Warm-up

Solve: (a) $\underbrace{\int_{-1}^{1} \sqrt{1-x^2}\, dx}_{\frac{\pi}{2}}$.

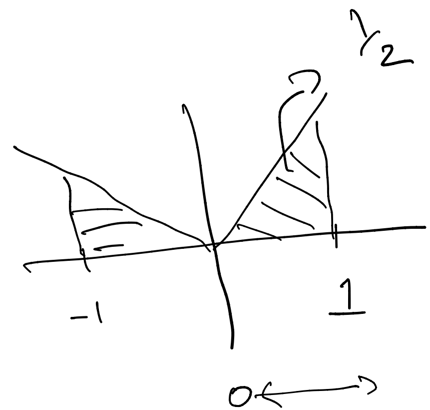
$f(x) = \sqrt{1-x^2}$

semi circle of radius 1

centred at $(0,0)$

(b) $\int_{-1}^{1} |x|\, dx$.

$= 1$



Riemann sum and definite integral. (also called Riemann Integral)

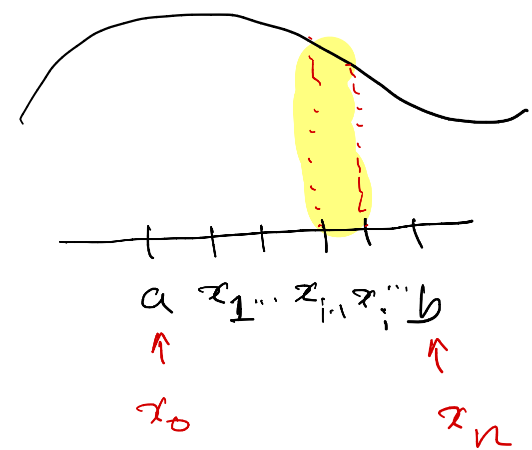
Q. How is Riemann sum and definite integral related?

Let f be a function from $\mathbb{R} \to \mathbb{R}$. The Rieman sum of f on $[a,b]$ is:

partition $[a,b]$: $a < x_1 < x_2 \cdots < x_{i-1} < x_i < \cdots < x_n = b$

$$x_i^* \in [x_{i-1}, x_i]$$

$$R_n = \sum_{i=1}^{n} \underbrace{f(x_i^*)} \Delta x$$



Definite Integral

Definite integral is limit of Riemann sum, if it exists.

Definite integral of f on $[a,b]$ is

$$\underbrace{\int_a^b f(x)\,dx}_{\text{area.}} = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{approx. area.}}$$

notation

$$\int f(x,y)\,dx\,dy$$

Warm-up

Let $f:[a,b]\to\mathbb{R}$.

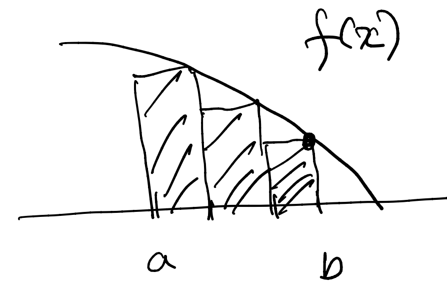
Let R_n be the Riemann sum of f on $[a,b]$ using right endpoint.

Let L_n be the Riemann sum of f on $[a,b]$ using left end-point.

Q. State a condition on f such that

$f(x)=c$

① $\int_a^b f(x)\,dx \geq R_n$



- $f'(x)\leq 0$ for all $x\in[a,b]$
- $f(x)\leq f(y)$ for all $x\geq y$
- $f(x)=c$ $\forall$ $x\in[a,b]$

② $\int_a^b f(x)\,dx \geq L_n$

- $f(x)\geq f(y)$ for all $x\geq y$

$$\int_a^b f(x)\,dx \leq R_n$$

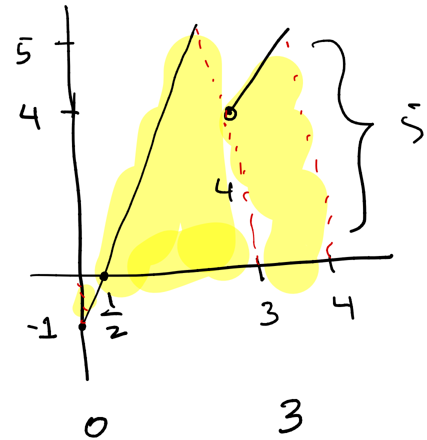
$$\int_a^b f(x)\,dx \leq L_n$$

Integrability of function.

Q. T/F: Integrable functions are continous?

Is $g(t) = \begin{cases} 2t-1 & \text{if } t<3 \\ t+1 & \text{if } t\geq 3 \end{cases}$ integrable on $[0, 4]$?

If yes evaluate $\int_0^4 g(t)\,dt$.



$$\int_0^4 g(t)\,dt = \int_0^3 g(t)\,dt + \int_3^4 g(t)\,dt$$

$$= \frac{9}{2}$$

$$\int_0^3 g(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

$[x_{i-1}, x_i]$

$\left[\frac{3(i-1)}{n}, \frac{3i}{n}\right]$

- $\Delta x = (3-0)/n = 3/n$, $x_i = 3i/n \Rightarrow$
- $R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum \left(2\left(\frac{3i}{n}\right) - 1\right)\frac{3}{n}$

Integrability of function (Contd)

$$R_n = \sum_{i=1}^{n} \left(\frac{2(3i)}{n} - 1 \right) \frac{3}{n} = \sum_{i=1}^{n} \frac{18i}{n^2} - \sum_{i=1}^{n} \frac{3}{n}$$

$$= \frac{18}{n^2} \sum_{i=1}^{n} i - \frac{3}{n} \sum_{i=1}^{n} 1$$

$$= \frac{18}{n^2} \frac{n(n+1)}{2} - 3$$

$$= 9 + \frac{18n}{n^2} - 3$$

$$\rightarrow 6 \text{ as } n \to \infty$$

$$\int_0^4 g(t)\, dt = 6 + \frac{9}{2} = \frac{21}{2}$$

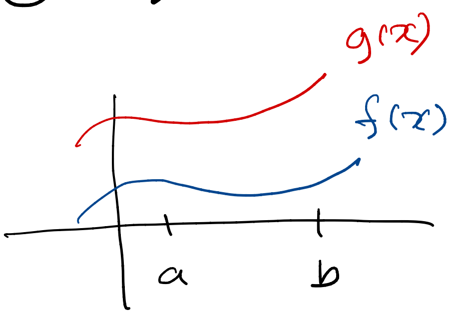
Some Properties of definite integral

Suppose f and g are integrable on $[a, b]$.

① If $f(x) \geq 0$ for all $x \in [a, b]$ then $\int_a^b f(x)\,dx \geq 0$

Remark: $\int_a^b f(x)\,dx$ is signed area.

② If $f(x) \leq g(x)$ for all $x \in [a, b]$ then



$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

$$\int_a^b (f(x) - g(x))\,dx \leq 0$$

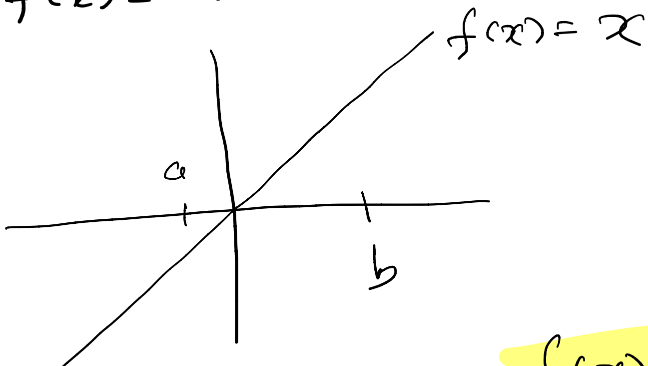
Some Properties of definite integral (contd)

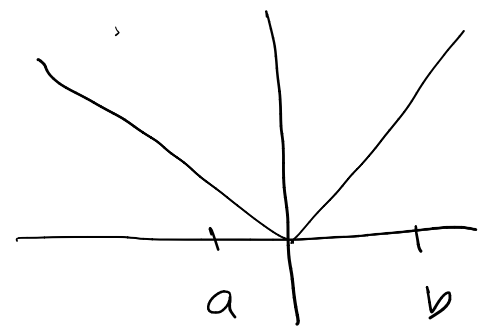
Suppose f and g are integrable on $[a, b]$.

③ $$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$ (triangle inequality)

$$|a+b| \leq |a| + |b|$$

$f(x) = x$





$|f(x)| = |x|$

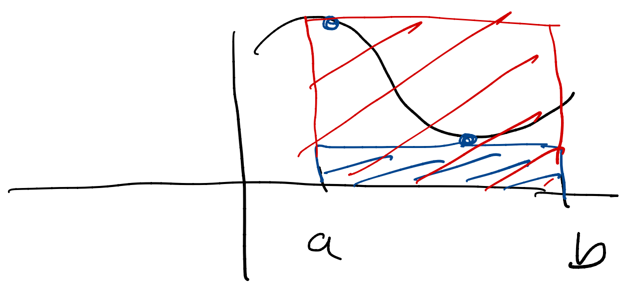
$f(x) \leq |f(x)|$

Some Properties of definite integral (contd)

Suppose f and g are integrable on $[a, b]$.

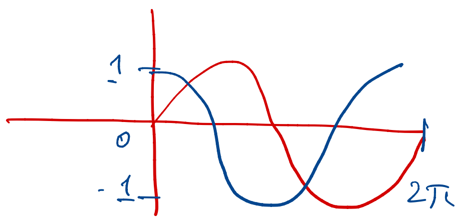
(4) Let M be the maximum of f on $[a,b]$.
Let m be the minimum of f on $[a,b]$.

Then.
$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$



Q: Find a lower bound and upper bound for

$$-1 \cdot 2\pi \leq \int_0^{2\pi} \sin(x)\cos(x)\,dx \leq 1 \cdot 2\pi$$



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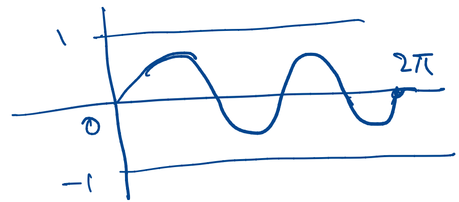
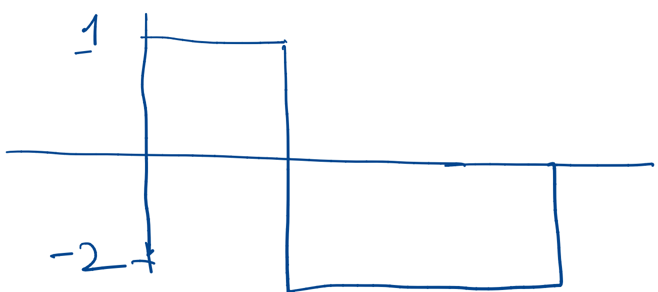
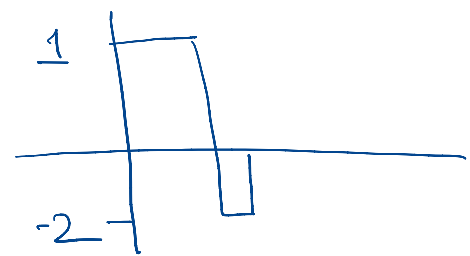


Q: T/F: If $|m| \geq |M|$, then $\int_a^b f(x)\,dx \leq 0$.





Linearity of definite integral.

ⓐ $$\int_a^b (f(x) + c g(x))\, dx = \int_a^b f(x)\, dx + c \int_a^b g(x)\, dx.$$

polynomials are easy to integrate. i.e $\int_a^b x^n dx$ is easy.

Let $p_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ for some $a_i \in \mathbb{R}$ $i = 0, 1, \dots, n$.

Thm (Wierstrass) For any continuous function $f(x)$, $\exists$ exists a polynomial $p_n(x)$ such that

$$p_n(x) \approx f(x)$$

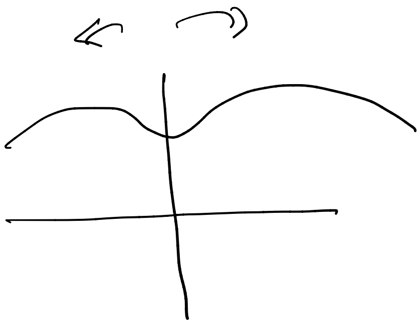
$$\int f(x)\, dx \approx \int p_n(x)\, dx$$

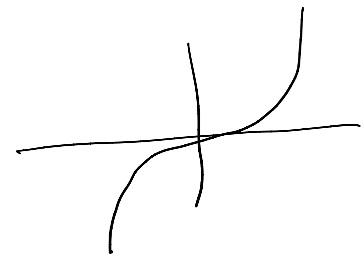
Symmetric (and anti-symmetric) integral.

Solve $\int_{-20}^{20} 2x^5+3x^3-x \, dx$.

even function: Let $f: \mathbb{R} \to \mathbb{R}$. f is even if

$f(x) = f(-x)$ for all x.

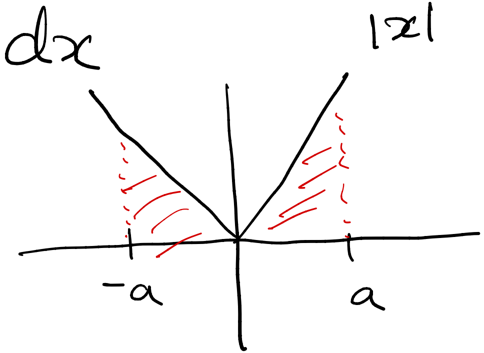




odd function: Let $f: \mathbb{R} \to \mathbb{R}$. f is odd if

$f(x) = -f(-x)$ for all x.

Even : $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



odd : $\int_{-a}^{a} f(x)\,dx = 0$.

$x^3 - x^2$

$\int_{a}^{b} x^n\,dx$

$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$