Integration by parts.

Product rule in reverse.

Let $u(x)$ and $v(x)$ be differentiable functions.

Then
$$\boxed{\frac{d}{dx}\Big(u(x)\cdot v(x)\Big) = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}}$$

Example: $u(x) = x$, $v(x) = e^x$

$$\frac{d}{dx}\Big(u(x)\cdot v(x)\Big) = \underbrace{x}_{x}\,\underbrace{e^x}_{\frac{d}{dx}(e^x)} + \underbrace{1}_{\frac{d}{dx}(x)}\cdot \underbrace{e^x}_{}$$

Product rule in reverse

$$\boxed{\frac{d}{dx}\Big(u(x)\cdot v(x)\Big) = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}}$$

So, $$u(x)\frac{dv}{dx} = \frac{d}{dx}\big(u(x)\cdot v(x)\big) - v(x)\frac{du}{dx}$$

or $$\int u(x)\frac{dv}{dx}\,dx = \int \frac{d}{dx}\big(u(x)\cdot v(x)\big)\,dx - \int v(x)\frac{du}{dx}\,dx.$$

$$\boxed{\int u(x)\frac{dv}{dx}\,dx = u(x)\cdot v(x) - \int v(x)\frac{du}{dx}\,dx.}$$

Integration by parts.

$$\boxed{\int u(x)\,\frac{dv}{dx}\,dx = u(x)\cdot v(x) - \int v(x)\,\frac{du}{dx}\,dx.}$$

Let's write this more compactly.

$$v'(x) = \frac{dv}{dx}$$

using substitution; replace $\frac{dv}{dx}\,dx \to dv$

replace $\frac{du}{dx}\,dx \to du$

$$dx \to \frac{dv}{v'(x)}$$

so,

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Example

$$\int x e^x \, dx$$

$$\int u \, dv = uv - \int v \, du \iff \int u(x) \frac{dv}{dx} dx = u(x) \cdot v(x) - \int v(x) \frac{du}{dx} dx$$

Let $u(x) = x$ $\qquad v'(x) = e^x$

$u'(x) = 1$ $\qquad v(x) = e^x$

$$\int x e^x \, dx = x \cdot e^x - \int e^x \, dx$$

$$= x e^x - (e^x + C)$$

$$= x e^x - e^x + C$$

Example 1 (Contd)

$$\int x e^x \, dx$$

Try: $u = e^x$ $\qquad$ $v' = x$

$u' = e^x$ $\qquad$ $v = \dfrac{x^2}{2}$

so, $\displaystyle \int x e^x \, dx = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x \, dx$

Challenging to integrate.

Remark

Integration by part is useful

- eliminate factors of x from integrand: xe^x.
- eliminate a $\log x$ from an integrand: $\frac{d}{dx}\log(x) = \frac{1}{x}$
- eliminate inverse trig functions: $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

$$\int u\,dv = uv - \int v\,du$$

Example 2

$$\int \log(x)\, dx = \log(x) \cdot x - x + C$$

$$\int x^n \log(x)\, dx, \quad n = 0, 1, 2, \dots$$

$$u(x) = \log(x) \qquad v' = x^n$$

$$u'(x) = \frac{1}{x} \qquad v = \frac{x^{n+1}}{n+1}$$

$$\int x^n \log(x)\, dx = \log(x) \cdot \frac{x^{n+1}}{n+1} - \int \overbrace{\frac{x^{n+1}}{n+1} \; \frac{1}{x}}^{\frac{x^n}{n+1}} dx.$$

$$= \frac{\log(x)\, x^{n+1}}{n+1} - \left(\frac{x^{n+1}}{(n+1)^2} + C \right)$$

$$= \frac{\log(x)\, x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C.$$

Example 3

$$\int \arctan(x)\,dx.$$

$u(x) = \arctan(x)$ $\qquad$ $v'(x) = 1$

$u' = \dfrac{1}{1+x^2}$ $\qquad$ $v(x) = x$

$$\int \arctan(x)\,dx = x\cdot\arctan(x) - \int \frac{x\cdot}{1+x^2}\,dx.$$

let $w(x) = 1+x^2$

$\dfrac{dw}{dx} = 2x \rightarrow dx \rightarrow \dfrac{1}{2x}\,dw$

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{w}\,dw\Bigg|_{w=1+x^2} = \frac{1}{2}\log|w|\Bigg|_{w=1+x^2} + C$$

$$= \frac{1}{2}\log(|1+x^2|) + C$$

$$\boxed{\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\log(1+x^2) + C}$$

So, $\int \arctan(x) = x\arctan(x) - \frac{1}{2}\log|1+x^2| + C$

Example 4

$$\int t^2 e^t \, dt$$

$u(t) = t^2$ $\qquad v'(t) = e^t$

$u'(t) = 2t$ $\qquad v(t) = e^t$

$$\int t^2 e^t \, dt = t^2 \cdot e^t - \int e^t \, 2t \, dt$$

$$= t^2 e^t - 2 \int t e^t \, dt$$

$w(t) = t$ $\qquad z'(t) = e^t$

$w'(t) = 1$ $\qquad z(t) = e^t$

$$\int t e^t \, dt = t e^t - \int e^t \, dt = t e^t - e^t + C$$

$$\boxed{\int t^2 e^t \, dt = t^2 e^t - 2\left(t e^t - e^t + C\right) = t^2 e^t - 2te^t - 2e^t + C}$$