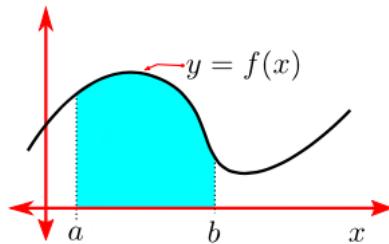


## Area under the curve

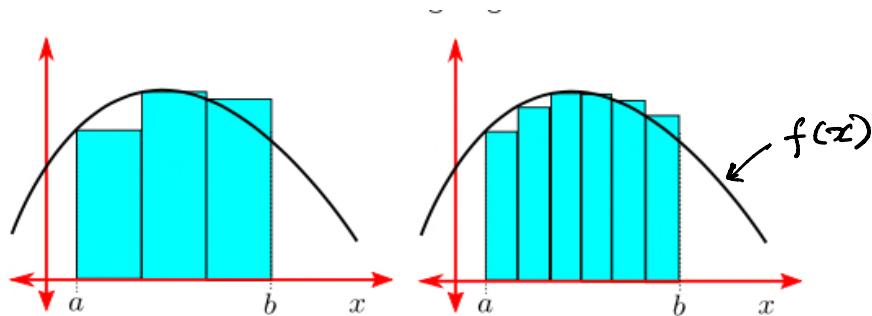


Goal :

- ① Approximate the area in the shaded region.
- ② Riemann sum.

## Sum of rectangles under curve.

We consider a function  $f: \mathbb{R} \rightarrow \mathbb{R}$ , i.e.  $f$  takes values in the set of real numbers and outputs values in the same set (self-map).



- Sum of area under rectangle  $\approx$  area under curve.
- Increase the number of rectangles to improve approx.

## Summation notation CLP section 1.1.3.

We use the symbol " $\sum$ " to denote sum (called sigma)  
for example:

- The sum of first 20 integers is

$$1+2+\dots+20 = \sum_{i=1}^{20} i$$

The sum reads : The sum of  $i$  from  $i = 1$  to 20.

Alternatively:  $1+2+\dots+20 = \sum_{\square=1}^{20} \square$

dummy variable.

- The sum of cubes is

$$1^3 + 2^3 + \dots + 20^3 = \sum_{i=1}^{20} i^3$$

## Summation notation CLP section 1.1.3. (contd.)

- Let  $f$  be any real valued function. A sum of a function evaluated at integer points.

$$f(1) + f(2) + \dots + f(20) = \sum_{i=1}^{20} f(i)$$

- A more formal sum

$$\sum_{i=1}^K a_i = a_1 + a_2 + \dots + a_K$$

$$a_i = i \quad , \quad a_i = i^3 \quad , \quad a_i = f(i)$$

---

## Properties of sum (Thm 1.1.5. in CLP)

An important property of sum is that it's a "linear operator".

- o Closed under scalar multiplication.

$$\text{For any } c \in R, \sum_{i=1}^n c a_i = c \left( \sum_{i=1}^n a_i \right)$$

constants can come outside

- o Closed under addition.

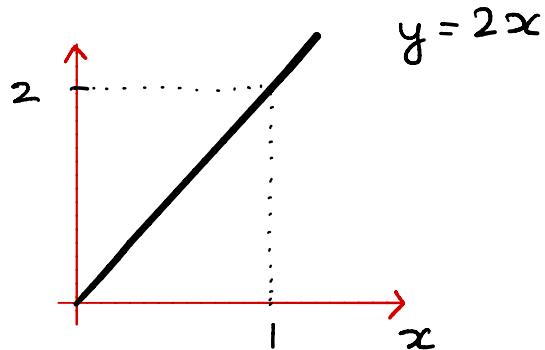
$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

## Special sums.

- First  $n$  integers :  $\sum_{k=1}^n k = \frac{n(n+1)}{2} \rightarrow$  in class.
  - First  $n$  integer square :  $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$
  - First  $n$  integer cube :  $\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$
- } see text.

## Back to areas and example

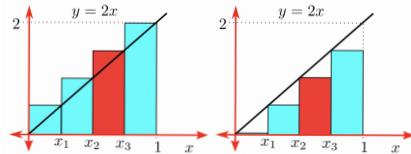
Q: Find the area between the curve  $y = 2x$  and the  $x$ -axis between  $x=0$  and  $x=1$ .



Idea:

- o Divide  $[a, b]$  in equal sized non-overlapping sub-intervals.
- o For each sub interval, approx using rectangle
- o Add the area.

## Example (contd)



## Approximation using left endpoint

- Partition the interval  $[0, 1]$  into  $n$  uniformly sized sub-intervals.

The width of each sub-interval is

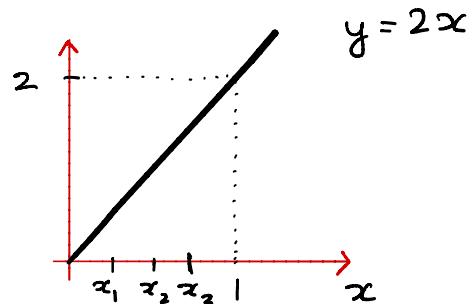
$$\Delta x = \frac{1}{n}.$$

- So,  $x_1 = 0$ ,  $x_2 = \frac{1}{n}$ ,  $x_3 = \frac{2}{n}$ , ...,  $x_n = \frac{n-1}{n}$

- The height of the  $i^{\text{th}}$  rectangle is  $f(x_i) = \frac{2(n-i)}{n}$

- The total area:

$$\begin{aligned} L_n &= \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \frac{2(n-i)}{n} \frac{1}{n} = \frac{2}{n^2} \left( \sum_{i=1}^n n - \sum_{i=1}^n i \right) \\ &= \frac{2}{n^2} \left( \frac{n(n+1)}{2} - n \right) \end{aligned}$$



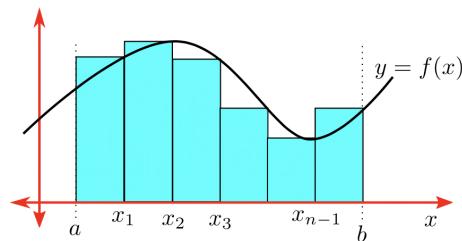
## Approximation using left endpoint (Contd)

- Now, to get exact area we take  $n \rightarrow \infty$

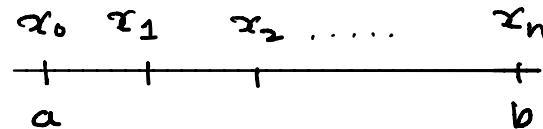
$$\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} \frac{2}{n^2} \left( \frac{n(n+1)}{2} - n \right) = ?$$

## Riemann sum

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be any function.

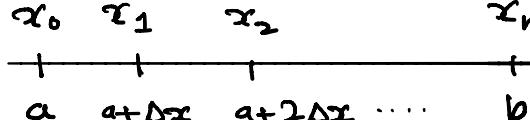


Riemann sum is the sum of area of  $n$  rectangles used to approximate area under curve from  $x=a$  and  $x=b$ .



o uniform width :  $\Delta x = (b-a)/n$

## Riemann sum (contd)

- the ordinate : 
$$x_i = a + i \Delta x \quad \text{for } i = 0, \dots, n$$

- the height :

$i^{\text{th}}$  rectangle using right endpoint :  $f(x_i)$

$i^{\text{th}}$  rectangle using left end point :  $f(x_{i-1})$

using midpoint :  $f\left(\frac{x_{i-1} + x_i}{2}\right)$

Riemann sum =  $\sum_{i=1}^n f(x_i^*) \Delta x$ ,  $x_i^* = x_i$  for right end point  
 $x_i^* = x_{i-1}$  for left.

## Riemann sum (contd)

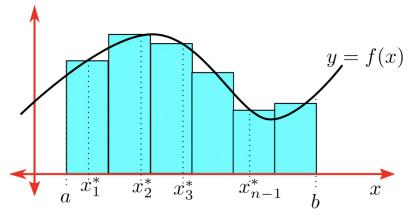
Consider the sub-interval  $[x_{i-1}, x_i]$

We can pick any point in  $[x_{i-1}, x_i]$  to get the height of the rectangle.

Say we pick  $x_i^* \in [x_{i-1}, x_i]$ .

$$R_n = \Delta x f(x_1^*) + \Delta x f(x_2^*) + \dots + \Delta x f(x_n^*)$$

And area =  $\lim_{n \rightarrow \infty} R_n$ .



## Riemann sum (CLP Definition 1.1.11)

Let  $a, b$  be two real numbers. Let  $n$  be a positive integer and  $f(x)$  be defined on  $[a, b]$ .

Set  $\Delta x = (b-a)/n$  and then (as above) divide the interval  $[a, b]$  into  $n$  even sub-intervals  $[x_{k-1}, x_k]$  and let  $x_k^*$  be any point in  $[x_{k-1}, x_k]$ . Then the sum

$$\sum_{k=1}^n f(x_k^*) \Delta x$$

is called a Riemann sum.

