11 Infinite sequences and series

11.9 Representing a function as a power series

Lets go back to our easiest power series

$$\sum_{n=0}^{\infty} x^n$$

This is also a geometric series, and so, provided $|x| < 1$ we know that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

So provided $|x| < 1$ we can *represent* the function $1/(1-x)$ as the power series $\sum x^n$. When $|x| > 1$ the sum does not converge and this equality no longer holds and we cannot represent it this way (though there are other representations).

Similarly we can write

$$\frac{1}{1+2x} = \sum_{n=0}^{\infty} (-2x)^n$$

and

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} -x^{2n}$$

A little more work gives

$$\frac{2}{3-x} = \frac{2/3}{1-x/3} = \frac{2}{3}\sum (x/3)^n = \sum \frac{2}{3^{n+1}} x^n$$
$$\frac{2x^5}{3-x} = \sum \frac{2}{3^{n+1}} x^{n+5}$$

And all of these hold when inside the radius of convergence of the power series. Okay — this is a silly party trick (suitable for only very dull parties) — just messing about with geometric series.

More generaly, the ratio test tells us that inside the radius of convergence, a power series is absolutely convergent — which is a very strong property. While conditionally convergent series are very dellicate, absolutely convergent series very robust and we can mess about with the series in different ways and it will still stay convergent. In particular we can differentiate and integrate

Theorem. *Let $\sum c_n(x-a)^n$ be a power series with radius of convergence $R > 0$, and let $f(x) = \sum c_n(x-a)^n$. Then on the interval $x \in (a-R, a+R)$, f is continuous and differentiable. Further*

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \sum nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$
$$\int f(x)\mathrm{d}x = C + \sum \frac{c_n}{n+1}(x-a)^{n+1} = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \dots\dots$$

The radius of convergence for these two series is still R.

So this tells us we can differentiate term-by-term and integrate term-by-term.

Simple example

$$\begin{aligned}\frac{1}{1-x} &= 1 + x + x^2 + \dots \\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + \dots = \sum nx^{n-1} \\ -\log(1-x) &= C + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = C + \sum \frac{x^n}{n}\end{aligned}$$

Think a bit to work out C — set $x = 0$

$$= \sum \frac{x^n}{n}.$$

So this definitely gets us something interesting. Another easy one — work out the series for $\arctan(x)$.

Remember $\frac{\mathrm{d}}{\mathrm{d}x} \arctan(x) = 1/(1+x^2)$ — what is its power series expansion, then integrate term-by-term

$$\begin{aligned}\frac{1}{1+x^2} &= \sum_{n=0}^{\infty} (-1)^n x^(2n) \\ \arctan(x) = \int \frac{\mathrm{d}x}{1+x^2} &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\ &= C + x - x^3/3 + x^5/5 - x^7/7 + \dots\end{aligned}$$

remember $\arctan(0) = 0$, so $C = 0$

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