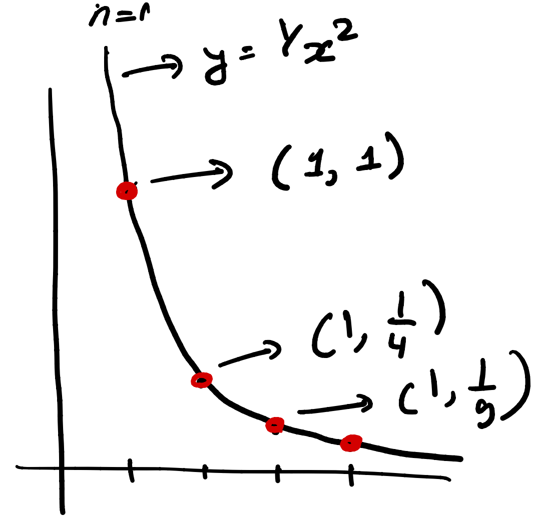
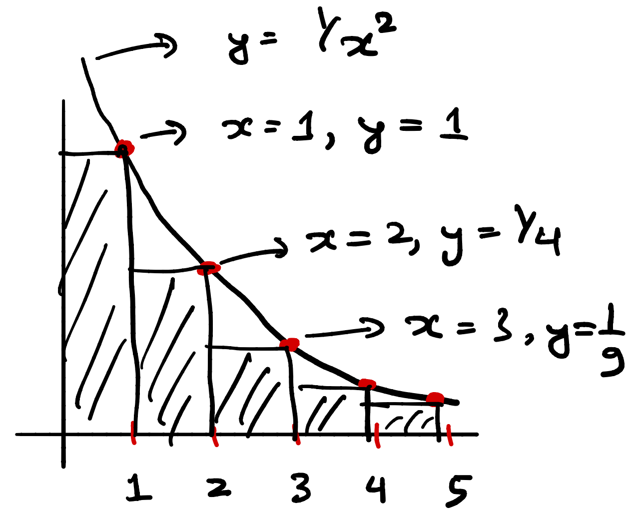
Integral test

Last time we showed harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

What about $\sum_{n=1}^{\infty} \frac{1}{n^2}$?



Integral test (contd)



Integral test (contd).

Notice that this method only provides a bound on the series and not a specific value. It provides a test for convergence.

In Math 316, you learn that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < 2$.

Idea: We anticipate that $\sum_{n=1}^{\infty} a_n \left(= \sum_{n=1}^{\infty} \frac{1}{n^2}\right)$ converges and $a_n > 0$ for all n. So, we write

$$\sum_{n=1}^{\infty} a_n < \text{integral that converges.}$$

Harmonic series.

What about series $\sum_{n=1}^{\infty} a_n$ that we expect diverges?

For example: $\sum_{n=1}^{\infty} \frac{1}{n}$. Write $\sum_{n=1}^{\infty} \frac{1}{n} >$ integral that diverge.

Integral test:

Theorem (CCLP 3.3.5): Let N_0 be a positive integer. Let $f(x)$ be a continuous function for all $x \geq N_0$. Furthermore, aussume that

1. $f(x) \geq 0$ for all $x \geq N_0$,
2. $f(x)$ decreases as x increases, and
3. $f(n) = a_n$ for all $n \geq N_0$.

Then, (I) If $\int_{N_0}^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(II) If $\int_{N_0}^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof of theorem.

Proof (contd)

Example

Does $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ diverge or converge?

Sol^n: define $f(x) = \frac{1}{x^{3/2}}$. Then for $x \geq 1$, $f(x) \geq 0$ and

$f(x)$ is decreasing.

observe that $\int_1^{\infty} \frac{1}{x^{3/2}} dx$ is convergent. So

$\sum_{i=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

p-test:

For what values of $p > 0$ does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge or diverge?

Sol^n:

Remark:

Integral test:

Another way to write integral test is:

Theorem: If $f(x)$ is continuous, positive, and decreasing on $[N_0, \infty)$ and $f(n) = a_n$, then

I) If $\int_{N_0}^{\infty} f(x)\,dx$ is convergent, so is $\sum_{n=N_0}^{\infty} a_n$

II) If $\int_{N_0}^{\infty} f(x)\,dx$ is divergent, so is $\sum_{n=N_0}^{\infty} a_n$.

Remark:

Example. 1

Discuss convergence / divergence of $\sum_{n=1}^{\infty} n e^{-n^2}$.

sol^n:

Example 2

For what values of p with $p > 0$ does $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converge?

Sol^n:

Estimating Remainders.

Suppose $f(x)$ is decreasing on $x \geq N$, $f \geq 0$ on $x \geq N$ and $f(n) = a_n$. Suppose that $S = \sum_{n=1}^{\infty} a_n < \infty$. Define R_N by

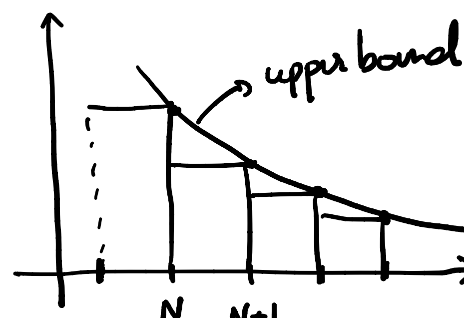
$$R = S - S_N \quad , \quad S_n = \sum_{n=1}^{N} s_n .$$

Then we have

$${}_N = a_{N+1} + a_{N+2} + \cdots$$

and the estimate

$$\int_{N+1}^{\infty} f(x)\,dx$$



$$\int_{N}^{\infty} f(x)\,dx$$

Estimating Remainder (contd).

$$\boxed{\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_N^{\infty} f(x)\,dx}$$

Example: $\sum_{n=1}^{100} \frac{1}{n^2} = 1.634$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Determine a bound for the remainder $\sum_{n=101}^{\infty} \frac{1}{n^2}$.

Sol^n: Here $a_n = \frac{1}{n^2}$, $N = 100$, and we want to estimate R_N.

$$\int_{101}^{\infty} \frac{1}{x^2}\,dx < R_{100} < \int_{100}^{\infty} \frac{1}{x^2}\,dx$$

so, $$\frac{1}{101} < R_{100} < \frac{1}{100}$$