

Series

Geometric series: $\sum_{n=1}^{\infty} ar^{n-1}$ (or $\sum_{n=0}^{\infty} ar^n$). $\sum_{n=3}^{\infty} ar^n$

Theorem: Consider the geometric series $\sum_{n=1}^{\infty} ar^n$. If

$|r| < 1$, then it converges and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$ and $a \neq 0$ then the series diverges.

Example

Calculate $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}}$

$$\sum_{n=1}^{\infty} ar^{n-1}$$

$$= (a + ar + ar^2 + \dots)$$

$$S = \sum_{n=1}^{\infty} \left(\frac{-1}{2^2}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n = \left(-\frac{1}{4}\right)^1 + \left(-\frac{1}{4}\right)^2 + \dots$$

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{4}\right) \left(-\frac{1}{4}\right)^{n-1}$$

$$S = \frac{a}{1-r}$$

$$= -\frac{1}{4} \cdot \frac{1}{1-(-\frac{1}{4})}$$

$$|r| = \left|-\frac{1}{4}\right| < 1 \Rightarrow S \text{ converges.}$$

$$= -\frac{1}{4} \cdot \frac{4}{5}$$

$$= -\frac{1}{5}$$

Counter Example

Notice that both $\sum_{n=1}^{\infty} \left(-\frac{4}{3}\right)^n$ and $\sum_{n=1}^{\infty} \frac{g^n}{5 \cdot 3^{2n}}$ diverges.

$$1^{\text{st}} \text{ problem: } \sum_{n=1}^{\infty} \left(-\frac{4}{3}\right)^n = \sum_{n=1}^{\infty} \left(-\frac{4}{3}\right) \left(\frac{-4}{3}\right)^{n-1}$$

$$|r| = \left|-\frac{4}{3}\right| > 1 \Rightarrow \text{it doesn't converge.}$$

$$2^{\text{nd}} \text{ problem: } \sum_{n=1}^{\infty} \frac{g^n}{5 \cdot 3^{2n}} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{g^n}{(3^2)^n} = \frac{1}{5} \sum_{n=1}^{\infty} (1)^n$$

$$= \frac{1}{5} \underbrace{\sum_{n=1}^{\infty} (1)^{n-1}}$$

$\nearrow N^{\text{th}}$ partial sum.

$$\text{consider } S_N = \frac{1}{5} \sum_{n=1}^N (1)^{n-1} \rightarrow \infty \text{ as } N \rightarrow \infty.$$

$$\boxed{r^n \rightarrow 0 \text{ if } |r| < 1.}$$

Example

Write the repeating decimal $0.\overline{1234} = 0.123434\dots$ as a ratio of integers.

$$\begin{aligned}
 \underline{\text{Soln}}: \quad 0.\overline{1234} &= \frac{12}{100} + \frac{34}{10000} + \frac{34}{(100)^3} + \frac{34}{(100)^4} + \dots \\
 &= \frac{12}{100} + \sum_{n=2}^{\infty} \frac{34}{(100)^n} \\
 &= \frac{12}{100} + \frac{1}{(100)^2} \left(34 + \frac{34}{100} + \frac{34}{(100)^2} + \dots \right) \\
 &= \frac{12}{100} + \frac{1}{100^2} \left(\frac{34}{1 - \frac{1}{100}} \right) \\
 &= \frac{12}{100} + \frac{1}{100^2} \cdot \frac{100 \cdot 34}{99} = \frac{1222}{9900}
 \end{aligned}$$

Telescoping sum.

$$S_n = a + ar + ar^2 + ar^3 + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots = \sum_{n=1}^{\infty} ar^n$$

And $S_n - rS_n = \sum_{n=1}^{\infty} (ar^{n-1} - ar^n)$ - type of telescoping sum.

Thm: Suppose $S = \sum_{n=1}^{\infty} (a_n - a_{n+1})$ with $\lim_{n \rightarrow \infty} a_n = L$.

Then $S = a_1 - L$ and so S is convergent.

Example 6:

$$\text{Calculate } \sum_{n=3}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right) \right) = \sum_{n=3}^{\infty} (\alpha_n - \alpha_{n+1})$$

Let $\alpha_n = \cos\left(\frac{\pi}{n}\right)$. So, N^{th} partial sum starting from 3:

$$S_N = \sum_{n=3}^N \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right) \right)$$

$$= \left(\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{4}\right) \right) + \left(\cancel{\cos\left(\frac{\pi}{4}\right)} - \cancel{\cos\left(\frac{\pi}{5}\right)} \right) \dots$$

$$+ \cancel{\left(\cos\left(\frac{\pi}{N}\right) - \cos\left(\frac{\pi}{N+1}\right) \right)}$$

$$= \cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{N+1}\right).$$

$$S_N \rightarrow ? \quad \lim_{N \rightarrow \infty} S_N = \cos\left(\frac{\pi}{3}\right) - \lim_{N \rightarrow \infty} \cos\left(\frac{\pi}{N+1}\right)$$

$$= \cos\left(\frac{\pi}{3}\right) - \cos\left(\lim_{N \rightarrow \infty} \frac{\pi}{N+1}\right)$$

$$= \cos\left(\frac{\pi}{3}\right) - \cos(0) = \frac{1}{2} - \frac{1}{2} = \underline{\underline{\frac{1}{2}}}$$

Example

Calculate

$$\sum_{n=2}^{\infty} \left(\frac{2^{n+1}}{3^n} + \frac{1}{2n-1} - \frac{1}{2n+1} \right) \quad (a_n - a_{n+1})$$

Solution: $= \sum_{n=2}^{\infty} \frac{2^{n+1}}{3^n} + \sum_{n=2}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$.

Geometric part: $r = \frac{2}{3}$

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{2^{n+1}}{3^n} &= \sum_{n=2}^{\infty} 2 \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} 2 \left(\frac{2}{3}\right)^2 \left(\frac{2}{3}\right)^{n-1} \\ &= 2 \left(\frac{2}{3}\right)^2 \cdot \frac{1}{1 - \frac{2}{3}} \\ &= \frac{8}{9} \cdot \frac{3}{1} = \underline{\underline{\frac{8}{3}}} \end{aligned}$$

Example (contd.)

Telescoping sum

$$\sum_{n=2}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

$$\text{let } a_n = \frac{1}{2n-1}, \quad a_{n+1} = \frac{1}{2(n+1)-1} = \frac{1}{2n+1}.$$

$$\Rightarrow \sum_{n=2}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) = \frac{1}{4-1} - 0 = \frac{1}{3}.$$

$$\sum_{n=2}^{\infty} \left(\frac{2^{n+1}}{3^n} + \frac{1}{2n-1} - \frac{1}{2n+1} \right) = \frac{8}{3} + \frac{1}{3} = \underline{\underline{3}}$$

Divergence test

- In general, it is very difficult to calculate a convergent infinite sum.
- We need criteria for establishing whether an infinite series converges or diverges.

Consider $\sum_{n=1}^{\infty} 2^{-n}$, $\sum_{n=1}^{\infty} ar^n$ with $|r| < 1$, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

note that $a_n \rightarrow 0$ in all cases:

Lemma: If $\sum a_n$ is convergent then $a_n \rightarrow 0$

Divergence test

Consider a statement :

If (he is Shakespeare) then (he is dead)

$\begin{matrix} P \\ n \text{ is even} \end{matrix}$

$\begin{matrix} Q \\ n \text{ is not odd.} \end{matrix}$

converse: If (he is dead) then (he is Shakespeare).

$\begin{matrix} Q \\ n \text{ is not odd} \end{matrix}$ $\begin{matrix} P \\ n \text{ is even.} \end{matrix}$

contra-positive: If (he is not dead) then (he is not Shakespeare)

$\begin{matrix} \neg Q \\ \neg P \end{matrix}$

Original statement is equivalent to its contra-positive.

Original statement may not be equivalent to its converse.

Divergence test

Lemma: If a series $\sum_{n=0}^{\infty} a_n$ is convergent then
 $a_n \rightarrow 0$.

Contrapositive: $a_n \not\rightarrow 0 \quad \sum a_n \not\rightarrow L$

Theorem: If $\{a_n\}$ does not converge to 0 then $\sum a_n$ is not convergent.

$\sum_{n=1}^{\infty} \frac{n^2+2}{2n^2+1}$ does not converge because

$$\left\{ \frac{n^2+2}{2n^2+1} \right\} \not\rightarrow 0.$$

Harmonic Series

Lemma: If a series $\sum_{n=0}^{\infty} a_n$ is convergent then $a_n \rightarrow 0$.

Converse of this statement is:

If $a_n \rightarrow 0$ then the series $\sum_{n=0}^{\infty} a_n$ is convergent.

harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Converges

Harmonic series

Let's compute the 2^n partial sum of $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$S_{2^0} \quad S_{2^1} \quad S_{2^2} \quad S_{2^3} \quad S_{2^4} \quad \dots$$

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_2 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} \geq 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2 \cdot \frac{1}{2}$$

$$S_{2^3} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{5} \right)}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} \right)}_{\geq \frac{4}{8}} \geq 1 + 3 \cdot \frac{1}{2}$$

$$\text{So, } S_{2^n} \geq 1 + n \cdot \frac{1}{2}$$

• - odd
○ - even

$$S_{2^n} \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

So, S is divergent.



