TAYLOR SERIES MANIPULATIONS

THE GENERAL FORM FOR A TAYLOR SERIES OF A SMOOTH FUNCTION $f(x)$ ABOUT $x = x_0$ IS

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \quad (*)$$

WHERE $f^{(n)}(x_0)$ DENOTES nth. derivative OF $f(x)$ AT $x = x_0$ (WITH $f^{(0)}(x_0) \equiv f(x_0)$)

THE SPECIAL CASE OF $(*)$ WHERE $x_0 = 0$ IS CALLED A MacLAURIN SERIES. THERE ARE FOUR BASIC FUNCTIONS THAT YOU SHOULD KNOW THE MacLAURIN SERIES FOR:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (2)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!} \quad (3)$$

THESE THREE TAYLOR SERIES CONVERGE FOR ALL x, i.e. $R = \infty$.

FINALLY, RECALL THE GEOMETRIC SERIES

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, \quad \text{VALID FOR } |x| < 1, \quad (4).$$

WE CAN DERIVE MANY OTHER SERIES EXPANSIONS, SUCH AS $\log(1-x)$ AND $\arctan x$ BY MANIPULATING (integrating) the geometric series, using substitution ETC..

REMARK THE DERIVATION OF (1) - (3) IS SIMPLE. FOR (2), WE LET $f(x) = \sin x$ AND CALCULATE $f'(x) = \cos x$, $f'' = -\sin x$, $f''' = -\cos x$, $f^{(iv)} = \sin x$ WHICH THEN REPEATS.

HENCE $f(0) = 0$ AND $f^{(2n)}(0) = 0$ FOR $n = 1, 2, 3, \ldots$

THEN $f'(0) = f^{(v)}(0) = f^{(9)}(0) = 1$

$f'''(0) = f^{(7)}(0) = f^{(11)}(0) = -1$

SO $$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots\cdots$$

NOW USE RATIO TEST WITH $a_n = \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$ SO THAT

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \; \frac{(2n+1)!}{x^{2n+1}} \right| = \frac{|x|^2}{(2n+3)(2n+2)}$$

NOW FOR ANY FIXED $|x|$ WE HAVE $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.$

THUS THE Maclaurin series FOR $\sin x$ converges FOR ALL x.

THE FINAL USEFUL APPROXIMATION IS THE LEADING TERM AS $x \to 0$ FOR $f(x) = (1+x)^p$ WITH p ANY REAL NUMBER. SINCE $f(0) = 0$ AND $f'(0) = p$, THE TANGENT LINE APPROXIMATION IS

$$(1+x)^p \cong 1 + px + \cdots \quad \text{AS} \quad x \to 0.$$

IN SUMMARY, OUR KEY RESULTS THAT SHOULD BE COMMITTED TO MEMORY ARE

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{VALID FOR ALL } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots \qquad \text{VALID FOR ALL } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots \qquad \text{VALID FOR ALL } x$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \text{VALID FOR } |x| < 1$$

$$(1+x)^p \cong 1 + px + \cdots \qquad \text{FOR } x \to 0.$$

THESE SERIES CAN BE INTEGRATED AND DIFFERENTIATED IN THEIR DOMAINS

OF CONVERGENCE. WE WILL CONSIDER A SERIES OF EXAMPLES TO ILLUSTRATE THE USE OF SUCH SERIES.

EXAMPLE 1 LET $f(x) = \log(1+2x^2)$ FOR $|x| < 1/\sqrt{2}$.

(i) FIND Maclaurin series FOR $f(x)$.

(ii) calculate $\lim\limits_{x\to 0} \dfrac{\log(1+2x^2)}{3x^2}$.

(iii) calculate $f^{(8)}(0)$.

SOLUTION

(i) WE RECALL $\dfrac{1}{1-y} = 1 + y + y^2 + \dots$ FOR $|y| < 1$.

INTEGRATE WRT y AND SET integration CONSTANT TO ZERO:

$$-\log(1-y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \dots$$

SO
$$\log(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} \dots \quad \text{FOR } |y|<1.$$

NOW PUT $y = -2x^2$. THEN FOR $2|x^2| < 1$, OR $|x| < 1/\sqrt{2}$

$$\log(1+2x^2) = -(-2x^2) - \frac{(-2x^2)^2}{2} - \frac{(-2x^2)^3}{3} - \frac{(-2x^2)^4}{4} \dots$$

SO
$$\log(1+2x^2) = 2x^2 - \frac{(2x^2)^2}{2} + \frac{(2x^2)^3}{3} - \frac{(2x^2)^4}{4} \dots \quad (*)$$

$$\log(1+2x^2) = \sum_{n=1}^{\infty} \frac{(2x^2)^n}{n}(-1)^{n+1} = \sum_{n=1}^{\infty} \frac{2^n(-1)^{n+1}}{n} x^{2n} \quad \text{FOR } |x| < 1/\sqrt{2}$$

(ii) WE HAVE $\log(1+2x^2) = 2x^2 - 2x^4 + \dots$ FROM $(*)$

THUS
$$\lim_{x\to 0} \frac{\log(1+2x^2)}{x^2} = \lim_{x\to 0} \frac{2x^2 - 2x^4 + \dots}{x^2} = \lim_{x\to 0}(2 - 2x^2 + \dots) = 2.$$

(iii) LET $f(x) = \log(1+2x^2)$. FORMULA $(*)$ YIELDS

$$(+) \quad f(x) = 2x^2 - (2x^2)^2/2 + (2x^2)^3/3 - (2x^2)^4/4 + \dots \quad \text{FOR } |x| < 1/\sqrt{2}.$$

NOW A GENERAL MACLAURIN SERIES HAS FORM

$$f(x) = f(0) + f'(0)x + f^{(2)}(0)\frac{x^2}{2!} + \dots + f^{(8)}(0)\frac{x^8}{8!} + \dots \quad (++)$$

MATCHING THE x^8 TERMS IN (+) AND (++) WE GET

$$\frac{f^{(8)}(0)}{8!} = -\frac{2^4}{4} = -\frac{16}{4} = -4.$$

SOLVING FOR $f^{(8)}(0)$ WE GET $f^{(8)}(0) = -4 \cdot 8!$.

EXAMPLE 2 CALCULATE $\lim\limits_{x\to 0} \dfrac{\cos(x^2) - (1 - x^4/2)}{x^8}$.

SOLUTION WE HAVE $\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} \dots$ VALID FOR ALL y.

SET $y = x^2$, SO $\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \dots$

REARRANGING GIVES $\cos(x^2) - 1 + \frac{x^4}{2} = \frac{x^8}{4!} - \frac{x^{12}}{6!} \dots$

DIVIDING BY x^8 AND TAKING THE LIMIT, WE GET

$$\frac{\cos(x^2) - 1 + x^4/2}{x^8} = \frac{1}{4!} - \frac{1}{6!}x^4 + \dots$$

THUS $\lim\limits_{x\to 0} \dfrac{\cos(x^2) - 1 + x^4/2}{x^8} = \dfrac{1}{4!}$.

EXAMPLE 3 LET $f(x) = e^{-x^2}$. USE MACLAURIN SERIES TO CALCULATE $f^{(8)}(0)$.

SOLUTION RECALL $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots$ VALID FOR ALL y.

NOW SET $y = -x^2$ TO GET $e^{-x^2} = 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \dots$

THIS BECOMES

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

NOW LET $f(x) = e^{-x^2}$. A GENERAL MACLAURIN SERIES HAS

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots + f^{(8)}(0)\frac{x^8}{8!} + \dots$$

COMPARING THE x^8 TERMS WE GET $\dfrac{f^{(8)}(0)}{8!} = \dfrac{1}{4!}$.

THUS $f^{(8)}(0) = \dfrac{8!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5$.

EXAMPLE 4 LET $f(x) = x^3 \sin(x^2)$. IN THE MACLAURIN SERIES $f(x) = \sum_{n=0}^{\infty} c_n x^n$ IDENTIFY THE COEFFICIENTS C_5, C_9, C_{11}, C_{13}, C_{17}.

SOLUTION $\sin y = y - \dfrac{y^3}{3!} + \dfrac{y^5}{5!} - \dfrac{y^7}{7!} + \dots$

SO $\sin(x^2) = (x^2) - \dfrac{1}{3!}(x^2)^3 + \dfrac{1}{5!}(x^2)^5 - \dfrac{1}{7!}(x^2)^7 + \dots$

MULTIPLY BY x^3:

$$f(x) = x^3 \sin(x^2) = x^3\left[x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \frac{1}{7!}x^{14} + \dots\right]$$

$$f(x) = x^5 - \frac{1}{3!}x^9 + \frac{1}{5!}x^{13} - \frac{1}{7!}x^{17} + \dots$$

COMPARING WITH $f(x) = C_0 + C_1 x + C_2 x^2 + \dots + C_5 x^5 + \dots + C_9 x^9 + \dots + C_{13} x^{13} + \dots + C_{17} x^{17} + \dots$

WE IDENTIFY

$$C_5 = 1, \quad C_9 = -\frac{1}{3!}, \quad C_{13} = \frac{1}{5!}, \quad C_{11} = 0, \quad C_{17} = -\frac{1}{7!}.$$

EXAMPLE 5 DEFINE $F(x)$ BY $F'(x) = \frac{\sin x}{x}$ WITH $F(0) = 0$.

CALCULATE A MACLAURIN SERIES FOR $F(x)$.

SOLUTION WE CAN WRITE $F(x) = \int_0^x \frac{\sin y}{y}\, dy$ FOR THEN $F(0) = 0$

AND BY FTC, $F'(x) = \frac{\sin x}{x}$.

WE HAVE $$\frac{\sin y}{y} = \frac{1}{y}\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} + \dots \right] = \frac{1}{y} \sum_{n=0}^{\infty} \frac{y^{2n+1}(-1)^n}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{y^{2n}(-1)^n}{(2n+1)!}$$

WE INTEGRATE TO GET
$$F(x) = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n+1)!}\, dy = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^x y^{2n}\, dy$$

SO $$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{y^{2n+1}}{2n+1} \Big|_0^x \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,(2n+1)!}$$

WRITING OUT A FEW TERMS GIVES
$$F(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} \dots \quad \text{VALID FOR ALL } x.$$

EXAMPLE 6 CALCULATE $\lim_{x \to \infty} \sqrt{x}\left[\sqrt{x+4} - \sqrt{x+1} \right]$.

SOLUTION WE CALCULATE $\sqrt{x+4} - \sqrt{x+1}$ FOR LARGE x USING

OUR FORMULA $(1+y)^p \cong 1 + py + \dots$ FOR SMALL y.

WE WRITE $$\sqrt{x+4} - \sqrt{x+1} = \sqrt{x(1 + 4/x)} - \sqrt{x(1 + 1/x)} \quad \text{FOR } x > 0$$
$$= \sqrt{x}\left((1 + 4/x)^{1/2} - (1 + 1/x)^{1/2} \right)$$
$$\cong \sqrt{x}\left(1 + \frac{2}{x} + \dots - \left(1 + \frac{1}{2x} + \dots\right)\right) \cong \sqrt{x}\left(\frac{3}{2x} + \dots\right)$$

THUS $\sqrt{x}\left[\sqrt{x+4} - \sqrt{x+1} \right] \cong x\left(\frac{3}{2x} + \dots \right)$ FOR LARGE x.

THIS GIVES $\lim\limits_{x \to \infty} \sqrt{x}\left[\sqrt{x+4} - \sqrt{x+1}\right] = \dfrac{3}{2}$.

EXAMPLE 7 DEFINE I AS THE IMPROPER INTEGRAL

$$I = \int_0^\infty \left(1 - \frac{x}{\sqrt{x^2+1}}\right) dx.$$

EXPLAIN WHY I IS FINITE AND CALCULATE IT EXPLICITLY.

SOLUTION LET $f(x) = 1 - \dfrac{x}{\sqrt{x^2+1}}$. ESTIMATE $f(x)$ AS $x \to +\infty$

USING $(1+y)^p \cong 1 + py + \ldots$ FOR SMALL y.

WRITE
$$f(x) = 1 - \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} \quad \text{for } x > 0$$
$$= 1 - \left(1 + \frac{1}{x^2}\right)^{-1/2} \qquad (\text{NOW SET } p = -1/2,\ y = 1/x^2)$$
$$\cong 1 - \left(1 - \frac{1}{2x^2} + \ldots\right) \quad \text{for large } x$$

SO $f(x) \cong \dfrac{1}{2x^2} + \ldots$ AS $x \to \infty$.

THUS $f(x)$ HAS SUFFICIENT DECAY AS $x \to \infty$ FOR CONVERGENCE OF IMPROPER INTEGRAL (RECALL $\int_1^\infty \frac{1}{x^p}\,dx$ IS FINITE IF AND ONLY IF $p > 1$).

NOW CALCULATE
$$I = \lim_{L \to \infty} \int_0^\infty \left(1 - \frac{x}{\sqrt{x^2+1}}\right) dx = \lim_{L \to \infty} \left[\left(x - (x^2+1)^{1/2}\right)\Big|_0^L\right]$$

$\uparrow$

USE $u = x^2+1$ SUBSTITUTION

$$\text{SO } I = \lim_{L \to \infty}\left(L - (L^2+1)^{1/2}\right) - (0 - 1) = 1 + \lim_{L \to \infty}\left(L - (L^2+1)^{1/2}\right). \quad (*)$$

BUT $L - (L^2+1)^{1/2} = L - L\left(1 + \frac{1}{L^2}\right)^{1/2} = L\left[1 - \left(1 + \frac{1}{L^2}\right)^{1/2}\right] \cong L\left[1 - \left(1 + \frac{1}{2L^2} + \ldots\right)\right]$ for large L.

WE GET $\lim_{L\to\infty}\left(L-(L^2+1)^{1/2}\right) = \lim_{L\to\infty} L\left(-\frac{1}{2L^2}+\dots\right) = 0.$

THUS FROM (*), $I = 1$.

<u>EXAMPLE 8</u> FIND THE VALUE OF C FOR WHICH THE FOLLOWING INTEGRAL EXISTS:

$$I = \int_1^\infty \left(\frac{CX}{\sqrt{4X^4+1}} - \frac{1}{X}\right) dX.$$

<u>SOLUTION</u> DEFINE $f(X) = \dfrac{CX}{\sqrt{4X^4+1}} - \dfrac{1}{X}$.

FOR THE INTEGRAL TO EXIST $f(X)$ MUST DECAY <u>FASTER</u> THAN $1/X$ AS $X\to\infty$.

WE ESTIMATE USING $(1+y)^p \simeq 1+py+\dots$ AS $y\to 0$ THAT

$$f(X) = \frac{CX}{\sqrt{4X^4}\sqrt{1+\frac{1}{4X^4}}} - \frac{1}{X} = \frac{CX}{2X^2}\frac{1}{\sqrt{1+\frac{1}{4X^4}}} - \frac{1}{X}$$

$$f(X) = \frac{C}{2X}\left(1+\frac{1}{4X^4}\right)^{-1/2} - \frac{1}{X}$$

NOW FOR LARGE X, $\left(1+\dfrac{1}{4X^4}\right)^{-1/2} \simeq 1 - \dfrac{1}{8X^4}\dots$

$$\text{SO} \quad f(X) \simeq \frac{C}{2X} - \frac{1}{X} + \frac{C}{2X}\left(-\frac{1}{8X^4}\right) + \dots$$

TO ELIMINATE THE $1/X$ BEHAVIOR AS $X\to+\infty$ SET $C=2$.

THEN $f(X) \simeq -\dfrac{1}{8X^5}$ AS $X\to+\infty$ AND THE INTEGRAL WILL EXIST.

RECALL A SIMILAR, BUT SLIGHTLY EASIER WEBWORK PROBLEM WHEN WE WERE DOING IMPROPER INTEGRALS.