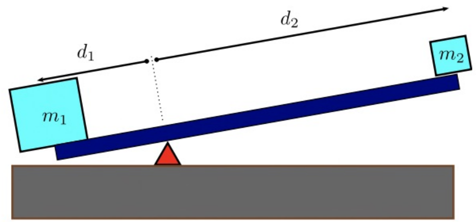
Center of mass.

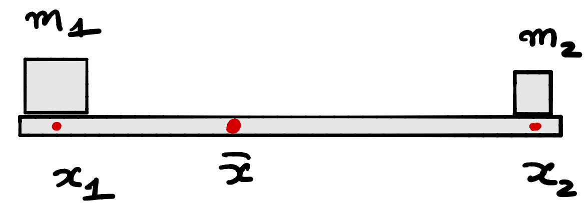


The see-saw perfectly balances if the Lever is at center of mass. Balance torque exerted by m_2 and m_2 to computed center of mass.

Torque, $T = F \times d$, F is the gravitational force
(rotational force) d is the distance from Lever.

How do we calculate center of mass?

Centre of mass (1D and discrete).



Assume mass m_1 is at x_1.

Assume mass m_2 is at x_2.

If we try to balance at $\bar{x}$ then.

The torques are
$$T_1 = m_1 g(\bar{x} - x_1)$$
$$T_2 = m_2 g(x_2 - \bar{x})$$

so, $T_1 = T_2 \Rightarrow m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}_1)$

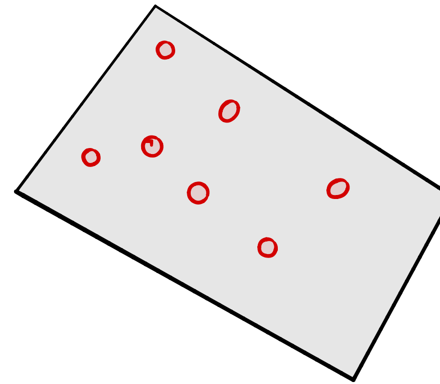
$$\Rightarrow \bar{x} = \frac{(m_1 x_1 + m_2 x_2)}{m_1 + m_2} = \frac{\text{Total moment}}{\text{total mass}}.$$

Let m be total mass. If there are N masses:

$$\boxed{\bar{x} = \frac{1}{m}\sum_{i=1}^{N} m_i x_i \quad , m = \sum_{i=1}^{N} m_i .}$$

$m_i x_i$ is the moment of i^{th} mass.

Center of mass (2D and discrete).



In a 2D scenario with point masses $m_1, m_2, \ldots, m_N$ centered at $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$, the center of mass are given by

$$\boxed{\bar{x} = \frac{1}{m} \sum_{i=1}^{N} m_i x_i \quad , \quad \bar{y} = \frac{1}{m} \sum_{i=1}^{N} m_i y_i}$$

1D continuous case.

If a body consists of mass distributed along a straight line with density $\rho(x)$ (kg/m) with $a \le x \le b$, the center of mass $\bar{x}$ is:

$$\bar{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx} = \frac{\text{Moment about origin}}{\text{Total mass.}} \quad \text{——} \ (1)$$

Remark: use Riemann sum to show the center of mass is (1). Go from discrete to continuous

Example.

A metal rod is 50 cm long. Its linear density at point x from left end is $\rho(x) = \dfrac{1}{100-x}$ (gm/cm). Find the mass and center of mass of the rod.

Sol^n Total moment $= \displaystyle\int_0^{50} \frac{x}{100-x}\,dx = \int_0^{50} \frac{x-100}{100-x}\,dx + \int_0^{50} \frac{100}{100-x}\,dx$

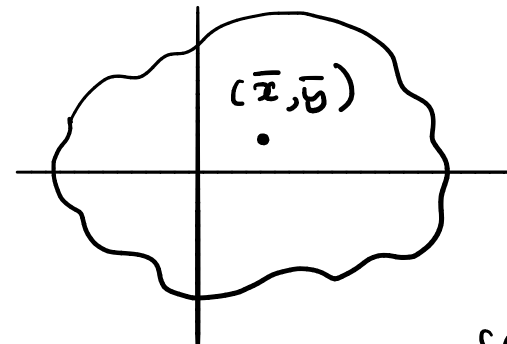
$$= -50 + 100\int_0^{50} \frac{1}{100-x}\,dx.$$

Total mass $= \displaystyle\int_0^{50} \rho(x)\,dx = \int_0^{50} \frac{1}{100-x}\,dx$

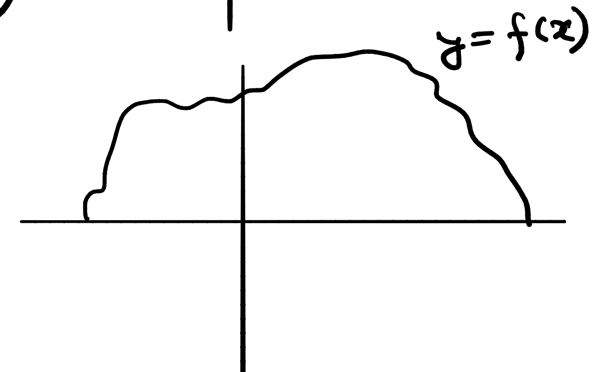
Centroid of a lamina.

Lamina is a thin "plate" which occupies some area in $\mathbb{R}^2$.
We will assume that the density $\rho(x,y)$ is constant.

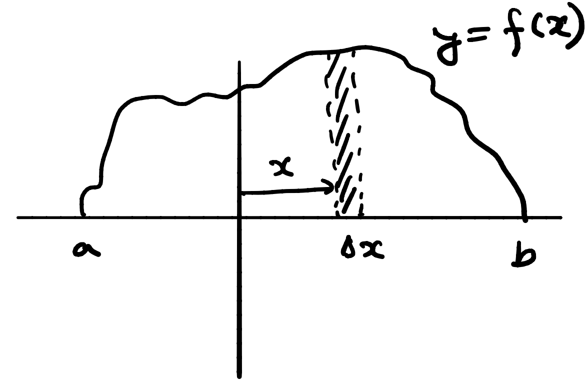
We want to calculate the centroid $(\bar{x}, \bar{y})$, i.e. the center of mass.



Case 1. Consider a lamina with constant density ρ whose lower boundary is the x-axis and the upper boundary is $y = f(x)$.



Case 1 lower bound is x-axis.



We first find moment about y-axis:

- Take a chunck Δx at a signed distance x from y-axis.
- The area of strip $\approx f(x)\cdot\Delta x$.
- Mass of strip $\approx (f(x)\cdot\Delta x)\rho$.
- Moment of strip about y-axis, labeled ΔM_y:

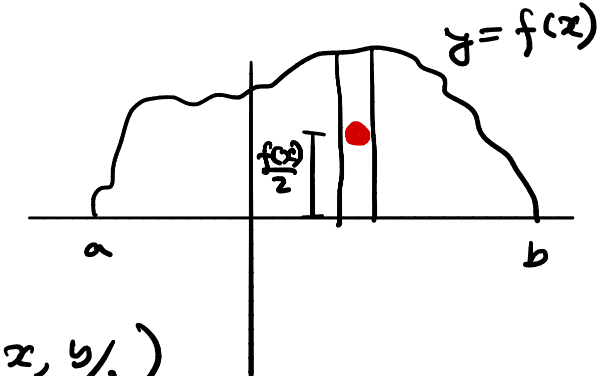
$$\Delta M_y = x\,(f(x)\cdot\Delta x)\,\rho \quad (g).$$

- Then integrating over all strips:

$$\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x\, f(x)\,\rho\, dx}{\int_a^b f(x)\,\rho\, dx} = \frac{\int_a^b x\, f(x)\, dx}{\int_a^b f(x)\, dx}$$

Case 1 lower bound is x-axis (contd.)

Now, we find moment about x-axis.



The center of mass of the strip of width Δx is $y = \frac{f(x)}{2}$ and can be thought of as concentrated at $(x, y/_2)$

Again, mass of strip is $(f(x) \cdot \Delta x)\rho$.

So, moment about x-axis is:

$$M_x = \int_a^b \frac{f(x)}{2} \cdot f(x) \rho \, dx.$$

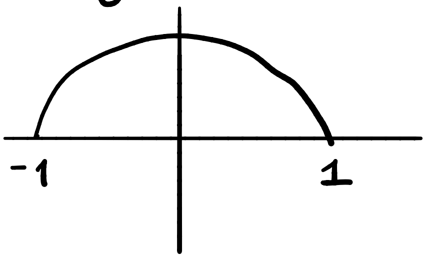
The y-ordinate of center of mass $\bar{y}$ is:

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{2}\int_a^b f(x)^2 \rho \, dx}{\int_a^b f(x) \rho \, dx} = \frac{\frac{1}{2}\int_a^b f(x)^2 \, dx}{\int_a^b f(x) \, dx}$$

Example:

Find the center of mass of a parabolic plate $y = 1 - x^2$ above $y = 0$ and $-1 \le x \le 1$. Assume constant density.

Solⁿ:



- By symmetry, we should have $\bar{x} = 0$.

$$M_y = \int_{-1}^{1} x f(x) \rho \, dx$$

$$= \rho \int_{-1}^{1} x (1 - x^2) \, dx \qquad = 0$$

so, $\bar{x} = \dfrac{M_y}{M} = 0$.

Example contd.

Now, $M_x = \frac{1}{2}\int_{-1}^{1} \rho f(x)^2 dx = \frac{\rho}{2}\int_{-1}^{1} (1-x^2)^2 dx = \frac{\rho}{2}\int_{-1}^{1} 1-2x^2+x^4 dx$

So, $M_x = \frac{\rho}{2}\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} = \frac{\rho}{2}\cdot\left[1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5}\right]$

$= \frac{\rho}{2}\cdot\left[\frac{30-20+6}{15}\right] = \frac{8\rho}{15}$

and $M = \int_{-1}^{1} f(x)\rho\, dx = \rho\int_{-1}^{1} 1-x^2 dx = \rho\left[x - \frac{x^3}{3}\right]_{-1}^{1}$

$= \rho\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right]$

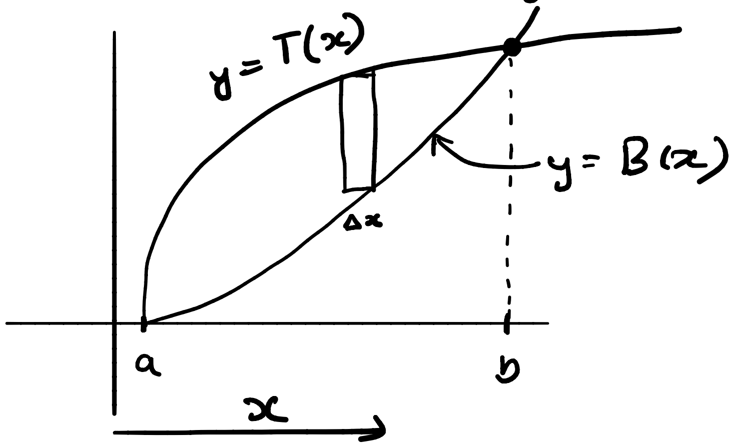
$= \frac{4}{3}\rho$

So, $\bar{y} = \frac{8}{15}\cdot\frac{3}{4} = \frac{2}{5}$

Case 2 centroid of general lamina.

Next we develop the centroid of lamina defined by

$a \le x \le b$, $B(x) \le y \le T(x)$.



Again, assume constant density.

The total mass M is

$$M = \int_a^b \rho [T(x) - B(x)]\, dx$$

Moment about y-axis of slice : $\Delta M_y = x[T(x) - B(x)] \rho \, \Delta x$

So, total moment about y-axis:

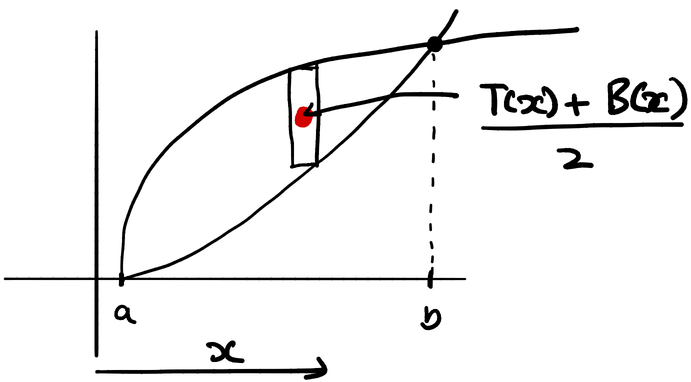
$$M_y = \int_a^b x[T(x) - B(x)] \rho \, dx.$$

Thus, x-coordinate of centroid $\bar{x}$ is:

$$\bar{x} = \left(\int_a^b x\,[T(x) - B(x)]\,dx\right) \Big/ \left(\int_a^b [T(x) - B(x)]\,dx\right)$$

area.

Now, to find moment about x-axis, observe that the center of mass of a slice is $\frac{T(x)+B(x)}{2}$ and we can put all mass of the slice at this point.



So, moment about x-axis of a slice is:

$$\Delta M_x = \left(\frac{T(x) + B(x)}{2}\right)\left(T(x) - B(x)\right)\rho\,\Delta x$$

Summing over all slices:

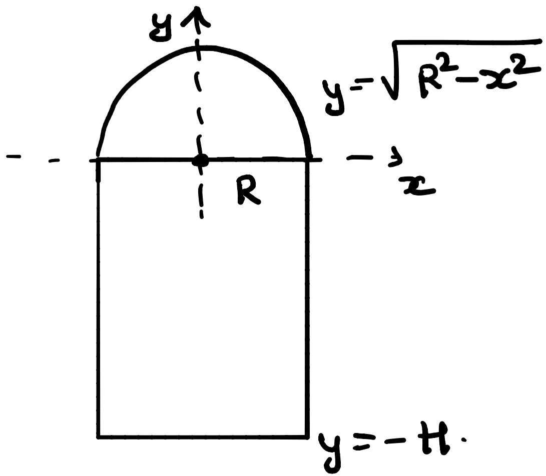
$$\boxed{\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{2}\int_a^b \left(T(x)^2 - B(x)^2\right)dx}{\int_a^b \left(T(x) - B(x)\right)dx}}$$

and recall:

$$\boxed{\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x\left(T(x) - B(x)\right)dx}{\int_a^b \left(T(x) - B(x)\right)dx}}$$

Example

Find the centroid of a region consisting of a rectangle of width $2R$ and height H which has a semicircle of radius R on one end. The picture is



By symmetry $\bar{x}=0$. centroid must lie on the y-axis.

$$\text{Area} = 2RH + \frac{1}{2}\pi R^2.$$

$$T(x) = \sqrt{R^2-x^2}, \quad B(x) = -H.$$

and $$\bar{y} = \frac{1}{2\,\text{Area}} \int_{-R}^{R} \left(T(x)^2 - B(x)^2 \right) dx$$

Example (contd.)

so, $$\bar{y} = \frac{2}{2\,\text{Area}} \int_0^R \left(R^2 - x^2 - H^2\right) dx = \frac{1}{2\,\text{Area}} \left[R^2 x - \frac{x^3}{3} - H^2 x\right]_0^R$$

$$\Rightarrow \bar{y} = \frac{1}{\text{Area}} \left[R^3 - \frac{R^3}{3} - RH^2\right] = \frac{\frac{2R^3}{3} - RH^2}{2RH + \frac{1}{2}\pi R^2} = \frac{4R^3 - 6RH^2}{12RH + 3\pi R^2}$$

so, $$\bar{y} = \frac{4R^2 - 6H^2}{12H + 3\pi R}$$

observe: if $R \ll H$, $\bar{y} \approx \frac{-6H^2}{12H} = -\frac{H}{2}$

if $R \gg H$, $\bar{y} \approx \frac{4R^2}{3\pi R} = \frac{4R}{3\pi}$

Question :

Find the centroid of a semi circle of radius R as shown.