Partial fractions.

Goal: Integrate rational functions.

Definition: A function $f(x)$ is called a rational function if and only if it can be written in the form.

$$f(x) = \frac{P(x)}{Q(x)}, \qquad \begin{array}{l} P(x) - \text{polynomial} \\ Q(x) - \text{polynomial.} \end{array}$$

Rational functions. (Why is it easy to integrate).

Rational function = sum of simpler functions.

Easier to integrate: $\dfrac{A}{x+a}$, $\dfrac{A}{x^2+ax+b}$, $\dfrac{Ax}{x^2+ax+b}$

Structure of the decomposition:

$$\frac{A}{x-a} + \frac{B}{x-b} = \frac{A(x-b) + B(x-a)}{(x-a)(x-b)} = \frac{x(A+B) - (Ab + Ba)}{(x-a)(x-b)}.$$

Steps of partial fraction.

1.

2.

Steps of partial fraction.

3.

4.

5.

Polynomial division.

Practice polynomial division.

eg: $\dfrac{x^3+x}{x-1}$

$$
\begin{array}{r}
x^2+x+2 \\
x-1 \overline{\big)\, x^3 \qquad +x} \\
-x^3+x^2 \\
\hline
x^2+x \\
-x^2+x \\
\hline
2x \\
-2x+2 \\
\hline
2
\end{array}
$$

so, ans $= x^2+x+2+\dfrac{2}{x-1}$

$$(x^3+x) = (x-1)(ax^2+bx+c)+d$$

$$= ax^3+x^2(b-a)+x(c-b)$$

$$+\, d-c$$

so, $a=1$, $b=1$, $c=2$, $d=2$

so,

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

Step 3: How to split up the fraction?

denominator factor	partial fraction expansion	covered in this course
$(x-a)$	$\frac{A}{x-a}$	$\checkmark$
$(x-a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r}$	$\checkmark$
(x^2+bx+c)	$\frac{Bx+C}{x^2+bx+c}$	$\checkmark$
$(x^2+bx+c)^r$	$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \frac{B_3x+C_3}{(x^2+bx+c)^3} + \cdots$	$\times$ (phew)

Example 1

Solve $\int \frac{3x+1}{x^2+2x-3}\,dx$.

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o

o

Example 1 (contd.)

o

Example 1 (contd.)

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Example 2.

$$\int \frac{x^2 - 9x + 17}{(x-2)^2 (x+1)} \, dx.$$

o

o

o

Example 2 (contd)

Example 3 (contd.)

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Example 3.

$$\int \frac{x+1}{x^2-2x+5}\,dx.$$

o

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