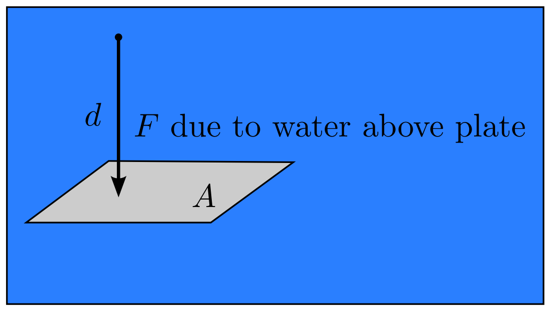
8 Further applications on integration

8.3 Applications to physics and engineering

Hydrostatic pressure and force

Definition. Pressure is the force per unit area. Measured in Pascals = Newtons / per square metre. $Pa = N/m^2$.

When you dive deep in a pool your ears begin to hurt because of all the water above you pressing down on your eardrums. The deeper you go the more it hurts.



If there was a difference in pressure then there would be a difference in forces and the liquid would be moving.

Important principle — at any point in a liquid (at rest) the pressure is the same in all directions. The pressure on your ears doesnt change when you get close to the wall of the pool.

Computing the pressure

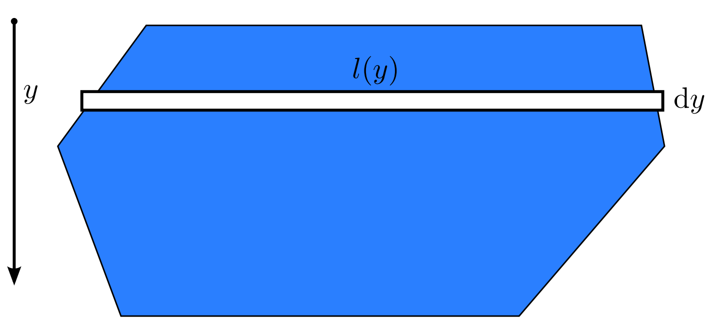
$$\begin{aligned} F &= mg & g &= \text{acc. due to gravity} \\ &= \rho A d g & \rho &= \text{density of water} = 1000 kg/m^3 \end{aligned}$$

So

$$P = F/A = \rho g d$$

Note that the pressure on your ears is equal to that of the pressure on the wall of the pool (at the same depth) but the total force on your ears is not the same as the total force on the wall of the pool

Consider the flat wall of a pool which has varying width $l(y)$. Compute the total force on the wall



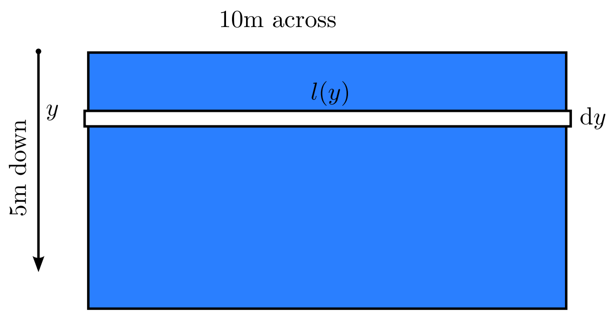
Use the Riemann sum idea. What is the force on the little strip $l(y)\Delta y$?

$$\begin{aligned}\Delta F(y) &= P(y)\Delta A(y)\\ &= (\rho g y)(l(y)\Delta y)\end{aligned}$$

Sum them and take limits

$$F = \int_0^{y_{max}} \rho g y l(y) \mathrm{d}y$$

Simple example — viewing window in a whale tank which is 5m deep and 10m across.



What is the force due to water on the window?

$$P(y) = \rho g y \qquad \text{depends only on depth}$$

Hence

$$\begin{aligned}\Delta F(y) &= P(y)\Delta A(y)\\ &= \rho g y 10\Delta y\\ &= 1000 \times 9.8 \times y \times 10\Delta y\end{aligned}$$

Sum and take limits

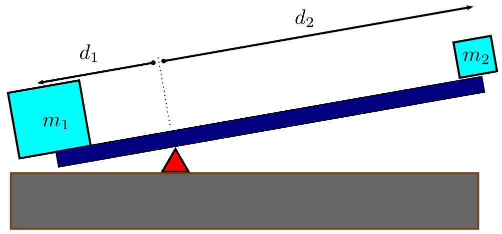
$$\begin{aligned}F &= \int_0^5 9.8 \times 10^4 y \mathrm{d}y\\ &= 9.8 \times 10^4 [y^2/2]_0^5\\ &= 9.8 \times 10^4 \times 25/2\\ &= 1.225 \times 10^6 N\end{aligned}$$

This is a big force!

Centre of mass and moments

When you sit on a see-saw (teeter-totter?) the torques need to be in balance. This will be the case if

$$m_1 d_1 = m_2 d_2.$$



More generally everything will balance if the lever is balanced at the "centre of mass"

$$\begin{aligned} m_1 \underbrace{(\bar{x} - x_1)}_{\text{to the left}} &= m_2 \underbrace{(x_2 - \bar{x})}_{\text{to the right}} \\ (m_1 + m_2)\bar{x} &= m_1 x_1 + m_2 x_2 \\ \bar{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \end{aligned}$$

More generally

$$\bar{x} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{m}$$

where m is the total mass and $m_i x_i$ is called the moment of mass i.

Now let us do this for a continuous system with density $\rho(x)$. You get a similar equation

$$\bar{x} = \frac{1}{m} \int x \rho(x) \mathrm{d}x$$

Note how this looks like the mean of a distribution. $\bar{x}$ is the mean position of mass.

Now consider a flat 2-d plate of constant density ρ whose lower boundary is the x-axis and upper boundary is $y = f(x)$. Let us compute the moment about the y-axis.

A vertical strip at position x has mass $\rho f(x) \Delta x$. This has moment $x \rho f(x) \Delta x$. Adding up these strips gives

$$M_y = \rho \int x f(x) \mathrm{d}x$$

What about its moment about the x-axis. This is a bit harder. What does a column at position x contribute now? It is a strip of width Δx and length $f(x)$. Its mass is $\rho f(x) \Delta x$. The average position of mass in this strip (from the x-axis) is $f(x)/2$. Hence it contributes $\frac{1}{2} \rho f(x)^2 \Delta x$.

Adding these up and taking the limit gives

$$M_x = \frac{1}{2} \int \rho f(x)^2 \mathrm{d}x$$

Dividing both of these by the total mass (density times area)

$$m = \int \rho f(x) \mathrm{d}x$$

gives the coordinates of the centre of mass

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int x f(x) \mathrm{d}x}{\rho \int f(x) \mathrm{d}x} = \frac{\int x f(x) \mathrm{d}x}{\int f(x) \mathrm{d}x} = \frac{\int x f(x) \mathrm{d}x}{A}$$
$$\bar{y} = \frac{M_x}{m} = \frac{1}{A} \int \frac{1}{2} f(x)^2 \mathrm{d}x$$

This coordinate is called the "centroid" of the area. Notice that it is independent of the density (provided the density is constant).

Example: Find the centre of mass of a parabolic plate $y = 1 - x^2, y = 0, -1 \leq x \leq 1$.

$$\begin{aligned} A &= \int_{-1}^{1} y \mathrm{d}x = \int_{-1}^{2} (1 - x^2) \mathrm{d}x \\ &= [x - x^3/3]_{-1}^{1} = (1 - 1/3) - (-1 + 1/3) = 4/3 \end{aligned}$$

Now use the formulas

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_{-1}^{1} x f(x) \mathrm{d}x \\ &= \frac{1}{A} \int_{-1}^{1} (x - x^3) \mathrm{d}x \\ &= 0 && \text{function is odd} \end{aligned}$$

and

$$\begin{aligned} \bar{y} &= \frac{1}{2A} \int_{-1}^{1} (1 - x^2)^2 \mathrm{d}x \\ &= \frac{3}{8} \int_{-1}^{1} (1 - 2x^2 + x^4) \mathrm{d}x \\ &= \frac{3}{8} [x - 2x^3/3 + x^5/5]_{-1}^{1} \\ &= \frac{3}{8} \left((1 - 2/3 + 1/5) - (-1 + 2/3 - 1/5)\right) \\ &= \frac{3}{8} \times \left(\frac{8}{15} + \frac{8}{15}\right) \\ &= \frac{2}{5} \end{aligned}$$

So centre of mass is at $(0, 2/5)$.

You don't need the theorem of pappas

9 Differential equations

9.1 Modelling with DEs

This section is not officially part of the syllabus, but it helps to read through it before we do chapter 9.3. So — we need some notations and stuff from this chapter.

This is a very grand sounding chapter, but it is really an introduction to the idea of using derivatives to model things. Perhaps the simplest model one can start with is a population model — how does the population change with time? You looked at this in 180 or 100 last term.

Start by defining variables — this is always the first step in modelling things.

- t = time (measured in seconds, minutes, years, ...)
- P = the size of the population — had better be ≥ 0.

Now the "rate of change" of the population is $\frac{\mathrm{d}P}{\mathrm{d}t}$. This is going to be *roughly* proportional to the size of the population. *ie* the number of children born and number of people who die in a given time is going to be proportional to the number of people in the population. This is quite a gross assumption but let us start with something simple. Writing this as an equation gives

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

where k is the constant of proportionality.

This is a *differential equation* — an equation involving a function, P, and its derivatives. In fact it is easy to solve this model exactly:

$$P = Ce^{kt} \qquad\qquad \frac{\mathrm{d}P}{\mathrm{d}t} = kCe^{kt} = kP.$$

Now — the behaviour of the solution of this equation is going to tell us about the behaviour of the population. This behaviour is critically dependant on k.

- What happens if $P = 0$? Well then $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$ — *ie* Population is constant — makes sense.
- If $P > 0$ *and* $k > 0$ then $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$ — *ie* the population is increasing with time.
- If $P > 0$ *and* $k < 0$ then $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$ — *ie* the population is decreasing with time.

So now as $t \to \infty$ either $P \to \infty$ (if $k > 0$) or $P \to 0$ (if $k < 0$). It says our population either explodes or dies out. This model (I think) was first introduced by Malthus — the "Malthusian catastrophe" is his prediction that population would explode and overtake food supply basically leading to a lot of misery. He made the first mathematical models of populations.

Okay — how did we arrive at this solution? The easiest way — and perhaps the most instructive way (for other applications) is by treating this equation as a "separable equation".

First let us talk a bit more generally about DEs.

- A differential equation is an equation that involves a function and one or more of its derivatives.
- The order of the equation is the order of the highest derivative in the equation.
- In the previous examples P was the dependent variable and t was the independent variable

- Consider the equation

$$y' = xy$$

  It is understood that y is really a function of x.

- A function $f(x)$ is a solution of the differential equation if when you substitute $y = f(x)$ the equation is satisfied.

$$f'(x) = xf(x)$$

- When we solve a differential equation it is important to find *all* the solutions. Just like finding an indefinite integral — you have to find all the anti-derivatives.

- In the above example

$$\begin{aligned} y &= Ce^{x^2/2} && \text{for any } C \\ y' &= Ce^{x^2/2} \cdot x = xy \end{aligned}$$

- In our population model $y' = ky$ we had the solution $y = Ce^{kx}$ for any C.

- Now to find this constant we need more information — such as the initial value of y.

$$y(0) = P_0 \qquad \text{initial population} = Ae^{k0} = A$$

  Hence $A = P_0$.

- This is an example of an *initial value problem* — a differential equation together with an initial condition.

10 Separable equations — 9.3

- How do we solve a differential equation.

- For some very simple ones we can (almost) do it by inspection.

- In general it is quite difficult — like integration there are some families of differential equations that can be solved nicely.

- In fact most integration problems we have done can be considered differential equations:

$$\begin{aligned} F'(x) &= \text{some function of } x && \text{so} \\ F(x) &= \int (\text{some function of } x)\mathrm{d}x \end{aligned}$$

- This seems a bit simplistic, but it in fact will allow us to solve quite a large family of equations.

- A *separable equation* is one of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)g(x)$$

  ie the LHS is just $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the RHS can be *factored* into a function of x and a function of y.

- We can now separate this equation — pull all the y-stuff to the left and leave all the x-stuff on the right.

$$\frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$

  The RHS really needs to be $f(y)g(x)$ *not* $f(y)+g(x)$ — that is a very common error.

- Now integrate both sides wrt x:

$$\int \frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int g(x)\mathrm{d}x$$

- But the LHS is exactly the form of a substitution integral — so we can rewrite this as

$$\int \frac{1}{f(y)}\mathrm{d}y = \int g(x)\mathrm{d}x$$

- So we can now integrate both sides to get a solution.

Let us do an example

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -x/y && \text{separate} \\
y\frac{\mathrm{d}y}{\mathrm{d}x} &= -x && \text{integrate} \\
\int y\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x &= -\int x\mathrm{d}x \\
\int y\mathrm{d}y &= c - \frac{1}{2}x^2 && \text{don't forget the constant} \\
\frac{1}{2}y^2 &= c - \frac{1}{2}x^2 && \text{rearrange — cannot always do this} \\
y^2 &= 2c - x^2 \\
y &= \pm\sqrt{2c - x^2}
\end{aligned}$$

Solution is a circle!

Another slightly harder example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy \qquad \text{separate}$$

Provided $y \neq 0$ — we will come back to $y = 0$ later

$$\begin{aligned}
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= x && \text{integrate} \\
\int y^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x &= \int x\mathrm{d}x \\
\int y^{-1}\mathrm{d}y &= x^2/2 + c \\
\log|y| &= x^2/2 + c \\
|y| &= e^c e^{x^2/2} \\
y &= \pm e^c e^{x^2/2} \\
y &= Ae^{x^2/2}
\end{aligned}$$

where $A = \pm e^c$. Now $e^c > 0$, so $\pm e^c$ is any real number *except* 0. So now we check $A = 0$ separately:

$$y = 0e^{x^2/2} = 0\frac{\mathrm{d}y}{\mathrm{d}x} \qquad = 0 = x \times 0$$

So $y = 0$ is a solution, so A is any real number. Hence our general solution is

$$y = Ae^{x^2/2}$$

Please read through the stuff on "orthogonal trajectories".

A standard problem for these is a "mixing" problem.

- A tank contains 20kg of salt dissolved in 5000L of water. Brine that contains 0.03kg of salt per litre entres the tank at a rate of 25L/min. The solution is kept thoroughly mixed and drains out at the same rate. How much salt remains in the tank after half an hour?
- Let y be the amount salt in the tank in kg.
- Let t be time in minutes.
- We are told $y(0) = 20$, and we need to find $y(30)$.
- Write down a differential equation for y.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (\text{rate in}) - (\text{rate out}) = kg/min$$

- Now rate in/out is (concentration)×(flow)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= (0.03kg/L)(25L/min) - (y/volumekg/L)(25L/min) \\
&= 0.75 - \frac{25y}{5000} = \frac{3}{4} - \frac{y}{200} \\
&= \frac{150 - y}{200}
\end{aligned}$$

- Now solve it — separable

$$\begin{aligned}\int \frac{1}{150-y}\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t &= \int \frac{1}{200}\mathrm{d}t \\ \int \frac{1}{150-y}\mathrm{d}y &= \frac{t}{200} + c \\ -\log|150-y| &= \frac{t}{200} + c\end{aligned}$$

- Inital condition $y(0) = 20$:

$$-\log|130| = c$$

- Hence we have

$$\begin{aligned}-\log|150-y| &= \frac{t}{200} - \log 130 \\ \log|150-y| &= 130 - \frac{t}{200} \\ |150-y| &= 130e^{-t/200} \\ 150-y &= \pm 130e^{-t/200} \\ y &= 150 \pm 130e^{-t/200}\end{aligned}$$

- Which of these is correct — well since $y(0) = 20$, we must take the negative-branch

$$y = 150 - 130e^{-t/200}$$

- Hence $y(30) = 150 - 130e^{-30/200}$.
- Also note that as $t \to \infty$ $y(t) \to 150$. If the tank is completely full of the incoming brine, then it would contain $5000 \times 0.3 = 150kg$ of salt.