3 Infinite sequences and series

3.2 Series

Consider the sequence $\{a_n\}$. Now add up the terms

$$a_1 + a_2 + a_3 + \cdots + a_n + \dots$$

This is called an infinite series. We denote it by

$$\sum_{n=1}^{\infty} a_n$$

But this infinite sum does not always make sense. How can we understand

$$1 + 2 + 3 + \dots \qquad \text{or } \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \dots$$

Lets think about the first one — clearly this sum just keeps getting larger and larger as we add more terms. If we consider the partial sums:

$$\begin{aligned} s_1 &= 1 = 1 \\ s_2 &= 1 + 2 = 3 \\ s_3 &= 1 + 2 + 3 = 6 \\ s_4 &= 1 + 2 + 3 + 4 = 10 \\ s_n &= 1 + \cdots + n = \frac{n(n+1)}{2} \end{aligned}$$

So we can think about what happens to these partial sums s_n as $n \to \infty$. Clearly they diverge as $n \to \infty$. Hence the sum $1 + 2 + 3 + \dots$ does not make sense.

What about the other one

$$\begin{aligned} s_1 &= \frac{1}{2} \\ s_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\ s_4 &= \frac{1}{2} + \cdots + \frac{1}{16} = \frac{15}{16} \\ s_n &= \frac{1}{2} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} = 1 - 2^{-n} \end{aligned}$$

Now it is clear that $s_n \to 1$ as $n \to \infty$. In this case we write

$$\sum_{n=1}^{\infty} 2^{-n} = 1$$

In both cases we determine whether or not the infinite series exists by considering its partial sums. In particular whether the sequence of partial sums converges. So we have reduced the convergence of these infinite series to a problem of convergence of sequences.

Anyone want to take a guess at the "harmonic series"

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

Let us make this more general with a definition

Definition. Let $a_1 + a_2 + \dots$ be an infinite series, and let

$$s_n = a_1 + \dots a_n$$

denote its n^{th} partial sum. If $s_n \to s$ for some finite real number s we write

$$a_1 + a_2 + a_3 + \cdots = s \qquad\qquad \sum_{n=1}^{\infty} a_n = s$$

and say that the series is convergent. If the sequence of partial sums does not converge we say that the series is divergent.

Again we have defined this nice mathematical object, we can see how it works together with our standard arithmetic. Not quite as nice as sequences, but still pretty good

Theorem (Arithmetic of series — CLP Theorem 3.2.9)**.** *Let $\sum a_n$ and $\sum b_n$ be convergent sequences with $\sum a_n = A$ and $\sum b_n = B$. Further let c be some real number. Then*

$$\sum_{n=1}^{\infty} ca_n = cA$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = A + B$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = A - B$$

Unfortunately $\sum(a_n b_n)$ and $\sum {}^{a_n}/_{b_n}$ are not so simple to evaluate.

There are lots of ways of testing if a series converges or not, but unlike the case of sequences — actually working out what the series converges to is quite difficult and might be impossible in general. There are some nice exceptions to this where it is relatively easy to work things out.

Theorem (Geometric series — CLP Lemma 3.2.5)**.** *The series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is called a geometric series. If $|r| < 1$ then it converges and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| \geq 1$ (and $a \neq 0$) then the series diverges.

Easy proof —

$$\begin{aligned} s_n &= a + ar + ar^2 + \ldots ar^{n-1} \\ rs_n &= ar + ar^2 + ar^3 + \ldots ar^n \\ s_n - rs_n &= (1-r)s_n = a - ar^n \\ s_n &= a\frac{1-r^n}{1-r} \end{aligned}$$

Now if $|r| < 1$ we know that $r^n \to 0$ and so the result follows. And if $|r| > 1$ then things clearly diverge. What about if $r = 1$, then $s_n = na$ which diverges. And if $r = -1$ then $s_n = a - a + a - a + a \ldots$ which is either 0 or a depending on whether n is even or odd.

Please work out

$$9 - \frac{27}{5} + \frac{81}{25} - \frac{243}{125} + \ldots$$

Note — it is a geometric series — so you need to work out a and r. How can you do that?

$$\begin{aligned} a &= 9 && \text{first term} \\ r &= \frac{(-27/5)}{9} = \frac{-3}{5} && \text{ratio of terms} \end{aligned}$$

Since $|r| < 1$, the series converges and it sums to

$$\frac{a}{1-r} = \frac{9}{1-(-3/5)} = \frac{9 \cdot 5}{5+3} = \frac{45}{8}$$

There are very few other families of series that are relatively easy to work out (without using calculus) — but one such class is "telescoping" series named after the old fashioned collapsing telescopes. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Lets write out the first few partial sums

$$\begin{aligned} s_1 &= \frac{1}{2} \\ s_2 &= \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \\ s_3 &= \frac{1}{2} + \cdots + \frac{1}{12} = \frac{9}{12} = \frac{3}{4} \\ s_4 &= \frac{1}{2} + \cdots + \frac{1}{20} = \frac{16}{20} = \frac{4}{5} \end{aligned}$$

Looks like $s_n = \frac{n}{n+1}$ which is surprisingly nice. Why does this happen. Partial fractions helps

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

So

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Lots of cancellations

$$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

The sequence of partial sums converges to 1 and hence the series equals 1. In general this sort of thing will work when you are summing a_n and $a_n = b_n - b_{n+1}$.

Indeed, more generally, if $a_n = b_n - b_{n-1}$ the n^{th} partial sum will be

$$\begin{aligned} s_n &= \sum_{k=1}^{n} a_n = \sum_{k=1}^{n} (b_n - b_{n-1}) \\ &= \sum_{k=1}^{n} b_n - \sum_{k=2}^{n+1} b_n \\ &= b_1 - b_{n+1} \end{aligned}$$

The converge depends on whether or not b_n converges to a constant. In the example abovce $b_n \to 0$, but this need not always be the case. The text does a good example —

$$\sum_{n=1}^{\infty} \log(1 + 1/n)$$

Lets write the partial sum carefully to decide its convergence:

$$\begin{aligned} \sum_{k=n}^{\infty} \log(1 + 1/k) &= \sum_{k=1}^{n} \log \frac{k+1}{k} \\ &= \sum_{k=1}^{n} \log(k+1) - \log(k) \\ &= \sum_{k=2}^{n+1} \log k - \sum_{k=1}^{n} \log k \\ &= \log(n+1) - \log 1 \\ &= \log(n+1) \end{aligned}$$

So when $n \to \infty$ the partial sums diverge to $+\infty$. Hence the series does not coverge.

Just because it is telescoping that does not mean that the series will converge. But it does mean that the partial sums are very simple to compute.

Unfortunately for series in general, computing partial sums tends to be very difficult. And indeed, computing the series example tends only to be possible in special cases (such as the telescoping and geometric series examples above) and using tricks (eg Taylor polynomials play an interesting role in that).

3.3 Convergence tests

One thing we can do, however, is answer the simpler problem — does the series converge or diverge.

- If the series diverges, then end of story — we cannot do more
- If the series converges, then we can (perhaps) compute it exactly, or estimate it, or bound it, or compute it numerically (eg just sum up the first n terms), or ...

So testing for convergence is an important problem. There are a vast array of different convergence tests and in this course we will cover some of the main ones.

3.3.1 Divergence test

The first convergence test that we'll look at is (arguably) the simplest to apply. It is based on a simple observation about convergent series. Consider any of the convergent series we've seen so far

$$\sum_{n=1}^{\infty} 2^{-n} \qquad \sum_{n=1}^{\infty} ar^n \text{ with } |r| < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

In all these cases the summand $a_n \to 0$.

Indeed it is easy to prove that this must happen for a convergent series.

- If $\sum a_n$ converges, then (by defn) the partial sums $s_n = \sum_{k=1}^{n} a_n$ converge to, say, S
- Now, we can also write

$$a_n = s_n - s_{n-1}$$

- So as $n \to \infty$, we know that $s_n \to S$ and also $s_{n-1} \to S$, so $a_n \to 0$.

This observation by itself isn't so useful — if we know a series is convergent then we also know its summands converge to 0. But what we really want is to be able to look at the summands and decide whether it converges or not. To convert our observation into something useful we form its "contrapositive" which is a logically equivalent statement.

What we have is a statement of the form

If P is true, then Q is true

The contrapositive of this statement is

If Q is false, then P is false

Note that the above is logically equivalent to our original statement.

Let's use a simple example

If (he is Shakespeare) then (he is dead)

Its contrapositive is

If (he is not dead) then (he is not Shakespeare)

You can see that these statements are equivalent. Note that people frequenly assume the converse of a statement is equivalent. The converse of our original statement is

If Q is true then P is true If (he is dead) then (he is Shakespeare)

Clearly this does not mean the same as the original.

Anyway — with that quick excursion into propositional logic out of the way — we can rearrange our observation into the following theorem

Theorem (Divergence test — CLP 3.3.1)**.** *If the sequence $\{a_n\}_{n=1}^{\infty}$ does not coverge to zero as $n \to \infty$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

This often gives a quick simple test to see if a series diverges. For example, the series

$$\sum \frac{n+2}{2n+3} \text{ diverges}$$

The sequence of summands converges to $1/2$.

Note that this test can only be used to test for divergence.

- If $a_n \not\to 0$ then $\sum a_n$ diverges, but
- just because $a_n \to 0$ it *does not mean* that $\sum a_n$ converges.

Consider the following two series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

We cannot determine whether or not they converge using the above test. In both cases $a_n \to 0$. However the first one — called the "Harmonic series" diverges (as I'll show in a moment) while the second converges (its equal to $\pi^2/6$ — solved, but not quite proved, first by Euler).

To see that the Harmonic series diverges, let us group the summands together carefully and compute the 2^n-th partial sums:

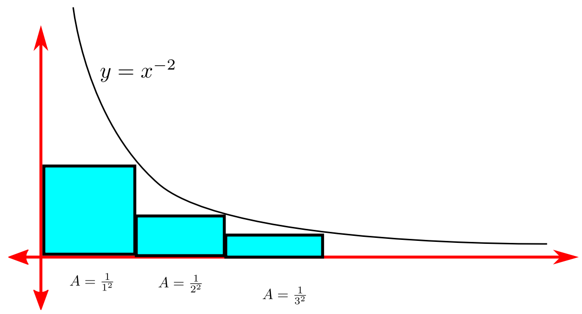
$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{2}{4}} \geq 1 + \frac{2}{2} \\
s_8 &= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{2}{4}} + \underbrace{\left(\frac{1}{5} + \cdots + \frac{1}{8}\right)}_{\geq \frac{4}{8}} > 1 + \frac{3}{2} \\
s_{16} &= 1 + \frac{1}{2} + \cdots + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{\geq \frac{8}{16}} > 1 + \frac{4}{2} \\
s_{2^n} &> 1 + \frac{n}{2}
\end{aligned}$$

Hence the sequence of partial sums keeps getting bigger, but very slowly. So it does not converge. The text book has a bit more on the Harmonic series and $\sum n^{-2}$ in the optional section 3.3.9.

The next test we discuss is sufficiently sensitive to determine the convergence of the above two series (but not their values) — it is based on a simple geometric observation.

3.4 Integral test

Let us represent the series $\sum n^{-2}$ pictorially:

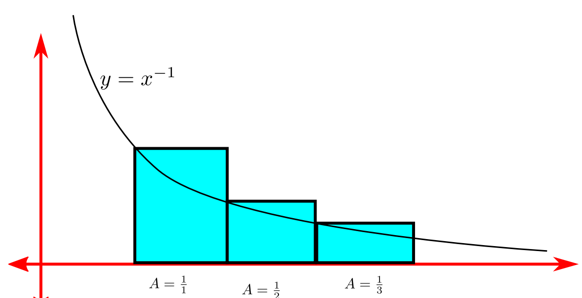


The series is then just the area of the little rectangles. But we can compute an upper bound on that area using a simple integral. We need to be a little careful about $x = 0$, but with that in mind we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &< 1 + \int_1^{\infty} x^{-2}\mathrm{d}x \\ &= 1 + \lim_{b\to\infty} \left[\frac{-1}{x}\right]_1^b = 2\end{aligned}$$

Hence the series cannot diverge because it is a bounded above by 2. With some more work we can show that it must converge to a finite number. Notice that the sequence of partial sums increases with n (since we keep adding non-negative numbers). We have already established that the partial sums cannot diverge to $+\infty$ — so they must converge to a finite number.

What about doing something like this for the harmonic series:



So the harmonic series is the sum of the areas of these little rectangles, but the rectangles are clearly bigger than the area under the curve, so

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &> \int_1^{\infty} \frac{1}{x}\mathrm{d}x \\ &= \lim_{b\to\infty} [\log x]_1^b\end{aligned}$$

which just gets bigger and bigger. Hence the series just gets bigger and bigger and so does not converge.

More generally we have

Theorem (Integral test — CLP 3.3.5)**.** *Let N_0 be a positive integer and let f be a continuous function for all $x \geq N_0$. Further, assume that*

- *$f(x) \geq 0$ for all $x \geq N_0$, and*
- *$f(x)$ decreases as x increases, and*
- *$f(n) = a_n$ for all $n \geq N_0$.*

Then

- *If $\int_{N_0}^{\infty} f(x)\mathrm{d}x$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.*
- *If $\int_{N_0}^{\infty} f(x)\mathrm{d}x$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Note that if the series converges it need not be equal to the integral (as we saw above).

A very useful example. For what values of p does the sum $\sum \frac{1}{n^p}$ converge? This series is called a p-series.

- If $p < 0$ then the summands do not converge to 0 and so the series diverges.
- If $p = 0$ then we have $\sum 1$ which also diverges.
- If $0 < p$ then we use the integral test

$$\int_1^{\infty} x^{-p}\mathrm{d}x$$

We did this example before — it diverges if $0 < p \leq 1$ and converges for $p \geq 1$.

Hence the series converges when $p > 1$ and otherwise diverges.

Let us look back at our sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

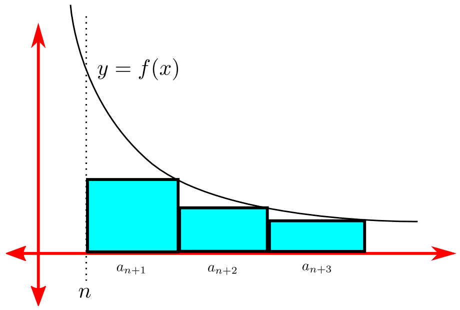
we know this converges by the above. Further one can prove (not in this course) that it converges to $\pi^2/6$. So we could *estimate* the sum by say

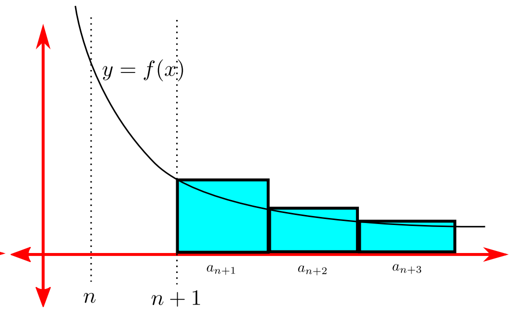
$$\sum_{n=1}^{100} \frac{1}{n^2} \approx 1.634984$$

and this should give us a rough estimate of $\pi^2/6$. But how can we estimate the remainder?

$$R_n = s - s_n = \sum_{k=n+1}^{\infty} a_k$$

We can do this (under certain circumstances) using the same ideas we just used for convergence testing.





So the remainder it just the area of all those squares — provided f is a nice function.

Theorem (Remainder estimate — CLP 3.3.5 (continued))**.** *Let $a_k = f(k)$ where f is a continuous, positive and decreasing function for $x \geq n$ and $\sum a_k = s$ is convergent. If we write $R_n = s - s_n$ then*

$$\int_{n+1}^{\infty} f(x)\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\mathrm{d}x$$

So in the case we just looked at above we have

$$\int_{n}^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{1}{n}$$
$$\int_{n+1}^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{1}{n+1}$$

Thus $\frac{1}{101} \leq R_{100} \leq \frac{1}{100}$. How good is this estimate?

$$\frac{1}{101} = 0.009900990099 \leq R_n = 0.009950168 \leq \frac{1}{100} = 0.01$$

So — pretty good!

A silly example: Consider the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$$

Let's apply the integral test.

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-1}}$$

Trig subs with $x = \sec\theta, x' = \sec\theta\tan\theta$

$$= \int \frac{\sec\theta\tan\theta\mathrm{d}\theta}{\sqrt{\sec\theta^2 - 1}}$$
$$= \int \sec\theta\mathrm{d}\theta$$
$$= \log|\sec\theta + \tan\theta| + C$$
$$= \log|x + \sqrt{x^2-1}|$$

Now, when we take the definite integral we will be taking the limit of the above as $x \to \infty$ which diverges to $+\infty$. So the series diverges. Oof!

Notice we can do this more easily by noting that

$$\begin{aligned} n^2 - 1 &\leq 4n^2 && \text{say} \\ \frac{1}{\sqrt{n^2-1}} &\geq \frac{1}{2n} \end{aligned}$$

So because $\sum \frac{1}{2n}$ diverges, so must our series.

Doing this sort of computation more carefully forms our next convergence test.