u- Substitution

Goals:

- Integrate using u- substitution.
- Chain rule.

Integration operation is linear

Recall :

$$\int_a^b \left(f(x) + g(x)\right) dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

$$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx.$$

- integration plays nicely with + and scalar multiplication.

- integrations like $\int e^{-x^2} dx$, $\int 9 \sin^8 x \cos x\, dx$ are difficult even though they are well-behaved.

Chain rule

Recall: If $y = F(u)$ and $u = u(x)$, then

$$\frac{d}{dx} F(u(x)) = \frac{dF}{du} \cdot \frac{du}{dx}$$

Ex: Differentiate $f(x) = \sin^9 x$

ans: using chain rule. let $\left.\begin{array}{l} F(u) = u^9 \\ u(x) = \sin(x). \end{array}\right\} \Rightarrow f(x) = F(u(x))$

$$\frac{df}{dx} = \frac{dF}{du} \frac{du}{dx} = \underbrace{\frac{d}{du}(u^9)}_{\substack{F'(u(x)) \\ = 9u^8}} \underbrace{\frac{du}{dx}}_{\substack{u'(x) \\ = \cos(x)}} = 9\sin^8 x \cos x$$

Chain rule (contd)

$u(x) = \sin x, \quad F(u) = u^9.$

So, $\sin^9(x)$ is anti derivative of $9\sin^8 x \cos x$.

$$\Rightarrow \int \underbrace{9\sin^8 x}_{F'(u(x))} \underbrace{\cos x}_{u'(x)} \, dx = \underbrace{\sin^9 x}_{F(u(x))} + C.$$

$$\Rightarrow \int \left(9u^8 \Big|_{u=\sin x} \cdot \cos(x) \right) dx = u^9 \Big|_{u=\sin(x)} + C$$

$$\Rightarrow \int \left(9u^8 \Big|_{u=\sin(x)} \cos(x) \right) dx = \left(\int 9u^8 \, du \right) \Bigg|_{u=\sin x}.$$

u - substitution

Thm (The substitution rule - CLP 1.4.2)

Let f be an integrable function and let $u = u(x)$ be a differentiable function. Then

$$\int f(u(x)) \frac{du}{dx} \, dx = \int f(u) \, du \Big|_{u = u(x)}$$

Theorem proof idea

$$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du \Big|_{u=u(x)}$$

- Let $F(u)$ be antiderivative of $f(u)$

  $\Rightarrow \quad \int f(u)\, du = F(u) + C$

- Assume $u = u(x)$. Evaluate at $u = u(x)$:

  $$\int f(u)\, du \Big|_{u=u(x)} = F(u(x)) + C$$

- Differentiate both sides: $f(u(x))\, u'(x) = F'(u(x))\, u'(x)$
- So, $F(u(x))$ is antiderivative of $f(u(x))\, u'(x)$
- So, $\int f(u(x))\, u'(x)\, dx = F(u(x)) + C$.

Main idea

Integrating $\int f(u)\,du$ might be easier than integrating $\int f(u(x))\,u'(x)\,dx$.

$u(x) = \cos(x)$

$u'(x) = -\sin(x)$.

$\int -9 \sin^8 x \cos(x) \cdot u'(x)\,dx$

Example

- $\int 9 \sin^8 x \cos x \, dx \quad -①$

If $u = \sin(x) \Rightarrow u'(x) = \cos(x)$

So, $① = \int 9u^8\,du \Big|_{u=\sin(x)} = u^9 + c \Big|_{u=\sin(x)} = \sin^9 x + c$

$\int 9u^8\Big|_{u=\sin(x)} \cdot u'(x)\,dx =$

Example (contd)

$$\int f(u(x))\cdot u'(x)\,dx = \int f(u)\,du \Big|_{u=u(x)}$$

- $\int 2x\, e^{x^2}\, dx$.

let $u(x) = x^2$

$\frac{du}{dx} = 2x \Rightarrow \int u'(x)\cdot e^{u(x)}\, dx = \int e^u\, du \Big|_{u=x^2} = e^u + c \Big|_{u=x^2}$

$= e^{x^2} + C$

- $\int 3x^2 (x^3+4)\, dx$

$= \int (3x^5 + 12x^2)\, dx$

$= \int 3x^5 dx + \int 12x^2 dx$

$= \frac{3x^6}{6} + C_1 + \frac{12x^3}{3} + C_2$

$= \frac{x^6}{2} + \frac{x^3}{4} + C$

let $u(x) = x^3+4 \Rightarrow \frac{du}{dx} = 3x^2$

$\Rightarrow \int u'(x)\, u(x)\, dx$

$= \int u\, du \Big|_{u=x^3+4}$

$= \frac{u^2}{2} + C \Big|_{u=x^3+4} = \frac{(x^3+4)^2}{2} + C$

Example (contd)

$u = \sin(x)$ $\qquad$ $\int u'(x) f(u(x))\,dx$

$\frac{du}{dx} = \cos(x)$

- $\int \cot x \, dx$

$= \int \frac{\cos(x)}{\sin(x)} \, dx$

$= \int u'(x) \cdot (u(x))^{-1} \, dx = \int u^{-1} \, du \Big|_{u=\sin(x)}$

$= \ln|u| + c \Big|_{u=\sin(x)} = \ln|\sin(x)| + c$

- $\int \sqrt{2x+1} \, dx$

$u = 2x+1$

$\frac{du}{dx} = 2$

$\frac{1}{2} \int 2\sqrt{2x+1} \, dx$

$= \frac{1}{2} \int u'(x) \sqrt{u(x)} \, dx = \frac{1}{2} \int \sqrt{u} \, du \Big|_{u=2x+1}$

$u^{1/2}$

$= \frac{1}{\cancel{2}} \frac{\cancel{2} u^{3/2}}{3} + c \Big|_{2x+1}$

$= \frac{(2x+1)^{3/2}}{3} + C.$

Linear Substitution

$\boxed{\int f(u)\,du = F(u) + C}$

<u>Thm</u> Let $F(u)$ be any antiderivative of $f(u)$ and let a, b be constants. Then

$$\int f(ax+b)\,dx = \frac{1}{a} F(ax+b) + C.$$

- Choose $u(x) = ax+b$. Then $\underbrace{\frac{du}{dx} = u'(x)}$ $= a$
- Substitute $dx \to \frac{1}{u'(x)}\,du$ $\qquad dx = \frac{1}{u'}\,du$
- integrate $\frac{1}{a}\int f(u)\,du = \frac{1}{a} F(ax+b) + C.$

Linear Substitution (Contd)

More generally,

$$\int f(u(x))\, u'(x)\, dx \qquad\qquad \frac{du}{dx} = u'(x)$$

By substituting $dx \to \frac{1}{u'(x)} du$, we get

$$\int f(u(x))\, \cancel{u'(x)}\, \frac{1}{\cancel{u'(x)}}\, du$$

$$= \int f(u)\, du$$

Note: $dx \neq \frac{1}{u'(x)} du$.

Revisiting example

- $\int \sqrt{2x+1}\, dx$

$u(x) = 2x + 1$

$$\int \sqrt{2x+1}\, dx = \frac{1}{2} \int \sqrt{u}\, du$$

$$= \frac{1}{2} \frac{2u^{3/2}}{3} + C$$

$$= \frac{u^{3/2}}{3} + C$$

Substitution for definite integrals.

Thm (CLP 1.4.6) Let f be an integrable function and let $u = u(x)$ be a differentiable function, then

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

Q: Let $F(x)$ be antiderivative $f(x)$. i.e $\frac{d}{dx} F(x) = f(x)$.

What is a antiderivative of $f(u(x))\, u'(x)$

A: The antiderivative of $f(u(x))\, u'(x)$ is $F(u(x))$

By FTC, $\displaystyle\int_a^b f(u(x))\, u'(x)\, dx = F(u(x))\Big|_a^b = F(u)\Big|_{u=u(a)}^{u=u(b)}$

$$= F(u(b)) - F(u(a))$$

Example:

Evaluate $\int_0^1 \frac{u^3}{u^2+1}\, du$

$$x(u) = u^2+1 \Rightarrow u^2 = x(u)-1$$

$$\frac{dx}{du} = 2u$$

$$du \to \frac{1}{2u}\, dx$$

$$\int \frac{u^3}{u^2+1}\, du = \int \frac{u^3}{x} \frac{1}{2u}\, dx$$

$$= \frac{1}{2}\int \frac{u^2}{x}\, dx$$

$$= \frac{1}{2}\int_{x(0)}^{x(1)} \frac{x-1}{x}\, dx$$

$$= \frac{1}{2}\int_1^2 \left(1 - \frac{1}{x}\right) dx$$

$$= \frac{1}{2}\left(x - \ln|x|\right)\Big|_1^2$$

$$= \frac{1}{2}\left[\left(2-\ln(2)\right) - \left(1 - 0\right)\right]$$

$$= \frac{1}{2}\left[1 - \ln(2)\right]$$