3 Infinite sequences and series

3.5 Power series

Lets go back to our easiest power series

$$\sum_{n=0}^{\infty} x^n$$

This is also a geometric series, and so, provided $|x| < 1$ we know that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

So provided $|x| < 1$ we can *represent* the function $1/(1-x)$ as the power series $\sum x^n$. When $|x| > 1$ the sum does not converge and this equality no longer holds and we cannot represent it this way (though there are other representations).

Similarly we can write

$$\frac{1}{1+2x} = \sum_{n=0}^{\infty} (-2x)^n$$

and

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} -x^{2n}$$

A little more work gives

$$\frac{2}{3-x} = \frac{2/3}{1-x/3} = \frac{2}{3}\sum (x/3)^n = \sum \frac{2}{3^{n+1}} x^n$$
$$\frac{2x^5}{3-x} = \sum \frac{2}{3^{n+1}} x^{n+5}$$

And all of these hold when inside the radius of convergence of the power series. Okay — this is a silly party trick (suitable for only very dull parties) — just messing about with geometric series.

More generaly, the ratio test tells us that inside the radius of convergence, a power series is absolutely convergent — which is a very strong property. While conditionally convergent series are very dellicate, absolutely convergent series very robust and we can mess about with the series in different ways and it will still stay convergent. In particular we can differentiate and integrate

Theorem. *Let $\sum c_n(x-a)^n$ be a power series with radius of convergence $R > 0$, and let $f(x) = \sum c_n(x-a)^n$. Then on the interval $x \in (a-R, a+R)$, f is continuous and differentiable. Further*

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \sum nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$
$$\int f(x)\mathrm{d}x = C + \sum \frac{c_n}{n+1}(x-a)^{n+1} = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \dots\dots$$

The radius of convergence for these two series is still R.

So this tells us we can differentiate term-by-term and integrate term-by-term.

Simple example

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + \dots \\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + \dots = \sum n x^{n-1} \\ -\log(1-x) &= C + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = C + \sum \frac{x^n}{n} \end{aligned}$$

Think a bit to work out C — set $x = 0$

$$= \sum \frac{x^n}{n}.$$

So this definitely gets us something interesting. Another easy one — work out the series for $\arctan(x)$.

Remember $\frac{\mathrm{d}}{\mathrm{d}x} \arctan(x) = 1/(1+x^2)$ — what is its power series expansion, then integrate term-by-term

$$\begin{aligned} \frac{1}{1+x^2} &= \sum_{n=0}^{\infty} (-1)^n x^(2n) \\ \arctan(x) = \int \frac{\mathrm{d}x}{1+x^2} &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\ &= C + x - x^3/3 + x^5/5 - x^7/7 + \dots \end{aligned}$$

Remember $\arctan(0) = 0$, so $C = 0$.

Since it is so easy to integrate and differentiate, it is clear that power series will be very useful for playing around with integrals and differential equations.

A quick aside — $\tan(\pi/4) = 1$, so $\arctan(1) = \pi/4$. Hence we can approximate

$$\frac{\pi}{4} \approx s_n = 1 - 1/3 + 1/5 - 1/7 + \dots + \frac{(-1)^n}{2n+1}$$

And we can get a handle on error of this approximation using the stuff we know about alternating series. $R_n = |b_{n+1}| = 1/(2n+3)$. Very slow convergence.

3.6 Taylor series

So now we come to the important bit. Taylor and Maclaurin series. Previously we have been finding the power series representation of a function by some careful manipulations of special cases (ie geometric series). What about more generally — given some function $f(x)$ find a representation of it about $x = a$?

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

How can we work out the value of c_n?

- c_0 is easy — just substitute $x = a$ to get $f(a) = c_0$.

- What about c_1? — well we can differentiate both sides to get
$$f'(x) = c_1 + 2c_2(x-a)^1 + 3c_3(x-a)^2 + \cdots$$
Substituting $x = a$ into this gives $c_1 = f'(a)$.
- To find c_2 we repeat — differentitate again and set $x = a$.
$$f''(x) = 2c_2 + 3 \cdot 2c_3(x-a)^1 + \cdots$$
So $2c_2 = f''(a)$.
- Keep going to get $n!c_n = f^{(n)}(a)$.

Now all of this works provided we are inside the radius of convergence (otherwise we cannot differentiate like that).

Theorem. *If f has a power series expansion about $x = a$, ie*
$$f(x) = \sum c_n(x-a)^n \qquad\qquad |x-a| < R$$
then
$$c_n = \frac{f^{(n)}(a)}{n!}$$
so that
$$f(x) = \sum \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad\qquad |x-a| < R$$
This is the Taylor series of f about $x = a$. The Maclaurin series of f is just its Taylor series about $x = 0$.

So to work out the Taylor series of e^x about $x = 0$ we need to know its derivatives (easy! — they are all the same)
$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = e^x \qquad\qquad f^{(n)}(0) = 1$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
And the ratio test tells us
$$\frac{n!x^{n+1}}{(n+1)!x^n} = \frac{x}{n+1} \to 0$$
So it is convergent for all finite x. Since this is a convergent series, the summands must converge to zero, so:
$$\frac{x^n}{n!} \to 0 \text{ for any real } x$$

Thats an easy one — what about cosine?

$$\begin{aligned} f(x) &= \cos(x) & f(0) &= 1 \\ f'(x) &= -\sin(x) & f'(0) &= 0 \\ f''(x) &= -\cos(x) & f(0) &= -1 \\ f'''(x) &= \sin(x) & f'(0) &= 0 & &\text{etc} \end{aligned}$$

So all the odd powers are zero and the even powers alternate ± 1. Thus

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

We can also work out $\sin(x)$ by integrating this

$$\begin{aligned} \sin(x) &= \int \cos(x) \mathrm{d}x = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \mathrm{d}x \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$

and of course $C = 0$ since $\sin(0) = 0$.

What if we want to expand $\cos(x)$ about a different point — eg $x = \pi/4$? In general we cannot use information about the expansion around the original point — we have to recompute everything.

$$\begin{aligned} f(x) &= \cos(x) & f({}^{\pi}\!/_{4}) &= 1/\sqrt{2} \\ f'(x) &= -\sin(x) & f'({}^{\pi}\!/_{4}) &= -1/\sqrt{2} \\ f''(x) &= -\cos(x) & f({}^{\pi}\!/_{4}) &= -1/\sqrt{2} \\ f'''(x) &= \sin(x) & f'({}^{\pi}\!/_{4}) &= 1/\sqrt{2} & &\text{etc} \end{aligned}$$

So

$$\cos(x) = \frac{1}{\sqrt{2}} \left(1 - \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} - - + + - - \right)$$

So we have now amassed

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n & R &= 1 \\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} & R &= \infty \\ \cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} & R &= \infty \\ \sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} & R &= \infty \\ \arctan(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} & R &= 1 \end{aligned}$$

What about our motivating example of $\int e^{-x^2}\mathrm{d}x$? If we have to analyse all the derivatives it is a pain — but we can use a nice trick that works for Maclaurin series — you can produce the series for $f(u)$ and then substitute $u = x^k$ (this is not on the exam).

$$\begin{aligned}
e^u &= \sum_{n=0}^{\infty} \frac{u^n}{n!} \\
e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \\
\int e^{-x^2}\mathrm{d}x &= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}
\end{aligned}$$

So if we want to compute

$$\begin{aligned}
\int_0^1 e^{-x^2}\mathrm{d}x &= \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\right]_0^1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}
\end{aligned}$$

Its an alternating series, so we can use our alternating series stuff to work out the error in an estimate

$$R_n = |s - s_n| = b_{n+1} \qquad\qquad = \frac{1}{(2n+1)n!}$$

which gets very small very quickly. eg $R_{10} = \frac{1}{21 \cdot 10!} \approx 1.3 \times 10^{-8}$

Another nice "trick" (which could be on the exam) that one can do with power series (when inside their radius of convergence) is that you can add, multiply and divide them

$$\begin{aligned}
x^7 e^x &= x^7 \cdot \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+7}}{n!} \\
\cos(x)e^x &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\
&= 1 + x + x^2(1/2! - 1/2!) + x^3(1/3! - 1/2!) + \dots \\
&= 1 + x - x^3/3 + \dots
\end{aligned}$$

Urgh - clearly this is painful. But you can get a computer to do it quickly.

Well — what if you dont want to get a computer to do and you just want to stop after a few terms (like we did above). Well the result of truncating a Taylor series is a Taylor polynomial and they are useful for approximating functions. If we stop at x^n we get the n^{th} degree Taylor polynomial which is usually denoted T_n.

$$f(x) \approx T_n = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

Immediately we would like to get some idea of the error in building such an approximation. So for f and T_n defined above, we define the remainder of the Taylor series to be

$$R_n(x) = f(x) - T_n(x)$$

One can show that inside the radius of convergence of f we have $R_n \to 0$ as $n \to \infty$.

Computing $R_n(x)$ exactly is usually pretty hard — it is equivalent to knowing $f(x)$ exactly (which we are trying to approximate) but there is a nice theorem that bounds the size of the remainder.

Theorem (Taylor's inequality)**.** *If we can bound $|f^{(n+1)}(x)| \leq M$ for all $|x-a| \leq d$ then the remainder $R_n(x)$ of the Taylor series of f is bounded by*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for all } |x-a| \leq d$$

This is a consequence of the Lagrange Remainder formula.

So if we approximate

$$\cos(x) \approx T_2(x) = 1 - \frac{x^2}{2}$$

then to work out the error in this we need to bound the third derivative of f

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\cos(x) = \sin(x)$$

So the $|f^{(3)}(x)| \leq 1$. So the remainder is bounded

$$|R_2(x)| \leq \frac{1}{3!}|x|^3$$