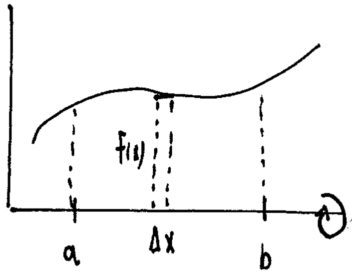
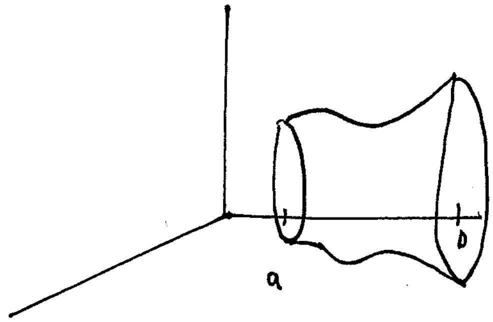
VOLUMES OF REVOLUTION

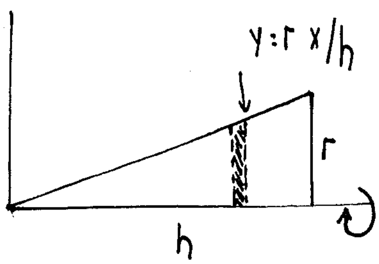


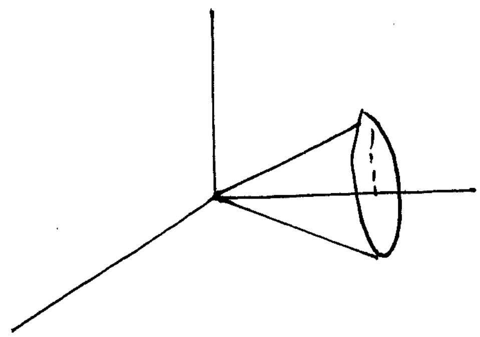


$$\Delta V = \underbrace{\pi [f(x)]^2}_{A(x)} \Delta x$$

SO $$V = \pi \int_a^b (f(x))^2 \, dx .$$

EXAMPLE (VOLUME OF A CONE)



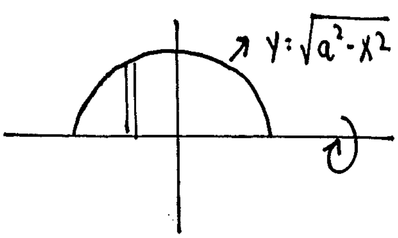


$$f(x) = \frac{r x}{h}$$

SO $$V = \pi \int_0^h (f(x))^2 \, dx = \int_0^h \left(\frac{r^2 x^2}{h^2} \right) dx = \pi \frac{r^2}{h^2} \, \frac{1}{3} x^3 \Big|_0^h = \frac{\pi r^2 h}{3} .$$

THUS $$V = \frac{\pi}{3} r^2 h .$$

XAMPLE (VOLUME OF A SPHERE)



$y = \sqrt{a^2 - x^2} = f(x)$.

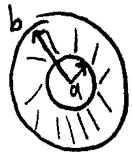
SO $$V = \pi \int_{-a}^{a} (f(x))^2 \, dx$$

$$V = \pi \int_{-a}^{a} (a^2 - x^2) \, dx = 2\pi \int_0^a (a^2 - x^2) \, dx$$

OR $$V = 2\pi \left(a^3 - \frac{1}{3} x^3 \Big|_0^a \right) = 2\pi \left(a^3 - \frac{1}{3} a^3 \right)$$

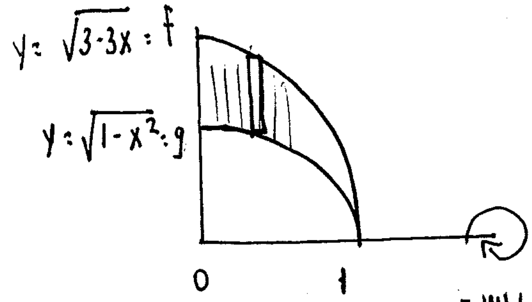
OR $$V = \frac{4\pi}{3} a^3 .$$

EXAMPLE (CAREFUL!)



$A = \pi b^2 - \pi a^2$ <u>NOT</u> $A = \pi (b-a)^2$.

THEN WE WANT TO CALCULATE VOLUME OBTAINED BY ROTATING SHAPE SHOWN BETWEEN TWO CURVES.



$dV = \pi \left[f^2 - g^2 \right] \Delta x$

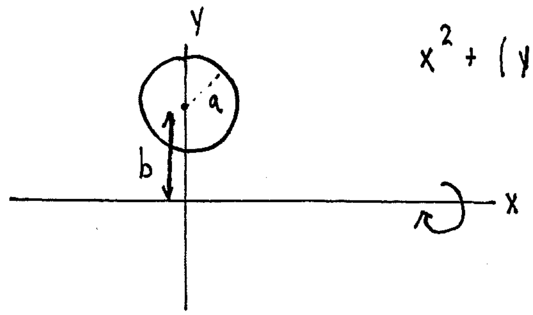
SO $V = \pi \int_0^a (f^2 - g^2)\, dx$

THUS $V = \pi \int_0^1 (3 - 3x - (1 - x^2))\, dx$

OR $V = \pi \int_0^1 \left[2 - 3x + x^2 \right] dx = \pi \left(2 - \frac{3}{2} + \frac{1}{3} \right) = \pi \left(\frac{12}{6} - \frac{9}{6} + \frac{2}{6} \right)$

OR $V = \frac{5\pi}{6}$.

EXAMPLE (TORUS)



$x^2 + (y - b)^2 = a^2$ WITH $b > a$

NOW $(y - b)^2 = a^2 - x^2$. $y = b \pm \sqrt{a^2 - x^2}$.

LET $f(x) = b + \sqrt{a^2 - x^2}$, $g(x) = b - \sqrt{a^2 - x^2}$.

THEN $V = \pi \int_{-a}^{a} (f^2 - g^2)\, dx = \pi \int_{-a}^{a} \left[\left(b^2 + (a^2 - x^2) + 2b\sqrt{a^2 - x^2} \right) - \left(b^2 - 2b\sqrt{a^2 - x^2} + (a^2 - x^2) \right) \right]$

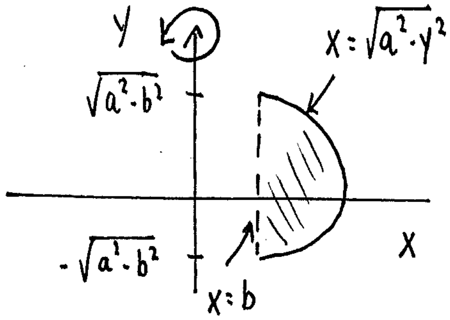
SO $V = \pi 4b \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = 4\pi b \left(\frac{1}{2} \pi a^2 \right) = 2\pi^2 b a^2$.

$V = 2\pi^2 b a^2$ IS VOLUME OF TORUS.

EXAMPLE CONSIDER THE SPHERE $x^2+y^2+z^2 \leq a^2$.

A CYLINDRICAL OBJECT $x^2+y^2 \leq b^2$ WITH $b < a$ IS DRILLED THROUGH THE SPHERE. HOW MUCH OF THE SPHERE REMAINS?

SOLUTION



CONSIDER A CIRCLE $x^2+y^2 \leq a^2$ AND line $x=b$ IN THE x-y plane as shown

IF WE ROTATE SHADED REGION AS SHOWN ABOUT y AXIS WE GET THE DESIRED VOLUME.

SO LET $x_1(y) = b$, $x_2(y) = \sqrt{a^2-y^2}$.

THEN
$$V = \pi \int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} (x_2^2 - x_1^2)\, dy = 2\pi \int_0^{\sqrt{a^2-b^2}} (x_2^2 - x_1^2)\, dy$$

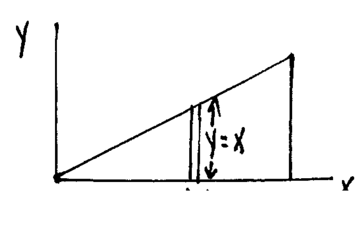
$$V = 2\pi \int_0^{\sqrt{a^2-b^2}} \left((a^2-y^2) - b^2 \right) dy = 2\pi \left[(a^2-b^2)\sqrt{a^2-b^2} - \frac{y^3}{3} \Big|_0^{\sqrt{a^2-b^2}} \right.$$

THEN
$$V = 2\pi \left[(a^2-b^2)^{3/2} - \frac{1}{3}(a^2-b^2)^{3/2} \right] = \frac{4\pi}{3}(a^2-b^2)^{3/2}.$$

check if $b=0$ get $V = \frac{4\pi a^3}{3}$ (whole sphere).

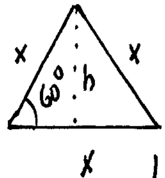
EXAMPLE THE BASE OF A 3-D OBJECT IS THE TRIANGULAR-SHAPED REGION $y \leq x$ WITH $0 \leq x \leq 1$. EACH CROSS-SECTION OF THE OBJECT AT POSITION x IN $0 \leq x \leq 1$ IS AN EQUILATERAL TRIANGLE. FIND THE VOLUME OF THE SOLID.

SOLUTION THE "FLOOR" OF THIS TENT-SHAPED SOLID WITH A SLOPING "ROOF" IS AS SHOWN



THE AREA OF THE CROSSECTION $A(x)$ IS THE AREA OF

AN EQUILATERAL TRIANGLE OF SIDE-LENGTH x,



SO $h = \frac{\sqrt{3}}{2} x$ AND $A(x) = \frac{1}{2} x h = \frac{\sqrt{3}}{4} x^2$.

THUS $$V = \int_0^1 A(x)\,dx = \int_0^1 \frac{\sqrt{3}\,x^2}{4}\,dx = \frac{\sqrt{3}}{4} \frac{x^3}{3}\Big|_0^1 = \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}.$$

EXAMPLE (ROTATION ABOUT ANOTHER AXIS)

CONSIDER $y = e^{-x}$ FOR $0 \le x \le 1$. FIND THE FOLLOWING:

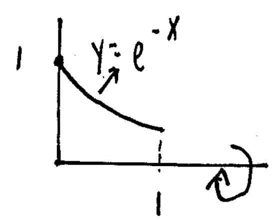
(i) FIND VOLUME WHEN ROTATED ABOUT X-AXIS.

ii) FIND VOLUME WHEN ROTATED ABOUT THE HORIZONTAL LINE $y = -1$

iii) WRITE AN INTEGRAL FOR VOLUME OBTAINED BY ROTATING ABOUT Y-AXIS.

SOLUTION

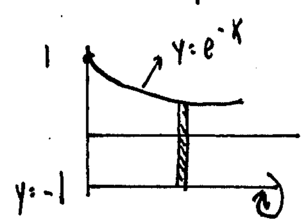
(i)



$$V = \pi \int_0^1 (y(x))^2\,dx = \pi \int_0^1 e^{-2x}\,dx = -\frac{\pi}{2} e^{-2x}\Big|_0^1$$

SO $V = \frac{\pi}{2}(1 - e^{-2})$.

ii)

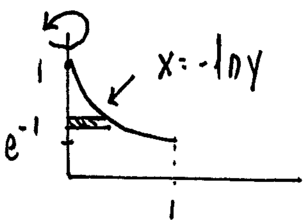


NOTICE $A(x) = \pi[e^{-x} - (-1)]^2$ IS THE AREA OF THIN DISK OBTAINED AFTER ROTATION.

SO $$V = \pi \int_0^1 (e^{-x} + 1)^2\,dx = \pi \int_0^1 (e^{-2x} + 2e^{-x} + 1)\,dx$$

$$V = \pi\left(-\frac{1}{2} e^{-2x}\Big|_0^1 - 2e^{-x}\Big|_0^1 + 1\right) = \pi\left(\frac{7}{2} - 2e^{-1} - \frac{e^{-2}}{2}\right)$$

iii)



NOW $$V = \pi \int_{e^{-1}}^1 (x(y))^2\,dy$$

SO $$V = \pi \int_{e^{-1}}^1 (\ln y)^2\,dy$$ which we will learn later how to calculate