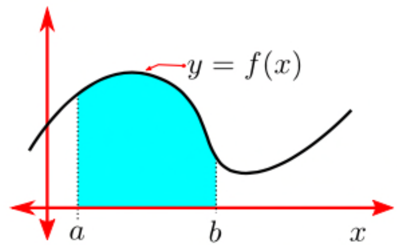
Area under the curve



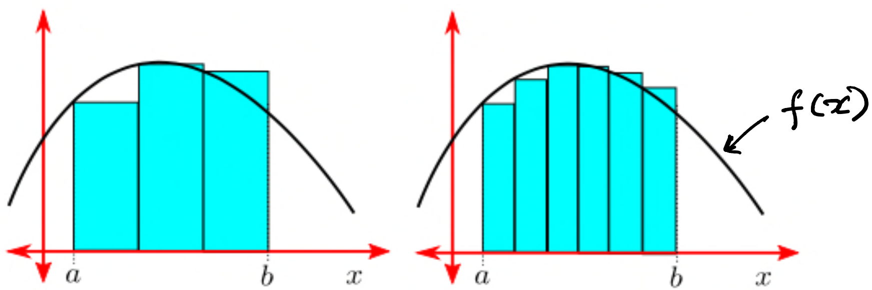
Goal:

① Approximate the area in the shaded region.

② Riemann sum.

Sum of rectangles under curve.

We consider a function $f : \mathbb{R} \to \mathbb{R}$, i.e. f takes values in the set of real numbers and outputs values in the same set (self-map).



- Sum of area under rectangle $\approx$ area under curve.

- Increase the number of rectangles to improve approx.

Summation notation CLP section 1.1.3.

We use the symbol "Σ" to denote sum (called sigma)

For example:

- The sum of first 20 integers is

$$1+2+\ldots+20 = \sum_{i=1}^{20} i.$$

The sum reads: The sum of i from $i=1$ to 20.

Alternatively: $1+2+\ldots+20 = \sum_{\square=1}^{20} \square$

dummy variable.

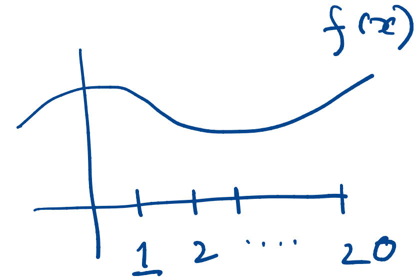
- The sum of cubes is

$$1^3+2^3+\ldots+20^3 = \sum_{i=1}^{20} i^3 = \sum_{j=1}^{20} j^3$$

Summation notation CLP section 1.1.3. (contd.)

- Let f be any real valued function. A sum of a function evaluated at integer points.

$$f(1) + f(2) + \cdots + f(20) = \sum_{i=1}^{20} f(i)$$



- A more formal sum

$$\sum_{i=1}^{K} a_i = a_1 + a_2 + \cdots + a_K$$

$$a_i = i \quad , \quad a_i = i^3 \quad , \quad a_i = f(i)$$

Properties of sum (Thm 1.1.5. in CLP)

An important property of sum is that it's a "linear operator".

- Closed under scalar multiplication.

For any $c \in \mathbb{R}$, $\sum_{i=1}^{n} c a_i = c\left(\sum_{i=1}^{n} a_i\right)$ constants can come outside

- Closed under addition.

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i + (-b_i))$$

$$\sum_{i=1}^{4} (i^3 + i)$$

$$\gamma(a+b) = \gamma a + \gamma b$$

$$(1^3+1) + (2^3+2) + (3^3+3) + (4^3+4)$$

Special sums.

- First n integer : $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ → in class.
- First n integer square : $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- First n integer cube : $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

see text.

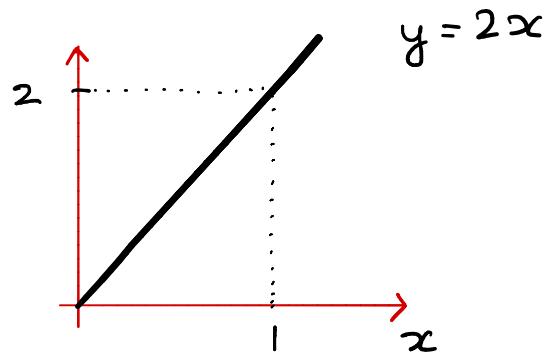
$$S = \sum_{k=1}^{n} k = 1+2+3+\dots+n$$

$$S = \sum_{k=1}^{n} k = n+(n-1)+\dots+1$$

$$2S = (n+1)+(n+1)+\dots+(n+1) = n(n+1)$$

Back to areas and example

Q: Find the area between the curve $y = 2x$ and the x-axis between $x=0$ and $x=1$.

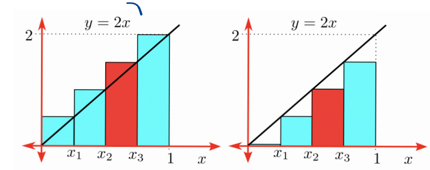


Idea:

- Divide $[a, b]$ n equal sized non-overlapping sub-intervals.
- For each sub interval, approx using rectangle
- Add the area.

Example (contd)

$0 \leftrightarrow 1$ $= 1/2$



$x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$

$x_i = a + i\Delta x$, $x_n = b$, $x_0 = a$

- Subinterval width: $\Delta x = 1/n$
- subinterval: $a=0$, $b=1$; $x_1 \; x_2 \; \cdots \; x_n$

$x_1 = a + \Delta x$
$x_2 = a + 2\Delta x$
$x_3 = a + 3\Delta x$ $\rightarrow x_i = a + i\Delta x$

$\Rightarrow$ i^{th} subinterval is $[x_{i-1}, x_i]$

- height (right end point): $f(x_i)$ is the height of i^{th} rectangle.
- Riemann sum: $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$f(x_i) = 2i/n$

$f(x) = 2x$

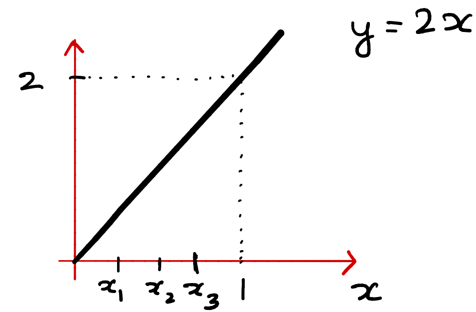
$x_i = i/n$

$$= \sum_{i=1}^{n} \underbrace{\frac{2i}{n}}_{} \cdot \frac{1}{n} = \frac{2}{n^2}\sum_{i=1}^{n} i = \frac{2}{n^2}\,\frac{n(n+1)}{2}$$

- limit as $n \to \infty$: $R_n = \dfrac{2n^2 + 2n}{2n^2} = 1$ as $n \to \infty$

$1/n \to 0$ as

Approximation using left endpoint:



- Partition the interval $[0, 1]$ into n uniformly sized sub-intervals. The width of each sub-interval is $\Delta x = \frac{1}{n}$.

- So, $x_0 = 0$, $x_1 = \frac{1}{n}$, $x_2 = \frac{2}{n}$, $\ldots$, $x_n = b$

- The height of the i^{th} rectangle is $f(x_{i-1}) = \frac{2(n-1)}{n}$

- The total area:

$$L_n = \sum_{i=1}^{n} f(x_{i-1}) \Delta x = \sum_{i=1}^{n} \frac{2(n-1)}{n} \frac{1}{n} = \frac{2}{n^2}\left(\sum_{i=1}^{n} n - \sum_{i=1}^{n} 1\right)$$

$$= \frac{2}{n^2}\left(\frac{n(n+1)}{2} - n\right)$$

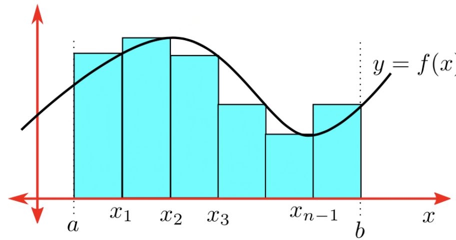
Approximation using left endpoint (contd)

- Now, to get exact area we take $n \to \infty$

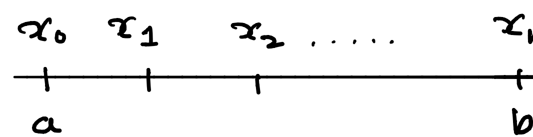
$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{2}{n^2}\left(\frac{n(n+1)}{2} - n\right) = ?$$

Riemann sum

Let $f : \mathbb{R} \to \mathbb{R}$ be any function.



Riemann sum is the sum of area of n rectangles used to approximate area under curve from $x = a$ and $x = b$.



- uniform width : $\Delta x = (b-a)/n$

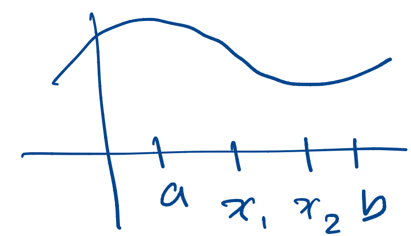
Riemann sum (contd)

- the ordinate :

$x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_n$

$a \quad a+\Delta x \quad a+2\Delta x \quad \cdots \quad b = a + n\Delta x$

$$x_i = a + i\,\Delta x \quad \text{for} \quad i = 0, \dots, n$$



- The hieght :

ith rectangle using right endpoint : $f(x_i)$

ith rectangle using left end point : $f(x_{i-1})$

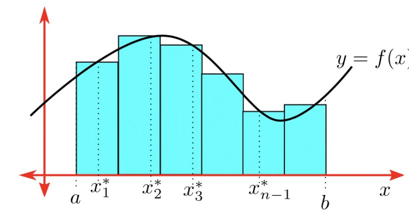
using midpoint : $f\left(\frac{x_{i-1}+x_i}{2}\right)$

$$\text{Riemann sum} = \sum_{i=1}^{n} f(x_i^*)\,\Delta x, \quad x_i^* = x_i \text{ for right end point}$$

$$x_i^* = x_{i-1} \text{ for left.}$$

$$x_i^* \in [x_{i-1}, x_i] \quad , \quad f(x_i^*)$$

Riemann sum (contd)



Consider the sub-interval $[x_{i-1}, x_i]$

We can pick any point in $[x_{i-1}, x_i]$ to get the height of the rectangle.

Say we pick $x_i^* \in [x_{i-1}, x_i]$.

$$R_n = \Delta x\, f(x_1^*) + \Delta x\, f(x_2^*) + \dots + \Delta x\, f(x_n^*)$$

And area $= \lim_{n \to \infty} R_n$.

Riemann sum (CLP Definition 1.1.11)

Let a, b be two real numbers. Let n be a positive integer and $f(x)$ be defined on $[a, b]$.
Set $\Delta x = (b-a)/n$ and then (as above) divide the interval $[a, b]$ up to n even sub-intervals $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$. Then the sum

$$\sum_{k=1}^{n} f(x_k^*) \Delta x$$

is called a Riemann sum.