1 Integration

1.4 Substitution

Up until now we have been doing very simple integrals. Basically we have just been using a lookup table from the derivatives we know, and using the fact that integrals play nicely with addition, subtraction and multiplication by constants

$$\begin{aligned}\int_a^b (f(x)+g(x))\,\mathrm{d}x &= \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x \\ \int_a^b (f(x)-g(x))\,\mathrm{d}x &= \int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x \\ \int_a^b cf(x)\mathrm{d}x &= \int_a^b f(x)\mathrm{d}x\end{aligned}$$

This was how we started out with derivatives too. So now we need some tools for building more complicated integrals. When we did this for derivatives we built up the product and chain (and quotient) rules

$$\begin{aligned}\frac{\mathrm{d}fg}{\mathrm{d}x} &= f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x} \\ \frac{\mathrm{d}}{\mathrm{d}x}f(u(x)) &= \frac{\mathrm{d}f}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\end{aligned}$$

These will be our starting points for finding some nice integration rules. In particular we find integration rules by anti-differentiating them.

Unfortunately things are definitely more difficult for integration — for example, the (perfectly well defined and behaved) indefinite integral

$$\int e^{-x^2}\mathrm{d}x$$

cannot be written down as any finite combination of the standard functions we know. (the easiest way to express it is as an infinite sum — one of the reasons we look at series in this course). This is not some mathematical oddity — this function turns up all the time in (for example) statistics — it is a key part of the Gaussian distribution — which appears all over the place due to something called the "central limit theorem". The interested student should do some search-engining.

What does the chain rule tell us:

Theorem (The chain rule — CLP Theorem 1.4.1)**.** *If $y = F(u)$ and $u = u(x)$, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}F(u(x)) = F'(u(x))\cdot u'(x)$$

We will get our "integrate the composition of functions" by integrating both sides of the chain rule (with respect to x)

Applying this is not as simple as applying the chain rule — we need to have the integrand in just the right shape to make it work. Recognising this structure (or massaging things into shape) comes with practice. Before we state it, let us practice "the chain rule in reverse".

$$\int 9\sin^8 x \cos x \mathrm{d}x = \sin^9 x + c$$

How did we find this — the chain rule in reverse. We can verify things by differentiating the right-hand side. Let us do so

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\sin^9 x &= \left(\frac{\mathrm{d}}{\mathrm{d}u}u^9\right)\cdot\frac{\mathrm{d}u}{\mathrm{d}x} & u = \sin x \\ &= 9u^8 \cdot \cos x \\ &= \underbrace{9\sin^8 x}_{F'(u(x))} \cdot \underbrace{\cos x}_{\frac{\mathrm{d}u}{\mathrm{d}x}} \end{aligned}$$

The key to making this work more generally is to recognise the form of the chain rule — $F'(u(x))u'(x)$ — in the integrand. Practice practice practice.

So — more generally — consider a function $f(u)$ with antiderivative $F(u)$. Then we know that

$$\int f(u)\mathrm{d}u = F(u) + C$$

But now, take this equation and substitute $u = u(x)$ into it

$$\int f(u)\mathrm{d}u\bigg|_{u=u(x)} = F(u(x)) + C$$

Focus on the RHS — it is a function of x and so we can (with the chain rule) differentiate it

$$\frac{\mathrm{d}}{\mathrm{d}x}F(u(x)) = F'(u(x)) \cdot u'(x)$$

This tells us that $F(u(x))$ is an antiderivative of $F'(u(x)) \cdot u'(x) = f(u(x)) \cdot u'(x)$. Hence

$$\int f(u)\mathrm{d}u\bigg|_{u=u(x)} = F(u(x)) + C = \int f(u(x)) \cdot u'(x)\mathrm{d}x.$$

Theorem (The substitution rule — CLP theorem 1.4.2)**.** *Let f be an integrable function and let $u = u(x)$ be a differentiable function, then*

$$\int f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int f(u)\mathrm{d}u\bigg|_{u=u(x)}$$

In order to apply this rule successfully we have to massage our integrand into the form of the LHS — this takes practice — but there are also some simple tricks that might help. The key is to try to pick a $u = u(x)$.

- Factor the integrand and choose one of the factors to be $u'(x)$ — for this to work you must be able to easily antidifferentiate the chosen factor.

$$\int 9\sin^8 x \cos x \mathrm{d}x \qquad\qquad u(x) = \cos x?$$

- Look for a factor in the integrand that is a function (like sine or exp) with an argument that is more complicated than just "x" — choose that argument to be $u(x)$

$$\int 2xe^{x^2}\mathrm{d}x \qquad\qquad u(x) = x^2$$

In both cases it helps to be able to think about "If I choose this then what happens next?"

Simple illustrations of the above. First up — choose $u'(x)$:

$$\int 3x^2(x^3+4)\mathrm{d}x$$

- Factors into 2 nice pieces — we could choose either $u' = (3x^2)$ or $u' = (x^3+4)$.

$$\begin{aligned} u' &= x^3+4 & &\Rightarrow u(x) = \frac{1}{4}x^4 + 2x^2 & &??? \\ u' &= 3x^2 & &\Rightarrow u(x) = x^3 & &\checkmark \end{aligned}$$

- Clearly we should choose $u' = 3x^2, u = x^3$ since it makes things easier. With this choice we have

$$\begin{aligned} \int 3x^2 \cdot (x^3+4)\mathrm{d}x &= \int u'(x)\cdot(u(x)+4)\mathrm{d}x \\ &= \int (u+4)\mathrm{d}u \\ &= \frac{u^2}{2} + 4u + C \\ &= \frac{1}{2}x^6 + 4x^3 + C \end{aligned}$$

- Notice we could also have chosen $u = x^3 + 4$ and arrived at the same answer:

$$\begin{aligned} \int 3x^2 \cdot (x^3+4)\mathrm{d}x &= \int u(x)\cdot u'(x)\mathrm{d}x \\ &= \int u\mathrm{d}u \\ &= \frac{u^2}{2} + C = \frac{1}{2}(x^3+4)^2 + C \\ &= \frac{1}{2}x^6 + 4x^3 + 8 + C \end{aligned}$$

  (since C is an arbitrary constant these answers are the same).

And — of course — you can (and should?) always check by differentiating your answer.

And the second idea — choose $u(x)$:

$$\int \underbrace{4x^3}\underbrace{\sin x^4}\mathrm{d}x$$

- Since we have x^4 sitting inside sine this really points in this direction. Otherwise we might try choosing x^3?

$$\begin{array}{lll} u(x) = x^4 & u'(x) = 4x^3 & \checkmark \\ u(x) = 4x^3 & u'(x) = 12x^2 & ??? \end{array}$$

- Lets go with the former $u = x^4, u' = 4x^3$:

$$\begin{aligned} \int \underbrace{4x^3}\underbrace{\sin x^4}\mathrm{d}x &= \int u'(x) \cdot \sin(u(x))\mathrm{d}x \\ &= \int \sin u \mathrm{d}u \\ &= -\cos u + c = -\cos x^4 + c \end{aligned}$$

- What if we tried the other "chunk" — ie $u = x^3$? Well the other chunk would be $\sin u^{4/3}$ — ugly! It is always a good idea if things are getting ugly, to take a step back and check if you made the right choice.

And — of course — you can (and should?) always check by differentiating your answer.

A trig flavoured example

$$\int \cot x \mathrm{d}x = \int \frac{\cos x}{\sin x}\mathrm{d}x$$

- Again, there are 2 obvious choices for u — $u = \sin x$ and $u = \cos x$.
- If we choose $u = \sin x$ then $u' = \cos x$ and things are looking good.

$$\begin{aligned} \int \cot x \mathrm{d}x &= \int \frac{\cos x}{\sin x}\mathrm{d}x \\ &= \int \frac{\cos x}{u}\mathrm{d}x & u = \sin x \qquad u' = \cos x \\ &= \int \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \\ &= \int \frac{1}{u}\mathrm{d}u \\ &= \log|u| + c = \log|\sin x| + c \end{aligned}$$

- On the other hand if we choose $u = \cos x$, so $u' = \sin x$ — so both terms are present in the integrand. But the integral becomes $\int \frac{u}{u'}\mathrm{d}x$ — which is not of the form of our rule.

Another example — here there is only a single factor:

$$\int \sqrt{2x+1}\mathrm{d}x$$

So the obvious thing to try is setting $u(x) = 2x + 1$. This requires $u'(x) = 2$ — but no such factor is there. But consider the following:

$$\begin{aligned}
\int \sqrt{2x+1}\mathrm{d}x &= \int \sqrt{u}\mathrm{d}x & u = 2x+1 \qquad u'(x) = 2 \\
&= \int \sqrt{u} \cdot \left(2 \cdot \frac{1}{2}\right) \mathrm{d}x \\
&= \int \frac{1}{2}\sqrt{u} \cdot u'(x)\mathrm{d}x \\
&= \int \frac{1}{2}u^{1/2}\mathrm{d}u = \frac{1}{2}\frac{u^{3/2}}{3/2} + c \\
&= \frac{1}{3}(2x+1)^{3/2} + c
\end{aligned}$$

We can repeat this reasoning for any linear substitution:

Theorem (CLP Theorem 1.4.10)**.** *Let $F(u)$ be any antiderivative of $f(u)$ and let a, b be constants. Then*

$$\int f(ax+b)\mathrm{d}x = \frac{1}{a}F(ax+b) + C.$$

This is quite specialised — it only works for a linear substitution $u = ax + b$ (we had a sneak-peek of this last week).

An easier way to view what we just did is to use the following scheme (which makes more general substitutions easier):

- Choose $u(x)$ and then compute $u'(x)$
- Substitute $\mathrm{d}x \mapsto \frac{1}{u'(x)}\mathrm{d}u$.

In so doing, we are really just doing:

$$\int f(u(x))u'(x)\mathrm{d}x = \int f(u(x))u'(x) \cdot \frac{1}{u'(x)}\mathrm{d}u = \int f(u)\mathrm{d}u$$

which is precisely our substitution rule.

We should be a little careful with the above we are not saying that $\mathrm{d}x$ equals $\frac{1}{u'(x)}\mathrm{d}u$ — since $\mathrm{d}x$ does not have a meaning outside of the integral (you can give it a meaning, but it is outside the scope of the course). That being said, this way of looking at things does give you a useful mnemonic:

$$\int f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int f(u(x))\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int f(u)\mathrm{d}u$$

"Cancel the $\mathrm{d}x$" — but remember the derivative $\frac{\mathrm{d}f}{\mathrm{d}u}$ is not a fraction! It is an operation you do on a function defined by a limit. You are not dividing $\mathrm{d}f$ by $\mathrm{d}u$ — this is just a memory aid.

Redoing things:

$$\int \sqrt{2x+1}\,\mathrm{d}x \qquad\qquad \text{set } u = 2x+1 \text{ so } u' = 2 \text{ and } \mathrm{d}x = \frac{1}{2}\mathrm{d}u$$

$$\begin{aligned}
&= \int \sqrt{u} \cdot \frac{1}{2}\mathrm{d}u \\
&= \frac{1}{2}\frac{u^{3/2}}{3/2} + c \\
&= \frac{1}{3}(2x+1)^{3/2} + c
\end{aligned}$$

We'll do some more, but first we should go back to definite integrals:

Theorem (Substitution rule for definite integrals — CLP theorem 1.4.6)**.** *Let f be an integrable function and let $u = u(x)$ be a differentiable function, then*

$$\int_a^b f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\mathrm{d}u$$

For example

$$\int_0^1 \frac{3x}{(x^2+1)^2}\mathrm{d}x$$

Looking at this, it makes sense to set $u = x^2+1$. This gives $u' = 2x$ and so we will sub $\mathrm{d}x = \frac{1}{2x}\mathrm{d}u$. Along the way we have to take care of the terminals — $u(0) = 1$ and $u(1) = 2$:

$$\begin{aligned}
\int_0^1 \frac{3x}{(x^2+1)^2}\mathrm{d}x &= \int_0^1 \frac{3x}{u^2}\mathrm{d}x && \text{terminals still wrt } x \\
&= \int_{u(0)}^{u(1)} u^{-2}\frac{3x}{2x}\mathrm{d}u && \text{terminals now wrt } u \\
&= \int_1^2 u^{-2}\frac{3}{2}\mathrm{d}u \\
&= \frac{3}{2}[-1/u]_1^2 = \frac{3}{2}(-1/2+1) = 3/4.
\end{aligned}$$

Another example

$$\int_0^\pi \sin(x)\exp(\cos x)\mathrm{d}x$$

Again

- Pretty clear we should choose $u = \cos x$.
- This gives us $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$, $\mathrm{d}x = -\frac{1}{\sin x}\mathrm{d}u$.
- Mapping the terminals gives $u(0) = 1, u(\pi) = -1$.

- Now we put it together

$$
\begin{aligned}
\int_0^\pi \sin(x)\exp(\cos x)\mathrm{d}x &= \int_0^\pi e^u \cdot \sin(x)\mathrm{d}x \\
&= \int_{u(0)}^{u(\pi)} e^u \cdot \frac{\sin x}{-\sin x}\mathrm{d}u && \text{careful of sign errors} \\
&= -\int_1^{-1} \exp(u)\mathrm{d}u && \text{and bad puns} \\
&= -\int_1^{-1} e^u \mathrm{d}u = -[e^u]_1^{-1} \\
&= -e^{-1} + e^1
\end{aligned}
$$

Practice practice practice!

Before we get to the integral equivalent of the product rule, we will do some not-quite-applications. These will give us practice integrating functions and getting used to messing around with converting a problem to a Riemann sum (don't worry — only easy ones) and then into an integral.

1.5 Areas between curves

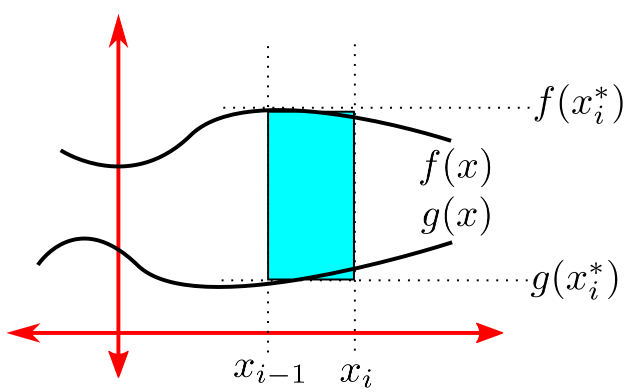
To practice our definite integrals, lets go back to the interpretation of the definite integral as the area under a curve.

$$\int_a^b f(x)\mathrm{d}x = \text{ area under the curve}$$

By this we mean the area bounded by the lines $x = a, x = b, y = 0$ and the curve $y = f(x)$.

Perhaps the first and easiest generalisation is to consider the area between two curves $f(x)$ and $g(x)$. Indeed we can then think of the orginial interpretation as just the specialisation $g(x) = 0$.

So we started with Riemann sums — lets do that again. Draw a picture and write down the appropriate sum



Take the interval $[a, b]$ and divide it up into n subintervals of width Δx. Choose our x_i and the x_i^*.

Previously the height of the rectangle was $f(x_i^*)$, but now it is the difference between the heights of the curves — $f(x_i^*) - g(x_i^*)$. And so the Riemann sum is

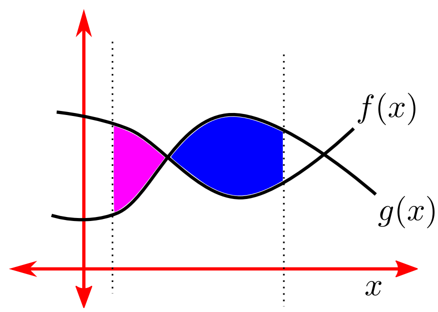
$$\sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x = \sum_{i=1}^{n} h(x_i^*)\Delta x$$

where $h(x) = f(x) - g(x)$.

So really there is nothing new going on here. This is just the definite integral of h. Hence the area is simply

$$\begin{aligned} \text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} h(x_i^*)\Delta x \\ &= \int_a^b h(x)\mathrm{d}x = \int_a^b (f(x) - g(x))\mathrm{d}x \end{aligned}$$

Of course we still have to be careful, because this is the signed-area!

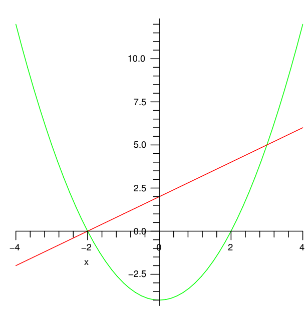


So we need to be careful about where the curves intersect — it gets a little tedious, but it is a good way to practice your algebra, solving equations, and integrating things.

Lets look at some examples.

Compute the finite area between the curves $f(x) = x + 2$ and $g(x) = x^2 - 4$.

- Notice that there is an infinite region trapped between the curves — extending left and right to infinity. The question specifies the finite region. Generally with these questions they will want you to find the finite region between the curves — otherwise the answer is very dull and doesn't really test your integrating skills.
- First step is to plot the graphs — work out intersection points and which is on top.



- They intersect at $x = -2, +3$.

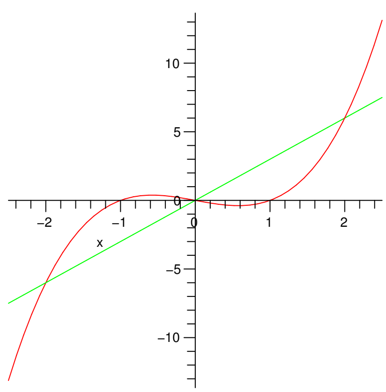
$$\begin{aligned} x + 2 &= x^2 - 4 \\ 0 &= x^2 - x - 6 = (x-3)(x+2) \end{aligned}$$

- In that region $x+2$ is above $x^2 - 4$ — you can test the functions at a point in the interval to be sure.
- Hence the area will be given by the integral

$$\begin{aligned} A &= \int_{-2}^{3} ((x+2) - (x^2 - 4))\mathrm{d}x \\ &= \int_{-2}^{3} (6 + x - x^2)\mathrm{d}x \\ &= [6x + x^2/2 - x^3/3]_{-2}^{3} \\ &= (18 + 9/2 - 9) - (-12 + 2 + 8/3) = 27/2 + 22/3 \\ &= 81/6 + 44/6 = 125/6 \end{aligned}$$

Another example — maybe you guys do this one. Find the total area (not the signed area) between the curves $y = x^3 - x$ and $y = 3x$.

- The first curve is a cubic $y = x(x+1)(x-1)$.
- They intersect at $x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2)$ — $x = 0, \pm 2$.
- Sketch them



Again you can check which is on top in which region by testing them at a point in each region. AND you can make sure they really do intersect at those point by testing the y-values at the points.

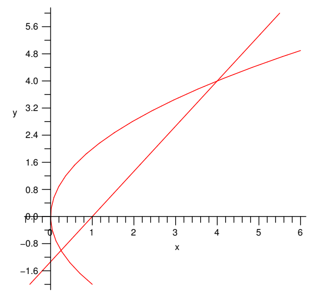
- So the total area is

$$\begin{aligned} A &= \int_{-2}^{0} ((x^3 - x) - 3x)\mathrm{d}x + \int_{0}^{2} (3x - (x^3 - x))\mathrm{d}x \\ &= \int_{-2}^{0} (x^3 - 4x)\mathrm{d}x + \int_{0}^{2} (4x - x^3)\mathrm{d}x \\ &= [x^4/4 - 2x^2]_{-2}^{0} + [2x^2 - x^4/4]_{0}^{2} \\ &= 0 - (4 - 8) + (8 - 4) - 0 = 8 \end{aligned}$$

You can do the similar problems but with $x \leftrightarrow y$ interchanged (there are some in the CLP problem set). A simple example of this is...

Find the area between the curves $y^2 = 4x$ and $4x - 3y = 4$.

- Draw a picture



- So we do this in the same way — the roles of x and y are reversed. Remember — they are just variables.
- In our Riemann sum, we split an interval in y and have y_i^* and Δy etc.
- So what is the range of y? The curves intersect at

$$\begin{aligned} y^2 &= 4x = 4 + 3y \\ y^2 - 3y - 4 &= 0 \\ (y-4)(y+1) &= 0 \qquad\qquad y = -1, +4 \end{aligned}$$

- So the area between the curves is

$$\begin{aligned} A &= \int_{-1}^{4} (x_r - r_l)\mathrm{d}y = \int_{-1}^{4} ((1 + 3y/4) - y^2/4)\mathrm{d}y \\ &= [y + 3y^2/8 - y^3/12]_{-1}^{4} \\ &= (4 + 48/8 - 64/12) - (-1 + 3/8 + 1/12) = (96 + 144 - 128)/24 - (-24 + 9 + 2)/24 \\ &= (112 + 13)/24 = 125/24 \end{aligned}$$

You could also do this *carefully* by vertical strips. You have to be careful where the curves intersect.

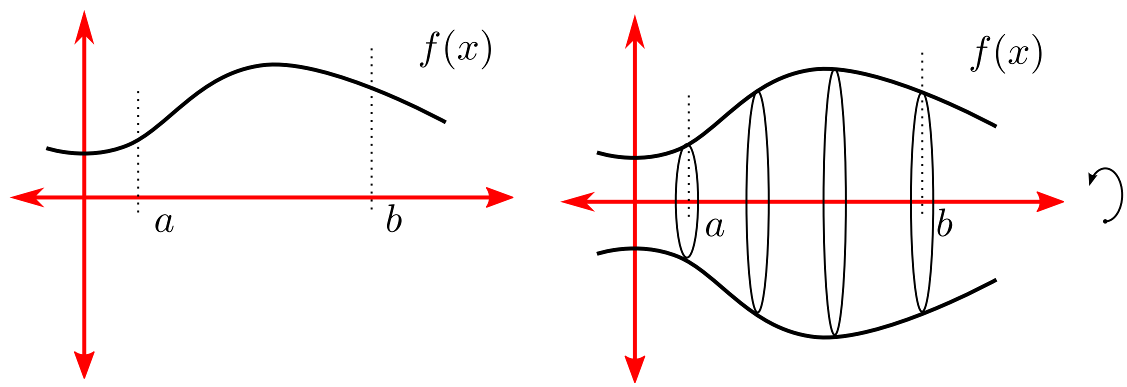
- To the left of $(1/4, -1)$ the vertical strip runs between $y^2 = 4x$ and itself.
- To the right of $(1/4, -1)$ the vertical strip runs between $y^2 = 4x$ and $4x - 3y = 4$.

Painful.

1.6 Volumes

Another good integration work-out is to compute volumes of some simple solids. You all (I hope) learned the formulas for the volumes of spheres, cones etc. These can be computed without integration (they were done by the ancient greeks — predating calculus by many many years), but they are much easier with calculus (see CLP examples 1.6.1 and 1.6.2 and we'll do them below).

For general solids computing volumes can be quite involved and might entail multivariate integration (typically a topic of calculus-3 course). However there are some families of solids which can be treated using the integration techniques we have already learned. Consider the graph of some function $y = f(x)$.

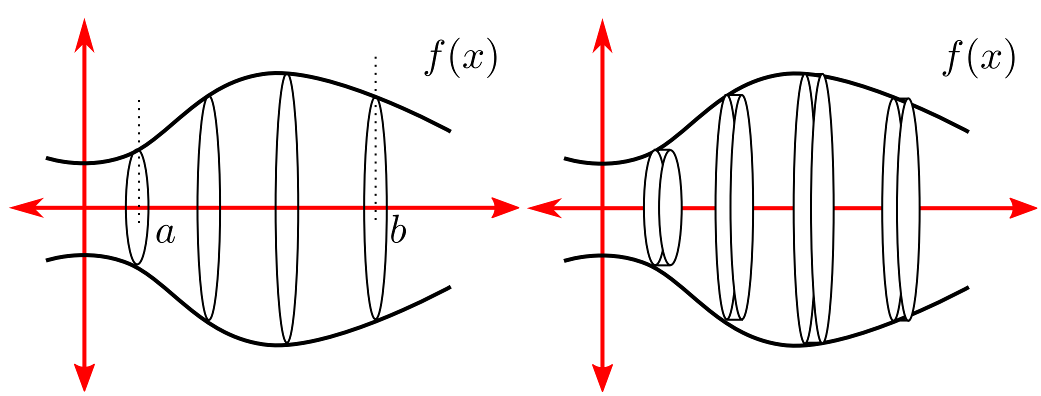


Now think of rotating it about the line $y = 0$ — this defines a 3-D surface. How do we find volume enclosed by this surface and the planes $x = a$ and $x = b$?

Again our starting point is Riemann sums. Split the interval $[a, b]$ into segments $[x_{i-1}, x_i]$ (just as before). Again we ask — how much does each of these intervals contribute to the total.

Before each segment contributed a rectangle to the area $f(x_i^*)\Delta x$, but now each segment contributes a slice of the volume. What does this slice look like?

Each slice is a cylinder of radius $f(x_i^*)$ and width Δx.



The volume is just the volume of a cylinder

$$\text{vol of cylinder } = \pi r^2 w = \pi f(x_i^*)^2 \Delta x$$

So our Riemann sum is

$$\begin{aligned} V &\approx \sum_{i=1}^{n} \pi f(x_i^*)^2 \Delta x \\ &= \sum_{i=1}^{n} g(x_i^*) \Delta x & g(x) = \pi f(x)^2 \end{aligned}$$

And as $n \to \infty$ this Riemann sum becomes an integral and we are left with

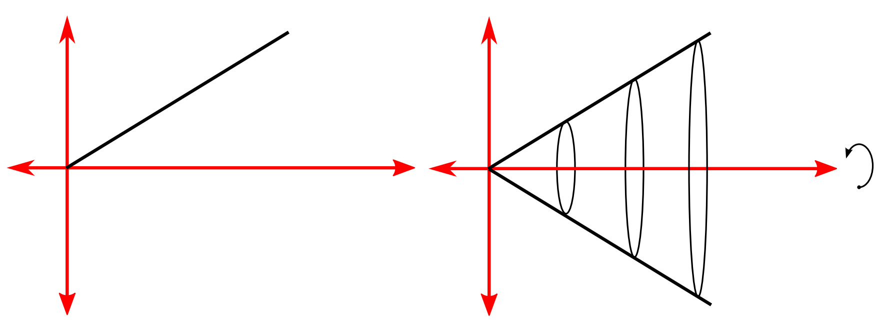
$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*) \Delta x & = \int_a^b g(x)\mathrm{d}x \\ &= \pi \int_a^b f(x)^2 \mathrm{d}x \end{aligned}$$

If we consider a more general shape - the contribution from any given "slice" to the Riemann sum will be the cross-section area times its width

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*) \Delta x \\ &= \int_a^b A(x)\mathrm{d}x \end{aligned}$$

Example — volume of a cone.

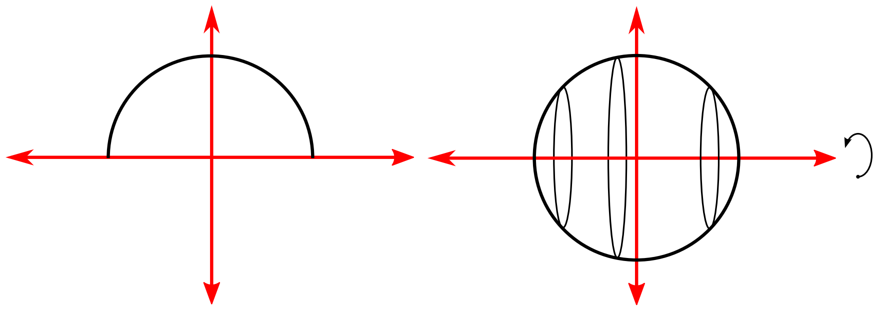
- Consider the line $y = \frac{x}{2}$ on the inteval $[0, 6]$.
- Rotate it about the x-axis to make a cone



- So the volume is

$$\begin{aligned} V &= \int_0^6 \pi f(x)^2 \mathrm{d}x \\ &= \pi \int_0^6 \frac{x^2}{4} \mathrm{d}x \\ &= \left[\frac{\pi x^3}{12}\right]_0^6 = \frac{\pi 6^3}{12} = 18\pi \end{aligned}$$

Example — volume of a sphere. Draw a semi-circle of radius r on the interval $[-r, r]$. The required function is This function is $y = \sqrt{r^2 - x^2}$ (because $x^2 + y^2 = r^2$). Rotating this curve around the x-axis gives a sphere.



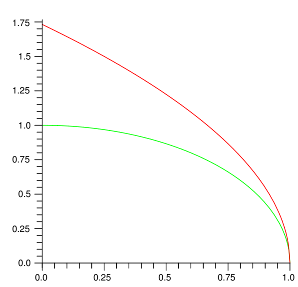
So the volume is given by

$$
\begin{aligned}
V &= \int_{-r}^{r} \pi f(x)^2 \mathrm{d}x \\
&= \pi \int_{-r}^{r} (r^2 - x^2) \mathrm{d}x \\
&= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \\
&= \pi \left((r^3 - r^3/3) - (-r^3 + r^3/3) \right) = \frac{4}{3}\pi r^3
\end{aligned}
$$

What about the volume of a "bowl"

- Upper curve defines outside of bowl $y = f(x) = \sqrt{3 - 3x}$
- Lower curve defines inside of bowl $y = g(x) = \sqrt{1 - x^2}$

So we plot things carefully and have a think.

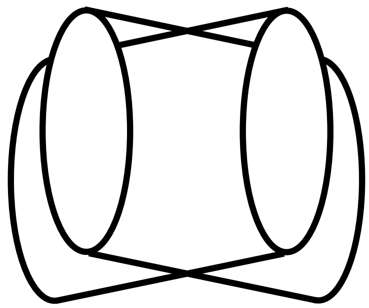


The volume of the bowl is given by

$$
\begin{aligned}
V &= \text{vol of outer} - \text{vol of hole} \\
&= \int_0^1 \pi f(x)^2 \mathrm{d}x - \int_0^1 \pi g(x)^2 \mathrm{d}x
\end{aligned}
$$

Just like area between curves.

What about a harder example? (only if time) Find the volume of the intersection of 2 perpendicular cylinders of unit radius.



- Cylinder 1 is all points such that $x^2 + z^2 \leq 1$
- Cylinder 2 is all points such that $y^2 + z^2 \leq 1$

Find volume by slicing perpendicular to the z-axis. Its hard to picture it, so we need to let the algebra guide us. Consider the slice at some fixed height z

- For cylinder 1, with z fixed, we have $x^2 \leq 1 - z^2$ or $-\sqrt{1-z^2} \leq x \leq \sqrt{1-z^2}$.
- For cylinder 2, with z fixed, we have $y^2 \leq 1 - z^2$ or $-\sqrt{1-z^2} \leq y \leq \sqrt{1-z^2}$.

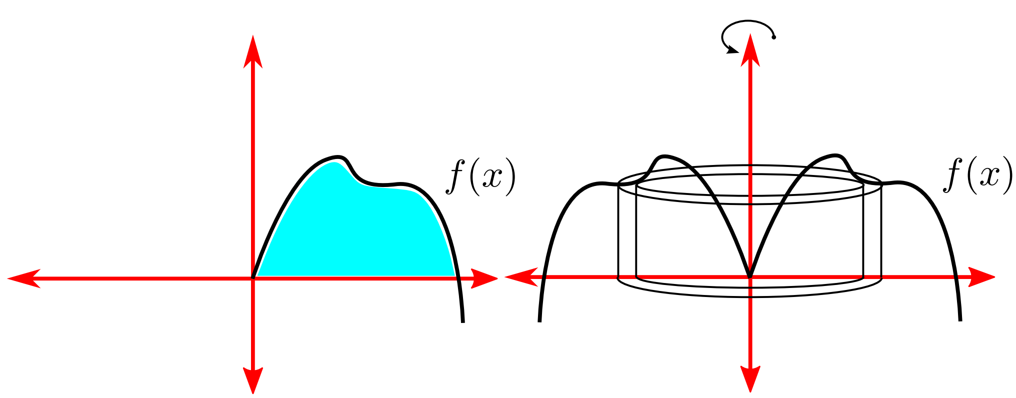
Hence at a fixed height z, the "slice" is just a square of side length $2\sqrt{1-z^2}$. The area of this square is $4(1-z^2)$ and the the volume of the slide is $4(1-z^2)\Delta z$. Riemann sums to definite integrals again:

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} 4(1 - z_i^{*2})\Delta z \\ &= \int_{-1}^{z} 4(1-z^2)\mathrm{d}z \\ &= [4z - 4z^3/3]_{-1}^{1} \\ &= (4 - 4/3) - (-4 + 4/3) = 16/3 \end{aligned}$$

oof!

Volumes by cylindrical shells — CLP 1.6.1

Another way one can compute volume is by rotating around a different axis. Before we rotated about the x-axis, but we would instead rotate about the y-axis.



Now when we slice up the interval, we get cylindrical shells. Take a look at the text — but we don't do this in this course. (though it used to be part of the course).

1.7 Integration by parts

Now that we've done some practice, its back to learning more techniques of integration. Much like the substitution rule was really just the chain rule in reverse, "integration by parts" is the product rule reverse.

- The product rule says

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f(x)\frac{\mathrm{d}g}{\mathrm{d}x} + g(x)\frac{\mathrm{d}f}{\mathrm{d}x}$$

- Now undo this by integrating

$$\int \frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x))\mathrm{d}x = f(x)g(x) = \int f(x)\frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x + \int g(x)\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x$$

- Rearrange this a bit to get the "integration by parts" rule

Theorem (Integration by parts — CLP theorem 1.7.2)**.**

$$\int f(x)\frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x = f(x)g(x) - \int g(x)\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x$$

Let $f(x) = u(x)$ and $g(x) = v(x)$ then the above becomes

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int v(x)u'(x)\mathrm{d}x$$

now using the substitution rule, this can be written in very compact notation

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$$

When you look at this, it is not immediately obvious why this is going to be any help with anything at all. You take one integral and turn it into another, and along the way we have to compute $g(x)$ from $\frac{\mathrm{d}g}{\mathrm{d}x}$. ie — we turn one integral into another by doing an integral.

But — it will help us if (1) it is easy to compute $g(x)$ from $\frac{\mathrm{d}g}{\mathrm{d}x}$ and (2) if the new integral is simpler than the original. This gives us a way to choose what should be $f \equiv u$ and what should $g' \equiv v'$. Consider the integral

$$\int xe^x\mathrm{d}x$$

- There are two factors in the integrand — one is going to be u and one is going to be v'.
- There are 2 obvious choices of how to split this up.
  - $u = x$ and $v' = e^x$, and then
  - $u = e^x$ and $v' = x$.
- Let us start with the choice — then this gives us $u' = 1$ and $v = e^x$.

- Sub into the formula

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int v(x)u'(x)\mathrm{d}x$$
$$\int xe^x\mathrm{d}x = xe^x - \underbrace{\int e^x \cdot 1\mathrm{d}x}_{\text{easier!}}$$
$$= xe^x - e^x + c$$

- Let us now try it the other way — $u' = e^x$ and $v = x^2/2$

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int v(x)u'(x)\mathrm{d}x$$
$$\int xe^x\mathrm{d}x = x^2e^x/2 - \underbrace{\int e^x \cdot x^2/2\mathrm{d}x}_{\text{uglier!}}$$

- The second way seems to make things harder rather than easier — this is your guide!

How should we write this up so the reader knows what is going on?

$$\int xe^x\mathrm{d}x$$

Use integration by parts. Set $u = x, v' = e^x$. Hence $u' = 1$ and $v = e^x$. Then:

$$= xe^x - \int e^x\mathrm{d}x$$
$$= xe^x - e^x + C.$$

Note that when we antidifferentiate $v' = e^x$ it doesn't matter if we use $v = e^x + C$ — that additive constant factor will cancel out — this is shown in the text.

Integration by parts is very useful

- to eliminate factors of x from an integrand like xe^x,
- to eliminate a $\log x$ from an integrand — using the fact that $\frac{\mathrm{d}}{\mathrm{d}x}\log x = \frac{1}{x}$,
- to eliminate inverse trig functions like $\arctan x$ — since $\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$.

The first is the most frequently seen (in undergrad calculus courses).

You should now do:

$$\int x\sin x\mathrm{d}x.$$

- Choose either $u = x, v' = \sin x$ — giving $u' = 1$ and $v = -\cos x$.

- Use the IBP formula:

$$\begin{aligned}\int u\mathrm{d}v &= uv - \int v\mathrm{d}u \\ &= -x\cos x - \int 1\cdot(-\cos x)\mathrm{d}x \\ &= -x\cos x + \sin x + C.\end{aligned}$$

Who used words in their solution?

An example in every calculus text

$$\int \log x\mathrm{d}x = ?$$

We only have a single function. How do we apply the integration by parts formula? Sometimes generalising a problem can help us — this is a good example of that. Since the derivative of $\log x$ is a power of x, it suggests starting with

$$\int x^n \log x\mathrm{d}x$$

- Choose $f = x^n$ and $g' = \log x$ then $f' = nx^{n-1}$ and $g = ?$. Doesnt work.
- Choose $f = \log x$ and $g' = x^n$ then $f' = 1/x$ and $g = x^{n+1}/(n+1)$

$$\begin{aligned}\int x^n \log x\mathrm{d}x &= \frac{x^{n+1}\log x}{n+1} - \int \frac{x^n}{n+1}\mathrm{d}x \\ &= \frac{x^{n+1}\log x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c\end{aligned}$$

So when $n = 0$, and we want $\int \log x$ we have

$$\int \log x\mathrm{d}x = \int 1\cdot\log x\mathrm{d}x$$

with $f = \log x$ and $g' = 1$.

$$\begin{aligned}\int \log x\mathrm{d}x &= x\log x - \int x/x\mathrm{d}x \\ &= x\log x - x + c\end{aligned}$$

Obviously the formula breaks when $n = -1$ — what then?

$$\int \log(x)/x\mathrm{d}x$$

This is just a substitution integral — $u = \log x$, so $\mathrm{d}u = \mathrm{d}x/x$.

$$\int \log(x)/x\mathrm{d}x = \int u\mathrm{d}u = u^2/2 + c = (\log|x|)^2 + c$$

Another good example along the same lines: Compute the integral

$$\int \arctan x\mathrm{d}x.$$

- Again — there is only 1 factor — suggests that we should choose $v' = 1$. This makes $u = \arctan x$, and $v = x, u' = \frac{1}{1+x^2}$.
- Putting things into the IBP formula:

$$\int \arctan x \mathrm{d}x. = x \arctan x - \int \frac{x}{1+x^2} \mathrm{d}x.$$

- This new integral is just a substitution integral with $w = 1 + x^2$ (not reusing "u")and so we set $\mathrm{d}x = \frac{1}{2x}\mathrm{d}w$:

$$\begin{aligned} \int \frac{x}{1+x^2} \mathrm{d}x &= \int \frac{1}{2w} \mathrm{d}w \\ &= \frac{1}{2} \log |w| + C \\ &= \frac{1}{2} \log |1+x^2| + C \end{aligned}$$

- Put things back together:

$$\int \arctan x \mathrm{d}x = x \arctan x - \frac{1}{2} \log |1+x^2| + C.$$

- Notice that we can simplify this a little further since $x^2 + 1 \geq 1$, we can write

$$\int \arctan x \mathrm{d}x = x \arctan x - \frac{1}{2} \log(1+x^2) + C.$$

What about $\int t^2 e^t \mathrm{d}t$?

- Apply integration by parts once
- Choose $f = t^2, g' = e^t$, so $f' = 2t$ and $g = e^t$

$$\int t^2 e^t \mathrm{d}t = t^2 e^t - \int 2te^t \mathrm{d}t$$

- Reduces integral of $t^2 e^t$ to integral of te^t.
- Apply integration by parts again to this integral.
- Choose $f = t, g' = e^t$ so $f' = 1$ and $g = e^t$:

$$\int te^t \mathrm{d}t = te^t - \int e^t \mathrm{d}t \qquad\qquad = te^t - e^t.$$

- So our original integral is

$$\begin{aligned} \int t^2 e^t \mathrm{d}t &= t^2 e^t - \int 2te^t \mathrm{d}t \\ &= t^2 e^t - 2(te^t - e^t) + C \\ &= e^t (t^2 - 2t + 2) + C \end{aligned}$$

Another example $\sin(t)e^t$

- Choose $f = \sin(t)$, $g' = e^t$, so $f' = \cos t$ and $g = e^t$

$$\int \sin(t)e^t \mathrm{d}t = \sin(t)e^t - \int \cos(t)e^t \mathrm{d}t$$

- Doesnt look much better — apply again
- Now $f = \cos(t)$ and $g' = e^t$, so $f' = -\sin(t)$ and $g = e^t$

$$\int \cos(t)e^t \mathrm{d}t = e^t \cos(t) + \int \sin(t)e^t \mathrm{d}t$$

- So we have expressed $\int \sin(t)e^t \mathrm{d}t$ in terms of itself?

$$\begin{aligned}\int \sin(t)e^t \mathrm{d}t &= \sin(t)e^t - \int \cos(t)e^t \mathrm{d}t \\ &= \sin(t)e^t - e^t \cos(t) - \int \sin(t)e^t \mathrm{d}t\end{aligned}$$

  So bring the $\int blah \mathrm{d}t$ all over to the left

$$\begin{aligned}2\int \sin(t)e^t \mathrm{d}t &= e^t(\sin(t) - \cos(t)) \\ \int \sin(t)e^t \mathrm{d}t &= \frac{1}{2}e^t(\sin(t) - \cos(t)) + c\end{aligned}$$

Another good one $\int \cos(x)\log(\sin(x))\mathrm{d}x$

- Choose $f = \cos(x)$ and $g' = \log(\sin(x))$ — urgh.
- Choose $f = \log(\sin(x))$ and $g' = \cos(x)$ — $f' = \cos(x)/\sin(x)$ and $g = \sin(x)$.

$$\begin{aligned}\int \cos(x)\log(\sin(x))\mathrm{d}x &= \sin(x)\log(\sin(x)) - \int \frac{\cos(x)}{\sin(x)}\sin(x)\mathrm{d}x \\ &= \sin(x)\log(\sin(x)) - \int \cos(x)\mathrm{d}x \\ &= \sin(x)\log(\sin(x)) - \sin(x) + C\end{aligned}$$