Homework 2 solution

```
using Random
using LinearAlgebra
using Convex
using ECOS
using Plots
using JLD
using CSV
using DataFrames
```

1. Since both $\mathbf{null}(A)$ and $\mathbf{null}(L)$ are subspaces, if $0 \neq u \in \mathbf{null}(A) \cap \mathbf{null}(L)$ then any scaling $\gamma u \in \mathbf{null}(A) \cap \mathbf{null}(L)$. So, if x^* solves the problem, then any $x^* + \gamma u$ also solves the problem, since $A(x^* + \gamma u) = Ax^*$ and $L(x^* + \gamma u) = Lx^*$.

   To show the other direction, suppose that the solution to the optimization problem is not unique. That is, there exists $x \neq y$ where

$$(A^T A + \lambda L^T L)x = A^T b \text{ and } (A^T A + \lambda L^T L)y = A^T b.$$

   Since we put no limitations on x and y (except that $x \neq y$) this implies that there exists some $u \neq 0$ where

$$(A^T A + \lambda L^T L)u = 0.$$

   This implies

$$u^T A^T A u + \lambda u^T L^T L u = \|Au\|_2^2 + \lambda \|Lu\|_2^2 = 0.$$

   Since both terms being added are nonnegative, both terms must be 0.

$$\|Au\|_2 = 0 \iff Au = 0, \quad \|Lu\|_2 = 0 \iff Lu = 0.$$

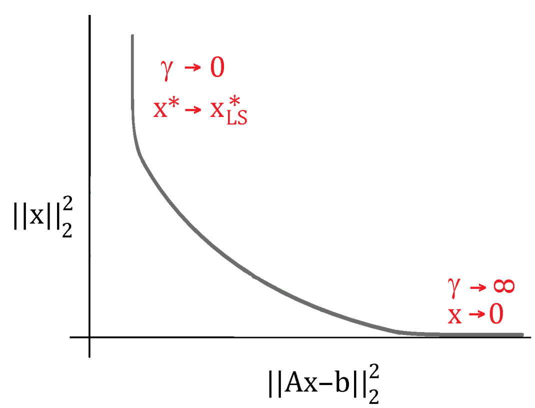
Therefore $u \in \mathbf{null}(A) \cap \mathbf{null}(L)$, and $u \neq 0$.

2. **Multiobjective problems**

   a. **2-norm regularization** The solution is $x = (A^TA + \gamma I)^{-1}A^Tb$. So,

$$\begin{aligned}\|x\|_2^2 =& \|(A^TA+\gamma I)^{-1}A^Tb\|_2^2 \\ =& \|(QDQ^T + \lambda QQ^T)^{-1}A^Tb\|^2 \\ =& \|(D-\lambda I)^{-1}Q^TA^Tb\|_2^2 \\ =& \sum_{i=1}^n \frac{1}{(d_i+\gamma)^2} g_i^2.\end{aligned}$$

   Sketch:



   b. **Sparsity** The following code provides accuray and sparsity measure for $\lambda = 1$.

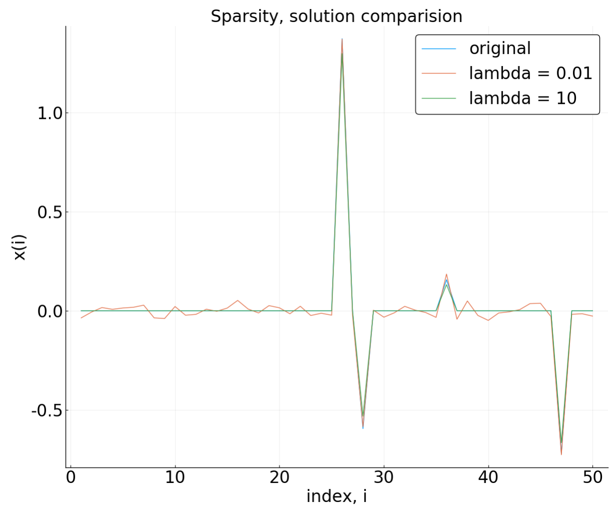
```
#part 2a.
# load data
A = load("hw2_p2_smooth_A.jld")["data"]
b = load("hw2_p2_smooth_b.jld")["data"]
# load signal
x0 = load("hw2_p2_smooth_signal.jld")["data"]

(m,n) = size(A)

x_var = Variable(n);
```

```
    loss = sumsquares(A*x_var-b)
    reg = sum(abs(x_var[2:end]-x_var[1:end-1]))

    λ = 1
    problem = minimize(loss+λ*reg)
    solve!(problem,ECOSSolver(verbose=false))
    xa = x_var.value;
    free!(x_var);#
    println("accuracy f1 = $(0.5*norm(A*xa-b,2)^2) and
 sparsity f2 = $(norm(xa,1))")
```

The value of $f_1(x) = 0.619$ and $f_2(x) = 3.332$.

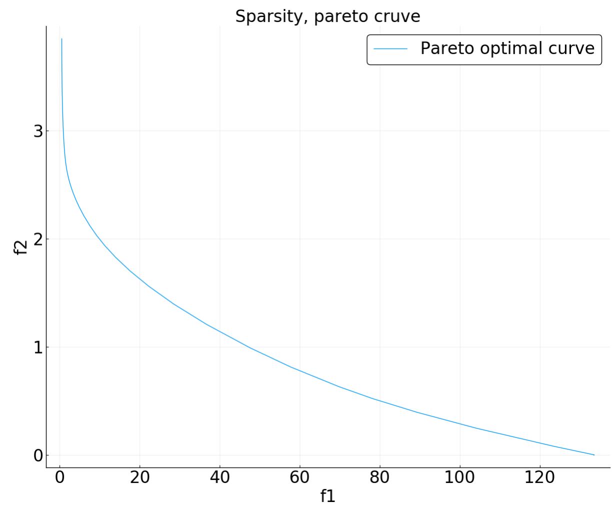
c. The following code plots the solution for $\lambda = 0.1$ and $\lambda = 10$.

```
    λ = .001
    problem = minimize(loss+λ*reg)
    solve!(problem,ECOSSolver(verbose=false))
    xb = x_var.value;
    free!(x_var)

    λ = 10
    problem = minimize(loss+λ*reg)
    solve!(problem,ECOSSolver(verbose=false))
    xc = x_var.value;
    free!(x_var)

    pyplot(size=(1200,1000), legend = true, legendfontsize=24,
 xguidefontsize=24, yguidefontsize=24,
    xtickfontsize=24, ytickfontsize=24, titlefontsize = 24)
    plot_temp = plot(1:n, [x0,xb,xc], label = ["original"
 "lambda = 0.01" "lambda = 10"], xlabel="index, i",
    ylabel="x(i)", title = "Sparsity, solution comparision")
    savefig(plot_temp, "figures/hw2_p2_c.png")
```



d. The following code plots the pareto optimal curve for the sparsity case.

```
    # generate logspace between -3,3
    Lambda = exp10.(range(-3, stop=3, length=100))
    f1_values = []
    f2_values = []
    for i in 1:100
        λ = Lambda[i]
        problem = minimize(loss+λ*reg)
        solve!(problem,ECOSSolver(verbose=0))
        xd = x_var.value;
        free!(x_var)
        push!(f1_values, 0.5*norm(A*xd-b,2)^2)
        push!(f2_values, norm(xd,1))

    end
    pyplot(size=(1200,1000), legend = true, legendfontsize=24,
xguidefontsize=24, yguidefontsize=24,
    xtickfontsize=24, ytickfontsize=24, titlefontsize = 24)
    plot_temp = plot(f1_values, f2_values, label = "Pareto
optimal curve", xlabel="f1", ylabel="f2", title = "Sparsity,
pareto cruve")
    savefig(plot_temp, "figures/hw2_p2_d.png")
```

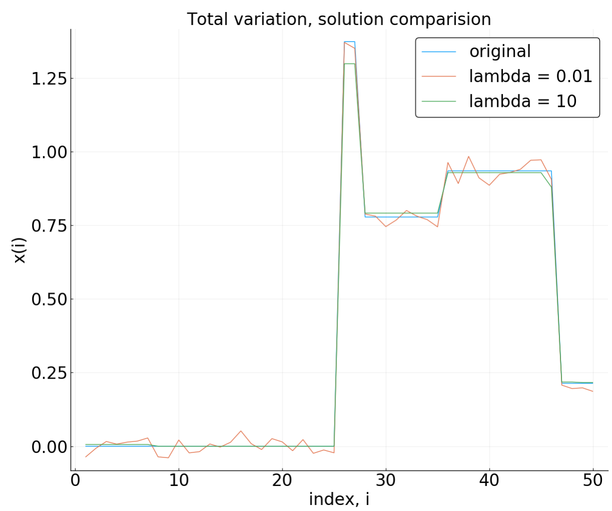


e. The following code plots the solution for $\lambda = 0.1$ and $\lambda = 10$. It also provides the accuray and total variation measure for $\lambda = 1$.

```
    # load data
    A = load("hw2_p2_smooth_A.jld")["data"]
    b = load("hw2_p2_smooth_b.jld")["data"]

    # load signal
    x0 = load("hw2_p2_smooth_signal.jld")["data"]
    x_var = Variable(n);

    (m,n) = size(A)

    loss = sumsquares(A*x_var-b)
    reg = sum(abs(x_var[2:end]-x_var[1:end-1]))

    λ = 1
    problem = minimize(loss+λ*reg)
    solve!(problem,ECOSSolver(verbose=false))
    xa = x_var.value;
    free!(x_var);#
    println("accuracy f1 = $(0.5*norm(A*xa-b,2)^2) and
sparsity f2 = $(norm(xa,1))")

    λ = .001
    problem = minimize(loss+λ*reg)
    solve!(problem,ECOSSolver(verbose=false))
    xb = x_var.value;
    free!(x_var)
```

```
    λ = 10
    problem = minimize(loss+λ*reg)
    solve!(problem,ECOSSolver(verbose=false))
    xc = x_var.value;
    free!(x_var)

    plot_temp = plot(1:n, [x0,xb,xc], label = ["original"
 "lambda = 0.01" "lambda = 10"], xlabel="index, i",
ylabel="x(i)",
    title = "Total variation, solution comparision")
    savefig(plot_temp, "figures/hw2_p2_e.png")
```

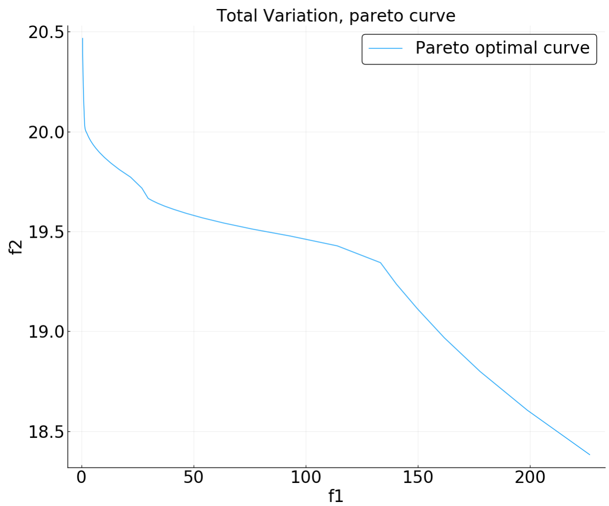
For $\lambda = 1$, the value of $f_1(x) = 0.652$ and $f_2(x) = 20.310$.



f. The following code plots the pareto optimal curve for the total vairation case.

```
    # generate logspace between -3,3
    Lambda = exp10.(range(-3, stop=3, length=100))
    f1_values = []
    f2_values = []
    for i in 1:100
        λ = Lambda[i]
        problem = minimize(loss+λ*reg)
        solve!(problem, ECOSSolver(verbose=false))
        xd = x_var.value;
        free!(x_var)
        push!(f1_values, 0.5*norm(A*xd-b,2)^2)
        push!(f2_values, norm(xd,1))
```

```
    end
    plot_temp = plot(f1_values, f2_values, label = "Pareto optimal curve", xlabel="f1", ylabel="f2",title = "Total Variation, pareto curve")
    savefig(plot_temp, "figures/hw2_p2_d.png")
```



3. **Non-linear least squares**

   a. The gradient is

$$\nabla f(x) = 4\sum_{i=1}^{m}(\|x-c_i\|_2^2 - d_i^2)(x-c_i).$$

   b. $$r(x) = \begin{bmatrix} \|x-c_1\|_2^2 - d_1^2 \\ \vdots \\ \|x-c_m\|_2^2 - d_n^2 \end{bmatrix}, \qquad J(x) = \begin{bmatrix} 2(x-c_1)^T \\ \vdots \\ 2(x-c_m)^T. \end{bmatrix}$$

   c. In the Gauss-Newton method, we attempt to solve at each iteration

$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|r(\bar{x}) + J(\bar{x})(x-\bar{x})\|_2^2$$

   where $\bar{x} = x^{(k)}$ the current iterate. Taking

$$A = J(\bar{x}) \in \mathbf{R}^{m\times n}, \quad b = J(\bar{x})\bar{x} - r(\bar{x})$$

the above problem becomes a least squares problem

$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|Ax-b\|_2^2$$

with normal equations

$$A^TAx = A^Tb.$$

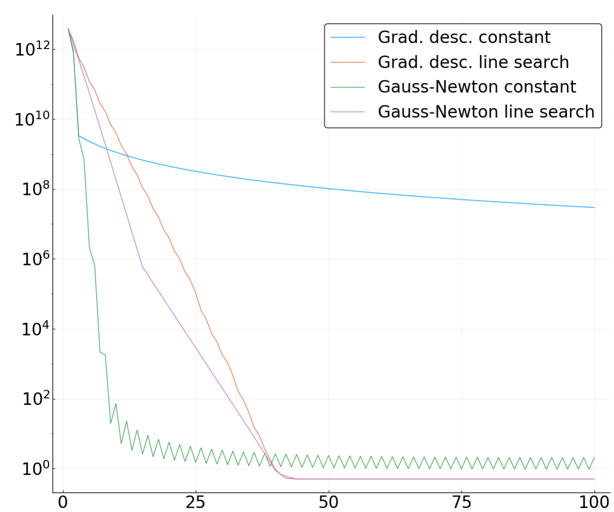
The solution to the normal equations is unique if and only if A^TA is invertible--that is, if A has full column rank. Since we are assuming $n=2$, we need only verify that $\mathbf{rank}(A)=2$.

Expanding

$$A = J(\bar{x}) = \begin{bmatrix} 2(\bar{x}-c_1)^T \\ \vdots \\ 2(\bar{x}-c_m)^T. \end{bmatrix}$$

which has rank 2 under our assumptions.

d. The plot of $f(x^{(k)})$ for $k=1,\ldots,100$, for all four solvers, is shown below:



Gradient descent: $\bar{\alpha} = 6.66\times 10^{-8}$, whereas Gauss-Newton $\bar{\alpha}=1$. (Anything super small for GD and 1 for Gauss-Newton is acceptable.)

(Any answers for qualitative observations is acceptable.) Possible observations:

i. Gauss Newton is much more tolerant of larger step sizes than gradient descent, suggesting better numerical stability.

ii. Overall, using line search greatly improves method performance. However, there is a complexity tradeoff, since line search requires checking a condition many times.

Here's the code:

```
    C = load("hw2_p3_C.jld")["data"]
    d = load("hw2_p3_d.jld")["data"]
    # load signal
    x0 = load("hw2_p3_signal.jld")["data"]

    s = 1
    α = 0.5
    β=0.5
    epsilon = 10^(-4)

    #Gradient descent with constant step size
    x = []
    obj_gd = []
    push!(x, [1000;-500])

    for iter in 1:100
        x_temp = x[iter]
        r = sum((C.-x_temp).^2,dims=1)' - d.^2;
        push!(obj_gd, norm(r,2)^2/2)
        g = 4*sum((ones(2,1)*r').*(x_temp.-C),dims = 2)
        push!(x, x_temp - g/15000000);
    end
    pyplot(size=(1200,1000), legend = true, legendfontsize=24,
 xguidefontsize=24, yguidefontsize=24,
    xtickfontsize=24, ytickfontsize=24, titlefontsize = 24)
    plot_temp = plot(obj_gd,yaxis=:log, label = "Grad. desc.
 constant")

    #Gradient descent with line search
    x = []
    obj_gd_line = []
    push!(x, [1000;-500])
```

```
    for iter in 1:100
        x_temp = x[iter]
        r = sum((C.-x_temp).^2,dims=1)' - d.^2;
        push!(obj_gd_line, norm(r,2)^2/2)
        g = 4*sum((ones(2,1)*r').*(x_temp.-C),dims = 2)

        global t = s
        global obj_cand = obj_gd_line[iter]

        while (obj_gd_line[iter] .- obj_cand .- α*t*(g'*g))[1]
< 1e-10
            global t = t*β
            xcand = x_temp .- t*g
            rcand = sum((C.-xcand).^2,dims=1)' .- d.^2;
            global obj_cand = norm(rcand,2)/2
        end
        push!(x, x_temp - t*g);
    end
    plot!(obj_gd_line,yaxis=:log, label = "Grad. desc. line
search")

    #Gauss newton with constant step size
    x = []
    obj_gd_gauss = []
    push!(x, [1000;-500])

    for iter in 1:100
        x_temp = x[iter]
        r = sum((C.-x_temp).^2,dims=1)' - d.^2
        push!(obj_gd_gauss, norm(r,2)^2/2)
        J = (x_temp.-C)'
        z = J \ r
        push!(x, x_temp-z)
    end
    plot!(obj_gd_gauss,yaxis=:log, label = "Gauss-Newton
constant")

    #Gauss newton with line search
    x = []
    obj_gd_gauss_line = []
    push!(x, [1000;-500])

    for iter in 1:100
        x_temp = x[iter]
```

```
        r = sum((C.-x_temp).^2,dims=1)' - d.^2
        push!(obj_gd_gauss_line, norm(r,2)^2/2)
        J = (x_temp.-C)'
        z = J \ r
        global t = s
        global obj_cand = norm(r,2)^2/2
        g = 4*sum((ones(2,1)*r').*(x_temp.-C),dims = 2)

        while (obj_gd_gauss_line[iter] .- obj_cand .- α*t*
(g'*z))[1] < 1e-10
            global t = t*β
            xcand = x_temp .- t*z
            rcand = sum((C.-xcand).^2,dims=1)' .- d.^2;
            global obj_cand = norm(rcand,2)/2
        end

        push!(x, x_temp-t*z)
    end
    plot!(obj_gd_gauss_line,yaxis=:log,label= "Gauss-Newton
line search")
    savefig(plot_temp, "figures/hw2_p3_d.png")
```