

Separable differential equation.

- A differential equation is an equation that involves a function and one or more of its derivatives.
- For example: $y' = xy$. — ①
 y is understood to be a function of x .
- The order of the differential equation is order of the highest derivative in the equation. In ①, the order is 1.
- A function $f(x)$ is solution of the differential equation if $f'(x) = xf(x)$.

Differentiable equations.

$$y' = xy$$

- The solution to differential equation is a family of functions - you have to find all anti derivatives.
- The solution to ① is $y = Ce^{x^2/2}$ for any C .
Check: $y' = Ce^{x^2/2} \cdot \frac{2x}{2} = Ce^{x^2/2} x = xy$.

Differential equation.

- Another example from last term (Math 100?) :
Population model, radioactive decay, etc.

$$\frac{dP}{dt} = KP$$

Solution: $P(t) = A \cdot e^{kt}$. How do we find A ?

- Assume at time $t=0$, the population $P(0) = P_0$.

then $P(0) = A e^{k \cdot 0} \Rightarrow A = P_0$

- $\frac{dP}{dt} = KP , P(0) = P_0$

This is an example of initial value problem.

A differential equation together with an initial condition.

How do we solve differential equations.

Consider some general differential equation:

$$F'(x) = \text{some function } x.$$

most integration problem we have done can be considered differential equation:

$$\int F'(x) dx = \int (\text{some function } x) dx$$

$$F(x) = \int (\text{some function of } x) dx.$$

Separable differential equation.

Definition: A separable differential equation is one of the form

$$\frac{dy}{dx} = f(y)g(x).$$

LHS: $\frac{dy}{dx}$

RHS: factored into a function of x and a function of y .

eg: o $\frac{dy}{dx} = x \cdot y$

o $\frac{dy}{dx} = \frac{xy}{x^2 + 1}$

o $x \frac{dy}{dx} + y = y^2$

Solving differential equations.

$$\frac{dy}{dx} = f(y)g(x).$$

- Note that RHS is of the form $f(y)g(x)$ and not $f(y)+g(x)$.

- Separate this equation - pull all y -stuff to the left and leave all x -stuff on the right.

$$\frac{1}{f(y)} \frac{dy}{dx} = g(x).$$

$$\frac{1}{f(y)} y' = g(x)$$

- Integrate both sides wrt x .

$$\int \frac{1}{f(y)} \frac{dy}{dx} dx = \int g(x) dx.$$

$$\int f(u) u' dx$$

$$= \int f(u) du$$

- Use u -substitution: $\int \frac{1}{f(y)} dy = \int g(x) dx.$

$$y = y(x).$$

Examples

Solve

$$\frac{dy}{dx} = -x/y = -x \cdot \frac{1}{y}$$

$$y' = x \cdot y$$

$$y(0) = \underline{\quad}$$

$$\Rightarrow y \frac{dy}{dx} = -x \quad \text{separate}$$

$$\Rightarrow \int y \frac{dy}{dx} dx = \int -x dx \quad \text{integrate both sides.}$$

$$\Rightarrow \int y dy = -\frac{1}{2}x^2 + C_1 \quad \text{— don't forget the constant.}$$

$$\Rightarrow \frac{1}{2}y^2 + C_2 = -\frac{1}{2}x^2 + C_1$$

$$\Rightarrow \frac{1}{2}y^2 = C_1 - C_2 - \frac{1}{2}x^2 \quad \text{— rearrange}$$

$$\Rightarrow y = \pm \sqrt{2(C_1 - C_2) - x^2}$$

$$\Rightarrow y = \pm \sqrt{C - x^2} \quad \text{— let } C = 2C_1 - 2C_2$$

Solution is a circle.

Example 2

$$\text{Solve } \frac{dy}{dx} = xy \quad \text{--- (2)}$$

For now, assume $y \neq 0$.

$$\frac{1}{y} \frac{dy}{dx} = x$$

$$\Rightarrow \int \frac{1}{y} \frac{dy}{dx} dx = \int x dx$$

$$\Rightarrow \int \frac{1}{y} dy = \int x dx$$

$$\Rightarrow \ln|y| = \frac{x^2}{2} + C$$

$$\Rightarrow |y| = e^{\frac{x^2}{2}} \cdot e^C \left(e^{x^2/2 + C} \right)$$

$$\Rightarrow y = \pm e^C e^{\frac{x^2}{2}}$$

$$y = A e^{\frac{x^2}{2}}$$

$$A = \pm e^C$$

$$y=0 \quad \frac{dy}{dx} = 0 = x \cdot 0 = xy$$

Note that $e^C \neq 0$ and

Check if $A=0$ in $y = A e^{\frac{x^2}{2}}$ satisfies (2)

$$y = 0 \cdot e^{\frac{x^2}{2}} = 0$$

$$\frac{dy}{dx} = 0 = 0 \cdot x.$$

So, $y=0$ is a solution and

the general solution is $y = A e^{\frac{x^2}{2}}$.

$$\frac{dy}{dx} = \frac{d}{dx} (A e^{\frac{x^2}{2}}) = xy$$

Example 3

Solve initial value problem: $\frac{dy}{dx} = \frac{xy}{x^2+1}$, $y(0) = 3$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{x}{x^2+1} \quad \text{--- separate}$$

$$\Rightarrow \int \frac{1}{y} dy = \int \frac{x}{x^2+1} dx \quad \text{--- integrate both sides wrt } x \\ \text{and substitution.}$$

$$\Rightarrow \ln|y| = \int \frac{1}{2u} du \quad \text{--- sub } u = x^2 + 1, \frac{du}{dx} = 2x.$$

$$\Rightarrow \ln|y| = \frac{1}{2} \ln|u| + C \quad \text{--- don't forget the constant.}$$

$$\Rightarrow \ln|y| = \frac{1}{2} \ln|x^2+1| + C \quad C \ln(a) = \ln(a^C)$$
$$= \ln|\sqrt{x^2+1}| + C$$

Example 3 contd.

Since $x^2+1 > 0$ for all x , we can remove absolute value:

$$\Rightarrow \ln|y| = \ln(\sqrt{x^2+1}) + C \Rightarrow e^{\ln|y|} = e^{\ln(\sqrt{x^2+1}) + C}$$

$$\Rightarrow |y| = e^C \sqrt{x^2+1} \Rightarrow y = \pm e^C \sqrt{x^2+1}$$

Since $y(0)=3$, we have $|y(0)| = e^C \sqrt{0^2+1}$
 $\Rightarrow e^C = 3$

so, $y = \pm 3 \sqrt{x^2+1}$ — this is not a single function.

but notice $y(0)=3$ so we take positive.

i.e. $y = 3 \sqrt{x^2+1}$ is the solution.

Example 4

Find a solution of $x \frac{dy}{dx} + y = y^2$, $y(1) = -1$.

First express the differential equation in a way
easy to separate:

$$\frac{dy}{dx} = \frac{y^2 - y}{x}.$$

$$\Rightarrow \int \frac{1}{y^2-y} \frac{dy}{dx} dx = \int \frac{1}{x} dx$$

$$\Rightarrow \int \frac{1}{y^2-y} dy = \ln|x| + C \quad - \text{ don't forget const!}$$

$$\Rightarrow \int \frac{1}{y(y-1)} dy = \ln|x| + C$$

↑
use partial fractions.

Example 5

using partial fraction:

$$\int \frac{1}{y(y-1)} dy = \int \frac{A}{y} dy + \int \frac{B}{y-1} dy.$$

$$\text{since } 1 = A(y-1) + By = y(A+B) - A$$

$$\text{we get } -A = 1 \text{ and } A+B=0$$

$$\Rightarrow A = -1, B = 1$$

$$\text{so, } \int \frac{1}{y^2-y} dy = \int \frac{-1}{y} dy + \int \frac{1}{y-1} dy$$

$$= -\ln|y| + \ln|y-1| + C$$

Example 4.

Put it all together: $\int \frac{1}{y(y-1)} dy = \int \frac{1}{x} dx$ $\ln a - \ln b$
 $= \ln\left(\frac{a}{b}\right)$

we get: $-\ln|y| + \ln|y-1| = \ln|x| + C$

$$\Rightarrow \ln\left|\frac{y-1}{y}\right| = C \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \ln|y-1| - \ln|y| - \ln|x| = C$$

$$\Rightarrow \left|\frac{y-1}{yx}\right| = e^C \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \frac{y-1}{yx} = \pm e^C = \pm 2$$

using initial value of $y(1) = -1$:

$$\left|\frac{y(1) - 1}{y(1) \cdot 1}\right| = e^C \Rightarrow \left|\frac{-2}{-1}\right| = e^C \Rightarrow e^C = 2$$

so, $\frac{y-1}{xy} = 2$ is the solution.

Example 4 contd.

Rearrange to get:

$$\begin{aligned}\frac{y-1}{xy} &= 2 \Rightarrow 2xy = y-1 \\ &\Rightarrow y(1-2x) = 1 \\ &\Rightarrow y = \frac{1}{1-2x} \quad . \quad \underline{\text{Done!}}\end{aligned}$$

How to check? **differentiate**

$$\frac{dy}{dx} = \frac{d}{dx} ((1-2x)^{-1}) = -1 (1-2x)^{-2} \cdot (-2) \quad \left| \begin{array}{l} \frac{x dy}{dx} + y = y^2 \\ \Rightarrow \frac{x dy}{dx} = y^2 - y \end{array} \right.$$

$$= \frac{2}{(1-2x)^2}$$

$$\text{and } y^2 - y = \frac{1}{(1-2x)^2} - \frac{1}{(1-2x)} = \frac{1 - (1-2x)}{(1-2x)^2} = \frac{2x}{(1-2x)^2}$$

Example 4 contd.

$$\text{So, } y^2 - y = x \frac{dy}{dx} .$$

Both is satisfied.

$$\text{Also, } y(1) = \frac{1}{1-2(0)} = -1$$