# POWER SERIES

**DEFINITION** AN EXPRESSION OF THE FORM $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$ IS CALLED A POWER SERIES CENTERED AT $x=0$. AN EXPRESSION OF THE FORM $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$ IS A POWER SERIES CENTERED AT $x=a$.

THE GEOMETRIC SERIES $\sum_{n=0}^{\infty} x^n = 1 + x + \ldots + x^n + \ldots$ IS A POWER SERIES CENTERED AT $x=0$. WE SHOWED EARLIER THAT IT CONVERGES TO $\frac{1}{1-x}$ FOR $|x|<1$, AND SO

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{FOR} \quad -1 < x < 1.$$

FOR A GENERAL POWER SERIES THERE ARE THREE POSSIBILITIES FOR $\sum_{n=0}^{\infty} c_n (x-a)^n$ WITH RESPECT TO CONVERGENCE.

**THEOREM (CONVERGENCE OF POWER SERIES)** FOR $\sum_{n=0}^{\infty} c_n (x-a)^n$, THERE ARE THREE POSSIBILITIES:

1. THERE IS A POSITIVE NUMBER $R$ SUCH THAT THE SERIES DIVERGES FOR $|x-a| > R$ BUT CONVERGES FOR $|x-a| < R$. THE SERIES MAY OR MAY NOT CONVERGE AT EITHER OF THE ENDPOINTS $x = a-R$ AND $x = a+R$.
2. THE SERIES CONVERGES FOR EVERY $x$ (I.E. $R = \infty$)
3. THE SERIES CONVERGES AT $x=a$ AND DIVERGES ELSEWHERE (I.E. $R=0$).

THE NUMBER $R$ IS CALLED THE <u>RADIUS OF CONVERGENCE</u> OF THE SERIES AND THE SET OF ALL VALUES OF $x$ FOR WHICH THE SERIES CONVERGES IS CALLED THE <u>INTERVAL OF CONVERGENCE</u>.

IN GENERAL, WE NEED TO EXAMINE (WHEN $R>0$) THE ENDPOINTS $x = a+R$ AND $x = a-R$ TO SEE IF THE SERIES CONVERGES THERE.
A CONVENIENT WAY TO FIND THE RADIUS OF CONVERGENCE IS THE RATIO TEST.

FIND THE RADIUS OF CONVERGENCE $R$ AND THE INTERVAL OF CONVERGENCE FOR EACH EXAMPLE BELOW.

**EXAMPLE 1** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \dots$

**SOLUTION** DEFINE $a_n = (-1)^{n-1}\dfrac{x^n}{n}$. TEST FOR ABSOLUTE CONVERGENCE.

USING RATIO TEST, $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1}\,\frac{n}{x^n}\right| = \frac{n}{n+1}|x|, \rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = |x|.$

BY RATIO TEST, SERIES IS ABSOLUTELY CONVERGENT (AND HENCE CONVERGENT) IF $|x|<1$ AND DIVERGENT IF $|x|>1$. THUS, $R=1$. NOW CHECK ENDPOINTS.

**AT $x=1$** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$. THIS IS Alternating series that is conditionally convergent.

**AT $x=-1$** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n} = -\sum_{n=1}^{\infty}\frac{1}{n} \rightarrow$ diverges (P-SERIES WITH $p=1$).

THUS, THE INTERVAL OF CONVERGENCE IS $-1 < x \leq 1$.

**EXAMPLE 2** $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$. DEFINE $a_n = \dfrac{x^n}{n!}$.

**SOLUTION** TEST FOR ABSOLUTE CONVERGENCE.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\,\frac{n!}{x^n}\right| = \frac{|x|}{n+1}$$

FOR convergence WE NEED BY RATIO TEST THAT $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\lim_{n\to\infty}\left(\frac{1}{n+1}\right) < 1 \implies$ SINCE $\displaystyle\lim_{n\to\infty}\frac{1}{n+1} = 0$ WE HAVE THAT THIS HOLDS FOR ALL $|x|$. THUS, $R=\infty$ AND SERIES CONVERGES FOR $-\infty < x < \infty$.

**EXAMPLE 3** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$

**SOLUTION** LET $a_n = (-1)^{n-1}\dfrac{x^{2n-1}}{2n-1}$. TEST FOR ABSOLUTE CONVERGE.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2(n+1)-1}}{2(n+1)-1}\,\frac{(2n-1)}{x^{2n-1}}\right| = \left|\frac{2n-1}{2n+1}\right||x|^2.$$

NOW $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|^2\lim_{n\to\infty}\left|\frac{2n-1}{2n+1}\right| = |x|^2.$

BY RATIO TEST, SERIES CONVERGES ABSOLUTELY IF $|x|^2 < 1 \longrightarrow |x|<1$ AND DIVERGES IF $|x|>1$. THUS, $R=1$.

NOW TEST THE ENDPOINTS:

AT $X=1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n X^{2n-1}}{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$. THIS SERIES CONVERGES CONDITIONALLY BY ALT. SERIES TEST.

AT $X=-1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^{2n-1}}{2n-1} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ SINCE $(-1)^{2n-1} = -1$ FOR ALL $n$. THUS CONVERGES CONDITIONALLY.

THE INTERVAL OF CONVERGENCE IS $-1 \leq X \leq 1$.

EXAMPLE 4 $\displaystyle\sum_{n=0}^{\infty} n!\, X^n = 1 + X + 2!\, X^2 + 3!\, X^3 + \cdots$

SOLUTION LET $a_n = n!\, X^n$. WE CALCULATE $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\, X^{n+1}}{n!\, X^n}\right| = (n+1)\,|X|$.

BY RATIO TEST SERIES CONVERGES IF AND ONLY IF $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |X| \lim_{n\to\infty}(n+1) < 1$.

SINCE $\displaystyle\lim_{n\to\infty}(n+1) = \infty \longrightarrow$ SERIES CONVERGES ONLY WHEN $X=0$. THUS, $R=0$.

EXAMPLE 5 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (X-1)^n}{2^n (n+2)}$.

SOLUTION LET $\displaystyle a_n = \frac{(-1)^n (X-1)^n}{2^n (n+2)}$. TEST FOR ABSOLUTE CONVERGENCE.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(X-1)^{n+1}}{2^{n+1}(n+3)} \cdot \frac{2^n (n+2)}{(X-1)^n}\right| = \frac{|X-1|}{2}\left(\frac{n+2}{n+3}\right)$$

NOW $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|X-1|}{2}\lim_{n\to\infty}\left(\frac{n+2}{n+3}\right) = \frac{|X-1|}{2}$.

THUS BY RATIO TEST SERIES CONVERGES ABSOLUTELY IF $\dfrac{|X-1|}{2} < 1$

AND DIVERGES IF $\dfrac{|X-1|}{2} > 1$. THUS $R=2$. NOW TEST ENDPOINTS $X=3,-1$.

AT $X=3$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (X-1)^n}{2^n (n+2)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}$ WHICH CONVERGES CONDITIONALLY.

AT $X=-1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (-2)^n}{2^n (n+2)} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n+2} = \sum_{n=1}^{\infty} \frac{1}{n+2}$ WHICH DIVERGES.

THUS, THE INTERVAL OF CONVERGENCE IS $-1 < X \leq 3$.

NEXT, WE STATE A RESULT FOR TERMWISE DIFFERENTIATION AND INTEGRATION OF A POWER SERIES WITHIN ITS INTERVAL OF CONVERGENCE.

THEOREM (DIFFERENTIATION) IF $\sum_{n=0}^{\infty} c_n (x-a)^n$ CONVERGES IN $a-R < x < a+R$ FOR SOME $R>0$ (possibly $R=\infty$), IT DEFINES A FUNCTION $f$ VIA:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{FOR} \quad a-R < x < a+R.$$

THEN WE HAVE,

$$f'(x) = \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1}, \qquad f''(x) = \sum_{n=2}^{\infty} n(n-1)\, c_n (x-a)^{n-2} \quad \square$$

THUS, WE ARE ALLOWED TO DIFFERENTIATE $f(x)$ TERM-BY-TERM IN $|x-a| < R$. NOTICE THAT FOR $f'(x)$ THE SERIES STARTS AT $n=1$ SINCE $f(x) = c_0 + c_1(x-a) + \dots$ IMPLIES $f'(x) = c_1 + 2c_2(x-a) + \dots = \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1}$. SAME FOR $f''(x)$ WHERE WE START FROM $n=2$.

THEOREM (INTEGRATION) SUPPOSE $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ CONVERGES IN $a-R<x<a+R$. THEN

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} + C \quad \text{IN} \quad a-R < x < a+R \quad \square$$

HERE $C$ IS A CONSTANT FIXED BY SOME VALUE OF $\int f\,dx$. THIS MEANS WE CAN INTEGRATE TERM BY TERM.

WE NOW CONSIDER SOME PROBLEMS RELATED TO THESE TWO KEY RESULTS.

WE RECALL THE GEOMETRIC SERIES

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \quad \text{FOR} \quad |x| < 1.$$

NOW INTEGRATING WE GET

$$-\log(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}. \qquad -1 < x < 1$$

SINCE $\log(1) = 0$ WE EVALUATE AT $x=0$ TO GET $C=0$.

THUS $\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} \dots$ VALID FOR $-1<x<1$.

NOW REPLACE $x$ BY $-x$ IN THIS FORMULA. WE GET

$$\log(1+x) = -(-x) - \frac{(-x)^2}{2} - \frac{(-x)^3}{3} \dots = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$$

SO $\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ VALID FOR $-1<x<1$.

NOW DERIVE A POWER SERIES REPRESENTATION FOR $\arctan x$.

FIRST RECALL $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ FOR $|x|<1$.

NOW LET $x = -y^2$. $\longrightarrow$ $\frac{1}{1+y^2} = 1 - y^2 + y^4 - y^6 + y^8 \dots$

REPLACE $y$ BY $x$ $\longrightarrow$ $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \dots$ FOR $|x|<1$.

NOW INTEGRATE BOTH SIDES AND SET $\arctan 0 = 0$. THIS GIVES

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)} (-1)^n \quad \text{FOR } |x|<1.$$

WE NOW CONSIDER A FEW SIMPLE EXAMPLES INVOLVING POWER SERIES MANIPULATIONS.

<u>EXAMPLE</u> THE INTERVAL OF CONVERGENCE OF $\sum_{n=0}^{\infty} n x^{n+1}$ IS $(-1,1)$. (EASY TO SHOW WITH RATIO TEST AND EXAMINING ENDPOINTS AT $x = \pm 1$). FIND A COMPACT FORMULA FOR $\sum_{n=0}^{\infty} n x^{n+1}$.

<u>SOLUTION</u> WE WANT $S = \sum_{n=0}^{\infty} n x^{n+1} = x^2 + 2x^3 + 3x^4 + \dots$ FOR $-1<x<1$

WE START WITH $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ IN $-1<x<1$

DIFFERENTIATE WRT $x$: $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 \dots$

MULTIPLY BY $x^2$: $x^2/(1-x)^2 = x^2 + 2x^3 + 3x^4 + \dots$ ✓. THUS $S = \frac{x^2}{(1-x)^2}$.

**EXAMPLE 2** FIND A SIMPLE COMPACT FORMULA FOR $\sum_{n=0}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$ IN $-1 < x < 1$. P6

**SOLUTION**

RECALL FROM EXAMPLE 1 THAT

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

MULTIPLY BY $x$: $\dfrac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$ IN $|x|<1$

NOW DIFFERENTIATE: $\dfrac{d}{dx}\left[\dfrac{x}{(1-x)^2}\right] = 1 + 4x + 9x^2 + 16x^3 + \cdots$

THUS $\dfrac{d}{dx}\left[\dfrac{x}{(1-x)^2}\right] = 1 + 4x + 9x^2 + 16x^3 + \cdots$

MULTIPLY BY $x$: $x\left[\dfrac{d}{dx}\left(\dfrac{x}{(1-x)^2}\right)\right] = x + 4x^2 + 9x^3 + 16x^4 + \cdots$ IN $|x|<1$.

$$x\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = x\frac{d}{dx}\left[x(1-x)^{-2}\right] = x\left[(1-x)^{-2} - 2x(1-x)^{-3}\right]$$

$$= x(1-x)^{-3}\left[(1-x) - 2x\right]$$

$$= \frac{x(1-3x)}{(1-x)^{+3}}$$

SO $\dfrac{x(1-3x)}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + \cdots = \sum_{n=0}^{\infty} n^2 x^n$ IN $-1 < x < 1$.

**EXAMPLE 3** CALCULATE $L = \lim_{x\to 0} \dfrac{2x - \log(1+2x)}{x^2}$

**SOLUTION** RECALL $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$ FOR $|x|<1$.

REPLACE $x \mapsto -x$: $\dfrac{1}{1+x} = 1 - x + x^2 + \cdots$

INTEGRATE $\log(1+x) = x - x^2/2 + \cdots$

REPLACE $x \mapsto 2x$ $\log(1+2x) = 2x - (2x)^2/2 + \cdots$

THUS $\dfrac{2x - \log(1+2x)}{x^2} = \dfrac{2x - (2x - (2x)^2/2 + \cdots)}{x^2} = \dfrac{2x^2 + \cdots}{x^2}$

THUS GIVES $\displaystyle \lim_{x\to 0} \frac{2x-\log(1+2x)}{x^2} = \lim_{x\to 0}\frac{2x^2+\cdots}{x^2} = 2.$

EXAMPLE 4 CALCULATE $\displaystyle \lim_{x\to\infty} x\left[\log(x+3)-\log(x+1)\right] \doteq L$

SOLUTION FOR LARGE $x$ WE WRITE
$$\log(x+3)-\log(x+1) = \log\left[x\left(1+\frac{3}{x}\right)\right]-\log\left[x\left(1+\frac{1}{x}\right)\right]$$
$$= \log x + \log\left(1+\frac{3}{x}\right) - \log x - \log\left(1+\frac{1}{x}\right)$$

THUS WE HAVE $\log(x+3)-\log(x+1) = \log\left(1+\frac{3}{x}\right)-\log\left(1+\frac{1}{x}\right).$ (*)

NOW RECALL $\frac{1}{1-h} = 1+h+h^2+\cdots$ FOR $|h|<1$.

REPLACE $h\mapsto -h$: $\frac{1}{1+h} = 1+(-h)+(-h)^2+\cdots = 1-h+h^2+\cdots$ FOR $|h|<1$

INTEGRATE: $\log(1+h) = 1-h+h^2\cdots$ FOR $|h|<1$.

NOW SETTING $h=3/x$ AND $h=1/x$ FOR $x$ large we get FROM (*) THAT
$$\log(x+3)-\log(x+1) = \log\left(1+\frac{3}{x}\right)-\log\left(1+\frac{1}{x}\right) = \left(1+\frac{3}{x}+\cdots\right)-\left(1+\frac{1}{x}+\cdots\right) = \frac{2}{x}.$$

THUS $\displaystyle \lim_{x\to\infty} x\left(\log(x+3)-\log(x+1)\right) = \lim_{x\to\infty} x\left(\frac{2}{x}+\cdots\right) = 2.$ SO $L=2$.

EXAMPLE 5 (HARD) SHOW THAT THE IMPROPER INTEGRAL $I = \int_1^\infty \left(\log(x^2+1)-2\log x\right)dx$ IS CONVERGENT AND CALCULATE ITS VALUE.

SOLUTION LET $f(x) \equiv \log(x^2+1) - 2\log x$. WE NEED TO SHOW THAT $f(x)\to 0$ AS $x\to\infty$ "FAST ENOUGH".

WE WRITE $f(x) = \log\left[x^2\left(1+\frac{1}{x^2}\right)\right] - 2\log x = \log x^2 + \log\left(1+\frac{1}{x^2}\right) - 2\log x$

SO $f(x) = \log\left(1+\frac{1}{x^2}\right)$ EXACTLY.

NOW RECALL $\log(1+h) = h - \frac{h^2}{2} + \cdots$ FOR $|h|<1$

THEREFORE FOR $x\to+\infty$, $f(x) \cong \frac{1}{x^2} - \frac{1}{2x^4}+\cdots$ SINCE $f(x) \simeq 1/x^p$ WITH $p=2$ FOR large $x \to$

WE MUST HAVE THAT $\int_1^\infty f(x)\,dx < \infty$.



NEXT, WE CALCULATE DIRECTLY:

$$u = \log x \quad du = \tfrac{1}{x}\,dx$$
$$dv = dx \quad v = x$$

$$I = \lim_{L\to\infty} \int_1^L \left(\log(x^2+1) - 2\log x\right) dx.$$

USE INTEGRATION BY PARTS: IN FIRST INTEGRAL

$$u = \log(x^2+1) \to du = \frac{2x}{x^2+1}\,dx$$
$$dv = dx \longrightarrow v = x$$

THUS
$$I = \lim_{L\to\infty}\left[ x\log(x^2+1)\Big|_1^L - 2\int_1^L \frac{x^2}{x^2+1}\,dx - 2\left(x\log x - x\right)\Big|_1^L \right]$$

$$= \lim_{L\to\infty}\left[ x\log(x^2+1)\Big|_1^L - 2\int_1^L \left(1 - \frac{1}{x^2+1}\right) dx - 2\left(x\log x - x\right)\Big|_1^L \right]$$

$$I = \lim_{L\to\infty}\left[ x\log(x^2+1)\Big|_1^L - 2\left(x - \arctan x\Big|_1^L\right) - 2x\log x\Big|_1^L + 2x\Big|_1^L \right]$$

WE NOW CANCEL TERMS:

$$I = \lim_{L\to\infty}\left[ L\log(L^2+1) - \log 2 - 2(L-1) + 2\arctan x\Big|_1^L - 2L\log L + 2(L-1)\right]$$

$$= \lim_{L\to\infty}\left[ L\log(L^2+1) - 2L\log L + 2\arctan L - 2\arctan(1)\right]$$

$$= \lim_{L\to\infty}\left[ L\log\left(\frac{L^2+1}{L^2}\right) + 2\arctan L - 2\arctan 1\right]$$

$$= \lim_{L\to\infty}\left[ L\log\left(1+\frac{1}{L^2}\right) + 2\arctan L - 2\arctan 1\right]$$

$$= \lim_{L\to\infty}\left[ L\left(1+\frac{1}{L^2}+\cdots\right)\right] + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= 0 + \pi/2.$$

THUS $I = \pi/2$.