

## Integral test

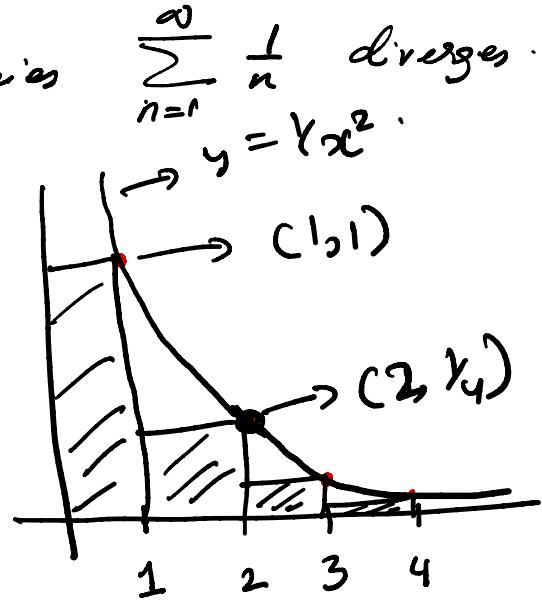
$$\int_1^\infty \frac{1}{x^p} dx$$

Last time we showed harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

What about  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ?

Notice that  $\sum_{i=1}^{\infty} \frac{1}{n^2}$  is sum of height

of  $f(x) = \frac{1}{x^2}$  at  $x = 1, 2, 3, \dots$



Moreover, we can draw rectangles

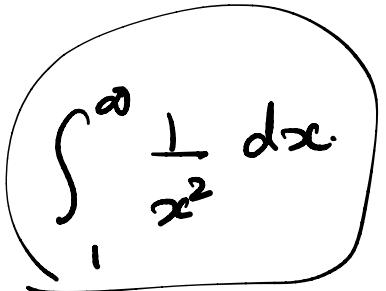
of width 1:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \text{sum of area of rectangles.}$$

## Integral test (contd)

Notice that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2} dx.$$



$$\sum_{n=1}^{\infty} \frac{1}{n}$$

because

$f(x) = \frac{1}{x^2}$  goes through top right corner of each rectangle.

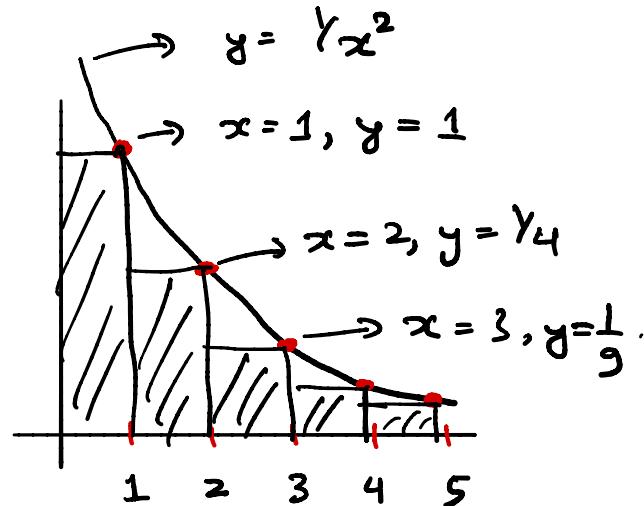
so,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \lim_{b \rightarrow \infty} \left[ -x^{-1} \right]_1^b$$

$$= 1 + 1$$

$$= 2 \quad \Rightarrow$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$



$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

so, converges

## Integral test (contd).

Notice that this method only provides a bound on the series and not a specific value. It provides a test for convergence.

In Math 316, you learn that  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < 2$ .

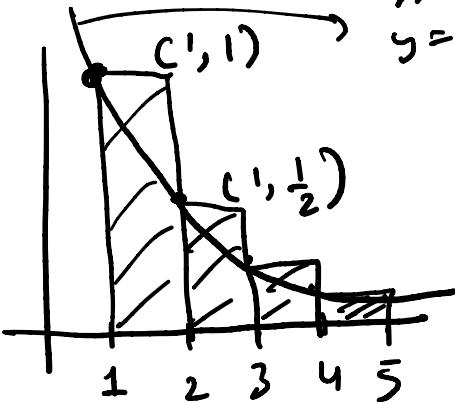
Idea We anticipate that  $\sum_{n=1}^{\infty} a_n \left( = \sum_{n=1}^{\infty} \frac{1}{n^2} \right)$  converges and  $a_n > 0$  for all  $n$ . So, we write

$$\sum_{n=1}^{\infty} a_n < \text{integral that converges.}$$

## Harmonic series.

What about series  $\sum_{n=1}^{\infty} a_n$  that we expect diverges?

For example:  $\sum_{n=1}^{\infty} \frac{1}{n}$ . Write  $\sum_{n=1}^{\infty} \frac{1}{n} > \text{integral that diverges}$



so,  $\sum_{n=1}^{\infty} \frac{1}{n} = \text{Area of rectangles}$ .

$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x} dx \\ = \lim_{b \rightarrow \infty} \left[ \ln|x| \right]_1^b = \infty$$

so,  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

## Integral test

Theorem (CLP 3.3.5): Let  $N_0$  be a positive integer.

Let  $f(x)$  be a continuous function for all  $x \geq N_0$ .

Furthermore, assume that

1.  $f(x) \geq 0$  for all  $x \geq N_0$ ,

2.  $f(x)$  decreases as  $x$  increases, and

3.  $f(n) = a_n$  for all  $n \geq N_0$ .

$$\sum_{n=1}^{N_0-1} a_n + \sum_{n=N_0}^{\infty} a_n$$



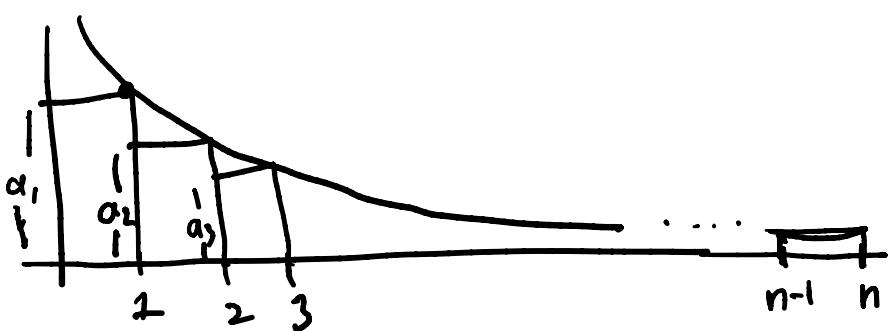
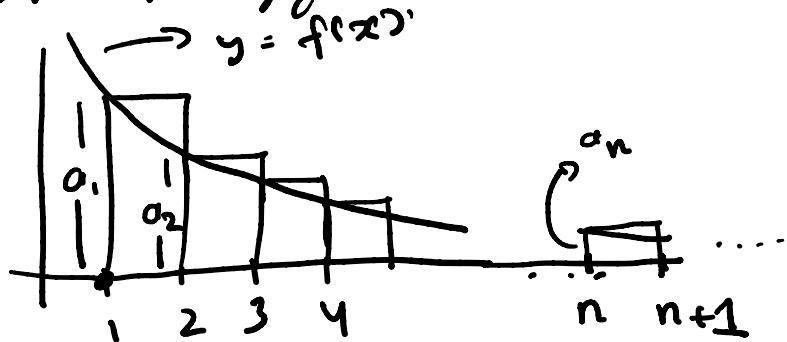
Then, (I) If  $\int_{N_0}^{\infty} f(x) dx$  is convergent, then  $\sum_{n=1}^{\infty} a_n$  is convergent.

(II) If  $\int_{N_0}^{\infty} f(x) dx$  diverges, then  $\sum_{n=1}^{\infty} a_n$  diverges.

## Proof of theorem.

We suppose  $N_0 = 1$ . since all we need is to estimate the tail behaviour of  $\sum_{n=1}^{\infty} a_n$ . (convergence or divergence).

draw two figures:



Area of  $n$  rectangles

$$a_1 + a_2 + a_3 + \dots + a_n \geq \int_1^{n+1} f(x) dx$$

$\curvearrowleft$   $n^{\text{th}}$  partial sum.  $\rightarrow$  (I)

Area of  $n$  rectangles:

$$a_1 + a_2 + \dots + a_n \leq a_1 + \int_1^n f(x) dx$$

$\curvearrowleft$  (II)

## Proof (contd)

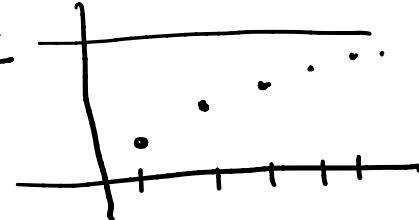
Combining ① and ②

$$\int_1^{n+1} f(x) dx \leq a_1 + a_2 + \dots + a_n \leq a_1 + \int_1^n f(x) dx$$

$\underbrace{\phantom{a_1 + a_2 + \dots + a_n}_{S_n}}$

MCT: A bounded and monotone sequence is

convergent.



Divergence : easy!

$$\int_1^\infty f(x) dx \text{ diverges} \xrightarrow{\text{so}} \sum_{n=1}^\infty a_n \text{ diverges}$$

Convergence : bounded:

$$S_n \leq a_1 + \int_1^\infty f(x) dx$$

converges

increasing:  $a_n \geq 0$

so, by Monotone convergence theorem we have

$$\sum_{n=1}^\infty a_n$$

### Example

Does  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  diverge or converge?

Sol<sup>n</sup>: Define  $f(x) = \frac{1}{x^{3/2}}$ . Then for  $x > 1$ ,  $f(x) \geq 0$  and  $f(x)$  is decreasing.

observe that  $\int_1^{\infty} \frac{1}{x^{3/2}} dx$  is convergent. So

$\sum_{i=1}^{\infty} \frac{1}{n^{3/2}}$  is convergent.

### p-test

For what values of  $p > 0$  does  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converge or diverge?

Sol<sup>n</sup>: Define  $f(x) = \frac{1}{x^p}$ .  $f(x) \geq 0$ ,  $f(x)$  is decreasing on  $x \geq 1$ .

Then  $\int_1^{\infty} \frac{1}{x^p} dx$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

So,  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

↪ p-test of series.

Remark: By p-test of series,  $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$  is convergent

and  $\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$  is divergent.

## Integral test

Another way to write integral test is:

Theorem: If  $f(x)$  is continuous, positive, and decreasing on  $[N_0, \infty)$  and  $f(n) = a_n$ , then

I) If  $\int_{N_0}^{\infty} f(x) dx$  is convergent, so is  $\sum_{n=N_0}^{\infty} a_n$

II) If  $\int_{N_0}^{\infty} f(x) dx$  is divergent, so is  $\sum_{n=N_0}^{\infty} a_n$ .

Remark:  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N_0-1} a_n + \sum_{n=N_0}^{\infty} a_n$ . and so, above theorem



applies to  $\sum_{n=1}^{\infty} a_n$ .

Example 1

Discuss convergence / divergence of  $\sum_{n=1}^{\infty} n e^{-n^2}$ .

Sol: Define  $f(x) = xe^{-x^2}$   
 $f'(x) = e^{-x^2}(1 - 2x^2) < 0 \quad \text{if } 1 - 2x^2 < 0$   
or  $x > \frac{1}{\sqrt{2}}$

so, we have  $f'(x) < 0 \quad \text{if } x \geq 1 \rightarrow \text{decreasing}$   
and  $f(x) > 0 \quad \text{if } x \geq 1 \rightarrow \text{positivity}$

using integral test:

$$\int_1^{\infty} xe^{-x^2} dx = \lim_{b \rightarrow \infty} \left[ -\frac{1}{2} e^{-x^2} \right]_1^b = \lim_{b \rightarrow \infty} \left( -\frac{1}{2} e^{-b^2} + \frac{1}{2} e^{-1} \right)$$
$$= \frac{1}{2} e^{-1} < \infty.$$

so,  $\sum_{n=1}^{\infty} n e^{-n^2}$  is convergent

### Example 2

For what values of  $p$  with  $p > 0$  does  $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$

converge?

Soln: Define  $f(x) = \frac{1}{x(\log x)^p}$ .  $f'(x) = -x^2(\log x)^{-p} - px^2 \cdot \frac{1}{(\log x)^{p+1}}$

$$\text{so, } f'(x) = x^2(\log x)^{-p} \left( -1 - p(\log x)^{-1} \right). < 0 \text{ if}$$

$$-1 - p(\log x)^{-1} < 0 \Rightarrow -1 < \frac{p}{\log x} \Rightarrow x > -e^p$$

$$x > 0$$

so,  $f(x)$  is decreasing on  $[2, \infty)$

$f(x)$  is positive on  $[2, \infty)$  (also continuous).

$$\int_2^{\infty} \frac{1}{x(\log x)^p} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\log x)^p} dx = \lim_{b \rightarrow \infty} \int_{\log 2}^{\log b} \frac{1}{u^p} du.$$

## Estimating Remainders.

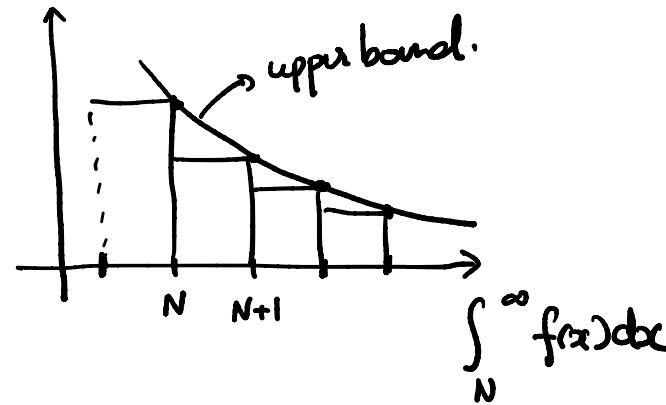
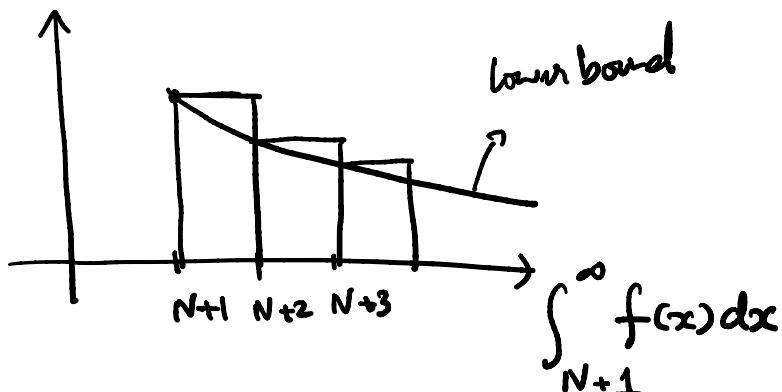
Suppose  $f(x)$  is decreasing on  $x \geq N$ ,  $f > 0$  on  $x \geq N$  and  $f(n) = a_n$ . Suppose that  $S = \sum_{n=1}^{\infty} a_n < \infty$ . Define

$R_N$  by

$$R_N = S - S_N \quad , \quad S_n = \sum_{n=1}^N a_n . \quad ? < R_N < ?$$

Then we have

$$R_N = a_{N+1} + a_{N+2} + \dots \quad \text{and the estimate}$$



## Estimating Remainder (contd).

$$\int_{N+1}^{\infty} f(x) dx < R_N < \int_N^{\infty} f(x) dx$$

Example:  $\sum_{n=1}^{100} \frac{1}{n^2} = 1.634$  and  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ .

Determine a bound for the remainder  $\sum_{n=101}^{\infty} \frac{1}{n^2}$ .

Sol: Here  $a_n = \frac{1}{n^2}$ ,  $N = 100$ , and we want to estimate  $R_N$ .

$$\int_{101}^{\infty} \frac{1}{x^2} dx < R_{100} < \int_{100}^{\infty} \frac{1}{x^2} dx$$

$$\text{so, } \frac{1}{101} < R_{100} < \frac{1}{100}$$

