Recall (Ratio test)

The series $\sum_{n=1}^{\infty} a_n$

① converges absolutely if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$.

② diverges if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.

Recall (absolute/conditional convergence)

① $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

② $\sum_{n=1}^{\infty} a_n$ converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Ratio test and examples.

- The ratio test is always inconclusive for ratios of polynomials since. For example, consider $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+3n}{n^3+4n^2}$ with $a_n = (-1)^n \frac{n^2+3n}{n^3+4n}$. Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

Example: $\sum_{n=1}^{\infty} \frac{n}{5^n}$. We could use ratio test with $f(x) = \frac{x}{5^x}$ or we could use ratio test.

Let $a_n = \frac{n}{5^n}$. Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n}\right|$

$$= \lim_{n\to\infty} \left|\frac{n+1}{n} \cdot \frac{1}{5}\right|$$

$$= \frac{1}{5} < 1,$$ series converges.

Example 2

$\sum_{n=1}^{\infty} \frac{n^n}{n!}$. Let $a_n = \frac{n^n}{n!}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right| = \lim_{n\to\infty} \left|\frac{1}{n+1} \frac{(n+1)^{n+1}}{n^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)^n}{n^n}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = 2.71\ldots > 1$$

so, $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

Why is $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$? Take log of $\left(1+\frac{1}{n}\right)^n$.

$$\lim_{n\to\infty} n \log\left(1+\frac{1}{n}\right) = \lim_{x\to 0} \frac{1}{x} \log(1+x) = 1 \quad \text{(why?)}$$

so, $\left(1+\frac{1}{n}\right)^n = e^1$ as $n \to \infty$.

Example

$$\sum_{n=1}^{\infty} \frac{n^n}{3^n n!} . \quad \text{Here} \quad a_n = \frac{n^n}{3^n n!}$$

so, $$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n^n} = \frac{1}{3} \cdot \frac{1}{n+1} \cdot \frac{(n+1)^{n+1}}{n^n} = \frac{1}{3}\left(1+\frac{1}{n}\right)^n .$$

so, $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{3} \left(1+\frac{1}{n}\right)^n = \frac{1}{3} . e = \frac{2.71..}{3} < 1 .$$

Thus, series converges absolutely.

Example

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n^2+1)(n!)^2}.$$ Let $a_n = \dfrac{(2n)!}{(n^2+1)(n!)^2}$

So, $$\frac{a_{n+1}}{a_n} = \frac{(2(n+1))!}{((n+1)^2+1)((n+1)!)^2} \cdot \frac{(n!)^2\, n^2+1}{2n!}$$

$$= \frac{(2(n+1))!}{(2n)!} \cdot \frac{1}{(n+1)^2} \cdot \frac{n^2+1}{(n^2+1)^2+1}$$

$$= \frac{(2n+1)(2n+2)}{(n+1)^2} \cdot \frac{n^2+1}{(n^2+1)^2+1}$$

So, $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 4 > 1.$ So, series diverges.

Power Series.

Def^n: An expression of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ is a power series centered at $x=0$. An expression of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x=a$.

For example: Geometric series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ is a power series centered at $x=0$. Recall that geometric series converges to $\frac{1}{1-x}$ for $|x|<1$, and so

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad |x|<1.$$

Convergence of power series.

For a general power series, there are three possibilities for convergence of $\sum_{n=0}^{\infty} c_n (x-a)^n$.

Theorem (convergence of power series): For $\sum_{n=0}^{\infty} a_n (x-a)^n$, one of the following holds:

1. There is a positive number R such that $\sum_{n=0}^{\infty} a_n (x-a)^n$ diverges for $|x-a|>R$ and converges for $|x-a|<R$. The series may or may not converge for $x=a\pm R$.
2. $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for all x (i.e. $R=\infty$).
3. $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges at $x=a$ (i.e. $R=0$).

Radius of convergence

Def^n: The number R is called radius of convergence of $\sum_{n=0}^{\infty} a_n (x-a)^n$ and the set of all values for which the series converges is called interval of convergence

For $\sum_{n=0}^{\infty} x^n$, radius of convergence, $R = 1$

interval of convergence if $R \in (-1, 1)$.

- In general, we need to examine (for $R>0$) the endpoints $x = a \pm R$ to see if it converges there.

- A good way to find radius of convergence is by ratio test.

Example 1

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

Sol^n: Define $a_n = \frac{(-1)^{n-1} x^n}{n}$. $\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} = \frac{xn}{n+1}$

so, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = |x|$

By ratio test, series is absolutely convergent if $|x| < 1$ and divergent if $|x| > 1$. Now, check end points.

$x = 1$, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1^n}{n}$. This series is conditionally convergent

$x = -1$ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = -1 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$. This

is divergent by p-test.

So, Radius of convergence, $R = 1$

interval of convergence, $(-1, 1]$.

Example 2

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots \qquad \text{Define } a_n = \frac{x^n}{n!}$$

Then. $$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}$$

so, $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0$$

so, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for all x.

Thus, radius of convergence is ∞ and interval of convergence is $(-\infty, \infty)$.

Example 3.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots .$$ Let $a_n = \frac{(-1)^{n-1} x^{2n-1}}{2n-1}$

So, $$\frac{a_{n+1}}{a_n} = \frac{(-1)^n x^{2n+1}}{2n+1} \frac{2n-1}{(-1)^{n-1} x^{2n-1}} = -1 \cdot x^2 \cdot \frac{2n-1}{2n+1}$$

So, $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|^2 < 1 \quad \text{if } |x|<1.$$

So, By ratio test, series is absolutely convergent if $|x|<1$ and divergent if $|x|>1$. Now check endpoints.

at $x = 1$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (1)^{2n-1}}{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$ which is conditionally convergent.

at $x = -1$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-1)^{2n-1}}{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot (-1)^{-1}}{2n-1}$$ which is conditionally convergent.

so, $R = 1$

interval of convergence $= \underline{\underline{[-1, 1]}}$

Example 4

$$\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\,x^2 + \ldots \qquad \text{let } a_n = n!\, x^n$$

so, $$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\, x^{n+1}}{n!\, x^n} = (n+1)\,x$$

So, $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty \quad \text{if } x \neq 0$$

$\Rightarrow$ series is divergent for all $x \neq 0$.

Radius of convergence is 0.

interval of convergence is $\{0\}$.