(based on this idea) — it is the energy expended acting against a force. eg — the energy expended moving a weight against gravity.

We need some definitions

Definition.

- Time t — measured in seconds
- Position s — measured in metres
- Mass m — measured in grams of kilograms
- Newton's second law

$$\text{Force } = \text{mass} \times \text{acceleration} \qquad\qquad F = m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

  Force is measured in Newtons $= kg.m/s^2$

- Work at constant force measures energy required to act against a force

$$\text{Work } = \text{Force} \times \text{displacement} \qquad\qquad W = Fd$$

  Measured in Newton-metres = Joules

So if the force is constant, then the work is simply the force times the distance moved against the force — eg moving a heavy weight up off the floor. How much work is done moving a 1kg book from the floor to the top of a 2m heigh shelf?

- Acceleration due to gravity $= 9.8m/s^2$. Force due to gravity $= ma = 9.8N$.
- Work done against force is $9.8 \times 2 = 19.6J$.

Very easy. But what happens when the force is not constant? If it varies with distance — eg a spring — then we approximate the work by a Riemann sum.

- Let $f(x)$ be the force acting on an object at position x.
- To compute the work done in order to move the object from $x = a$ to $x = b$ we cut up the interval $[a, b]$ into n segments $[x_{i-1}, x_i]$ each of width $(b - a)/n$.
- We approximate the varying force in the interval $[x_{i-1}, x_i]$ by a constant force $f(x_i^*)$ where $x_i^* \in [x_{i-1}, x_i]$ — just as we approximated the area in a Riemann sum.
- The work done in this interval $[x_{i-1}, x_i]$ is then approximately $f(x_i^*)\Delta x$.
- So the total work is

$$W \approx \sum_{i=1^n} f(x_i^*)\Delta x$$

  which is exactly a Riemann sum.

- Hence as $n \to \infty$ we have

$$W = \int_a^b f(x)\mathrm{d}x$$

Definition (Work — clp 2.1.1)**.** The word done by a force $F(x)$ moving an object from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\mathrm{d}x$$

Note that if $F(x) = c$ is constant then $W = c(b - a)$.

Hooke's law relates the force exterted by a string, F, to the distance it has been stretched x:

$$F = kx \qquad\qquad k = \text{spring constant}$$

Holds for lots of materials provided x isnt too large.

A standard example: A spring has natural length of 20cm. If a 25N force is required to keep it stretched at a length of 30cm how much work is required to stretch it from 20cm to 25cm?

- Careful of units — we need newtons and metres.
- First work out the spring constant:

$$\begin{aligned} F &= kx \\ 25 &= k(0.30 - 0.20) \\ k &= 25/0.1 = 250N/m \end{aligned}$$

- So we can now work out the work

$$\begin{aligned} W &= \int_0^{0.05} F(x)\mathrm{d}x \\ &= \int_0^{0.05} 250x\mathrm{d}x \\ &= [125x^2]_0^{0.05} = 125 \times 0.0025 = 0.3125J \end{aligned}$$

Another slightly less standard example: A chain lying on the ground is 10m long and weighs 80kg. How much work is required to raise one end to a height of 6m?

- Assume that the chain will be "L" shaped when it has been lifted, with 4m left on the ground. Also assume no friction and constant density of the chain = 8kg/m. (A picture might help)
- Let us do this with a Riemann sum — split the chain into segments $[x_{i-1}, x_i]$ and work out how much work is done lifting each segment.
- Let x be the distance (in m) from the top of the chain. The piece of chain at x is lifted $6 - x$ m. Hence the segment $[x_{i-1}, x_i]$ is lifted $6 - x_i^*$ m.
- The segment weighs $8\Delta x$, so experiences $8 \times 9.8\Delta x = 78.4\Delta x$ gravitational force.
- So the Riemann sum is given by

$$W \approx \sum_{i=1}^{n} (6 - x_i^*)78.4\Delta x$$

- So the work is given (limit of $n \to \infty$)

$$\begin{aligned} W &= 78.4 \int_0^6 (6-x)\mathrm{d}x \\ &= 78.4[6x - x^2/2]_0^6 = 78.4 \times 18 = 1411.2J \end{aligned}$$

What if the chain was dangling from the roof and we were to lift the far end?

- Let x be the distance from the middle of the chain.
- The piece of chain at x is lifted a distance of $2x$
- Hence the Riemann sum is

$$W \approx \sum_{i=1}^{n} 2x_i^* 78.4\Delta x$$

- So the work done is

$$\begin{aligned} W &= \int_0^5 156.8x\mathrm{d}x \\ &= 156.8[x^2/2]_0^5 = 156.8 \times 12.5 = 1960 \end{aligned}$$

Similarly if we calculate the work done pumping water from a tank — compute the work done pumping out each "slice" of water.

A very standard example:

- The tank is shaped like an inverted cone — height = 10m, radius = 4m.
- Filled to height of 8m.
- Find work pumping water out of top.
- Density of water is $1000kg/m^3$.

How do we do this?

- Draw a picture.
- How much work done to remove each "slice" of water?
- Let x be distance from bottom of the tank. The slice at x has volume

$$\begin{aligned} V(x) &= \pi r(x)^2 \Delta x \\ &= \pi \left(\frac{2x}{5}\right)^2 \Delta x \end{aligned}$$

- The weight of this slice is $1000V(x)$. So the gravitation force acting on the slice is $1000V(x) \times 9.8$.
- We need to move the slice at x up $10 - x$ metres.

- So the work done is

$$\begin{aligned} W &= \int_0^8 9800V(x)(10-x)\mathrm{d}x \\ &= 1568\pi \int_0^8 x^2(10-x)\mathrm{d}x \\ &\approx 3.36 \times 10^6 J \end{aligned}$$

Finally — let is look at how work relates to some of newton's laws of motion. To do this assume that we are moving an object against a force, $F(x)$, and that the position of the object is given by $x(t)$. Then the work is given by

$$\begin{aligned} W &= \int_a^b F(x)\mathrm{d}x && \text{sub } x = x(t) \\ &= \int_{t=\alpha}^{t=\beta} F(x(t))\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t \end{aligned}$$

Newton to the rescue with his laws $F = ma$:

$$\begin{aligned} &= \int_\alpha^\beta m\frac{\mathrm{d}^t 2}{\mathrm{d}x^t}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t \\ &= m\int_\alpha^\beta v'(t)v(t)\mathrm{d}t && \text{a little chain rule} \\ &= m\int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v(t)^2\right)\mathrm{d}t \\ &= m\left[\frac{1}{2}v(t)^2\right]_\alpha^\beta \\ &= \frac{1}{2}mv(\beta)^2 - \frac{1}{2}mv(\alpha)^2 \end{aligned}$$

This new function $\frac{m}{2}v^2$ is the *kinetic energy* — so this is telling us that the work done is equal to the difference in kinetic energy. (this is related to the concept of "conservation of energy").