ABSOLUTE AND CONDITIONAL CONVERGENCE: THE RATIO TEST

RECALL THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ CONVERGES BY ALTERNATING SERIES TEST BUT $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES BY INTEGRAL TEST. YET $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ AND $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGE.

WE NOW INTRODUCE TWO DEFINITIONS.

<u>DEFINITION</u> (<u>ABSOLUTE</u> AND <u>CONDITIONAL</u> CONVERGENCE). CONSIDER THE SERIES $\sum_{n=1}^{\infty} a_n$.

(I) IF THE SERIES $\sum_{n=1}^{\infty} |a_n|$ CONVERGES THEN WE SAY THAT $\sum_{n=1}^{\infty} a_n$ IS <u>ABSOLUTELY</u> <u>CONVERGENT</u>.

(II) IF THE SERIES $\sum_{n=1}^{\infty} |a_n|$ diverges, BUT $\sum_{n=1}^{\infty} a_n$ converges, THEN WE SAY THAT $\sum_{n=1}^{\infty} a_n$ IS <u>CONDITIONALLY CONVERGENT</u>.

A KEY PROPERTY THAT WE ESTABLISHED EARLIER IS THE FOLLOWING.

<u>THEOREM</u> IF $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. THUS, IF A SERIES IS ABSOLUTELY CONVERGENT (IN THE SENSE OF THE DEFINITION), THEN THE SERIES CONVERGES.

<u>PROOF</u> NOTICE THAT $0 \leq a_n + |a_n| \leq 2|a_n|$. THUS BY COMPARISON TEST IF $2\sum_{n=1}^{\infty} |a_n|$ CONVERGES, THEN SO DOES $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converge. SUBTRACTING THE CONVERGENT SERIES $\sum_{n=1}^{\infty} |a_n|$ THEN IMPLIES THAT $\sum_{n=1}^{\infty} a_n$ converges. $\blacksquare$

A KEY TEST FOR ABSOLUTE CONVERGENCE OF A SERIES IS THE RATIO TEST, WHICH WE DESCRIBE NOW.

THE RATIO TEST IS REALLY USEFUL WHEN THE SERIES HAS EXPONENTS IN n, FACTORIAL OF n, ETC.. LIKE

$$\sum_{n=1}^{\infty} \frac{n\, 2^n}{5^n}, \quad \sum_{n=1}^{\infty} \frac{e^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{n^2 e^n}{4^n} \quad \text{ETC..}$$

IT TURNS OUT TO <u>NOT</u> BE USEFUL FOR SERIES OF THE FORM $\displaystyle\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$ WHERE $P(n)$ AND $Q(n)$ ARE POLYNOMIALS IN n.

<u>THEOREM (RATIO TEST)</u> LET $N > 0$ BE AN INTEGER AND ASSUME THAT $a_n \neq 0$ FOR ALL $n \geq N$. THEN,

(I) IF $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, WE HAVE $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges $\rightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(II) IF $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ OR $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$ THEN $\displaystyle\sum_{n=1}^{\infty} |a_n|$ diverges.

<u>REMARK</u> (i) IN (I), IF $L < 1$ THEN THIS MEANS $\sum a_n$ IS ABSOLUTELY CONVERGENT.

(ii) IN (II) THIS MEANS THAT $\displaystyle\sum_{n=1}^{\infty} a_n$ IS <u>NOT</u> ABSOLUTELY CONVERGENT (I.E. $\sum |a_n|$ diverges). IT STILL COULD HAPPEN THAT $\sum a_n$ CONVERGES, I.E. THAT $\displaystyle\sum_{n=1}^{\infty} a_n$ IS <u>CONDITIONALLY</u> <u>CONVERGENT</u>.

(iii) <u>IMPORTANT</u>. THERE IS <u>NO</u> <u>CONCLUSION</u> FROM THE RATIO TEST IF $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. (I.E. $L = 1$). THEN $\sum |a_n|$ MAY OR MAY NOT CONVERGE. ANOTHER TEST IS NEEDED.

<u>PROOF (OUTLINE)</u> OF RATIO TEST.

WE FIRST PROVE (I). SINCE $L < 1$ WE CAN PICK AN R WITH, $0 < L < R < 1$. THEN $\exists M$ SO THAT FOR $n \geq M$ WE MUST HAVE $\left|\frac{a_{n+1}}{a_n}\right| \leq R$ (SINCE $\left|\frac{a_{n+1}}{a_n}\right| \to L < R$ AS $n \to \infty$).

THUS $|a_{n+1}| < R\,|a_n|$ FOR ALL $n \geq M$

THIS MEANS
$$|a_{M+1}| < R\,|a_M|$$
$$|a_{M+2}| < R\,|a_{M+1}| < R^2\,|a_M|$$
$$\vdots$$
$$|a_{M+k}| < R^k\,|a_M|.$$

THEN $\displaystyle\sum_{k=1}^{\infty} |a_{M+k}| < |a_M| \sum_{k=1}^{\infty} R^k = |a_M| \frac{R}{1-R}$ SINCE $0 < R < 1$.

WE CONCLUDE THAT $\sum_{l=1}^{\infty} |a_{M+l}| < \infty$.

THIS MEANS THAT $\sum_{n=M+1}^{\infty} |a_n|$ CONVERGES AND SO $\sum_{n=1}^{\infty} |a_n| = \underbrace{\sum_{n=\text{[redacted]}}^{M} |a_n|}_{\text{— FINITE}} + \underbrace{\sum_{n=M+1}^{\infty} |a_n|}_{<\infty}$

IMPLIES THAT $\sum_{n=1}^{\infty} |a_n|$ converges. THIS PROVES (I).

NOW TO PROVE (II), PICK R IN $1 < R < L$. THEN FOR $n \geq M$, WE HAVE

$$\left| \frac{a_{n+1}}{a_n} \right| \geq R \rightarrow |a_{n+1}| \geq R|a_n| \quad \text{FOR ALL } n \geq M.$$

THUS $|a_{n+1}| \geq |a_M| > 0$. SINCE $|a_{n+1}| \not\to 0$ AS $n \to \infty$ WE HAVE BY BASIC DIVERGENCE TEST THAT $\sum_{l=1}^{\infty} |a_{n+l}|$ IS DIVERGENT. THUS $\sum_{n=1}^{\infty} |a_n|$ diverges.

WE NOW DO MANY EXAMPLES TO ILLUSTRATE ABSOLUTE CONVERGENCE, CONDITIONAL CONVERGENCE, AND THE RATIO TEST.

<u>EXAMPLE 1</u> CONSIDER $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ WHERE $a_n = (-1)^{n-1} \frac{1}{n}$.

BY ALTERNATING SERIES TEST $\sum_{n=1}^{\infty} a_n$ CONVERGES. BUT WE HAVE $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES. THUS BY OUR DEFINITION IN (II) ON PAGE R1, WE CONCLUDE THAT $\sum_{n=1}^{\infty} a_n$ IS <u>CONDITIONALLY CONVERGENT</u>.

<u>EXAMPLE 2</u> CONSIDER $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ WHERE $a_n = (-1)^{n-1} \frac{1}{n^2}$.

BY ALT. SERIES TEST, WE KNOW THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ CONVERGES. NEXT, WE TEST $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$

WE OBTAIN THAT $\sum_{n=1}^{\infty} 1/n^2$ CONVERGES BY INTEGRAL TEST. THUS $\sum_{n=1}^{\infty} |a_n|$ CONVERGES.

BY (I) OF THE DEFINITION ON PAGE R1 WE SAY THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ IS <u>ABSOLUTELY CONVERGENT</u>.

EXAMPLE 3 DISCUSS ABSOLUTE, CONDITIONAL CONVERGENCE, OR DIVERGENCE OF EACH OF THE FOLLOWING:

(i) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$. (ii) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{6n+5}\right)$ (iii) $\sum_{n=1}^{\infty} (-1)^n \frac{(n^2+1)}{n^4+5}$.

(iv) $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$ (v) $\sum_{n=1}^{\infty} \frac{n \sin(n)}{n^3+1}$. (vi) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

SOLUTION

(i) LET $a_n = (-1)^n b_n$ WITH $b_n = \frac{n}{n^2+1}$. SINCE b_n IS DECREASING FOR n LARGE enough (easily shown) AND $b_n > 0$ WITH $b_n \to 0$ AS $n \to \infty$, BY alternating series test $\sum_{n=1}^{\infty} (-1)^n b_n$ converges. BUT $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges BY USING LIMIT COMPARISON test WITH $1/n$. HENCE WE CONCLUDE THAT $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ IS CONDITIONALLY CONVERGENT.

(ii) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{6n+5}\right)$. LET $a_n = (-1)^n \left(\frac{2n+1}{6n+5}\right)$. FOR $n \to \infty$, $a_n \approx \frac{(-1)^n}{3}$.
SINCE $a_n \not\to 0$ AS $n \to \infty$, BY BASIC divergence test $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{6n+5}\right)$ diverges.

(iii) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+1}{n^4+5}\right)$. LET $a_n = (-1)^n \left(\frac{n^2+1}{n^4+5}\right)$. LET'S FIRST TEST $\sum_{n=1}^{\infty} |a_n|$.
WE HAVE $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^2+1}{n^4+5}$. SINCE $\frac{n^2+1}{n^4+5} \approx \frac{1}{n^2}$ FOR n large AND $\sum 1/n^2$ converges, WE EXPECT convergence. USE LIMIT COMPARISON TEST WITH $1/n^2$. WE CALCULATE $\lim_{n\to\infty} \frac{(n^2+1)/(n^4+5)}{1/n^2} = \lim_{n\to\infty} \frac{n^2(n^2+1)}{n^4+5} = 1$.
BY LIMIT COMPARISON TEST, $\sum |a_n|$ converges. THUS,

$\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+1}{n^4+5}\right)$ IS Absolutely convergent (AND $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+1}{n^4+5}\right)$ MUST converge).

(R5)

(iv) $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$ WITH $a_n = (-1)^n n e^{-n^2}$.

LET'S FIRST TEST $\sum_{n=1}^{\infty} |a_n|$. WE CONSIDER THEN $\sum_{n=1}^{\infty} n e^{-n^2}$.

WE HAVE TWO APPROACHES:

RATIO TEST SINCE $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)}{n} \frac{e^{-(n+1)^2}}{e^{-n^2}}\right| = \lim_{n\to\infty} \left|\frac{(n+1)}{n} \frac{e^{-(n^2+2n+1)}}{e^{-n^2}}\right|$

$= \lim_{n\to\infty} \left|\frac{n+1}{n} e^{-(2n+1)}\right| = (1)(0) = 0$

WE HAVE $L = 0$ AND SO BY (I) OF RATIO TEST, $\sum_{n=0}^{\infty} |a_n|$ converges.

WE CONCLUDE THAT $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$ IS ABSOLUTELY CONVERGENT.

INTEGRAL TEST CONSIDER $\sum_{n=1}^{\infty} n e^{-n^2}$ BY INTEGRAL TEST. DEFINE $b_n = f(n)$ WITH $f(x) = xe^{-x^2}$. CLEARLY $f(x) > 0$ ON $x > 1$ AND $f'(x) = e^{-x^2}(1-2x^2) < 0$ FOR $x \geq 1$.

SINCE $b_n > 0$ AND $b_{n+1} < b_n$ (decreasing sequence) WE HAVE THAT $\sum_{n=1}^{\infty} n e^{-n^2}$ converges OR diverges when $\int_1^{\infty} f(x)\,dx$ converges OR diverges, respectively.

WE calculate $\int_1^{\infty} xe^{-x^2} dx = \lim_{L\to\infty} \int_1^{L} xe^{-x^2} dx = \lim_{L\to\infty} \left(-\frac{1}{2} e^{-x^2} \Big|_1^{L}\right) = \frac{1}{2} < \infty$.

THUS $\sum |a_n|$ converges AND SO $\sum a_n$ converges absolutely.

(v) $\sum_{n=1}^{\infty} \frac{n \sin(n)}{n^3+1}$. WE EXPECT CONVERGENCE SINCE $\frac{n \sin(n)}{n^3+1} \doteq \frac{\sin(n)}{n^2}$ FOR large n.

HOWEVER, WE CAN'T USE INTEGRAL TEST SINCE $f(x) = \frac{(x)(\sin x)}{x^3+1}$ IS NOT DECREASING FOR LARGE x. NOR CAN WE COMPARE WITH $\frac{1}{n^2}$ IN THE LIMIT COMPARISON TEST SINCE $\lim_{n\to\infty} \left(\frac{(n)(\sin(n))}{n^3+1}\right) \Big/ \left(\frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{n^3(\sin n)}{n^3+1}$ DOES NOT EXIST.

INSTEAD WE USE BASIC COMPARISON TEST WITH $|\sin n| < 1$.

WE HAVE $0 \leq \left|\frac{n \sin n}{n^3+1}\right| \leq \frac{n}{n^3+1}$

SINCE $n^3+1 \geq n^3$ WE HAVE $\left|\frac{n \sin n}{n^3+1}\right| \leq \frac{1}{n^2}$ AND $\sum \frac{1}{n^2}$ converges.

THUS SINCE $\sum |a_n|$ converges, a_n IS SAID TO CONVERGE ABSOLUTELY.

(vi) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$. THE TERMS ARE POSITIVE AND WE CAN TEST FOR CONVERGENCE WITH THE RATIO TEST. LET $a_n = \frac{2^n}{n!}$ AND CALCULATE $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!}$

SINCE $\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} 2 = \left(\frac{n \cdot (n-1) \cdots 1}{(n+1) \cdot n \cdot (n-1) \cdots 1} \right) 2 = \frac{2}{n+1}$ WE HAVE $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2}{n+1} = 0.$

WE CONCLUDE BY (I) OF THE RATIO TEST THAT $\sum_{n=0}^{\infty} a_n$ CONVERGES. (No need to write absolute convergence since $a_n > 0$ FOR ALL n).

FURTHER REMARKS AND EXAMPLES:

REMARK THE RATIO TEST IS ALWAYS INCONCLUSIVE FOR RATIOS OF POLYNOMIALS SINCE IF $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+3n}{n^3+4n^2}$ WE CALCULATE WITH $|a_n| = \frac{n^2+3n}{n^3+4n^2}$ THAT $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

SINCE $L=1$ THE RATIO TEST IS INCONCLUSIVE.

EXAMPLE $\sum_{n=1}^{\infty} \frac{n}{5^n}$. COULD USE INTEGRAL TEST WITH $f(x) = \frac{x}{5^x} = \frac{x}{e^{x \ln 5}} = x e^{-(\ln 5)x}$

OR USE RATIO TEST. LET $a_n = \frac{n}{5^n}$. WE WRITE $\frac{a_{n+1}}{a_n} = \frac{(n+1)/5^{n+1}}{n/5^n} = \left(\frac{n+1}{n} \right) \frac{1}{5}$.

SINCE $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{5} \lim_{n\to\infty} \left(\frac{n+1}{n} \right) = \frac{1}{5} < 1$, THE SERIES CONVERGES.

EXAMPLE (HARDER) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$. WE HAVE $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}/(n+1)!}{n^n/(n!)} = \frac{n!}{(n+1)!} \frac{(n+1)^{n+1}}{n^n}$.

BUT $\frac{n!}{(n+1)!} = \frac{1}{n+1}$ SO $\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)} \frac{(n+1)^{n+1}}{n^n} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n} \right)^n$.

NOW LET $n \to \infty$, $\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n = e = 2.7128\ldots > 1$. BY (II) OF RATIO TEST, $\sum n^n/n!$ DIVERGES. TO SEE WHY $\left(1+\frac{1}{n}\right)^n \to e$ AS $n \to \infty$ TAKE $\log$ OF $\left(1+\frac{1}{n}\right)^n$. CONSIDER $\lim_{n\to\infty} n \log\left(1+\frac{1}{n}\right) = \lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{(1/1+x)}{1} = 1$ BY L'HOPITALS RULE. THUS $(1+1/n)^n \to e^1$ AS $n \to \infty$.

EXAMPLE (HARDER)

$$\sum_{n=1}^{\infty} \frac{n^n}{3^n n!} . \quad \text{HERE} \quad a_n = \frac{n^n}{3^n n!} .$$

WE CAN PROCEED AS IN PREVIOUS EXAMPLE TO GET

$$\frac{a_{n+1}}{a_n} = \frac{1}{3}\left(1+\frac{1}{n}\right)^n .$$

NOW $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{3} e = \frac{2.7128\ldots}{3} < 1$. THUS, THE SERIES NOW CONVERGES BY THE RATIO TEST.

EXAMPLE (HARDER) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n^2+1)(n!)^2}$

WE DEFINE $a_n = \frac{(2n)!}{(n^2+1)(n!)^2} = \frac{(2n)(2n-1)(2n-2)(2n-3)\cdots 1}{(n^2+1)[n(n-1)\cdots 1]^2}$

NOW $\frac{a_{n+1}}{a_n} = \frac{[2(n+1)]!}{[(n+1)^2+1][(n+1)!]^2} \frac{(n!)^2(n^2+1)}{(2n)!} = \frac{(n^2+1)}{(n+1)^2+1} \frac{(2n+2)(2n+1)}{(n+1)^2}$

NOW LET $n\to\infty$. WE CALCULATE $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4n^4+\cdots}{n^4+\cdots} = 4 > 1.$

BY RATIO TEST, SINCE $L = 4 > 1$, $\sum_{n=1}^{\infty} \frac{(2n)!}{(n^2+1)(n!)^2}$ DIVERGES.

EXAMPLE (HARDER) SUPPOSE THAT $\sum_{n=1}^{\infty} \frac{n a_n - 2n + 1}{n+1}$ CONVERGES.

THEN CALCULATE $S = -\log a_1 + \sum_{n=1}^{\infty} \log\left(\frac{a_n}{a_{n+1}}\right)$.

SOLUTION DEFINE $b_n = \frac{n a_n - 2n + 1}{n+1}$. IF THE SERIES CONVERGES WE MUST HAVE THAT $b_n \to 0$ AS $n\to\infty$. WE WRITE $b_n = \frac{n(a_n-2)}{n+1} + \frac{1}{n+1}$. THIS MEANS WE MUST HAVE $a_n \to 2$ AS $n\to\infty$. NOW S IS A TELESCOPING SUM. THE N^{TH} PARTIAL SUM IS $S_N = -\log a_1 + \sum_{n=1}^{N}(\log a_n - \log a_{n+1}) = -\log a_1 + (\log a_1 - \log a_2) + (\log a_2 - \log a_3) \ldots + (\log a_N - \log a_{N+1})$

WE GET $S_N = -\log a_{N+1}$ WITH THIS CANCELLATION. THUS $S = -\log\left(\lim_{N\to\infty} a_{N+1}\right) = -\log 2$.