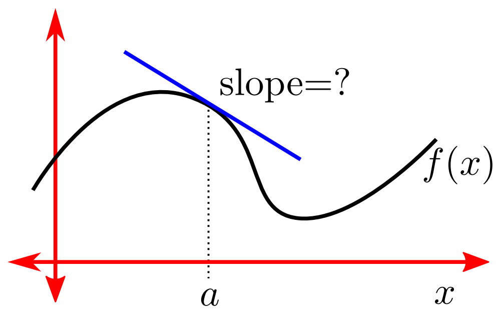
5 Integrals

5.1 Areas and distances

There are 2 basic problems in Calculus

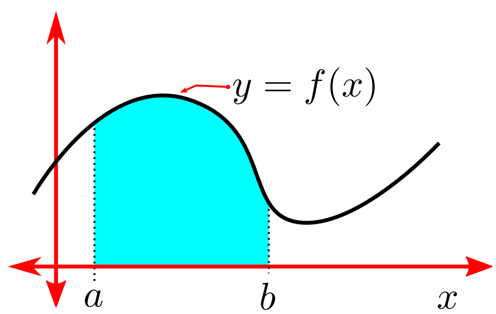
1. Tangent problem — Define the tangent to a curve and calculate its slope.



$$\text{slope} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{\mathrm{d}f}{\mathrm{d}x} = f'(x)$$

- This is Differential Calculus and is covered in Maths100.
- This is useful, not to compute the tangent line, but because it gives you the rate of change — allows you to compute maxima, minima and optimise things.

2. Area problem — Calculate the area between the x-axis and the curve $y = f(x)$.

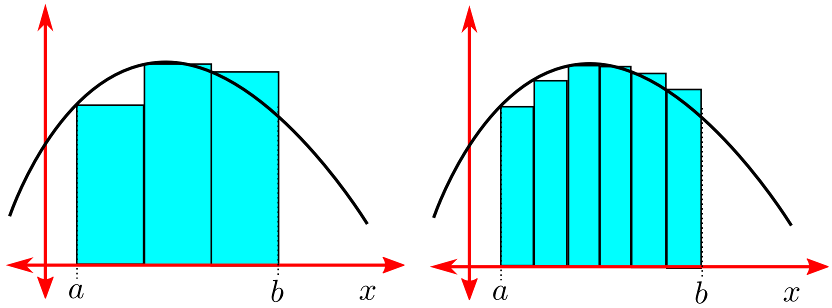


- This is Integral Calculus and is covered in this course.
- Integration is useful for far more things than computing areas. In particular it helps us solve differential equations which turn up everywhere!
- One of the main reasons that mathematics is so useful (both of itself and to other subjects).

Calculus (both differential and integral) was discovered independently by Newton and Leibniz around the same time. They spent the rest of their lives fighting over who really invented it. It caused much resentment between english-speaking mathematicians on Newton's side and European mathematicians on Leibniz's side — because it is so useful and fundamental the fight became so big.

It is not immediately obvious that these two problems are related, however the Fundamental Theorem of Calculus shows that they linked. Before we get to this theorem we need some more machinery.

In particular we will write the area under a curve as a sum of areas of rectangles under the curve. Then we let the widths of the rectangles get smaller and smaller.



Now here we have made sure that the top-left corner of the rectangle is touching the curve. We could have done the same but with the top-right corner and gotten a different approximation. However, you can see that as we make the width of the rectangles smaller, we get more rectangles, but we get a better approximation of the area. Indeed we will have a theorem that tells us that (provided $f(x)$ is a nice function) as the width of the rectangles goes to zero, the sum of those rectangles will be exactly the area under the curve.

Now I want to

Summation notation

We need to learn how to use the $\sum$ symbol — it is a capital "sigma". It is a very useful shorthand for writing sums of many terms. Some examples

- The sum of the first 20 integers is

$$1+2+3+\cdots+19+20=\sum_{i=1}^{20} i=\sum_{k=1}^{20} k$$

  i and k are "dummy variables" we could even write

$$\sum_{\spadesuit=1}^{20} \spadesuit$$

  At each stage we replace $\spadesuit$ by the integers from 1 to 20 and sum the result.

- The sums of cubes

$$\sum_{k=4}^{7} k^3 = 4^3+5^3+6^3+7^3$$

- A sum of a function evaluated at integer points

$$\sum_{k=3}^{6} f(k) = f(3)+f(4)+f(5)+f(6)$$

- A sum of constants

$$\sum_{i=1}^{n} C = \underbrace{C+C+\cdots+C}_{n \text{ terms}} = nC$$

- A more formal sum

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

The sum symbol is a "linear operator"

$$\begin{aligned}
\sum_{i=1}^{n} ca_i &= c\sum_{i=1}^{n} a_i && \text{move the constant out the front} \\
\sum_{i=1}^{n}(a_i + b_i) &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i && \text{split the sum into two sums} \\
\sum_{i=1}^{n}(a_i - b_i) &= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i
\end{aligned}$$

You can prove this by writing out both sides of each equation. There are also some special sums that you should know

- First n integers: $\displaystyle\sum_{k=1}^{n} k = \frac{k(k+1)}{2}$

  $$\begin{aligned}
  S &= 1 + 2 + \cdots + (n-1) + n \\
  S &= n + (n-1) + \cdots + 2 + 1
  \end{aligned}$$

  Now add these together. . .

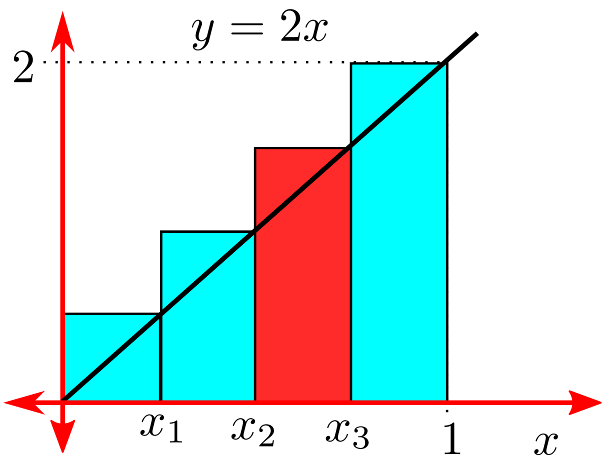
  $$\begin{aligned}
  2S &= (n+1) + (n+1) + \cdots + (n+1) + (n+1) \\
  &= n(n+1)
  \end{aligned}$$

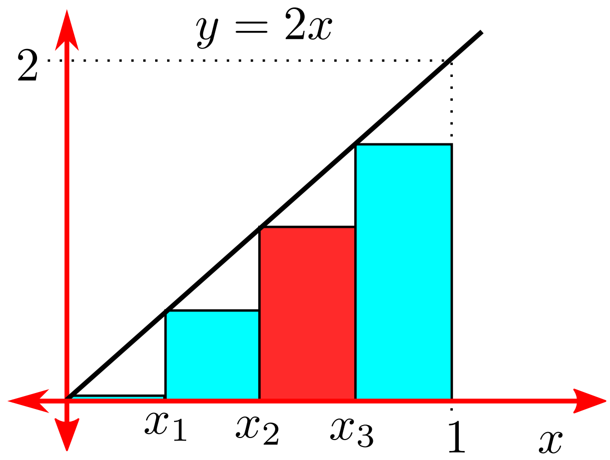
- Sum of first n squares $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- Sum of first n cubes $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

You need induction to prove these last two formulas.

Back to areas and an example

The text does a nice example of the area under the curve $y = x^2$ between $x = 0, x = 1$. Now, rather than doing this example in class (you can read the text instead), let us do something a little simpler that gives us the same idea. Find the area between the curve $y = 2x$ and the x-axis between $x = 0$ and $x = 1$ — of course the area should come out to be 1.





- Let us use n rectangles determined by the right-endpoints. This will overestimate the area. To make things much easier, make them all have the same width $\Delta x = 1/n$.
- So the x-ordinates are $1/n, 2/n, 3/n, \dots, n/n = 1$.
- The heights of the i^{th} rectangle is therefore $f(x_i) = 2x_i = 2i/n$.
- The area of the i^{th} rectangle is $f(x_i)\Delta x = \frac{2i}{n} \cdot \frac{1}{n}$.
- So the total area of the n rectangles is

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \frac{2i}{n^2} \\ &= \frac{2}{n^2} \sum_{i=1}^{n} i && \text{take out constants} \\ &= \frac{2}{n^2} \cdot \frac{n(n+1)}{2} \end{aligned}$$

- Now, in the limit as $n \to \infty$ we get

$$A = \lim_{n\to\infty} R_n = \frac{2}{2} \cdot \lim_{n\to\infty} \frac{n(n+1)}{n^2} = 1$$

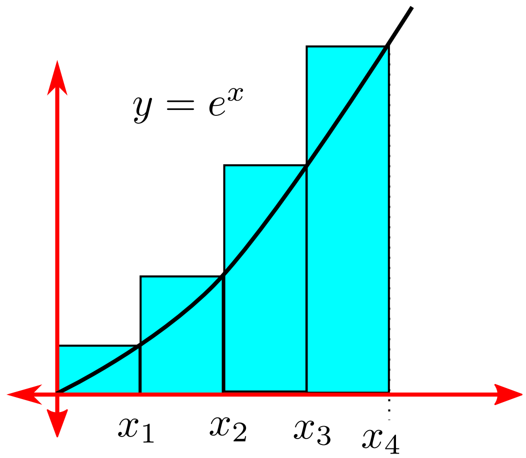
- Now if we repeat using the underestimate (left-endpoint) — rectangles of height $f(x_{i-1})$ we obtain the same answer $A = 1$ — you can check it at home.

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} \frac{2}{n^2} \cdot (i-1) \cdot \frac{1}{n} \qquad = \frac{2}{n^2} \underbrace{\sum_{i=1}^{n}(i-1)}_{=\sum_{j=0}^{n-1} j}$$

$$= \frac{2}{n^2} \cdot \frac{(n-1)(n+1-1)}{2} = \frac{n(n-1)}{n^2}$$

- If we let $n \to \infty$ we obtain the same answer $A = 1$.

We can also use these sums to find a numerical approximation of the area. For example — lets approximate the area under the curve $y = e^x$ between $x = 0$ and $x = 1$ using 4 rectangles defined by their right-endpoints.



$$\Delta x = \frac{1}{4}$$

$$x_1 = 1/4 \qquad x_2 = 1/2 \qquad x_3 = 3/4 \qquad x_4 = 1$$

So our approximation of the area is just

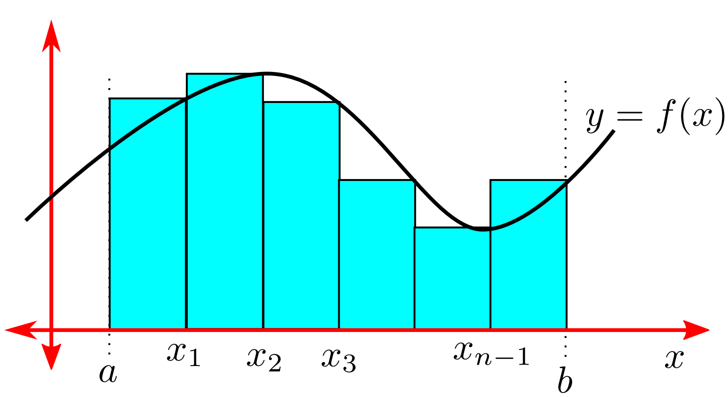
$$\begin{aligned} R_4 &= f(1/4)\Delta x + f(3/2)\Delta x + f(3/4)\Delta x + f(1)\Delta x \\ &= \frac{1}{4}\left(e^{1/4} + e^{1/2} + e^{3/4} + e^1\right) & \approx 1.94 \end{aligned}$$

The actual area is $e - 1 \approx 1.72$ — so we are not so far off.

More generally

Now everything worked out very very nicely in our first example because things ended in such a simple sum — and we knew the sum exactly. More generally (as happened in the second example) we are going to be dealing with much more general functions and sums that we won't know. That being said we can still do pretty much the same thing.

If we want the area under a curve from $x = a$ to $x = b$, we can approximate things by cutting that area up into n rectangles and summing them. Such sums are called Riemann sums.



Here I used right-endpoints, but I could have used left-endpoints too. It is easy to work out the little rectangles.

- All have width $\Delta x = (b-a)/n$.
- The ordinate $x_i = a + i\Delta x$.
- The height of the i^{th} rectangle is just $f(x_i)$

So our approximation of the area is just

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

And if we let $n \to \infty$ we get the area

Definition. The area under the curve $y = f(x)$ between $x = a$ and $x = b$ is the limit of the sum of areas of the approximating rectangles

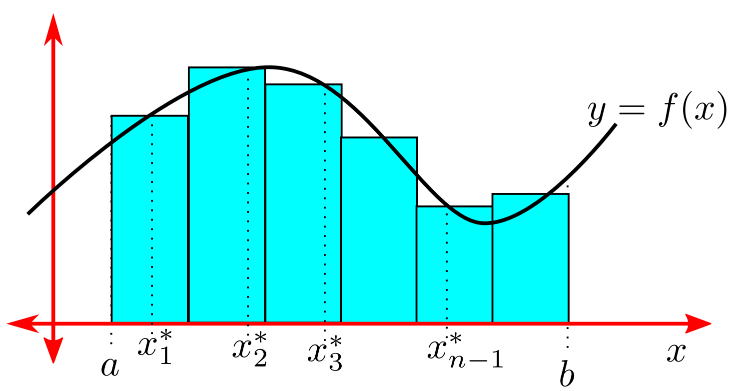
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

Now provided $f(x)$ is a continuous function one can prove that the above limit always exists — but this doesnt tell us what it actually is. We need some more calculus to do that.

We could have also defined things in terms of the left-endpoint rectangles L_n.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x$$

And we can go a bit further still. Keep the x_i as we have defined them — very nicely spaced and the rectangle widths all the same Δx. Now let x_j^* be *any* point in $[x_{j-1}, x_j]$ and make the height of the j^{th} rectangle be $f(x_j^*)$ — so we have a pic something like



$$A = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_j^*)\Delta x$$

Distances

Back in differential calculus we discussed distance and velocity. If velocity is constant then

$$\text{distance} = \text{velocity} \times \text{time}$$

Notice that this relationship is very similar to

$$\text{area} = \text{height} \times \text{width}$$

Of course the area under a curve is not as simple as this because the height varies as we move along the x-axis, but we approximated the area using lots of little rectangles. We can do the same thing with distances — the velocity is not constant as time increases, but we can approximate the total distance in the same way we approximated area.

Say we know the velocity of a car at regular times

time (s)	0	10	20	30	40	50	60
velocity (m/s)	3	7	8	11	13	12	11

Then during the first 10 seconds it travelled (using right-endpoints) $7 \times 10 = 70m$. Similarly for the next 10 seconds and the next etc etc.

$$\begin{aligned} \text{distance} &= 7 \times 10 + 8 \times 10 + 11 \times 10 + 13 \times 10 + 12 \times 10 + 11 \times 10 \\ &= (7 + 8 + 11 + 13 + 12 + 11) \times 10 = 620m \end{aligned}$$

5.2 The definite integral

We have now seen how we can find the area under the curve by summing lots of small rectangles and using a Riemann sum

$$A = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_j^*)\Delta x$$

The above form is useful not just for finding areas — indeed it still works when $f(x)$ is negative (why is this a problem for areas?) Because it is so useful we give this limit a special name — the definite integral.

Definition. Let f be a function defined on $[a, b]$. We divide the interval $[a, b]$ into n subintervals each of width $\Delta x = (b - a)/n$. Hence the endpoints of the intervals are $x_i = a + i\Delta x$. Further let $x_i^* \in [x_{i-1}, x_i]$ be a *sample point* from the i^{th} subinterval. Then the *definite integral of f from a to b* is

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided the limit exists. If it does exist then we say that f is integrable on $[a, b]$.

Note that

- $\int$ was introduced by Leibniz (it is a big "S") called "integral sign".

- $f(x)$ is called the "integrand"
- a and b are called the limits of integration
- dx does not have meaning by itself — it only means something as part of the whole $\int_a^b f(x)\mathrm{d}x$.
- $\int_a^b f(x)\mathrm{d}x$ is a number — it does not depend on x.

How do we know when the definite integral exists — there is a nice simple theorem?

Theorem (3)**.** *If f is continuous on $[a, b]$ or has only a finite number of jump discontinuities, then f is integrable on $[a, b]$.*

Okay — so with what we know presently we can compute some definite integrals, but not many. The first step is to express the integral as a Riemann sum. Then we have to take a limit. So lets do this for a simple polynomial function.

Let us compute the definite integral $\int_0^4 (x^2 - 3x)\mathrm{d}x$.

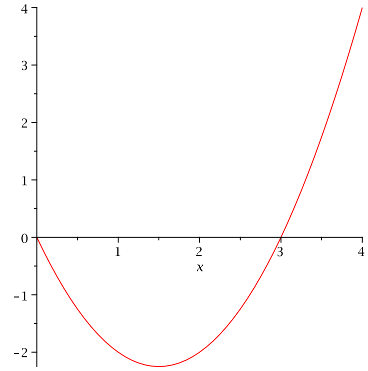
- Turn it into a riemann sum
  - $a = 0, b = 4$ so $\Delta x = 4/n$ and $x_i = \frac{4i}{n}$.
  - The Riemann sum (right-endpoints) is then

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{16i^2}{n^2} - \frac{12i}{n}\right)\frac{4}{n} \\
&= \sum_{i=1}^{n} \frac{64i^2}{n^3} - \sum_{i=1}^{n} \frac{48i}{n^2} \\
&= \frac{64}{n^3}\sum_{i=1}^{n} i^2 - \frac{48}{n^2}\sum_{i=1}^{n} i \\
&= \frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{48}{n^2}\frac{n(n+1)}{2}
\end{aligned}$$

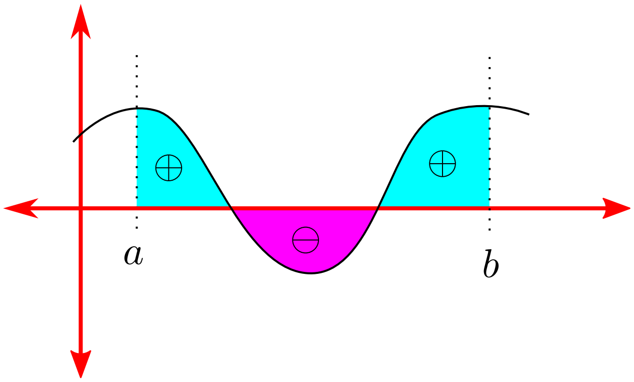
- So in the limit as $n \to \infty$ we have

$$\begin{aligned}
\int_0^3 f(x)\mathrm{d}x &= \lim_{n\to\infty} R_n \\
&= \lim_{n\to\infty} \frac{64}{6} \cdot \frac{n(n+1)(2n+1)}{n^3} - \lim_{n\to\infty} \frac{48}{2}\frac{n(n+1)}{n^2} \\
&= \frac{64}{3} - 24 = -\frac{8}{3}
\end{aligned}$$

So there are 2 things to notice here — firstly this is not such an easy way of doing things and we would really like a nicer way of doing it. Secondly — the answer is negative! What is going on here. Plot the graph



Since the function is negative between 0 and 3, those rectangles, $f(x)\Delta x$ are negative. Thus the definite integral computes what is called a "signed area".



So the definite integral computes the area of all the pieces above the axis and then subtracts the area of all the pieces below the axis.

Why don't you try $\int_0^1 (x^3 + 2x)\mathrm{d}x$.

- $a = 0, b = 1, \Delta x = 1/n, x_i = i/n$.
- Riemann sum is

$$
\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \sum_{i=1}^{n} \left(\frac{i^3}{n^3} + \frac{2i}{n} \right) \frac{1}{n} \\
&= \frac{1}{n^4} \sum_{i=1}^{n} i^3 + \frac{2}{n^2} \sum_{i=1}^{n} i \\
&= \frac{n^2(n+1)^2}{4n^4} + \frac{2n(n+1)}{2n^2}
\end{aligned}
$$

- Take limits as $n \to \infty$ to get

$$
\begin{aligned}
A &= \lim_{n\to\infty} R_n \\
&= \frac{1}{4} + 1 = \frac{5}{4}
\end{aligned}
$$

Some useful properties

Now when we looked at derivatives of functions we saw some useful properties that helped us compute things — eg

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)+g(x)\right) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}$$

And we will see that integrals have similar properties, but not quite as nice. Lets start with some nice ones

Theorem (Properties of the definite integral)**.** *Let f and g be integrable on $[a,b]$ and let c be a constant. Then*

$$\begin{gathered}
\int_a^a f(x) = 0 \\
\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x \\
\int_a^b c\mathrm{d}x = c(b-a) \\
\int_a^b \left(f(x)+g(x)\right)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x \\
\int_a^b cf(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x \\
\int_a^b \left(f(x)-g(x)\right)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x
\end{gathered}$$

These all follow quite directly from the Riemann sum definition. For example, in the first, $\Delta x = 0$. In the second Δx changes from $(a-b)/n$ to $(b-a)/n$.

Theorem (Further properties)**.** *Let f, g be integrable functions on $[a,b]$.*

1. *Let $c \in (a,b)$, then*

$$\int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$$

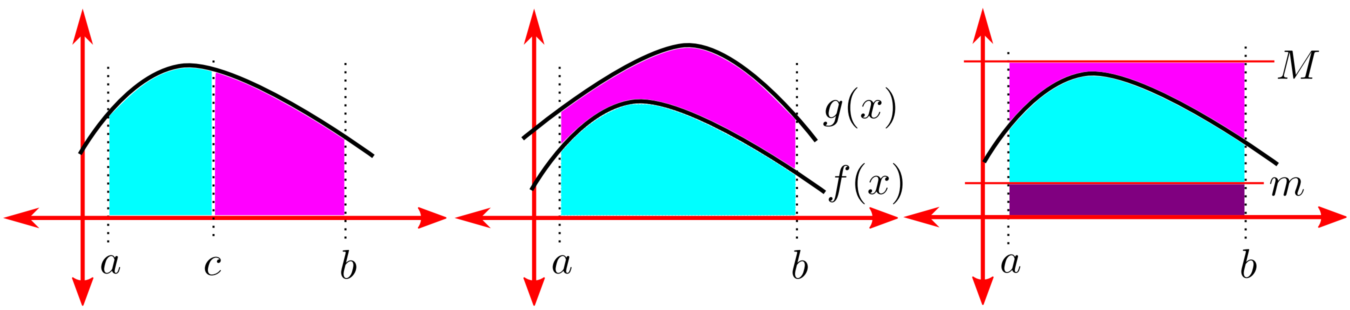
2. *If $f(x) \geq 0$ for all $x \in [a,b]$ then $\int_a^b f(x)\mathrm{d}x \geq 0$.*
3. *If $f(x) \leq g(x)$ for all $x \in [a,b]$ then*

$$\int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$$

4. *If $m \leq f(x) \leq M$ $\forall x \in [a,b]$, then*

$$m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$$

We won't prove these — they follow from carefully playing with the Riemann sums. The ideas behind the proofs are actually quite simple and we can sketch them. For #1, #3 and #4 we have



So we can use this to get an idea of the size of definite integrals. For example

$$\int_1^4 \sqrt{x}\mathrm{d}x$$

Now $\sqrt{x}$ is an increasing function — so it takes its minimum at $x = 1$ with a value of 1 and its maxmimum at 4 with a value of 2. Thus we have

$$1 \cdot (4-1) \leq \int_1^4 \sqrt{x}\mathrm{d}x \leq 2 \cdot (4-1)$$

That is — its value lies between 3 and 6. Its actual value is $14/3 = 4.666\dots$.

5.3 Fundamental Theorem of Calculus

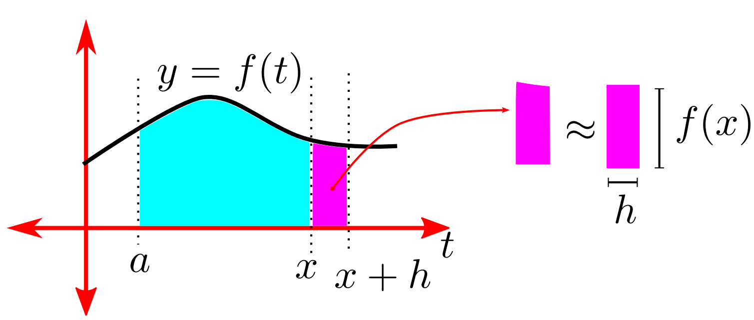
So I think I have now convinced you that computing definite integrals using Riemann sums is quite painful. There is a better way, and that is through a link to the calculus you did last term. The fundamental theorem of calculus links integral calculus to differential calculus. It is a remarkable result.

It says (approximately) that "differentiating undoes integrating."

Theorem (Fundamental Theorem of Calculus Part 1)**.** *Let f be continuous on $[a, b]$.*

- *Definte $F(x) = \int_a^x f(t)\mathrm{d}t$ for $x \in [a, b]$.*
- *Then $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$.*

Proof. Consider the following picture



So what happens to the area when we move x to $x+h$?

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}F(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\\
&= \lim_{h\to 0}\left(\int_a^{x+h} f(t)\mathrm{d}t - \int_a^{x} f(t)\mathrm{d}t\right)/h\\
&= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t \qquad \text{splitting the interval}
\end{aligned}$$

So now think of what this definite integral is — it is a tiny area that is very nearly a rectangle of height $f(x)$ and width h. So we expect the definite integral to be approximately $f(x)\cdot h$. Indeed we can show (if we are careful — see proof in text) that as h gets closer and closer to zero, the definite integral $\int_x^{x+h} f(t)\mathrm{d}t$ gets closer and closer to $f(x)h$. Hence as $h \to 0$ we get

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$$

Some examples:

- Let $F(x) = \int_0^x \cos(t)\mathrm{d}t$. Then $F'(x) = \cos(x)$.
- Let $F(x) = \int_1^x (t^2+\sqrt{t+1})\mathrm{d}t$. Then $F'(x) = x^2+\sqrt{x+1}$.
- Let $F(x) = \int_1^{x^2} \cos(t)\mathrm{d}t$ Then find $F'(x)$.
  - We need to use the chain rule.
  - Write $G(u) = \int_1^u \cos(t)\mathrm{d}t$, with $u = x^2$.
  - Then using the chain rule we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}G(u(x)) &= \underbrace{\frac{\mathrm{d}G}{\mathrm{d}u}}_{FTC1}\frac{\mathrm{d}u}{\mathrm{d}x}\\
&= \cos u \cdot 2x = 2x\cos(x^2)
\end{aligned}$$

While this is cute, it isnt so useful. What we really want to do is given $f(x)$ compute the integral of f. The FTC tells us that it is undone by differentiating. Let us make this more precise

Definition. Given a function f, then any function F with $F'(x) = f(x)$ is called an anti-derivative of f.

For example, $F(x) = x^2$ is an anti-derivative of $f(x) = 2x$. But so is $F(x) = x^2+3$ and $F(x) = x^2 - 7.35$. We can check by differentiating. What other anti-derivatives are out there? How are they related to each other? — thankfully it is not hard.

Theorem. *If F is an anti-derivative of f, then any other anti-derivative of f is of the form $F(x)+c$, where c is a constant.*

Proof. A common proof trick when trying to show that 2 functions are the same (or nearly so) is to show that their difference is zero (or nearly so).

- Suppose F and G are anti-derivatives of f on the interval $[a,b]$.
- Let $H = F - G$.
- By the FTC (part 1) we know that $H' = F' - G' = 0$ for all $x \in [a,b]$.
- Hence $H(x)$ is a constant.
- So $F(x) = G(x) + c$.

□

When we studied derivatives, we started by memorising (and maybe proving) the derivatives of some very simple functions — eg

$f(x)$	$\frac{\mathrm{d}f}{\mathrm{d}x}$	$f(x)$	$\frac{\mathrm{d}f}{\mathrm{d}x}$
x^n	nx^{n-1}	e^x	e^x
$\sin(x)$	$\cos(x)$	$\cos(x)$	$-\sin(x)$

We then learnt how to work out the derivatives of more complicated functions by piecing together the derivatives of these simple functions using rules such as

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)+g(x)) = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}g}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}cf(x) = c\frac{\mathrm{d}f}{\mathrm{d}x}$$

and the chain rule and the product rule. We will do something very similar for anti-derivatives. But before we get there, I want to finish off the FTC2. The second part of the theorem links anti-derivatives to the area under the curve $\int_a^b f(x)\mathrm{d}x$. It seems very surprising that these two things are linked — somehow differentiating (and its inverse) are very symbolic operations — we juggle the symbols around — and yet they are intimately linked with the very tangible geometric idea of the area under a curve.

Theorem (Fundamental Theorem of Calculus Part 2)**.** *Let f be continuous on $[a,b]$ and let F be any anti-derivative of f. Then*

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a)$$

So we saw above that an anti-derivative of $f(x) = 2x$ is $F(x) = x^2$. So if we want to compute

$$\int_0^1 2x\mathrm{d}x = F(1) - F(0) = 1^2 - 0^2 = 1$$

just as we got by the much more difficult method of Riemann sums before. Let us prove the above theorem

Proof.
- By FTC1 we know that $\displaystyle\int_a^x f(t)\mathrm{d}t$ is an anti-derivative of $f(x)$.

- Hence we write $\displaystyle\int_a^x f(t)\mathrm{d}t = F(x) + c$ for some constant c.
- What is c? what extra information do we have?
- When $x = a$ we have

$$\int_a^a f(t)\mathrm{d}t = F(a) + c = 0$$

- So $c = -F(a)$.
- Now when $x = b$ we get

$$\int_a^b f(t)\mathrm{d}t = F(b) + c = F(b) - F(a)$$

- Note, it doesnt matter which anti-derivative we choose, because the constant "c" cancels.

$\square$

So by FTC 1 and 2, we see that integration and differentiation are inverse operations — they undo each other.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\mathrm{d}t\right) = f(x)$$
$$\int_a^x \left(\frac{\mathrm{d}}{\mathrm{d}t}F(t)\right)\mathrm{d}t = F(x) - F(a)$$

Definition. Because we will be doing lots of integrals, let us make some notation:

$$F(x)|_a^b = [F(x)]_a^b = F(b) - F(a)$$

Hence $\int_a^b f(t)\mathrm{d}t = [F(x)]_a^b$.

So let us start by considering the anti-derivative of x^n. The derivative is easy nx^{n-1}. But we need to do the opposite of this.

- When we differentiate the exponent decreases by 1, so the anti-derivative must be something like $const \cdot x^{n+1}$.
- Differentiate this to get $const \cdot (n+1) \cdot x^n$.
- So the constant must be $1/(n+1)$.

Thus the anti-derivative of x^n is $\dfrac{1}{n+1}x^n$. But this is just one anti-derivative — more generally this plus any constant will do. So we write

$$\frac{1}{n+1}x^n + C$$

Note that this breaks down when $n = -1$ — ie for the anti-derivative of $1/x$ — then we get logarithms.

Armed we this result we can recompute

$$\begin{aligned}\int_0^1 (x^2 - 3x)\mathrm{d}x &= \int_0^1 x^2\mathrm{d}x - \int_0^1 3x\mathrm{d}x \\ &= \int_0^1 x^2\mathrm{d}x - 3\int_0^1 x\mathrm{d}x \\ &= [x^3/3]_0^1 - 3[x^2/2]_0^1 \\ &= \frac{1}{3} - 3 = -\frac{8}{3}\end{aligned}$$

Much much easier. Why don't you redo the other example

$$\int_0^1 x^3 + 2x\mathrm{d}x = \left[\frac{x^4}{4} + x^2\right]_0^1 \qquad = 1/4 + 1 = 5/4$$

What happens if we change the terminals —

$$\begin{aligned}\int_1^3 x^3 + 2x\mathrm{d}x &= \left[\frac{x^4}{4} + x^2\right]_1^3 \\ &= \left(\frac{81}{4} + 9\right) - \left(\frac{1}{4} + 1\right) \\ &= \frac{117}{4} - \frac{5}{4} = 112/4 = 28\end{aligned}$$

So by repeating what we did above with x^n, we can quickly construct a table of simple anti-derivatives (which we can check by differentiating). This table comes from chapter 5.4 of the text.

f	F	
x^n	$\frac{1}{n+1} \cdot x^n + c$	$n \neq 1$
$1/x$	$\log\lvert x\rvert + c$	don't forget the $\lvert\cdot\rvert$
e^{ax}	$\frac{1}{a}e^{ax} + c$	
a^x	$\frac{1}{\log a}a^x + c$	
$\cos(ax)$	$\frac{1}{a}\sin(ax) + c$	
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + c$	
$\sec^2(ax)$	$\frac{1}{a}\tan(ax) + c$	

Only do if lots of time? — from text chpt 3.6 Integral of $1/x$ — why is it $\log|x|$? We check this by cases — namely $x > 0$ and then $x < 0$.

- Assume $x > 0$ and let $y = \log|x| = \log x$ — hence $x = e^y$.
- Differentiate both sides with respect to x:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x &= \frac{\mathrm{d}}{\mathrm{d}x}e^y \\ 1 &= \frac{\mathrm{d}e^y}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} && \text{chain rule} \\ 1 &= e^y\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

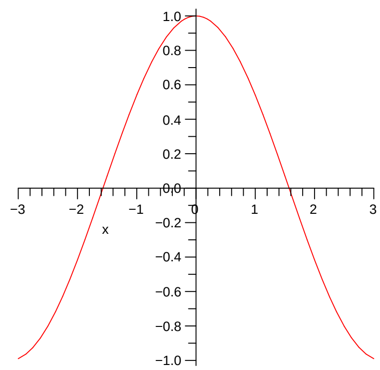
- Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x$ — as required.
- Now assume $x < 0$ — $y = \log|x| = \log(-x)$.
- Set $u = -x$, $y = \log u$ and use the chain rule to differentiate

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{u}\cdot(-1) = (-1/x)\cdot(-1) = 1/x \end{aligned}$$

  again, as required.

Another simple example for you to do.

- Area under the curve $y = \cos x$ between $x = \pm\pi/2$.



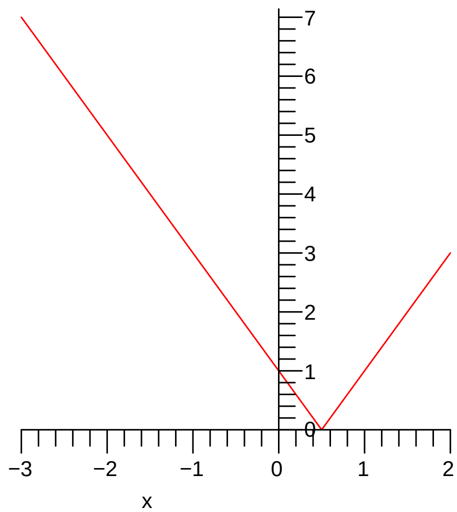
- Area is given by

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2}\cos(x)\mathrm{d}x \\ &= [\sin(x)]_{-\pi/2}^{\pi/2} \\ &= \sin(\pi/2) - \sin(-\pi/2) = 1 - (-1) = 2. \end{aligned}$$

Another example in which we have to be a bit more careful.

$$\int_{-2}^{1} |2x-1| \mathrm{d}x$$

The integrand is continuous, but the $|.|$ is a bit worrying. So first step is to plot things.



We could just compute areas of triangles, but let us do it using integration.

- Split the integral to take into account

$$|q| = \begin{cases} q & q \geq 0 \\ -q & q < 0 \end{cases}$$

- We need to split things where $2z-1=0$, namely at $z=1/2$.

$$|2z-1| = \begin{cases} 2z-1 & z \geq 1/2 \\ 1-2z & z < 1/2 \end{cases}$$

- Hence the integral becomes

$$\begin{aligned} \int_{-2}^{1} |2z-1| \mathrm{d}z &= \int_{-2}^{1/2} (1-2z) \mathrm{d}z - \int_{1/2}^{1} (2z-1) \mathrm{d}z \\ &= [z-z^2]_{-2}^{1/2} - [z^2-z]_{1/2}^{1} \\ &= ((1/2-1/4)-(-2-4)) - ((1/4-1/2)-(1-1)) \\ &= 1/4+6+1/4-0 = 6+1/2 = 13/2 \end{aligned}$$

5.4 The indefinite integral and net change

So far we have talked about definite integrals — where we "integrate" over a fixed region:

$$\int_a^b f(x)\mathrm{d}x = \text{definite integral of } f(x) \text{ from } a \text{ and } b = \text{ a number}$$

For things that follow it will help if we can talk about a slightly more general object — namely the indefinite integral.

Definition. $\underbrace{\int}_{\text{no terminals}} f(x)\mathrm{d}x = \text{the general anti-derivative of } f.$

Note — as opposed to a specific anti-derivative — it represents a family of functions.

We will use this notation a lot.
Some examples

- $\int x^2\mathrm{d}x = x^3/3 + c$ — we don't know what c is.
- $\int (z^2+1)^3\mathrm{d}z = \int (z^6 + 3z^4 + 3z^2 + 1)\mathrm{d}x = z^7/7 + 3z^5/5 + z^3 + z + c$
- $\int \sin(x)\mathrm{d}x = -\cos x + c.$

See table in chapter 5.4. LEARN IT!
Now the FTC tells us that

- if f is continuous on $[a,b]$,
- and F is any anti-derivative of f, then

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a)$$

- We can rewrite this as

$$\int_a^b F'(t)\mathrm{d}t = F(b) - F(a)$$

So $F'(t)$ is the rate of change of F wrt t, and $F(b) - F(a)$ is the difference between the final value of F and the initial value. Hence the integral of the rate of change gives us the net-change.

Theorem (Net change theorem)**.** *The integral of the rate of change is the net change*

$$\int_a^b F(t)\mathrm{d}t = F(b) - F(a).$$

For example

- An object moves along a straight line with position function $x(t)$.
- The velocity is the rate of change of position — so $v(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$.

- The net change in position — the displacement from time t_1 to time t_2, is therefore

$$\int_{t_1}^{t_2} v(t)\mathrm{d}t = x(t_2) - x(t_1)$$

- Note that this is not the total distance traveled, as sometimes it might be moving backwards and sometimes forward. The total distance would be

$$\int_{t_1}^{t_2} |v(t)|\mathrm{d}t$$

A preview of things to come

Do not do unless there is a lot of time.

- As discussed earlier, one of the main reasons for learning integration is to solve differential equations.
- Lets look at a simple example of an "initial value problem" — this is a differential equation together with an "initial condition".
- In a physical setting, the differential equation usually tells us how the system evolves in time, while the initial condition tells us the initial state of the system.
- An example would be

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t) \qquad\qquad y(a) = y_0$$

  where we are told $f(t)$ and y_0.
- In this case we can see directly that $y(t)$ is some anti-derivative of $f(t)$.
- Hence $y(t)$ is given by

$$y(t) = \underbrace{\int_a^t f(z)\mathrm{d}z}_{\text{note dummy variable}} + c$$

  and we need to work out what c is from the initial condition.
- If we set $t = a$, then we know that the integral is zero and that $y(a) = y_0$, hence $c = y_0$.
- So our solution is

$$y(t) = y_0 + \int_a^t f(z)\mathrm{d}z$$

An example

- Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec^2 x$ with $y(0) = 1$.

- We know from the above that y is

$$
\begin{aligned}
y(x) &= 1 + \int_0^x \sec^2(t)\mathrm{d}t \\
&= 1 + [\tan t]_0^x \\
&= 1 + \tan x - \underbrace{\tan 0}_{=0} = 1 + \tan x
\end{aligned}
$$

- Note - you can always check your answer

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(1+\tan x) = \sec^2 x \\
&(1+\tan x)_{x=0} = 1 + \tan 0 = 1.\checkmark
\end{aligned}
$$

When we do differential equations properly, then we will follow roughly the same steps — integrate and then fit the initial conditions.

5.5 Substitution rule

Up until now we have been doing very simple integrals. Basically we have just been using a lookup table from the derivatives we know, and using the fact that integrals play nicely with addition, subtraction and multiplication by constants

$$
\begin{aligned}
\int_a^b (f(x)+g(x))\,\mathrm{d}x &= \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x \\
\int_a^b (f(x)-g(x))\,\mathrm{d}x &= \int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x \\
\int_a^b cf(x)\mathrm{d}x &= \int_a^b f(x)\mathrm{d}x
\end{aligned}
$$

This was how we started out with derivatives too. So now we need some tools for building more complicated integrals. When we did this for derivatives we used the product and chain (and quotient) rules

$$
\begin{aligned}
\frac{\mathrm{d}fg}{\mathrm{d}x} &= f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x} \\
\frac{\mathrm{d}}{\mathrm{d}x}f(u(x)) &= \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}
\end{aligned}
$$

These will be our starting points for finding some nice integration rules. In particular we find integration rules by anti-differentiating them

What does the chain rule tell us:

Theorem (The chain rule)**.** *If $y = F(u)$ and $u = u(x)$, then*

$$
\frac{\mathrm{d}}{\mathrm{d}x}F(u(x)) = F'(u(x)) \cdot u'(x)
$$

Consider the following example

$$\int 9\sin^8 x \cos x \mathrm{d}x = \sin^9 x + c$$

How did we find this — the chain rule in reverse. We can verify by differentiating the right-hand side. Let us do so

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\sin^9 x &= \left(\frac{\mathrm{d}}{\mathrm{d}u}u^9\right)\cdot\frac{\mathrm{d}u}{\mathrm{d}x} & u = \sin x \\ &= 9u^8\cdot\cos x \\ &= \underbrace{9\sin^8 x}_{F'(u(x))}\cdot\underbrace{\cos x}_{\frac{\mathrm{d}u}{\mathrm{d}x}} \end{aligned}$$

We can do this more generally if we can write the integrand as $F'(u(x))u'(x)$. Let us see

- Consider a function $F(u)$.
- Let $u = u(x)$ — so we have $F(u(x))$.
- Differentiate wrt x — gives $F'(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}$.
- Integrate wrt x

$$\int F'(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = F(u(x)) + c$$

- But if we go back to the original function $F(u)$.
- Differentiate wrt u, we get $F'(u)$.
- Integrate it wrt u

$$\int F'(u(x))\mathrm{d}u = F(u(x)) + c$$

- If we write $F'(u) = f(u)$, we have shown that

$$F(u(x)) + c = \int f(u)\mathrm{d}u = \int f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

Theorem (The substitution rule)**.**

- *Let $u = u(x)$ is a differentiable function whose range is some interval I.*
- *Let f be a function that is continuous on I.*
- *Then we have*

$$\int f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int f(u)\mathrm{d}u$$

An example

$$\begin{aligned}
\int \underbrace{4x^3}\,\underbrace{\sin x^4}\,\mathrm{d}x &= \int 4x^3 \sin u \mathrm{d}x \qquad\qquad u = x^4 \qquad u' = 4x^3 \\
&= \int \sin u \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \\
&= \int \sin u \mathrm{d}u \\
&= -\cos u + c = -\cos x^4 + c
\end{aligned}$$

There were two obvious "chunks" to this expression — we chose the correct substitution. What if we tried the other "chunk" — ie $u = x^3$? Well the other chunk would be $\sin u^{4/3}$ — ugly! It is always a good idea if things are getting ugly, to take a step back and check if you made the right choice.

Another example

$$\begin{aligned}
\int \cot x \mathrm{d}x &= \int \frac{\cos x}{\sin x}\mathrm{d}x \\
&= \int \frac{\cos x}{u}\mathrm{d}x \qquad\qquad u = \sin x \qquad u' = \cos x \\
&= \int \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \\
&= \int \frac{1}{u}\mathrm{d}u \\
&= \log|u| + c = \log|\sin x| + c
\end{aligned}$$

- Again, we had 2 obvious choices for u — we made the right one.
- What happens if we choose $u = \cos x$, so $u' = \sin x$ — looks okay so far.
- The integral becomes $\int \frac{u}{u'}\mathrm{d}x$ — which is not of the form of our rule.

WARNING

Another example — here the $\frac{\mathrm{d}u}{\mathrm{d}x}$ is not present.

$$\begin{aligned}
\int \sqrt{2x+1}\mathrm{d}x &= \int \sqrt{u}\mathrm{d}x \qquad\qquad u = 2x+1 \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \\
&= \int \sqrt{u}\,\underbrace{\left(\frac{\mathrm{d}u}{\mathrm{d}x}/\frac{\mathrm{d}u}{\mathrm{d}x}\right)}_{=1}\mathrm{d}x \\
&= \int \frac{1}{2}\sqrt{u}\,\underbrace{\frac{\mathrm{d}u}{\mathrm{d}x}}_{=2}\,\mathrm{d}x \\
&= \int \frac{1}{2}u^{1/2}\mathrm{d}u = \frac{1}{2}\frac{u^{3/2}}{3/2} + c \\
&= \frac{1}{3}(2x+1)^{3/2} + c
\end{aligned}$$

- Some people, including the textbook, teach you that this rule is somehow equivalent to cancel the "dx":

$$\int f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int f(u(x))\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int f(u)\mathrm{d}u$$

- This is a useful mnemonic, however *I* think that it encourages sloppy thinking.
- The derivative $\frac{\mathrm{d}f}{\mathrm{d}u}$ is not a fraction! It is an operation you do on a function defined by a limit. You are not dividing df by du.
- Be careful with this.
- A better way:

$$\begin{aligned}\int blah\mathrm{d}x &= \int blah\left(\frac{\mathrm{d}u}{\mathrm{d}x}/\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x\\ &= \int blah\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x\\ &= \int blah\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1}\mathrm{d}u\end{aligned}$$

What about when we have terminals? ie a definite integral?

Theorem (Substitution rule for definite integrals)**.**

- *Let $u = u(x)$ is a differentiable function whose range is some interval I.*
- *Let f be a function that is continuous on I.*
- *Then we have*

$$\int_a^b f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\mathrm{d}u$$

For example

$$\begin{aligned}\int_0^1 \frac{3x}{(x^2+1)^2}\mathrm{d}x &= \int_0^1 \frac{3x}{u^2}\mathrm{d}x && u = x^2+1 \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x\\ &= \int_{u(0)}^{u(1)} u^{-2}3x(2x/2x)\mathrm{d}x && \text{or } \mathrm{d}x = \mathrm{d}u/2x\\ &= \int_{u(0)}^{u(1)} u^{-2}\frac{3}{2}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x\\ &= \frac{3}{2}\int_1^2 u^{-2}\mathrm{d}u\\ &= \frac{3}{2}[-1/u]_1^2 = \frac{3}{2}(-1/2+1) = 3/4.\end{aligned}$$

Another example

$$\int_0^{\pi} \sin(x)\exp(\cos x)\mathrm{d}x$$

Let $u = \cos x$ so $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$ and $u(0) = 1$ and $u(\pi) = -1$.

$$\begin{aligned}\int_0^{\pi} \sin(x)\exp(\cos x)\mathrm{d}x &= \int_{u(0)}^{u(\pi)} -(-\sin(x))\exp(u)\mathrm{d}x \\ &= \int_1^{-1} -\exp(u)\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \\ &= -\int_1^{-1} e^u \mathrm{d}u = -[e^u]_1^{-1} \\ &= -e^{-1} + e^1\end{aligned}$$

Symmetry

Symmetries play an important role in mathematics. They are very helpful to simplify problems.

For integration there are two useful symmetries to look for

Definition. A function $f(x)$ is called "even" if it obeys

$$f(-x) = f(x)$$

While it is called "odd" if it obeys

$$f(-x) = -f(x)$$

Why "even" and "odd"? because of monomials

$$\begin{aligned}(-x)^{even} &= (-1)^{even}x^{even} = x^{even} \\ (-x)^{odd} &= (-1)^{odd}x^{odd} = -x^{odd}\end{aligned}$$

Also $\sin(x)$ is odd and $\cos(x)$ is even. Hence $\tan(x)$ is odd.

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

So what does this have to do with integrating?

Theorem. *Suppose f is continuous on $[-a, a]$, then*

- *If f is even, then $\int_{-a}^{a} f(t)\mathrm{d}t = 2\int_0^a f(t)\mathrm{d}t$.*
- *If f is odd, then $\int_{-a}^{a} f(t)\mathrm{d}t = 0$.*

Proof.
- Split the integral into 2 pieces and massage it a bit.

$$\begin{aligned}\int_{-a}^{a} f(t)\mathrm{d}t &= \int_{-a}^{0} f(t)\mathrm{d}t + \int_0^a f(t)\mathrm{d}t \\ &= \underbrace{-\int_0^{-a} f(t)\mathrm{d}t}_{I_1} + \underbrace{\int_0^a f(t)\mathrm{d}t}_{I_2}\end{aligned}$$

- Look at the first of these and substitute $t = -u$:

$$-\int_0^{-a} f(t)\mathrm{d}t = -\int_0^{-a} f(-u)(\frac{\mathrm{d}u}{\mathrm{d}t}/\frac{\mathrm{d}u}{\mathrm{d}t})\mathrm{d}t = -\int_0^{-a} f(-u)(\frac{\mathrm{d}u}{\mathrm{d}t}/(-1))\mathrm{d}t$$
$$= -\int_0^{a} f(-u)(-1)\mathrm{d}u = \int_0^{a} f(-u)\mathrm{d}u$$

- Now if $f(-u) = f(u)$, then this integral is the same as I_2.
- If $f(-u) = -f(u)$ then $I_1 = -I_2$.

$\square$

An example, let $f(x) = \sin(x)\exp(\cos(x))$. Now - we could integrate this (we did so before). But we don't always have to.

$$\begin{aligned} f(-x) &= \sin(-x)\exp(\cos(-x)) \\ &= -\sin(x)\exp(\cos(x)) && \text{is odd, so} \\ \int_{-\pi}^{\pi} f(x) &= 0 \end{aligned}$$

Another example

$$\begin{aligned} f(x) &= \frac{\sin x}{1+x^2+x^4} && \text{is odd} \\ f(-x) &= \frac{\sin(-x)}{1+(-x)^2+(-x)^4} = -\frac{\sin(x)}{1+x^2+x^4} \\ \int_{-1}^{1} f(x)\mathrm{d}x &= 0 \end{aligned}$$