

Absolute and conditional convergence:

Recall that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by alternating series test

but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test.

Also, observe that $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge.

Definition Consider the series $\sum_{n=1}^{\infty} a_n$.

I. If $\sum_{n=1}^{\infty} |a_n|$ converges then we say the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

II. If the series $\sum_{n=1}^{\infty} |a_n|$ is divergent but $\sum_{n=1}^{\infty} a_n$ is convergent, we say $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Comparison test:

A key property we established in comparison test was:

Theorem: If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

proof idea: let $S_N = \sum_{n=1}^N a_n$ and $T_N = \sum_{n=1}^N |a_n|$.

① $T_N - S_N$ is bounded above by $2T_N < \infty$

② $T_N - S_N \geq 0 \quad \forall N \Rightarrow$ monotonic.

so, By Monotonic Convergence theorem $T_N - S_N$ is convergent

Ratio test

A key test for absolute convergence of a series
is the ratio test.

Thm: Let $N > 0$ be an integer and assume $a_n \neq 0$
for all $n \geq N$. Then

- ① If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, we have $\sum_{n=1}^{\infty} |a_n|$
converges. (and so does $\sum_{n=1}^{\infty} a_n$)
- ② If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ then
 $\sum_{n=1}^{\infty} |a_n|$ diverges.

Ratio test

Remark:

- a. Ratio test is useful when the series has exponent n , factorials, etc... like

$$\sum_{n=1}^{\infty} \frac{n^2}{5^n}, \quad \sum_{n=1}^{\infty} \frac{e^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{n^2 e^n}{4^n}, \text{ etc.}$$

- b. It turns out ratio test is not as useful for series of the form $\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$, $P(n)$ & $Q(n)$ are polynomial in n .

- c. In ①, If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Remark (contd)

- d. In ②, the series is not absolutely convergent.
But $\sum_{n=1}^{\infty} |a_n|$ could be convergent, i.e. $\sum_{n=1}^{\infty} |a_n|$ could be conditionally convergent.
- e. **Important** There is no conclusion if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$
(i.e. L=1). Then $\sum_{n=1}^{\infty} |a_n|$ may or may not be convergent.
Further test is required.

Proof outline of ratio test

We first prove ①. Since $L < 1$, we can pick a $R \in \mathbb{R}$ that satisfy $0 < L < R < 1$. Then there exists M so that for all $n \geq M$ we have $\left| \frac{a_{n+1}}{a_n} \right| < R$ because

$$\left| \frac{a_{n+1}}{a_n} \right| \rightarrow L \text{ as } n \rightarrow \infty.$$

Thus, $|a_{n+1}| < R |a_n|$ for all $n \geq M$.

$$\text{So, } |a_{M+1}| < R |a_M|$$

$$|a_{M+2}| < R |a_{M+1}| < R^2 |a_M|$$

$$\vdots$$

$$|a_{M+p}| < R^p |a_M|$$

$$\text{Then } \sum_{p=1}^{\infty} |a_{M+p}| < \sum_{p=1}^{\infty} R^p |a_M| = \sum_{p=1}^{\infty} |a_M| R^{p-1} = \frac{|a_M|R}{1-R}$$

proof (contd)

So, $\sum_{p=1}^{\infty} |a_{N+p}| < \infty$ and $\sum_{n=M+1}^{\infty} |a_n|$ converges.

So, $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^M |a_n| + \sum_{n=M+1}^{\infty} |a_n| < \infty$. This proves ①.
finite

Now to prove ②, pick R in $1 < R < L$. Then for $n \geq M$, we must have

$$\left| \frac{a_{n+1}}{a_n} \right| \geq R \rightarrow |a_{n+1}| \geq R |a_n| \quad \forall n \geq M.$$

Thus $|a_{n+1}| \geq |a_M|$ for $n \geq M$ since $|a_{n+1}| \rightarrow 0$

by basic divergence test. So, $\sum_{p=M+1}^{\infty} |a_{n+p}|$ is divergent.

Example 1

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ where $a_n = (-1)^n \frac{1}{n}$.

By alternating series test $\sum_{n=1}^{\infty} a_n$ converges.

But we have $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So,

by our definition we conclude that $\sum_{n=1}^{\infty} a_n$ is
conditionally convergent.

Example 2

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ where $a_n = (-1)^n \frac{1}{n}$.

By alternating series test, we know that $\sum_{n=1}^{\infty} a_n$

converges.

By integral test, we know that

So, by definition we say that

absolutely convergent.

$\sum_{n=1}^{\infty} |a_n|$ converges.

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ is

Example

Discuss absolute / conditional convergence or divergence of each of the following:

$$\textcircled{a} \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1} \quad \textcircled{b} \sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{6n+5} \right) \quad \textcircled{c} \sum_{n=1}^{\infty} \frac{(-1)^n (n^2+1)}{n^4+5}$$

$$\textcircled{d} \sum_{n=1}^{\infty} (-1)^n n e^{-n^2} \quad \textcircled{e} \sum_{n=1}^{\infty} \frac{n \sin(n)}{n^3+1}$$

Example

$$\textcircled{a} \quad \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}.$$

Solⁿ First check $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$. By comparison limit test (compare to $\frac{1}{n}$), we have that $\sum_{n=1}^{\infty} |a_n|$ is divergent. So, $\sum_{n=1}^{\infty} |a_n|$ is not absolutely convergent.

Now, check $\sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n = \frac{n}{n^2+1}$.

Notice that b_n is decreasing for large n and $b_n > 0$ with $b_n \rightarrow 0$. Thus, by alternating test $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.

Hence, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ is conditionally convergent.

Example

⑥ $\sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{6n+5}$.

Solⁿ Let $a_n = (-1)^n \frac{2n+1}{6n+5}$. Notice that

$$a_n \approx (-1)^n \frac{1}{3} \text{ for large } n. \text{ So, } a_n \not\rightarrow 0$$

as $n \rightarrow \infty$. Thus, $\sum_{n=1}^{\infty} a_n$ is divergent.

Example

$$\textcircled{c} \quad \sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+1}{n^4+5} \right)$$

Solⁿ: First check $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^2+1}{n^4+5}$.

Since $\frac{n^2+1}{n^4+5} \approx \frac{1}{n^2}$ for large n , we expect convergence.

Let $b_n = \frac{1}{n^2}$. Notice that $b_n > 0 \forall n$.

$$\text{and } \lim_{n \rightarrow \infty} \frac{n^2+1}{n^4+5} \cdot n^2 = \lim_{n \rightarrow \infty} \frac{n^4+n^2}{n^4+5} = 1$$

Since limit exists and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+5}$

is convergent by limit comparison test

So, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{n^4+5}$ is absolutely convergent.

Example

① $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$ with $a_n = (-1)^n n e^{-n^2}$.

Solⁿ: First test $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} n e^{-n^2}$.

By ratio test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1) e^{-(n+1)^2}}{n e^{-n^2}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1) e^{-n^2 - 2n - 1}}{n e^{-n^2}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} e^{-2n-1} \right| \\ &= 1 \cdot 0 = 0. \end{aligned}$$

So, by ratio test, we conclude that $\sum_{n=1}^{\infty} (-1)^n n e^{-n^2}$ is absolutely convergent.

Example

e) $\sum_{n=1}^{\infty} \frac{n \sin(n)}{n^3 + 1}$.

Note that we expect $\sum_{n=1}^{\infty} \frac{n \sin(n)}{n^3 + 1}$ to converge because $\frac{n \sin(n)}{n^3 + 1} \approx \frac{1}{n^2}$ for large n .

Note that we can't use integral test because $\frac{n \sin(n)}{n^3 + 1}$ is not always decreasing.

Also, we can't use limit comparison test because $\lim_{n \rightarrow \infty} \frac{n \sin(n)}{n^3 + 1} \cdot n^2 = \lim_{n \rightarrow \infty} \frac{n^3 \sin(n)}{n^3 + 1}$ does not exist.

Example.

We can use simple comparison test:

Notice

$$|\sin n| < 1$$

so,

$$0 < \left| \frac{n \sin n}{n^3 + 1} \right| < \frac{n}{n^3 + 1}$$

$$\downarrow$$

$$0$$

$$\downarrow$$

$$0$$

so, $\left| \frac{n \sin n}{n^3 + 1} \right| \rightarrow 0$ as $n \rightarrow \infty$. Thus, absolutely convergent

