Differentiation and integration of power series.

A power series has the form $\sum_{n=0}^{\infty} c_n (x-a)^n$.

If a power series converges, we can provide a meaningful interpretation of differentiation and integration of power series.

For example:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if} \quad |x|<1.$$

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) \quad \text{if} \quad |x|<1$$

$$\int \sum_{n=0}^{\infty} x^n \, dx = \int \frac{1}{1-x} \, dx \quad \text{if} \quad |x|<1.$$

Differentiation of power series

Thm: If $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $x \in (a-R, a+R)$ for some $R>0$ (possibly $R=\infty$), we can define a function $f(x)$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for} \quad x \in (a-R, a+R)$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Then we have

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}, \quad f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}.$$

- We are allowed to differentiate term-by-term inside interval of convergence
- Note that $f^{(k)}(x)$ is also a power series. The first non-zero term of $f^{(k)}(x)$ starts at index $n = k$.

Integration of power series.

Thm: Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges in $x \in (a-R, a+R)$

Then
$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C \quad \text{in } x \in (a-R, a+R).$$

Like in differentiation, we can integrate term-by-term.

Important results.

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \qquad |x| < 1$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}(-1)^n \qquad |x| < 1.$$

Log (1+x)

Recall the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \quad |x|<1$$

$$= \sum_{n=0}^{\infty} x^n$$

integrate both sides:

$$-\log(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \quad |x|<1.$$

Since $\log(1) = 0$, we get $c=0$ (use $x=0$)

So, $$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad |x|<1$$

$$\Rightarrow \log(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad |x|<1$$

subsitute $x \to -x$.

$$\Rightarrow \log(1+x) = -\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \quad //.$$

Arctan(x)

Recall $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad , \quad |x|<1$

Let $x = -y^2$: $\frac{1}{1+y^2} = \sum_{n=0}^{\infty} (-y^2)^n = \sum_{n=0}^{\infty} (-1)^n y^{2n} \quad , \quad |y|<1$

$|-y^2|<1$

replace y with x : $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad , \quad |x|<1 .$

integrate both sides:

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad , \quad |x|<1$$

$$\int \frac{1}{1+x^2}\, dx = \arctan(x)$$

Example 1

The interval of convergence of $\sum_{n=0}^{\infty} n x^{n+1}$ is $(-1, 1)$ (use ratio test).

Find a compact formula for $\sum_{n=0}^{\infty} n x^{n+1}$.

Sol^n: We start with

$$\frac{1}{1-x} = 1+ x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \quad , \quad |x|<1 .$$

differentiate wrt x.

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\, x^{n-1} \quad , \quad |x|<1 .$$

multiply by x^2:

$$0 \cdot x^1 + \sum_{n=1}^{\infty} n x^{n+1}$$

$$\frac{x^2}{(1-x)^2} = x^2 \cdot \sum_{n=1}^{\infty} n\, x^{n-1} = \sum_{n=1}^{\infty} n\, x^{n+1} = \sum_{n=0}^{\infty} n\, x^{n+1} , \quad |x|<1$$

Example 2

Find a simple compact formula for $\sum_{n=0}^{\infty} n^2 x^n$.

Recall from example 1:

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} n x^{n-1} , \quad |x|<1$$

multiply by x:

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} n x^n , \quad |x|<1$$

now differentiate:

$$(1-x)^{-2} - 2x(1-x)^{-3} = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

Example 2 contd.

multiply by x:

$$\left((1-x)^{-2} - 2x(1-x)^{3}\right) \cdot x = \sum_{n=1}^{\infty} n^2 x^n = \sum_{n=0}^{\infty} n^2 x^n$$

so, $$\frac{\left((1-x)-2x\right)x}{(1-x)^3} = \frac{(1-3x)x}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n, \quad |x|<1$$

Example 3.

Calculate $L = \lim_{x \to 0} \dfrac{2x - \log(1+2x)}{x^2}$

Replace $\dfrac{1}{1-x} = 1 + x + x^2 + \dots \qquad |x| < 1.$

Replace $x \to -x$: $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \dots, \quad |x| < 1$

integrate both sides:

$$\log(1+x) = x - x^2/2 + x^3/3 - \dots$$

replace $x \to 2x$

$$\log(1+2x) = 2x - 4x^2/2 + 9x^3/3 - \dots$$

so, $\dfrac{2x - \log(1+2x)}{x^2} = (2x^2 + 9x^3/3 - \dots)/x^2$

so, $\lim_{x \to 0} \dfrac{2 \cdot x - \log(1+2x)}{x^2} = 2 + \cancel{\lim_{x \to 0} \left(\dfrac{9x}{3} - \dfrac{16x^2}{3} + \dots \right)} = 2$

Example 4.

Calculate $\lim_{x\to\infty} x\left[\log(x+3) - \log(x+1)\right] = 2$.

Sol^n Note that for large x,

$$\log(x+3) - \log(x+1) = \log\left(x\left(1+\frac{3}{x}\right)\right) - \log\left(x\left(1+\frac{1}{x}\right)\right)$$

$$= \log(x) + \log\left(1+\frac{3}{x}\right) - \log(x) - \log\left(1+\frac{1}{x}\right)$$

$$= \log\left(1+\frac{3}{x}\right) + \log\left(1+\frac{1}{x}\right).$$

Recall:

$$\frac{1}{1+h} = 1 + h + h^2 + \dots \qquad \text{for} \quad |h|<1$$

$$\& \quad \log(1+h) = h + \frac{h^2}{2} + \frac{h^3}{3} + \dots \qquad |h|<1.$$

Example 4. (contd)

$\log(1+h) = h + \frac{h^2}{2} + \frac{h^3}{3} + \cdots \quad |h|<1.$

replace $h \to \frac{1}{x}$: $\log\left(1+\frac{1}{x}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \cdots$

replace $h \to \frac{3}{x}$: $\log\left(1+\frac{3}{x}\right) = \frac{3}{x} + \frac{9}{2x^2} + \frac{27}{3x^3} + \cdots$

so, $x\left(\log(x+3) - \log(x+1)\right)$

$= x\left(\log\left(1+\frac{3}{x}\right) - \log\left(1+\frac{1}{x}\right)\right)$

$= x\left(\frac{3}{x} - \frac{1}{x}\right) + x\left(\frac{9}{2x^2} + \frac{27}{3x^3} + \cdots\right) - x\left(\frac{1}{2x^2} + \frac{1}{3x^3} + \cdots\right)$

$\to 2$ as $x \to \infty$.

Taylor series:

Previously, we found power series representation of $f(x)$ by manipulation of special cases (geometric series).

What about for a general function $f(x)$?

Find power series representation of $f(x)$ near $x=a$.

So, $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ — 1

How do we find c_n, $n = 1, 2, 3, \ldots$?

Remark: By writing $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ we are assuming x is in the interval of convergence.

Taylor series.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots \quad — ①$$

For c_0 : substitute $x=a$ in ①

so, $f(a) = c_0 + (a-a) + (a-a)^2 + \dots \Rightarrow c_0 = f(a).$

For c_1: differentiate both sides and use $x=a$

$$\Rightarrow f'(x) = c_1 + 2c_2(x-a) + 3c_2(x-a)^2 + \dots$$

$$\Rightarrow f'(a) = c_1 + 2c_2(a-a) + 3c_2(a-a) + \dots \Rightarrow c_1 = \frac{f'(a)}{2}$$

So, assuming f is n-times differentiable, we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

Taylor series.

Theorem: If $f(x)$ has a power series expansion about $x=a$, i.e.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad x \in (a-R, a+R)$$

then $c_n = \dfrac{f^{(n)}(a)}{n!}$ so that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \quad x \in (a-R, a+R)$$

- $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ is the Taylor series expansion of f at $x=a$.
- Maclaurin series of f is just Taylor series of f at $x=0$.

$a=0$

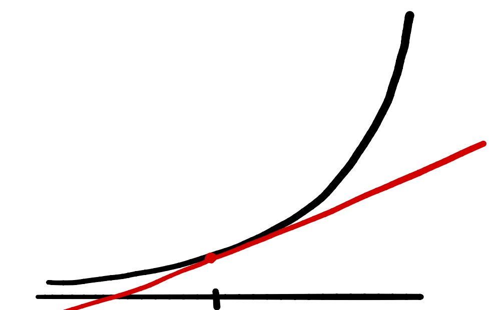
Example:

What is the Taylor series of $f(x)=e^x$ at $x=0$.

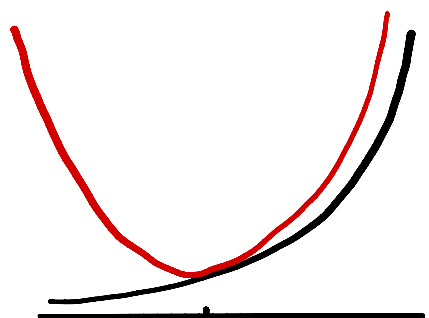
Sol^n: The Taylor series is $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$.

Notice that $\frac{d^n}{dx^n}(e^x)=e^x \Rightarrow f^{(n)}(0)=1$.

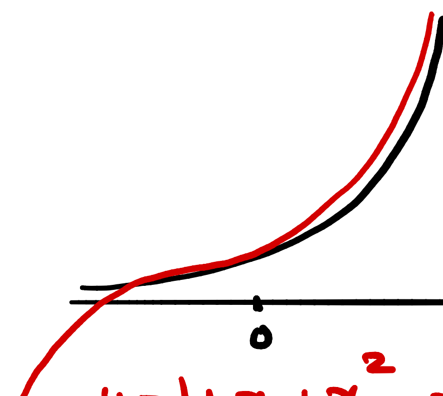
So, $e^x = 1 + 1\cdot x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$



$y=1+x$



$y=1+x+\frac{x^2}{2}$



$y=1+x+\frac{x^2}{2}+\frac{x^3}{3}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots$$

$$f(x) = e^x$$