6 Application of integration

6.1 Areas between curves

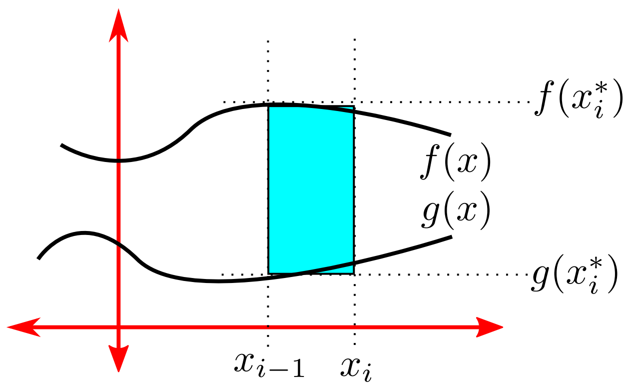
So let us now go back to the interpretation of the definite integral as the area under a curve.

$$\int_a^b f(x)\mathrm{d}x = \text{ area under the curve}$$

By this we mean the area bounded by the lines $x = a, x = b, y = 0$ and the curve $y = f(x)$.

Perhaps the first and easiest generalisation is to consider the area between two curves $f(x)$ and $g(x)$. Indeed we can then think of the orginial interpretation as $g(x) = 0$.

So we started with Riemann sums — lets do that again. Draw a picture and write down the appropriate sum



Take the interval $[a, b]$ and divide it up into n subintervals of width Δx. Choose our x_i and the x_i^*.

Previously the height of the rectangle was $f(x_i^*)$, but now it is the difference between the heights of the curves — $f(x_i^*) - g(x_i^*)$. And so the Riemann sum is

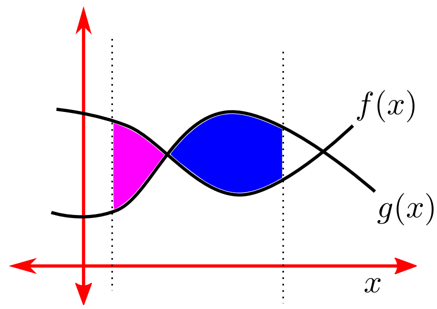
$$\sum_{i=1}^{n}(f(x_i^*) - g(x_i^*))\Delta x = \sum_{i=1}^{n} h(x_i^*)\Delta x$$

where $h(x) = f(x) - g(x)$.

So really there is nothing new going on here. This is just the definite integral of h. Hence the area is simply

$$\begin{aligned}\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n} h(x_i^*)\Delta x \\ &= \int_a^b h(x)\mathrm{d}x = \int_a^b (f(x) - g(x))\mathrm{d}x\end{aligned}$$

Of course we still have to be careful, because this is the signed-area!

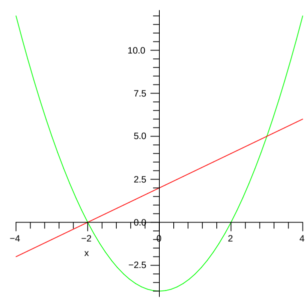


So we need to be careful about where the curves intersect.

Lets look at some examples.

Compute the (finite) area between the curves $f(x) = x + 2$ and $g(x) = x^2 - 4$.

- First step is to plot the graphs — work out intersection points and which is on top.



- So $x + 2$ is above $x^2 - 4$. They intersect at $x = -2, +3$.

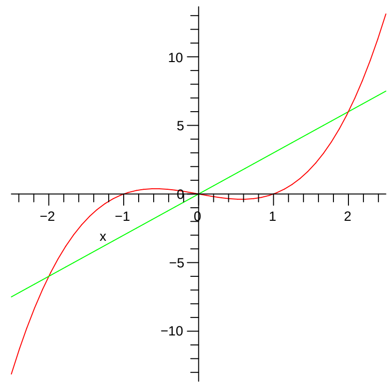
$$\begin{aligned} x + 2 &= x^2 - 4 \\ 0 &= x^2 - x - 6 = (x - 3)(x + 2) \end{aligned}$$

- So the area will be given by the integral

$$\begin{aligned} A &= \int_{-2}^{3} ((x + 2) - (x^2 - 4))\mathrm{d}x \\ &= \int_{-2}^{3} (6 + x - x^2)\mathrm{d}x \\ &= [6x + x^2/2 - x^3/3]_{-2}^{3} \\ &= (18 + 9/2 - 9) - (-12 + 2 + 8/3) = 27/2 + 22/3 \\ &= 81/6 + 44/6 = 125/6 \end{aligned}$$

Another example — maybe you guys do this one. Find the total area (not the signed area) between the curves $y = x^3 - x$ and $y = 3x$.

- The first curve is a cubic $y = x(x + 1)(x - 1)$.
- They intersect at $x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2)$ — $x = 0, \pm 2$.
- Plot them

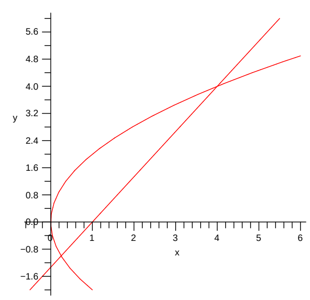


- So the total area is

$$
\begin{aligned}
A &= \int_{-2}^{0} ((x^3 - x) - 3x)\mathrm{d}x + \int_{0}^{2} (3x - (x^3 - x))\mathrm{d}x \\
&= \int_{-2}^{0} (x^3 - 4x)\mathrm{d}x + \int_{0}^{2} (4x - x^3)\mathrm{d}x \\
&= [x^4/4 - 2x^2]_{-2}^{0} + [2x^2 - x^4/4]_0^2 \\
&= 0 - (4 - 8) + (8 - 4) - 0 = 8
\end{aligned}
$$

In the text they also do the similar problems but with $x \leftrightarrow y$ interchanged. A simple example of this is... Find the area between the curves $y^2 = 4x$ and $4x - 3y = 4$.

- Draw a picture



- So we do this in the same way — the roles of x and y are reversed. Remember — they are just variables.
- In our Riemann sum, we split an interval in y and have y_i^* and Δy etc.
- So what is the range of y? The curves intersect at

$$
\begin{aligned}
y^2 &= 4x = 4 + 3y \\
y^2 - 3y - 4 &= 0 \\
(y - 4)(y + 1) &= 0 \qquad\qquad y = -1, +4
\end{aligned}
$$

- So the area between the curves is

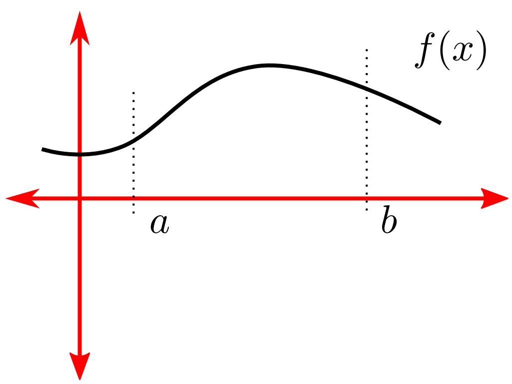
$$
\begin{aligned}
A &= \int_{-1}^{4} (x_r - r_l)\mathrm{d}y = \int_{-1}^{4} ((1+3y/4) - y^2/4)\mathrm{d}y \\
&= [y + 3y^2/8 - y^3/12]_{-1}^{4} \\
&= (4 + 48/8 - 64/12) - (-1 + 3/8 + 1/12) = (96 + 144 - 128)/24 - (-24 + 9 + 2)/24 \\
&= (112 + 13)/24 = 125/24
\end{aligned}
$$

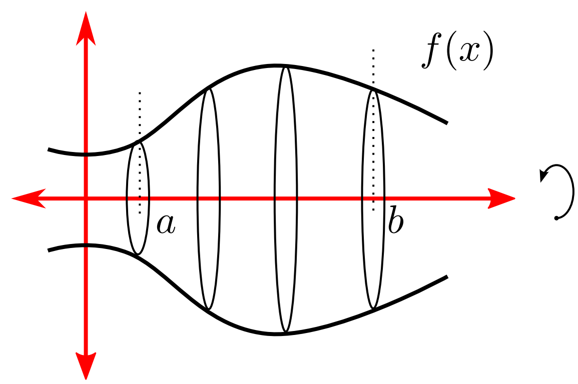
You could also do this *carefully* by vertical strips. You have to be careful where the curves intersect.

- To the left of $(1/4, -1)$ the vertical strip runs between $y^2 = 4x$ and itself.
- To the right of $(1/4, -1)$ the vertical strip runs between $y^2 = 4x$ and $4x - 3y = 4$.

6.2 Volumes

Consider the graph of some function $y = f(x)$.



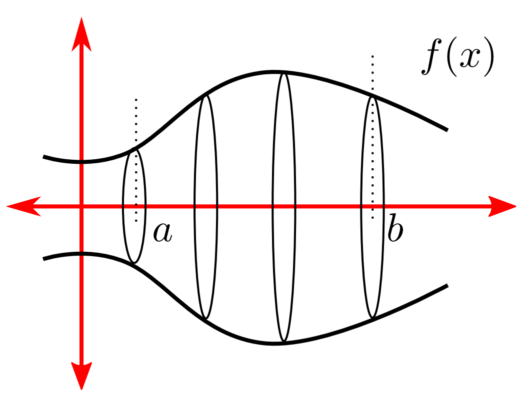


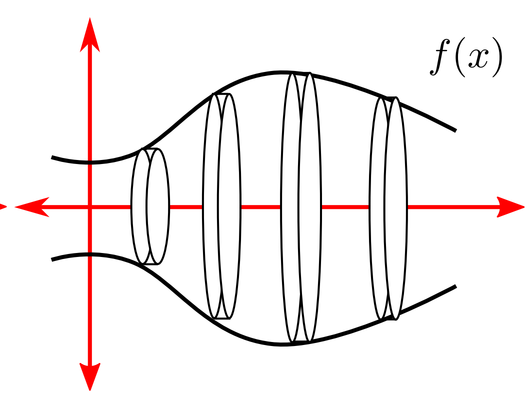
Now think of rotating it about the line $y = 0$ — this defines a 3-D surface. How do we find volume enclosed by this surface and the planes $x = a$ and $x = b$?

Again our starting point is Riemann sums. Split the interval $[a, b]$ into segments $[x_{i-1}, x_i]$ (just as before). Again we ask — how much does each of these intervals contribute to the total.

Before each segment contributed a rectangle to the area $f(x_i^*)\Delta x$, but now each segment contributes a slice of the volume. What does this slice look like?

Each slice is a cylinder of radius $f(x_i^*)$ and width Δx.





The volume is just the volume of a cylinder

$$\text{vol of cylinder } = \pi r^2 w = \pi f(x_i^*)^2 \Delta x$$

So our Riemann sum is

$$\begin{aligned} V &\approx \sum_{i=1}^{n} \pi f(x_i^*)^2 \Delta x \\ &= \sum_{i=1}^{n} g(x_i^*) \Delta x && g(x) = \pi f(x)^2 \end{aligned}$$

And as $n \to \infty$ this Riemann sum becomes an integral and we are left with

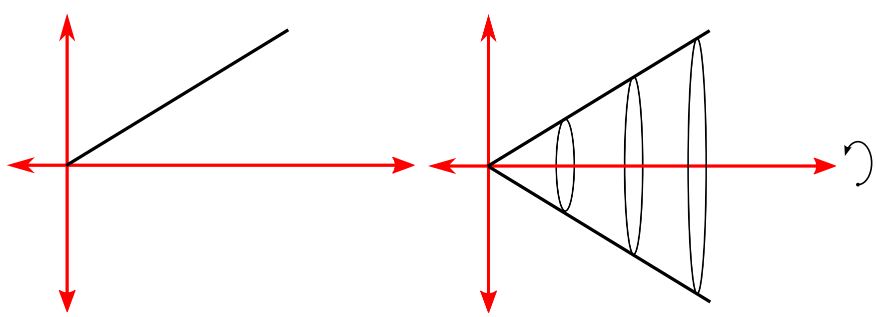
$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*) \Delta x && = \int_a^b g(x) \mathrm{d}x \\ &= \pi \int_a^b f(x)^2 \mathrm{d}x \end{aligned}$$

If we consider a more general shape - the contribution from any given "slice" to the Riemann sum will be the cross-section area times its width

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*) \Delta x \\ &= \int_a^b A(x) \mathrm{d}x \end{aligned}$$

Example — volume of a cone.

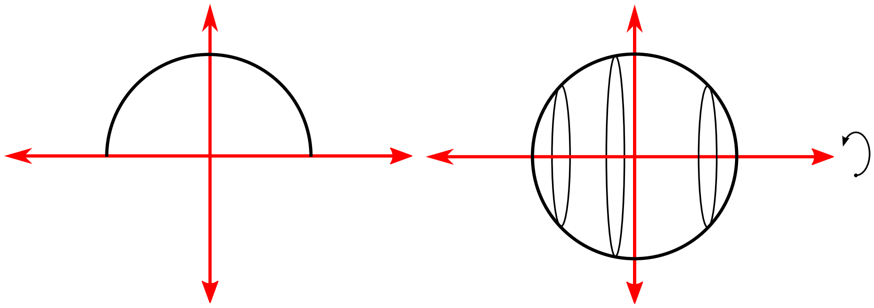
- Consider the line $y = \frac{x}{2}$ on the inteval $[0, 6]$.
- Rotate it about the x-axis to make a cone



- So the volume is

$$\begin{aligned} V &= \int_0^6 \pi f(x)^2 \mathrm{d}x \\ &= \pi \int_0^6 \frac{x^2}{4} \mathrm{d}x \\ &= \left[\frac{\pi x^3}{12} \right]_0^6 = \frac{\pi 6^3}{12} = 18\pi \end{aligned}$$

Example — volume of a sphere. Draw a semi-circle of radius r on the interval $[-r, r]$. The required function is This function is $y = \sqrt{r^2 - x^2}$ (because $x^2 + y^2 = r^2$). Rotating this curve around the x-axis gives a sphere.



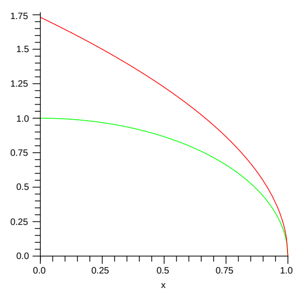
So the volume is given by

$$\begin{aligned} V &= \int_{-r}^{r} \pi f(x)^2 \mathrm{d}x \\ &= \pi \int_{-r}^{r} (r^2 - x^2) \mathrm{d}x \\ &= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \\ &= \pi \left((r^3 - r^3/3) - (-r^3 + r^3/3) \right) = \frac{4}{3} \pi r^3 \end{aligned}$$

What about the volume of a "bowl"

- Upper curve defines outside of bowl $y = f(x) = \sqrt{3 - 3x}$
- Lower curve defines inside of bowl $y = g(x) = \sqrt{1 - x^2}$

So we plot things carefully and have a think.

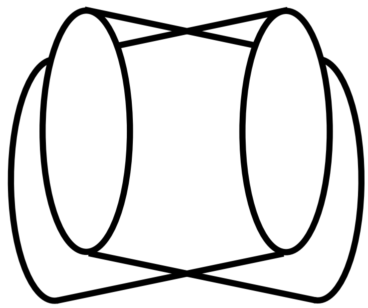


The volume of the bowl is given by

$$\begin{aligned} V &= \text{vol of outer} - \text{vol of hole} \\ &= \int_0^1 \pi f(x)^2 \mathrm{d}x - \int_0^1 \pi g(x)^2 \mathrm{d}x \end{aligned}$$

Just like area between curves.

What about a harder example — Q66(??) in 6.2. Find the volume of the intersection of 2 perpendicular cylinders of unit radius.



- Cylinder 1 is all points such that $x^2 + z^2 \leq 1$
- Cylinder 2 is all points such that $y^2 + z^2 \leq 1$

Find volume by slicing perpendicular to the z-axis. Its hard to picture it, so we need to let the algebra guide us. Consider the slice at some fixed height z

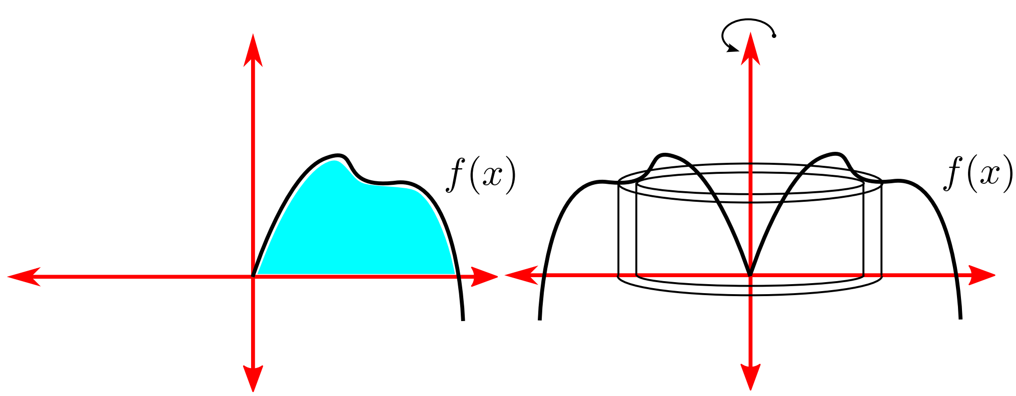
- For cylinder 1, with z fixed, we have $x^2 \leq 1 - z^2$ or $-\sqrt{1-z^2} \leq x \leq \sqrt{1-z^2}$.
- For cylinder 2, with z fixed, we have $y^2 \leq 1 - z^2$ or $-\sqrt{1-z^2} \leq y \leq \sqrt{1-z^2}$.

Hence at a fixed height z, the "slice" is just a square of side length $2\sqrt{1-z^2}$. The area of this square is $4(1-z^2)$ and the the volume of the slide is $4(1-z^2)\Delta z$. Riemann sums to definite integrals again:

$$
\begin{aligned}
V &= \lim_{n\to\infty} \sum_{i=1}^{n} 4(1 - z_i^{*2})\Delta z \\
&= \int_{-1}^{z} 4(1-z^2)\mathrm{d}z \\
&= [4z - 4z^3/3]_{-1}^{1} \\
&= (4 - 4/3) - (-4 + 4/3) = 16/3
\end{aligned}
$$

6.3 Volumes by cylindrical shells

Another way one can compute volume is by rotating around a different axis. Before we rotated about the x-axis, but we would instead rotate about the y-axis.



Now when we slice up the interval, we get cylindrical shells. Take a look at the text — but we don't do this in this course. (though it used to be part of the course).

6.4 Work

In everyday english the word "work" means something like "exertion or effort directed to produce or accomplish something". In physics (and mathematics) it has a precise meaning (based on this idea) — it is the energy expended acting against a force. eg — the energy expended moving a weight against gravity.

We need some definitions

Definition.

- Time t — measured in seconds
- Position s — measured in metres
- Mass m — measured in grams of kilograms
- Newton's second law

$$\text{Force } = \text{mass} \times \text{acceleration} \qquad\qquad F = m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

  Force is measured in Newtons $= kg.m/s^2$

- Work at constant force measures energy required to act against a force

$$\text{Work } = \text{Force} \times \text{displacement} \qquad\qquad W = Fd$$

  Measured in Newton-metres = Joules

So if the force is constant, then the work is simply the force times the distance moved against the force — eg moving a heavy weight up off the floor. How much work is done moving a 1kg book from the floor to the top of a 2m heigh shelf?

- Acceleration due to gravity $= 9.8m/s^2$. Force due to gravity $= ma = 9.8N$.
- Work done against force is $9.8 \times 2 = 19.6J$.

Very easy. But what happens when the force is not constant? If it varies with distance — eg a spring — then we approximate the work by a Riemann sum.

- Let $f(x)$ be the force acting on an object at position x.
- To compute the work done in order to move the object from $x = a$ to $x = b$ we cut up the interval $[a, b]$ into n segments $[x_{i-1}, x_i]$ each of width $(b - a)/n$.
- We approximate the varying force in the interval $[x_{i-1}, x_i]$ by a constant force $f(x_i^*)$ where $x_i^* \in [x_{i-1}, x_i]$ — just as we approximated the area in a Riemann sum.
- The work done in this interval $[x_{i-1}, x_i]$ is then approximately $f(x_i^*)\Delta x$.
- So the total work is

$$W \approx \sum_{i=1^n} f(x_i^*)\Delta x$$

  which is exactly a Riemann sum.

- Hence as $n \to \infty$ we have

$$W = \int_a^b f(x)\mathrm{d}x$$

Hooke's law relates the force exterted by a string, F, to the distance it has been stretched x:

$$F = kx \qquad\qquad k = \text{spring constant}$$

Holds for lots of materials provided x isnt too large.

A question from 6.4 "A spring has natural length of 20cm. If a 25N force is required to keep it stretched at a length of 30cm how much work is required to stretch it from 20cm to 25cm?"

- Careful of units — we need newtons and metres.
- First work out the spring constant:

$$\begin{aligned} F &= kx \\ 25 &= k(0.30 - 0.20) \\ k &= 25/0.1 = 250N/m \end{aligned}$$

- So we can now work out the work

$$\begin{aligned} W &= \int_0^{0.05} F(x)\mathrm{d}x \\ &= \int_0^{0.05} 250x\mathrm{d}x \\ &= [125x^2]_0^{0.05} = 125 \times 0.0025 = 0.3125J \end{aligned}$$

Q14 from 6.4 "A chain lying on the ground is 10m long and weighs 80kg. How much work is required to raise one end to a height of 6m?"

- Assume that the chain will be "L" shaped when it has been lifted, with 4m left on the ground. Also assume no friction and constant density of the chain = 8kg/m.
- Let us do this with a Riemann sum — split the chain into segments $[x_{i-1}, x_i]$ and work out how much work is done lifting each segment.
- Let x be the distance (in m) from the top of the chain. The piece of chain at x is lifted $6 - x$ m. Hence the segment $[x_{i-1}, x_i]$ is lifted $6 - x_i^*$ m.
- The segment weighs $8\Delta x$, so experiences $8 \times 9.8\Delta x = 78.4\Delta x$ gravitational force.
- So the Riemann sum is given by

$$W \approx \sum_{i=1}^{n} (6 - x_i^*) 78.4\Delta x$$

- So the work is given (limit of $n \to \infty$)

$$\begin{aligned} W &= 78.4 \int_0^6 (6-x)\mathrm{d}x \\ &= 78.4[6x - x^2/2]_0^6 = 78.4 \times 18 = 1411.2J \end{aligned}$$

What if the chain was dangling from the roof and we were to lift the far end?

- Let x be the distance from the middle of the chain.
- The piece of chain at x is lifted a distance of $2x$
- Hence the Riemann sum is

$$W \approx \sum_{i=1}^{n} 2x_i^* 78.4\Delta x$$

- So the work done is

$$\begin{aligned} W &= \int_0^5 156.8x\mathrm{d}x \\ &= 156.8[x^2/2]_0^5 = 156.8 \times 12.5 = 1960 \end{aligned}$$

Similarly if we calculate the work done pumping water from a tank — compute the work done pumping out each "slice" of water.

Example 5 from chpt 6.4.

- The tank is shaped like an inverted cone — height = 10m, radius = 4m.
- Filled to height of 8m.
- Find work pumping water out of top.
- Density of water is $1000kg/m^3$.

How do we do this?

- Draw a picture.
- How much work done to remove each "slice" of water?
- Let x be distance from bottom of the tank. The slice at x has volume

$$\begin{aligned} V(x) &= \pi r(x)^2 \Delta x \\ &= \pi \left(\frac{2x}{5}\right)^2 \Delta x \end{aligned}$$

- The weight of this slice is $1000V(x)$. So the gravitation force acting on the slice is $1000V(x) \times 9.8$.
- We need to move the slice at x up $10 - x$ metres.

- So the work done is

$$\begin{aligned} W &= \int_0^8 9800V(x)(10-x)\mathrm{d}x \\ &= 1568\pi \int_0^8 x^2(10-x)\mathrm{d}x \\ &\approx 3.36 \times 10^6 J \end{aligned}$$

6.5 Average value of a function

Integration is a very useful tool in probability and statistics.

- Given a "probability density function" compute the average and variance.
- A well known example would be the "bell curve", $f(x) = e^{-x^2}$.
- Let us look at a simpler version of this problem — compute the average value of a function over a given interval.
- Again we do it by Riemann sum arguments.

Compute the average value of the function $f(x)$ over the interval $[a, b]$.

- Split the interval up as per Riemann sum x_i and $\Delta x = (b-a)/n$.
- The average value is approximately the average of the values at each point x_i^* ($i = 1, 2, \ldots, n$)

$$\text{average of } f \approx \frac{1}{n}\sum_{i=1}^{n} f(x_i^*)$$

- But this doesn't look like a Riemann sum — no Δx.
- What happens as $n \to \infty$? There are n terms in the sum (presumably bounded?) and the sum is divided by n. So it is plausible that the limit exists.
- The Δx is there — it is hidden in the $\frac{1}{n}$.

$$\begin{aligned} \Delta x &= \frac{b-a}{n} \\ \frac{\Delta x}{b-a} &= \frac{1}{n} \end{aligned}$$

- So we can rewrite our sum to look like a Riemann sum

$$\begin{aligned} \text{average of } f &\approx \frac{1}{n}\sum_{i=1}^{n} f(x_i^*) \\ &= \frac{\Delta x}{b-a}\sum_{i=1}^{n} f(x_i^*) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x \end{aligned}$$

- So now as $n \to \infty$ our approximation becomes exact (you can prove this rigorously if you have to) and the sum becomes a definite integral.

Definition. The average value $\bar{y}$ of $y = f(x)$ for x in the interval $[a, b]$ is

$$\bar{y} = \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x$$

Examples

- Find the average value of the function $y = x^n$ (with $n \geq 0$) on $[0, 1]$

$$\begin{aligned}\bar{y} &= \frac{1}{1-0} \int_0^1 x^n \mathrm{d}x \\ &= \frac{1}{1} \left[\frac{1}{n+1} x^{n+1} \right]_0^1 \\ &= 1/(n+1)\end{aligned}$$

- Find the average value of the function $y = \sin x$ on the interval $[0, \pi/2]$

$$\begin{aligned}\bar{y} &= \frac{1}{\pi/2 - 0} \int_0^{\pi/2} \sin x \mathrm{d}x \\ &= \frac{2}{\pi} \left[-\cos x \right]_0^{\pi/2} \\ &= \frac{2}{\pi}(-(0) - (-1)) = \frac{2}{\pi}\end{aligned}$$

Unlike for a discrete system, in a continuous system, the function must actually take its average value at some point

Theorem (Average value theorem for integrals)**.**

- *Let f be a continuous function on the interval $[a, b]$.*
- *Let $\bar{y}$ be the average value of f on this interval*

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x$$

- *Then there exists a point $z \in [a, b]$ such that $f(z) = \bar{y}$.*

7 Techniques of integration

7.1 Integration by parts

Much like the substitution rule was really just the chain rule in reverse, "integration by parts" is the product rule reverse.

- The product rule says

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f(x)\frac{\mathrm{d}g}{\mathrm{d}x} + g(x)\frac{\mathrm{d}f}{\mathrm{d}x}$$

- Now undo this by integrating

$$\int \frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x))\mathrm{d}x = f(x)g(x) = \int f(x)\frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x + \int g(x)\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x$$

- Rearrange this a bit to get the "integration by parts" rule

Theorem (Integration by parts)**.**

$$\int f(x)\frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x = f(x)g(x) - \int g(x)\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x$$

- When you look at this, it is not immediately obvious why this i13418s going to be any help with anything at all.
- You take one integral and turn it into another, and along the way we have to compute $g(x)$ from $\frac{\mathrm{d}g}{\mathrm{d}x}$.
- Lets look at an example — $\displaystyle\int xe^x\mathrm{d}x$.
- There are 2 obvious choices of how to split this up.
  - $f(x) = x$ and $\frac{\mathrm{d}g}{\mathrm{d}x} = e^x$
  - $f(x) = e^x$ and $\frac{\mathrm{d}g}{\mathrm{d}x} = x$.
- Let us start with the first: $f' = 1$ and $g = e^x$.
- Sub into the formula

$$\begin{aligned} \int f(x)\frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x &= f(x)g(x) - \int g(x)\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x \\ \int xe^x\mathrm{d}x &= xe^x - \underbrace{\int e^x \cdot 1\mathrm{d}x}_{\text{easier!}} \\ &= xe^x - e^x + c \end{aligned}$$

- Let us now try it the other way — $f' = e^x$ and $g = x^2/2$

$$\begin{aligned} \int f(x)\frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x &= f(x)g(x) - \int g(x)\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x \\ \int xe^x\mathrm{d}x &= x^2e^x/2 - \underbrace{\int e^x \cdot x^2/2\mathrm{d}x}_{\text{uglier!}} \end{aligned}$$

- The second way seems to make things harder rather than easier — this is your guide!

An example in every calculus text

$$\int \log x \mathrm{d}x = ?$$

We only have a single function. How do we apply the integration by parts formula? Let us first start with

$$\int x^n \log x \mathrm{d}x$$

- Choose $f = x^n$ and $g' = \log x$ then $f' = nx^{n-1}$ and $g = ?$. Doesnt work.
- Choose $f = \log x$ and $g' = x^n$ then $f' = 1/x$ and $g = x^{n+1}/(n+1)$

$$\begin{aligned} \int x^n \log x \mathrm{d}x &= \frac{x^{n+1} \log x}{n+1} - \int \frac{x^n}{n+1} \mathrm{d}x \\ &= \frac{x^{n+1} \log x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c \end{aligned}$$

So when $n = 0$, and we want $\int \log x$ we have

$$\int \log x \mathrm{d}x = \int 1 \cdot \log x \mathrm{d}x$$

with $f = \log x$ and $g' = 1$.

$$\begin{aligned} \int \log x \mathrm{d}x &= x \log x - \int x/x \mathrm{d}x \\ &= x \log x - x + c \end{aligned}$$

Obviously the formula breaks when $n = -1$ — what then?

$$\int \log(x)/x \mathrm{d}x$$

This is just a substitution integral — $u = \log x$, so $\mathrm{d}u = \mathrm{d}x/x$.

$$\int \log(x)/x \mathrm{d}x = \int u \mathrm{d}u = u^2/2 + c = (\log |x|)^2 + c$$

From a previous exam paper

$$\int x^3 \log x \mathrm{d}x = ?$$

- Choose $f = x^3$ and $g' = \log x$ then $f' = 2x^2$ and $g = ?$ — doesnt work.
- Choose $f = \log x$ and $g' = x^3$ then $f' = 1/x$ and $g = x^4/4$

$$\begin{aligned} \int x^3 \log x \mathrm{d}x &= \frac{1}{4} x^4 \log x - \int \frac{1}{4} x^3 \mathrm{d}x \\ &= \frac{1}{4} x^4 \log x - \frac{1}{16} x^4 + C \end{aligned}$$

What about $\int t^2 e^t \mathrm{d}t$?

- Apply integration by parts once
- Choose $f = t^2, g' = e^t$, so $f' = 2t$ and $g = e^t$

$$\int t^2 e^t \mathrm{d}t = t^2 e^t - \int 2te^t \mathrm{d}t$$

- Reduces integral of $t^2 e^t$ to integral of te^t.
- Apply integration by parts again to this integral.
- Choose $f = t, g' = e^t$ so $f' = 1$ and $g = e^t$:

$$\int te^t \mathrm{d}t = te^t - \int e^t \mathrm{d}t \qquad\qquad = te^t - e^t.$$

- So our original integral is

$$\begin{aligned}\int t^2 e^t \mathrm{d}t &= t^2 e^t - \int 2te^t \mathrm{d}t \\ &= t^2 e^t - 2(te^t - e^t) + C \\ &= e^t(t^2 - 2t + 2) + C\end{aligned}$$

Another example $\sin(t)e^t$

- Choose $f = \sin(t)$, $g' = e^t$, so $f' = \cos t$ and $g = e^t$

$$\int \sin(t)e^t \mathrm{d}t = \sin(t)e^t - \int \cos(t)e^t \mathrm{d}t$$

- Doesnt look much better — apply again
- Now $f = \cos(t)$ and $g' = e^t$, so $f' = -\sin(t)$ and $g = e^t$

$$\int \cos(t)e^t \mathrm{d}t = e^t \cos(t) + \int \sin(t)e^t \mathrm{d}t$$

- So we have expressed $\int \sin(t)e^t \mathrm{d}t$ in terms of itself?

$$\begin{aligned}\int \sin(t)e^t \mathrm{d}t &= \sin(t)e^t - \int \cos(t)e^t \mathrm{d}t \\ &= \sin(t)e^t - e^t \cos(t) - \int \sin(t)e^t \mathrm{d}t\end{aligned}$$

So bring the $\int blah \mathrm{d}t$ all over to the left

$$\begin{aligned}2\int \sin(t)e^t \mathrm{d}t &= e^t(\sin(t) - \cos(t)) \\ \int \sin(t)e^t \mathrm{d}t &= \frac{1}{2}e^t(\sin(t) - \cos(t)) + c\end{aligned}$$

Another good one $\int \cos(x)\log(\sin(x))\mathrm{d}x$

- Choose $f = \cos(x)$ and $g' = \log(\sin(x))$ — urgh.
- Choose $f = \log(\sin(x))$ and $g' = \cos(x)$ — $f' = \cos(x)/\sin(x)$ and $g = \sin(x)$.

$$\begin{aligned}\int \cos(x)\log(\sin(x))\mathrm{d}x &= \sin(x)\log(\sin(x)) - \int \frac{\cos(x)}{\sin(x)}\sin(x)\mathrm{d}x \\ &= \sin(x)\log(\sin(x)) - \int \cos(x)\mathrm{d}x \\ &= \sin(x)\log(\sin(x)) - \sin(x) + C\end{aligned}$$