Delayed feedback stabilization with and without symmetry

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She asked him what his father's books were about. 'Subject and object and the nature of reality', Andrew had said. And when she said Heavens, she had no notion what that meant, 'Think of a kitchen table then', he told her, 'when you're not there'.

Virginia Woolf, *To the Lighthouse*

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Chapter 1

Introduction

This thesis is concerned with stabilizing effects of time delays in dynamical systems. In our daily lives, unwarranted time delays are often perceived to have a de-stabilizing effect. In this thesis, we deliberately introduce time delays in a way that preserves a prescribed motion, with the intention that such time delays then have a stabilizing effect, rather than a de-stabilizing one.

1.1 Feedback stabilization with and without symmetry

Methods that aim at stabilizing a motion by introducing an artificial time delay, are often called delayed feedback control schemes. In [Pyr92], Kestitutis Pyragas introduced a delayed feedback control scheme (now known as 'Pyragas control') that aims to stabilize periodic motion. Pyragas describes the system without feedback by an ordinary differential equation (ODE)

$$\dot{x}(t) = f(x(t)) \tag{1.1}$$

with state $x(t) \in \mathbb{R}^N$. For the feedback system, Pyragas then writes

$$\dot{x}(t) = f(x(t)) + K\left[x(t) - x(t-p)\right] \tag{1.2}$$

with time delay $p > 0$ and *gain matrix* $K \in \mathbb{R}^{N\times N}$. So the control term measures the difference between the current state and the state time p ago, and feeds this difference (multiplied by the gain matrix K) back into the system. For a periodic solution with period p, the difference between the current state and the state time p ago is zero. So a p-periodic solution of the original system (1.1) is also a solution of the feedback system (1.2). However, the overall dynamics of the systems without and with feedback are radically different, and an unstable periodic solution of the original system (1.1) can be a stable solution of the feedback system (1.2).

If the original system (1.1) has built-in symmetries, its periodic solutions can satisfy additional spatial-temporal relations. In this case, we can adapt the Pyragas control scheme so that it vanishes on a prescribed spatial-temporal pattern. As in [NU98a; FFGHS08], we write the feedback system as

$$\dot{x}(t) = f(x(t)) + K\left[x(t) - hx(t-\tilde{p})\right]. \tag{1.3}$$

with time delay $\tilde{p} > 0$ and $h \in \mathbb{R}^{N\times N}$ a linear, spatial transformation. The control term now feeds back the difference between the current state and a spatial-temporal transformation of the state, and vanishes on solutions that satisfy the spatial-temporal relation $hx(t) = x(t+\tilde{p})$. The feedback scheme (1.3) has the advantage that it is able to select a spatial-temporal pattern amongst a family of periodic solutions with the same period, and in such situations can indeed be more successful in stabilizing a specific pattern than Pyragas control [Sch16; FFS10]. Since symmetries of the original system (1.1) are often described in terms of equivariance (see Section 1.3 below), we refer to the control scheme (1.3) as *equivariant (Pyragas) control*.

Implementation of Pyragas control requires knowledge of the period of the targeted solution, but uses no additional information on the original system (1.1). This 'model-independence' makes Pyragas control widely applicable; for example in semiconductor lasers [SHWSH06; SWH11], CO_2-lasers [BDG94] and enzymatic reactions [LFS95]; the paper [Pyr92] has currently (June 2021) more than 2500 citations. The equivariant control scheme (1.3) has recently been applied to networks of chemical oscillators [HGTS21].

Despite its many experimental realizations, mathematical results on success or failure of stabilization with (equivariant) Pyragas control are relatively rare. This is mainly because, from a mathematical perspective, the feedback systems (1.2) and (1.3) generate infinite dimensional dynamical systems. Systems (1.2) and (1.3) are examples of *delay differential equations* (DDE), i.e. differential equations that depend explicitly on the past state variable. In order to associate to (1.2) (resp. (1.3)) a well-posed initial value problem, we should provide a function on the interval $[-p, 0]$ (resp. $[-\tilde{p}, 0]$) as initial condition. So the state space is a function space (that has to be specified more precisely) and the associated dynamical system is infinite dimensional. Although the abstract theory of DDE is well developed [HV93; DGVW95], the infinite dimensional nature of DDE remains challenging when we want to perform an explicit stability analysis in concrete examples.

This thesis aims to contribute to the mathematical understanding of (equivariant) Pyragas control by proving

1. *limitations* to Pyragas control in systems without symmetry;
2. *stabilization* of a class of spatial-temporal patterns using equivariant control in systems with symmetry.

We introduce both topics in more detail below.

1.2 Limitations to Pyragas control

When we aim to stabilize a periodic solution of an ODE, we should of course first determine whether this periodic solution is unstable to begin with; we do so using the principle of *linearized stability* and *Floquet multipliers*. To introduce these concepts, we consider a wider class of ODE than before (i.e. than in equation (1.1)); we now consider an ODE

$$\dot{x}(t) = f(x(t), t) \tag{1.4}$$

that is time-periodic with period $p > 0$, i.e. $f(x, t + p) = f(x, t)$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. So system (1.4) can have an explicit, periodic time dependence but it can also be autonomous (in which the periodicity condition is trivially satisfied). Given a p-periodic solution x_* of (1.4), the equation

$$\dot{y}(t) = \partial_1 f(x_*(t), t) y(t) \tag{1.5}$$

is a p-periodic, linear ODE. Let $Y(t) \in \mathbb{R}^{N\times N},\ t \geq 0$ be its *fundamental solution* with $Y(0) = I$, i.e. $Y(t)$ is the matrix-valued solution of the initial value problem

$$\frac{d}{dt}Y(t) = \partial_1 f(x_*(t), t)Y(t), \qquad Y(0) = I.$$

The operator

$$Y(p) : \mathbb{R}^N \to \mathbb{R}^N,$$

that captures how the linear ODE (1.5) evolves under a time step p, is called the *monodromy operator*. The eigenvalues of this monodromy operator are called the *Floquet multipliers* of system (1.5). Somewhat sloppily, we sometimes also refer to the eigenvalues of $Y(p)$ as the Floquet multipliers of the periodic solution x_* of (1.4). If there exists a Floquet multiplier $\mu > 1$, then the linear ODE (1.5) has a solution $y(t+p) = \mu y(t)$ and thus the origin of this system is asymptotically unstable (see Lemma A.3 in the appendix for more details). By the principle of linearized stability, the periodic solution x_* of (1.4) is then unstable as well (see Theorem 7 in the appendix).

One of the most general analytical results on Pyragas control is due to Nakajima [Nak97]. Nakajima considers the p-periodic ODE (1.4) and assumes it is 'truly nonautonomous' in the sense that $\partial_2 f \not\equiv 0$. He argues that if a p-periodic solution of (1.4) has an odd number of Floquet multipliers $\mu > 1$, then Pyragas control fails to stabilize, i.e. the periodic solution is an unstable solution of the controlled system

$$\dot{x}(t) = f(x(t), t) + K\left[x(t) - x(t-p)\right]$$

for every gain matrix $K \in \mathbb{R}^{N\times N}$. This result is known as *the odd number limitation.*

In [Nak97], Nakajima explicitly does *not* consider the situation where system (1.4) is autonomous. But he suggests in a footnote that 'all results in what follows can be also proved about stabilizing UPOs [Unstable Periodic Orbits] in autonomous systems with a slight revision, though we will deal with non-autonomous systems' [Nak97]. From this suggestion the odd number limitation was generally also believed to be true for autonomous systems [NU98a; NU98b]. However, in [FFGHS08] the authors give an example of an automous system which has a periodic solution with one Floquet multiplier $\mu > 1$. They then successfully stabilize this periodic solution using Pyragas control, thereby disproving the odd number limitation for autonomous systems.

In Chapter 2 of this thesis, we derive the following invariance principle for both autonomous and non-autonomous systems:

Invariance principle. Given a p-periodic solution of the p-periodic ODE (1.4), the geometric multiplicity of the Floquet multiplier 1 of this solution is preserved under Pyragas control.

From this invariance principle, we then derive the following limitations to Pyragas control:

1. For non-autonomous systems, we give a new proof of the odd number limitation.

2. For systems (either autonomous or non-autonomous) where the periodic orbit has a Floquet multiplier $\mu > 1$, we show that Pyragas control fails if the gain matrix has real spectrum and commutes with the linearization (so in particular, Pyragas control with *scalar* control gain fails). We refer to this result as the 'any number limitation'.

3. For autonomous systems, we derive a necessary condition on the gain matrix for Pyragas control to be successful.

The new proof of the odd number limitation makes a clear connection to the invariance principle, and also clarifies why the odd number limitation does *not* hold in autonomous systems. Failure of Pyragas control with scalar control gain has been observed before in examples [FFGHS08]; here we give a rigorous proof for a wide class of systems. The necessary condition on the gain matrix in autonomous systems is similar to the condition found in [HA12]; the proof presented in Chapter 2 gives a natural interpretation of this rather technical statement in the light of the invariance principle. So although some of the results in Chapter 2 have been proved before, the novelty of the approach here lies in the new invariance principle, which gives a clear and unifying understanding of the limitations to Pyragas control.

1.3 Stabilization in the presence of symmetries

The second part of this thesis (Chapters 3–9) is concerned with stabilization of periodic orbits in autonomous systems with symmetries. We describe symmetries of an autonomous ODE using the framework of *group equivariance*, as for example discussed in [GS02; GSS88]. Given the autonomous ODE (1.1) and a subgroup $\Gamma \subseteq GL(N, \mathbb{R})$ of the general linear group, we say that (1.1) is *equivariant* with respect to Γ if

$$f(\gamma x) = \gamma f(x) \quad \text{for all } \gamma \in \Gamma \text{ and } x \in \mathbb{R}^N.$$

If now $x(t)$ is a solution of (1.1) and γ is an element of Γ, then $\gamma x(t)$ is a solution of (1.1) as well. So the group Γ is indeed a group of symmetries of the solutions of system (1.1).

Equivariance of the ODE (1.1) naturally induces two symmetry groups on the periodic solutions. Indeed, let x_* be a periodic solution of (1.1) with minimal period $p > 0$; let $\mathcal{O} = \{x_*(t) \mid t \in \mathbb{R}\}$ be its orbit. Then the elements of the group

$$H_* := \{\gamma \in \Gamma \mid \gamma\mathcal{O} = \mathcal{O}\} \tag{1.6a}$$

leave the orbit $\mathcal{O}$ invariant as a set. Elements of the group

$$K_* := \{\gamma \in \Gamma \mid \gamma x_*(0) = x_*(0)\} \tag{1.6b}$$

leave the initial condition $x_*(0)$ invariant. (In the literature, the group (1.6a) is usually denoted by H, and the group (1.6b) is usually denoted by K, but we have added a subscript to avoid confusion with the gain matrix $K \in \mathbb{R}^{N\times N}$ in (1.2).) If $k \in K_*$, then $kx_*(t)$ and $x_*(t)$ are two solutions of (1.1) with the same initial condition, and therefore

$$kx_*(t) = x_*(t) \tag{1.7}$$

for all $t \in \mathbb{R}$. So in fact elements of K_* leave the orbit fixed pointwise and therefore we refer to K_* as the group of *spatial symmetries* of x_*. Similarly, if $h \in H_*$, then $hx_*(0) = x_*(\Theta(h)p)$ for some $\Theta(h) \in [0, 1)$. But then $hx_*(t)$ and $x_*(t + \Theta(h)p)$ are both solutions of (1.1) with the same initial condition and hence

$$hx_*(t) = x_*(t + \Theta(h)p) \tag{1.8}$$

for all $t \in \mathbb{R}$. Since every element $h \in H_*$ induces a spatial-temporal relation (1.8) on the periodic solution, we refer to the group H_* as the group of *spatial-temporal symmetries* of x_*.

If $h_1, h_2 \in H_*$ are two spatial-temporal symmetries of x_*, then

$$h_1h_2x_*(t) = x_*(t + \Theta(h_1)p + \Theta(h_2)p)$$

and hence $\Theta(h_1 h_2) = \Theta(h_1) + \Theta(h_2) \mod 1$. Thus the map

$$\Theta : H_* \to S^1 \simeq \mathbb{R}/\mathbb{Z}.$$

is a group homomorphism and $K_* = \ker \Theta$ is a normal subgroup of H_*. Therefore $H_*/K_* \simeq \mathrm{im}\ \Theta$ and im Θ is a subgroup of S^1. This implies that

$$\begin{cases} H_*/K_* \simeq \mathbb{Z}_n & \text{for some } n \in \mathbb{N}, \text{ or} \\ H_*/K_* \simeq S^1, \end{cases}$$

where $\mathbb{Z}_n$ denotes the cyclic group of order n. If $H_*/K_* \simeq S^1$, we say that the periodic solution x_* is a *rotating wave*; if $H_*/K_* \simeq \mathbb{Z}_n$ (i.e. the group of spatial-temporal symmetries modulo the group of purely spatial ones is a finite group), we say that the periodic solution x_* is a *discrete wave*.

If the ODE (1.1) is equivariant with respect to a group $\Gamma \subseteq GL(N, \mathbb{R})$ and has a periodic solution x_* with minimal period $p > 0$, then x_* is also a solution of the feedback system

$$\dot{x}(t) = f(x(t)) + K\left[x(t) - x(t - \Theta(h)p)\right] \tag{1.9}$$

with $h \in H_*$ a spatial-temporal symmetry of the periodic orbit and with gain matrix $K \in \mathbb{R}^{N\times N}$. Feedback schemes of the form (1.9), which go under the name *equivariant (Pyragas) control schemes*, were first introduced in an attempt to overcome the odd number limitation to Pyragas control. Indeed, in [NU98a] Nakajima and Ueda apply equivariant control to a non-autonomous system, and show numerically that equivariant control can stabilize a periodic solution with an odd number of Floquet multipliers larger than one. In [FFS10; SB16; PBS13], equivariant control is applied to systems of coupled oscillators, in which several periodic orbits with different spatial-temporal patterns coexist. In this situation the control scheme (1.9) is *pattern selective*, because it vanishes on only one spatial-temporal pattern. Also here, equivariant control is able to stabilize patterns that cannot be stabilized with Pyragas control.

Chapters 3–9 of this thesis address equivariant control of *discrete waves*, culminating in a *positive stabilization result* in Theorem 3. The methods and results presented in Chapter 3–9 have three main features:

1. Theorem 3 provides a positive stabilization result for a broad class of discrete waves and the necessary conditions for stabilization are formulated in relatively concrete properties of the uncontrolled system.

2. Theorem 3 deals with equivariant control with *scalar control gain*, i.e. the gain matrix $K \in \mathbb{R}^{N\times N}$ in (1.9) is of the form $K = kI$ with $k \in \mathbb{R}$. For Pyragas control, the 'any number limitation' (Theorem 2) gives an obstruction to stabilization with scalar control gain. But the positive stabilization result in Theorem 3 shows that, in some cases, equivariant control is able to overcome this obstruction.

3. In the literature so far, most analytical results on successful equivariant control and successful Pyragas control are either close to a bifurcation point [HKRH19; FLRSSW20; WV17; HBKR17] or concern periodic orbits that can be transformed to equilibria of an autonomous system [PPK14; FFGHS08; SB16; FFS10]. Close to a bifurcation point, one can determine the stability of the periodic orbit from an implicit function theorem argument; if periodic orbits

become equilibria after a coordinate transformation, one can avoid Floquet theory altogether. The proof of Theorem 3, however, uses an explicit analysis of the Floquet multipliers far from the bifurcation point.

Our explicit analysis of the Floquet multipliers is based on *equivariant Floquet theory* and the use of *characteristic matrix functions*. In systems without symmetries, we determine the stability of a p-periodic orbit using the monodromy operator, which involves solving the linearized equation over a timestep p. But in equivariant settings, we can work with the so-called *reduced monodromy operator*, which involves solving the linearized equation over only a fraction of the period. For system (1.9), we then show that the eigenvalues of the (infinite dimensional) reduced monodromy operator can be computed from a matrix-valued function, called the characteristic matrix function (cf. [KV21; KV92]). In fact, a large part of this thesis is concerned with the reduced monodromy operator (Chapters 4 and 6) and with characteristic matrix functions (Chapters 5 and 7). But we stress that this work pays off in the end: we obtain a concrete stabilization result, applicable to a relatively broad class of solutions, and gain structural insight in the workings of equivariant control along the way.

1.4 Grasshopper's guide and outline to the thesis

Chapter 2 discusses limitations to Pyragas control, and can be read independently from the rest of the thesis. Chapter 3 gives the main result on equivariant control of discrete waves (Theorem 3), stated in mathematically precise form. The definitions needed for this formulation are also introduced in Chapter 3, so that this chapter can be read independently from the rest of the thesis. The discussion section in Chapter 10 discusses the relation between successful stabilization with equivariant control and limitations to Pyragas control, and thus provides a connection between Chapters 2 and 3.

To prove Theorem 3 on stabilization of discrete waves, we first discuss equivariant Floquet theory for ODE in Chapter 4 and the theory of characteristic matrix functions in Chapter 5. Chapter 5 is an expository chapter and can be skipped by the reader already familiar with characteristic matrix functions. In Chapter 6 we develop equivariant Floquet theory for DDE, and give an heuristic argument why the notion of 'reduced monodromy operator' is especially relevant in the context of equivariant control. Chapter 7 then applies the concept of a characteristic matrix function to systems with equivariant control: we rigorously prove that, for systems with equivariant control, we can compute the eigenvalues of the reduced monodromy operator as roots of a characteristic equation. We analyze this characteristic equation in Chapter 8, and obtain a positive stabilization result for Pyragas control along the way. We then finally prove Theorem 3 in Chapter 9, where we also comment briefly on the structure of the group of spatial-temporal symmetries.

1.5 Some comments on notation

Throughout the thesis, we use some notational conventions that we establish here.

1. For $r > 0$ and $N \in \mathbb{N}$, we denote by $C\left([-r,0],\mathbb{R}^N\right)$ the space of continuous functions from $[-r,0]$ to $\mathbb{R}^N$, i.e.
$$C\left([-r,0],\mathbb{R}^N\right) = \{\phi : [-r,0] \to \mathbb{R}^N \mid \phi \text{ is continuous}\}.$$

For $\phi \in C\left([-r,0],\mathbb{R}^N\right)$, we define its norm $\|\phi\|$ as

$$\|\phi\| = \sup_{\theta\in[-r,0]} |\phi(\theta)| \, .$$

So we equip the space $C\left([-r,0],\mathbb{R}^N\right)$ with the supremum-norm, thereby making it into a Banach space.

2. Let X be a real Banach space and $T : X \to X$ a bounded linear operator. To study the spectrum of T, we first have to *complexify* both the space X and the operator T via a canonical procedure as detailed in, for example, [DGVW95, Chapter III.7]. We do not make the complexification explicit in notation, i.e. we write X for both the real Banach space and its complexification, and write T for both the operator on the real Banach space and complexified operator on the complexified Banach space.

3. In this thesis, we study stability of periodic orbits using the principle of linearized stability, which determines the *asymptotic* (in)stability of the periodic orbit (see Section A.3 in the appendix). However, we drop the word 'asymptotic' in the terminology, so '(in)stability of the periodic orbit' is meant to be understood as 'asymptotic (in)stability of the periodic orbit'.

4. Throughout this thesis, the fundamental solution of a time dependent ODE has *one* time argument, whereas the fundamental solution of a time dependent DDE has *two* time arguments. We explain this seeming discrepancy in notation here.

   Consider the time dependent ODE

$$\dot{y}(t) = A(t)y(t) \tag{1.10}$$

   with $A(t) \in \mathbb{R}^{N\times N}$. We let $Y(t) \in \mathbb{R}^{N\times N}$ be the fundamental solution of (1.10) with $Y(0) = I$, i.e. $Y(t)$ is the matrix-valued solution of

$$\frac{d}{dt}Y(t) = A(t)Y(t), \qquad Y(0) = I.$$

   For $x \in \mathbb{R}^N$, $y(t) = Y(t)x$ is the solution of (1.10) with initial condition $y(0) = x$ at time zero. Given any other starting time $s \in \mathbb{R}$, $y(t) = Y(t)Y(s)^{-1}x$ is the solution of (1.10) with initial condition $y(s) = x$ at time s. The interpretation of this is that we first use $Y(s)^{-1}$ to flow backwards from s to 0 and then use $Y(t)$ to flow forwards from 0 to t.

   For DDE, however, one can in general not flow backwards in time [HV93]. Therefore, we use a fundamental solution with *two* time arguments to be able to specify the initial condition at any given time. More precisely, consider the time-dependent DDE

$$\dot{y}(t) = L(t)y_t \tag{1.11}$$

   where $L(t) : C\left([-r,0],\mathbb{R}^N\right) \to \mathbb{R}^N$ is a bounded linear operator for every time $t \in \mathbb{R}$, and the *history segment* $y_t \in C\left([-r,0],\mathbb{R}^N\right)$ is defined as $y_t(\theta) = y(t+\theta)$, $\theta \in [-r,0]$. The fundamental solution is now a two-parameter family of operators

$$U(t,s) : C\left([-r,0],\mathbb{R}^N\right) \to C\left([-r,0],\mathbb{R}^N\right), \qquad t \geq s$$

   with the property that $y_t = U(t,s)\phi$ is the solution of (1.11) with initial condition $y_s = \phi \in C\left([-r,0],\mathbb{R}^N\right)$ at time s, see also [HV93, Chapter 8] and [DGVW95, Chapter 13].

Chapter 2

Limitations to Pyragas control

In 1995, Nakijima proved a qualitative constraint for stabilization via Pyragas control: in time periodic ODE with period $p > 0$, p-periodic solutions with an odd number of Floquet multipliers larger than 1 cannot be stabilized using Pyragas control [Nak97]. Although this result, which goes under the name *odd number limitation*, was proven for non-autonomous systems only, it was generally believed to also hold for autonomous systems. However, in [FFGHS07] Fiedler et al. gave a counterexample in the autonomous case, by stabilizing a periodic solution with one unstable multiplier larger than 1. Subsequently, Hooton and Amann [HA12] also addressed the autonomous case and gave a necessary condition on the gain matrix for stabilization of periodic solutions with an odd number of Floquet multipliers larger than 1.

In this chapter, we derive an invariance principle for Pyragas control which gives a unifying view of the above mentioned results. We show in Section 2.1 that, given a periodic orbit, the geometric multiplicity of the Floquet multiplier 1 is preserved under Pyragas control. For non-autonomous systems (where, generically, periodic orbits do not have a Floquet multiplier 1), this invariance principle leads to the odd number limitation, as discussed in Section 2.2. For autonomous systems (where periodic orbits always have a trivial Floquet multiplier 1), we *can* change the algebraic multiplicity of the Floquet multiplier 1. This leads to both the necessary condition on the gain matrix presented in [HA12] (cf. Section 2.4) and the positive stabilization result in [FFGHS07] (cf. Section 2.5).

In addition to revisiting the results in [Nak97; FFGHS07; HA12], we prove a new *any number limitation* in Section 2.3. We show that if the uncontrolled system has a Floquet multiplier larger than 1, and the gain matrix has real spectrum and commutes with the linearized ODE, then Pyragas control fails to stabilize. This in particular excludes stabilization with scalar control gain.

The invariance principle introduced in this chapter holds for a larger class of noninvasive control terms than Pyragas control only. But by discussing all possible generalizations from the start, we run the risk of losing clarity on the main application to Pyragas control. Therefore, this chapter solely focusses on Pyragas control; we return more general control terms in the outlook of this thesis (Chapter 10). The contents of Section 2.1 and Section 2.2 have also appeared in [WS21] in joint work with Isabelle Schneider.

2.1 An invariance principle

Consider the time-dependent ODE

$$\dot{x}(t) = f(x(t), t), \qquad t \geq 0 \tag{2.1}$$

with $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$. We make the following assumption on system (2.1):

Hypothesis 1.

1. *The function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is a C^2-function.*
2. *The function f is periodic with (not necessarily minimal) period $p > 0$ in its time argument, i.e. $f(x, t + p) = f(x, t)$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.*
3. *The ODE (2.1) has a periodic solution $x_*(t)$ with (not necessarily minimal) period p.*

We denote by $Y(t),\ t \in \mathbb{R}$ the fundamental solution of the linearized ODE

$$\dot{y}(t) = \partial_1 f(x_*(t), t) y(t) \tag{2.2}$$

with $Y(0) = I$, i.e. $Y(t) \in \mathbb{R}^{N \times N}$ is the unique matrix-valued function satisfying

$$\frac{d}{dt} Y(t) = \partial_1 f(x_*(t), t) Y(t), \qquad Y(0) = I.$$

We refer to the map $Y(p)$ as the **monodromy operator** and to its eigenvalues as the **Floquet multipliers** of system (2.2). We apply Pyragas control and write the controlled system as

$$\dot{x}(t) = f(x(t), t) + K\left[x(t) - x(t - p)\right]$$

with gain matrix $K \in \mathbb{R}^{N \times N}$. For $t \geq s$, we denote by $U(t, s) : X \to X,\ X := C\left([-p, 0], \mathbb{R}^N\right)$ the fundamental solution of the DDE

$$\dot{y}(t) = \partial_1 f(x_*(t), t) y(t) + K\left[y(t) - y(t - p)\right]. \tag{2.3}$$

with $U(s, s) = I$ (note that, in our notation, the fundamental solution of an ODE has one time argument, whereas the fundamental solution of a DDE has two time arguments, see Section 1.5). The monodromy operator $U(p, 0)$ of (2.3) is compact and hence all its non-zero spectrum consists of isolated eigenvalues of finite multiplicity, which we call the Floquet multipliers of system (2.3).

The following proposition compares the Floquet multiplier 1 of the ODE (2.2) with the Floquet multiplier 1 of the DDE (2.3). If 1 is a Floquet multiplier of the ODE (2.2) (i.e. if $1 \in \sigma(Y(p))$, we define its **geometric multiplicity** as the dimension of the space

$$\ker\left(I - Y(p)\right) = \{y \in \mathbb{C}^N \mid Y(p) y = y\}. \tag{2.4a}$$

If 1 is a Floquet multiplier of the DDE (2.3) (i.e. if $1 \in \sigma_{pt}(U(p, 0))$, we define its geometric multiplicity as the dimension of the space

$$\ker\left(I - U(p, 0)\right) = \{\varphi \in C\left([-p, 0], \mathbb{C}^N\right) \mid U(p, 0)\varphi = \varphi\}; \tag{2.4b}$$

the dimension of the space (2.4b) is finite since the operator $U(p, 0)$ is compact. If 1 is *not* a Floquet multiplier of the ODE (2.2) (resp. the DDE (2.3)), we say that it has geometric multiplicity zero.

Proposition 2.1 (Invariance principle for Pyragas control)**.** *The geometric multiplicity of the Floquet multiplier* 1 *is preserved under Pyragas control. That is, the geometric multiplicity of the Floquet multiplier* 1 *of the ODE* (2.2) *equals the geometric multiplicity of the Floquet multiplier* 1 *of the DDE* (2.3)*.*

Proof. We show that there is a one-to-one correspondence between elements of the space (2.4a) and elements of the space (2.4b). The vector $y \in \mathbb{C}^N \backslash \{0\}$ is an element of (2.4a) if and only if the ODE (2.2) has a solution that satisfies

$$\begin{cases} y(t+p) &= y(t), \qquad t \geq 0; \\ y(0) &= y; \end{cases} \tag{2.5}$$

(see Lemma A.3 in the Appendix). Similarly, $\varphi \in X\backslash\{0\}$ is an element of (2.4b) if and only if the DDE (2.3) has a solution that satisfies

$$\begin{cases} y(t+p) &= y(t), \qquad t \geq 0 \\ y(t) &= \varphi(t), \qquad t \in [-p, 0]. \end{cases} \tag{2.6}$$

But since the term $K[y(t) - y(t-p)]$ vanishes on p-periodic functions, the ODE (2.2) has a solution of the form (2.5) if and only if the DDE (2.3) has a solution of the form (2.6). Thus, there is a one-to-one correspondence between non-zero elements of (2.4a) and non-zero elements of (2.4b). This shows that the geometric multiplicity of the Floquet multiplier 1 of the ODE (2.2) equals the geometric multiplicity of the Floquet multiplier 1 of the DDE (2.3). □

2.2 The odd number limitation for Pyragas control

Using the invariance principle in Proposition 2.1, we give a new proof of the odd number limitation for Pyragas control. The original statement of the odd number limitation in [Nak97] is formulated for periodic solutions of non-autonomous system. Instead, we formulate the odd number limitation for non-degenerate periodic orbits: we say that x_* is a **non-degenerate** solution of the ODE (2.1) if 1 is not a Floquet multiplier of the linearization (2.2). By shifting the focus from (non-)autonomous systems to (non-)degenerate orbits, we (hopefully) clarify the confusion in the literature regarding the odd number limitation.

Theorem 1 (cf. Nakajima, '97)**.** *Consider the ODE* (2.1) *satisfying Hypothesis 1. Assume that* x_* *is a non-degenerate solution of* (2.1) *and that the linearized ODE* (2.2) *has an odd number (counting algebraic multiplicities) of Floquet multipliers larger than one.*

Then, for every gain matrix $K \in \mathbb{R}^{N \times N}$*,* x_* *is an unstable solution of*

$$\dot{x}(t) = f(x(t), t) + K\left[x(t) - x(t-p)\right]. \tag{2.7}$$

Proof. Fix a matrix $K \in \mathbb{R}^{N \times N}$ and introduce the homotopy parameter $\alpha \in [0, 1]$:

$$\dot{x}(t) = f(x(t), t) + \alpha K\left[x(t) - x(t-p)\right].$$

For $\alpha \in [0, 1]$, we denote by $U_\alpha(p, 0)$ the monodromy operator of the linearized DDE

$$\dot{y}(t) = \partial_1 f(x_*(t), t) y(t) + \alpha K\left[y(t) - y(t-p)\right];$$

in particular $U_0(p,0)$ is the monodromy operator of the uncontrolled system and $U_1(p,0)$ is the monodromy operator of the controlled system.

The operator $U_\alpha(p,0)$ is compact for all $\alpha \in [0,1]$ and the map $\alpha \mapsto U_\alpha(p,0)$ is continuous; therefore the eigenvalues of $U_\alpha(p,0)$ depend continuously on α (in the sense of [Kat95]). For all $\alpha \in [0,1]$, we can bound the spectral radius of $U_\alpha(p,0)$ by

$$\sup_{\alpha\in[0,1]} \|U_\alpha(p,0)\| < \infty.$$

Therefore, the number of eigenvalues outside the unit circle cannot change by eigenvalues "coming from infinity", but can only change by an eigenvalue crossing the unit circle. Moreover, if μ is an eigenvalue of $U_\alpha(p,0)$, then $\overline{\mu}$ is an eigenvalue as well, so non-real eigenvalues of $U_\alpha(p,0)$ appear in pairs.

For $\alpha \in [0,1]$, let n_α be the number of eigenvalues (counting algebraic multiplicities) of $U_\alpha(p,0)$ on the open half line

$$\{\mu \in \mathbb{C} \mid \mu \in (1,\infty)\}.$$

Then by the previous remarks, the parity of n_α only changes (as a function of α) if an eigenvalue crosses the point $1 \in \mathbb{C}$. However, by the assumption that x_* is non-degenerate, 1 is not an eigenvalue of $U_0(p,0)$; Proposition 2.1 implies that 1 is not an eigenvalue of $U_\alpha(1,0)$ for all $\alpha \in [0,1]$. Therefore, the parity of n_α does not change when varying α. By assumption, $n_{\alpha=0}$ is odd and hence n_α is odd for all $\alpha \in [0,1]$. So in particular, the linearized system

$$\dot{y}(t) = \partial_1 f(x_*(t))y(t) + K\left[y(t) - y(t-p)\right]$$

has at least one Floquet multiplier larger than one, and x_* is an unstable solution of (2.7). □

The conditions of Theorem 1 are not satisfied if the ODE (2.1) is autonomous. Indeed, suppose that

$$\partial_t f(x,t) = 0 \qquad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R} \tag{2.8}$$

and that the periodic orbit x_* is not an equilibrium. Then we can differentiate the relation

$$\dot{x}_*(t) = f(x_*(t),t)$$

with respect to t to see that $\dot{x}_*(t)$ is a p-periodic solution of the linearized ODE (2.2). So in this case (2.2) has a Floquet multiplier 1, called the **trivial Floquet multiplier**. Proposition 2.1 states that the geometric multiplicity of this trivial Floquet multiplier is preserved under control, but its *algebraic* multiplicity can change, and this indeed what happens in the positive stabilization result [FFGHS07]. We explore this in more detail in Section 2.4 and Section 2.5.

2.3 Any number limitation for commuting gain matrices with real spectrum

In this section, we discuss failure of Pyragas control for a class of gain matrices. We make no assumption on the Floquet multiplier 1, and hence the result applies to both autonomous and non-autonomous system. Moreover, we only assume that there *exists* a real Floquet multiplier larger than 1, but we do not make any assumptions on the number of Floquet multipliers larger than 1. Therefore, we refer to the result as the *any number limitation.*

Theorem 2 (Any number limitation)**.** *Consider the ODE* (2.1) *satisfying Hypothesis 1. Assume that the linearized ODE* (2.2) *has at least one Floquet multiplier larger than one. Moreover, let* $K \in \mathbb{R}^{N \times N}$ *be such that*

1. *K has only real eigenvalues, i.e. $\sigma(K) \subseteq \mathbb{R}$;*

2. *K commutes with the linear term in* (2.2)*, i.e.*

$$K\partial_1 f(x_*(t), t) = \partial_1 f(x_*(t), t)K \qquad \textit{for all } t \in \mathbb{R}. \tag{2.9}$$

Then x_* is an unstable solution of

$$\dot{x}(t) = f(x(t), t) + K\left[x(t) - x(t-p)\right]. \tag{2.10}$$

Proof. We divide the proof into three steps:

Step 1: We use the commutativity property (2.9) to find a common eigenvector for the unstable eigenvalue of the monodromy operator and the gain matrix K. More precisely, let $Y(p)$ be the monodromy operator of the linearized ODE (2.2); by assumption, there exists a $\mu_* \in \sigma(Y(p))$ such that $\mu_* > 1$. Moreover, assumption (2.9) implies that $Y(p)K = KY(p)$. If $y \in \mathbb{C}^N$ satisfies $(\mu_* I - Y(p))y = 0$, then Ky satisfies

$$(\mu_* I - Y(p))Ky = K(\mu_* I - Y(p))y = 0$$

so the space

$$\ker(\mu_* I - Y(p)) := \{y \in \mathbb{C}^N \mid (\mu_* I - Y(p))y = 0\} \tag{2.11}$$

is invariant under K. Hence there exists a non-zero element of the space (2.11) which is also an eigenvector of K, i.e. there exists an $y_0 \in \mathbb{C}^N \backslash \{0\}$ and $k \in \sigma(K) \subseteq \mathbb{R}$ such that

$$Y(p)y_0 = \mu_* y_0 \qquad \text{and} \qquad Ky_0 = ky_0. \tag{2.12}$$

Step 2: Let $U(p, 0)$ be the monodromy operator of the linearized DDE

$$\dot{y}(t) = \partial_1 f(x_*(t), t)y(t) + K\left[y(t) - y(t-p)\right]. \tag{2.13}$$

We show that

$$\left\{\mu \in \mathbb{C}\backslash\{0\} \mid \mu - \mu_* e^{k(1-\mu^{-1})p} = 0\right\} \subseteq \sigma_{pt}(U(p,0))\backslash\{0\} \tag{2.14}$$

i.e. roots of the *scalar* equation $z - \mu_* e^{k(1-z^{-1})p} = 0$ are Floquet multipliers of the controlled system. Indeed, let $\mu \in \mathbb{C}\backslash\{0\}$ satisfy

$$\mu - \mu_* e^{k(1-\mu^{-1})p} = 0 \tag{2.15}$$

and define

$$y(t) := Y(t)e^{K(1-\mu^{-1})t}y_0$$

with y_0 as in (2.12) and with $Y(t)$ the fundamental solution of the ODE $\dot{y}(t) = \partial_1 f(x_*(t), t)y(t)$. Assumption (2.9) implies that $y(t)$ solves the ODE

$$\dot{y}(t) = \partial_1 f(x_*(t), t)y(t) + K\left[1 - \mu^{-1}\right]y(t). \tag{2.16}$$

Moreover, since $Y(t+p) = Y(t)Y(p)$ (see (A.3) in the appendix), it holds that

$$y(t+p) = Y(t)Y(p)e^{K(1-\mu^{-1})(t+p)}y_0 \tag{2.17}$$

$$= \mu_* e^{k(1-\mu^{-1})p} y(t) = \mu y(t) \tag{2.18}$$

where in the last step we used (2.15). So $y(t)$ solves the ODE (2.16) and satisfies $y(t+p) = \mu y(t)$. But this means that $\mu^{-1}y(t) = y(t-p)$; hence $y(t)$ also solves the DDE

$$\dot{y}(t) = \partial_1 f(x_*(t), t)y(t) + K\left[y(t) - y(t-p)\right] \tag{2.19}$$

and satisfies $y(t+p) = \mu y(t)$. Since solutions of (2.19) with $y(t+p) = \mu y(t)$ correspond to eigenvectors of $U(p, 0)$ with eigenvalue μ, we conclude that μ is an eigenvalue of $U(p, 0)$, i.e. (2.14) holds.

Step 3: Define

$$f : \mathbb{R}_{>0} \to \mathbb{R}, \qquad f(x) = x - \mu_* e^{k(1-x^{-1})p}.$$

Then

$$f(1) = 1 - \mu_* < 0 \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \infty$$

and by the intermediate value theorem there exists a $\mu \in (1, \infty)$ with $f(\mu) = 0$. Then the relation (2.14) implies that the monodromy operator $U(p, 0)$ has an eigenvalue $\mu > 1$, and therefore x_* is an unstable solution of (2.10). □

In Step 2 of the proof, we showed with somewhat ad hoc methods that roots of a certain scalar-valued equation are eigenvalues of the monodromy operator, yielding the inclusion (2.14). In fact, it turns out for the linear DDE (2.13) *every* non-zero eigenvalue of the monodromy operator can be found as a root of a scalar-valued equation, which means a significant dimension reduction of the eigenvalue problem. We return to this issue at length in Chapter 5 and Chapter 7.

If $K = kI$, with $k \in \mathbb{R}$ and I the identity matrix, then $\sigma(K) \subseteq \mathbb{R}$ and (2.9) is trivially satisfied. Hence Theorem 2 implies that if the uncontrolled system has a Floquet multiplier larger than 1, then *Pyragas control with scalar control gain always fails*:

Corollary 2.2 (Any number limitation for scalar control gain)**.** *Consider system* (2.1) *satisfying Hypothesis 1. Assume that the linearized ODE* (2.2) *has at least one Floquet multiplier larger than one. Then, for any $k \in \mathbb{R}$, x_* is unstable as a solution of the controlled system*

$$\dot{x}(t) = f(x(t), t) + k\left[x(t) - x(t-p)\right].$$

2.4 Stabilization of autonomous systems

In [HA12], Hooton and Amann consider autonomous systems where the periodic orbit has an odd number of Floquet multiplier larger than 1, and give a necessary condition on the gain matrix for successful stabilization. The proof uses that the Floquet multipliers of the controlled system can be found as roots of a scalar-valued equation. This scalar-valued equations 'counts' the algebraic multiplicities of the Floquet multipliers, but not their geometric multiplicities. In this section, we given an alternative proof to the result in [HA12]. The proof presented here heavily relies on the distinction between algebraic/geometric multiplicity, which is crucial in the light of Proposition 2.1. In this way, the proof shows that the rather technical condition on the gain matrix found in [HA12] has a natural interpretation in terms of Jordan chains of the controlled system.

Proposition 2.3 (cf. [HA12]). *Consider the autonomous ODE*

$$\dot{x}(t) = f(x(t)), \qquad t \geq 0 \tag{2.20}$$

with $f : \mathbb{R}^N \to \mathbb{R}^N$ *a* C^1*-function and assume that* (2.20) *has a periodic solution* x_* *with (not necessarily minima) period* $p > 0$*. Denote by* $Y(t)$ *the fundamental solution of the linear ODE* $\dot{y}(t) = f'(x_*(t))y(t)$ *with* $Y(0) = I$*. Suppose that the monodromy operator* $Y(p)$ *satisfies the following two conditions:*

1. *The trivial eigenvalue* $1 \in \sigma(Y(p))$ *is algebraically simple.*
2. $Y(p)$ *has an odd number (counting algebraic multiplicities) of real eigenvalues strictly larger than 1.*

Moreover, let $y_0 \in \mathbb{C}^N \backslash \{0\}$ *be such that*

$$Y(p)^T y_0 = y_0 \qquad \text{and} \qquad \langle y_0, \dot{x}(0) \rangle = 1. \tag{2.21}$$

If $K \in \mathbb{R}^{N \times N}$ *satisfies*

$$\left\langle y_0, \int_0^p Y(s)^{-1} K Y(s) ds \dot{x}_*(0) \right\rangle < 1, \tag{2.22}$$

then x_* *is an unstable solution of*

$$\dot{x}(t) = f(x(t)) + K\left[x(t) - x(t-p)\right]. \tag{2.23}$$

Proof. As in the proof of the odd number limitation (Theorem 1), we introduce the homotopy parameter $\alpha \in [0, 1]$

$$\dot{x}(t) = f(x(t)) + \alpha K\left[x(t) - x(t-p)\right]$$

and linearize around the solution x_*

$$\dot{y}(t) = f'(x_*(t)) + \alpha K\left[y(t) - y(t-p)\right]. \tag{2.24}$$

From here, we divide the proof of the proposition in three steps:

Step 1: Suppose that for all $\alpha \in [0, 1]$ the Floquet multiplier 1 of (2.24) is algebraically simple. We prove that then x_* is unstable as a solution of (2.23).

For $\alpha \in [0, 1]$, denote by n_α the number of Floquet multipliers (counting algebraic multiplicities) of (2.24) on the half-line $(1, \infty)$. If for all $\alpha \in [0, 1]$, the Floquet multiplier 1 of (2.24) is algebraically simple, the parity of n_α is constant (cf. proof of Theorem 1). Since by assumption $n_{\alpha=0}$ is odd, this means that $n_{\alpha=1}$ is odd and hence there is at least one Floquet multiplier on the half-line $(1, \infty)$. This implies that x_* is an unstable solution of (2.23).

Step 2: For fixed $\alpha \in [0, 1]$, we show that the Floquet multiplier 1 of (2.24) has algebraic multiplicity strictly larger than 1 if and only if

$$\alpha \left\langle y_0, \int_0^p Y(s)^{-1} K Y(s) ds \dot{x}_*(0) \right\rangle = 1. \tag{2.25}$$

By assumption, the eigenvalue $1 \in \sigma(Y(p))$ is algebraically (and therefore also geometrically) simple; Proposition 2.1 implies that the Floquet multiplier 1 of (2.24) is geometrically simple for all $\alpha \in [0,1]$. So the Floquet multiplier 1 of (2.24) has algebraic multiplicity strictly larger than 1, if and only if we have a Jordan chain of at least length 2, i.e. if and only the linear DDE (2.24) has a solution of the form

$$y(t) = z(t) + t\dot{x}_*(t), \qquad z(t+p) = z(t). \tag{2.26}$$

Making the Ansatz (2.26) in the linar DDE (2.24) gives

$$\dot{z}(t) = f'(x_*(t))z(t) + \alpha K p\dot{x}_*(t) - \dot{x}_*(t). \tag{2.27}$$

Via Variation of Constants, we solve this inhomogeous ODE for z as

$$z(t) = Y(t)z(0) + \int_0^t Y(t)Y(s)^{-1}\left[\alpha Kp - I\right] Y(s)\dot{x}_*(0)ds.$$

Here $Y(t)$ is the fundamental solution of $\dot{y}(t) = f'(x_*(t))y(t)$ with $Y(0) = I$ and we used that $\dot{x}_*(s) = Y(s)\dot{x}_*(0)$. Thus, we find that (2.24) has a solution of the form (2.26) if and only if $z(0)$ satisfies

$$Y(p)z(0) + Y(p)\int_0^p Y(s)^{-1}[\alpha Kp - I]Y(s)\dot{x}_*(0)ds = z(0)$$

i.e. if and only if

$$\left(I - Y(p)\right)z(0) = Y(p)\int_0^p Y(s)^{-1}\alpha KpY(s)ds\dot{x}_*(0) - p\dot{x}_*(0). \tag{2.28}$$

Since by assumption the eigenvalue $1 \in \sigma(Y(p))$ is algebraically simple, it holds that

$$\text{range}(I - Y(p)) \simeq \left(\ker\left(I - Y(p)^T\right)\right)^{\perp} \simeq \left(\text{span}(y_0)\right)^{\perp}.$$

Thus, (2.28) is solvable for $z(0)$ if and only if

$$p\left\langle y_0, Y(p)\int_0^p Y(s)^{-1}\alpha KY(s)ds\dot{x}_*(0) - \dot{x}_*(0)\right\rangle = 0$$

i.e. if and only if (2.25) holds. We conclude that the Floquet multiplier 1 of the linear DDE (2.24) has algebraic multiplicity strictly larger than 1 if and only if (2.25) holds.

Step 3: Suppose that the gain matrix $K \in \mathbb{R}^{N\times N}$ satisfies condition (2.22) and consider the function

$$g : [0,1] \to \mathbb{R}, \qquad g(\alpha) = \alpha\left\langle y_0, \int_0^p Y(s)^{-1}KY(s)ds\dot{x}_*(0)\right\rangle - 1.$$

Condition (2.22) implies that $g(1) < 0$, moreover $g(0) = -1 < 0$ and, since g is an affine function, $g(\alpha) < 0$ for all $\alpha \in [0,1]$. Hence Step 2 implies that the Floquet multiplier 1 of the linear DDE (2.24) is algebraically simple for all $\alpha \in [0,1]$; Step 1 then implies that x_* is an unstable solution of (2.23). □

2.5 Example: Hopf normal form

In this section we consider the ODE

$$\dot{x}(t) = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} x(t) + \|x(t)\|^2 \begin{pmatrix} 1 & -\gamma \\ \gamma & 1 \end{pmatrix} x(t) \tag{2.29}$$

with $x(t) \in \mathbb{R}^2$ and parameters $\gamma, \lambda \in \mathbb{R}$. The ODE (2.29) is the normal form of the Hopf bifurcation [Kuz95] and is also referred to as the *Stuart-Landau oscillator*. In [FFGHS07], the authors use Pyragas control to succesfully stabilize periodic solutions of (2.29). This result serves as a counterexample to the claim made in [Nak97] that the odd number limitation also holds for autonomous systems.

An attractive feature of system (2.29) is that one can explicitly compute its periodic solutions and their Floquet multipliers, and can therefore explicitly compute the condition in Proposition 2.3; indeed, the authors of [HA12] state the result of this calculation. In this section, we do this computation in more detail, both for completeness of this chapter and as an illustrative example.

For $\theta \in \mathbb{R}$, denote by $M(\theta) \in \mathbb{R}^{2\times 2}$ the matrix

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

If $\lambda < 0$, then

$$x_*(t) := M(\omega t) r_* \tag{2.30}$$

with

$$\omega := 1 - \gamma\lambda, \qquad r_* := \begin{pmatrix} \sqrt{-\lambda} \\ 0 \end{pmatrix}$$

is a periodic solution of the ODE (2.29). Moreover, if we write

$$F(x) := \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} x + \|x\|^2 \begin{pmatrix} 1 & -\gamma \\ \gamma & 1 \end{pmatrix} x \tag{2.31}$$

then (2.29) is *equivariant* in the sense that

$$F(M(\theta)x) = M(\theta)F(x) \tag{2.32}$$

for all $x \in \mathbb{R}^2$, $\theta \in \mathbb{R}$. Due to this equivariance, we can explicitly compute the fundamental solution of the linearization of (2.29) around x_*:

Lemma 2.4. *Consider the system*

$$\dot{y}(t) = F'(x_*(t))y(t) \tag{2.33}$$

with F given in (2.31) *and x_* given in* (2.30)*. Then the fundamental solution of the linear ODE* (2.33) *is given by*

$$Y(t) = M(\omega t)e^{At} \tag{2.34}$$

with

$$A = \begin{pmatrix} -2\lambda & 0 \\ -2\gamma\lambda & 0 \end{pmatrix}. \tag{2.35}$$

Proof. We apply the coordinate transformation

$$z(t) := M(-\omega t)y(t)$$

with inverse transformation $y(t) = M(\omega t)z(t)$. Then $z(0) = y(0)$ and moreover

$$\dot{z}(t) = -\omega J z(t) + M(-\omega t)F'(M(\omega t)r_*)M(\omega t)z(t) \tag{2.36}$$

where

$$J := \left.\frac{d}{d\theta}\right|_{\theta=0} M(\theta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Differentiating the equivariance relation (2.32) with respect to x implies that

$$F'(M(\theta)x)M(\theta) = M(\theta)F'(x) \tag{2.37}$$

for all $x \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$. Therefore we can rewrite (2.36) and conclude that z satisfies

$$\begin{cases} \dot{z}(t) & = \left(-\omega J + F'(r_*)\right) z(t), \\ z(0) & = y(0). \end{cases} \tag{2.38}$$

An explicit calculation gives that $-\omega J + F'(r_*) = A$ and hence (2.38) is solved by $z(t) = e^{At}y(0)$. Therefore, (2.33) is solved by

$$y(t) = M(\omega t)e^{At}y(0)$$

and the fundamental solution of (2.33) is given by $Y(t) = M(\omega t)e^{At}$, as claimed. □

The minimal period $p > 0$ of (2.30) is given by

$$\begin{cases} p = \frac{2\pi}{\omega} & \text{if } \omega > 0; \\ p = -\frac{2\pi}{\omega} & \text{if } \omega < 0. \end{cases} \tag{2.39}$$

Hence the monodromy operator $Y(p)$ is given by

$$Y(p) = M(\pm 2\pi)e^{Ap} = e^{Ap}.$$

Since $\sigma(A) = \{0, -2\lambda\}$, the eigenvalues of $Y(p)$ are $\{1, e^{-2\lambda p}\}$:

Corollary 2.5. *The Floquet multipliers of system* (2.33) *are given by*

$$\mu_1 = 1, \qquad \mu_2 = e^{-2\lambda p}. \tag{2.40}$$

Therefore, for $\lambda < 0$, the periodic solution (2.30) *is an unstable solution of the ODE* (2.29).

To stabilize the periodic solution (2.30), we apply Pyragas control and write the controlled system as

$$\dot{x}(t) = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} x(t) + \|x(t)\|^2 \begin{pmatrix} 1 & -\gamma \\ \gamma & 1 \end{pmatrix} x(t) + K\left[x(t) - x(t-p)\right] \tag{2.41}$$

with $K \in \mathbb{R}^{2\times 2}$ and p as in (2.39). We apply Proposition 2.3 and find the following necessary condition on K for the control to be successful:

Proposition 2.6 (cf. [HA12])**.** *If* $K \in \mathbb{R}^{2\times 2}$ *satisfies*

$$\int_0^p \begin{pmatrix} -\gamma, & 1 \end{pmatrix} M(-\omega t) K M(\omega t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt < 1 \tag{2.42}$$

then the periodic solution (2.30) *is an unstable solution of* (2.41)*.*

In particular, suppose that K *is of the form*

$$K = kM(\beta) \tag{2.43}$$

with $k \in \mathbb{R}$ *and* $\beta \in [0, 2\pi]$*. Then condition* (2.42) *amounts to*

$$kp\,(\cos\beta + \gamma \sin\beta) < 1. \tag{2.44}$$

Proof. For x_* as in (2.30), it holds that

$$\begin{aligned} \dot{x}_*(0) &= \omega J r_* \\ &= \omega\sqrt{-\lambda} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Define

$$y_0 := \frac{1}{\omega\sqrt{-\lambda}} \begin{pmatrix} -\gamma, & 1 \end{pmatrix}.$$

Then y_0 satisfies

$$\begin{cases} y_0 A = 0; \\ \langle y_0, \dot{x}_*(0) \rangle = 1. \end{cases}$$

Since $Y(p) = e^{Ap}$, this implies that

$$\begin{cases} y_0 Y(p) = y_0; \\ \langle y_0, \dot{x}_*(0) \rangle = 1, \end{cases}$$

i.e. y_0 satisfies the conditions (2.21). Since $y_0 A = 0$, it holds that

$$y_0 Y(t)^{-1} = y_0 e^{-At} M(-\omega t) = y_0 M(-\omega t).$$

This yields

$$\begin{aligned} \left\langle y_0, \int_0^p Y(t)^{-1} K Y(t) ds \dot{x}_*(0) \right\rangle &= \int_0^p \langle y_0 M(-\omega t), K M(\omega t) \dot{x}_*(0) \rangle \, dt \\ &= \int_0^p \begin{pmatrix} -\gamma, 1 \end{pmatrix} M(-\omega t) K M(\omega t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt. \end{aligned}$$

So if (2.42) holds, then Proposition 2.3 implies that x_* is an unstable solution of (2.41).

If K is given by (2.43), then $M(-\omega t)K = KM(\omega t)$ and hence

$$\begin{aligned} \int_0^p \begin{pmatrix} -\gamma, 1 \end{pmatrix} M(-\omega t) K M(\omega t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt &= \int_0^p \begin{pmatrix} -\gamma, 1 \end{pmatrix} kM(\beta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt \\ &= kp\,(\cos\beta + \gamma\sin\beta). \end{aligned}$$

Thus, if K is of the form $K = kM(\beta)$, then condition (2.42) amounts to $kp(\cos\beta + \gamma\sin\beta) < 1$. □

In [FFGHS07], the authors consider Pyragas control of the Hopf normal form with gain matrices of the form (2.43). In agreement with Theorem 2, they show that control fails if $\beta = 0$, i.e. if the control matrix is a scalar multiple of the identity. Additionally, they show that if

$$\beta \neq 0 \qquad \text{and} \qquad kp\,(\cos\beta + \gamma \sin\beta) > 1$$

then for small $\lambda < 0$, (2.30) is stable as a solution of (2.41). In other words, Theorem 2 and Proposition 2.6 give necessary conditions for stabilization, and [FFGHS07] shows that these conditions are actually sufficient close to a Hopf bifurcation.

Chapter 3

Equivariant control of discrete waves

We adress equivariant control of discrete waves and state this thesis' main result on this topic. We first introduce the relevant notions for this result in Section 3.1 and proceed to state the result in Section 3.2 (Theorem 3). Since the majority of the rest of this thesis is concerned with the proof of Theorem 3, we give an outline of the proof in Section 3.3.

3.1 Equivariance and discrete waves

Consider the ODE

$$\dot{x}(t) = f(x(t)), \qquad t \geq 0 \tag{3.1}$$

with $f : \mathbb{R}^N \to \mathbb{R}^N$. We make the following assumptions on (3.1):

Hypothesis 2.

1. *$f : \mathbb{R}^N \to \mathbb{R}^N$ is a C^2 function.*
2. *system* (3.1) *has a periodic solution x_* with* minimal *period $p > 0$;*
3. *system* (3.1) *is equivariant with respect to a* compact *group $\Gamma \subseteq GL(N, \mathbb{R})$, i.e.*

$$f(\gamma x) = \gamma f(x) \qquad \textit{for all } x \in \mathbb{R}^N,\ \gamma \in \Gamma.$$

As in Section 1.3, we define the group of *spatial symmetries* of x_* as

$$K_* := \{\gamma \in \Gamma \mid \gamma x_*(0) = x_*(0)\}.$$

and the group of *spatial-temporal symmetries* of x_* as

$$H_* := \{\gamma \in \Gamma \mid \gamma O = O\},$$

where $O = \{x_*(t) \mid t \in \mathbb{R}\}$ denotes the orbit of x_*. Denote by $\Theta : H_* \to S^1$ the map with the property that

$$hx_*(t) = x_*(t + \Theta(h)p) \tag{3.2}$$

holds for all $h \in H_*$ and all $t \in \mathbb{R}$. Since $\Theta : H_* \to S^1$ is in fact a group homomorphism with $K_* = \ker \Theta$, K_* is a normal subgroup of H_* and

$$H_*/K_* \simeq \begin{cases} \mathbb{Z}_n & \text{for some } n \in \mathbb{N}, \text{ or} \\ S^1 & \end{cases}$$

where $\mathbb{Z}_n$ denotes the cyclic group of order n. As in Section 1.3, we say that x_* is a **rotating wave** if $H_*/K_* \simeq S^1$ and we say that x_* a **discrete wave** if $H_*/K_* \simeq \mathbb{Z}_n$ for some $n \in \mathbb{N}$. Moreover, we make the following definition:

Definition 3.1. Let $Y(t)$ be the fundamental solution of

$$\dot{y}(t) = f'(x_*(t))y(t) \tag{3.3}$$

with $Y(0) = I$. For $h \in H_*$, we define the **reduced monodromy operator (associated to h)** as

$$Y_h : \mathbb{R}^N \to \mathbb{R}^N, \qquad Y_h := h^{-1}Y(\Theta(h)p). \tag{3.4}$$

Note that $\dot{x}_*(t)$ is a solution of (3.3), i.e.

$$Y(t)\dot{x}_*(0) = \dot{x}_*(t)$$

for $t \geq 0$. Moreover, differentiating (3.2) gives that

$$h^{-1}Y(\Theta(h)p)\dot{x}_*(0) = h^{-1}\dot{x}_*(\Theta(h)p) = \dot{x}_*(0).$$

So $\dot{x}_*(0)$ is an eigenvector of Y_h with eigenvalue 1; we call this eigenvalue 1 the **trivial eigenvalue** of Y_h. Moreover, we prove in Chapter 4 that if μ is an eigenvalue of Y_h, then (3.3) has a solution satisfying

$$y(t + \Theta(h)p) = \mu h y(t).$$

In particular, if Y_h has an eigenvalue $|\mu| > 1$, then the zero equilibirum of (3.3) is unstable and x_* is unstable as a solution of (3.1).

3.2 Equivariant control of discrete waves: main result

We state this thesis' main result on equivariant control of discrete waves. We formulate the conditions of this result in terms of the reduced monodromy operator, as defined in (3.4).

Theorem 3. *Consider the ODE* (3.1) *satisfying Hypothesis 2. Assume that x_* is a discrete wave, i.e. $H_*/K_* \simeq \mathbb{Z}_n$ for some $n \geq 1$. Moreover, assume that there exists $h \in H_*$ such that the reduced monodromy operator Y_h defined in* (3.4) *has the following properties:*

1. *the eigenvalue $1 \in \sigma(Y_h)$ is algebraically simple and Y_h has no other eigenvalues on the unit circle.*

2. *if $\mu \in \sigma(Y_h)$ and $|\mu| > 1$, then*

$$-e^2 < \mu < -1. \tag{3.5}$$

Then, there exists an open interval $I \subseteq \mathbb{R}_{<0}$ such that for $k \in I$, x_* is a stable solution of

$$\dot{x}(t) = f(x(t)) + k\left[x(t) - hx(t - \Theta(h)p)\right]. \tag{3.6}$$

Theorem 3 assumes that any eigenvalue of the reduced monodromy operator Y_h that lies outside the unit circle, in fact lies on the negative real axis. However, Theorem 3 makes no assumptions on the eigenvalues of the monodromy operator $Y(p)$ of (3.3). It is possible that the reduced monodromy operator Y_h satisfies the conditions of Theorem 3, whereas the monodromy operator $Y(p)$ has an eigenvalue $\mu > 1$. For example, consider the Lorenz equation

$$\begin{cases} \dot{x}_1 = -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 = -x_1 x_3 + \lambda x_1 - x_2 \\ \dot{x}_3 = x_1 x_2 - b x_3 \end{cases} \tag{3.7}$$

for $\sigma = 10,\ b = 8/3$ and parameter $\lambda \in \mathbb{R}$. System (3.7) is equivariant with respect to the group $\mathbb{Z}_2 = \{e, h\}$, where e is the identity element and h acts on $\mathbb{R}^3$ as

$$(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3).$$

In [WS06], Wulff and Schebes show numerically that system (3.7) has a periodic solution with $H_* = \mathbb{Z}_2$ and $K_* = \{e\}$, so this periodic solution has a nontrivial spatial-temporal symmetry. For $\lambda \simeq 154$, this periodic solution undergoes a bifurcation where the monodromy operator $Y(p)$ has an eigenvalue $+1$, whereas the reduced monodromy operator Y_h has an eigenvalue -1. There exist parameter values close to this bifurcation point so that Y_h has an eigenvalue smaller than -1 and satisfies the assumptions of Theorem 3, whereas $Y(p)$ has an eigenvalue larger than 1. For these parameter values, equivariant control (3.6) is successful; however, by the 'any number limitation' (Theorem 2) Pyragas control with scalar control gain

$$\dot{x}(t) = f(x(t)) + k\left[x(t) - x(t-p)\right]$$

fails to stabilize. So in this situation, equivariant control with scalar control gain overcomes failure of Pyragas control with scalar control gain; see also the discussion in Chapter 10. The bifurcation that occurs in (3.7) is an example of a so-called 'flip-pitchfork bifurcation', as introduced by Fiedler in [Fie88] (see [WS06] for more examples).

3.3 Outline of the proof

The proof of Theorem 3 combines stability theory for equivariant dynamical systems with the notion of characteristic matrix functions. We introduce these concepts in Chapter 4 and Chapter 5, respectively. With the technical tools in our hands, we then proceed with the proof of Theorem 3:

- Chapter 6 discusses stability theory for equivariant DDE (Chapter 4 adresses the case of equivariant ODE). In particular, we show that the eigenvalues of the reduced monodromy operator determine the stability of the target periodic solution.

- In Chapter 7, we prove that the reduced monodromy operator has a characteristic matrix function. As a consequence, the eigenvalues of the reduced monodromy operator are roots of a finite-dimensional equation.

- In Chapter 8, we determine the roots of this finite-dimensional equation, and thus determine the eigenvalues of the reduced monodromy operator.
- Chapter 9 concludes by proving Theorem 3.

Theoretical background

Chapter 4

Equivariant Floquet theory for ODE

This chapter develops equivariant Floquet theory for ODE. 'Classical' (i.e. non-equivariant) Floquet theory shows that the stability of p-periodic solutions of ODE is determined by the eigenvalues of the monodromy operator, which captures how solutions of the linearized system evolve under a time step p. We show that in the presence of additional symmetry relations, the asymptotic stability of the periodic solution is determined by the eigenvalues of the *reduced monodromy operator*, which involves computing how solutions of the linearized system flow under a fraction of the period.

The equivariant Floquet theory introduced here is inspired by results by Fiedler in [Fie88] on bifurcations of periodic orbits with symmetries. In particular, the notion of a reduced monodromy operator is inspired by an equivariant version of the Poincaré map introduced in [Fie88, page 55]; the relation between the monodromy operator and reduced monodromy operator in Lemma 4.2 is inspired by [Fie88, Lemma 5.7]. The *name* for the reduced monodromy operator is inspired by the name 'reduced Poincaré map' introduced by Wulff and Schebesch in [WS06].

The main difference between the approach in [Fie88; WS06] and the approach here is that we work with the flow of the linearized ODE rather than with the Poincaré map. This choice is chiefly motivated by our interest in DDE: for the DDE studied in this thesis, we have a very explicit expression for the reduced monodromy operator (see Section 7.1), but not for the reduced Poincaré map.

This chapter is organized as follows: in Section 4.1, we show that (spatial-)temporal symmetries of the target periodic solution induce (spatial-)temporal relations on the linearized equation and its fundamental solution. In Section 4.2, we use these relations to give a decomposition of the monodromy operator and to give a decomposition of the state space. Section 4.3 then discusses how the eigenvalues of the reduced monodromy operator determine the stability of the target periodic orbit.

4.1 Relations on the fundamental solution

Consider the ODE

$$\dot{x}(t) = f(x(t)), \qquad t \geq 0 \tag{4.1}$$

with $f : \mathbb{R}^N \to \mathbb{R}^N$. As in Chapter 3, we make the following assumptions on system (4.1):

Hypothesis 3.

1. *$f : \mathbb{R}^N \to \mathbb{R}^N$ is a C^2-function.*

2. *system* (4.1) *has a periodic solution* x_* *with* minimal *period* $p > 0$*;*

3. *system* (4.1) *is equivariant with respect to a compact group* $\Gamma \subseteq GL(N, \mathbb{R})$*, i.e.*

$$f(\gamma x) = \gamma f(x) \qquad \textit{for all } x \in \mathbb{R}^N,\ \gamma \in \Gamma.$$

We consider the associated symmetry groups

$$\begin{aligned} K_* &= \{\gamma \in \Gamma \mid \gamma x_*(t) = x_*(t)\ \forall t\} \qquad &&\text{(spatial symmetries)}, \\ H_* &= \{\gamma \in \Gamma \mid \gamma \{x_*(t)\} = \{x_*(t)\}\} \qquad &&\text{(spatial-temporal symmetries)} \end{aligned} \tag{4.2}$$

and the induced map

$$\Theta : H_* \to S^1, \qquad h x_*(t) = x_*(t + \Theta(h)p).$$

We define $A(t) := f'(x_*(t))$ and denote by $Y(t)$ the fundamental solution of the linear ODE

$$\dot{y}(t) = A(t)y(t) \tag{4.3}$$

with $Y(0) = I$. Since

$$A(t + p) = A(t)$$

for all t, we have that

$$Y(t + p)Y(s + p)^{-1} = Y(t)Y(s)^{-1} \tag{4.4}$$

for $t \geq s$; see Section A.1 in the Appendix. The next lemma shows that the group of spatial symmetries K_* induces spatial relations on the fundamental solution; and the group of spatial-temporal symmetries H_* induces spatial-temporal relations on the fundamental solution.

Lemma 4.1. *Consider the ODE* (4.1) *satisfying Hypothesis 3. Let* $A(t) := f'(x_*(t))$ *and let* $Y(t)$ *be the fundamental solution of the linear ODE* $\dot{y}(t) = f'(x_*(t))y(t)$ *with* $Y(0) = I$*.*

If k *is an element of the group of spatial symmetries* K_**, it holds that*

$$kA(t) = A(t)k, \tag{4.5a}$$

$$kY(t) = Y(t)k \tag{4.5b}$$

for all $t \in \mathbb{R}$*. Moreover, if* h *is an element of the group of spatial-temporal symmetries* H_**, it holds that*

$$A(t + \Theta(h)p) = hA(t)h^{-1}, \tag{4.6a}$$

$$Y(t + \Theta(h)p)Y(s + \Theta(h)p)^{-1} = hY(t)Y(s)^{-1}h^{-1}, \tag{4.6b}$$

for all $t \in \mathbb{R}$ *and* $s \leq t$*.*

Proof. We prove relations (4.6a) and (4.6b), the relations (4.5a) and (4.5b) then follow since $\Theta(k) = 0$ for $k \in K_*$. Differentiating the relation

$$f(\gamma x) = \gamma f(x) \qquad \text{for all } x \in \mathbb{R}^N,\ \gamma \in \Gamma$$

with respect to x gives that

$$f'(\gamma x)\gamma = \gamma f'(x) \qquad \text{for all } x \in \mathbb{R}^N,\ \gamma \in \Gamma.$$

This implies that

$$\begin{aligned}A(t+\Theta(h)p)h &= f'(x_*(t+\Theta(h)p))h\\ &= f'(hx_*(t))h\\ &= hf'(x_*(t))\\ &= hA(t)\end{aligned}$$

which proves (4.6a).

Since $\partial_t Y(t)Y(s)^{-1} = A(t)Y(t)Y(s)^{-1}$, we find that

$$\begin{aligned}\partial_t(hY(t)Y(s)^{-1}h^{-1}) &= hA(t)Y(t)Y(s)^{-1}h^{-1}\\ &= A(t+\Theta(h)p)\left(hY(t)Y(s)^{-1}h^{-1}\right).\end{aligned}$$

So both $hY(t)Y(s)^{-1}h^{-1}$ and $Y(t+\theta(h)p)Y(s+\theta(h)p)^{-1}$ solve the matrix-valued initial value problem

$$\dot{W}(t) = A(t+\Theta(h)p)W(t), \qquad W(s) = I$$

and thus the equality (4.6b) follows from uniqueness of solutions. □

4.2 Decomposition

We use Lemma 4.1 to give a decomposition of the monodromy operator $Y(p)$ and a decomposition of the space $\mathbb{C}^N$. Fix $h \in H_*,\ h \notin K_*$ and let the **reduced monodromy operator** (associated to h) be as in Definition 3.1, i.e.

$$Y_h = h^{-1}Y(\Theta(h)p) \tag{4.7}$$

where $Y(t)$ is the fundamental solution of (4.3) with $Y(0) = I$. If x_* is a **discrete wave**, i.e. $H_*/K_* \simeq \mathbb{Z}_n$ for some $n \in \mathbb{N}$, then $h^n \in K_*$. So

$$x_*(t) = h^n x_*(t) = x_*(t + n\Theta(h)p)$$

and therefore $n\Theta(h) = 0 \mod 1$. This implies that there exists an integer m with $1 \le m < n$ such that

$$\Theta(h) = \frac{m}{n} \tag{4.8}$$

and we can write the reduced monodromy operator Y_h as

$$Y_h = h^{-1}Y\left(\frac{m}{n}p\right). \tag{4.9}$$

The next lemma provides a relation between the reduced monodromy operator Y_h and the monodromy operator $Y(p)$.

Lemma 4.2 (Decomposition of the monodromy operator, cf. Lemma 5.7 in [Fie88])**.** *Consider the ODE* (4.1) *satisfying Hypothesis 3. Assume that x_* is a discrete wave, i.e. $H_*/K_* \simeq \mathbb{Z}_n$ for some $n \in \mathbb{N}$. Let $Y(p)$ be the monodromy operator of the linear ODE* (4.3)*. For $h \in H_*,\ h \notin K_*$, let $1 \le m < n$ be as in* (4.8) *and let*

$$Y_h = h^{-1}Y\left(\frac{m}{n}p, 0\right)$$

be the reduced monodromy operator of the linear ODE (4.3)*. Then*

$$Y(p)^m = h^n Y_h^n. \tag{4.10}$$

Proof. Step 1: we first prove that

$$Y(mp) = Y(p)^m. \tag{4.11}$$

Iteratively applying (4.4) gives that

$$\begin{aligned}
&Y(2p)Y(p)^{-1} = Y(p)\\
&Y(3p)Y(2p)^{-1} = Y(2p)Y(p)^{-1} = Y(p)\\
&\quad\vdots\\
&Y(jp)Y((j-1)p)^{-1} = Y((j-1)p)Y((j-2)p)^{-1} = \ldots = Y(p)
\end{aligned}$$

for $j \geq 1$. This implies that

$$\begin{aligned}
Y(mp) &= \left[Y(mp)Y((m-1)p)^{-1}\right]\left[Y((m-1)p)Y((m-2)p)^{-1}\right]\ldots\left[Y(2p)Y(p)^{-1}\right]Y(p)\\
&= Y(p)Y(p)\ldots Y(p) = Y(p)^m
\end{aligned}$$

which proves (4.11).

Step 2: we prove that

$$Y(mp) = h^n\left(h^{-1}Y\left(\frac{m}{n}p\right)\right)^n. \tag{4.12}$$

Iteratively applying (4.6b) with $\Theta(h) = \frac{m}{n}$ gives that

$$\begin{aligned}
&Y\left(2\frac{m}{n}p\right)Y\left(\frac{m}{n}p\right)^{-1} = hY\left(\frac{m}{n}p\right)h^{-1}\\
&Y\left(3\frac{m}{n}p\right)Y\left(2\frac{m}{n}p\right)^{-1} = hY\left(2\frac{m}{n}p\right)Y\left(\frac{m}{n}p\right)^{-1}h^{-1} = h^2Y\left(\frac{m}{n}p\right)h^{-2}\\
&\quad\vdots\\
&Y\left(j\frac{m}{n}p\right)Y\left((j-1)\frac{m}{n}p\right)^{-1} = \ldots = h^{j-1}Y\left(\frac{m}{n}p\right)h^{1-j}
\end{aligned}$$

for $j \geq 1$. This implies that

$$\begin{aligned}
Y(mp) &= \left[Y(mp)\,Y\left(mp-\frac{m}{n}p\right)^{-1}\right]\ldots\left[Y\left(2\frac{m}{n}p\right)Y\left(\frac{m}{n}p\right)^{-1}\right]Y\left(\frac{m}{n}p\right)\\
&= h^{n-1}Y\left(\frac{m}{n}p\right)h^{1-n}h^{n-2}Y\left(\frac{m}{n}p\right)h^{2-n}\ldots h^{-1}Y\left(\frac{m}{n}p\right)\\
&= h^nh^{-1}Y\left(\frac{m}{n}p\right)\ldots h^{-1}Y\left(\frac{m}{n}p\right)\\
&= h^n\left(h^{-1}Y\left(\frac{m}{n}p\right)\right)^n,
\end{aligned}$$

which proves (4.12). Combining the equalities (4.11) and (4.12) now proves the claim. □

The next lemma decomposes the space $\mathbb{C}^N$ into eigenspaces of a fixed matrix $k \in K_*$. So every spatial symmetry $k \in K_*$ gives a decomposition of the space $\mathbb{C}^N$. This is relevant for the proof of Proposition 4.4, where we use this decomposition to relate the spectrum of the monodromy operator to the spectrum of the reduced monodromy operator.

Lemma 4.3 (Decomposition of the state space)**.** *Consider the ODE* (4.1) *satisfying Hypothesis 3. Let k be an element of the group of spatial symmetries K_*. Then the following statements hold:*

1. *All eigenvalues of the matrix $k \in \mathbb{R}^{N\times N}$ lie on the unit circle, i.e.*

$$\sigma(k) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}. \tag{4.13}$$

2. *The matrix $k \in \mathbb{R}^{N\times N}$ is diagonalizable, i.e. if $\lambda_1, \ldots, \lambda_d$ are the distinct eigenvalues of k then*

$$\mathbb{C}^N = X_1 \oplus \ldots \oplus X_d \qquad \textit{with} \qquad X_i := \ker(\lambda_i I - k), \ 1 \leq i \leq d. \tag{4.14}$$

3. *Let $Y(p)$ be the monodromy operator of the ODE* (4.3)*. Then the space X_i defined in* (4.14) *is invariant under $Y(p)$, i.e.*

$$Y(p)X_i \subseteq X_i \qquad \textit{for } 1 \leq i \leq d.$$

4. *For $h \in H_*$, let Y_h be the reduced monodromy operator defined in* (4.7)*. If the elements h and k commute, then the space X_i defined in* (4.14) *is invariant under Y_h, i.e.*

$$Y_h X_i \subseteq X_i \qquad \textit{for } 1 \leq i \leq d.$$

Proof. To prove the first point, we argue by contradiction. Suppose that k has an eigenvalue $|\lambda| > 1$ and let $x_0 \neq 0$ be an associated eigenvector. Then

$$\lim_{j\to\infty} \left\|k^j x_0\right\| = \lim_{j\to\infty} |\lambda|^j \|x_0\| = \infty.$$

Therefore

$$\lim_{j\to\infty} \left\|k^j\right\| \geq \lim_{j\to\infty} \left\|k^j x_0\right\| = \infty.$$

But since $k^j \in \Gamma$ for all $j \in \mathbb{N}$, this contradicts the assumption that the group $\Gamma \subseteq GL(N,\mathbb{R})$ is compact. If k has an eigenvalue $|\lambda| < 1$, then k^{-1} has an eigenvalue $\left|\frac{1}{\lambda}\right| > 1$, which similarly leads to a contradiction. So we conclude that (4.13) holds.

To prove the second point, assume by contradiction that k is not diagonalizable. Then there exists $\lambda \in \sigma(k)$ and $x_0, x_1 \neq 0$ such that

$$kx_0 = \lambda x_0, \qquad kx_1 = \lambda x_1 + x_0.$$

By induction we find that

$$k^j x_1 = (j-1)\lambda^{j-1} x_0 + \lambda^j x_1$$

for $j \geq 2$. This implies that

$$\lim_{j\to\infty} \left\|k^j\right\| = \infty$$

which contradicts compactness of Γ. So we conclude that k is diagonalizable, and therefore that we can write $\mathbb{C}^N$ as a direct sum of the eigenspaces as in (4.14).

To prove the third point, fix $1 \leq i \leq d$ and let $x \in X_i = \ker(\lambda_i I - k)$. Relation (4.5b) implies that $kY(p) = Y(p)k$. Therefore

$$(\lambda_i I - k)\, Y(p)x = Y(p)\,(\lambda_i I - k)\, x = 0$$

so $Y(p)x \in X_i$. We conclude that $Y(p)X_i \subseteq X_i$ for all $1 \leq i \leq d$.

To prove the fourth point, assume that $kh = kh$, then $kY_h = kh^{-1}Y(\Theta(h)p) = h^{-1}kY(\Theta(h)p)$. But (4.5b) implies that $h^{-1}kY(\Theta(h)p) = h^{-1}Y(\Theta(h)p)k$ and hence $kY_h = Y_h k$. Thus, if $1 \leq i \leq d$ and $x \in \ker X_i = \ker(\lambda_i I - k)$, then

$$(\lambda_i I - k)\, Y_h x = Y_h\, (\lambda_i I - k)\, x = 0.$$

So $Y_h x \in X_i$ and we conclude that $Y_h X_i \subseteq X_i$ for all $1 \leq i \leq d$. □

The first two statements of Lemma 4.3 give a pedestrian approach to well-known results from representation theory of compact Lie groups. For an abstract Lie group G, we call a group homomorphism

$$\rho : G \to GL(N, \mathbb{R})$$

a **representation** of G. In this section, we did not view the symmetry group Γ as an abstract group, but viewed it as a subgroup of $GL(N, \mathbb{R})$, so we directly picked a representation. Given a representation ρ of a compact Lie group G, one can construct an inner product

$$\langle . \,,\, . \rangle : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$$

so that the matrices $\rho(g) \in \mathbb{R}^{N\times N},\ g \in G$ are orthogonal with respect to this inner product; see [GSS88, p. 31]. Since orthogonal matrices are diagonalizable and have all their spectrum on the unit circle, the representations $\rho(g)$ have this property as well, cf. the 1st and 2nd statement of Lemma 4.3.

Compact Lie groups also induce a decomposition of the state space that block-diagonalizes any equivariant matrix. More precisely, given a compact Lie group G, there exists a decomposition

$$\mathbb{R}^N = V_1 \oplus \ldots \oplus V_m \tag{4.15}$$

such that for all $1 \leq i \leq m$ it holds that

1. $gV_i \subseteq V_i$ for all $g \in G$,
2. if $A \in \mathbb{R}^{N\times N}$ satisfies $gA = Ag$ for all $g \in G$, then $AV_i \subseteq V_i$.

The decomposition (4.15) is called the **isotypic decomposition** induced by G [GS02; GSS88]. In Lemma 4.3, the reduced monodromy operator Y_h does not necessarily commute with every element $k \in K_*$ (we have not assumed that the symmetry group Γ is abelian). Therefore, the isotypic decomposition induced by K_* does not necessarily block-diagonalize Y_h. However, for a discrete wave with $H_*/K_* \simeq \mathbb{Z}_n$, it holds that $h^n \in K_*$ for every $h \in H_*$, and h^n trivially commutes with h. So Lemma 4.3 implies that eigenspace decomposition induced by the single element h^n block-diagonalizes Y_h. This has the additional advantage that we can straightforwardly express the eigenvalues of $Y(p)$ in terms of the eigenvalues of h^n and Y_h, as we will do in the next section.

4.3 Stability

This section addresses the stability of the periodic solution x_* in system (4.1). The ODE (4.1) is autonomous and therefore $\dot{x}_*(t)$ is a solution of the linearization (4.3). Since this solution is p-periodic, $Y(p)$ has an eigenvalue 1, which we call its **trivial eigenvalue**. For $h \in H_*$, the solution $\dot{x}_*$ of (4.3) satisfies

$$h^{-1}\dot{x}_*(t + \Theta(h)p) = \dot{x}_*(t)$$

and therefore also the reduced mondromy operator Y_h has a trivial eigenvalue 1.

By 'classical' (i.e. non-equivariant) stability theory, the non-trivial eigenvalues of the monodromy operator $Y(p)$ determine the stability of the periodic orbit x_*. If $Y(p)$ has an eigenvalue strictly outside the unit circle, then x_* is unstable as a solution of (4.1). On the other hand, if the trivial eigenvalue $1 \in \sigma(Y(p))$ is algebraically simple, and all $N-1$ non-trivial eigenvalues of $Y(p)$ lie strictly inside the unit circle, then x_* is stable as a solution of (4.1) (see also Section A.3 in the Appendix).

The next proposition shows that we can also determine the stability of *discrete waves* from the eigenvalues of the *reduced* monodromy operator.

Proposition 4.4. *Consider the ODE* (4.1) *satisfying Hypothesis 3. Assume that x_* is a discrete wave and for $h \in H_*,\ h \notin K_*$, let*

$$Y_h = h^{-1} Y(\Theta(h)p)$$

be the reduced monodromy operator of the linear ODE (4.3)*.*

If Y_h has an eigenvalue strictly outside the unit circle, then x_* is an unstable solution of (4.1)*. If the trivial eigenvalue $1 \in \sigma(Y_h)$ is algebraically simple, and all $N-1$ non-trivial eigenvalues of Y_h lie strictly inside the unit circle, then x_* is a stable solution of* (4.1)*.*

Proof. We divide the proof into two steps:

Step 1: Let $n \in \mathbb{N}$ be such that $H_*/K_* \simeq \mathbb{Z}_n$ and let $1 \leq m \leq n-1$ be such that $\Theta(h) = \frac{m}{n}$. Then Lemma 4.2 implies that

$$Y(p)^m = h^n Y_h^n.$$

Since $H_*/K_* \simeq \mathbb{Z}_n$, h^n is a purely spatial symmetry, i.e. $h^n \in K_*$. Let $\lambda_1, \ldots, \lambda_d$ be the distinct eigenvalues of h^n. Then Lemma 4.3 implies that

$$\mathbb{C}^N = X_1 \oplus \ldots \oplus X_d$$

where $X_j = \ker(\lambda_j I - h^n)$. Moreover, Lemma 4.3 implies that the spaces X_j are invariant under both $Y(p)^m$ and Y_h, so we can decompose $Y(p)^m$ as

$$Y(p)^m = Y(p)^m|_{X_1} + \ldots + Y(p)^m|_{X_d} \tag{4.16a}$$

and we can decompose Y_h as

$$Y_h = Y_h|_{X_1} + \ldots + Y_h|_{X_d}.$$

Note that $h^n y = \lambda_j y$ if $y \in X_j$. Therefore

$$Y(p)^m|_{X_j} = \left(h^n Y_h^n\right)\Big|_{X_j} = \lambda_j \left(Y_h|_{X_j}\right)^n.$$

This implies that

$$\sigma\left(Y(p)^m|_{X_j}\right) = \lambda_j \sigma\left(\left(Y_h|_{X_j}\right)^n\right) = \lambda_j \sigma\left(Y_h|_{X_j}\right)^n \tag{4.16b}$$

where all equalities count algebraic multiplicities.

Step 2: Now assume that Y_h has an eigenvalue ν with $|\nu| > 1$ and let $1 \leq j \leq d$ be such that

$$\nu \in \sigma\left(Y_h|_{X_j}\right)$$

Then (4.16b) implies that

$$\lambda_j \nu^n \in \sigma(Y(p)^m).$$

Since $|\lambda_j| = 1$, it follows that $|\lambda_j \nu^n| > 1$. So $Y(p)^m$ has an eigenvalue outside the unit circle and hence x_* is an unstable solution of (4.1).

Vice versa, suppose that Y_h has $N-1$ eigenvalues strictly inside the unit circle. By Lemma 4.3, all eigenvalues $\lambda_1, \ldots, \lambda_d$ of h^n have norm one. Therefore the equalities (4.16a)–(4.16b) imply that $Y(p)^m$ has $N-1$ eigenvalues strictly inside the unit circle, and the trivial eigenvalue $1 \in \sigma(Y(p)^m)$ is algebraically simple. Therefore x_* is a stable solution of (4.1), as claimed. $\square$

For completeness, we give the following characterization of the eigenvalues of the reduced monodromy operator:

Lemma 4.5. *Consider the ODE* (4.1) *satisfying Hypothesis 3. Let* $h \in H_*,\ h \notin K_*$ *and let*

$$Y_h = h^{-1} Y(\Theta(h)p)$$

be the reduced monodromy operator of the linear ODE (4.3)*. Then* μ *is an eigenvalue of* Y_h *if and only if the linear ODE* (4.3) *has a non-zero solution of the form*

$$y(t + \Theta(h)p) = \mu h y(t). \tag{4.17}$$

Proof. Suppose that μ is an eigenvalue of Y_h with eigenvector $y_0 \neq 0$, i.e.

$$h^{-1} Y(\Theta(h)p, 0) y_0 = \mu y_0.$$

Then $y(t) := Y(t)y_0$ is a non-zero solution of (4.3) that satisfies

$$\begin{aligned} y(t + \Theta(h)p) &= Y(t + \Theta(h)p) Y(\Theta(h)p)^{-1} Y(\Theta(h)p) y_0 \\ &= h Y(t) h^{-1} Y(\Theta(h)p) y_0 \\ &= \mu h Y(t) y_0 \end{aligned}$$

where in the second step we used (4.6b). We conclude that $y(t) := Y(t)y_0$ is a non-zero solution of (4.3) that satisfies $y(t + \Theta(h)p) = \mu h y(t)$.

Vice versa, suppose that $y(t)$ is a non-zero solution of (4.3) with $y(t + \Theta(h)p) = \mu h y(t)$. Then

$$h^{-1} Y(\Theta(h)p) y(0) = h^{-1} y(\Theta(h)p) = \mu y(0)$$

and $y(0) \neq 0$ is an eigenvector of $Y_h = h^{-1} Y(\theta(h)p)$ with eigenvalue μ. $\square$

Chapter 5

Characteristic matrix functions

This chapter discusses the notion of characteristic matrix functions for bounded linear operators, as introduced in [KV21; KV92]. The spectral information of a finite dimensional linear operator (i.e. a matrix) is naturally described in terms of an analytic function. Indeed, let $A \in \mathbb{C}^{N \times N}$ be a matrix and consider the analytic function

$$\Delta(z) = zI - A.$$

Then $\mu \in \mathbb{C}$ is an eigenvalue of A if and only if $\det \Delta(\mu) = 0$; in this case, the algebraic multiplicity of μ equals the order of μ as a root of $\det \Delta(z) = 0$; the geometric multiplicity of μ equals the dimension of the linear space $\ker \Delta(\mu)$.

For certain classes of bounded linear operators, there exist matrix-valued functions which similarly capture the spectral information. The spectral problem of the (infinite dimensional) operator then reduces to the analysis of a finite dimensional function, which means a significant dimension reduction. The notion of a *characteristic matrix function* gives a rigorous underpinning to this idea.

This chapter is an expository chapter based on the article [KV92] and the book [KV21] by Kaashoek and Verduyn Lunel. Section 5.1 introduces the notion of Jordan chains for analytic operator-valued functions, with special attention to the usual notion of a Jordan chain for bounded linear operators. Section 5.2 gives the formal definition of a characteristic matrix function for a bounded linear operator; we show that a characteristic matrix function captures the spectral information of the associated bounded linear operator. Section 5.3 introduces a result from [KV21] which shows the existence of a characteristic matrix function for a certain class of bounded linear operators. We return to this result in Chapter 7, when we determine a characteristic matrix function for the reduced monodromy operator (as introduced in Chapter 4) of a DDE.

5.1 Jordan chains for analytic operator-valued functions

In this section, we consider analytic operator-valued functions and discuss several notions related to Jordan chains. Throughout the section, we let X be a complex Banach space, $\mathcal{L}(X, X)$ the set of bounded linear operators from X to X and $L : \mathbb{C} \to \mathcal{L}(X, X)$ an analytic function.

Definition 5.1. Let $\mu \in \mathbb{C}$ and let $x_0, \ldots, x_{k-1},\ x_0 \neq 0$, be an ordered set of vectors in X. We say that $x_0, \ldots, x_{k-1}$ is a **Jordan chain of length** k for L at μ if

$$L(z)\left[x_0 + (z-\mu)x_1 + \ldots (z-\mu)^{k-1}x_{k-1}\right] = \mathcal{O}\left((z-\mu)^k\right); \tag{5.1}$$

we call the vector x_0 an **eigenvector** of L at μ and we call the vectors $x_2, \ldots, x_{k-1}$ **generalized eigenvectors**. The maximal length of a Jordan chain starting with x_0 is called the **rank** of x_0; the rank is infinite if no maximum exists.

In the above definition, we do *not* require that the generalized eigenvectors are nonzero. We will illustrate this in Example 5.3 below. In Example 5.2, we consider the situation where $T : X \to X$ is a bounded linear operator and L is given by $L(z) = zI - T$. In this case, we show that the above definition of a Jordan chain coincides with the usual definition of a Jordan chain for a bounded linear operator.

If x_0 is an eigenvector of L at μ, then equation (5.1) implies that $L(\mu)x_0 = 0$. Vice versa, if $L(\mu)x_0 = 0$, then $L(z)x_0 = \mathcal{O}(z-\mu)$ and hence x_0 is an eigenvector of L (with at least rank 1). So x_0 is an eigenvector of L at μ if and only if x_0 is a non-zero element of the space

$$\ker L(\mu) = \{x \in X \mid L(\mu)x = 0\}. \tag{5.2}$$

We now consider the case in which the space (5.2) has finite dimension $n \in \mathbb{N}$ and all Jordan chains of L at μ have finite rank. We pick a basis $x^0, \ldots, x^n$ of $\ker L(\mu)$ and for $1 \leq j \leq n$, we let r_j be the rank of the eigenvector x^j. Then, if x is any eigenvector of L at μ, its rank has to be equal to one of the r_j. In particular, the set $\{r_1, \ldots, r_n\}$ does not depend on the choice of basis. We define the **algebraic multiplicity** of μ as the number

$$r_1 + \ldots + r_n$$

and we define the **geometric multiplicity** of μ as the dimension of $\ker L(\mu)$.

Example 5.2 (Jordan chains for bounded linear operators)**.** Let $T : X \to X$ be a bounded linear operator and consider the analytic operator-valued function

$$L : \mathbb{C} \to \mathcal{L}(X, X), \qquad L(z) = zI - T.$$

Let $\mu \in \mathbb{C}$ be an eigenvalue of T and let $x_0, \ldots, x_{k-1}$ be a Jordan chain in the sense of bounded operators, i.e.

$$Tx_0 = \mu x_0, \quad Tx_1 = \mu x_1 + x_0, \quad \ldots \quad Tx_{k-1} = \mu x_{k-1} + x_{k-2}. \tag{5.3}$$

Then

$$(zI - T)x_j = (z-\mu)x_j - x_{j-1}, \qquad 1 \leq j \leq k-1,$$

and hence

$$\begin{aligned}(zI - T)&\left[x_0 + (z-\mu)x_1 + \ldots + (z-\mu)^{k-1}x_{k-1}\right] \\ &= (z-\mu)x_0 + (z-\mu)\left[(z-\mu)x_1 - x_0\right] + \ldots + (z-\mu)^{k-1}\left[(z-\mu)x_{k-1} - x_{k-2}\right] \\ &= (z-\mu)^k x_{k-1}.\end{aligned}$$

So

$$(zI - T)\Big[x_0 + (z-\mu)x_1 + \ldots + (z-\mu)^{k-1}x_{k-1}\Big] = \mathcal{O}\Big((z-\mu)^k\Big) \tag{5.4}$$

and $x_0, \ldots, x_{k-1}$ is a Jordan chain for $zI - T$ in the sense of Definition 5.1. Vice versa, suppose the vectors $x_0, \ldots, x_k$ satisfy (5.4). Then evaluating the derivatives of (5.4) at $z = \mu$ yields the equalities (5.3) and hence $x_0, \ldots, x_{k-1}$ is a Jordan chain for the bounded operator T. Thus, the notion of a Jordan chain of the analytic function $z \mapsto zI - T$ coincides with the notion of a Jordan chain for the operator T.

For a general analytic function $z \mapsto L(z)$, Jordan chains of the function $z \mapsto L(z)$ at $z = \mu$ are not related to Jordan chains of the operator $L(\mu)$. We illustrate the difference in the following example.

Example 5.3 ($z \mapsto L(z)$ vs $L(\mu)$)**.** Let $\mu \in \mathbb{C}$ and let $B \in \mathbb{C}^{N\times N}$ be an invertible matrix. For $z \in \mathbb{C}$, we define the analytic function

$$L(z) : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}, \qquad L(z) = \begin{pmatrix} (z-\mu)^2 & 0 \\ 0 & B \end{pmatrix}.$$

Let $x = (1, 0, \ldots 0) \in \mathbb{C}^{N+1}$, then

$$L(z)x = (z-\mu)^2 x = \mathcal{O}\left((z-\mu)^2\right).$$

Hence the analytic function $z \mapsto L(z)$ has algebraic multiplicity at least two at μ (the generalized eigenvector is the zero vector). We now fix $z = \mu$ and consider the matrix

$$L(\mu) = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Since B is invertible, this matrix has an eigenvalue 0 with algebraic multiplicity 1. So the algebraic multiplicity of the function $z \mapsto L(z)$ at μ is different from the algebraic multiplicity of the eigenvalue 0 of the matrix $L(\mu)$.

The next lemma addresses how Jordan chains and algebraic multiplicities behave under conjugation. This is crucial for analytic *matrix*-valued functions (see Corollary 5.5 below) and for the discussion of characteristic matrix functions in the next section.

Lemma 5.4 (cf. Proposition 1.2 in [KV92])**.** *Given a complex Banach space X, let*

$$L, M : \mathbb{C} \to \mathcal{L}(X, X)$$

be analytic operator-valued functions, and let

$$E, F : \mathbb{C} \to \mathcal{L}(X, X)$$

be analytic operator-valued functions whose values are invertible operators. Suppose that

$$M(z) = F(z)L(z)E(z)$$

for all $z \in \mathbb{C}$. Then there is a one-to-one correspondence between Jordan chains for L and Jordan chains for M. In particular, for $\mu \in \mathbb{C}$, the algebraic multiplicity of L at μ equals the algebraic multiplicity of M at μ.

Proof. Let $x_0, \ldots, x_{k-1}$ be a Jordan chain for L at μ, i.e.

$$L(z)\left[x_0 + \ldots + (z-\mu)^{k-1} x_{k-1}\right] = \mathcal{O}\left((z-\mu)^k\right).$$

For $n \in \mathbb{N}$, let $y_n \in X$ be such that

$$E(z)^{-1}\left[x_0 + \ldots + (z-\mu)^{k-1} x_{k-1}\right] = \sum_{n=0}^{\infty} y_n (z-\mu)^n.$$

Then $y_0 \neq 0$ and

$$E(z)\left[y_0 + \ldots + (z-\mu)^{k-1} y_{k-1}\right] = x_0 + \ldots + (z-\mu)^{k-1} x_{k-1} - E(z) \sum_{n=k}^{\infty} y_n (z-\mu)^n$$

so $y_0, \ldots, y_{k-1}$ satisfy

$$\begin{aligned} M(z)\left[y_0 + \ldots + (z-\mu)^{k-1} y_{k-1}\right] &= F(z)L(z)E(z)\left[y_0 + \ldots + (z-\mu)^{k-1} y_{k-1}\right] \\ &= F(z)L(z)\left[x_0 + \ldots + (z-\mu)^{k-1} x_{k-1}\right] \\ &\qquad - F(z)L(z)E(z) \sum_{n=k}^{\infty} y_n (z-\mu)^n \\ &= \mathcal{O}\left((z-\mu)^k\right). \end{aligned}$$

So $y_0, \ldots, y_{k-1}$ is a Jordan chain for M at μ. Vice versa, every Jordan chain $y_0, \ldots, y_{k-1}$ of L at μ induces a Jordan chain for M at μ. So there is a one-to-one correspondence between Jordan chains for M and L. In particular, the algebraic multiplicity of L at μ equals the algebraic multiplicity of M at μ. □

For an analytic matrix-valued function Δ, the algebraic multiplicity of Δ at μ is related to the roots of the equation $\det \Delta(z) = 0$. More precisely, let $\Delta : \mathbb{C} \to \mathbb{C}^{N \times N}$ be an analytic matrix-valued function and fix $\mu \in \mathbb{C}$. Then there exist analytic matrix functions E, F whose values are invertible matrices, and unique non-negative integers $r_1, \ldots, r_N$ such that

$$\Delta(z) = F(z)D(z)E(z) \tag{5.5a}$$

with

$$D(z) = \begin{pmatrix} (z-\mu)^{r_1} & 0 & \ldots & 0 \\ 0 & (z-\mu)^{r_2} & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ldots & (z-\mu)^{r_N} \end{pmatrix} \tag{5.5b}$$

The expression (5.5a)–(5.5b) is called the *local Smith form* of Δ, see [KV92]. This leads to the following corollary:

Corollary 5.5 (cf. p. 485 in [KV92]). *Let $\Delta : U \to \mathbb{C}^{N \times N}$ be an analytic matrix-valued function. Then the algebraic multiplicity of Δ at $\mu \in \mathbb{C}$ equals the order of μ as a root of the equation*

$$\det \Delta(z) = 0.$$

Proof. Consider the local Smith form (5.5a)–(5.5b) for Δ at μ. Since $E(z), F(z)$ are invertible matrices, Lemma 5.4 implies that the algebraic multiplicity of Δ at μs equals the algebraic multiplicity of D at μ. But since D is in local Smith form, we see that the algebraic multiplicity of D at μ (and thus the algebraic multiplicity of Δ at μ) is given by $r_1 + \ldots + r_N$.

On the other hand, the determinant $\det \Delta$ equals

$$\det \Delta(z) = \det(F(z))\,(z-\mu)^{r_1} \ldots (z-\mu)^{r_N} \det(E(z)).$$

Since $F(z), E(z)$ are invertible matrices, $\det F(z) \neq 0$ and $\det E(z) \neq 0$ for all $z \in \mathbb{C}$. Therefore, the order of μ as a root of $\det \Delta(z) = 0$ is also given by $r_1 + \ldots + r_N$. We conclude that the algebraic multiplicity of Δ at μ equals the the order of μ as a root of $\det \Delta(z) = 0$, as claimed. □

5.2 Characteristic matrix functions

In this section, we introduce the notion of a characteristic matrix function. A characteristic matrix function captures the spectral information of an infinite dimensional operator in an analytic matrix-valued function. This powerful dimension reduction has a wide range of applications, such as completeness problems [KV21], numerical continuation [SGH06] and stability problems in DDE (as in this thesis).

A characteristic matrix function can be defined for both unbounded operators [KV92; SGH06; SS11] and bounded operators [KV21]. Since Chapters 7-9 of this thesis are concerned with characteristic matrix functions for monodromy operators of DDE, which are bounded operators, we discuss characteristic matrix functions for bounded operators here.

Definition 5.6. Let X be a complex Banach space, $T : X \to X$ a bounded linear operator and $\Delta : \mathbb{C} \to \mathcal{L}\left(\mathbb{C}^N, \mathbb{C}^N\right)$ an analytic matrix-valued function. We say that Δ is a **characteristic matrix function** for T if there exist analytic functions

$$E, F : \mathbb{C} \to \mathcal{L}(\mathbb{C}^N \oplus X, \mathbb{C}^N \oplus X)$$

such that $E(z), F(z)$ are invertible operators for all $z \in \mathbb{C}$ and such that

$$\begin{pmatrix} \Delta(z) & 0 \\ 0 & I_X \end{pmatrix} = F(z) \begin{pmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & I - zT \end{pmatrix} E(z) \tag{5.6}$$

for all $z \in \mathbb{C}$.

In the above definition, we first extend the operators $\Delta(z) : \mathbb{C}^N \to \mathbb{C}^N$ and $I - zT : X \to X$ to the operators

$$\begin{pmatrix} \Delta(z) & 0 \\ 0 & I_X \end{pmatrix} : \mathbb{C}^N \oplus X \to \mathbb{C}^N \oplus X, \qquad \begin{pmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & I - zT \end{pmatrix} : \mathbb{C}^N \oplus X \to \mathbb{C}^N \oplus X, \tag{5.7}$$

respectively. We need to work with the extension (5.7), because the operators $\Delta(z)$ and $I - zT$ have different dimensions and hence cannot be conjugated directly. However, the trivial extensions do not change the spectral information. Therefore, Jordan chains of $\Delta(z)$ are related to Jordan chains of $I - zT$, as we make precise in the following lemma:

Lemma 5.7 (Characterisation of eigenvalues, cf. Theorem 5.2.2 in [KV21])**.** *Let X be a complex Banach space, $T : X \to X$ a bounded linear operator and $\Delta : \mathbb{C} \to \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)$ a characteristic matrix function for T. Let $\mu \in \mathbb{C}\backslash\{0\}$, then*

1. *$\mu^{-1} \in \sigma_{pt}(T)$ if and only if $\det \Delta(\mu) = 0$, i.e.*
$$\sigma_{pt}(T)\backslash\{0\} = \{\mu^{-1} \in \mathbb{C} \mid \det \Delta(\mu) = 0\}.$$
2. *If $\mu^{-1} \in \sigma_{pt}(T)$, then the geometric multiplicity of μ^{-1} as an eigenvalue of T equals the dimension of the space*
$$\ker(\Delta(\mu)) = \{x \in \mathbb{C}^N \mid \Delta(\mu)x = 0\}.$$
3. *If $\mu^{-1} \in \sigma_{pt}(T)$, then the algebraic multiplicity of μ^{-1} as an eigenvalue of T equals the order of μ as a root of*
$$\det \Delta(z) = 0. \tag{5.8}$$

Proof. Let $\mu \neq 0$, then μ^{-1} is an eigenvalue of T if and only if $\mu^{-1}I - T = \mu^{-1}(I - \mu T)$ has a non-trivial kernel, i.e. if and only if $I - \mu T$ has a non-trivial kernel. We first show that there is a one-to-one correspondence between kernel vectors of $I - \mu T$ and kernel vectors of $\Delta(\mu)$. This then implies the first two statements of the lemma.

We write
$$M(z) = \begin{pmatrix} \Delta(z) & 0 \\ 0 & I_X \end{pmatrix}, \qquad L(z) = \begin{pmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & I - zT \end{pmatrix}, \tag{5.9}$$
then the kernels of the operators $L(\mu)$, $M(\mu)$ are given by
$$\ker M(\mu) = \ker \Delta(\mu) \oplus \{0\}, \qquad \ker L(\mu) = \{0\} \oplus \ker(I - \mu T).$$

Since Δ is a characteristic matrix function for T, there exists analytic functions $E, F : \mathbb{C} \to \mathcal{L}(\mathbb{C}^N \oplus X, \mathbb{C}^N \oplus X)$ so that $E(z), F(z)$ are invertible operators for all $z \in \mathbb{C}$ and such that $M(z) = F(z)L(z)E(z)$ for all $z \in \mathbb{C}$. In particular, the operator $E(\mu)$ maps the space $\ker M(\mu)$ in a one-to-one way to the space $\ker L(\mu)$. This implies that the map
$$\ker \Delta(\mu) \to \ker(I - \mu T), \qquad c \mapsto (0, I_X)E(\mu)\begin{pmatrix} c \\ 0 \end{pmatrix}$$
with inverse
$$\ker(I - \mu T) \to \ker \Delta(\mu), \qquad x \mapsto (I_{\mathbb{C}^N}, 0)E(\mu)^{-1}\begin{pmatrix} 0 \\ x \end{pmatrix}$$
is a bijection. So there is a one-to-one correspondence between elements of $\ker(I - \mu T)$ and elements of $\ker \Delta(\mu)$, which proves the first two statements of the lemma.

To prove the third statement of the lemma, let $\mu^{-1} \in \sigma_{pt}(T)$ and denote by $r \in \mathbb{N}$ the algebraic multiplicity of μ^{-1} as an eigenvalue of T. Then r is also equal to algebraic multiplicity of μ^{-1} for the function $z \mapsto L(z)$ defined in (5.9). Since equality (5.6) holds, Lemma 5.4 implies that r equals the algebraic multiplicity of μ^{-1} for the function $z \mapsto M(z)$ defined in (5.9), which in turn equals the algebraic multiplicity of μ^{-1} for $z \mapsto \Delta(z)$. But then Corollary 5.5 implies that r equals the order of μ^{-1} as a root of $\det \Delta(z) = 0$. $\square$

The equation (5.8) is often called the **characteristic equation**. Lemma 5.7 implies that the algebraic multiplicity of an eigenvalue μ^{-1} can be found from the characteristic equation. However, if we are additionally interested in the geometric multiplicity of an eigenvalue, we have to study the matrix-valued function $z \mapsto \Delta(z)$.

If a bounded linear operator allows a characteristic matrix function, then the bounded linear operator has to be a Riesz operator:

Lemma 5.8 (cf. Section 5.2 in [KV21])**.** *Let X be a complex Banach space and $T : X \to X$ a bounded linear operator. If there exists a characteristic matrix function $\Delta : \mathbb{C} \to \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)$ for T, then T is a Riesz operator, i.e. all the non-zero spectrum of T consist of isolated eigenvalues of finite algebraic multiplicity.*

Proof. Let $\mu \neq 0$ and assume that $\mu^{-1} \in \sigma(T)$, ie. $I - \mu T$ is not invertible. Equality (5.6) implies that $\Delta(\mu)$ is not invertible, so $\det \Delta(\mu) = 0$. But by Lemma 5.7, this implies that $\mu^{-1} \in \sigma_{pt}(T)$. So all non-zero spectrum of T consists of eigenvalues.

Since $z \mapsto \det \Delta(z)$ is analytic and $\det \Delta \not\equiv 0$ (for example, $\det \Delta(0) \neq 0$), the roots of $\det \Delta(z) = 0$ are isolated and of finite order. Therefore, Lemma 5.7 implies that all eigenvalues of T are isolated eigenvalues of finite multiplicity, which proves the claim. □

In Definition 5.6, we make a conjugation between the trivial extensions of $\Delta(z)$ and $I - zT$. As a result, roots of $\det \Delta(z)$ relate to *inverses* of eigenvalues of T. We could try to circumvent this by instead making a conjugation between the trivial extension of $\Delta(z)$ and $zI - T$. However, the analogue of Lemma 5.8 then implies that *all* spectral points of T are eigenvalues of finite multiplicity and can have no accumulation point. So by working with $I - zT$ instead of $zI - T$, an operator T with a characteristic matrix function can have essential spectrum at zero. This is especially important in view of the next section, which discusses the existence of characteristic matrix functions for a class of compact operators.

5.3 Characteristic matrix functions for a class of compact operators

Throughout this section, we let X be a complex Banach space and $V : X \to X$ be a **Volterra operator**, i.e. V is a compact operator with $\sigma(V) \subseteq \{0\}$. Moreover, we let $R : X \to X$ be a **finite rank operator**, i.e. the range of R is finite dimensional. If we let $N \in \mathbb{N}$ be the dimension of the range of R, then we can identify the range of R with $\mathbb{C}^N$ and write

$$R = DC \tag{5.10a}$$

with

$$C : X \to \mathbb{C}^N, \qquad D = \mathbb{C}^N \to X. \tag{5.10b}$$

We call a factorisation of the form (5.10a)–(5.10b) with N the dimension of the range of R a **minimal rank factorisation** of R. The next theorem from [KV21] gives a characteristic matrix function for the operator $V + R$. The proof of this theorem is beyond the scope of this thesis and hence we state the theorem without proof.

Theorem 4 (from [KV21])**.** *Let X be a complex Banach space and $T : X \to X$ a bounded linear operator. Assume that $T = V+R$, with $V : X \to X$ a Volterra operator and $R : X \to X$ an operator of finite rank N; let $R = DC$ be a minimal rank factorisation of R. Then the matrix-valued function*

$$\Delta(z) = I_{\mathbb{C}^N} - zC(I - zV)^{-1}D$$

is a characteristic matrix function for T.

Note that, since V is Volterra, the map $z \mapsto (I - zV)^{-1}$ is analytic on $\mathbb{C}$ and hence $z \mapsto \Delta(z)$ is an analytic matrix-valued function.

A compact operator $T : X \to X$ is said to have a *complete eigenbasis* if the span of eigenvectors and generalized eigenvectors of T is dense in X. In [KV21], Theorem 4 is used to show that operators of the form $T = V + R$ can have a complete eigenbasis. This is surprising, since the Volterra operator V has no eigenvectors (if X is infinite dimensional, the spectral point $0 \in \sigma(V)$ is not an eigenvalue) and the operator R has finite rank.

In Chapter 7, we consider a class of equivariant DDE where the reduced monodromy operator (as defined in Chapters 4 and 6) is of the form $T = V + R$. We use Theorem 4 to give an explicit characteristic matrix function for this reduced monodromy operator.

Proofs

Chapter 6

Equivariant Floquet theory for DDE

This chapter is concerned with equivariant Floquet theory for DDE and its applications to equivariant control. Equivariant Floquet theory for DDE is similar to equivariant Floquet theory for ODE (as discussed in Chapter 4), but the main difference is that for DDE the state space is infinite dimensional. We show that the main results from Chapter 4 carry over to this infinite dimensional setting, mainly due to the fact that the monodromy operator is a *Riesz operator*, i.e. its non-zero spectrum consists of isolated eigenvalues of finite algebraic multiplicity.

This chapter is structured as follows. In Section 6.1, we discuss equivariant Floquet theory for DDE, and show that the spectrum of the reduced monodromy operator determines the stability of the target periodic solution. The proofs of Lemma 6.1 and Lemma 6.2 are almost identical (up to suitable change of notation) to the proofs of Lemma 4.1 and Lemma 4.2, respectively. We include the proofs of Lemma 6.1 and Lemma 6.2 for completeness, but these proofs can be savely skipped by the reader who is familiar with both the contents Chapter 4 and with DDE. The proof of Proposition 6.3, however, is more subtle than the proof of its finite dimensional counterpart Proposition 4.4, since we now also have to consider the spectral properties of the reduced monodromy operator. Section 6.2 addresses *why* working with the reduced monodromy operator is useful in the context of Pyragas control. The arguments in Section 6.2 are mostly heuristic, but they serve as a motivation for the more rigorous (and more elaborate) discussion in Chapter 7.

6.1 Reduced monodromy operator

Consider the DDE

$$\dot{x}(t) = F(x_t), \qquad t \geq 0 \tag{6.1}$$

with state space $X = C\left([-r,0],\mathbb{R}^N\right)$, $r > 0$, $F : X \to \mathbb{R}^N$ and *history segment* $x_t \in X$ defined as $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$. We let $\Gamma \subseteq GL(N,\mathbb{R})$ be a compact group, which we represent on X as

$$\rho(\gamma) : X \to X, \qquad (\rho(\gamma)\phi)(\theta) = \gamma\phi(\theta) \quad \text{ for } \theta \in [-r, 0]$$

but we suppress the explicit choice of representation in the notation, i.e. we write $\gamma\phi$ instead of $\rho(\gamma)\phi$. We make the following assumptions on system (6.1):

Hypothesis 4.

1. *$F : X \to \mathbb{R}^N$ is a C^2-function.*

2. *System* (6.1) *is equivariant with respect to the compact group* Γ, *i.e.*

$$F(\gamma\phi) = \gamma F(\phi) \tag{6.2}$$

for all $\gamma \in \Gamma$ *and* $\phi \in X$.

3. *System* (6.1) *has a periodic solution* $x_*(t)$ *with* minimal *period* $p > 0$.

To the periodic orbit x_*, we associate the symmetry groups

$$\begin{aligned} K_* &= \{\gamma \in \Gamma \mid \gamma x_*(t) = x_*(t)\ \forall t\} \qquad &&\text{(spatial symmetries)},\\ H_* &= \{\gamma \in \Gamma \mid \gamma\{x_*(t)\} = \{x_*(t)\}\} \qquad &&\text{(spatial-temporal symmetries)} \end{aligned} \tag{6.3}$$

and the induced map

$$\Theta : H_* \to S^1, \qquad hx_*(t) = x_*(t + \Theta(h)p). \tag{6.4}$$

We define $L(t) : X \to X,\ L(t) := F'((x_*)_t)$ where the history segment $(x_*)_t \in X$ is defined as $(x_*)_t(\theta) = x_*(t+\theta),\ \theta \in [-r, 0]$. Let $U(t,s) : X \to X,\ t \geq s$ be the fundamental solution of the DDE

$$\dot{y}(t) = L(t)y_t \tag{6.5}$$

with $U(s,s) = I$ (the fundamental solution of the DDE (6.5) now has two time-arguments, whereas in our notation the fundamental solution of a linear ODE has only one, see Section 1.5). Since

$$L(t+p) = L(p),$$

for all $t \in \mathbb{R}$, we have that

$$U(t+p, s+p) = U(t,s) \tag{6.6}$$

for all $t \geq s$; see Section A.2 in the appendix. The next lemma shows that the group of spatial symmetries K_* induces spatial relations on the fundamental solution; the group of spatial-temporal symmetries H_* induces spatial-temporal relations on the fundamental solution.

Lemma 6.1. *Consider the DDE* (6.1) *satisfying Hypothesis 4. Let* $L(t) := F'((x_*)_t)$ *and let* $U(t,s),\ t \geq s$ *be the fundamental solution of the linearized DDE* (6.5) *with* $U(s,s) = I$.

If k *is an element of the group of spatial symmetries* K_*, *then*

$$kL(t) = L(t)k, \tag{6.7a}$$

$$kU(t,s) = U(t,s)k, \tag{6.7b}$$

for all $t \in \mathbb{R}$. *Moreover, if* h *is an element of the group of spatial-temporal symmetries* H_*, *it holds that*

$$L(t + \Theta(h)p) = hL(t)h^{-1}, \tag{6.8a}$$

$$U(t + \Theta(h)p, s + \Theta(h)p) = hU(t,s)h^{-1}, \tag{6.8b}$$

for $t \in \mathbb{R}$ *and* $s \leq t$.

Proof. We prove relations (6.8a) and (6.8b), the relations (6.7a) and (6.7b) then follow since $\Theta(k) = 0$ for $k \in K_*$. Differentiating the relation (6.2) with respect to ϕ gives that

$$F'(\gamma\phi)\gamma = \gamma F'(\phi) \qquad \text{for all } \phi \in X,\ \gamma \in \Gamma.$$

For $h \in H_*$, the relation (6.4) in particular implies that the history function $(x_*)_t$ satisfies

$$h(x_*)_t = (x_*)_{t+\Theta(h)p}$$

and hence

$$\begin{aligned} L(t + \Theta(h)p)h &= F'\left((x_*)_{t+\Theta(h)p}\right) h \\ &= F'(h(x_*)_t)h \\ &= hF'((x_*)_t) \\ &= hL(t) \end{aligned}$$

which proves (6.8a).

To prove (6.8b), fix $h \in H_*$, $\phi \in X$ and $s \in \mathbb{R}$. Then $y_t := hU(t,s)h^{-1}\phi$ satisfies

$$\begin{aligned} \dot{y}(t) &= h\partial_t U(t,s)h^{-1}\phi \\ &= hL(t)U(t,s)h^{-1}\phi \\ &= L(t + \Theta(h)p)hU(t,s)h^{-1}\phi \end{aligned}$$

where in the last step we used (6.8a). So $y_t := hU(t,s)h^{-1}\phi$ satisfies

$$\begin{cases} \dot{y}(t) &= L(t + \Theta(h)p)y_t, \qquad t \geq s \\ y_s &= \phi \end{cases}$$

and uniqueness of solutions implies that $hU(t,s)h^{-1}\phi = U(t + \Theta(h)p, s + \Theta(h)p)$. This proves (6.8b). $\square$

For $h \in H_*$, we define the **reduced monodromy operator** (associated to h) as

$$U_h = h^{-1}U(\Theta(h)p, 0) \tag{6.9}$$

where $U(t,s),\ t \geq s$ is the fundamental solution of the linear DDE (6.5) with $U(s,s) = I$. The next lemma provides a relation between the monodromy operator $U(p,0)$ and the reduced monodromy operator U_h.

Lemma 6.2. *Consider the DDE* (6.1) *satisfying Hypothesis 4. Assume that x_* is a discrete wave, i.e. $H/K \simeq \mathbb{Z}_n$ for some $n \in \mathbb{N}$. Let $U(p,0)$ be the monodromy operator of the linear DDE* (6.5)*. For $h \in H_*,\ h \notin K_*$, let $1 \leq m < n$ be such that $\Theta(h) = \frac{m}{n}$ and let*

$$U_h = h^{-1}U\left(\frac{m}{n}p, 0\right)$$

be the reduced monodromy operator of the linear DDE (6.5)*. Then*

$$U(p,0)^m = h^n U_h^n. \tag{6.10}$$

Proof. STEP 1: we first prove that

$$U(mp,0) = U(p,0)^m. \tag{6.11}$$

Iteratively applying (6.6) gives that

$$\begin{aligned}
&U(2p,p) = U(p,0)\\
&U(3p,2p) = U(2p,p) = U(p,0)\\
&\qquad\vdots\\
&U(jp,(j-1)p) = \ldots = U(p,0)
\end{aligned}$$

for $j \geq 1$. This implies that

$$\begin{aligned}
U(mp,0) &= U(mp,(m-1)p)U((m-1)p,(m-2)p)\ldots U(2p,p)U(p,0)\\
&= U(p,0)\ldots U(p,0)\\
&= U(p,0)^m
\end{aligned}$$

which proves (6.11).

STEP 2: we prove that

$$U(mp,0) = h^n \left(U\left(\frac{m}{n}p,0\right)\right)^n. \tag{6.12}$$

Iteratively applying (6.8b) with $\Theta(h) = \frac{m}{n}$ gives that

$$\begin{aligned}
&U\left(\frac{2m}{n}p,\frac{m}{n}p\right) = hU\left(\frac{m}{n}p,0\right)h^{-1}\\
&U\left(\frac{3m}{n}p,\frac{2m}{n}p\right) = hU\left(\frac{2m}{n}p,\frac{m}{n}p\right)h^{-1} = h^2U\left(\frac{m}{n}p,0\right)h^{-2}\\
&\quad\vdots\\
&U\left(\frac{jm}{n}p,\frac{(j-1)m}{n}p\right) = \ldots = h^{j-1}U\left(\frac{m}{n}p,0\right)h^{1-j}
\end{aligned}$$

for $j \geq 1$. This implies that

$$\begin{aligned}
U(mp,0) &= U\left(mp,mp-\frac{m}{n}p\right)\ldots U\left(\frac{2m}{n}p,\frac{m}{n}p\right)U\left(\frac{m}{n}p,0\right)\\
&= h^{n-1}U\left(\frac{m}{n}p,0\right)h^{1-n}h^{n-2}U\left(\frac{m}{n}p,0\right)h^{2-n}\ldots h^{-1}U\left(\frac{m}{n}p,0\right)\\
&= h^n h^{-1}U\left(\frac{m}{n}p,0\right)h^{-1}U\left(\frac{m}{n}p,0\right)\ldots h^{-1}U\left(\frac{m}{n}p,0\right)\\
&= h^n\left(h^{-1}U\left(\frac{m}{n}p,0\right)\right)^n
\end{aligned}$$

which proves (6.12). Combining the equalities (6.11) and (6.12) now proves the claim. □

Next, we address the stability of the periodic solution x_* in (6.1). Since the DDE (6.1) is autonomous, $\dot{x}_*$ is a solution of the linearization (6.5). Since this solution is p-periodic, the monodromy operator $U(p,0)$ has an eigenvalue 1, which we call its **trivial eigenvalue**. For $h \in H_*$, the solution $\dot{x}_*$ satisfies

$$h^{-1}\dot{x}_*(t+\Theta(h)p) = \dot{x}_*(t)$$

and therefore also the reduced monodromy operator U_h has a trivial eigenvalue 1.

Similar to the ODE case, the non-trivial eigenvalues of the monodromy operator $U(p,0)$ determine the stability of the periodic orbit x_*. If $j \in \mathbb{N}$ is such that $jp \geq r$ (i.e. j times the period is larger than the delay), then the operator $U(jp,0) = U(p,0)^j$ is compact [DGVW95, Chapter 8]; hence the non-zero spectrum of both $U(p,0)^m$ and $U(p,0)$ consists of isolated eigenvalues of finite algebraic multiplicity (cf. Lemma A.10 in the appendix). If all non-trivial eigenvalues of $U(p,0)$ lie strictly inside the unit circle, then x_* is a stable solution of (6.1). On the other hand, if $U(p,0)$ has an eigenvalue outside the unit circle, then x_* is an unstable solution of (6.1) (see Section A.3 in the appendix).

The next proposition shows that we can determine the stability of *discrete waves* from the non-trivial eigenvalues of the *reduced* monodromy operators U_h.

Proposition 6.3. *Consider the DDE* (6.1) *satisfying Hypothesis 4. Assume that x_* is a discrete wave and for $h \in H_*,\ h \notin K_*$, let*

$$U_h = h^{-1}U(\Theta(h)p, 0)$$

be the reduced monodromy operator of the linear DDE (6.5)*. Then the following statements hold:*

1. *The non-zero spectrum of U_h consists of isolated eigenvalues of finite algebraic multiplicity;*

2. *If U_h has an eigenvalue strictly outside the unit circle, then x_* is unstable as a solution of* (6.1)*. If the trivial eigenvalue $1 \in \sigma_{pt}(U_h)$ is algebraically simple and all other eigenvalues of U_h lie strictly inside the unit circle, then x_* is stable as a solution of* (6.1)*.*

Proof. The proof is similar in spirit to the proof of Proposition 4.4, but with two main differences:

1. We have to prove that the non-zero spectrum of U_h consists of isolated eigenvalues of finite algebraic multiplicity.

2. We have to make an extra step in the decomposition of the state space: we first use an element of K_* to give a decomposition of $\mathbb{C}^N$, which then induces a decomposition of the state space $X = C\left([-r,0], \mathbb{C}^N\right)$.

We divide the proof into three steps:

STEP 1: We first decompose decompose the state space X and show that the elements of this decomposition are invariant under both the monodromy operator $U(p,0)$ and the reduced monodromy operator U_h. Let $n \in \mathbb{N}$ be such that $H_*/K_* \simeq \mathbb{Z}_n$ and let $1 \leq m < n$ be such that $\Theta(h) = \frac{m}{n}$. Since $H_*/K_* \simeq \mathbb{Z}_n$, h^n is a purely spatial symmetry, i.e. $h^n \in K_*$. Let $\lambda_1, \ldots, \lambda_d$ be the distinct eigenvalues of h^n. Then Lemma 4.3 implies that

$$\mathbb{C}^N = Y_1 \oplus \ldots \oplus Y_d$$

where $Y_j = \ker\left(\lambda_j I - h^n\right)$. This induces a decomposition of the state space $X = C\left([-r,0], \mathbb{C}^N\right)$:

$$X = X_1 \oplus \ldots \oplus X_d$$

with $X_j = C\left([-r,0], Y_j\right)$ for $1 \leq j \leq d$.

Now fix $1 \leq j \leq d$ and let $\phi \in X_j$, i.e. $\lambda_j\phi - h^n\phi = 0$. Equality (6.7b) implies that

$$\lambda_j U(p,0)\phi - h^n U(p,0)\phi = U(p,0)(\lambda_j\phi - h^n\phi) = 0.$$

So $U(p,0)\phi \in X_j$ and hence the space X_j is invariant under the monodromy operator $U(p,0)$. Therefore we can decompose $U(p,0)^m$ as

$$U(p,0)^m = U(p,0)^m|_{X_1} + \ldots + U(p,0)^m|_{X_d}\,. \tag{6.13a}$$

Similarly, if $\phi \in X_j$, then

$$\lambda_j U_h \phi - h^n U_h \phi = U_h(\lambda_j \phi - h^n \phi) = 0.$$

So $U_h\phi \in X_j$ and hence the space X_j is also invariant under the reduced monodromy operator U_h. Therefore we can decompose U_h as

$$U_h = U_h|_{X_1} + \ldots + U_h|_{X_d}\,.$$

Note that $h^n\phi = \lambda_j\phi$ if $\phi \in X_j$. This implies that

$$U(p,0)^m|_{X_j} = \left(h^n U_h^n\right)\Big|_{X_j} = \lambda_j \left(U_h|_{X_i}\right)^n.$$

Step 2: We now prove the first statement of the proposition. For fixed $1 \le j \le d$, all non-zero spectrum of

$$U(p,0)^m|_{X_j} = \lambda_j \left(U_h|_{X_j}\right)^n$$

consists of isolated eigenvalues of finite algebraic multiplicity. Therefore, all non-zero spectrum of $U_h|_{X_j}$ consists of isolated eigenvalues of finite algebraic multiplicity and

$$\sigma_{pt}\left(U(p,0)^m|_{X_j}\right) = \lambda_j \sigma_{pt}\left(\left(U_h|_{X_j}\right)^n\right) = \lambda_j \sigma_{pt}\left(U_h|_{X_j}\right)^n \tag{6.13b}$$

where all equalities count algebraic multiplicities (cf. Lemma A.10 in the appendix). But since this holds on all of the invariant subspaces X_j, $1 \le j \le d$, it follows that all the non-zero spectrum of $U_h : X \to X$ consists of eigenvalues of finite algebraic multiplicity, which proves the first statement of the proposition.

Step 3: To prove the second statement of the proposition, first assume that the reduced monodromy operator U_h has an eigenvalue ν with $|\nu| > 1$; let $1 \le j \le d$ be such that

$$\nu \in \sigma_{pt}\left(U_h|_{X_j}\right)$$

The relation (6.13b) implies that

$$\lambda_j \nu^n \in \sigma_{pt}(U(p,0)^m).$$

Since $|\lambda_j| = 1$ (cf. Lemma 4.3), $|\lambda_j \nu^n| > 1$. So $U(p,0)^m$ has an eigenvalue outside the unit circle and hence x_* is an unstable solution of (6.1).

Vice versa, assume that the eigenvalue $1 \in \sigma_{pt}(U_h)$ is algebraically simple and all other eigenvalues of U_h lie inside the unit circle. Then the equalities (6.13a)–(6.13b) imply that $U(p,0)^m$ has exactly one eigenvalue (counting algebraic multiplicities) on the unit circle, and that all other eigenvalues of $U(p,0)^m$ lie inside the unit circle. Hence x_* is a stable solution of (6.1). $\square$

For completeness, we give a characterization of the eigenvalues of the reduced monodromy operator.

Lemma 6.4. *Let $h \in H_*,\ h \notin K_*$ and let*

$$Y_h = h^{-1}U(\Theta(h)p, 0)$$

be the reduced monodromy operator of the linear DDE (6.5)*. Then $\mu \neq 0$ is an eigenvalue of U_h if and only if* (6.5) *has a non-zero solution of the form*

$$y(t + \Theta(h)p) = \mu h y(t). \tag{6.14}$$

Proof. Suppose that $\mu \neq 0$ is an eigenvector of U_h with eigenfunction $\phi \neq 0$, i.e.

$$h^{-1}U(\Theta(h)p, 0)\phi = \mu\phi.$$

Then $y_t := U(t, 0)\phi$ is a non-zero solution of (6.5) that satisfies

$$\begin{aligned} y_{t+\Theta(h)p} &= U(t + \Theta(h)p, 0)\phi \\ &= U(t + \Theta(h)p, \Theta(h)p)U(\Theta(h)p, 0)\phi \\ &= hU(t, 0)h^{-1}U(\Theta(h)p, 0)\phi \\ &= \mu h U(t, 0)\phi \end{aligned}$$

where in the second step we used (6.8b). We conclude that $y_t := U(t, 0)\phi$ is a non-zero solution of (6.5) that satisfies $y(t + \Theta(h)p) = \mu h y(t)$.

Vice versa, suppose that $y(t)$ is a non-zero solution of (6.5) with $y(t + \Theta(h)p) = \mu h y(t)$. Let $y_0(\theta) = y(\theta),\ \theta \in [-r, 0]$, then $y_t = U(t, 0)y_0$. So if $y(t)$ satisfies $y(t + \Theta(h)p) = \mu h y(t)$, then in particular

$$h^{-1}U(\Theta(h)p, 0)y_0 = \mu y_0.$$

We conclude that $y_0 \neq 0$ is an eigenfunction of U_h with eigenvalue μ. □

6.2 Working with the reduced monodromy operator

In the context of equivariant control, working with the *reduced* monodromy operator (instead of with the monodromy operator) gives a considerable computational advantage. We prove this in Chapter 7, but give an heuristic argument here. We consider again the ODE

$$\dot{x}(t) = f(x(t)) \tag{6.15}$$

and assume that (6.15) is equivariant with respect to a compact group $\Gamma \subseteq GL(N, \mathbb{R})$ and possesses a periodic solution x_* with minimal period $p > 0$. We write the controlled system as

$$\dot{x}(t) = f(x(t)) + K\left[x(t) - hx(t - \Theta(h)p)\right], \tag{6.16}$$

where h is an element of the group of spatial-temporal symmetries H_*. We make the mild assumption that the gain matrix $K \in \mathbb{R}^{N\times N}$ satisfies $hK = Kh$; this condition is in particular satisfied if we choose the gain matrix K to be a scalar, i.e. if $K = kI$ with $k \in \mathbb{R}$. If $Kh = hK$, the reduced monodromy operator U_h (cf. (6.9)) of the linear DDE

$$\dot{y}(t) = f'(x_*(t))y(t) + K\left[y(t) - hy(t - \Theta(h)p)\right]. \tag{6.17}$$

determines whether the solution x_* of (6.16) is stable, see Proposition 6.3.

By Lemma 6.4, $\mu \in \mathbb{C}\backslash\{0\}$ is an eigenvalue of the reduced monodromy operator U_h if and only if (6.17) has a non-zero solution with spatial-temporal pattern

$$y(t + \Theta(h)p) = \mu h y(t). \tag{6.18}$$

The time shift in the spatial-temporal pattern (6.18) is equal to the time delay in (6.17). Therefore, the DDE (6.17) has a non-zero solution of the form $y(t + \Theta(h)p) = \mu h y(t)$ if and only if the ODE

$$\dot{y}(t) = f'(x_*(t))y(t) + K\left[1 - \mu^{-1}\right]y(t). \tag{6.19}$$

has a non-zero solution of the form $y(t + \Theta(h)p) = \mu h y(t)$. So instead of studying solutions of the infinite dimensional DDE (6.17), we now have to study solutions of the family of finite dimensional ODE (6.19), which is a significant dimension reduction.

In contrast, let $U(p, 0)$ be the monodromy operator of (6.17). Then $\nu \in \mathbb{C}\backslash\{0\}$ is an eigenvalue of $U(p, 0)$ if and only if (6.17) has a solution of the form

$$y(t + p) = \nu y(t). \tag{6.20}$$

But now the time shift p in the pattern (6.20) is strictly larger than the time delay $\Theta(h)p$ in (6.17). Therefore, making the Ansatz (6.20) in (6.17) does not yield a direct reduction to an ODE problem. This heuristic argument motivates why, in the context of equivariant control, working with the *reduced* monodromy operator is more attractive than working with the monodromy operator.

However, it is not clear if – and if so, how – the family of ODE (6.19) also captures the algebraic and geometric multiplicities of the eigenvalues of the reduced monodromy operator. In the next chapter, we apply the theory of characteristic matrix functions (introduced in Chapter 5) to the reduced monodromy operator to address this issue.

Chapter 7

Characteristic matrices for reduced monodromy operators

This chapter is concerned with linear DDE with time-dependent coefficients and one (discrete) time delay. If the coefficients of the DDE are periodic with period equal to the time delay, it is well known that there exists a characteristic matrix for the monodromy operator [HV93, Chapter 8], [KV92]. This result is especially useful in the context of Pyragas control, where indeed the linearized equation is periodic with period equal to the time delay.

For systems with equivariant control, the period of the linearized equation is not equal to the delay, but the linearized equation *does* satisfy a spatial-temporal relation with time shift equal to the delay. In this setting, we now rigorously prove the existence of a characteristic matrix for the *reduced* monodromy operator (we have defined the reduced monodromy operator in Section 6.1, specifically in equation (6.9)). The proof is similar in spirit to the argument in [KV21, Section 11.3], but the novelty is the application to *reduced* monodromy operator in *equivariant* control.

In fact, we consider a wider class of equations than equivariant Pyragas control only: we consider linear DDE that satisfy a spatial-temporal relation, whose time shift is equal to the delay of the DDE. In Section 7.1, we first address the case where the time delay is equal to one, and compute a characteristic matrix for the reduced monodromy operator. In Section 7.2, we perform a scaling argument to cover the case of an arbitrary time delay. Section 7.3 then applies the result to equivariant Pyragas control (Theorem 6).

7.1 Characteristic matrix for reduced monodromy operators: time drift 1

Consider the DDE

$$\begin{cases} \dot{y}(t) = A(t)y(t) + B(t)y(t-1), & t \geq s, \\ y(t) = \phi(t), & t \in [s-1, s], \end{cases} \tag{7.1a}$$

with $s \in \mathbb{R}$ and initial condition $\phi \in C\left([-1,0], \mathbb{R}^N\right)$. We assume that the matrix-valued maps $A, B : \mathbb{R} \to \mathbb{R}^{N \times N}$ are C^1. Moreover, we assume that there exists an $h \in GL(N, \mathbb{R})$ such that

$$hA(t)h^{-1} = A(t+1), \qquad hB(t)h^{-1} = B(t+1). \tag{7.1b}$$

So A, B satisfy some spatial-temporal pattern with time shift equal to the time delay of (7.1a).

Let $U(t,s), t \geq s$, be the fundamental solution of the DDE (7.1a). In this section, we use Theorem 4 to find a characteristic matrix for the operator $h^{-1}U(1,0)$. To do so, we make the following intermediate steps:

1. We give an explicit expression for $h^{-1}U(1,0)$ (Lemma 7.1) and write $h^{-1}U(1,0) = V + R$, with V an integral operator and R a finite rank operator.
2. We show that V is a Volterra operator and, for $z \in \mathbb{C}$, compute $(I - zV)^{-1}$ (Lemma 7.2).
3. We decompose the finite rank operator R as $R = DC$ and give a simplified expression for $(I - zV)^{-1}D$ (Lemma 7.3).
4. We apply Theorem 5 to find a characteristic matrix for $h^{-1}U(1,0)$ (Proposition 7.4).

Lemma 7.1. *Let $U(t,s),\ t \geq s$ be the fundamental solution of* (7.1a)*; for $t \geq -1$, let $Y_A(t)$ be the fundamental solution of the ODE*

$$\dot{y}(t) = A(t)y(t)$$

with $Y_A(0) = I$. Then the operator $h^{-1}U(1,0)$ is given by

$$(h^{-1}U(1,0)\phi)(\theta) = h^{-1}Y_A(1+\theta)\phi(0) + \int_{-1}^{\theta} Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\phi(s)ds. \tag{7.2}$$

Proof. For $s = 0$ and $t \in [0,1]$, the initial value problem (7.1a) becomes

$$\dot{y}(t) = A(t)y(t) + B(t)\phi(t-1), \qquad y(0) = \phi(0),$$

which we solve by the Variation of Constants formula as

$$y(t) = Y_A(t)\phi(0) + \int_0^t Y_A(t)Y_A(s)^{-1}B(s)\phi(s-1)ds. \tag{7.3}$$

Since $hA(t)h^{-1} = A(t+1)$, Lemma 4.1 implies that

$$hY_A(t)Y_A(s)^{-1}h^{-1} = Y_A(t+1)Y_A(s+1)^{-1}.$$

Thus, with $t = 1+\theta,\ \theta \in [-1,0]$, (7.3) becomes

$$\begin{aligned} y(1+\theta) &= Y_A(1+\theta)\phi(0) + \int_0^{1+\theta} Y_A(1+\theta)Y_A(s)^{-1}B(s)\phi(s-1)ds \\ &= Y_A(1+\theta)\phi(0) + \int_{-1}^{\theta} Y_A(1+\theta)Y_A(1+s)^{-1}B(1+s)\phi(s)ds \\ &= Y_A(1+\theta)\phi(0) + \int_{-1}^{\theta} hY_A(\theta)Y_A(s)^{-1}h^{-1}hB(s)h^{-1}\phi(s)ds. \end{aligned}$$

So $(h^{-1}U(1,0)\phi)(\theta) := h^{-1}y(1+\theta)$ is given by

$$(h^{-1}U(1,0)\phi)(\theta) = h^{-1}Y_A(1+\theta)\phi(0) + \int_{-1}^{\theta} Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\phi(s)ds,$$

which proves the lemma. $\square$

We decompose $h^{-1}U(1,0)$ in (7.2) as

$$h^{-1}U(1,0) = V + R \tag{7.4}$$

with the (suggestive) notation

$$V : X \to X, \qquad (V\phi)(\theta) = \int_{-1}^{\theta} Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\phi(s)ds, \tag{7.5a}$$

$$R : X \to X, \qquad (R\phi)(\theta) = h^{-1}Y_A(1+\theta)\phi(0). \tag{7.5b}$$

and with $X = C\left([-1,0],\mathbb{C}^N\right)$. The operator R has a finite-dimensional range; indeed, it factorizes as $R = DC$ with

$$C : X \to \mathbb{C}^N, \qquad C\phi = \phi(0) \tag{7.6a}$$

$$D : \mathbb{C}^N \to X, \qquad (Du)(\theta) = h^{-1}Y_A(1+\theta)u. \tag{7.6b}$$

Next, we show that the operator V is Volterra (i.e. V is compact and has no non-zero spectrum) and give an explicit expression of the resolvent $(I - zV)^{-1}$.

Lemma 7.2. *The operator V defined in* (7.5a) *is Volterra.*
Moreover, for $z \in \mathbb{C}$, let $F(t,z)$ be the fundamental solution of the ODE

$$\dot{y}(t) = \left[A(t) + zB(t)h^{-1}\right] y(t)$$

with $F(0,z) = I$. Then the resolvent

$$(I - zV)^{-1} : X \to X$$

is given by

$$\left((I - zV)^{-1}\phi\right)(\theta) = \phi(\theta) + zF(\theta,z)\int_{-1}^{\theta} F(s,z)^{-1}B(s)h^{-1}\phi(s)ds. \tag{7.7}$$

Proof. We start by proving that, for $z \in \mathbb{C}$, the operator $I - zV$ is invertible with inverse given by (7.7). To do so, we first compute $(I - zV)^{-1}V$ and then use the resolvent identity

$$(I - zV)^{-1} = I + z(I - zV)^{-1}V. \tag{7.8}$$

Let $\phi \in X$ and let $\psi = (I - zV)^{-1}V\phi$, then

$$\int_{-1}^{\theta} Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\phi(s)ds = \psi(\theta) - z\int_{-1}^{\theta} Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\psi(s)ds. \tag{7.9}$$

Since the left hand side of (7.9) is C^1, the right hand side has to be C^1 as well. Differentiating both sides with respect to θ gives

$$\begin{aligned} &A(\theta)\int_{-1}^{\theta} Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\phi(s)ds + B(\theta)h^{-1}\phi(\theta) \\ &\quad = \psi'(\theta) - zA(\theta)\int_{-1}^{\theta} Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\psi(s)ds - zB(\theta)h^{-1}\psi(\theta). \end{aligned} \tag{7.10}$$

Equality (7.9) implies

$$A(\theta)\int_{-1}^{\theta} Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\phi(s)ds = A(\theta)\psi(\theta) - zA(\theta)\int_{-1}^{\theta} Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}x(s)ds$$

and substituting this into (7.10) gives

$$A(\theta)\psi(\theta) + B(\theta)h^{-1}\phi(\theta) = \psi'(\theta) - zB(\theta)h^{-1}\psi(\theta).$$

So $\psi(\theta)$ satisfies the initial value problem

$$\begin{cases} \psi'(\theta) & = \left[A(\theta) + zB(\theta)h^{-1}\right]\psi(\theta) + B(\theta)h^{-1}\phi(\theta), \\ \psi(-1) & = 0, \end{cases}$$

which has as its unique solution

$$((I - zV)^{-1}V\phi)(\theta) = \psi(\theta) = F(\theta, z)\int_{-1}^{\theta} F(s, z)^{-1}B(s)h^{-1}\phi(s)ds.$$

The resolvent identity (7.8) implies that

$$\left((I - zV)^{-1}\phi\right)(\theta) = \phi(\theta) + zF(\theta, z)\int_{-1}^{\theta} F(s, z)^{-1}B(s)h^{-1}\phi(s)ds.$$

We conclude that, for $z \in \mathbb{C}$, the operator $I - zV$ is invertible and that the equality (7.7) holds.

Recall that the operator V is Volterra if it is compact and $\sigma(V) \subseteq \{0\}$. If $\phi \in C\left([-1, 0], \mathbb{R}^N\right)$, then (7.5a) implies that $V\phi \in C^1$ and hence by the Arzelà-Ascoli theorem V is compact. In particular, this implies that all non-zero spectrum of V consists of eigenvalues. But since $I - zV$ is invertible for all $z \in \mathbb{C}$, V has no non-zero eigenvalues and hence $\sigma(V) \subseteq \{0\}$. We conclude that V is Volterra, which proves the claim. □

Since V is Volterra, Theorem 4 implies that the matrix-valued function

$$\Delta(z) = I_{\mathbb{C}^N} - C\left(I - zV\right)^{-1}D \tag{7.11}$$

is a characteristic matrix for $h^{-1}U(1, 0)$. The next lemma gives a simplified expression for $(I - zV)^{-1}D$.

Lemma 7.3. *Let V be as in* (7.5a) *and D be as in* (7.6b)*. For $z \in \mathbb{C}$, let $F(t, z)$ be the fundamental solution of the ODE*

$$\dot{y}(t) = \left[A(t) + zB(t)h^{-1}\right]y(t)$$

with $F(0, z) = I$. Then

$$\left((I - zV)^{-1}D\right)(\theta) = h^{-1}F(\theta + 1, z), \qquad \theta \in [-1, 0].$$

Proof. The identity (7.7) and the definition of D as in (7.6b) imply that

$$\left((I - zV)^{-1}D\right)(\theta) = h^{-1}Y_A(1 + \theta) + zF(\theta, z)\int_{-1}^{\theta} F(s, z)^{-1}B(s)h^{-1}\left[h^{-1}Y_A(1 + s)\right]ds. \tag{7.12}$$

Differentiating the relation $F(\theta, z)F(\theta, z)^{-1} = I$ with respect to θ gives

$$\left[A(\theta) + zB(\theta)h^{-1}\right] F(\theta, z)F(\theta, z)^{-1} + F(\theta, z)\frac{d}{d\theta}F(\theta, z)^{-1} = 0$$

and hence

$$\frac{d}{d\theta}F(\theta, z)^{-1} = -F(\theta, z)^{-1}\left[A(\theta) + zB(\theta)h^{-1}\right].$$

So we can rewrite (7.12) as

$$\begin{aligned}\left((I - zV)^{-1}D\right)(\theta) = h^{-1}Y_A(1+\theta) &- F(\theta, z)\int_{-1}^{\theta}\left(\frac{d}{ds}F(s, z)^{-1}\right)\left[h^{-1}Y_A(1+s)\right]ds \\ &- F(\theta, z)\int_{-1}^{\theta}F(s, z)^{-1}A(s)\left[h^{-1}Y_A(1+s)\right]ds\end{aligned}$$

and integration by parts gives

$$\begin{aligned}\left((I - zV)^{-1}D\right)(\theta) = h^{-1}Y_A(1+\theta) &- h^{-1}Y_A(1+\theta) + F(\theta, z)F(-1, z)^{-1}h^{-1} \\ &+ F(\theta, z)\int_{-1}^{\theta}F(s, z)^{-1}h^{-1}A(1+s)Y_A(1+s)ds \\ &- F(\theta, z)\int_{-1}^{\theta}F(s, z)^{-1}A(s)\left[h^{-1}Y_A(1+s)\right]ds.\end{aligned} \tag{7.13}$$

Since $h^{-1}A(1+s) = A(s)h^{-1}$, the last two terms in (7.13) cancel and hence

$$\left((I - zV)^{-1}D\right)(\theta) = F(\theta, z)F(-1, z)^{-1}h^{-1}.$$

Since $hA(t)h^{-1} = A(t+1), hB(t)h^{-1} = B(t+1)$, Lemma 4.1 gives that

$$F(\theta, z)F(-1, z)^{-1}h^{-1} = h^{-1}F(\theta+1, z)F(0, z)^{-1}.$$

with $F(0, z) = I$. This implies that

$$\left((I - zV)^{-1}D\right)(\theta) = h^{-1}F(\theta+1, z),$$

as claimed. $\square$

We are now ready to use Theorem 4 and give an explicit expression for the characteristic matrix for $h^{-1}U(1, 0)$:

Proposition 7.4. *Let $U(t, s)$, $t \geq s$ be the fundamental solution of* (7.1a) *and let $h \in GL(N, \mathbb{R})$ be as in* (7.1b)*. For $z \in \mathbb{C}$, let $F(t, z)$ be the fundamental solution of the ODE*

$$\dot{y}(t) = \left[A(t) + zB(t)h^{-1}\right]y(t)$$

with $F(0, z) = I_{\mathbb{C}^N}$.

Then the matrix-valued function

$$\Delta(z) = I_{\mathbb{C}^N} - zh^{-1}F(1, z) \tag{7.14}$$

is a characteristic matrix for the operator $h^{-1}U(1, 0)$.

Proof. The operator $h^{-1}U(1,0)$ decomposes as $T = V + R$, where V defined in (7.5a) is a Volterra operator and R defined in (7.5b) is a finite rank operator. Therefore, if we let D, C be as in (7.6b)–(7.6a), Theorem 4 implies that

$$\Delta(z) = I_{\mathbb{C}^N} - zC(I - zV)^{-1}D$$

is a characteristic matrix for $h^{-1}U(1,0)$. Lemma 7.3 implies that

$$C(I - zV)^{-1}D = h^{-1}F(1,z)$$

and hence

$$\Delta(z) = I_{\mathbb{C}^N} - zh^{-1}F(1,z)$$

is a characteristic matrix for $h^{-1}U(1,0)$, as claimed. □

7.2 Characteristic matrix for reduced monodromy operators: arbitrary time drift

In this section, we again consider DDE of the form (7.1a)–(7.1b), but now allow the time delay to be equal to an arbitrary positive constant $r > 0$. More precisely, we consider the DDE

$$\begin{cases} \dot{y}(t) = A(t)y(t) + B(t)y(t-r), & t \geq s \\ y(t) = \phi(t), & t \in [s-r, s] \end{cases} \tag{7.15a}$$

with $r > 0$, $s \in \mathbb{R}$ and initial condition $\phi \in C\left([-r,0], \mathbb{R}^N\right)$. We assume that the matrix-valued maps $A, B : \mathbb{R} \to \mathbb{R}^{N \times N}$ are C^1 and that there exists an $h \in GL(N, \mathbb{R})$ such that

$$hA(t)h^{-1} = A(t+r), \qquad hB(t)h^{-1} = B(t+r). \tag{7.15b}$$

We use the results from Section 7.1 to find a characteristic matrix for the operator $h^{-1}U(r,0)$, where $U(t,s), t \geq s$ is now the fundamental solution of system (7.15a).

We bring system (7.15a)–(7.15b) in the form (7.1a)–(7.1b) by performing a time rescaling. On the level of the fundamental solution, the monodromy operator of (7.15a)–(7.15b) is related to the monodromy operator of (7.1a)–(7.1b) via conjugation with a bounded operator. Conjugation with a bounded linear operator does not change the spectral information of the monodromy operator, and hence we expect that it will also not change its characteristic matrix. We make this precise in the following lemma.

Lemma 7.5 (Characteristic matrix is invariant under conjugation)**.** *Consider two Banach spaces X and Y and a bounded linear operator*

$$s : Y \to X$$

with bounded inverse

$$s^{-1} : X \to Y.$$

Moreover, let

$$T_X : X \to X$$

be a bounded linear operator on X and consider the induced operator

$$T_Y := s^{-1} T_X s$$

on Y.

Then, if the matrix valued function $\Delta(z) : \mathbb{C}^N \to \mathbb{C}^N$ is a characteristic matrix for T_X, then $\Delta(z)$ is a characteristic matrix for T_Y as well.

Proof. Since Δ is a characteristic matrix for T_X, there exist holomorphic functions $E, F : \mathbb{C} \to \mathcal{L}(\mathbb{C}^N \oplus X)$ such that

$$\begin{pmatrix} \Delta(z) & 0 \\ 0 & I_X \end{pmatrix} = F(z) \begin{pmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & I_X - zT_X \end{pmatrix} E(z)$$

for all $z \in \mathbb{C}$ (see Definition 5.6). Since $T_X = sT_Y s^{-1}$, we have that

$$\begin{pmatrix} I_{\mathbb{C}_N} & 0 \\ 0 & I_X - zT_X \end{pmatrix} = \begin{pmatrix} I_{\mathbb{C}_N} & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} I_{\mathbb{C}_N} & 0 \\ 0 & I_Y - zT_Y \end{pmatrix} \begin{pmatrix} I_{\mathbb{C}_N} & 0 \\ 0 & s^{-1} \end{pmatrix}$$

and moreover

$$\begin{pmatrix} \Delta(z) & 0 \\ 0 & I_Y \end{pmatrix} = \begin{pmatrix} I_{\mathbb{C}_N} & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} \Delta(z) & 0 \\ 0 & I_X \end{pmatrix} \begin{pmatrix} I_{\mathbb{C}_N} & 0 \\ 0 & s \end{pmatrix}$$

Therefore, if we define

$$\tilde{E}(z) = \begin{pmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & s^{-1} \end{pmatrix} E(z) \begin{pmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & s \end{pmatrix}, \qquad \tilde{F}(z) = \begin{pmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & s^{-1} \end{pmatrix} F(z) \begin{pmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & s \end{pmatrix}$$

then $\tilde{E}, \tilde{F} : \mathbb{C} \to \mathcal{L}(\mathbb{C}^N \oplus Y)$ are holomorphic functions and

$$\begin{pmatrix} \Delta(z) & 0 \\ 0 & I_Y \end{pmatrix} = \tilde{F}(z) \begin{pmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & I_Y - zT_Y \end{pmatrix} \tilde{E}(z).$$

We conclude that $\Delta(z)$ is a characteristic matrix for T_Y. □

Next we give a characteristic matrix for the operator $h^{-1}U(r, 0)$, where $U(t, s)$ is the fundamental solution of (7.15a).

Proposition 7.6. *Let $U(t, s)$, $t \geq s$ be the fundamental solution of* (7.15a) *and let $h \in GL(N, \mathbb{R})$ be as in* (7.15b)*. For $z \in \mathbb{C}$, let $F(t, z)$ be the fundamental solution of the ODE*

$$\dot{y}(t) = \left[A(t) + zB(t)h^{-1}\right] y(t) \tag{7.16}$$

with $F(0, z) = I_{\mathbb{C}^N}$.

Then the matrix-valued function

$$\Delta(z) = I_{\mathbb{C}^N} - zh^{-1}F(r, z) \tag{7.17}$$

is a characteristic matrix for the operator $h^{-1}U(r, 0)$.

Proof. We divide the proof into three steps; Figure 7.1 gives a schematic summary of the first two steps.

Step 1 (Figure 7.1, top row): For $t \geq s$, let

$$U(t,s) : C\left([-r,0],\mathbb{R}^N\right) \to C\left([-r,0],\mathbb{R}^N\right)$$

be the fundamental of (7.15a) and let

$$V(t,s) : C\left([-1,0],\mathbb{R}^N\right) \to C\left([-1,0],\mathbb{R}^N\right)$$

be the fundamental solution of the DDE

$$\dot{w}(t) = rA(rt)w(t) + rB(rt)w(t-1) \tag{7.18}$$

Moreover, define the linear operator

$$s : C\left([-r,0],\mathbb{R}^N\right) \to C\left([-1,0],\mathbb{R}^N\right), \qquad (s\phi)(\theta) = \phi(r\theta) \tag{7.19}$$

with inverse

$$s^{-1} : C\left([-1,0],\mathbb{R}^N\right) \to C\left([-r,0],\mathbb{R}^N\right), \qquad (s^{-1}\phi)(\theta) = \phi(r^{-1}\theta).$$

We show that

$$h^{-1}U(r,0) = s^{-1}\left[h^{-1}V(1,0)\right]s, \tag{7.20}$$

i.e. the operators $h^{-1}U(r,0)$ and $h^{-1}V(1,0)$ are related via conjugation with s.

Indeed, let $\phi \in C\left([-r,0],\mathbb{R}^N\right)$ and let $\psi_1 = V(1,0)s\phi$. Then

$$\dot{\psi}_1(\theta) = rA(r\theta)\psi_1(\theta) + rB(r\theta)\phi(r\theta), \qquad \theta \in [-1,0].$$

Therefore $\psi = s^{-1}\psi_1$ satisfies

$$\dot{\psi}(\theta) = r^{-1}\dot{\psi}_1(r^{-1}\theta) = A(\theta)\psi(\theta) + B(\theta)\phi(\theta), \qquad \theta \in [-r,0].$$

Hence $\psi = U(r,0)\phi$ and we conclude that $U(r,0) = s^{-1}V(1,0)s$. Since

$$s^{-1}h^{-1}\phi = h^{-1}s^{-1}\phi$$

for all $\phi \in C\left([-1,0],\mathbb{R}^N\right)$, the equality (7.20) follows.

Step 2 (Figure 7.1, bottom row) For $z \in \mathbb{C}$, let $F(t,z)$ be the fundamental solution of the ODE

$$\dot{y}(t) = A(t)y(t) + zB(t)h^{-1}y(t)$$

with $F(0,z) = I$; moreover, let $G(t,z)$ be the fundamental solution of the ODE

$$\dot{w}(t) = rA(rt)w(t) + zrB(rt)h^{-1}w(t) \tag{7.21}$$

with $G(0,z) = I$. We show that $F(r,z) = G(1,z)$.

Indeed, $G(r^{-1}t, z)$ satisfies

$$\begin{aligned}\partial_t G(r^{-1}t, z) &= r^{-1}\left[rA(r^{-1}rt)G(r^{-1}t, z) + zrB(r^{-1}rt)G(r^{-1}t, z)\right] \\ &= A(t)G(r^{-1}t, z) + zB(t)G(r^{-1}t, z).\end{aligned}$$

Moreover, $G(r^{-1}t, z) = I$ for $t = 0$. So by uniqueness of solutions $G(r^{-1}t, z) = F(t, z)$ holds for all t and in particular $F(r, z) = G(1, z)$.

Step 3: The coefficients $A(rt), B(rt)$ in (7.18) satisfy

$$h^{-1}A(rt)h = A(r(t+1)), \qquad h^{-1}B(rt)h = B(r(t+1))$$

and hence (7.18) is of the form (7.1a)–(7.1b). As before, let $G(t, z)$ be the fundamental solution of (7.21), then Proposition 7.4 implies that the matrix valued function

$$\Delta(z) = I - zh^{-1}G(1, z) \tag{7.22}$$

is a characteristic matrix for $h^{-1}V(1, 0)$. Lemma 7.5 and the equality (7.20) imply that (7.22) is also a characteristic matrix for $h^{-1}U(r, 0)$. But by Step 2, $G(1, z) = F(r, z)$, where $F(t, z)$ is the fundamental solution of (7.16). We conclude that

$$\Delta(z) = I - zh^{-1}F(r, z)$$

is a characteristic matrix for $h^{-1}U(r, 0)$, as claimed. $\square$

$$\begin{tikzcd}
\begin{array}{c}U(t,s)\text{ fundamental solution of}\\ \dot{y}(t) = A(t)y(t) + B(t)y(t-r)\end{array} \arrow[r, leftrightarrow, "U(r,0) = s^{-1}V(1,0)s"] \arrow[d, dashed] & \begin{array}{c}V(t,s)\text{ fundamental solution of}\\ \dot{w}(t) = rA(rt)w(t) + rB(t)w(t-1)\end{array} \arrow[d, dashed] \\
\begin{array}{c}F(t,z)\text{ fundamental solution of}\\ \dot{y}(t) = A(t)y(t) + zB(t)h^{-1}y(t)\end{array} \arrow[r, leftrightarrow, "F(r,z) = G(1,z)"] & \begin{array}{c}G(t,z)\text{ fundamental solution of}\\ \dot{w}(t) = rA(rt)w(t) + rzB(rt)h^{-1}w(t)\end{array}
\end{tikzcd}$$

Figure 7.1: Schematic summary of the fundamental solutions used in the proof of Proposition 7.4. The systems in the left column satisfy a spatial-temporal relation with time shift equal to r; the systems in the right column satisfy a spatial-temporal relation with time shift equal to 1. The operator s on the top line is defined in (7.19).

7.3 Applications

We considered the equations (7.1b)–(7.15b) with in the back of our minds equivariant Pyragas control. But the equations also cover the special case $h = I$, in which case equation (7.1b) is periodic with period equal to the time delay. In this case, an application of Corollary 7.6 gives a characteristic matrix for the monodromy operator, in agreement with the familiar result from [KV92; HV93].

Theorem 5 (Period equal to delay, cf. [KV92; HV93])**.** *Consider the DDE*

$$\dot{y}(t) = A(t)y(t) + B(t)y(t-r) \tag{7.23}$$

with $r > 0$, $y \in \mathbb{R}^N$ and $A(t), B(t) \in \mathbb{R}^{N\times N}$ satisfying

$$A(t+r) = A(t), \qquad B(t+r) = B(t) \tag{7.24}$$

for all $t \in \mathbb{R}$. For $t \geq s$, let

$$U(t,s) : C\left([-r,0],\mathbb{R}^N\right) \to C\left([-r,0],\mathbb{R}^N\right)$$

be the fundamental solution of (7.23). For $z \in \mathbb{C}$, let $F(t,z)$ be the fundamental solution of the ODE

$$\dot{y}(t) = A(t)y(t) + zB(t)y(t)$$

with $F(0,z) = I_{\mathbb{C}^N}$.

Then the matrix-valued function

$$\Delta(z) = I_{\mathbb{C}^N} - zF(r,z) \tag{7.25}$$

is a characteristic matrix for the monodromy operator $U(r,0)$.

The next theorem considers equivariant Pyragas control, and gives a characteristic matrix for the reduced monodromy operator.

Theorem 6. *Consider the system*

$$\dot{x}(t) = f(x(t)) \tag{7.26}$$

with $f : \mathbb{R}^N \to \mathbb{R}^N$ a C^1 function. Assume that the system is equivariant with respect to a compact group $\Gamma \subseteq GL(N,\mathbb{R})$ and possesses a periodic solution x_* with minimal period $p > 0$. Consider the system with equivariant control

$$\dot{x}(t) = f(x(t)) + K\left[x(t) - hx(t - \Theta(h)p)\right]$$

where h is an element of the group of spatial-temporal symmetries H_* and where $K \in \mathbb{R}^{N\times N}$ satisfies $hK = Kh$. Let $U(t,s), t \geq s$, be the fundamental solution of the linear DDE

$$\dot{y}(t) = f'(x_*(t))y(t) + K\left[y(t) - hy(t - \Theta(h)p)\right] \tag{7.27}$$

and let

$$U_h := h^{-1}U(\Theta(h)p)$$

be the reduced monodromy operator.

For $z \in \mathbb{C}$, let $F(t,z)$ be the fundamental solution of the

$$\dot{y}(t) = f'(x_*(t))y(t) + k\,[y(t) - zy(t)]$$

with $F(0,z) = I_{\mathbb{C}^N}$. Then the matrix-valued function

$$\Delta(z) = I_{\mathbb{C}^N} - zh^{-1}F(\Theta(h)p, z) \tag{7.28}$$

is a characteristic matrix for the operator U_h.

Proof. Define

$$A(t) := f'(x_*(t)) + K, \qquad B(t) := -Kh.$$

Lemma 4.1 (applied to the ODE (7.26) implies that

$$h^{-1}f'(x_*(t))h = f'(x_*(t + \Theta(h)p).$$

Since by assumption $hK = Kh$, this implies that

$$h^{-1}A(t)h = A(t + \Theta(h)p), \qquad h^{-1}B(t)h = B(t + \Theta(h)p).$$

So (7.27) is of the form (7.15a)–(7.15b) with $r = \Theta(h)p$.

Let $U(t,s)$ is the fundamental solution of the DDE (7.27) and $F(t,z)$ is the fundamental solution of the ODE

$$\begin{aligned}\dot{y}(t) &= A(t)y(t) + B(t)h^{-1}y(t) \\ &= f'(x_*(t))y(t) + K\,[y(t) - zy(t)]\end{aligned}$$

with $F(0,z) = I$. Proposition 7.6 implies that (7.28) is a characteristic matrix for $U_h = h^{-1}U(\Theta(h)p, 0)$.

$\square$

Chapter 8

Scalar control gains and negative Floquet multipliers

In this chapter, we consider 'classical' (i.e. non-equivariant) Pyragas control of periodic orbits whose unstable Floquet multipliers are on the *negative* real axis. Under additional conditions on the size of the Floquet multipliers, one can successfully stabilize the periodic orbits using Pyragas control with *scalar* control gain, as was proven [MNS11].

We have two reasons to revisit the result from [MNS11] at this point in the thesis. First, we put the result from [MNS11] in the context of the 'any number limitation', which implies that Pyragas control with scalar control gain always fails if the uncontrolled system has an unstable Floquet multiplier on the *positive* real axis. Moreover, the proof of the result in [MNS11] involves the analysis of a class of transcedental equations; this analysis will be a technical tool for the stabilization of discrete waves in Chapter 9.

We reprove the main result from [MNS11] (which we state precisely in Proposition 8.7) in this chapter; however, the proof presented here differs from the proof in [MNS11] on two points. First of all, in the system with Pyragas control the time delay equals the period of the target periodic solution. Therefore, Theorem 5 directly and rigorously yields a characteristic equation for the Floquet multipliers. In contrast, the authors of [MNS11] derive the characteristic equation by making an Ansatz for the eigenfunctions of the monodromy operator; they then show by hand that this characteristic equation also captures the algebraic multiplicity. Secondly, we prove a new connection between the characteristic equation for the Floquet multipliers and a characteristic equation for a linear, *autonomous* DDE. This makes our arguments more systematic and simplifies the stability analysis compared to [MNS11].

This chapter is structured as follows: in Section 8.1, we explicitly compute a characteristic equation for the Floquet multipliers in the controlled system. We then analyze this equation in Section 8.2. We prove the successful stabilization result in Section 8.3, where we also compare this positive stabilization result with the 'any number limitation' (Proposition 2 on page 12 of this thesis).

8.1 Working with scalar control gains

We consider the ODE

$$\dot{x}(t) = f(x(t))$$

with $f : \mathbb{R}^N \to \mathbb{R}^N$ a C^2-map and assume that this ODE has a periodic solution x_* with period $p > 0$. We apply Pyragas control with *scalar* control gain, i.e. we write the controlled system as

$$\dot{x}(t) = f(x(t)) + k\,[x(t) - x(t-p)] \tag{8.1}$$

with $k \in \mathbb{R}$. The linear, time dependent DDE

$$\dot{y}(t) = f'(x_*(t))y(t) + k\,[y(t) - y(t-p)]\,. \tag{8.2}$$

is periodic with period p equal to the delay. Therefore, Theorem 5 implies the existence of a characteristic matrix function for the monodromy operator of (8.2). The next lemma gives an *explicit* expression of this characteristic matrix function in terms of the monodromy operator of the ODE $\dot{y}(t) = f'(x_*(t))y(t)$.

Lemma 8.1 (Characteristic equation for *scalar* control gain)**.** *Let $Y(p)$ be the monodromy operator of the ODE $\dot{y}(t) = f'(x_*(t))y(t)$ and denote by $\mu_1, \ldots, \mu_N$ the eigenvalues (counting algebraic multiplicities) of $Y(p)$, i.e.*

$$\{\mu_1, \ldots, \mu_N\} = \sigma(Y(p)).$$

Moreover, fix $k \in \mathbb{R}$ and let $U(p,0)$ be the monodromy operator of the DDE

$$\dot{y}(t) = f'(x_*(t))y(t) + k\,[y(t) - y(t-p)]\,. \tag{8.3}$$

Then the equation

$$d(z) = 0 \qquad \text{with} \qquad d(z) = \prod_{j=1}^{N} \left(1 - z\mu_j e^{k(1-z)p}\right) \tag{8.4}$$

is a characteristic equation for $U(p,0)$ and hence

1. *$\mu \neq 0$ is an eigenvalue of $U(p,0)$ if and only if μ^{-1} is a root of $d(z) = 0$, i.e.*

$$\sigma_{pt}(U(p,0))\backslash\{0\} = \{\mu \in \mathbb{C} \mid d(\mu^{-1}) = 0\}.$$

2. *If $\mu \neq 0$ is an eigenvalue of $U(p,0)$, then the algebraic multiplicity of μ as an eigenvalue of $U(p,0)$ equals the order of μ^{-1} as a root of $d(z) = 0$.*

Proof. Let $Y(t)$ be the fundamental matrix solution of the ODE $\dot{y}(t) = f'(x_*(t))y(t)$ with $Y(0) = I$. Then, for all $z \in \mathbb{C}$, $F(t,z) := Y(t)e^{k(1-z)t}$ is the fundamental matrix solution of the ODE

$$\dot{y}(t) = f'(x_*(t))y(t) + k\,[1-z]\,y(t)$$

with $F(0,z) = I$. Therefore, Theorem 5 implies that the matrix-valued function

$$\begin{aligned}\Delta(z) &= I - zF(p,z)\\ &= I - zY(p)e^{k(1-z)p}\end{aligned}$$

is a characteristic matrix function for $U(p,0)$. Since the multiplicative factor $e^{k(1-z)p}$ is a complex scalar, we have that

$$\det \Delta(z) = \prod_{j=1}^{N} \left(1 - z\mu_j e^{k(1-z)p}\right)$$

where $\mu_1, \ldots, \mu_N$ are the eigenvalues of $Y(p)$. Therefore, (8.4) is a characteristic equation for $U(p,0)$, and Lemma 5.7 implies statements 1. and 2. of the current lemma. $\square$

For Pyragas control with a general gain matrix $K \in \mathbb{R}^{N\times N}$, the linear DDE

$$\dot{y}(t) = f'(x_*(t))y(t) + K\left[y(t) - y(t-p)\right] \tag{8.5}$$

is periodic with period p equal to the delay. So also for system (8.5), Theorem 5 implies the existence of a characteristic matrix function for the monodromy operator. But computing this characteristic matrix function involves computing the solutions of the ODE

$$\dot{y}(t) = f'(x_*(t))y(t) + K\left[1 - z\right]y(t), \qquad z \in \mathbb{C}, \tag{8.6}$$

which is, for general $K \in \mathbb{R}^{N\times N}$, not straightforward. So in Lemma 8.1, the crucial assumption is that the control gain in (8.3) is scalar.

The function $z \mapsto d(z)$ defined in (8.4) is the product of N factors. Therefore, to determine the zeros of $d(z)$, it suffices to find the zeros of each of the individual factors, i.e. it suffices to find zeros of equations of the form

$$0 = 1 - z\mu_* e^{k(1-z)} \tag{8.7}$$

with $\mu_* \in \mathbb{C}$. In the next section, we analyze the equation (8.7) for the case $\mu_* = 1$, $\mu_* < -1$ and $|\mu_*| < 1$. We stress again that μ is an eigenvalue of the monodromy operator of (8.3) if and only if μ^{-1} is a root of (8.4); therefore, x_* is a stable solution of (8.1) if all the non-trivial solutions of (8.4) are *outside* the unit circle.

8.2 Analysis of the characteristic equation

To analyze equations of the form (8.7), we start by a (seeming) detour into the equation

$$0 = -z + a + be^{-z}. \tag{8.8}$$

with $a, b \in \mathbb{R}$. If

$$a + b = 0 \tag{8.9}$$

then $z = 0$ is a root of (8.8). Substituting $z = i\omega,\ 0 \le \omega < \pi$ into (8.8) and solving for a, b gives the curve

$$a = \frac{\omega\cos(\omega)}{\sin(\omega)}, \qquad b = -\frac{\omega}{\sin(\omega)}. \tag{8.10}$$

Note that for $\omega = 0$, the curve (8.10) intersects the line (8.9) at the point

$$a = 1, \qquad b = -1 \tag{8.11}$$

corresponding to $z = 0$ being a double root of (8.8). Equation (8.8) is studied in detail in, for example, [HV93, Appendix], [DGVW95, Chapter XI], [Hay50] from where we take the following proposition:

Proposition 8.2. *Consider the equation* (8.8) *with* $a, b \in \mathbb{R}$ *and* $\lambda \in \mathbb{C}$. *In the* (a, b)*-plane, consider the half-line*

$$R = \{(a, b) \mid a + b = 0,\ a < 1\} \tag{8.12}$$

and the curve

$$C = \left\{(a, b) \mid a = \frac{\omega \cos(\omega)}{\sin(\omega)},\ b = -\frac{\omega}{\sin(\omega)},\ 0 \leq \omega < \pi\right\}. \tag{8.13}$$

Denote by S *the region in the* (a, b)*-plane bounded by* R *and* C *(see Figure 8.1). It holds that*

1. *For* (a, b) *on the half-line* R*,* $z = 0$ *is a solution of* (8.8) *and all other solutions* λ *satisfy* $Re\,\lambda < 0$.
2. *For* (a, b) *on the curve* C*,* (8.8) *has two solutions on the imaginary axis and all other solutions* λ *satisfy* $Re\,\lambda < 0$.
3. *For* (a, b) *in the interior of* S*, all roots* λ *of* (8.8) *satisfy* $Re\,\lambda < 0$.
4. *For* $(a, b) \in \mathbb{C}\backslash S$*,* (8.8) *has at least one root* λ *satisfying* $Re\,\lambda > 0$.

The next lemma relates solutions of the equation

$$0 = -z + a + be^{-z}$$

to solutions of the equation

$$0 = 1 - ze^{a}e^{bz}. \tag{8.14}$$

Lemma 8.3. *Let* $a, b \in \mathbb{R}$ *and consider the analytic functions*

$$\begin{aligned} F &: \mathbb{C} \to \mathbb{C}, \qquad F(z) = 1 - ze^{a}e^{bz} \\ G &: \mathbb{C} \to \mathbb{C}, \qquad G(z) = -z + a + be^{-z}. \end{aligned}$$

Then $\mu \neq 0$ *is a solution of* $F(z) = 0$ *if and only if* $\mu = e^{-\lambda}$*, where* λ *is a solution of* $G(z) = 0$. *So*

$$\big\{\mu \in \mathbb{C}\backslash\{0\} \mid F(\mu) = 0\big\} = \big\{e^{-\lambda} \mid G(\lambda) = 0\big\}. \tag{8.15}$$

Proof. For $z \in \mathbb{C}$, it holds that

$$F(e^{-z}) = 1 - e^{G(z)}. \tag{8.16}$$

Therefore, the map

$$\begin{gathered} \big\{\lambda \in \mathbb{C} \mid G(\lambda) = 0\big\} \to \big\{\mu \in \mathbb{C}\backslash\{0\} \mid F(\mu) = 0\big\} \\ \lambda \mapsto e^{-\lambda} \end{gathered} \tag{8.17}$$

is well-defined. We show that the map (8.17) is bijective.

Injective: suppose that λ, ν satisfy $G(\lambda) = 0, G(\nu) = 0$ and $e^{-\lambda} = e^{-\nu}$. Then

$$\begin{aligned} \lambda &= a + be^{-\lambda} \\ \nu &= a + be^{-\nu} \end{aligned}$$

but since $e^{-\nu} = e^{-\mu}$, this implies that $\lambda = \nu$. Hence the map (8.17) is injective.

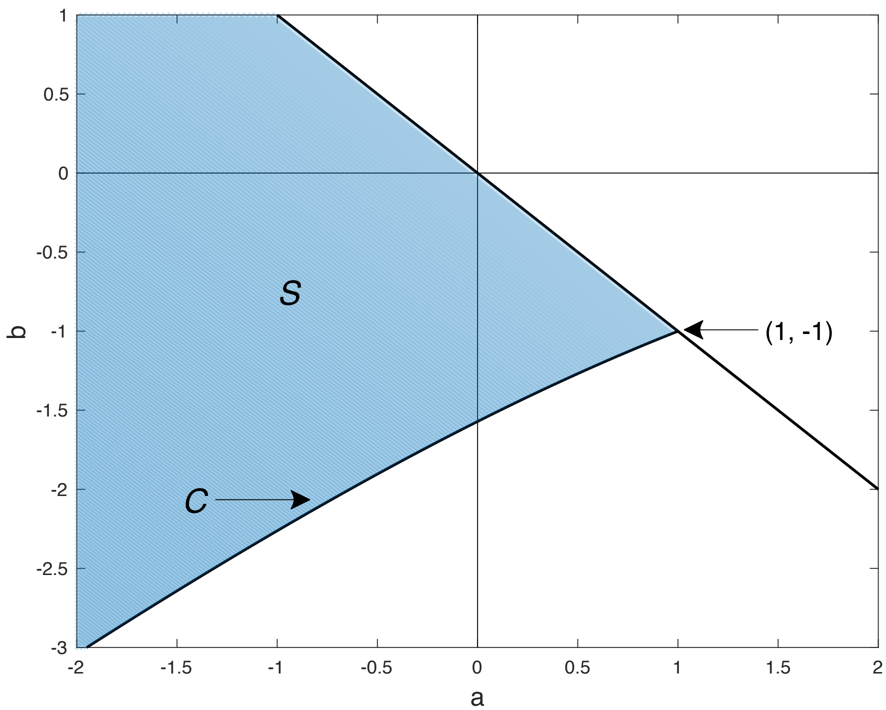

Figure 8.1: The (a, b) plane with the straight line $a = -b$, the curve C defined (8.13) and the region S (blue) defined in Lemma 8.1.

SURJECTIVE: let $\mu \in \mathbb{C}\backslash\{0\}$ be such that $F(\mu) = 0$ and let $\tilde{\lambda} \in \mathbb{C}$ be such that $e^{-\tilde{\lambda}} = \mu$. Then

$$1 = e^{-\tilde{\lambda}} e^{a} e^{be^{-\tilde{\lambda}}}$$

and hence

$$-\tilde{\lambda} + a + be^{-\tilde{\lambda}} = 2\pi i k$$

for some $k \in \mathbb{Z}$. But then $\lambda := \tilde{\lambda} + 2\pi i k$ satisfies

$$-\lambda + a + be^{-\lambda} = 0$$

and $e^{-\lambda} = \mu$. So the map (8.17) is surjective. We conclude that the map (8.17) is bijective and the equality (8.15) follows. $\square$

We are now ready to analyze the roots of the characteristic equation (8.4), or, equivalently, of the factors (8.7). Recall that $\mu \neq 0$ is a root of (8.4) if and only if μ^{-1} is an eigenvalue of the monodromy operator. Therefore, to prove that all non-trivial eigenvalues of the monodromy operator are *inside* the unit circle, we have to prove that all non-trivial solutions of (8.4) are *outside* the unit circle. Since in Proposition 8.7 and Theorem 3 we stabilize with a control gain $k < 0$, we also consider $k < 0$ in the analysis of the factors (8.7).

Corollary 8.4 (Case $\mu_* = 1$)**.** *Let $k < 0$. Then the equation*

$$1 - ze^{k(1-z)} = 0 \tag{8.18}$$

has a simple root $z = 1$; all other roots of (8.18) *lie strictly outside the unit circle.*

Proof. Equation (8.18) is of the form

$$1 - ze^a e^{bz} = 0$$

with

$$a(k) = k, \qquad b(k) = -k.$$

For $k < 0$, the point $(a(k), b(k))$ lies on the line R as defined in (8.12). Therefore, the equation

$$-z + a(k) + b(k)e^{-z} = 0$$

has a solution $z = 0$ and all other solutions lie in the strict left half plane. Therefore Lemma 8.3 implies that equation (8.18) has a solution $z = 1$ and that all other roots of (8.18) lie strictly outside the unit circle.

To prove that $z = 1$ is simple as a solution (8.18) for $k < 0$, we compute

$$\begin{aligned} \frac{d}{dz}\bigg|_{z=1} 1 - ze^{k(1-z)} &= -ze^{k(1-z)} + kze^{k(1-z)}\bigg|_{z=1} \\ &= -1 + k \neq 0 \end{aligned}$$

for $k < 0$. □

Corollary 8.5 (Case $\mu_* < -1$)**.** *Let $-e^2 < \mu_* < -1$ and consider the equation*

$$1 - \mu_* z e^{k(1-z)} = 0. \tag{8.19}$$

Then, there exists an open interval $I \subseteq \mathbb{R}_{<0}$ such that for $k \in I$, all solutions of (8.19) *lie strictly outside the unit circle.*

Proof. Let μ be a solution of (8.19), then $\nu := -\mu$ is a solution of

$$1 - z[-\mu_*]e^{k(1+z)} = 0. \tag{8.20}$$

Since by assumption $-e^2 < \mu_* < -1$, we can write $-\mu^* = e^{\lambda_*}$ with $0 < \lambda_* < 2$. So (8.20) is of the form

$$1 - ze^a e^{bz} = 0$$

with

$$a(k) = \lambda_* + k, \qquad b(k) = k. \tag{8.21}$$

The path (8.21) crosses the line R at the point

$$(a, b) = (\lambda_*/2, -\lambda_*/2)$$

with $\lambda_*/2 < 1$. Therefore, there exists a $k_* < -\lambda_*/2$ such that for k in the interval

$$I := (k_*, -\lambda_*/2)$$

the path (8.21) lies inside the set S as defined in Proposition 8.2. To give a more precise (but by no means sharp!) bound on the number k_*, note that the line$\{(a, b) \mid a < 1,\ b = -1\}$ lies in the region S (see Figure 8.2). Hence $k_* \leq -1$ and

$$(-1, -\lambda_*/2) \subseteq I. \tag{8.22}$$

Since for $k \in I$, the path (8.21) lies inside the set S, it holds that for $k \in I$ all solutions of

$$-z + a(k) + b(k)e^{-z} = 0$$

are in the strict left half of the complex plane (cf. Figure 8.2). Therefore Lemma 8.3 implies that, for $k \in I$, all solutions of (8.20) lie strictly outside the unit circle. But this implies that all solutions of (8.19) lie strictly outside the unit circle, as claimed. □

In the case where $e^{\mu_*} \in \mathbb{C}\backslash\mathbb{R}$, the equation (8.7) can be only brought into the form $1 - ze^a e^{bz} = 0$ by choosing a to be complex. So in this case, Proposition 8.2 does not give information on the roots of (8.7). However, for $|\mu_*| < 1$, we can analyze the roots by a direct estimate:

Corollary 8.6 (Case $|\mu_*| < 1$, statement (v) in Lemma 7.4 in [MNS11])**.** *Let $\mu_* \in \mathbb{C}$, $|\mu_*| < 1$ and $k < 0$. Then all solutions of*

$$1 - \mu_* z e^{k(1-z)} = 0 \tag{8.23}$$

lie strictly outside the unit circle.

Proof. Let $\mu \in \mathbb{C}$ be a solution of (8.23), then

$$1 = |\mu_*|\,|\mu|\,e^k\left|e^{-k\mu}\right|.$$

Because $k < 0$, it holds that $|-k\mu| = -k\,|\mu|$ and hence $\left|e^{-k\mu}\right| \leq e^{-k|\mu|}$. So μ satisfies the estimate

$$1 \leq |\mu_*|\,|\mu|\,e^k e^{-k|\mu|}. \tag{8.24}$$

Now assume by contradiction that $|\mu| \leq 1$. Since $k < 0$, we can estimate the right hand side of (8.24) as

$$\begin{aligned} |\mu_*|\,|\mu|\,e^k e^{-k|\mu|} &\leq |\mu_*|\,e^k e^{-k} \\ &\leq |\mu_*| < 1. \end{aligned}$$

But this contradicts the estimate (8.24). We conclude that if μ is a solution of (8.19), then $|\mu| \geq 1$. □

8.3 Stabilizing systems with a negative Floquet multiplier

The next proposition addresses the case where the uncontrolled periodic orbit has unstable Floquet multipliers on the *negative* real axis:

Proposition 8.7 (cf. [MNS11])**.** *Consider the system*

$$\dot{x}(t) = f(x(t)), \qquad t \geq 0 \tag{8.25}$$

with $f : \mathbb{R}^N \to \mathbb{R}^N$ a C^1-function. Assume that system (8.25) has a periodic solution x_* with period $p > 0$. Let $Y(p)$ be the monodromy operator of the ODE

$$\dot{y}(t) = f'(x_*(t))y(t) \tag{8.26}$$

and assume that $Y(p)$ satisfies the following properties:

1. *the eigenvalue* $1 \in \sigma(Y(p))$ *is algebraically simple and* $Y(p)$ *has no other eigenvalues on the unit circle;*

2. *if* $\mu \in \sigma(Y(p))$ *and* $|\mu| > 1$*, then*
$$-e^2 < \mu < -1.$$

Then, there exists an open interval $I \subseteq \mathbb{R}_{<0}$ *such that for* $k \in I$*,* x_* *is stable as a solution of*

$$\dot{x}(t) = f(x(t)) + k\left[x(t) - x(t-p)\right]. \tag{8.27}$$

Proof. Denote by
$$\{\mu_1, \ldots, \mu_N\} = \sigma(Y(p))$$
the eigenvalues, counting algebraic multiplicities, of $Y(p)$. Moreover, order the eigenvalues $\mu_1, \ldots, \mu_N$ in such a way that $\mu_1 = 1$, $-e^2 < \mu_2 \leq \ldots \leq \mu_l < -1$ are the eigenvalues strictly outside the unit circle and $\mu_{l+1}, \ldots, \mu_N$ are the eigenvalues strictly inside the unit circle. The characteristic equation in (8.4) in Lemma 8.1 becomes

$$\left(1 - ze^{k(1-z)}\right)\left(1 - \mu_2 ze^{k(1-z)}\right)\ldots\left(1 - \mu_N ze^{k(1-z)}\right) = 0. \tag{8.28}$$

For $k < 0$, Corollary 8.4 implies that

$$1 - ze^{k(1-z)} = 0$$

has a simple root $z = 1$ and all other solutions lie strictly outside the unit circle.

For fixed $2 \leq j \leq l$, let $\lambda_{*,j} := \ln|\mu_j|$. Then Corollary 8.5, together with the estimate (8.22), implies that for $k \in (-1, -\lambda_{*,j}/2)$ and for *fixed* $2 \leq j \leq l$, all solutions of

$$1 - \mu_j ze^{k(1-z)} = 0 \tag{8.29}$$

lie strictly outside of the unit circle. But since by assumption $|\mu_2| \geq \ldots \geq |\mu_l|$, it holds that $-\lambda_{*,2} \leq \ldots \leq -\lambda_{*,j}$. Hence for all $k \in (-1, -\lambda_{*,2}/2)$ and for all *all* $2 \leq j \leq l$, all solutions of (8.29) lie strictly outside the unit circle.

For $l + 1 \leq j \leq N$ and $k < 0$, Corollary 8.6 implies that all solutions of

$$1 - \mu_j ze^{k(1-z)} = 0$$

lie outside the unit circle.

We conclude that there exist an open interval $I \subseteq \mathbb{R}_{<0}$ such that for all $k \in I$, the equation (8.28) has a simple root $z = 1$ and all other solutions lie strictly outside the unit circle. Let $U(p, 0)$ be the monodromy operator of

$$\dot{y}(t) = f'(x_*(t))y(t) + k\left[y(t) - y(t-p)\right].$$

Then Lemma 8.1 implies that $U(p, 0)$ has eigenvalue $\mu = 1$, which is algebraically simple, and all eigenvalues of $U(p, 0)$ lie inside the unit circle. Therefore x_* is stable as a solution of

$$\dot{x}(t) = f(x(t)) + k\left[x(t) - x(t-p)\right],$$

as claimed. ∎

The result in Proposition 8.7 is in contrast with Corollary 2.2, which addressed the situation in which the uncontrolled periodic orbit has unstable Floquet multipliers on the *positive* real axis; Pyragas control then fails if the control gain is chosen to be a scalar.

In Proposition 8.7, the period $p > 0$ is not necessarily the minimal period of x_*. However, the choice of period is relevant for the properties of $Y(p)$. Indeed, let p_* be the minimal period of x_* and assume that the monodromy operator $Y(p_*)$ has an unstable, real eigenvalue. Then, for $p = np_*,\ n \in \mathbb{N}$, it holds that

$$Y(p) = Y(np_*) = Y(p_*)^n$$

and thus

$$\sigma(Y(p)) = \sigma(Y(p_*))^n.$$

So if $p = np_*$ with n *even*, then $Y(p)$ has eigenvalues on the *positive* real axis. So in this case $Y(p)$ does not satisfy the conditions of Proposition 8.7; moreover, Corollary 2.2 implies that control of the form (8.27) is always unsuccessful.

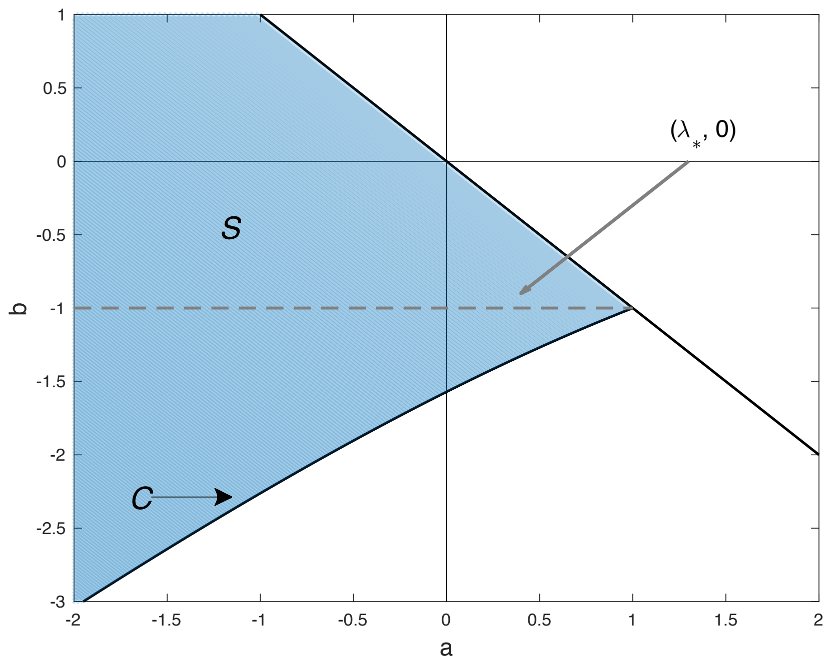

Figure 8.2: The (a, b) plane with the straight line $a = -b$, the curve C defined in (8.13) and the region S (blue) defined in Lemma 8.1. Note that the line $\{(a, b) \mid a < 1, b = -1\}$ (dotted gray) lies inside the region S. The gray arrow indicates the path (8.21); since $\lambda_* < 2$, this path enters the region S.

Chapter 9

Stabilization of discrete waves

Having assembled all the necessary ingredients, we prove Theorem 3 in Section 9.1.

The structure of the group H_* can be a useful guide to picking an 'optimal' element for equivariant Pyragas control. We discuss the relation between the structure of the group H_* and the choice of group element $h \in H_*$ in Section 9.2.

9.1 Proof of Theorem 3

We prove Theorem 3; we repeat its statement here for convenience.

Theorem 3. *Consider system* (3.1) *on page 20 and assume that Hypothesis 2 holds. Let x_* be a discrete wave, i.e. $H_*/K_* \simeq \mathbb{Z}_n$ for some $n \geq 1$. Assume that there exists a $h \in H_*$ such that the reduced monodromy operator Y_h defined in* (3.4) *has the following properties:*

1. *the eigenvalue $1 \in \sigma(Y_h)$ is algebraically simple and Y_h has no other eigenvalues on the unit circle.*

2. *if $\mu \in \sigma(Y_h)$ and $|\mu| > 1$, then*
$$-e^2 < \mu < -1.$$

Then, there exists an open interval $I \subseteq \mathbb{R}_{<0}$ such that for $k \in I$, x_* is stable as a solution of

$$\dot{x}(t) = f(x(t)) + k\left[x(t) - hx(t - \Theta(h)p)\right]. \tag{9.1}$$

Proof (of Theorem 3). Throughout the proof, we let $Y(t) \in \mathbb{R}^{N \times N}$ be the fundamental matrix solution of

$$\dot{y}(t) = f'(x_*(t))y(t) \tag{9.2}$$

with $Y(0) = I_{\mathbb{C}^N}$; we let

$$Y_h = h^{-1}Y(\Theta(h))$$

be the reduced monodromy operator of the ODE (9.3). For $t \geq s$, we let $U(t,s)$ be the fundamental solution of the DDE

$$\dot{y}(t) = f'(x_*(t))y(t) + k\left[y(t) - hy\left(t - \Theta(h)p\right)\right]; \tag{9.3}$$

on the state space $X = C\left([-\Theta(h)p, 0], \mathbb{R}^N\right)$ and with $U(s,s) = I_X$. We let

$$U_h = h^{-1}U(\Theta(h)p, 0)$$

be the reduced monodromy operator of the DDE (9.3). The proof assembles results from the previous three chapters, and we structure the proof accordingly.

Step 1 (Chapter 6) Since the controlled system (9.1) is equivariant with respect to the group Γ, Proposition 6.3 implies that the eigenvalues of the reduced monodromy operator U_h determine the stability of x_* as a solution of (9.1). In particular, if $1 \in \sigma_{pt}(U_h)$ is algebraically simple and all other eigenvalues of U_h lie strictly inside the unit circle, then x_* is a stable solution of (9.1).

Step 2 (Chapter 7) For $t \in \mathbb{R}$, let $Y(t)$ be the fundamental solution of the linearized ODE $\dot{y}(t) = f'(x_*(t))y(t)$ with $Y(0) = I$. Then, for $z \in \mathbb{C}$,

$$F(t,z) = Y(t)e^{k(1-z)t}$$

is the fundamental solution of the ODE

$$\dot{y}(t) = f'(x_*(t))y(t) + k\left[1-z\right]y(t)$$

with $F(0,z) = I$. Let $Y_h = h^{-1}Y(\Theta(h)p)$ reduced monodromy operator of the *ODE* (9.2), then Theorem 6 implies that

$$\Delta(z) = I - zY_h e^{k(1-z)\Theta(h)p} \tag{9.4}$$

is a characteristic matrix for the reduced monodromy operator U_h of the *DDE* (9.3).

Since $\Delta(z)$ defined in (9.4) is a characteristic matrix for U_h, $\mu^{-1} \in \mathbb{C}\backslash\{0\}$ is an eigenvalue of U_h if and only if $\det\Delta(\mu) = 0$. Moreover, if the root $z = 1$ of $\det\Delta(z) = 0$ is algebraically simple, and all other roots lie strictly *outside* the unit circle, then x_* is a *stable* solution of (9.1).

Step 3 (Chapter 8) Denote by

$$\{\mu_1, \ldots, \mu_N\} = \sigma(Y_h)$$

the eigenvalues, counting algebraic multiplicities, of the reduced monodromy operator Y_h. Moreover, order the eigenvalues such that $\mu_1 = 1$, $-e^2 < \mu_2 \leq \ldots \leq \mu_l < -1$ are the eigenvalues strictly outside the unit circle and $\mu_l, \ldots, \mu_N$ are the eigenvalues strictly inside the unit circle. Since the factor $e^{k(1-z)\Theta(h)p}$ in (9.4) is a complex scalar, it holds that

$$\det\Delta(z) = \prod_{j=1}^{N}\left(1 - z\mu_j e^{k(1-z)\Theta(h)p}\right). \tag{9.5}$$

Therefore, to find the roots of $\det\Delta(z) = 0$, it suffices to find the roots of each of the individual factors of (9.5).

The equation

$$1 - ze^{k\Theta(h)p(1-z)} = 0 \tag{9.6}$$

is of the form (8.18), with k replaced by $k\Theta(h)p$. Corollary 8.4 implies that for $k < 0$, $z = 1$ is a simple solution of (9.6) and all other solutions lie strictly outside the unit circle.

For fixed $2 \leq j \leq l$, let $\lambda_{*,j} := \ln |\mu_j|$. Then Corollary 8.5, together with the estimate (8.22), implies that for $k\Theta(h)p \in (-1, -\lambda_{*,j}/2)$ and for *fixed* $2 \leq j \leq l$, all solutions of

$$1 - \mu_j z e^{(1-z)k\Theta(h)p} = 0 \tag{9.7}$$

lie strictly outside of the unit circle. But since by the assumption that $|\mu_2| \geq \ldots \geq |\mu_l|$, it holds that $-\lambda_{*,2} \leq \ldots \leq -\lambda_{*,j}$. Hence for all $k\Theta(h)p \in (-1, -\lambda_{*,2}/2)$ and for all *all* $2 \leq j \leq l$, all solutions of (9.7) lie strictly outside the unit circle. So there exists an open interval $I \subseteq \mathbb{R}_{<0}$ such that for $k \in I$ and for all $2 \leq j \leq l$, all solutions of (9.7) lie strictly outside the unit circle.

For $l+1 \leq j \leq N$, the equation

$$1 - z\mu_j e^{k(1-z)\Theta(h)p} \tag{9.8}$$

is of the form (8.23) with k replaced by $k\Theta(h)p$. Corollary 8.6 implies that for $k < 0$, all solutions of (9.8) lie outside the unit circle.

CONCLUSION: There exists an open interval $I \subseteq \mathbb{R}_{<0}$ such that for $k \in I$, $z = 1$ is a simple root of (9.5) and all other roots lie strictly outside the unit circle. Therefore the eigenvalue $1 \in \sigma_{pt}(U_h)$ is algebraically simple and all other eigenvalues lie strictly *inside* the unit circle. This proofs that for $k \in I$, x_* is a stable solution of (9.1). $\square$

9.2 Drift symmetries and minimal time delay

For certain discrete waves, the group of spatial-temporal symmetries H_* can essentially be described by a single group element. Following Wulff [WS06], we call such a group element a **drift symmetry** (see Definition 9.1 below). In this section, we discuss the relation between the existence of a drift symmetry and the choice of group element $h \in H_*$ in Theorem 3. We start by giving a precise definition of a drift symmetry:

Definition 9.1. Let x_* be a discrete wave, i.e. $H_*/K_* \simeq \mathbb{Z}_n$ with $n \in \mathbb{N}$. We say that an element $g \in H_*$ is a **drift symmetry** of the periodic orbit x_* if g generates H_* and $\Theta(g) = \frac{1}{n}$.

For a general group H_*, a drift symmetry does not necessarily exist (for example, a group generated by two distinct elements has no drift symmetry). Moreover, if a drift symmetry exists, it is not necessarily unique, see [Fie88] (for the case of finite cyclic groups) and [LI99] (for the case of direct products of groups, and a more general discussion of group structure).

The next proposition discusses the the choice of spatial-temporal symmetry in Theorem 3 for the case where the group H_* has a drift symmetry. We consider the ODE

$$\dot{x}(t) = f(x(t)) \tag{9.9}$$

with $f : \mathbb{R}^N \to \mathbb{R}^N$ a C^2-function; we assume that the ODE (9.9) is equivariant with respect to a compact group $\Gamma \subseteq GL(N, \mathbb{R})$ and has a periodic solution x_* with *minimal* period $p > 0$. We assume that this periodic solution is a discrete wave, i.e. $H_*/K_* \simeq \mathbb{Z}_n$ for some $n \in \mathbb{N}$. For $h \in H_*$, we define the reduced monodromy operator Y_h associated to h as

$$Y_h = h^{-1}Y(\Theta(h)p) \tag{9.10}$$

where $Y(t) \in \mathbb{R}^{N\times N}$ is the fundamental matrix solution of $\dot{y}(t) = f'(x_*(t))y(t)$ with $Y(0) = I$. We assume H_* has a drift symmetry and that there exists $h \in H_*$ such that the reduced monodromy operator Y_h satisfies the conditions of Theorem 3. We show that the number $\Theta(h)$ 'decides' whether the element h is the optimal element for stabilization: if $\Theta(h) = \frac{j}{n}$ with $2 \le j < n-1$, then there exists an element $h_* \in H_*$ such that Y_{h_*} also satisfies the conditions of Theorem 3, but such that the unstable eigenvalues of Y_{h_*} have strictly smaller norm than the unstable eigenvalues of Y_h. For simplicity, we restrict the analysis to the situation where the reduced monodromy operator Y_h has only one (algebraically simple) unstable eigenvalue.

Proposition 9.2. *Consider the ODE* (9.9)*; assume that this ODE is equivariant with respect to a compact symmetry group $\Gamma \subseteq GL(N,\mathbb{R})$ and has a periodic solution x_* with* minimal *period $p > 0$. We assume that x_* is a discrete wave, i.e. $H_*/K_* \simeq \mathbb{Z}_n$ for some $n \ge 1$, and assume that the group H_* has a drift symmetry. Moreover, assume that there exists a $h \in H_*$ such that*

$$\Theta(h) = \frac{j}{n}$$

for some $1 \le j \le n-1$ and such that the reduced monodromy operator $Y_h = h^{-1}Y\left(\frac{j}{n}p\right)$ satisfies

1. *The trivial eigenvalue $1 \in \sigma(Y_h)$ is algebraically simple;*
2. *Y_h has exactly one eigenvalue $-\rho$ with $\rho \ge 1$ outside the unit circle and this eigenvalue is algebraically simple;*
3. *All other eigenvalues lie strictly inside the unit circle.*

Then there exists $h_* \in H$ such that

$$\Theta(h_*) = \frac{1}{n}$$

and such that the reduced monodromy operator $Y_{h_*} = h_*^{-1}Y\left(\frac{1}{n}p\right)$ satisfies

1. *The trivial eigenvalues $1 \in \sigma(Y_{h_*})$ is algebraically simple;*
2. *Y_{h_*} has exactly one eigenvalue $-\sqrt[j]{\rho}$ outside the unit circle and this eigenvalue is algebraically simple;*
3. *All other eigenvalues lie strictly inside the unit circle.*

Proof. Step 1: let $g \in H_*$ be a drift symmetry of H_* and let $m \in \mathbb{N}$ be such that $h = g^m$. Then $\Theta(h) = m\Theta(g) \mod 1$, i.e. $\frac{j}{n} = \frac{m}{n} \mod 1$ and hence $m = j + nk$ for some $k \in \mathbb{Z}$. So h and g are related as

$$h = g^{nk}g^j \tag{9.11}$$

for some $k \in \mathbb{Z}$. We next show that the reduced monodromy operators Y_h and Y_g are related as

$$Y_h = g^{-nk}\left(Y_g\right)^j. \tag{9.12}$$

Indeed, let $Y(t) \in \mathbb{R}^{N\times N}$ be the fundamental solution of the ODE $\dot{y}(t) = f'(x_*(t))y(t)$ with $Y(0) = I$; then (9.11) implies that

$$Y_h := h^{-1}Y\left(\frac{j}{n}p\right) = g^{-nk}g^{-j}Y\left(\frac{j}{n}p\right).$$

Lemma 4.1 implies that $g^{-1}Y\left(t+\frac{p}{n}\right)Y\left(s+\frac{p}{n}\right)^{-1}=Y(t)Y(s)^{-1}g^{-1}$ for all $t\geq s$, so

$$\begin{aligned}
g^{-j}Y\left(\frac{j}{n}p\right) &= g^{-j}Y\left(\frac{j}{n}p\right)Y\left(\frac{j-1}{n}p\right)^{-1}Y\left(\frac{j-1}{n}p\right)\dots Y\left(\frac{1}{n}p\right)\\
&= g^{-j}g^{j-1}Y\left(\frac{1}{n}p\right)g^{1-j}g^{j-2}\dots g^{-1}Y\left(\frac{1}{n}p\right)\\
&= \underbrace{g^{-1}Y\left(\frac{1}{n}p\right)\dots g^{-1}Y\left(\frac{1}{n}p\right)}_{j\text{ times}} = Y_g^j.
\end{aligned}$$

and (9.12) follows.

Step 2: Let $\lambda_1,\dots,\lambda_d$ be the distinct eigenvalues of g^{nk}. Then by Lemma 4.3, we can decompose $\mathbb{C}^N$ as

$$X=X_1\oplus\dots\oplus X_d$$

with $X_i=\ker\left(\lambda_i I-g^{nk}\right)$ for all $1\leq i\leq d$. Since $\Theta(g)=1/n \mod 1$, it holds that $\Theta(g^{nk})=0 \mod 1$ and therefore $g^{nk}\in K$. Lemma 4.3 then implies that the space X_i is invariant under both Y_h and Y_g, and the restrictions of $Y_h,\ Y_g$ to X_i satisfy

$$\sigma\left(Y_h|_{X_i}\right)=\lambda_i\sigma\left(Y_g|_{X_i}\right)^j,\qquad 1\leq i\leq d. \tag{9.13}$$

This implies that the trivial eigenvalue $1\in\sigma(Y_g)$ is algebraically simple; that Y_g has exactly one algebraically simple eigenvalue outside the unit circle and that all other eigenvalues of Y_g lie strictly inside the unit circle.

Step 3: Equality (9.13) particularly implies the eigenvalue $-\rho\in\sigma(Y_h)$ is of the form

$$-\rho=\lambda\mu^j$$

with $\lambda\in\sigma(g^{-nk})$ and $\mu\in\sigma(Y_g)$. By assumption, ρ is an algebraically simple eigenvalue, so both λ and μ have to be real. In particular, this implies that

$$\lambda=\pm1\quad\text{and}\quad\mu=\pm\sqrt[j]{\rho}.$$

If $\mu=-\sqrt[j]{\rho}$, the statement of the proposition follows for $h_*=g$. The situation $\mu=+\sqrt[j]{\rho}$ can only occur if $\lambda=-1$, so in this case there exists $x\in\mathbb{C}^N\backslash\{0\}$ such that

$$g^{-nk}x=-x\qquad\text{and}\qquad g^{-j}Y\left(\frac{p}{n}\right)x=\sqrt[j]{\rho}x.$$

But this implies that

$$g^{-nk}g^{-1}Y\left(\frac{p}{n}\right)x=-\sqrt[j]{\rho}x.$$

So if we choose $h_*=g^{nk}g$, then $\Theta(h_*)=\frac{1}{n} \mod 1$ and $Y_{h_*}=g^{-nk}g^{-1}Y\left(\frac{p}{n}\right)$ satisfies the statement of the proposition. □

The contents of this section come with a word of warning concerning the (possible) non-uniqueness of drift symmetries. In [Van87] Vanderbauwhede points out that it is possible that there exist two drift symmetries $g, g' \in H_*$, such that the unstable eigenvalue of Y_g is on the negative real axis, but the unstable eigenvalue of $Y_{g'}$ is on the positive real axis. See also [Fie88, Lemma 5.12] for a detailed discussion for the case where H_* is cyclic. So from the point of equivariant Pyragas control, not all drift symmetries are created equal. In Proposition 9.2, we essentially side-step these subtleties around the choice of drift symmetry, because we do *not* claim that h_* is a drift symmetry, just that $\Theta(h_*) = \frac{1}{n}$.

In equivariant control of discrete waves, the choice of symmetry $h \in H_*$ determines the time delay $\Theta(h)p$ in the controlled system. In Proposition 9.2, the symmetry h_* leads to a smaller time delay than the symmetry h. Simultaneously, the unstable eigenvalue of Y_{h_*} is smaller than the unstable eigenvalue of Y_h, which makes h_* the more suitable candidate for stabilization. So morally speaking, we have better hopes of stabilization with a smaller time delay, a phenomenon that was also observed by Fiedler in [Fie08].

Chapter 10

Discussion & Outlook

We conclude by putting the results of this thesis into perspective, and by outlining possible future lines of inquiry. In the discussion, we focus on the role that the symmetry plays in the stability analysis. In the outlook, we address open questions both regarding spectral properties and dynamical properties of feedback control.

10.1 Discussion

Limitations with and without symmetry

In this thesis, we discussed limitations to Pyragas control in systems without symmetry (Chapter 2) and positive stabilization results for systems with symmetry (Theorem 3). Here, we discuss how these two topics relate to each other, and argue that using the symmetry in the stability analysis is crucial for understanding the mechanisms of equivariant control.

A running theme in this thesis is that we do not only use the symmetry in the implementation of the control scheme, but also actively use the symmetry in the stability analysis. First, we consider an ODE

$$\dot{x}(t) = f(x(t)) \tag{10.1}$$

with periodic solution $x_*(t+p) = x_*(t)$ but without additional symmetry. If $Y(t)$ is the fundamental solution of the linearization $\dot{y}(t) = f'(x_*(t))y(t)$, then the *monodromy operator* $Y(p)$ determines the stability of the periodic solution x_*. In the context of stabilization of orbits with symmetry, we additionally assume that ODE (10.1) is equivariant with respect to a compact group Γ, and that its periodic orbit satisfies the spatial-temporal symmetry $hx_*(t) = x_*(t + \Theta(h)p)$ for some $h \in \Gamma$. Then we can determine its stability both from the monodromy operator $Y(p)$ and from the *reduced monodromy operator*

$$Y_h = h^{-1}Y(\Theta(h)p).$$

If we want to understand *how* equivariant control overcomes limitations to Pyragas control, the relation between the reduced monodromy operator and the monodromy operator is crucial, as we will discuss here.

In Chapter 2, we apply Pyragas control

$$\dot{x}(t) = f(x(t)) + K\left[x(t) - x(t-p)\right] \tag{10.2}$$

with $K \in \mathbb{R}^{N\times N}$ and prove the following two restrictions:

1. (Invariance priniciple) **Classical Pyragas control preserves the geometric multiplicity of the eigenvalue 1 of the monodromy operator.**

2. (Any number limitation) **Suppose the monodromy operator of the uncontrolled system has an eigenvalue larger than 1. Then classical Pyragas control with a scalar control gain always fails.**

If the periodic solution satisfies the spatial-temporal pattern $hx_*(t) = x_*(t+\Theta(h)p)$, we write system with equivariant control as

$$\dot{x}(t) = f(x(t)) + K\left[x(t) - hx(t-\Theta(h)p)\right]. \tag{10.3}$$

We make the mild assumption that $hK = Kh$, so that for the linearization of (10.3) the notion of a reduced monodromy operator is also well-defined. Somewhat suprisingly, if we determine the stability from the reduced monodromy operator, we find similar limitations as for 'classical' Pyragas control, as we outline here:

1. (Invariance principle) **Equivariant Pyragas control preserves the geometric multiplicity of the eigenvalue 1 of the reduced monodromy operator.** For the uncontrolled system, we count the geometric multiplicity of the eigenvalue 1 of Y_h by counting the number of distinct solution of
$$\dot{y}(t) = f'(x_*(t))y(t) \tag{10.4}$$
of the form $y(t+\Theta(h)p) = hy(t)$. For the controlled system, let $U(t,s)$, $t \geq s$ be the fundamental solution of the linearized DDE
$$\dot{y}(t) = f'(x_*(t))y(t) + K\left[y(t) - hy(t-\Theta(h)p)\right]. \tag{10.5}$$
and let $U_h = h^{-1}U(\Theta(h)p, 0)$ be the reduced monodromy operator of this linearized DDE. We count the geometric multiplicity of the eigenvalue 1 of U_h by counting the number of distinct solution of (10.5) of the form $y(t+\Theta(h)p) = hy(t)$. But since the control vanishes on all solutions of this form, $y(t+\Theta(h)p) = hy(t)$ is a solution of (10.4) if and only if it is a solution of (10.5). So the geometric multiplicity of the eigenvalue 1 of Y_h equals to geometric multiplicity of the eigenvalue 1 of U_h.

2. (Any number limitation) **Suppose the reduced monodromy operator of the uncontrolled system has an eigenvalue larger than 1. Then equivariant Pyragas control with a scalar control gain always fails.** Indeed, for a control gain $k \in \mathbb{R}$, the characteristic matrix function for U_h becomes
$$\Delta(z) = I - zY_h e^{k(1-z)};$$
see Theorem 6. Thus, if $\lambda_* > 1$ is an eigenvalue of Y_h, then every solution of
$$1 - z\lambda_* e^{k(1-z)} = 0 \tag{10.6}$$
is a solution of $\det \Delta(z) = 0$. By the Intermediate Value Theorem, (10.6) has a root $\mu^{-1} < 1$, and hence the monodromy operator U_h has an eigenvalue $\mu > 1$.

If the periodic orbit satisfies a spatial-temporal pattern $hx_*(t) = x_*(t + \Theta(h)p)$ with *rational* time shift $\Theta(h) = \frac{m}{n}$, $0 < m < n$, then the monodromy operator $Y(p)$ and the reduced monodromy operator Y_h are related as

$$Y(p)^m = h^n Y_h^n.$$

So there is no one-to-one relation between the eigenvalue 1 of the reduced monodromy operator Y_h and the eigenavalue 1 of the monodromy operator $Y(p)$:

1. **Equivariant Pyragas control does not preserve the geometric multiplicity of the eigenvalue 1 of the monodromy operator.** An eigenvalue -1 of the reduced monodromy operator Y_h can lead to an eigenvalue $+1$ on for the monodromy operator $Y(p)$. Since the eigenvalue -1 of the reduced monodromy operator Y_h is not preserved under equivariant control, the eigenvalue $+1$ of the monodromy operator $Y(p)$ is also not preserved under equivariant control.

2. **Equivariant Pyragas control with a scalar control gain can succeed, even if the monodromy operator has eigenvalues larger than** 1. If the monodromy operator $Y(p)$ has an eigenvalue larger than 1, but the unstable eigenvalues of the reduced monodromy operator Y_h are on the negative real axis, then equivariant control can stabilize (cf. Theorem 3).

So even when 'classical' Pyragas control is bound to fail, equivariant Pyragas control can still stabilize. Moreover, the above discussion indicates that for equivariant Pyragas control, the reduced monodromy operator (and not the monodromy operator) is the natural object to study. Philosophically speaking, if we actively use the symmetries in the control scheme, we should also actively use the symmetries in the stability analysis.

Reduced monodromy operator

The proof of Theorem 3 relied on a very explicit analysis of the eigenvalue problem. Also here, working with the reduced monodromy operator (rather then with the monodromy operator) is crucial at two points:

In Theorem 6, we give a N-dimensional characteristic matrix function for the reduced monodromy operator. However, we cannot so easily find a N-dimensional characteristic matrix function for the monodromy operator. This is because the time delay of the controlled system equals the time step of the spatial-temporal symmetry, whereas the time delay is smaller than the full period. The existence of a characteristic matrix function for the reduced monodromy operator implies that we can compute its eigenvalues as roots of a finite dimensional equation, which is of course advantageous from a computational point of view.

In the uncontrolled system in Theorem 3, the unstable eigenvalues of the reduced monodromy operator are on the negative real axis (whereas the unstable eigenvalues of the 'full' monodromy operator are possibly on the positive real axis). As a result we can work with *scalar* control gain and can explicitly compute the characteristic equation using only the reduced monodromy operator of the uncontrolled system (cf. proof of Theorem 3, Step 2). So also here working with the reduced monodromy operator is crucial.

10.2 Outlook

Time periodic and autonomous systems

The proof of the positive stabilization result in Theorem 3 uses a very explicit analysis of the Floquet multipliers. The fact that we work with a *scalar* control gain in Theorem 3 is crucial for this explicit analysis. We now discuss how a fundamental question on Floquet theory for DDE can contribute to the analytical understanding of Pyragas control with non-scalar control gain.

For linear, time-periodic ODE we can embed the monodromy operator in the flow of a linear, autonomous ODE (this statement is also known as Floquet's theorem, cf. Section A.1 in the appendix). So for an asymptotic stability analysis, we can work with the generator of the autonomous ODE instead of with the monodromy operator of the periodic ODE. For DDE, it seems natural to ask: can we embed the monodromy operator of a linear, time-periodic DDE into the semiflow of a linear, autonomous DDE? But this question is too restrictive if we are purely interested in asymptotic stability, as the following example illustrates:

Example. Consider the 1-periodic DDE

$$\dot{x}(t) = \left(\frac{1}{2} + \sin(2\pi t)\right) x(t-1). \tag{10.7a}$$

and let $U(t,s), t \geq s$ be its fundamental solution on the state space $X = C([-1,0],\mathbb{R})$. Moreover, let $T(t) : X \to X, t \geq 0$ be the solution semigroup of the autonomous DDE

$$\dot{x}(t) = \frac{1}{2} x(t-1); \tag{10.7b}$$

and let $A : \mathcal{D}(A) \subseteq X \to X$ be its generator.

It can be shown that the periodic system (10.7a) has so-called small solutions (solutions which decay faster than any exponential) but the autonomous system (10.7b) has not [KV21, Section 11.4]. Therefore, we cannot embed the monodromy operator of (10.7a) in the semi-flow of (10.7b).

By Theorem 5, the analytic function

$$\Delta(z) = 1 - z e^{z \int_0^1 \frac{1}{2} + \sin(2\pi s) ds} = 1 - z e^{\frac{1}{2} z}$$

is a characteristic function for the monodromy operator $U(1,0)$ of (10.7a). But also by Theorem 5, the function

$$\Delta(z) = 1 - z e^{z \int_0^1 \frac{1}{2} ds} = 1 - z e^{\frac{1}{2} z}$$

is also a characteristic function for the solution operator $T(1)$ of (10.7b). So the operators $U(1,0)$ and $T(1)$ have the same characteristic function and hence the same non-zero eigenvalues, and the equality

$$\sigma_{pt}(U(1,0)) = \sigma_{pt}(T(1)) = e^{\sigma_{pt}(A)}$$

holds.

In the above example, we can determine the stability of the origin of the periodic system (10.7a) from the spectrum of the autonomous system (10.7b), although we cannot embed the monodromy operator of the periodic system in the semigroup of the autonomous one. So if we are purely interested in asymptotic stability, it is better to ask:

Question. Consider the 1-periodic system

$$\dot{y}(t) = A(t)y(t) + B(t)y(t-1) \quad \text{with} \quad A(t+1) = A(t),\ B(t+1) = B(t) \tag{10.8a}$$

and let $U(t,s), t \geq s$ be its fundamental solution on the state space $X = C\left([-1,0], \mathbb{R}^N\right)$. Does there exists an autonomous system

$$\dot{y}(t) = Ly_t \tag{10.8b}$$

where $L : X \to \mathbb{R}^N$ is a bounded linear operator, $y_t \in X$ defined as $y_t(\theta) = y(t+\theta)$, and $A : \mathcal{D}(A) \subseteq X \to X$ is the generator of the semiflow of (10.8b), such that the equality

$$\sigma_{pt}(U(1,0)) = e^{\sigma_{pt}(A)}$$

holds?

The above question is especially relevant in the context of Pyragas control with non-scalar control gain. So far, most analytical results on successful stabilization with Pyragas control away from bifurcation points have been obtained for periodic orbits that can be transformed to equilibria of an autonomous system [PPK14; FFGHS08; SB16; FFS10] or for scalar control gain [Nak97]. In the spirit of the above question, we might try to prove that for Pyragas control with non-scalar control gain, we can compute the Floquet multipliers from an autonomous ODE with *distributed* delay. This would then allow for an explicit analysis of the the eigenvalues of the monodromy operator, and would thus make a significant contribution to the understanding of Pyragas control with non-scalar control gain.

Noninvasive control terms

In Chapter 2, we introduced an invariance principle for Pyragas control, from where we then derived a number of limitations to Pyragas control. We expect both the invariance principle and some of the resulting limitations to hold for a larger class of control terms, as we discuss here.

Pyragas control is often said to be 'noninvasive' because the control term $K\left[x(t) - x(t-p)\right]$ vanishes on the target periodic orbit. But more importantly, the control term vanishes on *every* p-periodic function (not just on the target periodic orbit), leading to the invariance principle in Proposition 2.1. So when extending the results in Chapter 2 to a larger class of feedback control schemes, the natural class to look at are control schemes that are 'noninvasive' in the sense that they vanish on *every* p-periodic function. However, when considering such noninvasive control terms, we have to keep the following two things in mind:

Not all feedback schemes fit into the functional analytical framework used in Chapter 2. In Chapter 2, we explicitly use that the monodromy operator is compact, and that all non-zero spectral points consists of eigenvalues of finite multiplicity. This property is not shared by all feedback control schemes. For example, the 'extended time delayed feedback control' (ETDFC) is a DDE of neutral type, and the Floquet multipliers (even the trivial one) can have infinite geometric multiplicity. So although some of the arguments used in Chapter 2 may be adapted to this setting (cf. [Sie16]), we have to be careful of the functional analytic framework before we draw any conclusions.

The analogues of Proposition 2 (any number limitation) and Proposition 2.3 (necessary condition for successful stabilization) will be different for different noninvasive control schemes. The proof of Proposition 2 explicitly uses the 'delay equals period' structure of Pyragas control. For a noninvasive control scheme where not every delay is a multiple of the period, one expects that the ‘any number limitation for scalar control gain’ does *not* hold. Proposition 2.3 has a (relatively straightforward) analogue for a larger class of noninvasive control types, but also here the explicit form of the necessary condition is different for different types of control.

Appendix A

Floquet theory for ODE and DDE

This appendix collects and proves the main results on Floquet theory for ODE and DDE. Section A.1 deals with Floquet theory for ODE; Section A.2 discusses Floquet theory for DDE. The main results on stability of periodic orbits are collected in Section A.3.

A.1 Floquet theory for ODE

Consider the linear, time-dependent ODE

$$\dot{y}(t) = A(t)y(t), \quad t \geq 0 \tag{A.1}$$

where $A(t) \in \mathbb{R}^{N\times N}$ is periodic with period $p > 0$, i.e. $A(t+p) = A(t)$. For $t \geq 0$, let $Y(t) \in \mathbb{R}^{N\times N}$ be the matrix-solution of (A.1) with $Y(0) = I$, so $Y(t)$ satisfies

$$\frac{d}{dt}Y(t) = A(t)Y(t) \text{ for } t \in \mathbb{R}, \qquad Y(0) = I. \tag{A.2}$$

We refer to $Y(t)$ as the **fundamental solution** of (A.1). The map $Y(p) \in \mathbb{R}^{N\times N}$ is called the **monodromy operator**; its eigenvalues are called the **Floquet multipliers** of (A.1).

The main result of Floquet theory is that the Floquet multipliers determine the stability of the origin of (A.1). In this section, we will address two (intertwined) approaches to this result:

1. via algebraic relations on the fundamental solution;

2. via a time-periodic time transformation of (A.1) to a linear system with constant coefficients.

Algebraic relations for the fundamental solution

Fix $s \in \mathbb{R}$, then $Y(t)Y(s)^{-1}$ solves the ODE

$$\frac{d}{dt}Y(t)Y(s)^{-1} = A(t)\left[Y(t)Y(s)^{-1}\right]$$

and $Y(t)Y(s)^{-1} = I$ for $t = s$. Similarly, $Y(t+p)Y(s+p)^{-1}$ solves the ODE

$$\begin{aligned}\frac{d}{dt}Y(t+p)Y(s+p)^{-1} &= A(t+p)\left[Y(t+p)Y(s+p)^{-1}\right]\\ &= A(t)\left[Y(t+p)Y(s+p)^{-1}\right]\end{aligned}$$

and $Y(t+p)Y(s+p)^{-1} = I$ for $t = s$. So uniqueness of solutions implies that

$$Y(t+p)Y(s+p)^{-1} = Y(t)Y(s)^{-1} \qquad \text{for } t, s \in \mathbb{R}. \tag{A.3}$$

By purely using (A.3) as an algebraic relation, we obtain furthermore:

Lemma A.1. *We have the following identities:*

$$Y(t+np)Y(s+np)^{-1} = Y(t)Y(s)^{-1}, \qquad \text{for } n \in \mathbb{N},\ t, s \in \mathbb{R}; \tag{A.4}$$

$$Y(np) = Y(p)^n, \qquad \text{for } n \in \mathbb{N}. \tag{A.5}$$

Proof. To prove (A.4), we repeatedly apply (A.3):

$$Y(t+np)Y(s+np)^{-1} = Y(t+(n-1)p)Y(s+(n-1)p)^{-1} = \ldots = Y(t)Y(s)^{-1}.$$

To prove (A.5), we decompose $Y(np)$ as

$$\begin{aligned} Y(np) &= \left[Y(np)Y((n-1)p)^{-1}\right]\left[Y((n-1)p)Y((n-2)p)^{-1}\right]\ldots\left[Y(2p)Y(p)^{-1}\right]Y(p) \\ &= Y(p)Y(p)\ldots Y(p) \end{aligned}$$

where in the last step we use (A.4). So we conclude that $Y(np) = Y(p)^n$, as claimed. □

The next lemma shows that the eigenvalues of the monodromy operator determine the asymptotic stability of the origin of (A.1).

Lemma A.2. *Let $Y(p)$ be the monodromy operator of the ODE* (A.1)*. If all the eigenvalues of $Y(p)$ are strictly inside the unit circle, then the origin of* (A.1) *is asymptotically stable. If $Y(p)$ has an eigenvalue strictly outside the unit circle, then the origin of* (A.1) *is asymptotically unstable.*

Proof. Suppose that all the eigenvalues of $Y(p)$ are strictly inside the unit circle. By the spectral radius formula

$$\lim_{n\to\infty} \|Y(p)^n\|^{1/n} = \max\{|\lambda| \mid \lambda \in \sigma(Y(p))\}$$

there exists a $n_0 \in \mathbb{N}$ such that $\|Y(p)^{n_0}\| < 1$. For $j \in \mathbb{N}$, the identities (A.4)–(A.5) imply that

$$\begin{aligned} Y(t+jn_0p) &= Y(t+jn_0p)Y(jn_0p)^{-1}Y(jn_0p) \\ &= Y(t)\left(Y(p)^{n_0}\right)^j \end{aligned}$$

and hence

$$\|Y(t+jn_0p)\| \le \|Y(t)\|\,\|Y(p)^{n_0}\|^j \to 0 \qquad \text{as } j \to \infty.$$

So $\|Y(s)\| \to 0$ as $s \to \infty$, which proves that the origin of (A.1) is asymptotically stable.

Suppose that $Y(p)$ has an eigenvalue μ with $|\mu| > 1$; let $y_0 \neq 0$ be the associated eigenvector. Then

$$\begin{aligned} Y(t+np)y_0 &= \left[Y(t+np)Y(np)^{-1}\right]Y(np)y_0 \\ &= Y(t)Y(p)^n y_0 \\ &= \mu^n Y(t)y_0. \end{aligned}$$

So $y(t) := Y(t)y_0$ is a solution of (A.1) with $\|y(t)\| \to \infty$ as $t \to \infty$, which proves that the origin of (A.1) is asymptotically unstable. □

The following lemma characterizes the eigenvalues of $Y(p)$ in terms of solutions of (A.1):

Lemma A.3. *Let $Y(p)$ be the mondromy operator of the ODE* (A.1)*. Then $\mu \in \mathbb{C}$ is an eigenvalue of $Y(p)$ if and only if* (A.1) *has a non-zero solution of the form $y(t+T) = \mu y(t)$.*

Proof. Let $\mu \in \mathbb{C}$ be an eigenvalue of $Y(p)$ and let $y_0 \neq 0$ be an associated eigenvector. Then

$$\begin{aligned} Y(t+p)y_0 &= Y(t+p)Y(p)^{-1}Y(p)y_0 \\ &= Y(t)\mu y_0 \end{aligned}$$

so $y(t) := Y(t)y_0$ is a solution of (A.1) with $y(t+p) = \mu y(t)$.

Vice versa, suppose that (A.1) has a solution of the form $y(t+p) = \mu y(t)$. Then in particular $y(p) = \mu y(0)$; since $y(t) = Y(t)y(0)$, this implies that μ is an eigenvalue of $Y(p)$ with eigenvector $y(0) \neq 0$. □

Time periodic transformation to system with contant coefficients

System (A.1) can be transformed into a linear system with constant coefficients via a time-periodic coordinate transform. This coordinate transformation then yields results that are equivalent to Lemma A.2 and Lemma A.3. Indeed, let $B \in \mathbb{C}^{N\times N}$ be such that

$$e^{Bp} = Y(p) \tag{A.6a}$$

and define

$$P(t) = Y(t)e^{-Bt} \tag{A.6b}$$

for $t \geq 0$.

Lemma A.4. *The matrix-valued function P in* (A.6b) *is periodic with period T, i.e. $P(t+p) = P(t)$. Moreover, the coordinate transform $y(t) = P(t)v(t)$ transforms the system* (A.1) *into*

$$\dot{v}(t) = Bv(t). \tag{A.7}$$

Proof. We first proof that P is p-periodic. We compute

$$\begin{aligned} P(t+p) &= Y(t+p)e^{-Bp}e^{-Bt} \\ &= Y(t+p)Y(p)^{-1}e^{-Bt} \\ &= Y(t)e^{-Bt} \end{aligned}$$

where in the last step we used (A.3). So indeed $P(t+p) = P(t)$.

Differentiating the relation (A.6b) gives

$$\dot{P}(t) = A(t)P(t) - P(t)B. \tag{A.8}$$

Let $y(t)$ be a solution and let $y(t) = P(t)v(t)$, then differentiating the identity $y(t) = P(t)v(t)$ gives that

$$A(t)P(t)v(t) = A(t)P(t)v(t) - P(t)Bv(t) + P(t)\dot{v}(t)$$

and (A.7) follows. □

Since the time-periodic coordinate transformation $y(t) = P(t)v(t)$ does not change the asymptotic stability of the origin, we find as a corollary to Lemma A.4:

Corollary A.5. *Consider the system* (A.1) *and let B be as in* (A.6a)*. If all the eigenvalues λ of B satisfy $Re\,\lambda < 0$, then the origin of* (A.1) *is asymptotically stable. If there exists an eigenvalue λ of B such that $Re\,\lambda > 0$, then the origin of* (A.1) *is unstable.*

The eigenvalues of B are referred to as the **Floquet exponents**. The matrix B satisfying (A.6a) is not unique. However, its eigenvalues are defined up to a shift by $i\frac{2\pi}{p}$ and hence the real part of the spectrum of B is independent of the choice of B.

If λ is a Floquet exponent, then $\mu = e^{\lambda p}$ is a Floquet multiplier (hence the name 'Floquet exponents'). Vice versa, if $\mu \in \sigma(Y(p))$ is a Floquet multiplier, then μ is of the form $\mu = e^{\lambda p}$ where λ is an eigenvalue of B. Hence the statements in Corollary A.5 and Lemma A.2 are equivalent.

We rewrite (A.6b) as

$$Y(t) = P(t)e^{Bt}. \tag{A.9}$$

This relationship is called the **Floquet decomposition**, giving a decomposition of the fundamental solution in a periodic and an exponential part. Equation (A.9) also gives a characterisation of the Floquet exponents: if λ is a Floquet exponent, and x_0 an associated eigenvector of B, then $Y(t)x_0 = P(t)e^{\lambda t}x_0$ is a solution of (A.1). Vice versa, if $q(t)e^{\lambda t}$, $q(t+p) = q(t)$ is a solution of (A.1), then $q(0)$ is an eigenvector of B with eigenvalue λ. So we conclude that

Corollary A.6. *The value $\lambda \in \mathbb{C}$ is an eigenvalue of B if and only if there exist a periodic $q(t+p) = q(t)$ such that*

$$y(t) = q(t)e^{\lambda t}$$

is a solution of (A.1)*.*

Since μ is an eigenvalue of $Y(p)$ if and only if it is of the form $\mu = e^{\lambda p}$, with $\lambda \in \sigma(B)$, the statements of Lemma A.3 and Corollary A.6 are equivalent.

A.2 Floquet theory for DDE

This sections summarizes the main results of Floquet theory for DDE. In the previous section, we saw that for periodic ODE one can embed the monodromy operator in the flow of an autonomous ODE (cf. (A.6a)). For periodic DDE one can in general not embed the monodromy operator in the semiflow of an autonomous DDE, see [DGVW95, Chapter 13] and the outlook of this thesis. But still we are able to obtain analogues of most of the 'main results' of Floquet theory for ODE, as we will show here. More details on Floquet theory for DDE can be found in [DGVW95, Chapter XIII] and [HV93, Chapter 8].

Consider the system

$$\dot{y}(t) = L(t)y_t \tag{A.10}$$

with state space $X = C([-r, 0], \mathbb{R}^n)$, history segment $y_t(\theta) := y(t+\theta),\ \theta \in [-r, 0]$ and $\{L(t)\}_{t\in\mathbb{R}}$ a periodic family of bounded linear operators. That is, $L(t) : X \to \mathbb{R}^n$ is bounded linear for all $t \in \mathbb{R}$ and there exists a $p > 0$ such that $L(t+p) = L(t)$ for all $t \in \mathbb{R}$.

We summarize the solution information to (A.10) in a two-parameter family of bounded linear operators. Given $t \geq s \in \mathbb{R}$ and $\phi \in X$, there is a unique solution $x_{s,\phi}(t)$ satisfying (A.10) for $t > s$ and with $x_{s,\phi}(s) = \phi$. We define the bounded linear operator $U(t, s)$ as

$$U(t,s) : X \to X, \qquad U(t,s)\phi = x_{s,\phi}$$

and refer to $U(t, s)$ as the **fundamental solution** of (A.10). The map $U(p, 0)$ is called the **monodromy operator** and its eigenvalues are referred to as the **Floquet multipliers** of (A.10).

By uniqueness of the initial value problem associated to (A.10), it follows that

$$U(t,\tilde{s})U(\tilde{s},s) = U(t,s) \qquad \text{for } t \geq \tilde{s} \geq s. \tag{A.11}$$

By time-periodicity of (A.10), it also holds that

$$U(t+p, s+p) = U(t,s) \qquad \text{for } t \geq s. \tag{A.12}$$

By purely using (A.11)–(A.12) as algebraic relations, we obtain furthermore:

Lemma A.7. *We have the following identities:*

$$U(t+np, s+np) = U(t,s) \qquad \textit{for } t \geq s \in \mathbb{R},\ n \in \mathbb{N} \tag{A.13}$$
$$U(np+s, s) = U(s+p, s)^n \qquad \textit{for } s \in \mathbb{R},\ n \in \mathbb{N} \tag{A.14}$$

Proof. To prove (A.13), we repeatedly apply (A.12):

$$U(t+np, s+np) = U(t+(n-1)p, s+(n-1)p) = \ldots = U(t,s).$$

To prove (A.14), we combine (A.11) with (A.13):

$$\begin{aligned} U(np+s, s) &= U(np+s, s+(n-1)p)\ldots U(s+p, s) \\ &= U(s+p, s)\ldots U(s+p, s) \\ &= U(s+p, s)^n \end{aligned}$$

which proves the claim. □

If $j \in \mathbb{N}$ is such that $jp \geq r$ (i.e. j times the period is larger than the delay), then the operator $U(jp, 0) = U(p, 0)^j$ is a compact operator [DGVW95, Chapter 13] and hence all its non-zero spectrum consists of isolated eigenvalues of finite multiplicities. Therefore, all non-zero eigenvalues of the **monodromy operator** $U(p, 0)$ consists of isolated eigenvalues of finite algebraic multiplicity as well (see Lemma A.10 on page 89). We call the non-zero eigenvalues of $U(p, 0)$ the **Floquet multipliers** of system (A.10). The next lemma shows that the Floquet multipliers of (A.10) determine the stability of the origin of (A.10).

Lemma A.8. *Let $U(p, 0)$ be the monodromy operator of* (A.10)*. If all eigenvalues of $U(p, 0)$ lie strictly inside the unit circle, then the origin of* (A.10) *is asympotically stable. If $U(p, 0)$ has an eigenvalue strictly outside the unit circle, then the origin of* (A.10) *is asymptotically unstable.*

Proof. Suppose that all the eigenvalues of $U(p,0)$ lie strictly inside the unit circle. By Gelfand's formula [Wer05, Theorem VI.1.6]

$$\lim_{n\to\infty} \|U(p,0)^n\|^{1/n} = \max\{|\lambda| \mid \lambda \in \sigma_{pt}(U(p,0))\}$$

there exists a $n_0 \in \mathbb{N}$ such that $\|U(p,0)^{n_0}\| < 1$. For $j \in \mathbb{N}$, the identities (A.4)–(A.5) imply that

$$\begin{aligned} U(t + jn_0p, 0) &= U(t + jn_0p, jn_0p)^{-1} U(jn_0p, 0) \\ &= U(t, 0)U(p, 0)^{n_0 j} \end{aligned}$$

and hence

$$\|U(t + jn_0p)\| \le \|U(t, 0)\| \, \|U(p, 0)^{n_0}\|^j \to 0 \qquad \text{as } j \to \infty.$$

So $\|U(s,0)\| \to 0$ as $s \to \infty$, which proves that the origin of (A.1) is asymptotically stable.

Suppose that $U(p,0)$ has an eigenvalue μ with $|\mu| > 1$; let $\varphi \neq 0$ be an associated eigenfunction. Then

$$\begin{aligned} U(t + np, 0)\varphi &= U(t + np, np)U(np, 0)\varphi \\ &= U(t, 0)U(p, 0)^n \varphi \\ &= \mu^n U(t, 0)\varphi. \end{aligned}$$

So $y_t := U(t,0)\varphi$ is a solution of (A.1) with $\|y_t\| \to \infty$ as $t \to \infty$, which proves that the origin of (A.1) is asymptotically unstable. $\square$

The next lemma gives a characterization of the Floquet multipliers of system (A.10).

Lemma A.9. *Let $U(p,0)$ be the monodromy operator of the DDE* (A.10). *Then $\mu \in \mathbb{C}$ is an eigenvalue of $U(p,0)$ if and only if* (A.10) *has a non-zero solution of the form $y(t+T) = \mu y(t)$.*

Proof. Let $\mu \in \mathbb{C}$ be an eigenvalue of $U(p,0)$ and let $\phi \neq 0$ be an associated eigenfunction. Then

$$\begin{aligned} U(t + p, 0)\varphi &= U(t + p, p)U(p, 0)\varphi \\ &= U(t, 0)\mu y_0 \end{aligned}$$

so $y_t := U(t,0)\varphi$ is a solution of (A.10) with $y(t+p) = \mu y(t)$.

Vice versa, suppose that (A.10) has a solution of the form $y(t+p) = \mu y(t)$. Then in particular the history segments y_0, y_p satisfy $y_p = \mu y_0$. Since $y_t = U(t,0)y_0$, this implies that μ is an eigenvalue of $U(p,0)$ with eigenvector $y_0 \neq 0$. $\square$

A.3 Stability of periodic orbits

One of the main applications of Floquet theory is determining the stability of periodic solutions in nonlinear systems. Consider the ODE

$$\dot{x}(t) = f(x(t)) \tag{A.15}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ a C^2-function. Suppose that (A.15) has a solution x_* with periodic $p > 0$, i.e. $x_*(t+p) = x_*(t)$.

Differentiating the relation $\dot{x}_*(t) = f(x_*(t))$ with respect to time gives that $\dot{x}_*(t) = \dot{x}_*(t+p)$ is a solution of the linearized system

$$\dot{y}(t) = f'(x_*(t))y(t); \tag{A.16}$$

if $\dot{x}_*$ is non-stationary, then $\dot{x}_* \neq 0$. So in this case Lemma A.2 implies that 1 is always a Floquet muliplier of (A.16), which we call the **trivial Floquet multiplier**. The stability of x_* as a solution of (A.15) is determined by the non-trivial Floquet multipliers of (A.16). This is summarized in the following theorem, which we state without proof, for a proof see for example [Hal80, Theorem 2.1, Chapter 6].

Theorem 7. *Suppose that the system* (A.15) *has a non-stationary periodic solution* $x_*(t+p) = x_*(t)$*; let* $Y(p)$ *be the monodromy operator of the linearized system* (A.16)*. If* $Y(p)$ *has an eigenvalue strictly outside the unit circle, then* x_* *is an unstable solution of* (A.15)*. If the trivial eigenvalue* $1 \in \sigma(Y(p))$ *is algebraically simple and all other eigenvalues of* $Y(p)$ *lie strictly inside the unit circle, then* x_* *is a stable solution of* (A.15)*.*

Now consider the DDE

$$\dot{x}(t) = F(x_t), \qquad t \geq 0 \tag{A.17}$$

with state space $X = C\left([-r,0], \mathbb{R}^N\right)$, $r > 0$, and $F : X \to \mathbb{R}^N$ a C^2-function. Suppose that (A.17) has a non-stationary periodic orbit x_* with period $p > 0$. For $t \geq 0$, denote by $(x_*)_t$ the history segment of x_*. Differentiation the relation $\dot{x}_*(t) = F((x_*)_t)$ with respect to t gives that $\dot{x}_*$ is a p-periodic solution of the linearized system

$$\dot{y}(t) = F'((x_*)_t)y(t). \tag{A.18}$$

So Lemma A.9 implies that 1 is a Floquet multiplier of (A.18), which we call the **trivial Floquet multiplier**. The stability of x_* as a solution of (A.17) is determined by the non-trivial Floquet multipliers of (A.18). This is summarized in the following theorem, which we state without proof from [DGVW95, Theorem 3, Chapter 14].

Theorem 8. *Suppose that the system* (A.17) *has a non-stationary periodic solution* $x_*(t+p) = x_*(t)$*; let* $U(p,0)$ *be the monodromy operator of the linearized system* (A.18)*. If* $U(p,0)$ *has an eigenvalue strictly outside the unit circle, then* x_* *is an unstable solution of* (A.17)*. If the trivial eigenvalue* $1 \in \sigma_{pt}(U(p,0))$ *is algebraically simple and all other eigenvalues of* $U(p,0)$ *lie strictly inside the unit circle, then* x_* *is a stable solution of* (A.17)*.*

A.4 Riesz operators

This section is concerned with Riesz operators. Given a Banach space X and a bounded linear operator $T : X \to X$, we say that T is a *Riesz operator* if all its non-zero spectrum consists of eigenvalues of finite algebraic multiplicity. The next lemma (cf. Exercise [DGVW95, Exercise 2.4.8, Appendix II]) states that if a power of T is a Riesz operator, then T itself is Riesz as well. In particular, this means that if a power of T is compact, then T is a Riesz operator.

Lemma A.10. *Let* X *be a Banach space and* $T : X \to X$ *a bounded linear operator. Assume that there exists a* $k \in \mathbb{N}$ *such that* $T^k : X \to X$ *is a Riesz operator, i.e. all non-zero spectrum of* T^k *consists of isolated eigenvalues of finite multiplicity. Then the following statements hold:*

1. *$T : X \to X$ is a Riesz operator, i.e. all non-zero spectrum of T consists of eigenvalues of finite algebraic multiplicity.*
2. *$\nu \in \sigma_{pt}(T^k)$ if and only if there exists a $\mu \in \sigma(T)$ with $\mu^k = \nu$, i.e.*
$$\sigma_{pt}(T^k) = \sigma_{pt}(T)^k. \tag{A.19}$$
3. *For $\mu \in \sigma(T)$, let $m(\mu, T)$ be the algebraic multiplicity of μ as an eigenvalue of T; for $\nu \in \sigma(T^k)$, let $m(\nu, T^k)$ be the algebraic multiplicity of μ as an eigenvalue of T^k. Then*
$$\sum_{\mu \in \sigma_{pt}(T),\ \mu^k = \nu} m(\mu, T) = m(\nu, T^k) \tag{A.20}$$
and hence the equality (A.19) *'counts algebraic multiplicities'.*

Proof. We divide the proof into two steps:

STEP 1: we start by proving the second and third item of the lemma. If $\mu \in \sigma_{pt}(T)$ and $x \in X \backslash \{0\}$ is an associated eigenvector, then $T^k x = \mu^k x$ and hence $\mu^k \in \sigma_{pt}(T^k)$. Vice versa, let $\nu \in \sigma_{pt}(T^k)$ and let $m \in \mathbb{N}$ be its ascent, i.e. m is the least integer such that $\ker \left(\left(\nu I - T^k\right)^{n+1} \right) = \ker \left(\left(\nu I - T^k\right)^n \right)$ (this integer is finite since T^k is a Riesz operator). We decompose the space X as
$$X = N \oplus M \qquad \text{with } N = \ker \left(\left(\nu I - T^k\right)^m \right) \text{ and } M = \operatorname{range} \left(\left(\nu I - T^k\right)^m \right).$$
So N is the generalized eigenspace of the eigenvalue of ν for T^k and M its orthogonal complement; therefore it holds that
$$\sigma \left(T^k \Big|_N \right) = \{\nu\}, \qquad \nu \notin \sigma \left(T^k \Big|_M \right).$$
Moreover, the spaces N, M are both invariant under T. Since T^k is a Riesz operator, the generalized eigenspace N is finite dimensional and the maps
$$T|_N : N \to N, \qquad T^k \Big|_N : N \to N$$
are finite dimensional maps. By going to Jordan canonical form we see that
$$\sigma_{pt} \left(T^k \Big|_N \right) = \sigma_{pt} \left(T|_N \right)^k.$$
So in particular there exists a $\nu \in \sigma_{pt}(T|_N) \subseteq \sigma_{pt}(T)$ such that $\nu^k = \mu$ and the equality (A.19) holds. Moreover,
$$\sum_{\substack{\mu \in \sigma(T|_N), \\ \mu^k = \nu}} m\left(\mu, T|_N\right) = m\left(\nu, T^k \Big|_N\right). \tag{A.21}$$
But $\nu \notin \sigma \left(T^k \big|_M \right)$, and if $\mu \in \sigma_{pt}\left(T|_M\right)$, then $\mu^k \neq \nu$. Therefore the equality (A.21) implies the equality (A.20).

STEP 2: we prove the first statement of the lemma. Fix $\mu \in \sigma(T) \backslash \{0\}$; we want to show that $\mu \in \sigma_{pt}(T)$. Since μ^k is an isolated point of the spectrum of T^k, the spectral mapping theorem for bounded operators
$$\sigma(T)^k = \sigma(T^k)$$

(see [GGK90, Theorem 1.3.3]) implies that μ is an isolated point of the spectrum of T. Therefore, we can decompose the space X as

$$X = N \oplus M$$

where N, M are both invariant under T, and

$$\sigma\left(T|_N\right) = \{\mu\}, \qquad \sigma\left(T|_M\right) = \sigma(T)\backslash\{\mu\};$$

(this decomposition can be constructed using the *Riesz projection*, cf. [GGK90, Theorem 1.2.2]). Since $\mu \in \sigma\left(T|_N\right)$, the spectral mapping theorem for bounded operators implies that $\mu^k \in \sigma\left(T^k\big|_N\right)$. But operator $T^k\big|_N$ is a Riesz operator, and hence $\mu^k \in \sigma_{pt}\left(T^k\big|_N\right)$. The second statement of the lemma (applied to $T|_N$) now implies that

$$\mu e^{2\pi i j/k} \in \sigma_{pt}\left(T|_N\right)$$

for some $0 \leq j \leq k-1$. However, since $\sigma\left(T|_N\right) = \{\mu\}$, the only option is $j = 0$, so $\mu \in \sigma_{pt}\left(T|_N\right)$. This shows that $\mu \in \sigma_{pt}(T)$, as desired. $\square$

Appendix B

Summary

This thesis is concerned with delayed feedback stabilization of periodic orbits. Pyragas introduced a feedback scheme (now known as '*Pyragas control*') that measures the difference between the current state and the state time p ago, and feeds the result back into the system. For p-periodic orbits, the difference between the current state and the state time p ago is zero; hence any p-periodic solution of the system without feedback is also a solution of the system with feedback. But the introduction of the feedback term changes the dynamics of the system, and periodic orbits that are unstable in the system without feedback can become stable in the system with feedback, as has been attested in many experiments.

Equivariant control follows the spirit of Pyragas control, but adapts it to situations where a periodic orbit satisfies a known spatial-temporal pattern. Equivariant control feeds back the difference between the current state and a spatial-temporal transformation of the state, in such a way that the difference vanishes on the known spatial-temporal pattern. Equivariant control is mainly used in systems with symmetries, where the symmetry of the system induces known spatial-temporal patterns on the periodic orbit.

The first part of this thesis addresses limitations to Pyragas control. The main novelty here is that the limitations follow from a new *invariance principle*, which gives a clear and unifying understanding why Pyragas control can fail to stabilize.

In the second part of the thesis, we consider periodic orbits where the group of 'genuine' spatial-temporal symmetries (i.e. spatial-temporal symmetries that are not actually spatial symmetries) is cyclic. Such periodic orbits are called *discrete waves*. We prove sufficient conditions under which equivariant control can stabilize discrete waves; this positive stabilization result is applicable to a broad class of discrete waves and the necessary conditions are formulated in terms of accessible information on the uncontrolled system.

A running theme in the thesis is that we actively use the symmetry of the system (if present) in the stability analysis. This 'equivariant stability analysis' is both crucial on a technical level and in our understanding of equivariant control: it clarifies the connection with limitations to Pyragas control, and shows why (and in which situations) equivariant control is able to overcome these limitations.

Anhang C

Zusammenfassung

Diese Arbeit befasst sich mit der verzögerten Rückkopplungsstabilisierung periodischer Lösungen. Pyragas führte eine Rückkopplungsmethode ein (jetzt als "Pyragas-Kontrolle" bekannt), welche die Differenz zwischen dem aktuellen Zustand und dem um die Zeit p verzögerten Zustand misst und das Ergebnis in das System zurückspeist. Für p-periodische Lösungen ist diese Differenz Null; daher ist jede p-periodische Lösung des Systems ohne Rückkopplung auch eine Lösung des Systems mit Rückkopplung. Die Einführung des Rückkopplungsterms ändert jedoch die Dynamik des Systems, und periodische Lösungen, die im System ohne Rückkopplung instabil sind, können im System mit Rückkopplung stabil werden, wie in vielen Experimenten bestätigt wurde.

Die äquivariante Kontrolle folgt im Geiste der Pyragas-Kontrolle, passt sie jedoch an Situationen an, in denen ein periodischer Orbit mit einem bekannten räumlich-zeitlichen Muster vorliegt: Sie speist die Differenz zwischen dem aktuellen Zustand und einer räumlich-zeitlichen Transformation des Zustands so in das System zurück, dass die Differenz auf dem bekannten räumlich-zeitlichen Muster verschwindet. Die äquivariante Kontrolle wird hauptsächlich in Systemen mit Symmetrien verwendet, bei denen die Symmetrie des Systems bekannte räumlich-zeitliche Muster auf den periodischen Lösungen induziert.

Der erste Teil dieser Arbeit befasst sich mit Einschränkungen der Pyragas-Kontrolle. Die Hauptneuheit hierbei ist, dass sich die Einschränkungen aus einem neuen Invarianzprinzip ergeben, welches ein klares und einheitliches Verständnis dafür liefert, warum die Pyragas-Kontrolle nicht (immer) stabilisieren kann.

Im zweiten Teil dieser Arbeit betrachten wir periodische Lösungen, bei denen die Gruppe der "echten" räumlich-zeitlichen Symmetrien (d. h. räumlich-zeitliche Symmetrien, die nicht auch räumliche Symmetrien sind) zyklisch ist. Solche periodischen Lösungen werden diskrete Wellen genannt. Wir beweisen notwendige Bedingungen, unter denen eine äquivariante Kontrolle diskrete Wellen stabilisieren kann; dieses positive Stabilisierungsergebnis ist auf eine breite Klasse diskreter Wellen anwendbar, und die erforderlichen Bedingungen beziehen sich auf zugängliche Informationen über das unkontrollierte System.

Ein wiederkehrendes Thema dieser Arbeit ist die aktive Nutzung der Symmetrie des Systems (falls vorhanden) für die Stabilitätsanalyse. Diese "Analyse der äquivarianten Stabilität" ist sowohl auf technischer Ebene als auch für unser Verständnis der äquivarianten Kontrolle von entscheidender Bedeutung: Sie verdeutlicht den Zusammenhang mit den Einschränkungen der Pyragas-Kontrolle und zeigt, warum (und in welchen Situationen) die äquivariante Kontrolle diese Einschränkungen überwinden kann.

Anhang D

Selbstständigkeitserklärung

Hiermit bestätige ich, Babette de Wolff, dass ich die vorliegende Dissertation mit dem Thema

Delayed feedback stabilization with and without symmetries

selbstständig angefertigt und nur die genannten Quellen und Hilfen verwendet habe. Die Arbeit ist erstmalig und nur an der Freien Universität Berlin eingereicht worden.

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