Taylor series.

The general form for a Taylor series of a smooth function $f(x)$ about $x = x_0$ is.

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.}$$

- $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$.
- We get Taylor series representation by differentiating general power series representation about $x = x_0$.

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

- Radius of convergence?

Taylor series of cos(x) at $x = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x-0)^n .$$

$$f^{(0)}(x) = \cos(x), \quad f^{(1)}(x) = \cdot \sin(x), \quad f^{(2)}(x) = -\cos(x), \quad f^{(3)}(x) = \sin(x)$$

$$f^{(0)}(0) = 1, \quad f^{(1)}(x) = 0, \quad f^{(2)}(0) = -1, \quad f^{(3)}(0) = 0 .$$

$$\cos(x) = \frac{1}{0!}(x-0)^0 + 0 + \frac{(-1)^2}{2!}(x-0)^2 + 0 + \frac{(-1)^4}{4!}(x-0)^4 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Maclaurin series of $\cos(x)$.

Radius of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}}{(2(n+1))!}\cdot\frac{(2n)!}{x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{(2n+1)(2n+2)}\right|$$

$$= 0.$$

so, the taylor series representation of $\cos(x)$ converges for all $x \in \mathbb{R}$.

The series $\sum a_n$ converges absolutely if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$$

Maclaurin series:

Some important Taylor series about $x=0$ are:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}$$

- The interval of convergence is $\mathbb{R}$, $(-\infty, \infty)$.
- Taylor series expansion about $x=0$, which is called Maclaurin series.

Geometric series

The geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$, is also a Maclaurin series.

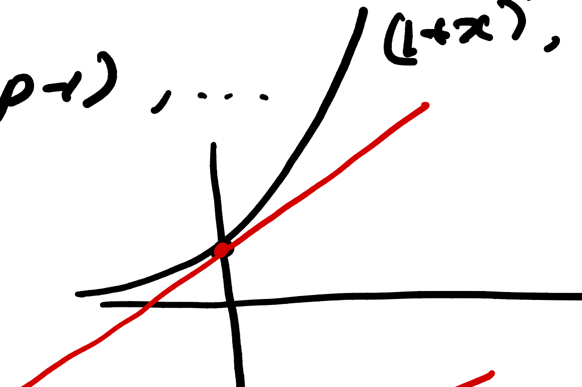
We can derive series representation of other functions like $\log(1-x)$ and $\arctan(x)$ by manipulating geometric series. See previous lecture

Example. Write the first two terms of Maclaurin series of $f(x) = (1+x)^p$ for $p \in \mathbb{R}$.

$$f^{(1)}(x) = p(1+x)^{p-1}, \quad f^{(2)}(x) = p(p-1)(1+x)^{p-2} \cdots$$

$$f^{(1)}(0) = p, \quad f^{(2)}(0) = p(p-1), \ldots$$

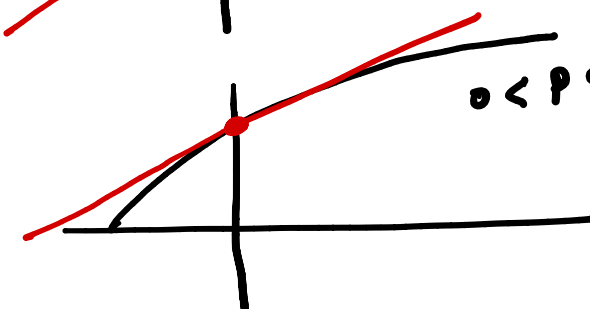
$(1+x)^p, \ p>1$



$$(1+x)^p = \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \ldots$$

$$= \underbrace{1 + px}_{\text{tangent approximation.}} + \frac{p(p-1)}{2}x^2 + \ldots$$

$0 < p < 1$



$(1+x)^p \simeq 1 + px$ for p small and x small

Summary

Our key results that should be memorized are:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Taylor series can be integrated, differentiated, added, multiplied and divided when inside interval of convergence.

Example 1

Let $f(x) = \log(1+2x^2)$ for $|x| < 1/\sqrt{2}$.

① Find the Maclaurin series of $f(x)$.

Sol^n One approach is $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ — hard.

Recall: $$\frac{1}{1-x} = 1 + x + x^2 + \cdots \quad . \qquad |x| < 1.$$

integrate both sides:

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad |x| < 1.$$

replace $x \to -2y^2$

$$\log(1+2y^2) = -\left((-2y^2) + \frac{(-2y^2)^2}{2} + \frac{(-2y^2)^3}{3} \cdots \right) \quad |y| < 1/\sqrt{2}$$

$$= -\left(\sum_{n=1}^{\infty} \frac{(-2)^n (y^2)^n}{n} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n y^{2n}}{n}$$

Example 1 (contd).

ii) compute $\lim_{x \to 0} \dfrac{\log(1+2x^2)}{x^2}$

Since $\log(1+2x^2) = 2x^2 - \dfrac{2^2 x^4}{2} + \dfrac{2^3 x^6}{3} - \dots$

$$\Rightarrow \frac{\log(1+2x^2)}{x^2} = 2 \text{ as } x \to 0.$$

iii) Compute $f^{(8)}(0)$.

Maclaurin series: $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

So, $\dfrac{f^{(8)}(0)}{8!} = -\dfrac{2^4}{4} \Rightarrow f^{(8)}(0) = -4 \cdot 8!$

Example 2.

Calculate $\lim_{x \to 0} \dfrac{\cos(x^2) - (1 - x^4/2)}{x^8}$

using Taylor expansion of $\cos(x)$.

- Taylor series of $\cos(x^2)$ from $\cos(x)$.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\Rightarrow \cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots$$

simplify $\cos(x^2) - (1 - x^4/2)$ then take limit of $\dfrac{\cos(x^2) - (1 - x^4/2)}{x^8} \longrightarrow \dfrac{1}{4!}$ as $x \to 0$.

Examples

ⓐ Compute $x^7 e^x$

$$x^7 e^x = \sum_{n=0}^{\infty} \frac{x^7 x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+7}}{n!}$$

ⓑ Compute $\cos x \, e^x$

$$\cos x \, e^x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= 1 + x - \frac{x^3}{3} + \dots$$

Integrating e^{-x^2}:

We can use Maclaurin series of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

$$\Rightarrow \quad e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}, \quad x \in \mathbb{R}$$

$$\Rightarrow \quad \int e^{-x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} + C$$

$$\Rightarrow \quad \int_0^1 e^{-x^2} dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$$

Integrating e^{-x^2} (contd).

So, we can estimate $\int_0^1 e^{-x^2}dx$ using N^{th} partial

$$S_N = \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)n!} \quad , \quad S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$$

$$R_N = |S_N - S| \leq b_{N+1} = \frac{1}{(2N+3)(N+1)!}.$$

$$R_{10} = \frac{1}{23 \cdot 11!} \approx 1.3 \times 10^{-8}.$$

Remainder Theorem.

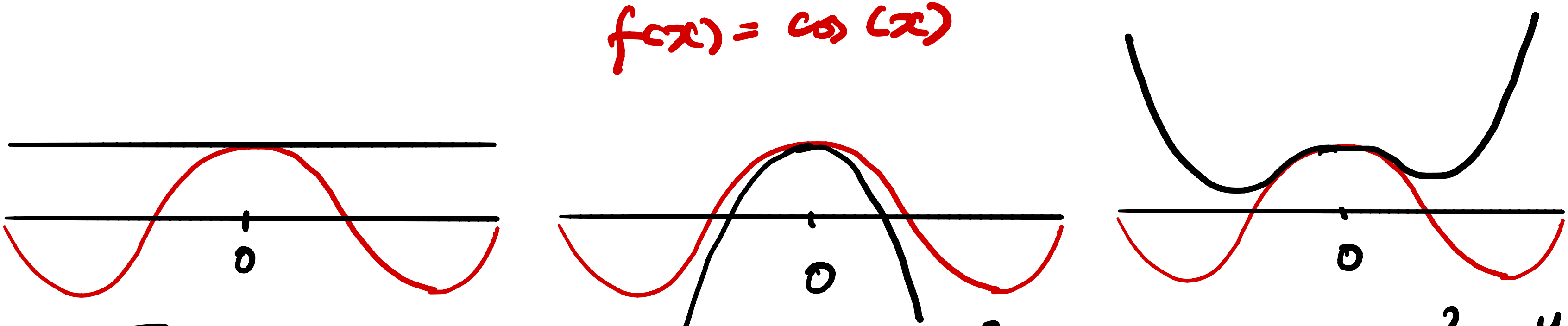
We can truncate Taylor series to get a Taylor polynomial that approximates the function.

The n^{th} Taylor approximation polynomial

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{n!} (x-a)^n.$$

What is the error $R_N = |T_N(x) - T(x)|$?

$f(x) = \cos(x)$



$T_0 = 1$ $\qquad$ $T_2 = 1 - \frac{x^2}{2}$ $\qquad$ $T_4 = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$

Taylor Remainder Theorem.

Thm: If we can bound $|f^{(n+1)}(x)| \leq M$ for all $|x-a| \leq d$ then the remainder $R_n(x)$ of Taylor series satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1},$$

ε^n, ε^{n+1}

for all $|x-a| < d$.

<u>Ex:</u> Say we approximate $\cos(x) \approx T_2(x) = 1 - \frac{x^2}{2}$

Note that $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$.

$f^{(3)}(x) = -\sin(x)$, $|f^{(3)}(x)| < 1$

$$\Rightarrow R_2 = |T_2 - T| \leq \frac{1}{3!}|x|^3 = \frac{1}{6}|x|^3$$