

Squeeze theorem.

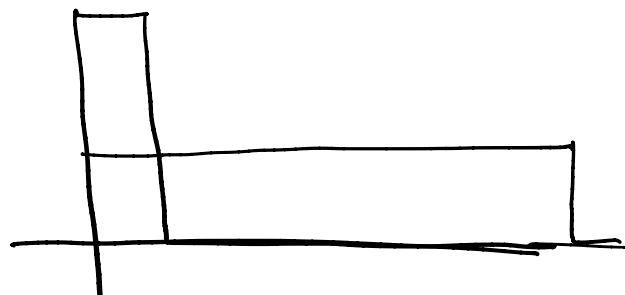
Theorem If $a_n \leq b_n \leq c_n$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$

then $\lim_{b_n \rightarrow \infty} = L$.

Let $b_n \rightarrow L$ and assume $a_n \leq b_n \leq c_n$. Is

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L ?$$

Assume $\int_a^b f(x) dx$ and $\int_a^b g(x) dx$ converge and
 $\int_a^b f(x) dx \leq \int_a^b g(x) dx$. Is $f(x) \leq g(x)$ for
all $x \in [a, b]$.



Squeeze theorem:

o Compute $\lim_{n \rightarrow \infty} \frac{n!}{n^n}$

$$a_n \leq b_n \leq c_n$$

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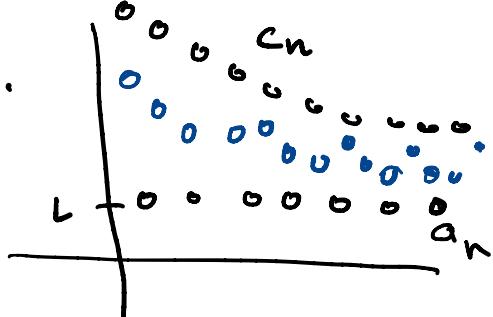
L L L

$$0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} \leq \frac{1}{n} \cdot \underbrace{1 \cdot 1 \cdots 1}_{n-1} = \frac{1}{n} \downarrow 0$$

↓ ↓ ↓

0 0 0

- True or False: let $a_n \leq b_n \leq c_n$ with $a_n \rightarrow L$ and $c_n \rightarrow L+1$. Then $\{b_n\}$ converges to a real number M that satisfy $L \leq M \leq L+1$.



Example

L

- Given that $|a_n| \rightarrow 0$, does $\{a_n\}$ converge?

$$-|a_n| \leq a_n \leq |a_n|$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$0 \quad 0 \quad 0$$

Sequences and continuous functions.

Theorem: If $a_n \rightarrow L$ and $f(x)$ is continuous at $x=L$ then $f(a_n) \rightarrow f(L)$.

Example

(1) Calculate $\lim_{n \rightarrow \infty} \sin\left(\frac{\pi n}{2n+1}\right)$

let $a_n = \frac{\pi n}{2n+1}$.. $\lim_{n \rightarrow \infty} \frac{\pi n}{2n+1} = \lim_{n \rightarrow \infty} \frac{\pi}{2 + \frac{1}{n}}$

by theorem:

$$\sin\left(\frac{\pi n}{2n+1}\right) \rightarrow \sin\left(\frac{\pi}{2}\right) = \cancel{\frac{1}{2}}$$

$r^n \rightarrow 0$
if $|r| < 1$.

$$= \frac{\lim_{n \rightarrow \infty} \pi}{\lim_{n \rightarrow \infty} \left(2 + \frac{1}{n}\right)} = \frac{\pi}{2}$$

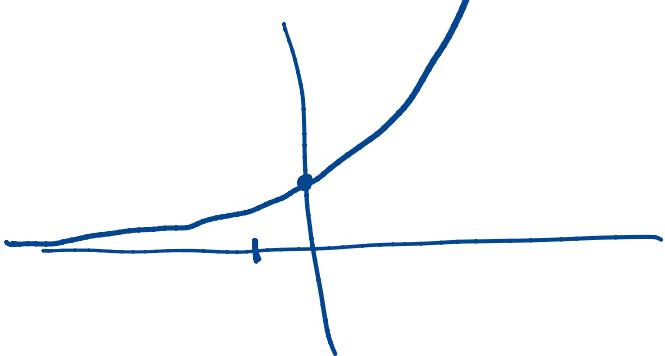
Example

ii) Calculate

$$\lim_{n \rightarrow \infty}$$

$$e^{-\left(\frac{n^2+2}{n^2+1}\right)}$$

inner sequence



$$a_n = -\frac{n^2+2}{n^2+1}$$

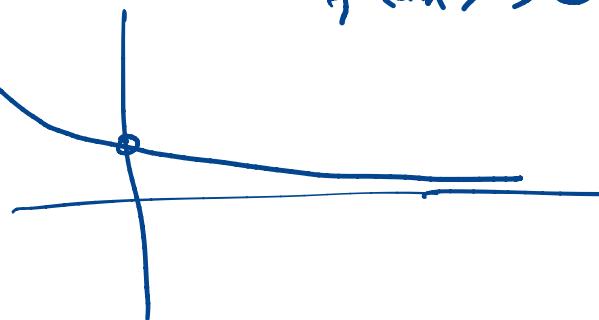
$$a_n = \frac{n^2+2}{n+1}$$

$$f(a_n) \rightarrow e^{-1}$$

$$\boxed{1 - f(x) = e^x}$$

$$1 - f(x) = e^{-x}$$

$$f(a_n) \rightarrow e^{-1}$$

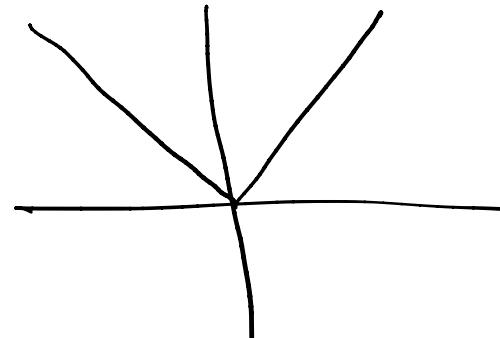


Example.

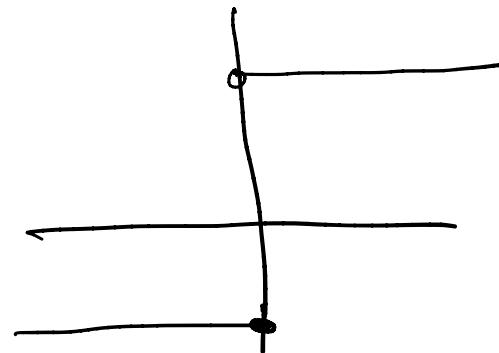
Let $f(x) = |x|$.

compute $\lim_{n \rightarrow \infty} f' \left(\frac{n^2+1}{(n+1)(n^2+2)} \right)$

$$\frac{n^2+1}{(n+1)(n^2+2)} \approx \frac{n^2}{n^3} = \frac{1}{n} \rightarrow 0$$



$$f(x) = |x|$$



$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x > 0 \end{cases}$$

Exponential function.

$$\{r^n\}_{n=0}^{\infty} = \{1, r, r^2, r^3, \dots\}$$

This is a very useful lemma that we will use later in next lecture.

Lemma: The sequence $a_n = r^n$ is convergent when $-1 < r \leq 1$ and otherwise is divergent.

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 1 & r = 1 \\ 0 & |r| < 1 \\ \text{divergent} & \text{otherwise.} \end{cases}$$

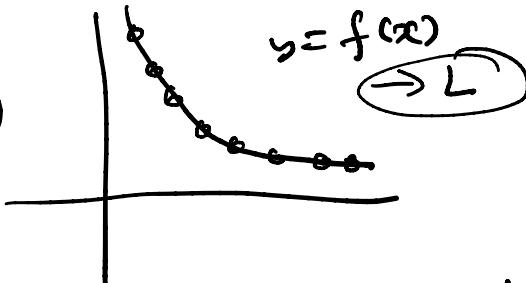
Recall:

Thm: If $\lim_{x \rightarrow \infty} f(x) = L$ and $a_n = f(n)$ for $n = 1, 2, 3, \dots$

then $\lim_{n \rightarrow \infty} a_n = L$.

Proof of Lemma.

- In the case $r=1$, obvious! ($r^n=1$)
- In the case $|r| < 1$. consider $f(x) = r^x$. Note that $r^x \rightarrow 0$ as $x \rightarrow \infty$ if $-1 < r < 1$
so, by using the theorem, $r^n \rightarrow 0$ if $|r| < 1$
- If $|r| > 1$. $r^n = \underbrace{r \cdot r \cdots r}_{n\text{-times}} \rightarrow \infty$



Bounded sequences.

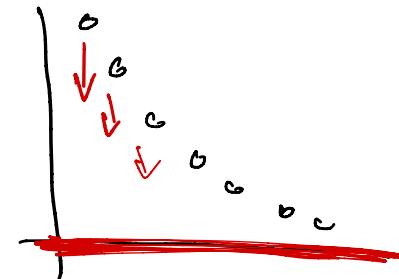
Sometimes it is useful to whether a sequence converges without computing the exact value.

consider the sequence $a_n = \frac{1}{n+2}$. Does $\{a_n\}$ converge?

We know:

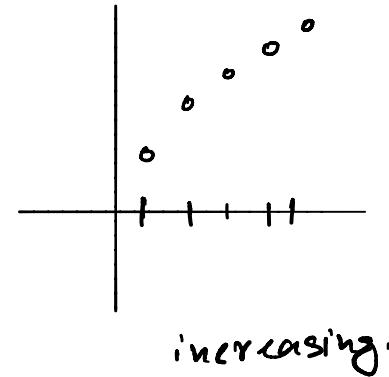
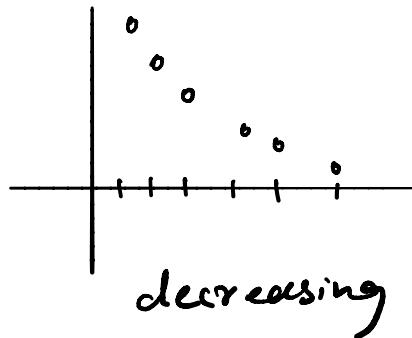
- $a_n > 0$ for all n - lower bounded.
- $a_{n+1} < a_n$ for all n - decreasing.

$$a_{n+1} - a_n = \frac{1}{n+3} - \frac{1}{n+2} = \frac{(n+2) - (n+3)}{(n+3)(n+2)} = \frac{-1}{(n+3)(n+2)} < 0$$



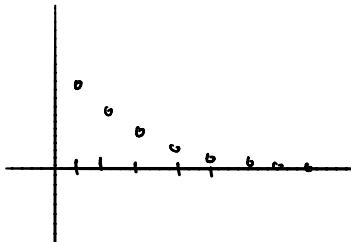
Definition:

- A sequence $\{a_n\}$ is decreasing if $a_n \geq a_{n+1}$ for all n .
- A sequence $\{a_n\}$ is increasing if $a_n \leq a_{n+1}$ for all n .
- A sequence is monotonic if it is increasing or decreasing



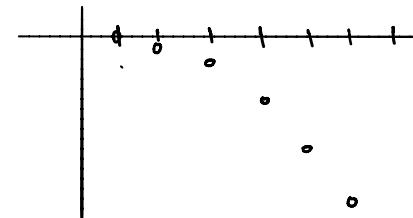
Definition.

- A sequence $\{a_n\}$ is bounded below if there is some number M s.t. $a_n \geq M$ for all $n \geq 1$.
- A sequence $\{a_n\}$ is bounded above if there is some number M s.t. $a_n \leq M$ for all $n \geq 1$.
- A sequence is bounded if it is bounded below and above.



$$a_n = e^{-n}$$

bounded below



$$a_n = \ln\left(\frac{1}{n}\right)$$

unbounded below.

Monotonic sequence theorem (Monotonic Convergence Theorem).

Thm Every bounded and monotonic sequence converges.

example : $a_n = \frac{n}{n+1}$

$$a_n - a_{n+1} = \frac{n}{n+1} - \frac{n+1}{n+2} = \frac{n(n+2) - (n+1)^2}{(n+1)(n+2)}$$

$$\Rightarrow a_n \leq a_{n+1} \text{ (increasing).} \quad = \frac{n^2 + 2n - n^2 - 2n - 1}{(n+1)(n+2)} \leq 0$$

(1)

$$\Rightarrow 0 \leq a_n \leq 10 \quad \text{(bounded)}$$

$\Rightarrow \{a_n\}$ converges.

Converse of MCT.

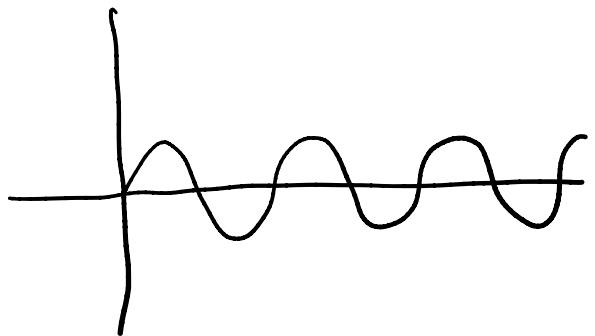
True / False: If a sequence is convergent then it is monotonic and bounded. False

Ex : $a_n = \frac{\sin(n)}{n} \rightarrow \underline{\text{not monotonic}}$

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$
$$\begin{matrix} \downarrow & & \downarrow \\ 0 & & 0 \\ \downarrow & & \downarrow \\ 0 & & 0 \end{matrix}$$

By squeeze theorem.

$\sin(x)$



$\frac{1}{x}$, $-\frac{1}{x}$

$\frac{\sin x}{x}$

