

Comparison test.

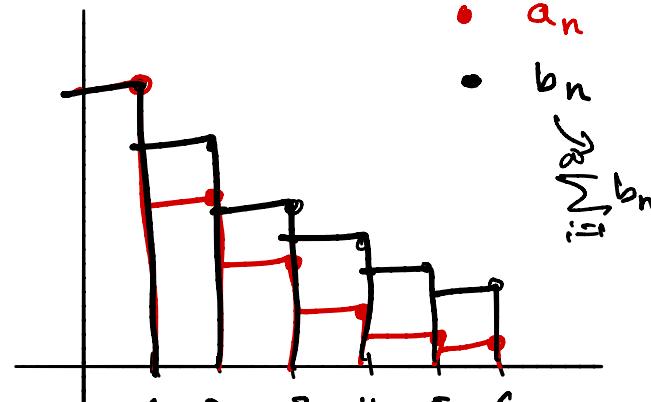
Recall from integral test that the "p-series" given by

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent if $p > 1$. It diverges if $p \leq 1$.

$\sum_{i=1}^{\infty} b_n$

Similar to comparison test for improper integral, we can compare series to test convergence / divergence.



black - $b_n = \frac{1}{n^3}$

red - $\sum a_n$ converges.

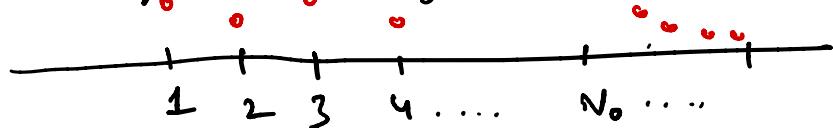
Comparison test (Theorem)

$$\sum_{n=1}^{\infty} a_n \quad \sum_{n=1}^{\infty} \cdot$$

- Theorem 1: (CLP Thm 3.3.8) . Let $N_0 > 0$ be an integer .
- a. If $|a_n| < c_n$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} c_n$ converges,
then $\sum_{n=0}^{\infty} a_n$ converges.
- b. If $a_n > d_n$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges.
then $\sum_{n=0}^{\infty} a_n$ diverges.

Remarks:

1. Notice that we only require $n \geq N_0$. Only the tail behaviour of the sequence is relevant.



Comparison test (Remark).

2. In (a), we require $|a_n| < c_n$. Why?

Give a counter example of a sequences that satisfy $a_n < c_n$ for $n \geq N_0$ and $\sum_{n=0}^{\infty} c_n$ converges but $\sum_{n=0}^{\infty} a_n$ does not.

$$-c_n \leq a_n \leq c_n$$

3. In (a), we have $|a_n| < c_n$. We can show $\sum_{n=0}^{\infty} (c_n - a_n)$ converges using monotonic convergence theorem.

4. Note that if $\sum_{n=0}^{\infty} |a_n|$ converges then so must $\sum_{n=0}^{\infty} a_n$.

$$a_n \leq |a_n|$$

Example 1

Does $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 3}$ converge or diverge?

$$\sum_{i=1}^{\infty} \frac{1}{n^p} < \infty \text{ if } p > 1.$$

Intuition for large n , we have

$\frac{1}{n^2 + 2n + 3} \approx \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges for $p = 2 > 1$.

Solution: Let $c_n = \frac{1}{n^2}$. notice that for $n \geq 1$.

$$\frac{1}{n^2 + 2n + 3} \leq \frac{1}{n^2} \quad \text{and} \quad \frac{1}{n^2 + 2n + 3} \geq 0 \geq -\frac{1}{n^2}$$

so, $|a_n| \leq c_n$ and by (a) of comparison test

we have $\sum_{n=0}^{\infty} \frac{1}{n^2 + 2n + 3}$ convergent. $(-c_n \leq a_n \leq c_n)$

Example 2.

$$|a_n| \leq c_n$$

Does $\sum_{n=1}^{\infty} \frac{1}{3n^2-5}$ converge or diverge?

Intuition: $\frac{1}{3n^2-5} \approx \frac{1}{3n^2}$ for large n . Since $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ converges by integral test, $\sum_{n=1}^{\infty} \frac{1}{3n^2-5}$ converges.

Solution: need $\frac{1}{3n^2-5} \leq \text{something}$ or $3n^2-5 \geq \text{something}$
notice $3n^2-5 \geq 2n^2 + (n^2-5) \geq 2n^2$ if $\frac{n^2-5 \geq 0}{n \geq 3}$

Also, need $-\frac{1}{2n^2} \leq \frac{1}{3n^2-5}$ for $n \geq N_0$. notice

$\frac{1}{3n^2-5} \geq 0$ if $3n^2-5 \geq 0$ or $n^2 \geq 5/3$, $n \geq 2$

so, $|\frac{1}{3n^2-5}| \leq \frac{1}{2n^2}$ for $n \geq 3$ and $\sum_{n=1}^{\infty} \frac{1}{3n^2-5}$ converges.

Example 3

Does $\sum_{n=1}^{\infty} \frac{2n+1}{6n^3-5}$ converge or diverge?

Intuition: $\frac{2n+1}{6n^3-5} \approx \frac{2n}{6n^3} = \frac{1}{3n^2}$ for large n .

so, $\sum_{n=1}^{\infty} \frac{2n+1}{6n^3-5} \approx \sum_{n=1}^{\infty} \frac{1}{3n^2}$ which converges.

Solution: Want $\frac{2n+1}{6n^3-5} \leq$ something

$$\text{notice } \frac{2n+1}{5n^3 + (n^3 - 5)} \leq \frac{2n+1}{5n^3} \quad \text{if } n^3 - 5 \geq 0 \\ n \geq 3$$

$$= \frac{3n - n + 1}{5n^3}$$

$$-\frac{3}{5n^2} \quad \nwarrow \leq \frac{3n}{5n^3} \quad \text{if } -n + 1 \leq 0 \\ n \geq 1$$

Example 3 contd

$$\frac{2n+1}{6n^3-5} \geq 0 \geq -\frac{3n}{5n^2}$$

Also, $-\frac{3n}{5n^3} < \frac{2n+1}{6n^3-5}$ if $6n^3-5 \geq 0$
or $n^3 \geq \frac{5}{6}$
 $n \geq 1.$

So, $\left| \frac{2n+1}{6n^3-5} \right| \leq \frac{3n}{5n^3} \underset{\frac{3}{5n^2}}{\cancel{>}} \text{ if } n \geq 3 \text{ and}$

$\sum_{n=1}^{\infty} \frac{2n+1}{6n^3-5}$ converges by comparison test.

Example 4.

Does $\sum_{n=1}^{\infty} \frac{2n+1}{6n^2-5}$ converge or diverge?

Intuition: $\frac{2n+1}{6n^2-5} \approx \frac{2n}{6n^2} = \frac{1}{3n}$ and
 harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. $\xrightarrow{\text{harmonic}}$

Solution: need $\frac{2n+1}{6n^2-5} \geq \text{something}$

$$\text{notice : } \frac{2n+1}{6n^2-5} \geq \frac{2n}{6n^2-5} \underset{\text{red}}{\geq} \frac{2n}{6n^2} = \frac{1}{3n}$$

for $n \geq 1$.

Since $\frac{1}{3n} \leq \frac{2n+1}{6n^2-5}$, $\sum_{n=1}^{\infty} \frac{2n+1}{6n^2-5}$ diverges by part
 (b) of comparison test.

Limit comparison test

In complicated examples, it gets tedious to find explicit bounds on n to ensure

$$|a_n| \leq c_n \quad \text{for } n \geq N_0$$

$$a_n \geq d_n \quad \text{for } n \geq N_0.$$

Comparison limit test

Theorem (CLP 3.3.11) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n . Assume that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$ exists. Then.

a) If $\sum_{n=1}^{\infty} b_n$ converges, we have that

$\sum_{n=1}^{\infty} a_n$ converges.

b) If $L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Remark.

In (b), $L \neq 0$ is essential.

Consider $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. We know
 $\sum a_n$ converges but $\sum b_n$ diverges.

However

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = 0. \quad \lim_{n \rightarrow \infty} \frac{1}{n}$$

so, if we ignored $L \neq 0$ condition we get wrong conclusion of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is divergent.

Example 5

Redo $\sum_{n=1}^{\infty} \frac{2n+1}{6n^3-5}$ using limit comparison test.

Intuition: For large n , $\frac{2n+1}{6n^3-5} \approx \frac{1}{3n^2}$

so, let $b_n = \frac{1}{3n^2}$ and $a_n = (2n+1)/(6n^3-5)$

$$\text{Notice } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2n+1}{6n^3-5} \cdot \frac{3n^2}{1} = \lim_{n \rightarrow \infty} \frac{6n^3 + 3n^2}{6n^3 - 5} = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is convergent,

$\sum_{n=1}^{\infty} a_n$ is also convergent by limit comparison test.

Easy!

Example 6

Does $\sum_{n=0}^{\infty} \frac{\sqrt{2n^2+1}}{n^2-2}$ converge or diverge

Intuition: $\frac{\sqrt{2n^2+1}}{n^2-2} \approx \frac{\sqrt{2n^2}}{n^2} \approx \frac{\sqrt{2}}{n}$ for large n .

so, expect divergence.

Solⁿ: Let $a_n = \frac{\sqrt{2n^2+1}}{n^2-2}$, $b_n = \frac{\sqrt{2}}{n}$.

$$\text{then } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{2n^2+1}}{\sqrt{n^2-1}} \cdot \frac{n}{\sqrt{2}} = 1.$$

Since $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1 \neq 0$ and $\sum_{n=1}^{\infty} b_n$ is divergent,

we have $\sum_{n=0}^{\infty} a_n$ is also divergent.

Example 7

Does $\sum_{n=2}^{\infty} \frac{n^2 + 2\sin(n)}{\sqrt{9n^8 + 1}}$ converge or diverge?

Intuition: $\frac{n^2 + 2\sin(n)}{\sqrt{9n^8 + 1}} \approx \frac{n^2}{3n^4} = \frac{1}{3n^2}$ for large n .

Since $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ converge, expect convergence.

Sol'n: Let $a_n = (n^2 + 2\sin(n)) / \sqrt{9n^8 + 1}$, $b_n = 1/3n^2$.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2 + 2\sin(n)}{\sqrt{9n^8 + 1}} \cdot \frac{3n^2}{1} = 1$$

Since $\lim_{n \rightarrow \infty} a_n/b_n$ exists and $\sum_{n=1}^{\infty} b_n$ is convergent, we have $\sum_{n=1}^{\infty} \frac{n^2 + 2\sin(n)}{\sqrt{9n^8 + 1}}$ is also convergent.

