D - 8B \end{aligned}$$

$$\begin{aligned} &A - 2B - 2D = 0 \\ -&(A - 2B + D = 0) \\ \hline &-3D = 0 \\ &D = 0 \end{aligned}$$

$$\begin{aligned} B + C &= 0 \quad \text{so } B = -C \\ A - 4C + D &= 1 \\ 2A + 4C - 4D &= 0 \\ 4A + 4D - 8B &= 0 \end{aligned} \Longrightarrow \begin{aligned} A + 4B + D &= 1 \\ A - 2B - 2D &= 0 \\ A + D - 2B &= 0 \end{aligned} \Longrightarrow \begin{aligned} &A + 4B = 1 \\ -&(A - 2B = 0) \\ \hline &6B = 1 \\ &B = 1/6 \\ &C = -1/6 \end{aligned}$$

$$A - 2 \cdot 1/6 = 0$$

$$A = 1/3$$

INVERSE DIFFERENTIATION

Oct 26 2022

CHAPTER 13, p. 147

$y = 3x \Rightarrow x = y/3$

$\frac{dy}{dx} = 3 \qquad \frac{dx}{dy} = \frac{1}{3}$

$y = 4x^2 \Rightarrow x = \frac{\sqrt{y}}{2}$

$\frac{dy}{dx} = 8x \qquad \frac{dx}{dy} = \frac{1}{2} \cdot \frac{y^{-1/2}}{2} = \frac{1}{4\sqrt{y}} = \frac{1}{8x}$

$y = \sqrt{\frac{3}{x} - 1} \Rightarrow x = \frac{3}{1+y^2} = 3(1+y^2)^{-1}$

$= \left(\frac{3}{x} - 1\right)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{3}{x} - 1\right)^{-1/2} \cdot -3x^{-2} \qquad \frac{dx}{dy} = -3(1+y^2)^{-2} \cdot 2y = -\frac{2}{3}\left(\frac{3}{x} - 1\right)^{1/2} x^2$

$y = (\theta+5)^{1/3} \Rightarrow y^3 = \theta + 5$

$y^3 - 5 = \theta$

$dy/d\theta = \frac{1}{3}(\theta+5)^{-2/3} \qquad d\theta/dy = 3y^2 = 3(\theta+5)^{2/3}$

Exer. I 5, 6, 7

5) $z = \sqrt[3]{v} \qquad \frac{dz}{dv} = \frac{1}{3}v^{-2/3}$

$z^3 = v$

$dv/dz = 3z^2$

$= 3v^{2/3}$

$dz/dv = (3v^{2/3})^{-1}$

6) $y = \sqrt[3]{x^{-5}} = x^{-5/3}$

$dy/dx = -5/3\, x^{-8/3}$

$y^3 = x^{-5} = 1/x^5$

$x^5 = 1/y^3$

$x = y^{-3/5}$

$dx/dy = -3/5\, x^{8/3}$

7) $v = x^{-8/5} \qquad dv/dx = -8/5\, x^{-13/5}$

$v^{-5/8} = x \qquad dx/dv = -5/8\, v^{-13/8}$

$= -5/8\, (x^{-8/5})^{-13/8}$

$= -5/8\, x^{13/5}$

Ch. 9, Ex. 1, 2, 4

1) $y = \sqrt{a+x}$

$dy/dx = \frac{1}{2}(a+x)^{-1/2}$

$y^2 = a + x$

$y^2 - a = x$

$dx/dy = 2y = 2(a+x)^{1/2}$

2) $y = (a+x^2)^{-1/2}$

$dy/dx = -\cancel{\tfrac{1}{2}}(a+x^2)^{-3/2} \cdot \cancel{2}x$

$y^{-2} = a + x^2$

$y^{-2} - a = x^2$

$(y^{-2} - a)^{1/2} = x$

$\cancel{1/2}(y^{-2} - a)^{-1/2} \cdot (-\cancel{2})y^{-3} = x$

$\left((a+x^2)^{-1/2 \cdot -2} - a\right)^{-1/2} \cdot (a+x^2)^{-1/2 \cdot -3}$

$= x^{-1} \cdot (a+x^2)^{3/2}$

4) $y = (x^3 - a^2)^{-1/2}$

$dy/dx = -3/2\,(x^3 - a^2)^{-3/2} \cdot x^2$

$y^{-2} = x^3 - a^2$

$y^{-2} + a^2 = x^3$

$x = (y^{-2} + a^2)^{1/3}$

$dx/dy = 1/3\,(y^{-2} + a^2)^{-2/3} \cdot (-2)\, y^{-3}$

$= (-2/3)\, x^{-2} (x^3 - a^2)^{3/2}$

Exer. 6, 1, 2, 3, 4

1) $y = (x^2+1)^{1/2}$

$dy/dx = \cancel{1/2}(x^2+1)^{-1/2}\cancel{2}x$

$y^2 = x^2+1$

$y^2 - 1 = x^2$

$\sqrt{y^2-1} = x$

$dx/dy = \cancel{\tfrac{1}{2}}(y^2-1)^{-1/2}\cancel{2}y = x^{-1}(x^2+1)^{1/2}$

2) $y = \sqrt{x^2+a^2}$ $\quad dy/dx = \cancel{\tfrac{1}{2}}(x^2+a^2)^{-1/2}\cancel{2}x$

$y^2 = x^2+a^2$

$\sqrt{y^2-a^2} = x$

$\frac{dx}{dy} = \cancel{\tfrac{1}{2}}(y^2-a^2)^{-1/2}\cdot\cancel{2}y = x^{-1}(x^2+a^2)^{1/2}$

3) $y = (a+x)^{1/2}$ $\quad y^2 = a+x$

$dy/dx = \frac{1}{2}(a+x)^{-1/2}$ $\quad y^2 - a = x$

$dx/dy = 2y = 2(a+x)^{1/2}$

4) $y = a(a-x^2)^{-1/2}$ $\quad \Rightarrow (a-x^2)^{1/2} = \frac{a}{y}$

$dy/dx = \cancel{-}\frac{a}{\cancel{2}}(a-x^2)^{-3/2}\cdot(\cancel{-}\cancel{2}x)$

$= ax(a-x^2)^{-3/2}$

$a - x^2 = a^2/y^2$

$a - a^2/y^2 = x^2$

$\sqrt{a - a^2/y^2} = x$

$\cancel{\tfrac{1}{2}}(a - a^2/y^2)^{-1/2}\cdot\cancel{2}a^2y^{-3} = dx/dy$

$a^{-1}x^{-1}(a-x^2)^{3/2}$

The Calculus of Growth

Calc Made Easy, ch. 14

Examples, p. 173

1) $y = \exp(-ax)$

$y' = -a\exp(-ax)$

$\ln y = -ax$

$y' = -ay$

2) $y = \exp(x^2/3)$

$y' = \exp(x^2/3)\ 2/3\, x$

OR

$\ln y = x^2/3$

$y' = y \cdot 2x/3$

3) $y = \exp(2x/(x+1))$

$$y' = \frac{2\exp(2x/(x+1))}{(x+1)^2}$$

$\ln y = 2x/(x+1)$

$$y' = y\left[\frac{2(x+1) - 2x}{(x+1)^2}\right] = \frac{y \cdot 2}{(x+1)^2}$$

4) $y = \exp(\sqrt{x^2+a})$

$$y' = \exp(\sqrt{x^2+a})\ \frac{1}{\cancel{2}}(x^2+a)^{-1/2} \cdot \cancel{2}x$$

$\ln y = \sqrt{x^2+a}$

$$y' = y \cdot \frac{1}{\cancel{2}}(x^2+a)^{-1/2} \cdot \cancel{2}x$$

5) $y = \ln(a+x^3)$

$$y' = \frac{3x^2}{a+x^3}$$

6) $y = \ln(3x^2 + \sqrt{a+x^2})$

$$y' = (3x^2 + \sqrt{a+x^2})^{-1}\left(6x + \frac{1}{\cancel{2}}(a+x^2)^{-1/2} \cdot \cancel{2}x\right)$$

7) $y = (x+3)^2\sqrt{x-2}$

$$y' = 2(x+3)\sqrt{x-2} + (x+3)^2\tfrac{1}{2}(x-2)^{-1/2}$$

$$\frac{4(x+3)(x-2) + (x+3)^2}{(x-2)^{1/2}} = \frac{5(x+3)(x-1)}{2(x-2)^{1/2}}$$

$\ln y = 2\ln(x+3) + \tfrac{1}{2}\ln(x-2)$

$$y' = y\left[\frac{2}{x+3} + \frac{1}{2(x-2)}\right]$$

$$= y\left[\frac{4x-8+x+3}{2(x-2)(x+3)}\right] = \frac{5y}{2}\,\frac{(x-1)}{(x-2)(x+3)}$$

8) $y = (x^2+3)^3(x^3-2)^{2/3}$

$$y' = 3(x^2+3)^2\, 2x\,(x^3-2)^{2/3} + (x^2+3)^3\,\frac{2}{\cancel{3}}(x^3-2)^{-1/3}\,\cancel{3}x^2$$

$$= 2x(x^2+3)^2(x^3-2)^{-1/3}\left[3(x^3-2) + (x^2+3)\cdot x\right]$$

$$\left[3x^3 - 6 + x^3 + 3x\right]$$

$$2x(x^2+3)^2(x^3-2)^{-1/3}(4x^3+3x-6)$$

OR

$\ln y = 3\ln(x^2+3) + 2/3\ln(x^3-2)$

$$y' = y\left[\frac{3}{x^2+3}\cdot 2x + \frac{2}{\cancel{3}(x^3-2)}\cdot\cancel{3}x^2\right]$$

$$= 2xy\left[\frac{3(x^3-2) + x(x^2+3)}{(x^2+3)(x^3-2)}\right]$$

$$= 2xy\,\frac{(4x^3+3x-6)}{(x^2+3)(x^3-2)}$$

9) $y = \frac{(x^2+a)^{1/2}}{(x^3-a)^{1/3}}$

$\ln y = \frac{1}{2}\ln(x^2+a) - \frac{1}{3}\ln(x^3-a)$

$y' = y\left[\frac{1}{\cancel{2}(x^2+a)}\cdot\cancel{2}x - \frac{1}{\cancel{3}(x^3-a)}\cancel{3}x^2\right]$

$= y\left[\frac{x(x^3-a) - x^2(x^2+a)}{(x^2+a)(x^3-a)}\right]$

$= y\left[\frac{-xa - x^2a}{(x^2+a)(x^3-a)}\right]$

$= \frac{-xay(1-x)}{(x^2+a)(x^3-a)}$

$= \frac{-ax(1-x)}{(x^2+a)^{1/2}(x^3-a)^{4/3}}$

10) $y = \frac{1}{\ln x}$

sometimes written $\ln^2 x$

$y' = \frac{-1}{(\ln(x))^2}\cdot\frac{1}{x}$

11) $y = (\ln x)^{1/3}$

$y' = 1/3\,(\ln x)^{-2/3}\,\frac{1}{x}$

$= \frac{1}{3x(\ln x)^{2/3}}$

12) $y = a^{-ax^2}$

$\ln y = -ax^2 \ln a$

$y' = y \cdot -2a\ln a\, x$

$y' = -2a^{1-ax^2}\ln a\, x$

Further Examples, p. 171

1) $C = \frac{E}{R}\left(1 - \exp(-Rt/L)\right)$

at $t = \frac{L}{R}$, C is $\left(1-\frac{1}{e}\right)$ times the original amount...

2) $I = I_0 \exp(-K\ell)$

$2 = \exp(-K\ell_{1/2})$

$\ell_{1/2} = \frac{\ln 2}{-K}$

3) $Q = Q_0 \exp(-\lambda t)$

$\frac{1}{2} = \exp(-\lambda t_{1/2})$

$\ln(1/2) = -\lambda t_{1/2}$

$t_{1/2} = \frac{\ln(1/2)}{-\lambda}$

Exponential Derivatives Practice

CALC MADE EASY, Exercises 12, p.166

1) $y = b(e^{ax} - e^{-ax})$

$$\frac{dy}{dx} = abe^{ax} + abe^{-ax}$$
$$= abe^{-ax}(e^{2ax} + 1)$$

2) $v = at^2 + 2\ln t$

$$\frac{dv}{dt} = 2at + 2t^{-1}$$
$$= 2t^{-1}(at^2 + 1)$$

3) $y = n^t \qquad \ln y = t\ln n$

$$\frac{d\ln y}{dt} = \ln n$$

4) $y = \frac{1}{b}\frac{a^{bx}}{\ln a}$

$$\frac{dy}{dx} = \frac{b\cdot\ln a\cdot a^{bx}}{b\cdot\ln a} = a^{bx}$$

5) $w = pv^n$

$$\frac{dw}{dv} = npv^{n-1}$$

6) $y = \ln x^n$

$$\frac{dy}{dx} = \frac{1}{x^n}\cdot n\cdot x^{n-1} = \frac{n}{x}$$

7) $y = 3e^{-\frac{x}{x-1}}$

$$\frac{dy}{dx} = 3e^{-\frac{x}{x-1}}\cdot\frac{-(x-1)+x}{(x-1)^2} = \frac{3}{(x-1)^2}e^{-\frac{x}{x-1}}$$

8) $y = (3x^2+1)e^{-5x}$

$$\frac{dy}{dx} = 6x\cdot e^{-5x} - 5e^{-5x}(3x^2+1)$$

9) $y = \ln(x^a + a)$

$$\frac{dy}{dx} = \frac{1}{x^a+a}\cdot a\cdot x^{a-1}$$

10) $y = (3x^2-1)(\sqrt{x}+1)$

$$= 3x^2\sqrt{x} + 3x^2 - \sqrt{x} - 1$$
$$\frac{dy}{dx} = 7.5x^{1.5} + 6x - \tfrac{1}{2}x^{-1/2}$$

11) $y = \frac{\ln(x+3)}{x+3}$

$$\frac{dy}{dx} = \frac{\frac{x+3}{x+3} - \ln(x+3)}{(x+3)^2} = \frac{1-\ln(x+3)}{(x+3)^2}$$

12) $y = a^x x^a$

$$= a^x\ln a\,x^a + a^x a x^{a-1}$$
$$= a^x x^{a-1}[x\ln a + a]$$

13) $s = ay^2\ln\left(\frac{1}{y}\right)$

$$\frac{ds}{dy} = a\left[2y\ln\left(\tfrac{1}{y}\right) - \cancel{y^2}\cdot y\cdot\cancel{y^{-2}}\right]$$
$$= ay[2\ln(1/y) - 1] = 0$$
$$\ln(1/y) = 1/2$$
$$1/y = \exp(1/2)$$
$$\boxed{y = e^{-1/2}}$$

14) $y = x^3 - \ln x$

$$\frac{dy}{dx} = 3x^2 - \frac{1}{x} = 0$$

$$3x^3 - 1 = 0$$

$$x^3 = 1/3$$

so min is $\hat{x} = (1/3)^{1/3} \approx 0.693$

15) $y = \ln(a x e^x)$

$$= \ln(a) + \ln(x) + \ln(e^x)$$

$$= \ln(a) + \ln(x) + x$$

$$\frac{dy}{dx} = \frac{1}{x} + 1$$

16) $y = (\ln(ax))^3$

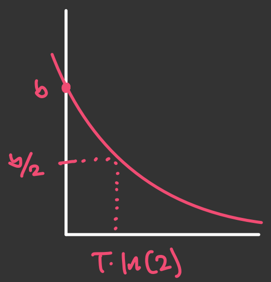
$$\frac{dy}{dx} = \frac{3\ln(ax)^2}{\cancel{a}x} \cdot \cancel{a} = 3/x \ln(ax)^2$$

More Growth Exercises

Calc Made Easy Ex. 13, p. 173

1) $y = b\exp(-t/T)$

$= b\exp\left(\frac{-t\ln(2)}{t_{1/2}}\right)$



2) $t_{1/2} = 24$

$y = b\exp(-t\ln(2)/24)$

$1/100 = \exp\left(-\frac{t_{1/100}\ln(2)}{24}\right)$

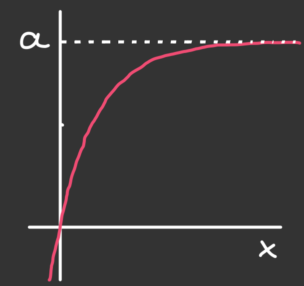
$\ln(10^{-2}) = -\frac{t_{1/100}\ln(2)}{24}$

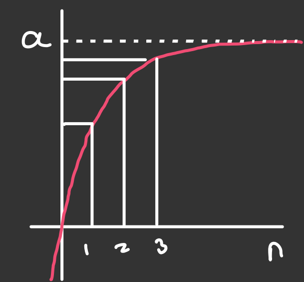
$t_{1/100} = 2\cdot 24\log_2(10) \approx 159.45$

3) $y = a(1-\exp(-2t))$

$= a(1-(1/2)^n)$

$n = \log_2(\exp(bx))$





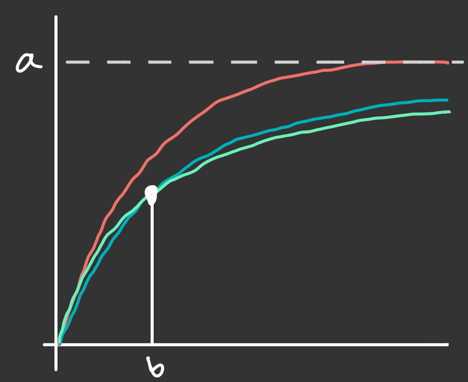
4) $y = \frac{2a}{\pi}\arctan(x/b)$

$y = a(1-\exp(-x/b))$

$= a(1-(1/2)^n)$; $x = \ln(1/2^n)b$

$y = \frac{ax}{x+b} = \frac{an}{n+1}$; $n = x/b$

Also see Ex. 14, #11



5) $y = x^x$

$= \exp(x\cdot\ln(x))$

$y' = x^x\cdot(\ln(x)+1)$

$y = (e^x)^x$
$= e^{x^2}$

$y' = e^{x^2}\cdot 2x$

$y = e^{x^x} = e^u$

$y' = e^u\cdot u' = e^{x^x}\cdot x^x(\ln(x)+1)$

6) $Q = Q_0\exp(-\lambda t)$

$\frac{1}{2}\cancel{Q_0} = \cancel{Q_0}\exp(-\lambda t)$

$\ln(1/2) = -\lambda t_{1/2}$

$t_{1/2} = \ln(1/2)/-\lambda$

if $\lambda = 5$, $t_{1/2} \approx 0.14$ sec

7) $V = V_0 \exp(-t/KR)$

$V_0 = 20$
$K = 4 \times 10^{-6}$
$R = 10^4$

$V(0.1) = 1.64$
$V(0.01) = 15.58$

8) $Q = Q_0 \exp(-\mu t)$

t	Q
0	20
10	16
31.1	10

$16 = 20 \exp(-\mu \cdot 10)$

$\ln\left(\frac{16}{20}\right) = -10\mu$

$\mu = \ln(4/5)/-10 \approx 0.022$

$\mu = 0.00037?$ answer key is wrong!

$\frac{1}{2}\cancel{Q_0} = \cancel{Q_0} \exp(-\mu t_{1/2})$

$\ln(1/2) = -\mu\, t_{1/2}$

$t_{1/2} = \dfrac{\ln(1/2) \cdot 10}{\ln(4/5)}$

9) $i = i_0 \exp(-\beta l)$ $\beta = 0.0114$

l if $i/i_0 = 0.08$

$0.08\cancel{i_0} = \cancel{i_0} \exp(-\beta l)$

$-\dfrac{\ln(0.08)}{0.0114} = l$

$221.6 \approx l$

$\alpha = i/i_0$ if $l = 40$

$\alpha \cancel{i_0} = \cancel{i_0} \exp(\beta \cdot l)$

$\alpha = \exp(-40 \cdot 0.0114)$

≈ 0.63

10) $p = p_0 \exp(-kh)$

$\dfrac{\ln\left(\frac{p}{760}\right)}{-h} = k$

h	p	k
0	760	
10	199.2	0.133
20	42.4	0.144
50	0.32	0.155

$\left.\begin{matrix}0.133\\0.144\\0.155\end{matrix}\right\} \Rightarrow \bar{k} = 0.144$

11) $y = x^x$

$y' = x^x(\ln x + 1) = 0$

$\hat{x} = \exp(-1)$

12) $y = x^{1/x} = \exp\left(\frac{1}{x}\ln x\right)$

$y' = \exp\left(\frac{1}{x}\ln x\right)\cdot\left(-x^{-2}\ln x + \frac{1}{x}\frac{1}{x}\right)$

$= x^{1/x}\, x^{-2}(1 - \ln x)$

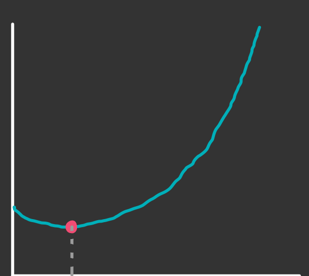
$\hat{x} = \exp(1)$

13) $y = x a^{1/x}$, $a > 1$

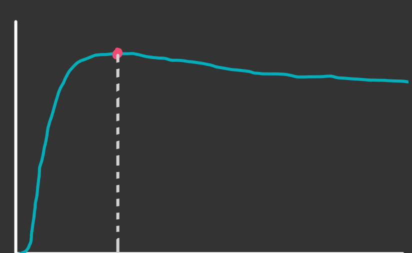
$y' = a^{1/x} - x a^{1/x} \ln a\, x^{-2}$

$= a^{1/x}(1 - x^{-1}\ln a)$

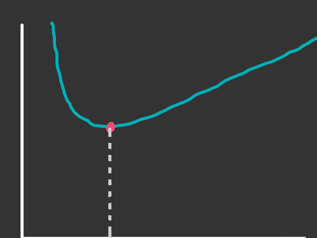
$\hat{x} = \ln a$



e^{-1}



e^1



$\ln a$

THE CALCULUS OF TRIGONOMETRY

Calc Made Easy Ch. 15, p. 175

1) $y = \arcsin x$

$x = \sin y$

$dx/dy = \cos y = \sqrt{1-x^2}$

$dy/dx = (1-x^2)^{-1/2}$

2) $y = \cos^3\theta$

$dy/dx = 3\cos^2\theta \cdot (-\sin\theta)$

3) $y = \sin(x+a)$

$dy/dx = \cos(x+a)$

4) $y = \ln \sin\theta$

$y' = \frac{1}{\sin\theta} \cdot \cos\theta = \cot\theta$

5) $y = \cot\theta = \frac{\cos\theta}{\sin\theta}$

?

$y' = \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} = -(1+\cot^2\theta) = -\csc^2\theta$

6) $y = \tan 3\theta$

$y' = \sec^2 3\theta \cdot 3$

7) $y = \sqrt{1+3\tan^2\theta}$

$y' = \frac{1}{2}(1+3\tan^2\theta)^{-1/2} \cdot 6\tan\theta \cdot \sec\theta$

8) $y = \sin x \cos x$

$y' = \cos x \sin x - \sin^2 x$

$= \cos^2 x - \sin^2 x$

$$\sin M - \sin N = 2\cos\left(\frac{M+N}{2}\right)\sin\left(\frac{M-N}{2}\right)$$

$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$\cos^2\theta + \sin^2\theta = 1$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \sinh x / \cosh x$

$\cosh^2 x - \sinh^2 x = 1$

$d(\sinh x)/dx = \cosh x$

$d(\cosh x)/dx = \sinh x$

$d(\tanh x)/dx = 1/\cosh^2 x$

$d \sin x/dx = \cos x$

$d \cos x/dx = -\sin x$

$d \tan x/dx = \sec^2 x$

$\sec x = 1/\cos x$

$\csc x = 1/\sin x$

$\cot x = 1/\tan x$

$\arcsin x = \sin^{-1} x$

$\arccos x = \cos^{-1} x$

$\arctan x = \tan^{-1} x$

if $\theta = \arccos x$, $\sin\theta = \sqrt{1-x^2}$ for $x \in [-1,1]$

if $\theta = \arcsin x$, $\cos\theta = \sqrt{1-x^2}$ for $x \in [-1,1]$

$\sec^2 x = 1 + \tan^2 x$

if $\theta = \arctan x$, then

$\sin\theta = \frac{x}{\sqrt{1+x^2}}$ & $\cos\theta = \frac{1}{\sqrt{1+x^2}}$

Trig Derivative Exercises

CALC MADE EASY Ex. 14, p.183

1) $y = A\sin\left(\theta - \frac{\pi}{2}\right)$

$y' = A\cos(\theta - \pi/2)$

$y = \sin^2\theta$

$y' = 2\cos\theta\,\sin\theta$

$= \sin\ 2\theta$

$y = \sin 2\theta$

$y' = 2\cos 2\theta$

$y = \sin^3\theta$

$= 3\sin^2\theta \cdot \cos\theta$

$y = \sin 3\theta$

$= 3\cos 3\theta$

2) $y = \sin\theta\cos\theta$

$y' = \cos^2\theta - \sin^2\theta = 0$ if $\sin\theta = \cos\theta$

or

$\tan\theta = 1$

$\hat{\theta} = \tan^{-1}(1) = \pi/4 + 2\pi N$

3) $y = \dfrac{\cos(2\pi n t)}{2\pi}$

$y' = \dfrac{-\sin(2\pi n t)\ \cancel{2\pi} n}{\cancel{2\pi}}$

4) $y = \sin(a^x)$

$y' = \cos(a^x)\, a^x \ln a$

5) $y = \ln(\cos(x))$

$y' = \dfrac{-\sin(x)}{\cos(x)} = -\tan(x)$

6) $y = a\sin(x+b)$

$y' = a\cos(x+b)$

7) $y = a\sin(x-b)$

$y' = a\cos(x-b) = 0$ if $x = \cos^{-1}(0) + b$
$= \pi/2 + b$

$y'' = -a\sin(x-b) = 0$ if $x = \sin^{-1}(0) + b$
$= b$

$y'(b) = a$

$a = 100 \quad b = 15°$

$x = 75°$

$y' = 100\cos(75-15) = 50$

8) $y = \sin\theta \sin 2\theta$

$y' = \cos\theta \sin 2\theta + \sin\theta \cos 2\theta \cdot 2$

9) $y = a\tan^n(\theta^n)$

$y' = an\tan^{n-1}(\theta^n)\sec^2(\theta)\cdot n\theta^{n-1}$

10) $y = e^x \sin^2 x$

$y' = e^x \sin^2 x + e^x 2\sin x \cos x$
$= e^x \sin x(\sin x + 2\cos x)$
$= e^x(\sin^2 x + \sin 2x)$

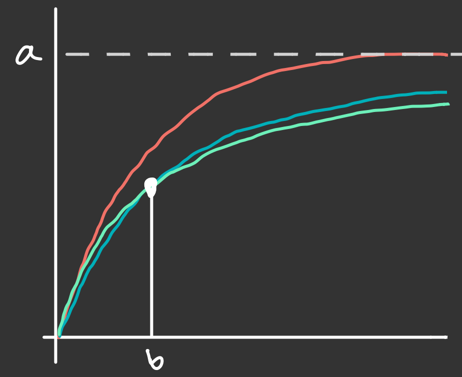
$\sin 2x = 2\sin x \cos x$

11) $y=\frac{2a}{\pi}\arctan(x/b)$

$y=a(1-\exp(-x/b))$

$=a(1-(\frac{1}{2})^n)$; $x=\ln(1/2^n)b$

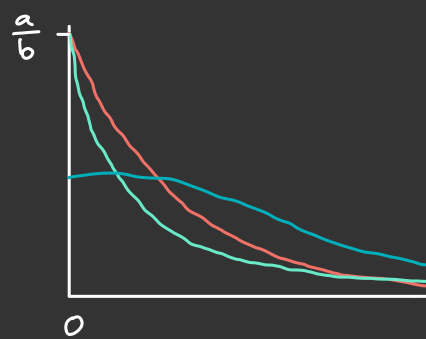
$y=\frac{ax}{x+b}=\frac{an}{n+1}$; $n=x/b$



$y'=\frac{2a}{\pi b(1+x^2)}$

$y'=a/b\exp(-x/b)$

$y'=\frac{ab}{(x+b)^2}$



12) $y=\sec x=\frac{1}{\cos x}$

$y'=\frac{\sin x}{\cos^2 x}$

$=\tan x\sec x$

$y=\arccos x$

$x=\cos y$

$\frac{dx}{dy}=-\sin y$

$y'=\frac{1}{-\sin y}=\frac{-1}{\sin(\arccos(x))}$

$=-(1-x^2)^{-1/2}$

$y=\operatorname{arcsec} x$

$x=\sec y$

$\frac{dx}{dy}=\tan y\cdot x$

$y'=\frac{1}{x\tan(\operatorname{arcsec} x)}$

$=\frac{1}{x\sqrt{x^2-1}}$

$y=\arctan x$

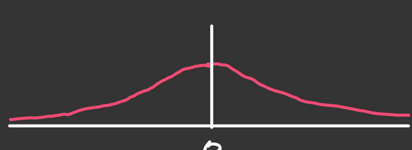
$x=\tan y$

$dx/dy=\sec^2 y$

$y'=\frac{1}{\sec^2 y}=\frac{1}{\sec^2(\arctan(x))}$

$=\cos^2(\arctan x)$

$=\frac{1}{x^2+1}$



$y=\tan x\sqrt{3\sec x}$

$y'=\sec^2 x\cdot\sec^{1/2}x\cdot\sqrt{3}$

$+\tan x\,\frac{1}{2}(3\sec x)^{-1/2}\cdot 3\tan x\sec x$

$=\sqrt{3}\cdot\sec^{5/2}x+\frac{3\tan^2 x\sec x}{\sqrt{3}\cdot 2\cdot\sec^{1/2}x}$

$=\sqrt{3}\sec^{1/2}x\left(\sec^2 x+\frac{\tan^2 x}{2}\right)$

$=\frac{\sqrt{3\sec x}\,(3\sec^2 x-1)}{2}$

using $\sec^2 x = 1+\tan^2 x$

13) $y = \sin(a\theta + b)^c$

$y' = c\sin(a\theta+b)^{c-1}\cos(a\theta+b)\,a$

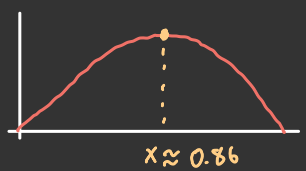
14) $y = \theta^3 + 3\sin(\theta+3) - 3^{\sin\theta} - 3^{\theta}$

$y' = 3\theta^2 + 3\cos(\theta+3) - 3^{\sin\theta}\ln 3 \cdot \cos\theta - 3^{\theta}\ln 3$

15) $y = \theta\cos\theta$

$y' = \cos\theta - \theta\sin\theta = 0$ if $\theta = \cot\theta$

$\theta \approx \pm 0.86$



PARTIAL DIFFERENTIATION

1) $w = 2ax^2 + 3bxy + 4cy^3$

$\partial w/\partial x = 4ax + 3by$

$\partial w/\partial y = 3bx + 12cy^2$

HENCE:

$dw = (4ax + 3by)\,dx + (3bx + 12cy^2)\,dy$

2) $z = x^y$ $\quad \partial z/\partial x = yx^{y-1}$ $\quad \partial z/\partial y = x^y \cdot \ln x$

so

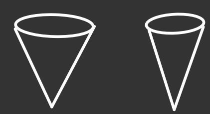
$dz = yx^{y-1} \cdot dx + x^y \ln x \, dy$

$= x^{y-1}(y\,dx + x \ln x \, dy)$

3) $V = \frac{1}{3}\pi r^2 h$

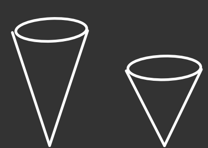
If radius varies for a fixed height:

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h$$



If height varies for a fixed radius:

$$\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$



If both height & radius change:

$$dV = \frac{\partial V}{\partial r} \cdot dr + \frac{\partial V}{\partial h} dh$$

$$= \frac{2}{3}\pi r h \, dr + \frac{1}{3}\pi r^2 dh$$

$$= \frac{1}{3}\pi r\left(2h\,dr + r\,dh\right)$$

$$\frac{dV}{dr} = \frac{1}{3}\pi r\left(2h + r\frac{dh}{dr}\right)$$

$$\frac{dV}{dh} = \frac{1}{3}\pi r\left(2h\frac{dr}{dh} + r\right)$$

4) $y = F(w) + f(v)$ generic, differentiable functions

$w = x + at$

$v = x - at$

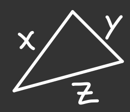
$$\frac{\partial y}{\partial x} = \frac{\partial F}{\partial w} + \frac{\partial f}{\partial v} = F'(w) + f'(v)$$

$$\frac{\partial y}{\partial t} = \frac{\partial F}{\partial w} \cdot a - \frac{\partial f}{\partial v} a$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 F}{\partial w^2} + \frac{\partial^2 f}{\partial v^2}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 F}{\partial^2 w} \cdot a^2 + \frac{\partial^2 F}{\partial^2 v} a^2$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 \cdot \frac{\partial^2 y}{\partial x^2}$$

5) 

$x+y+z=2s$, for some constant s

$$A=\sqrt{s(s-x)(s-y)(s-z)}$$
$$=\sqrt{s(s-x)(s-y)(s-(2s-x-y))}$$
$$=\sqrt{s(s-x)(s-y)(-s+x+y)}$$
$$P=(s-x)(s-y)(x+y-s)$$

max when: $\partial P/\partial x=0$ & $\partial P/\partial y=0$

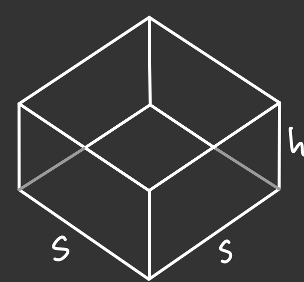
$$\frac{\partial P}{\partial x}=-(s-y)(x+y-s)+(s-x)(s-y)$$
$$=(s-y)\left[s-x-x-y+s\right]$$
$$=(s-y)(2(s-x)-y)=0$$
$$\frac{\partial P}{\partial y}=(s-x)(2(s-y)-x)=0$$

if $s=y$ or $y=2(s-x)$
if $s=x$ or $y=s-x/2$

$$2s-2x=s-\frac{x}{2}$$
$$-s \qquad -s$$
$$s=\frac{3x}{2}$$

$x=2s/3$
$y=2s/3$
$z=2s/3$

6) For a fixed V, what are the dimensions of a lidless box that minimize its surface area?



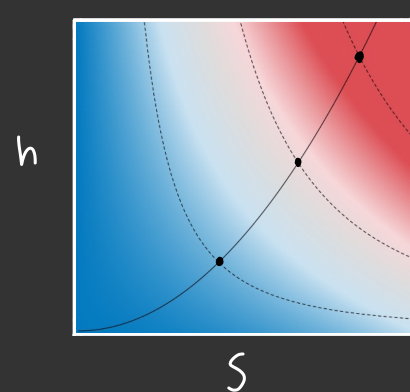
By symmetry, we can assume the base must be a square.

$V=s^2h$ iso-volumes: $h=V/s^2$

$A=4sh+s^2$ iso-areas: $h=\frac{A-s^2}{4s}$

$$A=4\frac{V}{s^{\cancel{2}}}\cdot\cancel{s}+s^2$$

$$\frac{\partial A}{\partial s}=-\frac{4V}{s^2}+s^2=0 \quad \text{if } s^2=2\sqrt{V}$$



← shading indicates surface area (which we want to minimize)

Which implies

$$h=\frac{V}{s^2}=\frac{V}{2\sqrt{V}}=\frac{\sqrt{V}}{2}$$

$$h=\left(\frac{s}{2}\right)^2$$

Dotted lines show iso-volumes. For any given volume, the min. surface area is at $h=(s/2)^2$.

PARTIAL DIFFERENTIATION PRACTICE

CALC MADE EASY EXERCISES 15, PAGE 189

1) $z = \frac{x^3}{3} - 2x^3y - 2y^2x + y/3$

$\partial z/\partial x = x^2 - 6x^2y - 2y^2$

$\partial z/\partial y = -2x^3 - 4yx + 1/3$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= (x^2 - 6x^2y - 2y^2)\,dx + (-2x^3 - 4yx + 1/3)\,dy$$

2) $A = x^2yz + xy^2z + xyz^2 + x^2y^2z^2$

$\partial A/\partial x = 2xyz + y^2z + yz^2 + 2xy^2z^2$

$\partial A/\partial y = x^2z + 2xyz + xz^2 + 2x^2yz^2$

$\partial A/\partial z = x^2y + xy^2 + 2xyz + 2x^2y^2z$

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz$$

3) $r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$

$\partial r/\partial x = \cancel{\tfrac{1}{2}}(r^2)^{-1/2} \cdot \cancel{2}(x-a) = \left(\frac{x-a}{r}\right)$

so by symmetry, the gradient is

$\nabla r = \left(\frac{x-a}{r}, \frac{y-b}{r}, \frac{z-c}{r}\right); r \neq 0$

gradients are vector fields

Hence the requested sum is

$\partial r/\partial x + \partial r/\partial y + \partial r/\partial z = r^{-1}\left((x-a) + (y-b) + (z-c)\right)$

Interpretation: the directional derivative along the vector $(1,1,1)$

The second partial derivatives:

$$\partial^2 r/\partial x^2 = \frac{r - (x-a)\frac{\partial r}{\partial x}}{r^2} = \frac{1}{r} - \frac{(x-a)^2}{r^3}$$

so

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{3}{r} - \frac{1}{r^3}\left((x-a)^2 + (y-b)^2 + (z-c)^2\right)$$

$$= \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

This expression is the Laplacian of r, a.k.a. $\nabla^2 r$ or Δr

∇ = "nabla" or "del"; $\nabla = \left(\partial/\partial x, \partial/\partial y, \dots\right)$

4) Find the total derivative of $y = u^v$.

$y = u^v$

$\frac{\partial y}{\partial u} = vu^{v-1}$ $\quad \frac{\partial y}{\partial v} = u^v \ln u$

$dy = vu^{v-1}\,du + u^v \ln u\,dv$;

hence $\frac{dy}{du} = vu^{v-1} + u^v \ln u \frac{dv}{du}$

$\frac{dy}{dv} = vu^{v-1}\frac{du}{dv} + u^v \ln u$

note: for some function $f(x_1, x_2, \dots, x_n)$ where each x_i is a function of t, the total derivative is

$$\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{dx_i}{dt}$$

a.k.a. the multivariate chain rule

conceptually: total derivative = gradient × velocity vector

not exactly the same as a directional derivative!

there's also a "total differential" form:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

5) $y = u^3 \sin v$ $\quad \dfrac{\partial y}{\partial u} = 3u \sin v$ $\quad \dfrac{\partial y}{\partial v} = u^3 \cos v$

$y = (\sin x)^u$ $\quad \dfrac{\partial y}{\partial u} = (\sin x)^u \ln(\sin(x))$ $\quad \dfrac{\partial y}{\partial x} = u(\sin x)^{u-1} \cos x$

$y = \dfrac{\ln u}{v}$ $\quad \dfrac{\partial y}{\partial u} = \dfrac{1}{uv}$ $\quad \dfrac{\partial y}{\partial v} = -\dfrac{\ln u}{v^2}$

6) $x \cdot y \cdot z = K$ $\quad \dfrac{\partial m}{\partial x} = 1$ $\quad \dfrac{\partial m}{\partial y} = 1$ $\quad \dfrac{\partial z}{\partial z} = 1$

$x + y + z = m$

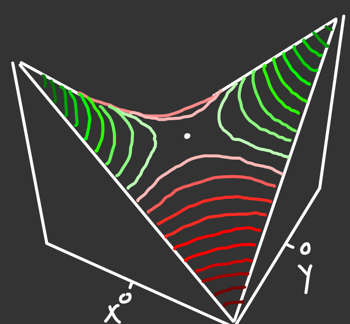
$dm = dx + dy + dz$; $\quad \dfrac{dm}{dx} = 1 + \dfrac{dy}{dx} + \dfrac{dz}{dx} = 1 - \dfrac{K}{zx^2} - \dfrac{K}{yx^2}$

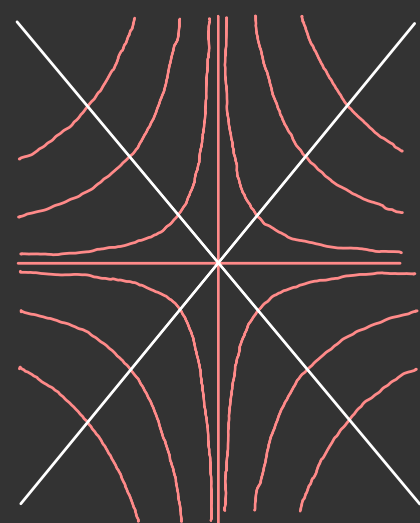
7) $u = -ax + xy - by$ $\quad$ isoclines at: $y = \dfrac{u + ax}{x - b}$

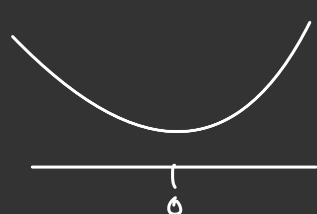
$\dfrac{\partial u}{\partial x} = -a + y = 0 \qquad y = a$

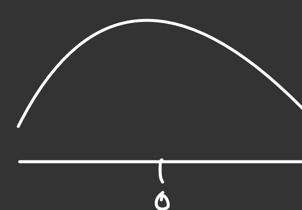
$\dfrac{\partial u}{\partial y} = -b + x = 0 \qquad x = b$

point (a,b) is a <u>saddle</u>

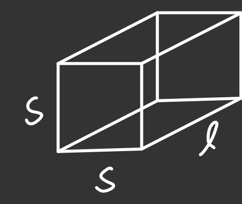








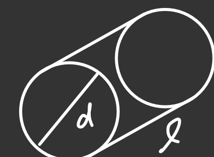
8) Maximize $V = s^2 \ell$ for a given $D = \ell + 4s$.



$$V = s^2 \ell = s^2 (D - 4s) \qquad 6s = \ell + 4s$$

$$\frac{\partial V}{\partial s} = 2sD - 12s^2 = 0 \quad \text{so} \quad 6s = D \;\;\&\;\; 3\ell = D$$
$$\ell = 2s$$

Maximize $V = \pi (d/2)^2 \ell$ for a given $S = \pi d + \ell$.

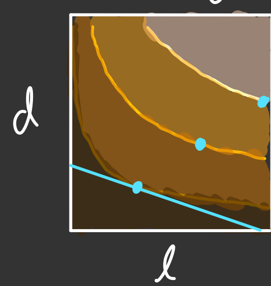


$$V = \frac{\pi d^2}{4}(S - \pi d) = \frac{\pi d^2 S}{4} - \frac{\pi^2 d^3}{4}$$

$$\frac{\partial V}{\partial d} = \frac{\pi d S}{2} - \frac{3\pi^2 d^2}{4} = 0 \quad \text{if } d = \frac{2S}{3\pi}$$

So $S = 3\ell$ & $\ell = \frac{\pi d}{2}$ or $\ell = \pi r$

For a given S, we find the maximum V along $d = 2\ell/\pi$



9) $\pi = a + b + c$
$s = \sin(a)\sin(b)\sin(c)$

$$\frac{\partial s}{\partial a} = \sin(a)\sin(2b + a) = 0 \quad \text{if } a = \{0, \pi\} \text{ or } 2b + a = \pi$$
$$a = \pi - 2b$$

$$\frac{\partial s}{\partial b} = \sin(b)\sin(2a + b) = 0 \quad \text{if } b = \{0, \pi\} \text{ or } 2a + b = \pi \quad a = \frac{\pi - b}{2}$$
$$b = \pi - 2a$$

$\pi - 2b = \frac{\pi - b}{2}$ $\qquad$ $2\pi - 4b = \pi - b$ $\qquad$ $\pi = 3b$ $\qquad$ $b = \pi/3$ so by symmetry, $a = b = c = \pi/3$

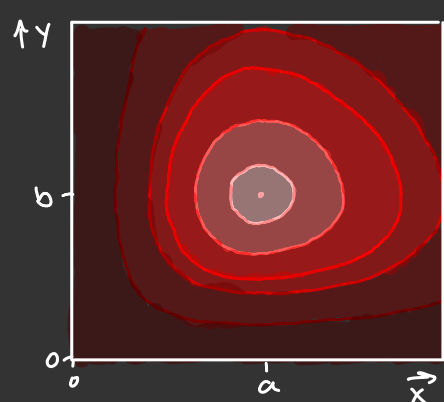
$-\pi + b \quad -\pi + b$

$\pi - 3b = 0$

10) $$u = \frac{\exp(x/a + y/b)}{xy}$$

$$\frac{\partial u}{\partial x} = \frac{(x-a)\exp(x/a + y/b)}{y x^2} = \frac{(x-a)}{x} \cdot u$$

$$\frac{\partial u}{\partial y} = \frac{(x-b)\exp(x/a + y/b)}{x y^2} = \frac{(y-b)}{y} \cdot u$$



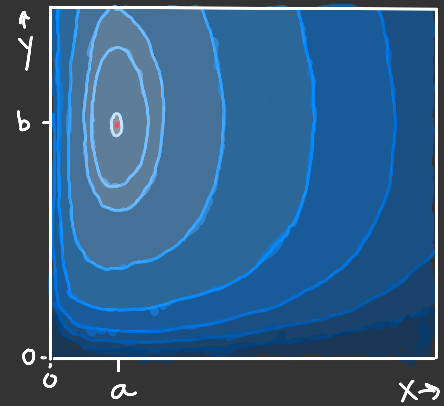
The critical point is (a, b), which by inspection is a minimum.

11) $u = y + \frac{x}{a} - b\ln y - \ln x$

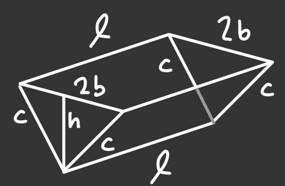
$$\frac{\partial u}{\partial x} = \frac{1}{a} - \frac{1}{x} = 0 \text{ if } x = a$$

$$\frac{\partial u}{\partial y} = 1 - \frac{b}{y} = 0 \text{ if } y = b$$

The point $(1/a, b)$ is a minimum of u.



12)



$S = 2lc + 2bh$
$V = bhl$

Our goal is to find the values of $b, h, \& \, l$ which minimize S, for a given V.

First, let's express l & c in terms of V, b & h:

$$c = \sqrt{b^2 + h^2} \qquad l = V/bh$$

So then:

$$S = \frac{2V}{bh}\sqrt{b^2+h^2} + 2bh$$

Because b & h are exchangable in S, by symmetry, $\hat{b} = \hat{h}$ at the joint min. Thus,

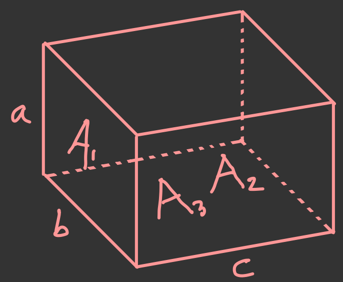
$$S = \frac{2V}{h^2}\sqrt{2h^2} + 2h^2$$
$$= \frac{2\sqrt{2}V}{h} + 2h^2$$

$$\frac{dS}{dh} = \frac{-2\sqrt{2}V}{h^2} + 4h = 0 \text{ if } \hat{h} = \left(\frac{\sqrt{2}V}{2}\right)^{1/3} = \hat{b}.$$

This implies $\hat{l} = \sqrt[3]{2V}$

Partial Differentiation Practice

Calc Made Easy, ch 16



$V = abc$
$A_1 = ab$
$A_2 = ac$
$A_3 = bc$

For any given V, we want to find the values of a, b, & c that minimize $2A_1 + 2A_2 + A_3$.

$$2A_1 + 2A_2 + A_3 \qquad c = \frac{V}{ab}$$

$$w = 2ab + 2ac + bc$$

$$= 2ab + \frac{2\cancel{a}V}{\cancel{a}b} + \frac{\cancel{b}V}{a\cancel{b}}$$

$$\frac{\partial w}{\partial a} = 2b - Vb^{-2} \rightarrow 2\hat{b} = V\hat{b}^{-2}$$

$$2\hat{b}^3 = V$$

$$\hat{b} = \sqrt[3]{V/2}$$

$$\frac{\partial w}{\partial b} = 2a - Va^{-2} \rightarrow 2\hat{a} = V\hat{a}^{-2}$$

$$2\hat{a}^3 = V$$

$$\hat{a}^3 = V/2$$

$$\hat{a} = \sqrt[3]{V/2}$$

$$V = \sqrt[3]{V/2}\,\sqrt[3]{V/2}\,\hat{c}$$

$$\frac{V}{V^{2/3}} = \frac{(V/2)^{2/3}\hat{c}}{V^{2/3}}$$

$$V^{1/3} = \hat{c}/2^{2/3}$$

$$\hat{c} = 2^{2/3}V^{1/3}\frac{2^{1/3}}{2^{1/3}} = 2\sqrt[3]{V/2}$$

So, the area-minimizing dimensions are

$$\left(\sqrt[3]{V/2},\ \sqrt[3]{V/2},\ 2\sqrt[3]{V/2}\right)$$

Integration Practice

Calc. Made Easy, Ch. 18

$$\frac{dy}{dx} = ax^b \quad y = \int \frac{dy}{dx}\,dx = \int ax^b\,dx = a\int x^b\,dx = \frac{a}{b+1}x^{b+1} + C$$

$$\frac{dy}{dx} = 24x^{11} \Rightarrow y = \int 24x^{11}\,dx = 2x^{12} + C$$

$$\int (a+b)(x+1)\,dx = \left(\frac{a+b}{2}\right)x^2 + (a+b)x + C$$

$$\int g t^{1/2}\,dt = \frac{2g}{3}t^{3/2} + C$$

$$\int (x^3 - x^2 + x)\,dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

$$\int ax^b\,dx = \left(\frac{a}{b+1}\right)x^{b+1}dx \Rightarrow \text{if } a = 9.75,\ b = 2.25,\ y = 3x^{3.75}$$

$$\int ax^{-1}\,dx = a\ln x + C \quad \text{if } x > 0$$

$$\int (x+1)(x+2)\,dx = \int (x^2 + 3x + 2)\,dx = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C$$

Since $y = x(\ln x - 1) \Rightarrow \frac{dy}{dx} = \ln x \cancel{+ \frac{x}{x} - 1}$
it follows that $\int \ln x\,dx = x(\ln x - 1)$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

Some Easy Series

Calc Made Easy, Ex. 16, p. 197

1. $S = 2/3 + 1/3 + 1/6 + 1/12 + 1/24 + \dots$

$$= 2/3\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) = 2/3\,A$$

$$= 2/3\left(1 + \frac{1}{2}\underbrace{\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)}_{A}\right)$$

$A = 1 + \frac{1}{2}A$

$A = 2$ so $S = 4/3$

2. $S = 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 - \dots =$

$$= \sum_{i=1}^{\infty} (-1)^{i-1} \cdot \frac{1}{i}$$

This series is convergent, b/c $|a_{n+1}| < |a_n|$, and $\lim_{n\to\infty} a_n = 0$.

In general, $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \ln(1+x)$

so here $x = 1$ & $S = \ln(2) \approx 0.69$

3. $\ln(1.3) = 0.3 - \frac{0.3^2}{2} + \frac{0.3^3}{3} - \frac{0.3^4}{4} + \dots$

≈ 0.2625

(actually ≈ 0.2624)

4. $y' = \frac{1}{4}x$

so $y = \frac{1}{8}x^2 + C$

5. $y' = 2x + 3$

$y = x^2 + 3x + C$

Integration by Power Rule

Calc Made Easy, Exercises 17, p. 209

1. $\int \sqrt{4ax}\,dx = \frac{4}{3}\sqrt{a}\,x^{3/2} + C$

2. $\int \frac{3}{x^4}\,dx = -x^{-3} + C$

3. $\int a^{-1}x^3\,dx = \frac{1}{4a}x^4 + C$

4. $\int (x^2+a)\,dx = \frac{1}{3}x^3 + ax + C$

5 $\int 5x^{-7/2}\,dx = -2x^{-5/2} + C$

6. $\int (4x^3+3x^2+2x+1)\,dx = x^4 + x^3 + x^2 + x + C$

7. $\int \left(\frac{ax}{2} + \frac{bx^2}{3} + \frac{cx^3}{4}\right)dx = \frac{a}{4}x^2 + \frac{b}{9}x^3 + \frac{c}{16}x^4 + C$

8. $\int \left(\frac{x^2+a}{x+a}\right)dx = \int \left(\frac{a^2+a}{x+a} - a + x\right)dx$

$= \ln|x+a|\,(a^2+a) - ax + \frac{1}{2}x^2 + C$

$$\begin{array}{r|l} & x - a \\ \hline x+a & x^2 + a \\ & -(x^2 + xa) \\ \hline & (a - xa) \\ & -(-xa - a^2) \\ \hline & a + a^2 \leftarrow \text{remainder} \end{array}$$

$\frac{x^2+a}{x+a} = x - a + \frac{a+a^2}{x+a}$

using algebraic long division

9. $\int (x+3)^3\,dx = \int (x^2+6x+9)(x+3)\,dx = \int (x^3+9x^2+27x+27)\,dx$

$= \frac{1}{4}x^4 + 3x^3 + \frac{27}{2}x^2 + 27x + C$

10. $\int (x+2)(x-a)\,dx = \int (x^2 + (2-a)x - 2a)\,dx = \frac{1}{3}x^3 + \frac{(2-a)}{2}x^2 - 2ax + C$

11. $\int (\sqrt{x} + \sqrt[3]{x})\,3a^2\,dx = 3a^2\left(\frac{3}{2}\right)x^{3/2} + \cancel{3}a^2\left(\frac{4}{\cancel{3}}\right)x^{4/3} + C$

12. $\int (\sin\theta - \tfrac{1}{2})\frac{d\theta}{3} = \frac{1}{3}\int \sin\theta\,d\theta - \frac{1}{6}\int d\theta = -\frac{1}{3}\cos\theta - \frac{1}{6}\theta + C$

13. $\int \cos^2 a\theta\,d\theta = \frac{1}{2}\int (1 + \cos(2a\theta)) = \frac{\theta}{2} + \frac{\sin(2a\theta)}{4a} + C$

14. $\int \sin^2\theta\,d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + c$

15. $\int \sin^2 a\theta\,d\theta = \frac{1}{2}\int (1 + \sin(2a\theta)) = \frac{\theta}{2} - \frac{\sin(2a\theta)}{4a} + C$

16. $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$

17. $\int \frac{dx}{1+x} = \ln|1+x| + c$

18. $\int \frac{dx}{1-x} = -\ln|1-x| + C$

$(\cos(x))^2 = \frac{1}{2}(\cos(2x) + 1)$

$(\sin(x))^2 = \frac{1}{2}(1 - \cos(2x))$

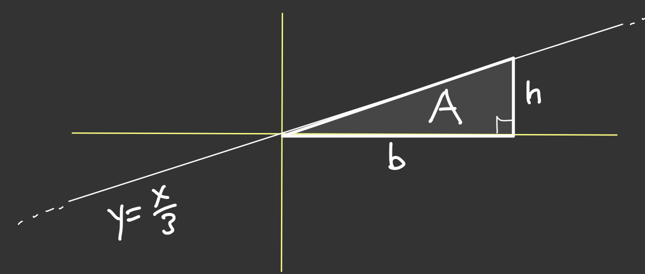
Area Integration Practice

Calculus Made Easy, Ch. 19

$$A = \int_0^b y\,dx = \int_0^b \frac{x}{3}\,dx = \frac{1}{3}\int_0^b x\,dx$$

$$= \frac{1}{3}\left[\frac{1}{2}x^2 + C\right]_0^b = \frac{1}{6}\cdot b^2 = \frac{1}{2}\cdot b\left(\frac{b}{3}\right)$$

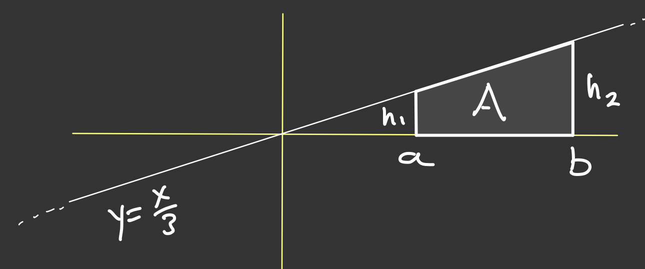
$$= \frac{1}{2}\,b\cdot h$$



$$A = \int_a^b y\,dx = \int_a^b x/3\,dx = \frac{1}{3}\left[\frac{1}{2}x^2 + C\right]_a^b$$

$$= \frac{1}{3}\left(\frac{1}{2}b^2 - \frac{1}{2}a^2\right) = \frac{1}{2}\left(b\cdot\frac{b}{3} - a\cdot\frac{a}{3}\right)$$

$$= \frac{1}{2}\left(b\cdot h_2 - a\cdot h_1\right)$$

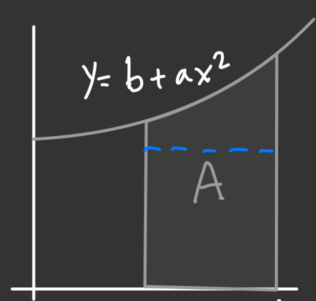


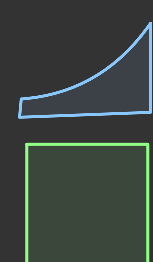
$$A = \int_m^n y\,dx = \int_m^n (b + ax^2)\,dx$$

$$= b\int_m^n dx + a\int_m^n x^2\,dx$$

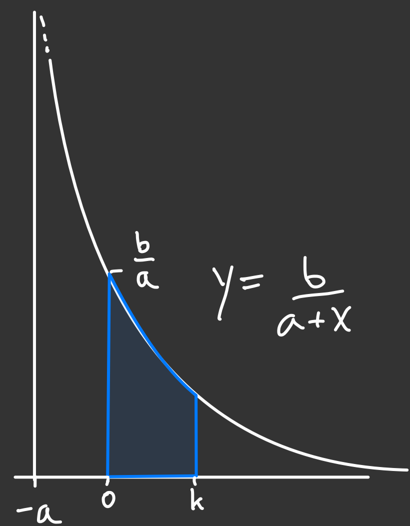
$$= b\left[x + C\right]_m^n + a\left[\frac{1}{3}x^3 + C\right]_m^n$$

$$= b\,(n - m) + a\left(\frac{1}{3}n^3 - \frac{1}{3}m^3\right)$$

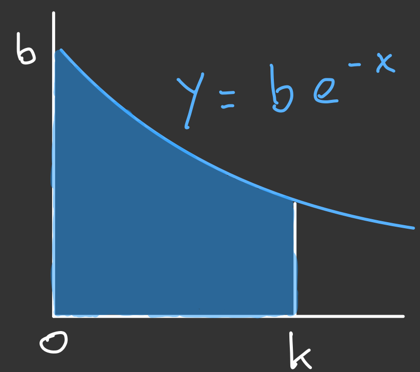




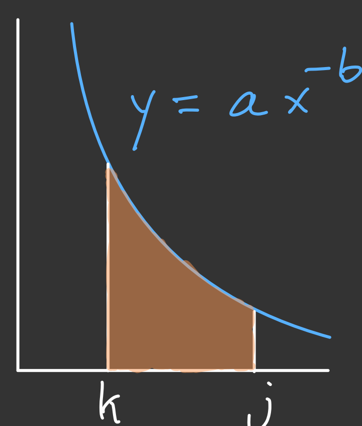
$$A = \int_0^k \frac{b}{a+x}\,dx = b\left[\ln(x+a) + C\right]_0^k$$
$$= b\ln\left(\frac{k+a}{a}\right)$$



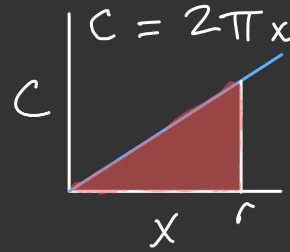
$$A = \int_0^k b e^{-x}\,dx$$
$$= b\left[-e^{-x} + C\right]_0^k$$
$$= b\left(1 - e^{-k}\right)$$



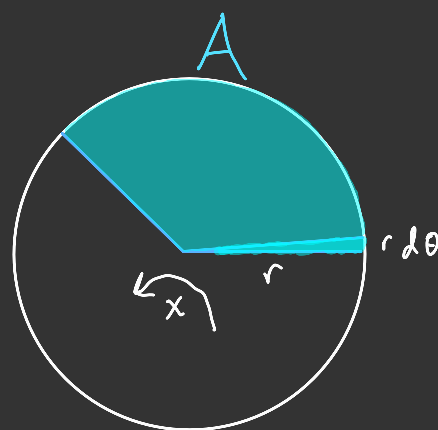
$$A = \int_k^j a x^{-b}\,dx$$
$$= a\left[\left(\frac{1}{-b+1}\right)\cdot x^{(-b+1)} + C\right]_K^j$$
$$= \left(\frac{a}{-b+1}\right)\left[j^{(-b+1)} - K^{(-b+1)}\right]$$



$$A = \int_0^r 2\pi x \, dx$$
$$= 2\pi\left[\tfrac{1}{2}x^2 + C\right]_0^r$$
$$= \pi r^2$$



$$A = \int_0^x \frac{r^2 \cdot d\theta}{2}$$
$$= \frac{r^2}{2}\left[\theta + C\right]_0^x$$
$$= \frac{r^2 x}{2}$$



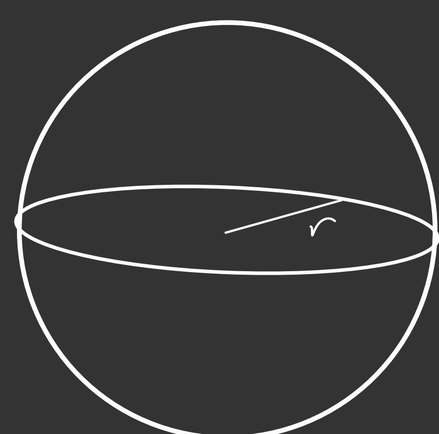
If $x = 2\pi$, $A = \dfrac{\cancel{2}\pi \cdot r^2}{\cancel{2}} = \pi r^2$

The surface area of a sphere with radius r is given by

$$A = 4\pi r^2$$

so the volume is

$$V = \int_0^a 4\pi r^2 \, dr =$$
$$= \left[\tfrac{4}{3}\pi r^3 + C\right]_0^a$$
$$= \tfrac{4}{3}\pi a^3$$

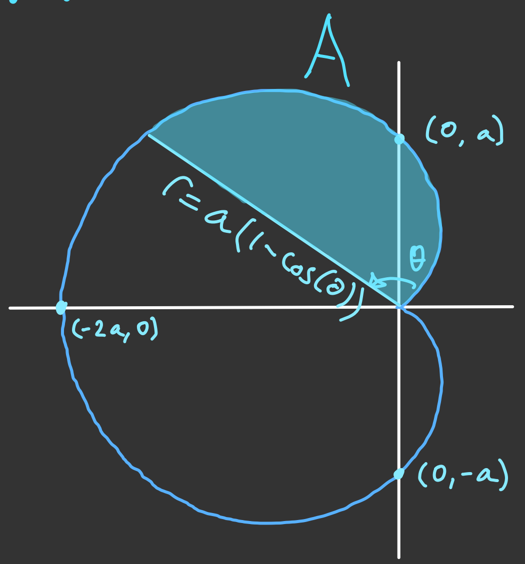


$$A = \int_a^b 2\pi r \, dr$$
$$= \left[\pi r^2 + C\right]_a^b$$
$$= \pi b^2 - \pi a^2$$

$$y = x - x^2 = -\left(\left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right)$$

$$A = \int_0^t (x - x^2)\, dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + C\right]_0^t$$
$$= \tfrac{1}{2}t^2 - \tfrac{1}{3}t^3$$

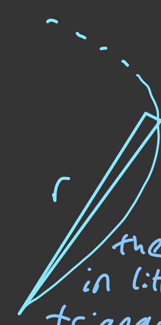
Area of half of a Cartioid



A cartioid (Greek for "heart") is defined by the equation

$$r(\theta) = a(1-\cos\theta)$$

for angle θ & distance r.



we compute the area in little triangular strips.

$0 \le \theta \le \pi$ (half the heart)

$$A = \int_0^\theta \frac{1}{2} r^2 dx = \frac{1}{2}\int_0^\theta a^2(1-\cos(x))^2 dx = \frac{a^2}{2}\int_0^\theta (1-\cos x)(1-\cos x)\, dx$$

$$= \frac{a^2}{2}\int_0^\theta (1-2\cos x+\cos^2 x)\, dx = \frac{a^2}{2}\left(\int_0^\theta dx - 2\int_0^\theta \cos x\, dx + \int_0^\theta \cos^2 x\, dx\right)$$

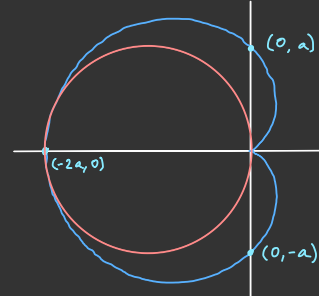
$$= \frac{a^2}{2}\left([x+c]_0^\theta - 2[\sin x + C]_0^\theta + \frac{1}{2}[x+\sin x\cos x]_0^\theta\right) = \frac{a^2}{2}\left(\theta - 2\sin\theta + \frac{1}{2}(\theta+\sin\theta\cos\theta)\right)$$

$$= \frac{a^2}{2}\left(\frac{3}{2}\theta - 2\sin\theta + \frac{\sin\theta\cos\theta}{2}\right)$$

if $\theta = \pi/2$, $A = \frac{a^2}{2}\left(\frac{3\pi}{4} - 2\right) = a^2\left(\frac{3\pi}{8} - 1\right)$

if $\theta = \pi$, $A = a^2 \cdot (3\pi/4)$

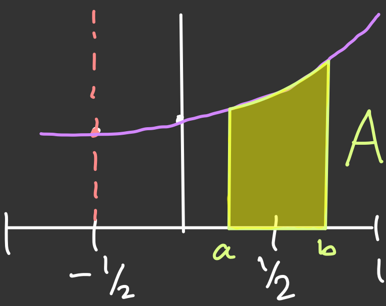
The area of the whole cartioid is thus $A = \frac{3}{2}\pi a^2$, or 1.5 times the area of a circle with radius a.



More Fun with Area Integration

Calc. Made Easy, Exercises 18, p. 225

1. $y = x^2 + x + 5$
$= (x + 1/2)^2 + 4\,3/4$

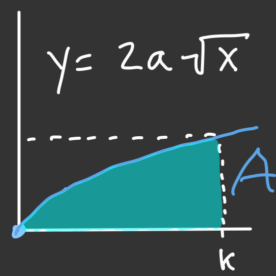


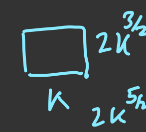
$$A = \int_a^b (x^2 + x + 5)\,dx$$
$$= \left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 5x + C\right]_a^b$$
$$= \tfrac{1}{3}b^3 + \tfrac{1}{2}b^2 + 5b \cancel{+C}$$
$$- \tfrac{1}{3}a^3 - \tfrac{1}{2}a^2 - 5a \cancel{-C}$$

If $a = 0$, $b = 6$, $A = 1/3\, 6^3 + 1/2\, 6^2 + 6 \cdot 5$
$= 6 \cdot 6 \cdot 2 + 6 \cdot 3 + 6 \cdot 5$
$= 6(6 \cdot 2 + 3 + 5)$
$= 6(12 + 3 + 5)$
$= 120$

$\bar{y} = 120/6 = 20$

2.



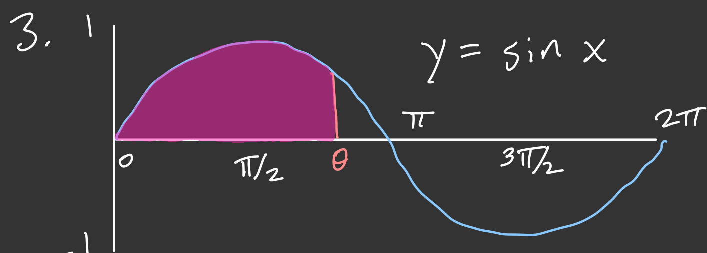


it's 2/3 of the rectangle

$$A = \int_0^k 2a\sqrt{x}\,dx$$
$$= \left[2a\,\tfrac{2}{3}\cdot x^{3/2} + C\right]_0^k$$
$$= \tfrac{4}{3}a\left(k^{3/2}\right)$$

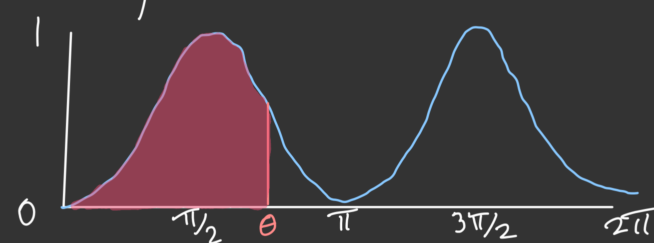
if $k = a$, $A = \frac{4}{3}a^{5/2} = \frac{2}{3}\,2k^{5/2}$

3. $y = \sin x$



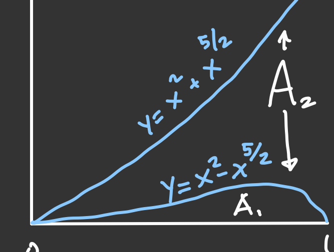
$$A = \int_0^\theta \sin x\,dx = \left[-\cos x + C\right]_0^\theta$$
$$= 1 - \cos\theta$$

4. $y = \sin^2 x$



$$A = \int_0^\theta \sin^2 x\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4} + C\right]_0^\theta$$
$$= \theta/2 - \frac{\sin 2\theta}{4}$$

5.

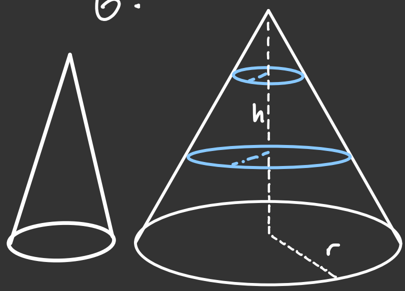


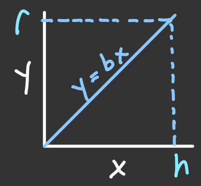
$$A_1 = \int_0^1 (x^2 - x^{5/2})\,dx$$
$$= \left[\tfrac{1}{3}x^3 - \tfrac{2}{7}x^{7/2} + C\right]_0^k$$
$$= \tfrac{1}{3}k^3 - \tfrac{2}{7}k^{7/2}$$

if $k = 1$, $A_1 = \frac{1}{21}$, $A_2 = 4/7$

$$A_2 = \int_0^k (x^2 + x^{5/2})\,dx - A_1$$
$$= \left[\tfrac{1}{3}x^3 + \tfrac{2}{7}x^{7/2}\right]_0^k - A_1$$
$$= \cancel{\tfrac{1}{3}k^3} + \tfrac{2}{7}k^{7/2} - \cancel{\tfrac{1}{3}k^3} + \tfrac{2}{7}k^{7/2}$$
$$= 4/7\, k^{7/2}$$

6.



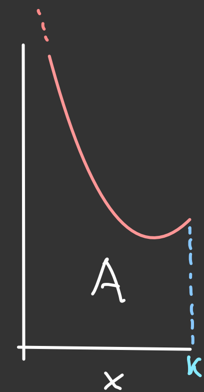


$$V=\int_0^r \pi y^2\,dx = b^2\int_0^H \pi x^2\,dx$$
$$= b^2\pi\left[\tfrac{1}{3}x^3\right]_0^h = \frac{b^2\pi h^3}{3} = \frac{r^2\pi h}{3}$$

7.

$$y = x^3 - \ln x$$
$$A = \int_0^k (x^3 - \ln x)\,dx$$
$$= \left[\tfrac{1}{4}x^4 - x(\ln x - 1) + C\right]_0^k$$
$$= \tfrac{1}{4}k^4 - k(\ln k - 1)$$

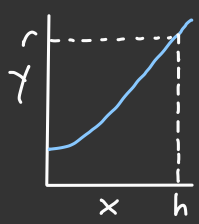
if $k=1$, $A=\frac{5}{4}$

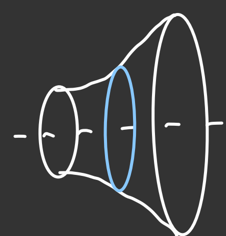


8.

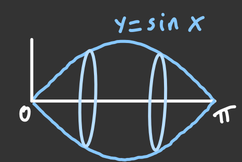
$$y=\sqrt{1+x^2}$$
$$V=\int_0^k \pi y^2\,dx$$
$$= \int_0^h \pi(1+x^2)\,dx$$
$$= \pi\left[x + \tfrac{1}{3}x^3 + C\right]_0^h$$
$$= \pi\left(h + \tfrac{1}{3}h^3\right) = \frac{h}{3}\pi(2+r^2) = \frac{2\pi h}{3} + \frac{\pi r^2 h}{3}$$

If $h=4$, $V = 4\cdot\pi + {}^{64}/_{3}\,\pi = \frac{76}{3}\pi$





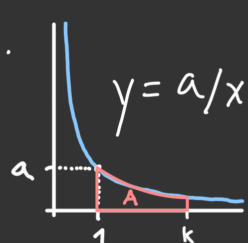
9.



$$V=\int_0^\theta \pi \sin^2 x\,dx$$
$$= \pi\left[\frac{x}{2} - \frac{\sin 2x}{4} + C\right]_0^\theta$$
$$= \frac{\pi}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)$$
$$= \frac{\pi}{2}\left(\theta - \sin\theta\cos\theta\right)$$

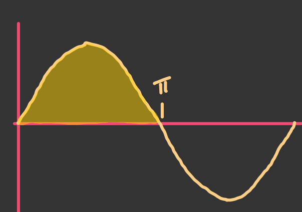
If $\theta=\pi$, $V=\pi^2/2$.

10.



$$A=\int_1^k a/x\,dx$$
$$= a\left[\ln x + C\right]_1^k$$
$$= a\ln k$$

11) $\int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi = -\left(\cos(\pi) - \cos(0)\right) = 2$



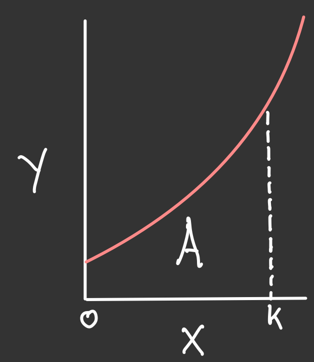
12. $y = x^2 + 3x + 2 = (x + 3/2)^2 - 1/4$

$$A = \int_0^k (x^2 + 3x + 2)\,dx$$

$$= \left[\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 2x + C\right]_0^k$$

$$= \tfrac{1}{3}k^3 + \tfrac{3}{2}k^2 + 2k$$

$$\bar{y} = \tfrac{1}{3}k^2 + \tfrac{3}{2}k + 2$$



$$\bar{q} = \sqrt{\frac{1}{k}\int_0^k y^2\,dx} \quad ?$$

$$y^2 = (x^2 + 3x + 2)^2$$

$$= x^4 + 3x^3 + 2x + 3x^3 + 9x^2 + 6x + 2x^2 + 6x + 4$$

$$= x^4 + 6x^3 + 11x^2 + 14x + 4$$

$$\frac{1}{k}\int_0^k y^2\,dx = k^{-1}\left[\tfrac{1}{5}x^5 + \tfrac{6}{4}x^4 + \tfrac{11}{3}x^3 + \tfrac{14}{2}x^2 + 4x\right]_0^k$$

so $\bar{q} = \left(\tfrac{1}{5}k^4 + \tfrac{3}{2}k^3 + \tfrac{11}{3}k^2 + 7k + 4\right)^{1/2}$

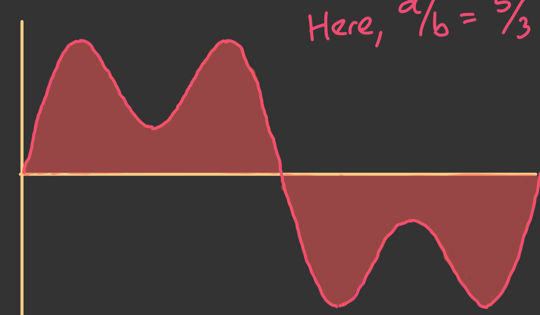
13. $y = a\sin x + b\sin 3x$

$$A = a\int_0^{2\pi} \sin x\,dx + b\int_0^{2\pi} \sin 3x\,dx$$

$$= a\left[-\cos x\right]_0^{2\pi} + b\left[\frac{-\cos 3x}{3}\right]_0^{2\pi}$$

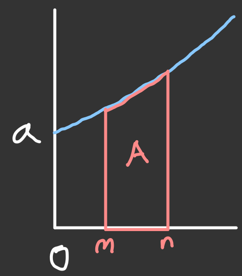
$$= -a[1-1] - \tfrac{b}{3}[1-1] = 0$$

Here, $a/b = 5/3$



also true for <u>any</u> integers multiplied by x!

14. $y = ae^{bx}$

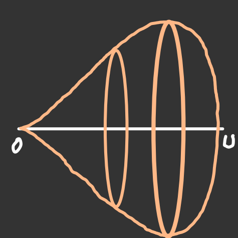


$$A = \int_m^n ae^{bx}\,dx$$
$$= a\left[\frac{1}{b}e^{bx} + C\right]_m^n$$
$$= a\left(\frac{1}{b}e^{bn} - \frac{1}{b}e^{bm}\right)$$
$$= \frac{a}{b}e^{bn} - \frac{a}{b}e^{bm}$$

$a = 3.42$ $b = 0.21$
$m = 2$ $n = 8$
$\bar{y} \approx 10.43$
$A \approx 62.60$

15. See "Area of a Cartioid"

16. $y = \pm ax\sqrt{x(u-x)}$



$$V = \int_0^k \pi a^2 x^2 (x(u-x))\,dx$$
$$= a^2\pi \int_0^k x^3 u - x^4\,dx$$
$$= a^2\pi\left[\frac{1}{4}x^4 u - \frac{1}{5}x^5\right]_0^k$$
$$= a^2\pi\left(\frac{1}{4}k^4 u - \frac{1}{5}k^5\right)$$
$$= a^2\pi k^4\left(\frac{1}{4}u - \frac{1}{5}k\right)$$

When $k = u$, $V = \frac{a^2}{20}\pi u^5$

INTEGRATION BY PARTS

CALC MADE EASY, CHAPTER 20, p. 227

INTEGRATION BY PARTS GOES BACKWARDS FROM THE PRODUCT RULE:

$$d(fg)/dx = f\,dg/dx + df/dx \cdot g$$

$$fg = \int f\,dg/dx \cdot dx + \int df/dx\, g\, dx$$

$$\int f\,dg/dx \cdot dx = fg - \int df/dx\, g\, dx$$

SO, FOR AN INTEGRATION OF A PRODUCT, WE PICK ONE TERM TO BE f AND ONE TO BE dg/dx

1) $y = \int w \sin w \, dw$

$= \int f g' = fg - \int f' g + C$

$f = w \quad g' = \sin w$

$f' = 1 \quad g = -\cos w$

$= w(-\cos w + c_1) - \int -\cos w \, dw + C$

$= -w \cos w + \sin w + C$

2) $\int x e^x dx$

$f = x \quad g' = e^x$

$f' = 1 \quad g = e^x$

$= xe^x - \int 1 \cdot e^x dx$

$= xe^x - e^x = e^x(x-1) + C$

3) $y = \int \cos^2\theta \, d\theta$

$f = \cos\theta \quad g' = \cos\theta$

$f' = -\sin\theta \quad g = \sin\theta$

$= \cos\theta \sin\theta + \int \sin^2\theta \, d\theta$

$= \cos\theta \sin\theta + \int (1 - \cos^2\theta)\, d\theta$

$y = \cos\theta \sin\theta + \theta - y$

$= \frac{1}{2}(\cos\theta \sin\theta + \theta) + C$

$f = x^2 \quad g' = \sin x$

$f' = 2x \quad g = -\cos x$

4) $\int x^2 \sin x \, dx$

$= -x^2 \cos x + 2\int x \cos x \, dx$ → REPEAT! →

$f = x \quad g' = \cos x$

$f' = 1 \quad g = \sin x$

$= -x^2 \cos x + 2\left(x \sin x - \int 1 \cdot \sin x \, dx\right)$

$= -x^2 \cos x + 2x \sin x - 2 \cdot (-\cos x) + C$

$= -x^2 \cos x + 2(x \sin x + \cos x) + C$

5) $\int \sqrt{1-x^2}\, dx = \int \sqrt{1-x}\sqrt{1+x}\, dx$

$f = \sqrt{1+x} \quad g' = \sqrt{1-x}$

$f' = \frac{1}{2\sqrt{1+x}} \quad g = \frac{-1}{2\sqrt{1-x}}$

$-\frac{\sqrt{1+x}}{2\sqrt{1-x}} - \int \frac{1}{2\sqrt{1+x}} \frac{1}{2\sqrt{1-x}}\, dx$

$-\frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\, dx$

$-\frac{\sqrt{1+x}}{2\sqrt{1-x}} - \frac{1}{2}\sin^{-1}(x) + C$

INTEGRATION BY SUBSTITUTION

CALC MADE EASY, CHAPTER 20, p. 230

1) $\int \sqrt{3+x}\,dx$ $\quad u = 3+x$, $du/dx = 1$

$= \int \sqrt{u}\,du$

$= \frac{2}{3}u^{3/2} + C$

$= \frac{2}{3}(3+x)^{3/2} + C$

2) $\int \frac{dx}{e^x + e^{-x}}$ $\quad u = e^x$, $du/dx = e^x$

$= \int \frac{du}{u \cdot (u + 1/u)} = \int \frac{du}{u^2+1} = \tan^{-1} u + C$

$= \tan^{-1} e^x + C$

3) $\int \frac{dx}{x^2+2x+3} = \int \frac{dx}{(x+1)^2 + (\sqrt{2})^2}$ $\quad u = x+1$, $du/dx = 1$

$= \int \frac{du}{u^2 + \sqrt{2}^2}$

$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$

Knowing that:

$$\left(\begin{array}{l} y = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \\ y' = \frac{1}{x^2+a^2} \end{array} \right)$$

PARTIAL FRACTIONS IN INTEGRATION

CALC MADE EASY, CHAPTER 20, p. 231

$$\int \frac{dx}{x^2+2x-3} = \frac{1}{4}\int\left(\frac{1}{x-1} - \frac{1}{x+3}\right)dx = \frac{1}{4}\left(\ln(x-1) - \ln(x+3)\right) + C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\int\left(\frac{1}{a-x} + \frac{1}{a+x}\right)dx = \frac{1}{2a}\left(\ln(a+x) - \ln(a-x)\right) + C$$

see Boole's <u>Differential Equations</u>

Bonus problem from Reddit:

$$\int \frac{5x^2}{x^2+4x+3}\,dx = \int \frac{5x^2}{(x+3)(x+1)}\,dx = \int \frac{(-45/2)}{x+3}\,dx + \int \frac{(5/2)}{x+1}\,dx + \int 5\,dx$$

$$= -45/2 \log(x+3) + (5/2)\log(x+1) + 5x + C$$

Practice With Differential Equations

CALC MADE EASY, EX. 20, p. 248

1) $\frac{dT}{d\theta} = \mu T$

$$\int \frac{dT}{T} = \mu \int d\theta$$

$$\ln T = \mu \theta$$

$$T = \exp(\mu \theta) \cdot T_0$$

2) $\frac{d^2 s}{dt^2} = a \qquad \frac{ds}{dt} = v$ if $t=0, s=0$

$$\frac{ds}{dt} = at + v$$

$$s = \frac{at^2}{2} + vt$$

3) $\frac{di}{dt} + 2i = \sin 3t \qquad i=0$ when $t=0$

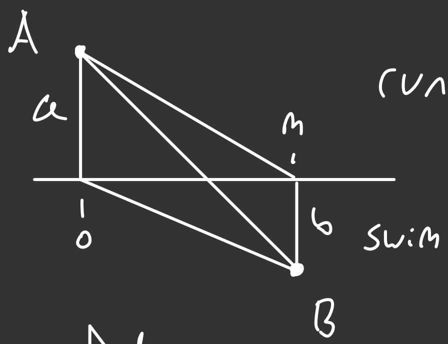
$$\frac{dv}{dt} = e^{2t}\frac{di}{dt} + 2e^{2t} \cdot i = e^{2t} \sin 3t$$

$$v = e^{2t} \cdot i$$

$$v = \int e^{2t} \sin 3t \, dt = \frac{\exp(2t)}{2^2 + 3^2}\left(2 \sin 3t - 3 \cos 3t\right) + C$$

$$i = 13^{-1}\left(2 \sin 3t - 3 \cos 3t\right) + \frac{3}{13}$$

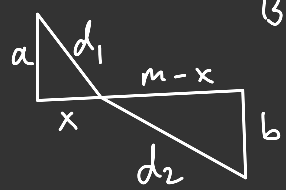
Path Minimization



$r = \alpha \cdot s$

$\alpha > 1$

$p = \dfrac{t_1}{t_1 + t_2}$

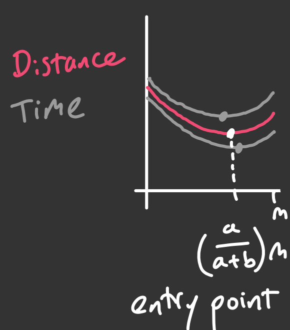


$$D = d_1 + d_2 = \sqrt{a^2 + x^2} + \sqrt{b^2 + (m-x)^2}$$

$$\hat{x} = \frac{a}{a+b} m$$

$$T = t_1 \cdot r + t_2 \cdot s = t_1 \alpha s + t_2 s$$

$$T = s(t_1 \alpha + t_2)$$



$$d_1 = r \cdot t_1 = \alpha s t_1 = \sqrt{a^2 + x^2}$$

$$\text{so } t_1 = \frac{\sqrt{a^2 + x^2}}{\alpha s}$$

$$d_2 = s t_2 = \sqrt{b^2 + (m-x)^2}$$

$$\text{so } t_2 = \frac{\sqrt{b^2 + (m-x)^2}}{s}$$

$$T = \cancel{s}\left(\frac{\sqrt{a^2 + x^2}}{\alpha \cancel{s}} + \frac{\sqrt{b^2 + (m-x)^2}}{\cancel{s}}\right)$$

$$= \alpha^{-1}\sqrt{\alpha^2 + x^2} + \sqrt{b^2 + (m-x)^2}$$

minimum has no analytic solution?

SHORTEST PATHS IN A RACE

FROM *CALC MADE EASY* PAGE 300

WHEN RACING AROUND A SHARP CORNER, WE HAVE THE OPTION TO CUT ACROSS ROUGH TERRAIN, AT A REDUCED SPEED. THE CORNER IS 1 UNIT AWAY, THE DESTINATION IS b UNITS FROM THE CORNER. IF WE TAKE A SHORTCUT x UNITS FROM THE CORNER, WE ONLY HAVE h UNITS LEFT, BUT AT A FRACTION γ OF OUR NORMAL SPEED. THE TOTAL TIME IS THUS

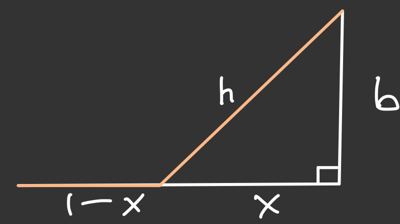


Fig. 1

$$t = \underbrace{(1-x)}_{\text{TIME BEFORE WE CUT}} + \underbrace{h/\gamma}_{\text{SHORTCUT TIME}}$$

DIFFERENTIATING,

$$\frac{dt}{dx} = \frac{x}{\gamma(x^2+b^2)^{1/2}} - 1$$

WHICH IS MINIMIZED AT

$$\hat{x} = b\sqrt{\frac{\gamma^2}{1-\gamma^2}}$$

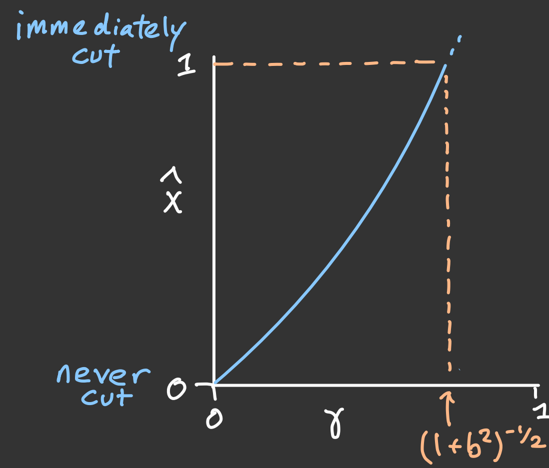


Fig. 2

THE BEST SHORTCUT IS THUS A FUNCTION OF b & γ. FOR ANY SPECIFIC b, WE CAN ADJUST γ TO MAKE $\hat{x} \to 0$ AS $\gamma \to 0$, SO NEVER CUT, TO $\hat{x}=1$ IF $\gamma \geq (1+b^2)^{-1/2}$, SO IMMEDIATELY CUT.

IN ***CALC MADE EASY***, $b = 3/4$, $\gamma = 3/4$, SO $\hat{x} = 0.85$, ABOUT 15% ALONG (MARKED IN FIG. 3)

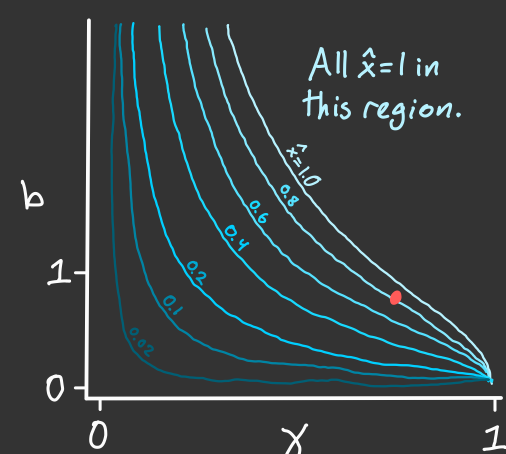


Fig. 3