IMPROPER INTEGRALS

THERE ARE FOUR TYPES OF IMPROPER INTEGRALS:

DEFINITION

(i) IF $f(x)$ IS CONTINUOUS ON $[a,\infty)$, THEN

$$\int_a^\infty f(x)\,dx = \lim_{L\to\infty}\int_a^L f(x)\,dx.$$

(ii) IF $f(x)$ IS CONTINUOUS ON $(a,b]$ THEN

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

(iii) IF $f(x)$ IS CONTINUOUS ON $[a,b)$ THEN

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

(iv) IF $f(x)$ IS CONTINUOUS ON $[a,c)\cup(c,b]$ THEN

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_b^c f(x)\,dx.$$

— TREATED THEN AS IN (ii) AND (iii) BY LIMITS —

KEY RESULT 1 CONSIDER $I = \int_1^\infty \frac{1}{x^p}\,dx$. THEN I IS FINITE IFF $p>1$.

PROOF DEFINE $I_L \equiv \int_1^L x^{-p}\,dx$. WE CALCULATE $I_L = \begin{cases} \left.\frac{x^{1-p}}{1-p}\right|_1^L, & p\neq 1 \\ \left.\ln x\right|_1^L, & p=1 \end{cases}$.

THIS GIVES
$$I_L = \begin{cases} \frac{L^{1-p}}{1-p} - \frac{1}{1-p}, & \text{IF } p\neq 1 \\ \ln L, & \text{IF } p=1 \end{cases}$$

NOW LET $L\to\infty$. WE HAVE $I_L \to$ FINITE VALUE IFF $p>1$ FOR THEN $L^{1-p}\to 0$. THUS, IF $p>1$, $I = \lim_{L\to\infty} I_L = \frac{1}{p-1}$.

QUALITATIVELY, A LIMIT EXISTS IFF $f(x) = x^{-p}$ DECAYS FAST ENOUGH (i.e. $p>1$) AS $x\to+\infty$.

KEY RESULT 2 CONSIDER AN EXAMPLE OF (ii) WHERE $I = \int_0^1 \frac{1}{x^p}\,dx$.
THEN I IS FINITE IFF $p<1$, i.e. IF x^{-p} BLOWS UP "SLOW ENOUGH" AS $x\to 0^+$. WE DEFINE $I_c \equiv \int_c^1 \frac{1}{x^p}\,dx$, AND ARE INTERESTED IN $\lim_{c\to 0^+} I_c$.

WE CALCULATE
$$I_C = \begin{cases} \frac{1}{1-p} x^{1-p} \Big|_c^1 = \frac{1}{1-p}\left(1 - c^{1-p}\right) & \text{IF } p \neq 1 \\ \ln x \Big|_c^1 = -\ln c & \text{IF } p = 1. \end{cases}$$

N ORDER FOR $\lim_{c \to 0^+} I_c$ TO EXIST WHEN $p \neq 1$ WE NEED $\lim_{c \to 0} c^{1-p} = 0$.

HIS ONLY OCCURS IF $p < 1$. THUS INTEGRAL CONVERGES IFF $p < 1$ AND IN THIS ASE
$$I = \lim_{c \to 0} I_c = \frac{1}{1-p} \quad \text{FOR } p < 1.$$

WITH THESE TWO BASIC KEY RESULTS WE CAN USE THEM IN ONJUNCTION WITH A STANDARD COMPARISON TEST TO PROVE CONVERGENCE R DIVERGENCE OF INTEGRALS.

<u>THEOREM (COMPARISON TEST)</u>

SUPPOSE $f(x)$ AND $g(x)$ ARE CONTINUOUS ON $[a, \infty)$ WITH $0 \le f(x) \le g(x)$ FOR $x \ge a$. THEN

(I) IF $\int_a^\infty g(x)\,dx < \infty \implies \int_a^\infty f(x)\,dx < \infty$.

(II) IF $\int_a^\infty f(x)\,dx$ diverges THEN SO DOES $\int_a^\infty g(x)\,dx$.

<u>ROOF OF (I)</u> SINCE $0 \le f(x) \le g(x)$ FOR $x \ge a$ WE HAVE
$$\int_a^L f(x)\,dx \le \int_a^L g(x)\,dx \quad \text{FOR ANY } L > a.$$

OW IF $\int_a^\infty g(x)\,dx$ IS FINITE WE HAVE SINCE $g(x) \ge 0$ THAT
$$\int_a^L f(x)\,dx \le \int_a^L g(x)\,dx \le \int_a^\infty g(x)\,dx < \infty.$$

LETTING $L \to \infty$ GIVES THE RESULT. (II) IS PROVED THE SAME WAY.

NEXT, WE DO SOME EXAMPLES WITH THE COMPARISON TEST AND UR TWO KEY RESULTS 1 AND 2.

EXAMPLE 1 LET $f(x) = \dfrac{\sin^2 x}{x^2}$ FOR $x \geq 1$.

DOES $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ CONVERGE? IF SO, GIVE A PROOF.

SOLUTION WE MUST FIND A COMPARISON FUNCTION. WE OBSERVE THAT SINCE $|\sin^2 x| \leq 1$ $\forall x$ THAT $\dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}$ FOR $x \geq 1$.

LET $g(x) = \dfrac{1}{x^2}$. THEN $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx < \infty$ SINCE $p = 2 > 1$ (KEY RESULT 1).

THUS, BY (I) OF COMPARISON THEOREM $\displaystyle\int_1^\infty f(x)\,dx \leq \int_1^\infty g(x) < \infty$ AND $\displaystyle\int_1^\infty f(x)\,dx < \infty$.

EXAMPLE 2 CONSIDER $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^2 - .5}}$. DOES THIS INTEGRAL CONVERGE OR DIVERGE?

WE FIRST GET SOME INTUITION: FOR x LARGE $\dfrac{1}{\sqrt{x^2 - .5}} \cong \dfrac{1}{x}$ AND $\displaystyle\int_1^\infty \frac{dx}{x}$ DIVERGES SO WE EXPECT DIVERGENCE. WE TRY TO IMPLEMENT (II) OF COMPARISON TEST.

NOTICE THAT $\sqrt{x^2 - .5} < \sqrt{x^2}$, $\forall x \geq 1$

THUS,
$$\frac{1}{\sqrt{x^2 - .5}} > \frac{1}{\sqrt{x^2}} = \frac{1}{x}.$$

DEFINE $g(x) = \dfrac{1}{x}$ AND $f(x) = \dfrac{1}{\sqrt{x^2 - .5}}$. WE HAVE $\displaystyle\int_1^\infty f(x)\,dx > \int_1^\infty g(x)\,dx$

AND $\displaystyle\int_1^\infty g(x)\,dx$ DIVERGES, BY (II) OF COMPARISON TEST $\displaystyle\int_1^\infty f(x)\,dx$ DIVERGES.

EXAMPLE 3 CONSIDER $\displaystyle\int_2^\infty \frac{x\,dx}{x^3 + x^2 + 1}$. DOES THE INTEGRAL CONVERGE OR DIVERGE? JUSTIFY YOUR ANSWER. <u>INTUITION</u>: $x^3 + x^2 \cong x^3$ FOR x LARGE, SO THAT
$$\frac{x}{x^3 + x^2 + 1} \cong \frac{x}{x^3} = \frac{1}{x^2}$$ AND $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ CONVERGES. SO WE EXPECT CONVERGENCE.

WE NOW MAKE A PROOF, USING (I) OF COMPARISON TEST.

WE OBSERVE $x^3 + x^2 + 1 \geq x^3$ ON $x \geq 2$. THUS $\dfrac{1}{x^3 + x^2 + 1} \leq \dfrac{1}{x^3}$ ON $x \geq 2$,

AND SO $\displaystyle\int_2^\infty \frac{x}{x^3 + x^2 + 1}\,dx \leq \int_2^\infty \frac{x}{x^3}\,dx = \int_1^\infty \frac{1}{x^2}\,dx$. LET $f(x) = \dfrac{x}{x^3 + x^2 + 1}$, $g(x) = \dfrac{1}{x^2}$

SINCE $\displaystyle\int_2^\infty g(x)\,dx < \infty \qquad (p = 2 > 1)$

THEN $\displaystyle\int_2^\infty f(x)\,dx < \infty.$

NOW CONSIDER IMPROPER INTEGRALS WITH AN INTERIOR SINGULARITY.

EXAMPLE 1 $\displaystyle I = \int_{-1}^{1} \frac{1}{x^2}\,dx.$ NOW $f(x) = \dfrac{1}{x^2}$ IS NOT CONTINUOUS AT $x = 0$ AND SO

WE CAN'T SIMPLY FIND ANTI-DERIVATIVE: i.e. $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx \neq -\frac{1}{x}\Big|_{-1}^{1} = -2.$

INSTEAD WE WRITE

$$I = \lim_{a\to 0^+} \underbrace{\int_a^1 \frac{1}{x^2}\,dx}_{\text{DIVERGES SINCE } p=2>1} + \lim_{a\to 0^-} \underbrace{\int_{-1}^{a} \frac{1}{x^2}\,dx}_{\text{diverges since } p=2>1.}$$

SO $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ IS DIVERGENT.

EXAMPLE 2 LET $\displaystyle I = \int_0^3 \frac{1}{(x-1)^{2/3}}\,dx.$ SINCE $f(x) = (x-1)^{-2/3}$ IS NOT CONTINUOUS AT $x = 1$

WE MUST WRITE

$$I = \lim_{a\to 1^-} \int_0^a \frac{1}{(x-1)^{2/3}}\,dx + \lim_{a\to 1^+} \int_a^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{a\to 1^-} 3(x-1)^{1/3}\Big|_0^a + \lim_{a\to 1^+} 3(x-1)^{1/3}\Big|_a^3$$

$$= 3 + 3\cdot 2^{1/3} = 3\left(2^{1/3}+1\right)$$ FINITE. $\to$ INTEGRAL CONVERGES.

EXAMPLE 3 $\displaystyle I = \int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx.$ IS IT CONVERGENT OR DIVERGENT.

SOLUTION WE WRITE $\displaystyle I = \int_1^2 \frac{x}{\sqrt{(x-1)(x+1)}}\,dx.$ LET $f(x) = \dfrac{x}{\sqrt{(x-1)(x+1)}}$

NEAR $x = 1$, $f(x) \approx \dfrac{1}{2\sqrt{x-1}}$ AND $\displaystyle\int_1^b \frac{dx}{2\sqrt{x-1}}$ EXISTS SINCE $p = 1/2 < 1.$

INTEGRAL CONVERGES!

1) WE KNOW THAT $\lim_{L\to\infty} \int_2^L \frac{1}{x}\,dx$ IS INFINITE. WHAT HAPPENS IF WE CHOOSE AN $f(x)$ THAT DECAYS <u>SLIGHTLY</u> MORE RAPIDLY AS $x \to \infty$. LET $p>0$ AND CONSIDER $f(x) = \frac{1}{x[\ln x]^p}$.

THEN USING SUBSTITUTION RULE $u = \ln x$

$$\int_2^L \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\ln L} u^{-p}\,du = \begin{cases} u^{1-p}/_{1-p}\Big|_{\ln 2}^{\ln L} & \text{if } p \neq 1 \\ \ln u\Big|_{\ln 2}^{\ln L} & \text{if } p = 1 \end{cases}$$

($u>0$ so $|\,|$ NOT needed.)

THUS $I_L \equiv \int_2^L \frac{1}{x(\ln x)^p}\,dx = \begin{cases} \frac{(\ln x)^{1-p}}{1-p}\Big|_2^L & \text{if } p \neq 1 \\ \ln(\ln x)\Big|_2^L & \text{if } p = 1 \end{cases}$

WE CONCLUDE THAT $\lim_{L\to\infty} I_L$ IS FINITE ONLY IF $p>1$.

SIMILARLY WE KNOW THAT IF $f(x) = \frac{1}{x}$ THAT $I_\varepsilon \equiv \int_\varepsilon^{1/2} \frac{1}{x}\,dx$ DIVERGES AS $\varepsilon \to 0^+$ SINCE $\frac{1}{x}$ BLOWS UP TOO FAST AS $x \to 0^+$.

WHAT IF WE MODIFY $f(x)$ SLIGHTLY TO $f(x) = \frac{1}{x[\ln x]^p}$, FOR $p>0$.

WE STILL HAVE $|f(x)| \to \infty$ AS $x \to 0^+$ $\left(\text{SINCE } \lim_{x\to 0^+} x[\ln x]^p = 0\right)$

BUT NOW IF WE DEFINE

$$I_\varepsilon \equiv \int_\varepsilon^{1/2} \frac{1}{x[\ln x]^p}\,dx$$ WE CALCULATE AS ABOVE THAT

$$I_\varepsilon = \begin{cases} \frac{(\ln x)^{1-p}}{1-p}\Big|_\varepsilon^{1/2} & \text{if } p \neq 1 \\ \ln(\ln x)\Big|_\varepsilon^{1/2} & \text{if } p = 1 \end{cases}$$

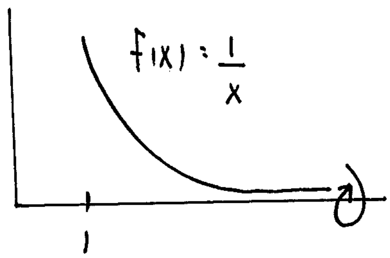
SINCE $(\ln \varepsilon)^{1-p} \to 0$ AS $\varepsilon \to 0^+$ IFF $p>1$ WE HAVE THAT

$\lim_{\varepsilon\to 0^+} I_\varepsilon$ IS <u>FINITE</u> IFF $p>1$.

2) VOLUMES OF REVOLUTION

WE LET $f(x) = \frac{1}{x}$ AND WE KNOW THAT $\int_1^\infty \frac{1}{x}\,dx$ IS INFINITE.

NOW SUPPOSE WE CALCULATED THE VOLUME OF REVOLUTION



WE GET $V = \pi \int_1^\infty (f(x))^2\,dx = \pi \int_1^\infty x^{-2}\,dx$

SO $V = \pi \left(-x^{-1}\right)\Big|_1^\infty = \pi$, WHICH IS FINITE.

(MORE PROPERLY $V_L = \pi \int_1^L (f(x))^2\,dx \to \pi$ AS $L \to \infty$.

THUS WE EXPECT THAT IF $f(x) \to 0$ AS $x \to \infty$ MORE slowly than $1/x$ WE CAN GET A FINITE INTEGRAL FOR THE VOLUME.

SUPPOSE $f(x) = 1/x^p$ FOR $p > 0$.

THEN $V(x) = \pi \lim_{L\to\infty} \int_1^L [f(x)]^2\,dx = \pi \lim_{L\to\infty} \int_1^L x^{-2p}\,dx = \pi \begin{cases} \frac{x^{1-2p}}{1-2p}\Big|_1^L, & \text{IF } p \neq 1/2 \\ \ln x \Big|_1^L, & \text{IF } p = \frac{1}{2} \end{cases}$

WE CONCLUDE THAT IFF $p > 1/2$ WE HAVE A FINITE INTEGRAL.

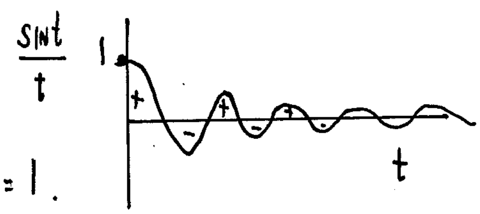
HENCE WE GET A <u>FINITE</u> VOLUME IF $f(x) \approx \frac{1}{x^p}$ WITH $p > 1/2$ AS $x \to \infty$

(slower decay is allowed than with calculating areas).

3) ONE MIGHT ASK whether WE CAN get A finite integral through "AREA CANCELLATION".

A FAMOUS SPECIAL FUNCTION IS

$$f(x) = \int_0^{x} \frac{\sin t}{t}\,dt$$



NOTICE $t = 0$ IS FINE SINCE $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

HOWEVER, DO WE GET A FINITE LIMIT AS $x \to \infty$ (IF NO $\sin t$ TERM THEN $\int_1^x \frac{1}{t}\,dt = \ln x \to +\infty$.).

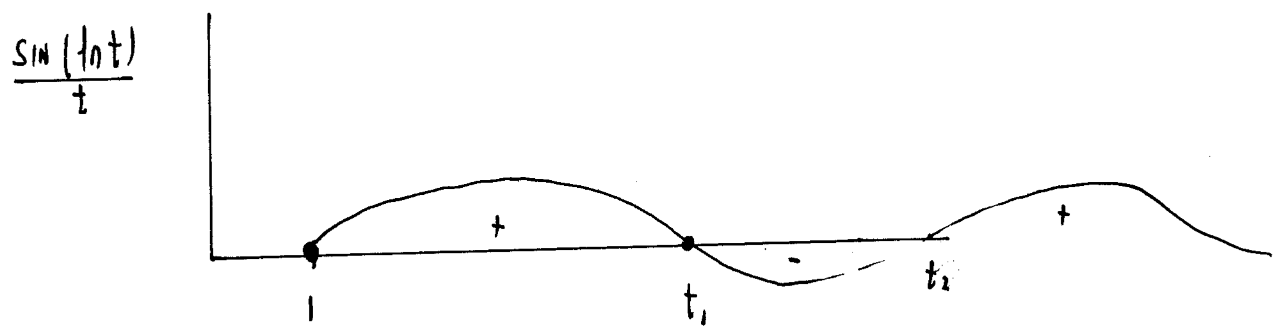
THIS IS A DIFFICULT QUESTION (M300) TO EVALUATE $\lim f(x)$ BUT WOLFRAM ALPHA GIVES $\lim_{x\to\infty} f(x) = \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

SO AN AREA CANCELLATION PROCEDURE MUST BE AT PLAY HERE.

TO SHOW THE SUBTLTY IN THIS CONSIDER MODIFYING $f(x)$ TO

$$\hat{f}(x) = \int_1^X \frac{\sin(\ln t)}{t}\,dt$$

THEN $\sin(\ln t) = 0$ WHEN $\ln t = n\pi \rightarrow t_n = e^{n\pi}$ AND SO WE STILL GET SOME AREA CANCELLATION, BUT $\sin(\ln t)$ VARIES REALLY SLOWLY IN t.



WE NOW CALCULATE USING $u = \ln t$ $\quad du = 1/t\,dt$

$$\int_1^X \frac{\sin(\ln t)}{t}\,dt = \int_0^{\ln X} \sin(u)\,du = -\cos u\Big|_0^{\ln X} = 1 - \cos(\ln X).$$

NOTICE THEN THAT $\hat{f}(X) \equiv 1 - \cos(\ln X)$.

AS $X \to \infty$ $\hat{f}(x)$ OSCILLATES Between 0 AND 2

AND DOES NOT APPROACH A LIMITING VALUE.

HENCE IMPROPER INTEGRALS CAN BE FINITE, INFINITE, OR OSCILLATORY DEPENDING ON SPECIFICS OF THE PROBLEM.

4) <u>REMARK</u> CONSIDER $I = \int_a^b \frac{g(x)}{f(x)}\,dx$.

IF $f(c) = 0$ FOR $a < c < b$ WITH $f'(c) \neq 0$ AND $g(c) \neq 0$ THE INTEGRAL DIVERGES SINCE BY TANGent LINE APPROXIMATION

$$\frac{g(x)}{f(x)} \simeq \frac{g(c)}{f'(c)(X-C)} \quad \text{NEAR } X = C.$$

5) FINALLY CONSIDER

$$I = \int_1^L \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} \right) dx .$$

WE KNOW THAT SEPARATELY $\int_1^\infty \frac{1}{x}$ AND $\int_1^\infty \frac{1}{\sqrt{x^2+1}}$ ARE INFINITE,

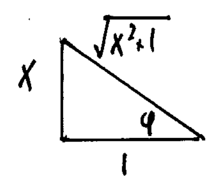
BUT CAN WE GET A FINITE INTEGRAL THROUGH CANCELLATION.

(WHEN WE DO TAYLOR SERIES WE NOTE THAT FOR $x \to +\infty$

$$\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} = \frac{1}{x(1+1/x^2)^{1/2}} - \frac{1}{x} = \frac{1}{x}\left(1+\frac{1}{x^2}\right)^{-1/2} - \frac{1}{x} \hat{=} \frac{1}{x}\left(1 - \frac{1}{2x^2} + \cdots\right) - \frac{1}{x}$$

$$\approx 1/2x^3 \quad \text{AS} \quad x \to \infty \quad \text{AND} \quad \int_A^\infty \frac{1}{2x^3}\,dx \text{ IS FINITE.}$$

SO WE EXPECT THAT $\lim_{L\to\infty} I_L$ IS FINITE.



$\sec\varphi = \sqrt{.}$

$\tan\varphi = x$

WE CALCULATE: $x = \tan\varphi \qquad dx = \sec^2\varphi \, d\varphi$ SO

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{\sec^2\varphi}{\sec\varphi}\,d\varphi = \int \sec\varphi = \log\left[\sec\varphi + \tan\varphi\right] + C$$

SO $$\int \frac{1}{\sqrt{x^2+1}}\,dx = \log\left[\sqrt{x^2+1} + x\right] + C .$$

THUS $$I_L \equiv \int_1^L \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x}\right) dx = \left(\log\left(\sqrt{x^2+1} + x\right) - \log x\right)\Big|_1^L$$

$$= \log\left(\frac{x+\sqrt{x^2+1}}{x}\right)\Big|_1^L = \log\left(1+\sqrt{1+\frac{1}{x^2}}\right)\Big|_1^L$$

SO $$I_L = \log\left(1+\sqrt{1+\frac{1}{L^2}}\right) - \log\left(1+\sqrt{2}\right)$$

NOW let $L \to \infty$ SO $$I_L \to \log\left(\frac{2}{1+\sqrt{2}}\right) = \int_1^\infty \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x}\right) dx .$$

SINCE $\sqrt{x^2+1} > x$ ON $1 < x \to \frac{1}{\sqrt{x^2+1}} - \frac{1}{x} < 0$ ON $x > 1$ SO $I_L < 0$.

INDEED $\frac{2}{1+\sqrt{2}} < 1$ SO $\log\left(\frac{2}{1+\sqrt{2}}\right) < 0$.

INTERESTINGLY WOLFRAM FAILS TO CALCULATE $\int_1^\infty \left(\frac{1}{\ldots} - \frac{1}{\ldots}\right) dx$

6) CONSIDER AN IMPROPER INTEGRAL BUT ONE FOR WHICH WE GET A FINITE VALUE; i.e.

$$I = \int_0^1 \frac{f(x)}{\sqrt{x}}\,dx \qquad \text{WHERE } f(x) \text{ IS CONTINUOUS ON } 0 \le x \le 1 \text{ WITH } f(0) \neq 0.$$

SINCE THE INTEGRAND BLOWS UP AS $x \to 0^+ \to$ STANDARD RIEMANN SUMS ARE NOT SO GOOD.

LET $u = x^{1/2}$ $\qquad du = \frac{1}{2} x^{-1/2}\,dx = \frac{1}{2u}\,dx$

$$\frac{dx}{\sqrt{x}} = \frac{2u}{u}\,du.$$

$x = 0 \to u = 0$

$x = 1 \to u = 1$

SO
$$I = \int_0^1 2 f(u^2)\,du$$

$$I = 2\int_0^1 f(u^2)\,du$$

$\Uparrow$ equivalent integral BUT better FOR numerical quadrature.