

Differential equation review.

Solve the differential equation $\frac{dy}{dx} = -xy^3$, $y(0) = -\frac{1}{4}$

$$\frac{1}{y^3} \frac{dy}{dx} = -x \quad \text{separating } x \text{ & } y.$$

$$\Rightarrow \int \frac{1}{y^3} \frac{dy}{dx} dx = \int -x dx. \quad \text{integrate both sides wrt } x.$$

$$\Rightarrow \int \frac{1}{y^3} dy = -\frac{x^2}{2} + C \quad \text{used u-substitution on LHS.}$$

$$\Rightarrow -\frac{y^{-2}}{2} = -\frac{x^2}{2} + C, \quad \text{absorbed constant into one.}$$

$$\Rightarrow \frac{1}{y^2} = x^2 + C, \quad \text{constant is arbitrary.}$$

$$\Rightarrow y^2 = \frac{1}{x^2 + C}.$$

From initial value, $y(0) = -\frac{1}{4}$, we get $| y^2 = \frac{1}{x^2+c}$

$$\left(-\frac{1}{4}\right)^2 = \frac{1}{0+c} \Rightarrow c = 16$$

so, $y^2 = \frac{1}{x^2+16}$

$$\Rightarrow y = \frac{1}{\sqrt{x^2+16}} \quad \text{or}$$

$$y = -\frac{1}{\sqrt{x^2+16}}$$

Sequences.

A sequence is a list of infinitely many numbers with a specified order. We write a sequence as

$$\{a_1, a_2, a_3, \dots, a_n, \dots\} \text{ or } \{a_i\}_{i=1}^{\infty} \text{ or } \{a_n\}$$

we call a_i the i^{th} element of the sequence. $\rightarrow N = \{0, 1, 2, 3, \dots\}$

Typically, for $n = \text{positive number}$, we can write
 $\{a_n = f(n)\}_{n=1}^{\infty}$. $f: N \rightarrow R$, $f(n) = a_n$.

examples: $\{a_n = \frac{1}{n}\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\} = \left\{\frac{1}{n}\right\}$

$$\{a_n = \frac{(-1)^n}{n}\}_{n=1}^{\infty} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\right\} = \left\{\frac{(-1)^n}{n}\right\}$$

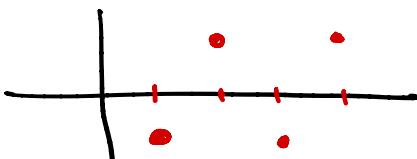
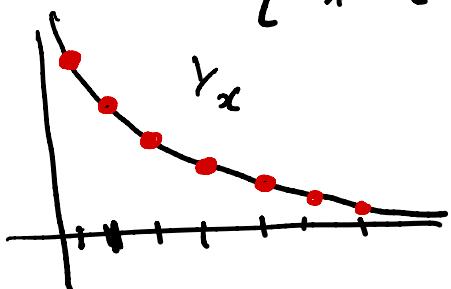
$$\{a_n = e^{-n}\}_{n=1}^{\infty} = \{e^{-1}, e^{-2}, e^{-3}, \dots\}.$$

limit of a sequence (not so rigorous).

Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to the limit L if a_n approaches L as $n \rightarrow \infty$. We write $\lim_{n \rightarrow \infty} a_n = L$ or $a_n \rightarrow L$ as $n \rightarrow \infty$ or $a_n \rightarrow L$.

Example: $\left\{ a_n = \frac{1}{n} \right\}_{n=1}^{\infty}$ converges to zero since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

$\left\{ a_n = (-1)^n \right\}_{n=1}^{\infty}$ diverges since a_n oscillates between -1 and 1 for all n .

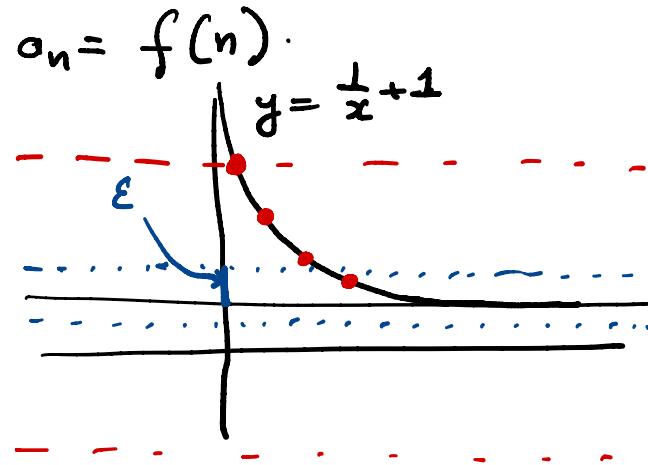


limit of a sequence (rigorous)

$$a_n = \left\{ 2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, \dots \right\}$$

$\rightarrow 1 ?$

Q: Fix $\epsilon > 0$. Is it possible to find an element of sequence a_n such that $|a_n - L| < \epsilon$?



Defn: A sequence $\{a_n\}$ converges to L if for every $\epsilon > 0$, there exist n_0 s.t.

$$|a_n - L| < \epsilon.$$

if the limit exists then the sequence converges to L
otherwise $\{a_n\}$ diverges.

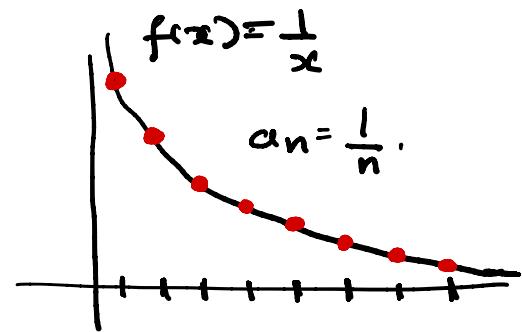
limit of sequences

Theorem: If $\lim_{x \rightarrow \infty} f(x) = L$ and $a_n = f(n)$ for $n = 1, 2, 3, \dots$

then $\lim_{n \rightarrow \infty} a_n = L$.

Example: Discuss convergence of

$$\left\{ \frac{n}{4n+5} \right\}_{n=1}^{\infty} \quad \text{and} \quad f \left\{ \frac{2n^2+3}{n^2+n} \right\}_{n=1}^{\infty}$$



- Define $f(x) = \frac{x}{4x+5} = \frac{x/x}{(4x+5)/x} = \frac{1}{4+\frac{5}{x}}$

$$\lim_{x \rightarrow \infty} f(x) = \frac{1}{4}$$

$$\Rightarrow \{a_n\} = \left\{ \frac{n}{4n+5} \right\} \rightarrow \frac{1}{4}$$

$$\begin{aligned} f(x) &= \frac{2x^2+3}{x^2+x} \\ &= \frac{2x^2+3}{x^2} \cdot \frac{1}{(x^2+x)/x^2} \end{aligned}$$

Limit laws.

Thm Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences with $a_n \rightarrow A$ and $b_n \rightarrow B$. Further let c be a constant. Then.

- o $\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = A + B$
- o $\lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n = A - B$.
- o $\lim_{n \rightarrow \infty} c a_n = c \cdot A$
- o $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$ provided $B \neq 0$.
- o $\lim_{n \rightarrow \infty} (a_n)^p = A^p$, provided $a_n > 0$ and $p > 0$.

Example.

Let's use these rules to compute limit of sequences.

Compute $\lim_{n \rightarrow \infty} \frac{3n}{2n+7}$

$$= \lim_{n \rightarrow \infty} \frac{\frac{3n}{n}}{\frac{2n+7}{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{3}{(2+\frac{7}{n})}$$

$$= \lim_{n \rightarrow \infty} 3 / \lim_{n \rightarrow \infty} (2 + \frac{7}{n})$$

$$= 3 / (2 + \lim_{n \rightarrow \infty} \frac{7}{n}) = 3/2$$

Example

Now, you compute $\lim_{n \rightarrow \infty} \frac{3n^2 + 4}{5n^2 + n + 7}$.

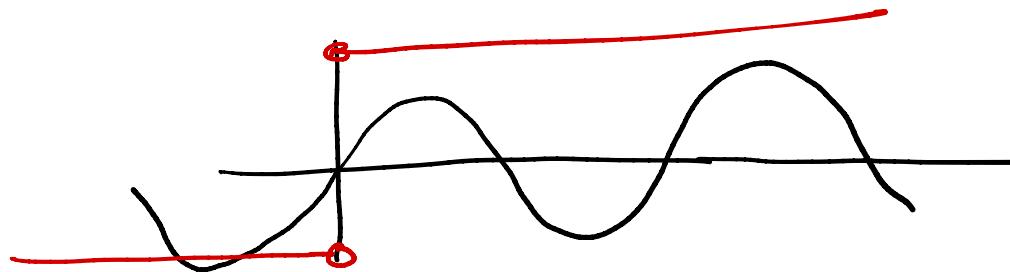
$$\lim_{n \rightarrow \infty} \frac{(3n^2 + 4)/n^2}{(5n^2 + n + 7)/n^2} = \dots = \frac{3}{5}$$

limit of sequence

What about $\lim_{n \rightarrow \infty} \sin(\pi/n)$?

we know $\pi/n \rightarrow 0$. Does $\sin(\pi/n) \rightarrow \sin(0)$?

Thm: If $a_n \rightarrow L$ and $f(x)$ is continuous at $x=L$ then $f(a_n) \rightarrow f(L)$.



Squeeze theorem.

Some sequences are very "messy" and may be challenging to analyze. We can squeeze it between two sequences.

Let π_n be the n^{th} decimal digit of $\pi = 3.14159265\dots$

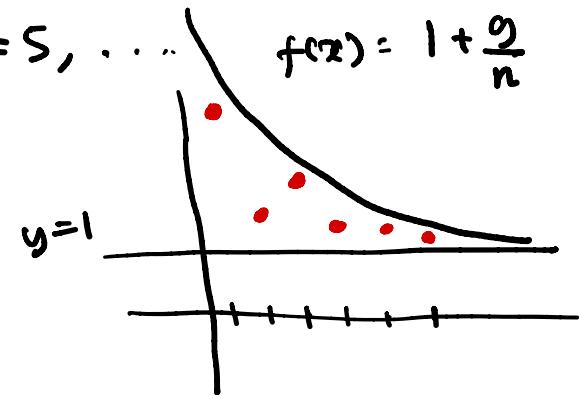
so, $\pi_1 = 1, \pi_2 = 4, \pi_3 = 1, \pi_4 = 5, \dots$

compute $\lim_{n \rightarrow \infty} \left(1 + \frac{\pi_n}{n} \right)$

$$1 \leq 1 + \frac{\pi_n}{n} \leq 1 + \frac{9}{n}$$

$\downarrow \text{as } n \rightarrow \infty$ $\downarrow \text{also.}$ $\downarrow \text{as } n \rightarrow \infty$

$$1 \leq 1 + \frac{9}{n} \leq 1$$



$$\frac{\pi_n}{n} \leq \frac{9}{n}$$

Squeeze theorem.

Theorem If $a_n \leq b_n \leq c_n$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$

then $\lim_{n \rightarrow \infty} b_n = L$.
 $b_n \rightarrow \infty$

Example.

- o Compute $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$.

$$n! = 1 \cdot 2 \cdot 3 \cdots n \leq n \cdot n \cdots n$$
$$n^n = n \cdot n \cdot n \cdots n$$

- o Let $a_n \leq b_n \leq c_n$ with $a_n \rightarrow L$ and $c_n \rightarrow L + 1$.

Is b_n convergent?

- o Given that $|a_n| \rightarrow 0$, does a_n converge?

If yes, what is $\lim_{n \rightarrow \infty} a_n$?

