Series.

Geometric series: $\sum_{n=1}^{\infty} a r^{n-1}$ $\left(\text{or } \sum_{n=0}^{\infty} a r^{n}\right)$.

Theorem: Consider the geometric series $\sum_{n=1}^{\infty} a r^{n}$. If $|r| < 1$, then it converges and

$$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$ and $a \neq 0$ then the series diverges.

Example

Calculate $$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}}$$

Counter Example

Notice that both $\sum_{n=1}^{\infty} \left(-\frac{4}{3}\right)^n$ and $\sum_{n=1}^{\infty} \frac{9^n}{5 \cdot 3^{2n}}$ diverges.

Example

Write the repeating decimal $0.12\overline{34} = 0.123434\ldots$ as a ratio of integers.

Sol^n:

Telescoping sum.

$$S_n = a + ar + ar^2 + ar^3 + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots = \sum_{n=1}^{\infty} ar^n$$

And $S_n - rS_n = \sum_{n=1}^{\infty} \left(ar^{n-1} - ar^n \right)$ — type of telescoping sum.

Thm: Suppose $S = \sum_{n=1}^{\infty} (a_n - a_{n+1})$ with $\lim_{n\to\infty} a_n = L$.

Then $S = a_1 - L$ and so S is convergent.

Examples.

Calculate $\sum_{n=3}^{\infty}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right)$

Example

Calculate $$\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{3^n}+\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

Solution:

Example (cont'd.)

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Divergence test

- In general, it is very difficult to calculate a convergent infinite sum.
- We need criteria for establishing whether an infinite series converges or diverges.

Divergence test:

Consider a statement:

If (he is Shakespeare) then (he is dead)

P Q

Divergence test:

Lemma: If a series $\sum_{n=0}^{\infty} a_n$ is convergent then $a_n \to 0$.

Contrapositive:

Harmonic Series

Lemma: If a series $\sum_{n=0}^{\infty} a_n$ is convergent then $a_n \to 0$.

Harmonic series

Let's compute the 2^n partial sum of $\sum_{n=1}^{\infty} \frac{1}{n}$.