U-substitution.

- Let $f(x) = e^x$. What is $\int f(x)\,dx$.?
- Let $g(x) = x^2$. what is $\int g(x)\,dx$?
- Solve $\int f(g(x))\,dx$.

Chain rule in reverse

$$\frac{d}{dx} F(u(x)) = F'(u(x))\, u'(x)$$

Let $f(x)$ be a continuous function.

Let $u(x)$ be a differentiable function.

Indefinite :

$$\boxed{\int f(u(x))\, u'(x)\, dx = \int f(u)\, du \Big|_{u=u(x)}.}$$

Definite:

$$\boxed{\int_a^b f(u(x))\, u'(x)\, dx = \int f(u)\, du \Big|_{u(a)}^{u(b)}}$$

Strategy for picking $u(x)$.

- **Practice**

- Look for $u(x)$ and $u'(x)$ in the integrand.

  For example: $\int e^{2x} \cos(e^{2x})\, dx$

  $$\frac{d}{dx} e^{2x} = 2e^{2x}. \quad \text{so, pick } u(x) = e^{2x}.$$

- Chain rule requires composition of functions.
  Choose $u(x)$ to be the inner function.

- Choose $u(x)$ to be the complicated argument.

- **Practice**

Example 1

Evaluate: $\displaystyle\int_0^1 \frac{x^2}{(x^3+1)^2}\,dx$

Example

Evaluate $\int \tan(x) \log(\cos x)\, dx$. (Q22 CLP Problem book)

- Choose inner function as $u(x)$.

  So, $\frac{du}{dx} = -\sin(x)$

- Replace $dx \to \frac{du}{-\sin(x)}$

So, $\int \tan(x) \log(\cos(x)) \frac{du}{-\sin(x)}$

$$= \int \frac{\sin(x)}{\cos(x)} \log(u) \frac{1}{-\sin(x)}\, du$$

$$= \int \frac{1}{u} \log(u)\, du$$

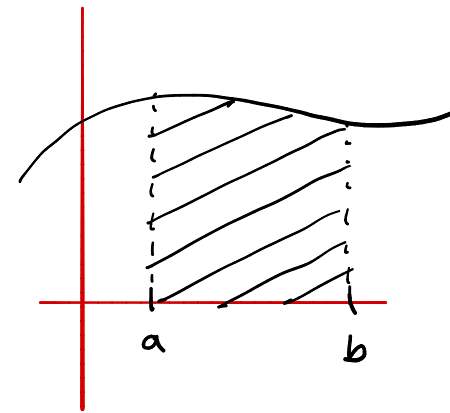
$$= \int \frac{1}{u} \log(u)\, du \Big|_{u=u(x)}$$

Example

Evaluate $\int \tan(x) \log(\cos x) \, dx$.

Area between curves: (CLP 1.5)

The definte integral $\int_a^b f(x)\,dx$ is the area under the curve.

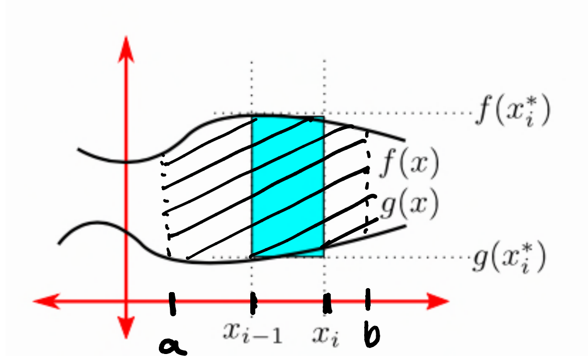


Precisely,

$\int_a^b f(x)$ is area bounded by

- line $x = a$
- line $x = b$
- line $y = 0$
- curve $y = f(x)$.

Area between curves:

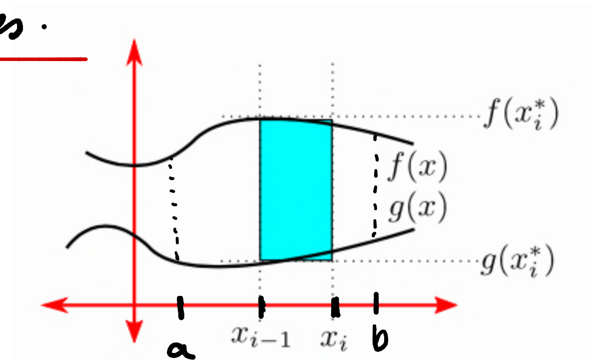


Find the area between two curves $y = f(x)$ and $y = g(x)$.

- Generalization of area below the curve.
  think $g(x) = 0$.

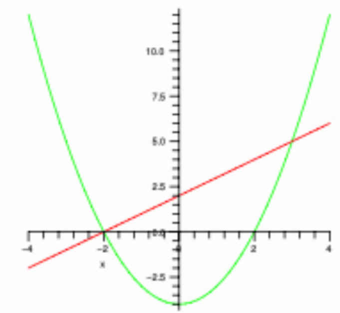
- Setup Riemann sum: Height of subrectangle will depend on $f(x)$ and $g(x)$.

Riemann sum for area between curves.



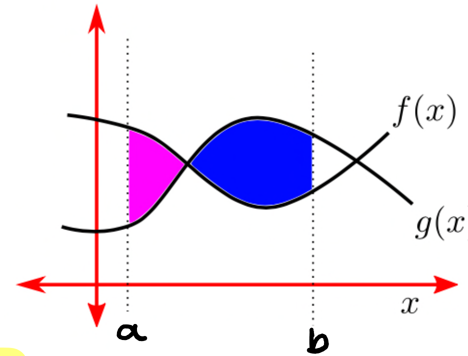
Example

Compute the finite area between the curves $f(x) = x+2$ and $g(x) = x^2 - 4$.



Example contd.

Separate domain of integration.



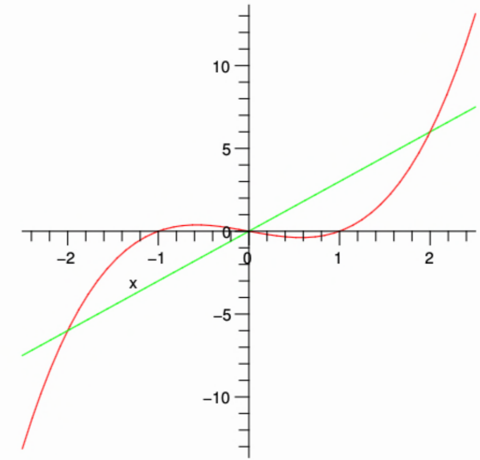
- If $y=f(x)$ and $y=g(x)$ cross, separate domain of integration.
- $f(x)-g(x)$ change sign at intersection point

Ex: Find the finite area bounded by the curves

$$y = x^3 - x \quad \text{and} \quad y = 3x.$$

Ex. solution

o



Interchange x and y.

For some problem, it's easier to integrate with respect to y.

For ex: Find the finite area bounded by $y^2=4x$ and $4x-3y=4$.

Reverse the role of x and y.

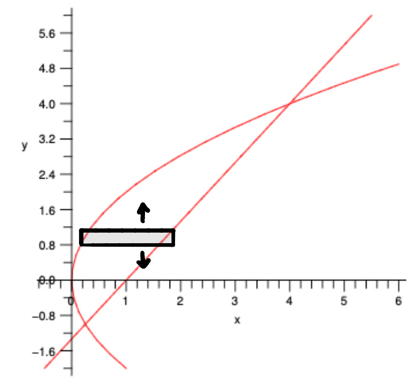
- Find intersection points:

$$y^2 = 4x = 3y+4$$

$$\Rightarrow y^2-3y-4=0$$

$$\Rightarrow y=-1,\ 4.$$

Check



$x = y^2/4$ — left

$x = \frac{1}{4}(3y+4)$ — right

Example contd.

- Riemann sum by splitting interval $[-1, 4]$ in y.

$\Delta y =$

$y_i =$

$$\text{Area} \approx \sum_{i=1}^{n} (\text{right} - \text{left})\, \Delta y$$

$$= \sum_{i=1}^{n} \left(\tfrac{3}{4} y + 1 - \tfrac{y^2}{4} \right) \Delta y$$

- $$A = \int_{-1}^{4} \left(1 + \frac{3}{4} y - \frac{y^2}{4} \right) dy$$

$$= y + \frac{3}{8} y^2 - \frac{y^3}{12} \Big|_{-1}^{4}$$

$$= \frac{125}{24}$$