Quiz 5 solutions

Solution 2: The divergence test applied to the series

$$\sum_{i=1}^{\infty} \frac{5k}{(2k+7)^5}$$

tells us that **further testing is needed** because $\lim_{n\to\infty} \frac{5k}{(2k+7)^5} = 0$.

Solution 3: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be divergent. If $a_n > b_n$ for all n, then by comparision test $\sum a_n$ **diverges**.

Solution 4: Consider the series $\sum a_n$. Using ratio test,

we conclude the series converges if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1,$

we cannot conclude convergence or divergence of $\sum a_n$ if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$ and

the series diverges if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1.$

Solution 5: In order to determine the convergence behaviour of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+8}$$

we can try to use ratio test. Using ratio test, we get

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+8}{n+9}\right| = 1.$$

So ratio test is inconclusive. So, let us use a different test. We see that the sequence $\{\frac{1}{n+8}\}$ is decreasing for all n, is positive and converges to 0. So by alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+8}$ converges. However, the series given by $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n+8}\right| = \sum_{n=1}^{\infty} \frac{1}{n+8}$ does not converge. To see that the series $\sum_{n=1}^{\infty} \frac{1}{n+8}$ does not converge, we use comparison test and compare the series to the harmonic series. So, the serise $\sum_{n=1}^{\infty} \frac{1}{n+8}$ is **conditionally convergent.**

Solution 6: We can use integral test to determine the values of p such that the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^{3p}}$$

converges. Let $f(x) = \frac{\ln x}{x^{3p}}$. Then $f'(x) = x^{-3p-1}\,(1 - 3p \ln x)$ and $f'(x) < 0$ for $x \geq 1$ if

$$\begin{aligned} &(1 - 3p \ln x) \leq 0 \\ \Rightarrow &x \geq \exp\left(\frac{1}{3p}\right). \end{aligned}$$

So, the sequence $\{\frac{\ln n}{n^{3p}}\}$ is decreasing for $n \geq N_0$, where N_0 is sufficiently large. Also, the sequence is positive and $f(x)$ is continuous for positive x. Thus, by integral test, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3p}}$ converges if the improper integral $\int_{N_0}^{\infty} \frac{\ln x}{x^{3p}}\, dx$ converges. Observe that

$$\int_{N_0}^{\infty} \frac{\ln x}{x^{3p}}\, dx = \lim_{b \to \infty} \int_{N_0}^{b} \frac{\ln x}{x^{3p}}\, dx$$

and

$$\int_{N_0}^{b} \frac{\ln x}{x^{3p}}\, dx = \left.\frac{x^{-3p+1} \ln x}{-3p+1}\right|_{N_0}^{b} + \frac{1}{-3p+1} \int_{N_0}^{b} \frac{1}{x^{3p}}\, dx$$

using integration by parts. So,

$$\begin{aligned} \int_{N_0}^{\infty} \frac{\ln x}{x^{3p}}\, dx &= \lim_{b \to \infty} \left.\frac{x^{-3p+1} \ln x}{-3p+1}\right|_{N_0}^{b} + \frac{1}{-3p+1} \lim_{b \to \infty} \int_{N_0}^{b} \frac{1}{x^{3p}}\, dx \\ &= \frac{N_0^{-3p+1} \ln N_0}{-3p+1} + \frac{1}{-3p+1} \underbrace{\lim_{b \to \infty} \int_{N_0}^{b} \frac{1}{x^{3p}}\, dx}_{I} \end{aligned}$$

and the improper integral I exists if $3p > 1$. So, the improper integral $\int_{N_0}^{\infty} \frac{1}{x^{3p}}\, dx$ converges if $p \in (\frac{1}{3}, \infty)$ and so, as a result, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3p}}$ converges if $p \in \left(\frac{1}{3}, \infty\right)$.

Solution 7: See WeBWorK