- $$\int_{-a}^{a} \left(x^7 \cos(x) e^{x^2} + 10 \right) dx$$

- $$\int_0^x \frac{t^2 - 4}{1 + x^2} \, dx$$

Application of integration.

Section 2.1

Work. Energy expended acting against a force.

eg: energy expended moving a weight against gravity.

Newton's second law of motion.

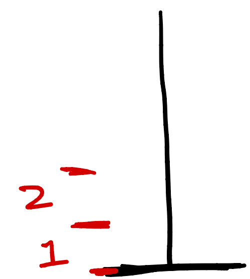
Force = mass × acceleration i.e. $F = m\dfrac{d^2s}{dt^2}$

Work at constant force

Work = Force × displacement $= F \times d$

Work at varying force:

$$W = \int_a^b F(x)\,dx.$$

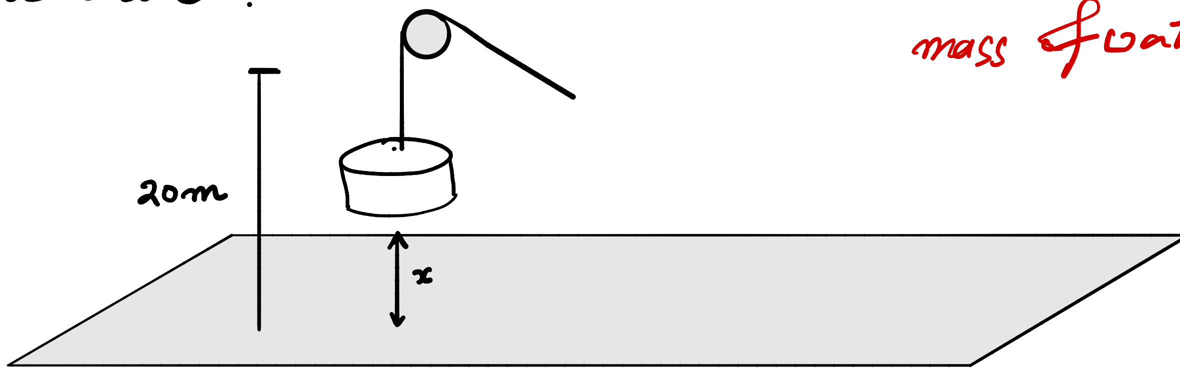


Example

A leaky bucket weighing 5N is lifted 20m into the air at constant speed. The bucket starts with 2N of water and leaks at constant rate. It finished draining just as it reaches the top. How much work was done lifting the water alone?

mass of water $= \dfrac{2N}{9.8\, m/s^2}$

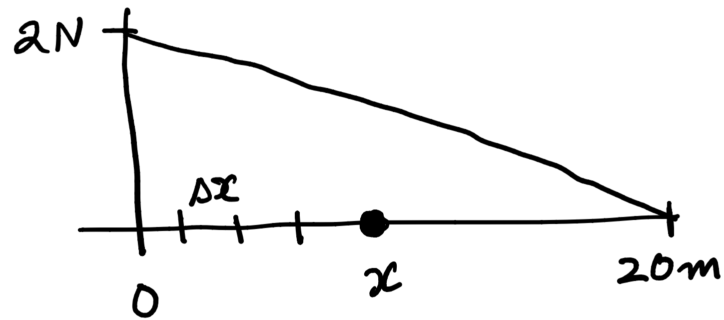
$g \downarrow$



$$W = \int f(x)\, dx$$

Example 1.

At height x, what is the weight of water in bucket.



$$F(x) = -\frac{1}{10}x + 2 \quad N$$

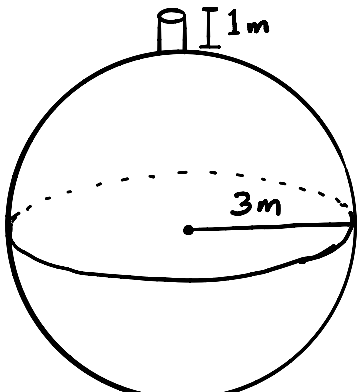
$$W \approx \sum_{i=1}^{n} F(x_i^*) \, \Delta x$$

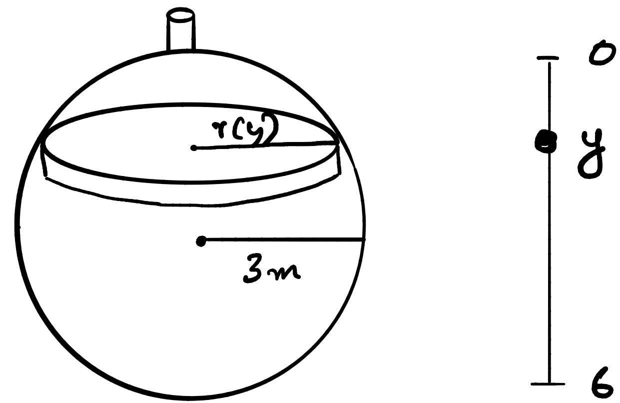
$$W = \int_0^{20} F(x)\,dx = \int_0^{20} \left(-\frac{1}{10}x + 2\right) dx$$

$$= \left[-\frac{x^2}{20} + 2x\right]_0^{20} = \underline{\underline{20\,J.}}$$

Example

A tank of dimension shown (see figure) is initially full of water. The density of water is 1000 kg/m^3. Find the work required to pump water out of the spout.

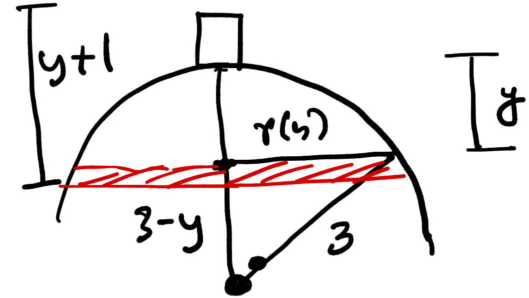




mass of slab of water at depth y.

Example cont'd.

work done to move Δ water at depth y:



$$\Delta V = \pi r^2 \Delta y = \pi(6y-y^2)\Delta y$$

$$\Delta m = \Delta V \cdot 1000 = 1000\pi(6y-y^2)\Delta y$$

$$\Delta F = 9800\pi(6y-y^2)\Delta y$$

$$\Delta w = (y+1)\,9800\pi(6y-y^2)\Delta y.$$

$$3^2 = r^2 + (3-y)^2$$

$$\Rightarrow r = \sqrt{6y-y^2}$$

$$W = \int_0^6 (y+1)\,9800\pi(6y-y^2)\,dy$$

$$= 9800\pi\int_0^6 (6y^2-y^3+6y-y^2)\,dy$$

$$= 9800\pi\left[\frac{5y^3}{3} - \frac{y^4}{4} + \frac{6y^2}{2}\right]_0^6 \approx 4.4\times10^6 \text{ Joules}$$

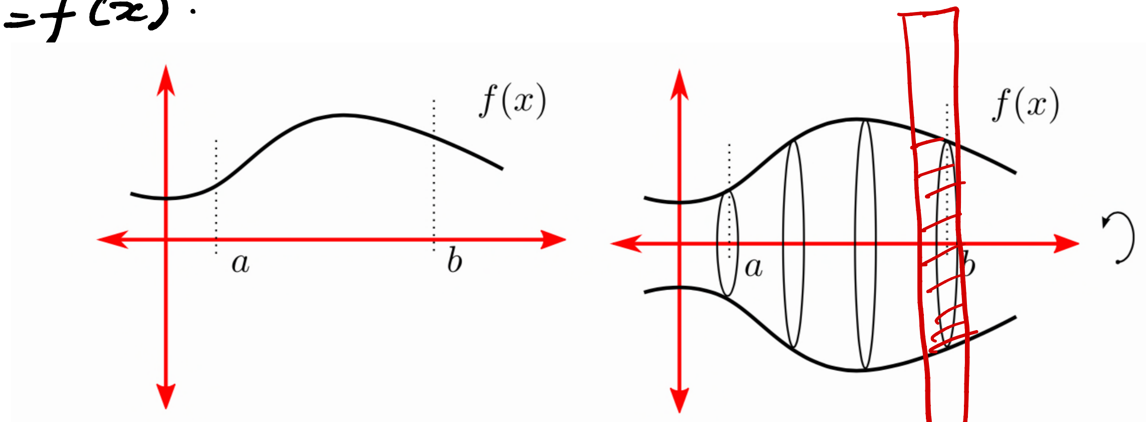
Volume

Section 1.6

Goal: ○ find area enclosed by a 3-D surface.

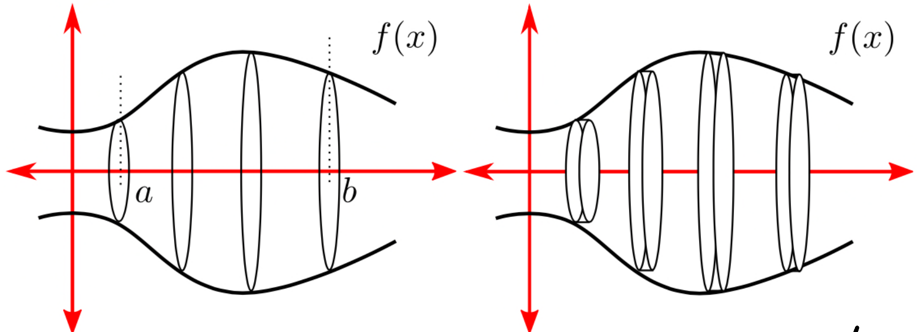
○ Rotate a curve about a line to get the 3-D surface.

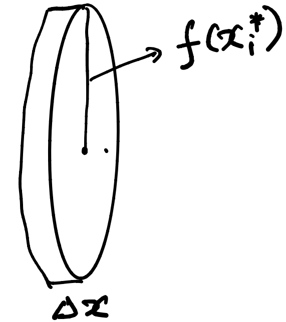
Let $y = f(x)$.



Find the enclosed area between $x = a$ and $x = b$.

Setup – Riemann sum.





- Split $[a,b]$ into $[x_{i-1}, x_i]$ sub-intervals.
- For each $[x_{i-1}, x_i]$, approximate volume with cylinder.

radius of a cylindrical slice $= f(x_i^*)$

width $= \Delta x$.

So, $V_i = \pi f(x_i^*)^2 \Delta x$, $V_i = A(x_i^*) \cdot \Delta x$

Setup - Riemann sum (contd.)

- Riemann sum.

$$V = \sum_{i=1}^{n} \pi \left(f(x_i^*)\right)^2 \Delta x$$

$$= \sum_{i=1}^{n} g(x) \Delta x, \qquad g(x) = \pi\left(f(x_i)\right)^2.$$

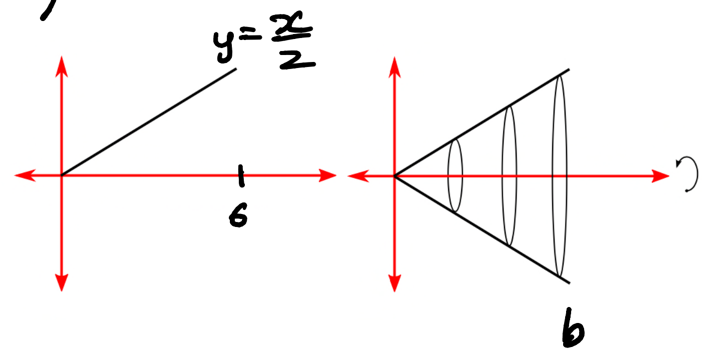
- limit as $n \to \infty$.

$$V = \int_a^b g(x)\,dx$$

more generally: $$V = \int_a^b A(x)\,dx.$$

Example (Volume of cone)

Consider the line $y = \frac{x}{2}$ on $[0, 6]$.
Rotate about x-axis and compute the enclosed volume.



$$V = \int_0^6 \pi \left(\frac{x}{2}\right)^2 dx$$

$$= \int_0^6 \pi \frac{x^2}{4} dx$$

$$= \frac{\pi}{4} \frac{x^3}{3} \Big|_0^6$$

$$= \underline{\underline{18\pi}}$$

more generally:

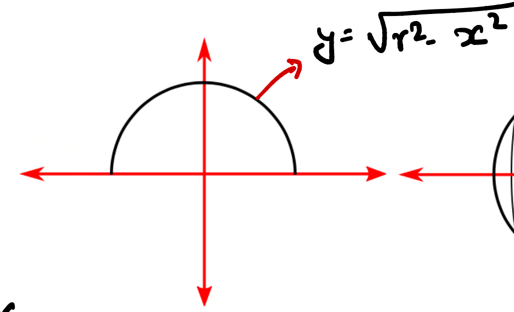
$$V = \int_0^b \pi \frac{x^2}{4} dx$$

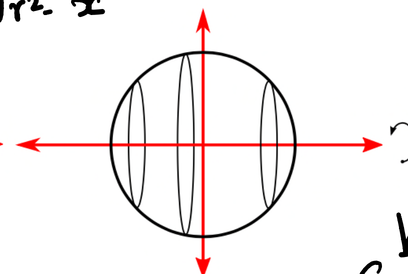
$$= \frac{\pi b^3}{12} = \frac{1}{3} \pi \cdot \left(\frac{b}{2}\right)^2 \cdot b.$$

Example

Let $y = \sqrt{r^2 - x^2}$ (semi circle of radius r)

Find the enclosed volume (rotate about x-axis).





$$\pi \left(f(x_i^*)\right)^2 \Delta x$$

$$\int_a^b \pi \left(f(x)\right)^2 dx$$

$$\pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$$

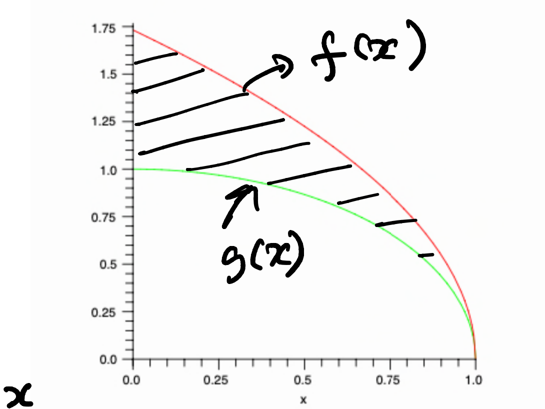
$$\pi \int_{-r}^{r} [r^2 - x^2]\, dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

$$= \pi \left[r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3} \right] = \frac{4}{3} \pi r^3$$

Example (volume of bowl).

Let $f(x) = \sqrt{3-3x}$

$g(x) = \sqrt{1-x^2}$



Find the volume of the bowl.

$$V_1 = \pi\int_0^1 f(x)^2\,dx \quad , \quad V_2 = \pi\int_0^1 g(x)^2\,dx$$

$$V = \pi \int_0^1 \left(f(x)^2 - g(x)^2\right)dx$$

like area between the curves.

$$= \pi \int_0^1 \left((3-3x) - (1-x^2)\right)dx$$

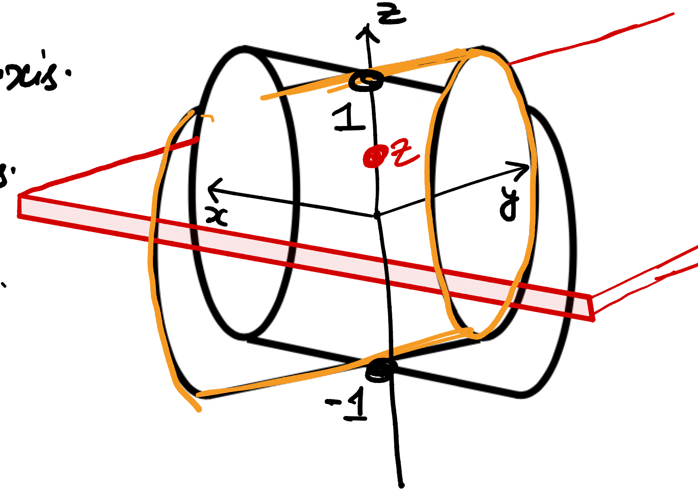
$$= \pi \left[3x - \frac{3x^2}{2} - x + \frac{x^3}{3} \right]_0^1 = \frac{5}{6}\pi$$

Example 4

Find volume of intersection of 2 perpendicular cones.

cylinder 1: $x^2+z^2 \leq 1 \rightarrow$ about y-axis.

cylinder 2: $y^2+z^2 \leq 1 \rightarrow$ about x-axis.



For a fixed z:

cylinder 1: $x^2 \leq 1-z^2$

$$\Rightarrow -\sqrt{1-z^2} \leq x \leq \sqrt{1-z^2}$$

cylinder 2: $y^2 \leq 1-z^2$ $\quad -a \leq \leq y \leq a$

$$-\sqrt{1-z^2} \leq y \leq \sqrt{1-z^2}$$

For a fixed z, we get a square of length $2\sqrt{1-z^2}$

Example 4 (contd.)

Volume of a slab: $\left(2\sqrt{1-z^2}\right)^2 \cdot \Delta z$

$$V = \int_{-1}^{1} \left(2\sqrt{1-z^2}\right)^2 dz$$

$$= 4\int_{-1}^{1} (1-z^2)\, dz$$

$$= 4\left[z - \frac{z^3}{3}\right]_{-1}^{1}$$

$$= \underline{\underline{\frac{16}{3}}}$$

$y^2 \leq a$, $a \geq 0$.

$\Rightarrow \quad y \in [-a, a]$.