AVERAGES, CENTER OF MASS, MOMENTS

CONSIDER DATA POINTS $f_1, .., f_N$. THE ARITHMETIC AVERAGE $\bar{f}_{ave}$ IS

$$\bar{f}_{ave} = \frac{1}{N}\left[f_1 + \ldots + f_N \right] = \frac{1}{N}\sum_{i=1}^{N} f_i .$$

OW SUPPOSE WE WANT TO DEFINE THE "AVERAGE" OF A FUNCTION $f(x)$ N $a \le x \le b$. WE PARTITION X-AXIS INTO N EQUAL SEGMENTS $[x_{i-1}, x_i]$ WITH $_0 = a$ AND $x_N = b$ AND $\Delta x = \frac{b-a}{N}$. LET x_i^* BE ANY POINT IN $[x_{i-1}, x_i]$.

HEN THE AVERAGE OF $f(x_1^*), .., f(x_N^*)$ IS SIMPLY

$$\bar{f}_{ave} = \frac{1}{N}\left[f(x_1^*) + \ldots + f(x_N^*)\right] = \frac{1}{N}\sum_{i=1}^{N} f(x_i^*).$$

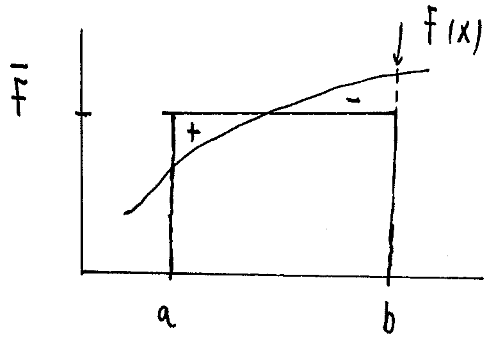
MULTIPLY BY $b-a$ TOP AND BOTTOM AND USE $\Delta x = \frac{b-a}{N}$.

SO
$$\bar{f}_{ave} = \frac{1}{b-a}\sum_{i=1}^{N} f(x_i^*)\,\Delta x .$$

OW LET $N \to \infty$, $\Delta x \to 0$ AND OBSERVING THAT WE HAVE A RIEMANN UM WE DEFINE

$$\bar{f} \equiv \frac{1}{b-a}\int_a^b f(x)\,dx. \qquad (1)$$

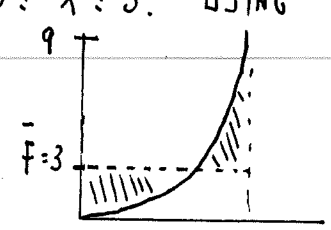
OR A POSITIVE FUNCTION $f(x)$ THIS YIELDS THE GEOMETRIC INTERPRETATION:



AREA OF RECTANGLE $\bar{f}(b-a)$ IS AREA $\int_a^b f(x)\,dx$ SO $-$, $+$ LOBES CANCEL IN AREA.

PROBLEM 1 FIND THE AVERAGE OF $f(x) = x^2$ ON $0 \le x \le 3$. USING (1) WE GET

$$\bar{f} = \frac{1}{3}\int_0^3 x^2\,dx = \frac{1}{9}x^3\Big|_0^3 = 3.$$



SHADED AREAS ARE THE SAME.

PROBLEM 2 LET $v(t)$ BE THE SPEED OF A PARTICLE ON A TIME INTERVAL $0 \le t \le T$. WHAT IS THE AVERAGE SPEED?

SOL'N $V_{ave} = \frac{1}{T-0} \int_0^T v(t)\,dt.$

BUT $v(t) = dx/dt$ SO THAT $V_{ave} = \frac{1}{T} \int_0^T \frac{dx}{dt}\,dt = \frac{1}{T} x(t)\Big|_0^T.$

$$\longrightarrow \quad V_{ave} = \frac{1}{T}\left[x(T) - x(0) \right] = \frac{\text{TOTAL distance}}{\text{TOTAL TIME}}.$$

ROBLEM 3 IF A CUP OF COFFEE HAS TEMPERATURE $95^\circ C$ IN A ROOM WHERE THE TEMPERATURE IS $20^\circ C$, THEN FROM NEWTON'S LAW OF COOLING THE TEMPERATURE OF THE COFFEE AFTER t MINUTES IS

$$T(t) = 20 + 75 e^{-t/50}.$$

WHAT IS THE AVERAGE TEMPERATURE OF THE COFFEE DURING THE FIRST HALF HOUR?

SOLUTION CONVERTING TO MINUTES WE WANT T_{ave} OVER $0 \le t \le 30$.

SO $$T_{ave} = \frac{1}{30} \int_0^{30} T(t)\,dt = \frac{1}{30} \int_0^{30} \left[20 + 75 e^{-t/50} \right] dt$$

$$= \frac{1}{30}\left[20(30) \right] + \frac{75(50)}{30} e^{-t/50} \Big|_0^{30} \Big] = 20 - \frac{75(50)}{30}\left(e^{-30/50} - 1 \right)$$

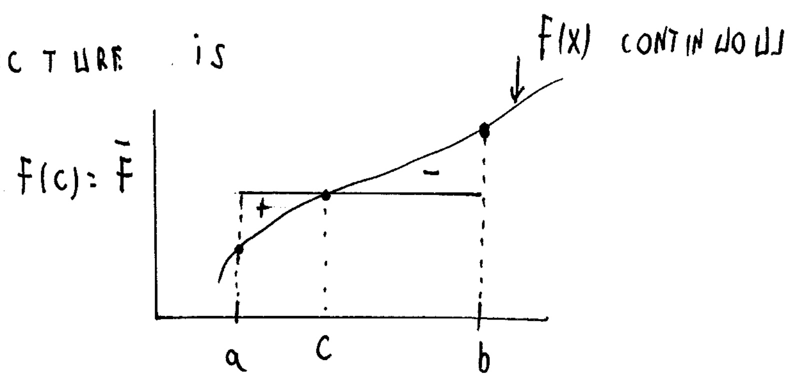
$$= 20 + \frac{(75)(5)}{3}\left(1 - e^{-3/5} \right) = 20 + 125\left(1 - e^{-3/5} \right) \cong 76.4^\circ C.$$

OPTIONAL (MEAN-VALUE THEOREM FOR INTEGRALS)

THEOREM IF $f(x)$ IS CONTINUOUS ON $[a, b]$ THEN THERE IS A NUMBER c IN $[a, b]$ SUCH THAT $\int_a^b f(x)\,dx = f(c)(b-a)$.

INTERPRETATION $f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$, i.e. IF $f(x)$ IS A CONTINUOUS FUNCTION THEN SOMEWHERE IN $[a, b]$ THE FUNCTION WILL TAKE ON ITS AVERAGE VALUE.

THE PICTURE IS



EXAMPLE (OPTIONAL) FIND THE NUMBER c SO THAT THE MVT FOR INTEGRALS IS SATISFIED FOR $f(x) = x^2 + 3x + 2$ ON $1 \le x \le 4$.

SOLUTION $\bar{f} = \frac{1}{4-1}\int_1^4 f(x)\,dx$. WANT $\bar{f} = f(c)$ FOR SOME c.

SO $c^2 + 3c + 2 = \frac{1}{3}\int_1^4 (x^2 + 3x + 2)\,dx = \frac{1}{3}\left(\frac{x^3}{3} + \frac{3}{2}x^2 + 2x\right)\Big|_1^4$.

THUS $3c^2 + 9c + 6 = \frac{1}{3}(4^3 - 1) + \frac{3}{2}(16 - 1) + 6 = \frac{1}{3}(63) + \frac{3(15)}{2} + 6$.

SO $3c^2 + 9c + 6 = 21 + \frac{45}{2} + 6 = \frac{42 + 45 + 12}{2} = \frac{99}{2}$

SO $3c^2 + 9c - \frac{87}{2} = 0 \rightarrow c_{\pm} = \frac{-3 \pm \sqrt{67}}{2}$

ONLY c_+ IS IN $[1, 4]$ SO $c_+ = \frac{-3 + \sqrt{67}}{2}$.