Warmup

Is $\displaystyle\int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx$ convergent or divergent?

Solution:

Improper integrals.

Consider $I = \int_0^1 \frac{1}{x^p}\,dx$. I converges iff $p<1$

Consider $I = \int_1^\infty \frac{1}{x^p}\,dx$. I converges iff $p>1$.

Does $I = \int_2^\infty \frac{1}{x}\,dx$ converge or diverge?

Does $I = \int_0^{1/2} \frac{1}{x}\,dx$ converge or diverge?

What happens if we choose a $f(x)$ that decays slightly faster as $x \to \infty$?

Decay rate

Consider $f(x) = \dfrac{1}{x(\ln x)^p}$. Is $\displaystyle\int_2^\infty f(x)\,dx$ convergent?

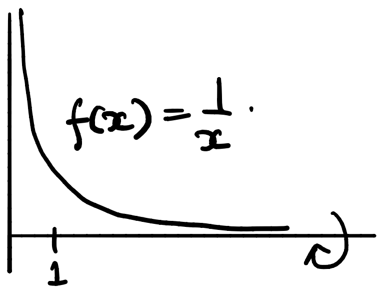
Decay rate (contd.)

Consider $f(x) = \dfrac{1}{x(\ln x)^p}$. Is $I = \int_0^{1/2} f(x)\,dx$ convergent?

Volume of revolution.

Let $f(x) = \frac{1}{x}$, we have $\int_1^\infty \frac{1}{x}\,dx$ is infinite.

What about volume of revolution of $f(x)$ about x-axis?



Volume of revolution (contd.)

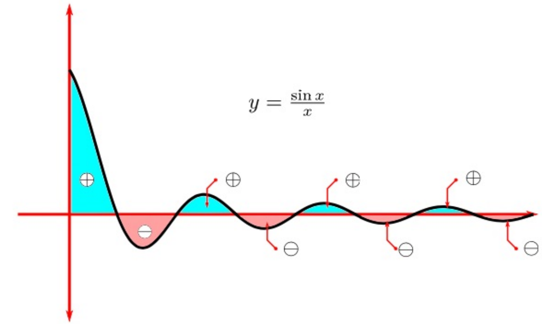
Suppose $f(x) = \frac{1}{x^p}$ for $p > 0$.

Find the range of p s.t. $V = \int_1^\infty \pi f(x)^2 \, dx$ is finite.

Area cancellation.

Area cancellation can lead to convergence.

A famous special function is the "sinc" function: $f(x) = \frac{\sin x}{x}$.

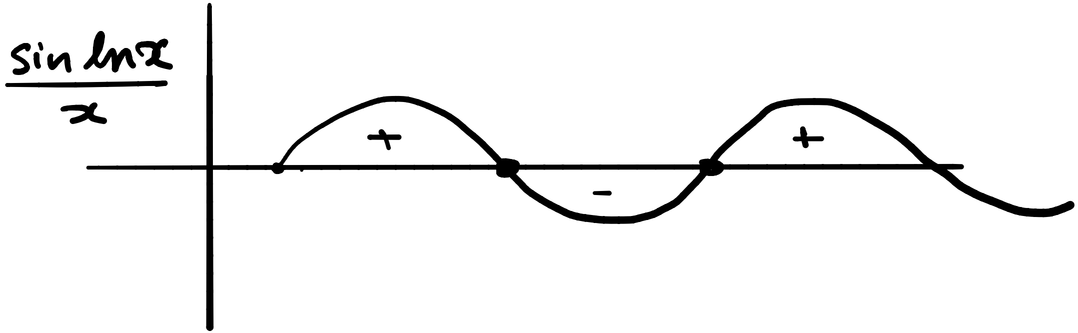


Is $\int_0^\infty \frac{\sin x}{x}\, dx$ convergent?

Area cancellation. (contd.).

Consider a slightly different function: $f(x) = \dfrac{\sin(\ln x)}{x}$

$$\int_1^\infty \frac{\sin(\ln x)}{x}\,dx$$



note that $\sin(\ln x) = 0$ if $\ln x = n\pi$

$\Rightarrow x = e^{n\pi}$.

$\sin(\ln x)$ varies very slowly. There is not enough cancellation for convergence.

Approximation by tangent line.

Consider $I = \int_a^b \frac{g(x)}{f(x)}\,dx$. Let $f(c)=0$ for some $c \in [a,b]$.

If $f'(c) \neq 0$ and $g(c) \neq 0$ then I diverges.

Example.

Consider $I = \int_1^\infty \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} \right) dx$. Is I convergent?

we know $\int_1^\infty \frac{1}{\sqrt{x^2+1}} dx$ and $\int_1^\infty \frac{1}{x} dx$ are divergent.

I is finite because of cancellation.

Example (contd.)

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