1 Integration

1.8 Trigonometric integrals

In this section we will learn to integrate combinations of trig functions.

$$\int \sin^a x \cos^b x \mathrm{d}x$$
$$\int \tan^a x \sec^b x \mathrm{d}x$$

You will need to remember

$$\sin^2 x + \cos^2 x = 1 \qquad \text{divide by cos} \qquad \tan^2 x + 1 = \sec^2 x$$

and some double angle formulas

$$\begin{aligned} \sin 2x &= 2 \sin x \cos x \\ \cos 2x &= 1 - 2\sin^2 x = 2\cos^2 x - 1 = \cos^2 x - \sin^2 x \end{aligned}$$

Sometimes it will be helpful to rewrite these isolating the sine or cosine squared term:

$$\begin{aligned} \sin^2 x &= \frac{1-\cos(2x)}{2} \\ \cos^2 x &= \frac{1+\cos(2x)}{2} \end{aligned}$$

These last two can be very helpful to rewrite higher powers of sine and cosine in terms of lower powers:

$$\begin{aligned} \sin^4 x &= \left(\frac{1-\cos(2x)}{2}\right)^2 \\ &= \frac{1}{4}\left(1 - 2\cos(2x) + \underbrace{\cos^2(2x)}_{again}\right) \\ &= \frac{1}{4}\left(1 - 2\cos(2x) + \frac{1+\cos(4x)}{2}\right) \\ &= \frac{1}{8}\left(3 - 4\cos(2x) + \cos(4x)\right) \end{aligned}$$

This is one basic approach — rewrite higher powers of trig functions in terms of lower powers using identities like the above. The other is to massage the integrand to make it look like a substitution by sine or cosine.

Integrating powers of sine and cosine — 1.8.1

Examples — not obviously a substitution integral

$$\begin{aligned} \int \sin^5 x \mathrm{d}x &= \int \sin x (1-\cos^2 x)^2 \mathrm{d}x \\ &= \int \sin x (1 - 2\cos^2 x + \cos^4 x) \mathrm{d}x \end{aligned}$$

now this is clearly a substitution integral with $u = \cos x$ and $u' = -\sin x$

$$\begin{aligned}
&= -\int (1-2u^2+u^4)\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = -\int (1-2u^2+u^4)\mathrm{d}u \\
&= -u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + c \\
&= -\cos x + \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x + c
\end{aligned}$$

So even powers of sine became $\cos^2 x$, leaving us with a single sine and a polynomial in cosine.

What about even powers? We need the double angle formulas

$$\begin{aligned}
\int \cos^4 x\mathrm{d}x &= \int (\cos^2 x)^2 \mathrm{d}x & 2\cos^2 x = \cos 2x + 1 \\
&= \int \left(\frac{\cos 2x + 1}{2}\right)^2 \mathrm{d}x \\
&= \int \left(\frac{\cos^2 2x}{4} + \frac{\cos 2x}{2} + \frac{1}{4}\right)\mathrm{d}x & 2\cos^2 2x = \cos 4x + 1 \\
&= \int \left(\frac{\cos 4x + 1}{8} + \frac{\cos 2x}{2} + \frac{1}{4}\right)\mathrm{d}x \\
&= \frac{1}{8}\int (\cos 4x + 4\cos 2x + 3)\,\mathrm{d}x \\
&= \frac{1}{8}\left(\frac{1}{4}\sin 4x + 2\sin 2x + 3x\right) + c
\end{aligned}$$

In the text this is not really a theorem, but more of an algorithm. Let us summarise it here.

Theorem (CLP section 1.8.1)**.** *To integrate* $\int \sin^a x \cos^b x\mathrm{d}x$

- *If power of cosine is odd, then hold onto 1 power of cosine, and turn all the others into sines using* $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned}
\int \sin^a x \cos^{2k+1} x\mathrm{d}x &= \int \sin^a x(\cos^2 x)^{2k}\cos x\mathrm{d}x \\
&= \int \sin^a x(1-\sin^2 x)^{2k}\cos x\mathrm{d}x
\end{aligned}$$

- *If power of sine is odd, then hold onto 1 power of sine, and turn all the others into cosines using* $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned}
\int \sin^{2k+1} x \cos^b x\mathrm{d}x &= \int \sin x(\sin^2 x)^{2k}\cos^b x\mathrm{d}x \\
&= \int \sin x(1-\cos^2 x)^{2k}\cos^b x\mathrm{d}x
\end{aligned}$$

- *If both powers of sine and cosine are even, then use*

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad\qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

You can do (if there is time)

$$\begin{aligned}
\int \cos^2 x \sin^5 x \mathrm{d}x &= \int \cos^2 x \sin^4 x \cdot \sin x \mathrm{d}x && \text{keep one sine}\\
&= \int \cos^2 x \left(1-\cos^2 x\right)^2 \cdot \sin x \mathrm{d}x && \text{set } u = \cos x \text{ and } \mathrm{d}u = -\sin x \mathrm{d}x\\
&= -\int u^2(1-u^2)^2 \mathrm{d}u\\
&= -\int (u^6 - 2u^4 + u^2)\mathrm{d}u\\
&= -\frac{u^7}{7} + \frac{2u^5}{5} - \frac{u^3}{3} + C && \text{rewrite in terms of } x\\
&= -\frac{1}{7}\cos^7 x + \frac{2}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C
\end{aligned}$$

Why does this work? Because $\sin x$ and $\cos x$ are derivatives of each other. This will generalise to other pairs of functions whose derivatives are related. The most obvious place to continue is tangent and secant (well — arguably).

Integrating powers of tangent and secant — 1.8.2

We can do similar things with sec and tan. Lets recall some derivatives (maybe make the class do this?)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \sec^2 x\\
\frac{\mathrm{d}}{\mathrm{d}x}\sec x &= \frac{\sin x}{\cos^2 x} = \sec x \tan x
\end{aligned}$$

and also the equivalent identity involving their squares:

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 && \text{divide by cosine}\\
\tan^2 x + 1 &= \sec^2 x.
\end{aligned}$$

Now — unfortunately the double-angle identities are not as nice. For example

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

so it is not so easy to rewrite powers of tangent and secant in terms of lower powers (like we did for sine and cosine).

The other tool available to us is to move and massage the integrand around until it looks like a subsitution of either $u = \tan x$ or $u = \sec x$. Again — precisely what to do depends on the parity of the powers of tangent and secant. Unfortunately this method won't work in all cases, but will cover many.

Let us start with an easy case where everything works smoothly — when we have an even power of secant:

$$\begin{aligned}\int \tan^2 x \sec^4 x \mathrm{d}x &= \int \tan^2 x \sec^2 x (\sec^2 x \mathrm{d}x) \\ &= \int \tan^2 x (1+\tan^2 x)(\sec^2 x \mathrm{d}x) && u = \tan x, \frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 \\ &= \int u^2(1+u^2)\mathrm{d}u && \text{and so on}\end{aligned}$$

What if the power of secant is odd?

$$\int \tan^3 x \sec^7 x \mathrm{d}x = \int \tan^3 x \sec^5 x (\sec^2 x \mathrm{d}x)$$

Urgh — we get stuck because of this odd power of secant that we cannot completely convert into tangents.

All is not lost for this integral because we have an odd power of tangent. Try substituting $u = \sec x$, so $u' = \tan x \sec x$

$$\begin{aligned}\int \tan^3 x \sec^7 x \mathrm{d}x &= \int \tan^2 x \sec^6 x (\sec x \tan x \mathrm{d}x) \\ &= \int (\sec^2 x - 1) \sec^6 x (\sec x \tan x \mathrm{d}x) \\ &= \int (u^2-1)u^6 \mathrm{d}u \\ &= \frac{u^9}{9} - \frac{u^7}{7} + C \\ &= \frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C\end{aligned}$$

You can see that this will work out okay provided we also have an odd power of tangent.

We can also make the "odd power of tangent" case work by rewriting in terms of sine and cosine:

$$\begin{aligned}\int \tan^3 x \sec^7 x \mathrm{d}x &= \int \left(\frac{\sin x}{\cos x}\right) \cdot \left(\frac{1}{\cos x}\right)^7 \mathrm{d}x \\ &= \int \frac{\sin^3 x}{\cos^{10} x}\mathrm{d}x && \text{now a sine cosine integral} \\ &= \int \frac{\sin^2 x}{\cos^{10} x}\sin x \mathrm{d}x && \text{remove even powers of sine} \\ &= \int \frac{1-\cos^2 x}{\cos^{10} x}\sin x \mathrm{d}x && \text{sub } u = \cos x, u' = -\sin x \\ &= \int u^{-8} - u^{-10}\mathrm{d}u \\ &= \frac{1}{9}u^{-9} - \frac{1}{7}u^{-7} + C \\ &= \frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C\end{aligned}$$

So this takes care of either odd powers of tangent or even powers of secant. But what if we have even powers of tangent and odd powers of secant? Unfortunately in those cases we have to think a bit — sometimes we might have to think a lot. But first, lets summarise what we have seen above (its not really a theorem but more of a guide)

Theorem (See CLP 1.8.2)**.** *To integrate* $\int \tan^a x \sec^b x \mathrm{d}x$

- *If power of secant is even the hold onto* $\sec^2 x$ *and turn other factors of* $\sec^2 x$ *into* $(1 + \tan^2 x)$

$$\begin{aligned}\int \tan^a x \sec^{2k} x \mathrm{d}x &= \int \tan^a x (\sec^2 x)^{k-1} \sec^2 x \mathrm{d}x \\ &= \int \tan^a x (1 + \tan^2 x)^{k-1} \sec^2 x \mathrm{d}x\end{aligned}$$

  then sub $u = \tan x$*.*

- *If power of tangent is odd, then hold onto 1 factor of* $\sec x \tan x$ *and turn remaining factors of* $\tan^2 x$ *into* $(\sec^2 x - 1)$

$$\begin{aligned}\int \tan^{2k+1} x \sec^b x \mathrm{d}x &= \int \tan^{2k} x \sec^{b-1} x \sec x \tan x \mathrm{d}x \\ &= \int (\sec^2 x - 1)^k \sec^{b-1} x \sec x \tan x \mathrm{d}x\end{aligned}$$

  then sub $u = \sec x$*.*

- *Othercases are harder and are usually handled in an ad-hoc manner.*

You should try this one:

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x} \mathrm{d}x && \text{sub } u = \cos x \\ &= -\log|\cos x| + c \equiv \log|\sec x| + c\end{aligned}$$

or equivalently

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \tan x \sec x \cdot \frac{1}{\sec x} \mathrm{d}x \\ &= \int \frac{1}{u} \mathrm{d}u && \text{using } u = \sec x, u' = \tan x \sec x \\ &= \log|u| + C \\ &= \log|\sec x| + C\end{aligned}$$

When we get odd powers of secant and even powers of tangent we just have to work hard (unfortunately). So even the integral of secant by itself is not so easy to do (with the techniques we know so far). In fact

$$\int \sec x \mathrm{d}x = \log|\frac{1 + \sin x}{\cos x}| + c = \log|\sec x + \tan x| + c$$

which is not so easy to do directly, but we can check it by taking derivatives:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\log|\sec x+\tan x| &= \frac{1}{\sec x+\tan x}\cdot\left(\sec x\tan x+\sec^2 x\right)\\ &= \sec x\frac{\sec x+\tan x}{\sec x+\tan x}=\sec x\checkmark\end{aligned}$$

but that doesn't really explain how we found it in the first place. Perhaps the easiest way to do that is using partial fractions — see CLP example 1.10.5.

Some more examples

$$\begin{aligned}\int\tan^4 x\mathrm{d}x &= \int\tan^2 x(1+\sec^2 x)\mathrm{d}x\\ &= \int(\tan^2 x\sec^2 x+\tan^2 x)\mathrm{d}x\\ &= \int\tan^2 x\sec^2 x\mathrm{d}x+\int(1+\sec^2 x)\mathrm{d}x\\ &= \frac{1}{3}\tan^3 x+x+\tan x+c\end{aligned}$$

A harder one

$$\int\sec^3 x\mathrm{d}x=\int\sec^2 x\sec x\mathrm{d}x$$

Integration by parts $f=\sec x$ and $g'=\sec^2 x$, so $g=\tan x$ and $f'=\sec x\tan x$

$$\begin{aligned}\int\sec^2 x\sec x\mathrm{d}x &= \sec x\tan x-\int\sec x\tan^2 x\mathrm{d}x\\ &= \sec x\tan x-\int\sec x(1+\sec^2 x)\mathrm{d}x\\ &= \sec x\tan x-\log|\sec x+\tan x|-\int\sec^3 x\mathrm{d}x\end{aligned}$$

So

$$\int\sec^3 x\mathrm{d}x=\frac{1}{2}\left(\sec x\tan x-\log|\sec x+\tan x|\right)+c$$

1.9 Trigonometric substitutions

So continuing on our trigonometric theme, we now look at simplifying integrals by making a subsitution of the form

$$x\mapsto\sin\theta$$

To see this in action, consider the problem of computing the area of a circle - or a semi-circle. We saw this back when we were interpretting the definite integral as the area under a curve and we realised that we could compute

$$\int_{-1}^{1}\sqrt{1-x^2}\mathrm{d}x=\frac{\pi}{2}$$

since the integrand describes a semi-circle. More generally

- The formula for a circle is $x^2 + y^2 = r^2$, so the top half is

$$y = \sqrt{r^2 - x^2}$$

- So the area is given by

$$\begin{aligned} A &= 4\int_0^r \sqrt{r^2 - x^2}\mathrm{d}x \\ &= \pi r^2 \end{aligned}$$

Of course, we know the π is there because of geometry, but how do we get to it algebraically and more systematically? To see how we are going to use some substitutionms. Recall the rule:

$$\int f(u)\mathrm{d}u = \int f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

Before we started with something like the right-hand side and tried to find a $\frac{\mathrm{d}u}{\mathrm{d}x}$ so that we could write it as the left-hand side.

But now instead of looking for $u(x)$, we will substitute $x = x(\theta)$ so that the integrand simplifies (maybe via some trig identities).

$$\int f(x)\mathrm{d}x = \int f(x(\theta))\frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta$$

ie we start with the left-hand side and rewrite it as the right-hand side.

Go back to the circle example and substitute $x = r\sin\theta$

$$\int_0^r \sqrt{r^2 - x^2}\mathrm{d}x = \int \sqrt{r^2 - r^2\sin^2\theta}\frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta$$

Now what happens to the terminals?

$$\begin{aligned} x = r &= r\sin\theta & \theta &= \pi/2 \\ x = 0 &= r\sin\theta & \theta &= 0 \end{aligned}$$

So - put this in:

$$\begin{aligned} \int_0^r \sqrt{r^2 - x^2}\mathrm{d}x &= \int_0^{\pi/2} r\sqrt{1 - \sin^2\theta} \cdot r\cos\theta\mathrm{d}\theta \\ &= \int_0^{\pi/2} r^2|\cos\theta|\cos\theta\mathrm{d}\theta \\ &= r^2\int_0^{\pi/2} \cos^2\theta\mathrm{d}\theta && \text{cos is positive on range} \\ &= r^2\int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\mathrm{d}\theta \\ &= \frac{r^2}{2}[\theta + (\sin 2\theta)/2]_0^{\pi/2} \\ &= \frac{r^2}{2}\frac{\pi}{2} = \pi r^2/4\checkmark \end{aligned}$$

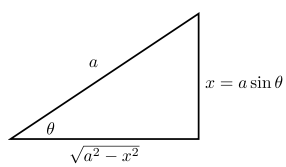
Why does this work? Because of trig-identities:

Identity	Expression	Substitution
$1-\sin^2\theta = \cos^2\theta$	$\sqrt{a^2-x^2}$	x $= a\sin\theta$
$\sec^2\theta - 1 = \tan^2\theta$	$\sqrt{x^2-a^2}$	x $= a\sec\theta$
$1+\tan^2\theta = \sec^2\theta$	$\sqrt{a^2+x^2}$ or $\dfrac{1}{a^2+x^2}$	x $= a\tan\theta$

We saw the first of these when we did $\int \sqrt{r^2-x^2}\mathrm{d}x$. But we did that one as a definite integral — let us redo it as an indefinite integral.

$$\begin{aligned}
\int \sqrt{a^2-x^2}\mathrm{d}x &= \int \sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\mathrm{d}\theta \\
&= a^2\int \cos^2\theta\mathrm{d}\theta && \text{I've assumed } \cos\theta \geq 0 \\
&= \frac{a^2}{2}\int 1+\cos 2\theta\mathrm{d}\theta \\
&= \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + c
\end{aligned}$$

So far this is the same, but we now need to re-express this in terms of x. Since we have $x = a\sin\theta$ we can draw the following triangle:



Since we know that $x = a\sin\theta$ this corresponds to a triangle with hypotenuse a and angle θ and opposite edge $x = a\sin\theta$. This tells us that the adjacent edge which is $a\cos\theta$ is (by pythagorous) $\sqrt{a^2-x^2}$. Hence $\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$. More algebraically we can do:

$$\begin{aligned}
x &= a\sin\theta & \theta &= \arcsin(x/a) \\
a\cos^2\theta &= a^2 - a^2\sin^2\theta & a\cos\theta &= \sqrt{a^2-x^2} \\
\sin 2\theta &= 2\underbrace{\sin\theta}\,\underbrace{\cos\theta} \\
&= 2\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}
\end{aligned}$$

So solution is

$$\int \sqrt{a^2-x^2}\mathrm{d}x = \frac{a^2}{2}\arcsin(x/a) + \frac{x}{2}\sqrt{a^2-x^2} + c$$

This is also done in the text — example 1.9.3 — but to find the area of a chunk of a circle. Take a look.

Another example — this one is tangenty:

$$\begin{aligned}\int \frac{1}{x^2+4x+7}\mathrm{d}x &= \int \frac{1}{(x+2)^2-4+7}\mathrm{d}x && \text{complete the square}\\ &= \int \frac{1}{(x+2)^2+3}\mathrm{d}x\end{aligned}$$

(Make sure people know completing the square).

So this is a $\tan\theta$ example — put $(x+2) = \sqrt{3}\tan\theta$, so $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sqrt{3}\sec^2\theta$

$$\begin{aligned}\int \frac{1}{x^2+4x+7}\mathrm{d}x = \int \frac{1}{(x+2)^2-4+7}\mathrm{d}x &= \int \frac{1}{(x+2)^2+3}\mathrm{d}x\\ &= \int \frac{1}{3\tan^2\theta+3}\cdot\sqrt{3}\sec^2\theta\mathrm{d}\theta\\ &= \int \frac{\sqrt{3}\sec^2\theta}{3(\sec^2\theta)}\mathrm{d}\theta\\ &= \frac{1}{\sqrt{3}}\int 1\mathrm{d}\theta\\ &= \frac{1}{\sqrt{3}}\theta + c\end{aligned}$$

Now $\frac{x+2}{\sqrt{3}} = \tan\theta$, so

$$\int \frac{1}{x^2+4x+7}\mathrm{d}x = \frac{1}{\sqrt{3}}\theta + c = \frac{1}{\sqrt{3}}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + c$$

Another one (secanty):

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-16}}$$

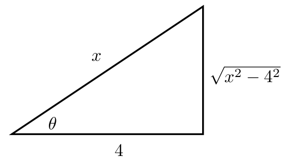
This contains $\sqrt{x^2-a^2}$ — so it is a $\sec\theta$ one. Substitute $x = 4\sec\theta$, so $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 4\sec\theta\tan\theta$.

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-16}} &= \int \frac{4\sec\theta\tan\theta\mathrm{d}\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}}\\ &= \int \frac{\tan\theta\mathrm{d}\theta}{4\sec\theta\sqrt{16\tan^2\theta}}\\ &= \int \frac{\tan\theta\mathrm{d}\theta}{4\sec\theta\cdot 4\tan\theta}\\ &= \int \frac{1}{16\sec\theta}\mathrm{d}\theta\\ &= \int \frac{\cos\theta}{16}\mathrm{d}\theta = \frac{1}{16}\sin\theta + C\end{aligned}$$

Now we need to convert this back to x. How? We can draw a similar triangle to the one we did above. Since

$$x = 4\sec\theta \qquad\qquad \text{so } \cos\theta = \frac{4}{x}$$

it corresponds to a triangle with angle θ, hypotenuse 4 and adjacent x:



Thus sine is simply

$$\sin\theta = opp/hyp = \frac{x^2-16}{x}$$

More algebraically we can do it as follows.

$$\begin{aligned}\sin^2\theta &= 1-\cos^2\theta = 1-\frac{1}{\sec^2\theta}\\ &= \frac{\sec^2\theta-1}{\sec^2\theta}\\ &= \frac{x^2/16-1}{x^2/16} = \frac{x^2-16}{x^2} && \text{so}\sin\theta = \frac{\sqrt{x^2-16}}{x}.\end{aligned}$$

where we have assumed that $\sin\theta > 0$.

Putting things back together we get

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-16}} = \frac{\sqrt{x^2-16}}{16x} + c$$

1.10 Partial fractions.

Recall that a rational function is the ratio of two polynomials

$$f(x) = \frac{polynomial(z)}{polynomial(z)}$$

These arise very frequently in mathematics (both pure and applied) and we will spend some time describing a method for integrating them. We already know how to integrate a couple of simple examples:

$$\begin{aligned}\int \frac{1}{x+a}\mathrm{d}x &= \log|x+a| + c\\ \int \frac{1}{x^2+a^2}\mathrm{d}x &= \frac{1}{a}\arctan(x/a) + c\end{aligned}$$

In general we want to rewrite a general rational function $\frac{P(x)}{Q(x)}$ as a sum of simpler pieces (like those above) that we can integrate easily.

$$\begin{aligned}\text{Rational function} &= \text{sum of simple bits}\\ \int \text{Rational function}\mathrm{d}x &= \int \text{sum of simple bits}\mathrm{d}x\end{aligned}$$

Indeed the simpler pieces we want are of the form

$$\frac{A}{x-a} \qquad\qquad \frac{A}{x^2+bx+c}$$

which we have already seen how to integrate using logarithms and arctangents.

The method relies on a simple observation (which of course needs to be justified, but this is given as an optional section — see CLP 1.10.3) that if we add together two simpler rational functions we get a more complicated one

$$\begin{aligned}\frac{1}{x-1} + \frac{2}{x+3} &= \frac{1(x+3)+2(x-1)}{(x-1)(x+3)} \\ &= \frac{3x+1}{x^2+2x-3}\end{aligned}$$

If we reverse this process (ie splitting up the rational function) then we can integrate things like

$$\begin{aligned}\int \frac{3x+1}{x^2+2x-3}\mathrm{d}x &= \int \left(\frac{1}{x-1} + \frac{2}{x+3}\right) \\ &= \log|x-1| + 2\log|x+3| + c\end{aligned}$$

More generally we might have something like:

$$\frac{A}{x-a} + \frac{B}{x-b} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)} = \frac{x(A+B)-(Ab+Ba)}{(x-a)(x-b)}$$

This suggests that if I wish to go backwards I need to look at the structure of the denominator of my rational function — what are the factors.

But also notice that in the nice functions I know how to integrate, the numerator has lower degree than the denominator. And when I add them together, then the numerator of the sum still has lower degree than the denominator of the sum.

Consequently the method of "partial fractions" requires a few steps. Assume we have a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

1. If $\deg(P) \geq \deg(Q)$ then rewrite as a proper fraction $S + R/Q$ with $\deg R < \deg Q$ — ie do polynomial division.

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1} \qquad \text{do the division}$$

Remember your polynomial division!! (many students have trouble with this)

$$\begin{array}{r|l} & x^2 \quad +x+2 \\ \hline x-1) & x^3 \qquad\quad +x \\ & -x^3+x^2 \qquad\quad \\ \hline & x^2 \quad +x \\ & -x^2 \quad +x \\ \hline & 2x \\ & -2x+2 \\ \hline & 2 \end{array}$$

Or "synthetic division" (many students have trouble with this)

$$\begin{aligned}
(x^3 + 0x^2 + x + 0) &= (ax^2 + bx + c)(x - 1) + d \\
&= ax^3 + x^2(b - a) + x(c - b) + (d - c) && \text{so } a = 1 \\
&= x^3 + x^2(b - 1) + x(c - b) + (d - c) && \text{so } b = 1 \\
&= x^3 + x^2(1 - 1) + x(c - 1) + (d - c) && \text{so } c = 2 \\
&= x^3 + x^2(1 - 1) + x(2 - 1) + (d - 2) && \text{so } d = 2\text{, and} \\
\frac{x^3 + x}{x - 1} &= x^2 + x + 2 + \frac{2}{x - 1}
\end{aligned}$$

2. Factor the denominator into linear $(x-a)$ and irreducible quadratic (x^2+bx+c) factors — some tricks for this are given in the appendix of the text. READ IT!!

3. Split the ratio into simpler pieces — depends on the denominator factors.

$$\frac{A}{(x-a)^n} \qquad\qquad \frac{Bx + C}{(x^2 + bx + c)^M}$$

   where $(x - a)$ and $(x^2 + bx + c)$ are factors of the denominator.

4. Find the constants

5. Integrate term by term.

How do we know how to split it up?

denominator factor	partial fraction expansion	covered in this course
$(x - a)$	$\frac{A}{x-a}$	✓
$(x - a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r}$	✓
$(x^2 + bx + c)$	$\frac{Bx+C}{x^2+bx+c}$	✓
$(x^2 + bx + c)^r$	$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \frac{B_3x+C_3}{(x^2+bx+c)^3} + \cdots$	× (phew)

Go back to our example

$$\frac{3x + 1}{x^2 + 2x - 3}$$

First step is okay — numerator has strictly lower degree than denominator. So move on to second step — factor the denominator

$$x^2 + 2x - 3 = (x - 1)(x + 3)$$

Do you recall tricks for this? eg — factor the constant term "3" and then try substituting in its factors

$$P(3), P(-3), P(1), P(-1)$$

If $P(a) = 0$ then we know $P(x) = Q(x)(x - a)$. READ THE TEXT!!

Now we are ready to split it up:

$$\frac{3x+1}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

How do we now find the constants? There are a few ways, but it is perhaps easiest to start by adding it back together:

$$\begin{aligned}\frac{3x+1}{(x+3)(x-1)} &= \frac{A}{x+3} + \frac{B}{x-1} \\ &= \frac{A(x-1)+B(x+3)}{(x+3)(x-1)} = \frac{x(A+B)+(-A+3B)}{(x+3)(x-1)}\end{aligned}$$

So we have 2 equations to solve (coming from the coefficients of x):

$$\begin{aligned}A+B &= 3 & -A+3B &= 1 \\ A &= 2 & B &= 1\end{aligned}$$

Alternatively, starting from here

$$\frac{3x+1}{(x+3)(x-1)} = \frac{A(x-1)+B(x+3)}{(x+3)(x-1)}$$

Comparing numerators, we have

$$3x+1 = A(x-1)+B(x+3)$$

Setting $x=1$ gives us $4=4B$, and setting $x=-3$ we have $-8=-3A$. Hence $A=2, B=1$.

In both cases:

$$\begin{aligned}\int \frac{3x+1}{(x+3)(x-1)} &= \int \frac{2}{x+3} + \frac{1}{x-1} \\ &= 2\log|x+3| + \log|x-1| + c \\ &= \log|(x+3)^2(x-1)| + c\end{aligned}$$

Another example — repeated factor. Numerator degree lower than denominator degree, so no poly-div required. Just leap into decomposition:

$$\begin{aligned}\frac{x^2-9x+17}{(x-2)^2(x+1)} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1} \\ &= \frac{A(x-2)(x+1)+B(x+1)+C(x-2)^2}{(x-2)^2(x+1)} \\ &= \frac{x^2(A+C)+x(-A+B-4C)+(-2A+B+4C)}{(x-2)^2(x+1)}\end{aligned}$$

Hence we have the system of equations

$$A+C=1 \qquad -A+B-4C=-9 \qquad -2A+B+4C=17$$

Solve this to get

$$A = -2 \qquad B = 1 \qquad C = 3$$

Alternatively, when we get to here

$$\frac{x^2 - 9x + 17}{(x-2)^2(x+1)} = \frac{A(x-2)(x+1) + B(x+1) + C(x-2)^2}{(x-2)^2(x+1)}$$

Evaluate the numerators at $x = 2$ to get

$$\begin{aligned} 4 - 18 + 17 &= 0A + 3B + 0C \\ 3 &= 3B & B = 1 \end{aligned}$$

Evaluate the numerator at $x = -1$ to get

$$\begin{aligned} 1 + 9 + 17 &= 0A + 0B + 9C \\ 27 &= 9C & C = 3 \end{aligned}$$

We cannot isolate A this way, so just pick another convenient point like $x = 0$ and the information we already have:

$$\begin{aligned} 17 &= -2A + B + 4C = -2A + 1 + 12 \\ 4 &= -2A & A = -2. \end{aligned}$$

So we get the same A, B, C (phew). Notice that at this point we should perhaps check our answer by putting things back together. Depends on how much time we have spare.

So we now have our decomposition and can integrate things.:

$$\begin{aligned} \int \frac{x^2 - 9x + 17}{(x-2)^2(x+1)} \mathrm{d}x &= \int \frac{-2}{x-2} \mathrm{d}x + \int \frac{1}{(x-2)^2} \mathrm{d}x + \int \frac{3}{x+1} \mathrm{d}x \\ &= -2 \log|x-2| - \frac{1}{x-2} + 3 \log|x+1| + c \\ &= \log\left|\frac{(x+1)^3}{(x-2)^2}\right| - \frac{1}{x-2} + c \end{aligned}$$

Another example — irreducible quadratic factor (ie — has complex roots, not real roots):

$$\int \frac{x+1}{x^2 - 2x + 5} \mathrm{d}x$$

- Start by completing the square $(x^2 - 2x + 5) = (x-1)^2 + 4$.
- This suggests setting $u = x - 1$, so $\mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned} \int \frac{x+1}{x^2 - 2x + 5} \mathrm{d}x &= \int \frac{u+2}{u^2+4} \mathrm{d}u \\ &= \int \frac{u}{u^2+4} + \frac{2}{u^2+4} \mathrm{d}u \\ &= \frac{1}{2} \log|u^2 + 4| + 2 \cdot \frac{1}{2} \arctan(u/2) + c & \int \frac{1}{x^2+a^2} = \frac{1}{a} \arctan(x/a) \\ &= \frac{1}{2} \log|(x-1)^2 + 4| + \arctan((x-1)/2) + c \end{aligned}$$

1.11 Numerical integration

- Exact integration is a very useful thing (especially for differential equations) — when we can do it.
- In order to do it we need to be able to find the anti-derivative of a function — we can't always do this.

$$\int e^{-x^2} \mathrm{d}x = ?$$

  We saw that we can write this down in terms of Taylor series, but we can also approximate it.
- There are several different ways of doing this — perhaps the most straight forward relies on interpretting the definite integral as "the area under the curve".
- By approximating the area we approximate the integral.
- We already know 1 way of approximating the integral — Riemann sums.

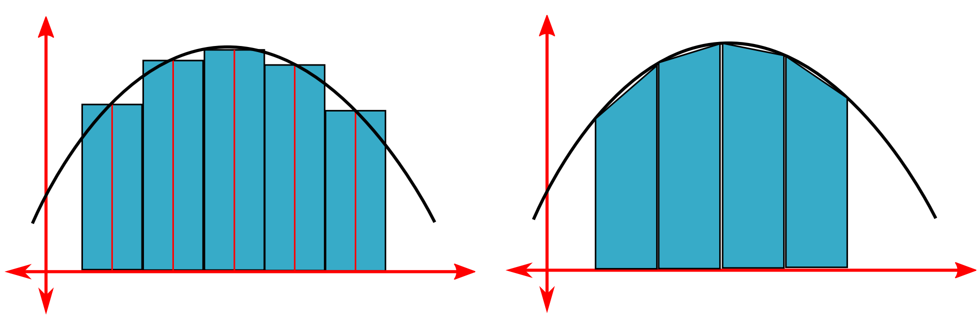
$$\int_a^b f(x)\mathrm{d}x \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

  with $x_i^* \in [x_{i-1}, x_i]$.
- It turns out that a good way to choose x_i^* is to take the mid-point of the interval $x_i^* = (x_{i-1} + x_i)/2$ (see Def 1.11.1)

  Theorem (Midpoint rule — CLP eqn 1.11.2)**.**

$$\int_a^b f(x)\mathrm{d}x \approx M_n = \sum f(\bar{x_i})\Delta x = \Delta x(f(\bar{x_1}) + f(\bar{x_1}) + f(\bar{x_2}) + \cdots + f(\bar{x_n}))$$

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x \qquad \bar{x_i} = (x_{i-1} + x_i)/2$$



  This is perhaps the simplest way of doing things — inside our little subinterval $[x_i, x_{i+1}]$ we approximate the function as being a constant.
- Way back at the start of the course, in our Riemann sum we approximated the area by little rectangles. Another method is to approximate it by little trapezoids — ie approximate the function on $[x_i, x_{i+1}]$ by a line. So approximate the whole function as a sequence of lines.

Now, the left-hand side of trapezoid has height $f(x_{i-1})$ while the right-hand side has height $f(x_i)$. The area is therefore

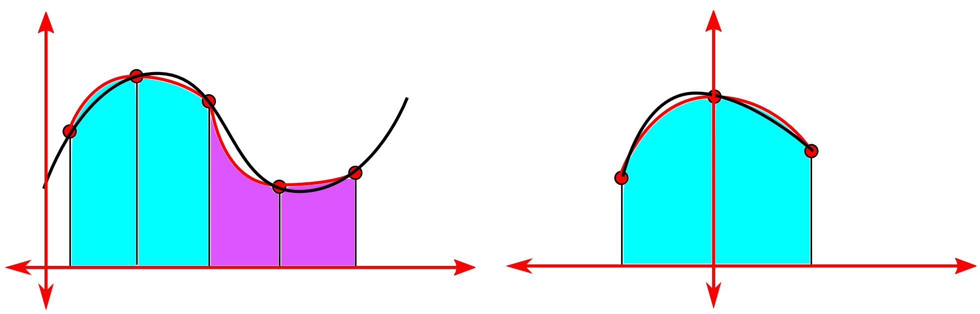
$$\Delta x \cdot (f(x_i) + f(x_{i-1}))/2$$

Theorem (Trapezoid rule — CLP eqn).

$$\int_a^b f(x)\mathrm{d}x \approx T_n = \sum (f(x_i) + f(x_{i-1}))/2 \cdot \Delta x$$
$$= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) \cdots + 2f(x_{n-1}) + f(x_n)\right)$$
$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x$$

To get the trapezoid rule, you approximate the function by a sequence of lines. A much better approximation is Simpson's rule. Approximate the curve by a sequence of parabolas — this is significantly more effective since it takes some change in slope into account.

- Divide $[a, b]$ into n segments with n even. Width is $\Delta x = (b-a)/n$.
- Each triple of points $(x_{i-1}, f(x_{i-1}), (x_i, f(x_i)), (x_{i+1}, f(x_{i+1})$ can be used to define a parabola. We then add up the area contributions from each parabola.



- Simplify things and look at a parabola going through 3 points $(-h, y_{-1}), (0, y_0), (h, y_1)$.
- The general equation for a parabola is $y = Ax^2 + Bx + C$, so the area under it is

$$\int_{-h}^{h} (Ax^2 + Bx + C)\mathrm{d}x = [Ax^3/3 + Bx^2/2 + Cx]_{-h}^{h}$$
$$= 2Ah^3/3 + 2Ch = \frac{h}{3}(2Ah^2 + 6C).$$

- Since we don't know A, B, C, we need to relate this to our 3 points.

$$y_{-1} = Ah^2 - Bh + Cy_0 \qquad = Cy_{-1} = Ah^2 + Bh + C$$

- Hence the area under the parabola is $\frac{h}{3}(y_{-1} + y_1 + 4y_0)$.

- Each triple of points gives us an area like this. Hence the total area is

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x &\approx \frac{h}{3}(y_0+4y_1+y_2)+\frac{h}{3}(y_2+4y_3+y_4)+\cdots+\frac{h}{3}(y_{n-2}+4y_{n-1}+y_n)\\ &= \frac{h}{3}(y_0+4y_1+2y_2+4y_3+2y_4+\cdots 2y_{n-2}+4y_{n-1}+y_n)\end{aligned}$$

Theorem (Simpson's rule)**.**

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x &\approx S_n\\ &= \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]\end{aligned}$$

Where $\Delta x = \frac{b-a}{n}$ and n is an even number.

So how good are these approximations — what is the error? Let us be more precise: The error is the difference between our approximation and the value of the integral — it is a function of n

$$Error(n) = \left|\int_a^b f(x)\mathrm{d}x - M_n\right|$$

We already know

- We know that as $n \to \infty$ the error goes to zero (since it is a riemann sum), but how quickly? The difference between a good method and a bad method.
- This depends on the function — in particular how "curvy" is the function — how much does its gradient change.
- If the gradient is constant — function is linear. Need to look at second derivative. But for simpson's rule, we'll need to look at higher order derivatives.
- Also — how wide are the slices — how big is Δx?

Theorem (Error bounds — CLP theorem 1.11.13)**.** *Suppose that $|f''(x)| < K$ on $a \le x \le b$ and $|f^{(4)}(x)| < L$ on $a \le x \le b$. If $E_M(n), E_T(n)$ and $E_S(n)$ are the errors of the midpoint, trapezoid and Simpson's rules respectively then*

$$|E_T(n)| \le \frac{K(b-a)^3}{12n^2} \qquad |E_M(n)| \le \frac{K(b-a)^3}{24n^2}$$

$$|E_S(n)| \le \frac{L(b-a)^5}{180n^4}$$

Midpoint rule is slightly better than trapezoid, but Simpson's rule is better than both.

Notice that we need to compute the same things for Trapezoid and Simpson's rules — $f(x_k)$ — so we should just use Simpson's rule (when given the choice).

An aside (NOT IN THE COURSE) — Note that there are nice variants on Simpson's rule using cubics, quartics etc. Eg Simpson's second rule or Simpson's 3/8 rule says

$$\int_a^b f(x)\mathrm{d}x \approx \frac{3\Delta x}{8}\Big[f(x_0)+3f(x_1)+3f(x_2)+2f(x_3)+2f(x_4)\\ +3f(x_5)+3f(x_6)+2f(x_7)+\cdots+f(x_n)\Big]$$

Notice that to compute the above we don't really need to do more work than either Simpson's rule (original) or the trapezoid rule. Surprisingly such "better" approximations don't always lead to lower errors — the interested student should look up "Runge's phenomena".

Some examples. Approximate the integral $\int_1^2 1/x\mathrm{d}x$ using the trapezoid rule and $n = 5$.

$$\int_a^b f(x)\mathrm{d}x \approx T_n = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2)\cdots + 2f(x_{n-1}) + f(x_n)\right)$$

$$a = 1, b = 2\Delta x = 0.2$$

$$\begin{aligned} T_n &= 0.1(f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)) \\ &= 0.1(1/1 + 2/1.2 + 2/1.4 + 2/1.6 + 2/1.8 + 1/2) = 0.6956\ldots \end{aligned}$$

Of course we know (well — the calculator knows) that the actual value of the integral is just

$$\int_1^2 \frac{\mathrm{d}x}{x} = [\log|x|]_1^2 = \log(2) = 0.6931471806$$

So our value is close to this (as it should be). We can be more precise (without a calculator) and form a rigorous bound on the error.

$$\begin{aligned} f'(x) &= -1/x^2 & f''(x) &= 2/x^3 \\ |f''(x)| &= 2|x^{-3}| \leq 2 & \text{since } 1 &\leq x \leq 2 \\ |E_T(5)| &\leq \frac{K(b-a)^3}{12n^2} = \frac{2\cdot 1^3}{12\cdot 25} = \frac{1}{150} \approx 0.00666 \end{aligned}$$

Note that the actual error is about 0.0025.

Let us turn the question around. How big do we need n to be in order for the error to be less than 10^{-6}?

$$\begin{aligned} 10^{-6} &\geq \frac{K(b-a)^3}{12n^2} = \frac{2}{12n^2} \\ 10^6 &\leq 6n^2 \\ \frac{10^6}{6} &\leq n^2 \\ \sqrt{\frac{10^6}{6}} &\approx 408.25 \leq n \end{aligned}$$

So we need n to be ≥ 409. (actually $n \approx 260$ will do)

Continue this previous example now using Simpson's rule — we want to the error in estimating $\int_1^2 1/x\mathrm{d}x$ to be less than 10^{-6}. The 4th derivative is $24/x^5$ and so is bounded by 24.

$$\begin{aligned} 10^{-6} &\geq \frac{K(b-a)^5}{180n^4} = \frac{24}{180n^4} \\ 10^6 &\leq \frac{15n^4}{2} \\ \frac{2}{15} \times 10^6 &\leq n^4 \\ 19.1 &\leq n \end{aligned}$$

Hence we need $n \geq 20$ (actually $n = 14$ will do). Compare this to $n \geq 409$ for the trapezoid rule. Oof!