Fundamental Theorem of Calculus:

Goal:

- FTC part 1 and part 2.

  "differentiating undoes integrating".

  "integrating undoes differentiation".

- Integrating without Riemann sum.

Some definite integrals.

- $\int_0^1 x\,dx = \frac{1}{2} = \frac{x^2}{2}\Big|_{x=1} - \frac{x^2}{2}\Big|_{x=0} = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$
- $\int_0^4 (x^2-3x)\,dx = -\frac{8}{3} = \left(\frac{x^3}{3} - \frac{3x^2}{2}\right)\Big|_{x=4} - \left(\frac{x^3}{3} - \frac{3x^2}{2}\right)\Big|_{x=0} = \frac{64}{3} - \frac{52}{2} = -\frac{8}{3}$
- $\int_0^1 \frac{x\,e^{x^2}}{2}\,dx = e^{x^2}\Big|_{x=1} - e^{x^2}\Big|_{x=0}$

Remark: $\int_a^b f(x)\,dx = F(b) - F(a)$

notation: $F(b) - F(a) := F(x)\Big|_{x=a}^{b}$

$$= F(x)\Big|_a^b$$

$$= \left[F(x)\right]_a^b$$

Fundamental theorem of calculus

Recall for any continous function f on $[a, b]$

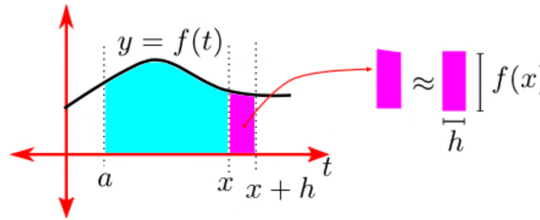
$$F(x) := \int_a^x f(t)\,dt \quad \text{is a function of } x. \quad \text{— (1)}$$

Thm (Fundamental Theorem of Calculus Part 1, CLP 1.3.1)

Let $F(x)$ be as defined in (1). Then

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

FTC Part 1



$$F(x) := \int_a^x f(t)\,dt.$$

Show: $\frac{d}{dx} F(x) = f(x)$.

Sketch: ∘ $$\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right) / h.$$

$$= \lim_{h \to 0} \left(\int_x^{x+h} f(t)\,dt \right) / h.$$

$$= \lim_{h \to 0} \left(f(x) \cdot h \right) / h = f(x).$$

Examples

- $F(x) = \int_0^x \cos(t)\,dt \Rightarrow \frac{d}{dx} F(x) = \cos(x).$

- $F(x) = \int_1^x (t^2 + \sqrt{t+1})\,dt \Rightarrow \frac{d}{dx} F(x) = x^2 + \sqrt{x+1}$

- $F(x) = \int_1^{x^2} \cos(t)\,dt.$ $\quad F(u(x))$ where $F(u) = \int_1^u \cos(t)\,dt.$

$F(u(x))$

if $\underbrace{u(x) = x^2}$

$$\frac{d}{dx} F(u(x)) = \frac{dF}{du} \cdot \frac{du}{dx}$$

$$= \cos(u) \cdot 2x$$

$$= \cos(x^2) \cdot 2x.$$

$$\frac{d}{dx} F(x) = f(x)$$

$$F(x) = \int_a^x f(t)\,dt.$$

Antiderivative

Def^n: Given any function f, then any function F such that $F'(x) = f(x)$ is called antiderivative of $f(x)$.

Example

- $F(x) = x^2$ is antiderivative of $f(x) = 2x$.
- $F(x) = \cos(x)$ is antiderivative of $f(x) = \sin(x)$
- $F(x) = \int_0^x \cos(t)\,dt$ is antiderivative of $f(x) = \cos(x)$.

$\int_0^x f(t)\,dt$ is an antiderivative of $f(x)$.

Is antiderivative unique?

Antiderivative is not unique.

For example: x^2 and x^2+3 are antiderivatives of $2x$.

Lemma (CLP 1.3.8) If F is an anti-derivative of f, then any other antiderivative is of f is of the form $F(x)+C$, where c is a constant.

Remark:

$\int_a^x f(t)\,dt + C$ is anti-derivative of $f(x)$ by FTC-1.

Proof of Lemma

- Suppose G is also an anti derivative of f.
  F

- Consider $H = F - G$

- By def^n, $F'(x) = f(x)$, $G'(x) = f(x)$

  $\Rightarrow H'(x) = 0 \quad \forall\, x \in$ Domain.

- So, $H(x) = c$, c is a constant.

- $G(x) = F(x) + c$.

Indefinite Integrals

Definition: The indefinite integral of $f(x)$ is denoted $\int f(x)dx$.
(without terminals)

Definition: $\int f(x)\,dx$ is the general antiderivative of $f(x)$.
In particular, $F(x)$ is the antiderivative of $f(x)$
then

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitary constant. (constant of integration)

$\int_a^x f(t)\,dt + C$ is antiderivative of $f(x)$

Examples

$f(x)$	$F(x) = \int f(x)\mathrm{d}x$
1	$x + C$
x^n	$\frac{1}{n+1}x^{n+1} + C$ provided $n \neq -1$
$\frac{1}{x}$	$\log\lvert x\rvert + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\frac{1}{1+x^2}$	$\arctan x + C$

Q: $\int \sin(e^x)\, e^x\, dx = ?$

Note that $\frac{d}{dx}\cos(e^x) = \sin(e^x)\, e^x$

so, $\int \sin(e^x)\, e^x\, dx = \cos(e^x) + C.$

Fundamental theorem of Calculus - Part 2

Thm (CLP 1.3.1) Let f be any continous function on $[a,b]$. Let F be antiderivitive of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

ex: $\displaystyle\int_1^2 \frac{1}{x}\,dx$

The antiderivative of $\frac{1}{x}$ is $\ln(x)$: $\displaystyle\int \frac{1}{x}\,dx = \ln(x) + c$.

So, $\displaystyle\int_1^2 \frac{1}{x}\,dx = \ln(x)\Big|_{x=1}^{2} = \ln(2)$

ex:

Evaluate $\int_{-\frac{\pi}{2}}^{\pi/2} \cos(x)\,dx$.

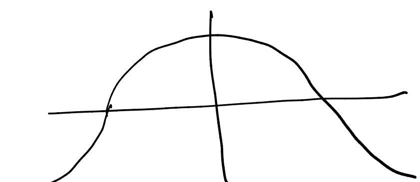
note: $\int \cos(x)\,dx = \sin(x) + C$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \cos(x)\,dx$$

$$= \sin(x)\Big|_{-\pi/2}^{\pi/2}$$

$$= \sin(\pi/2) - \sin(-\pi/2)$$

$$= 2$$



or, since $\cos(x)$ is even function.

$$2\int_0^{\pi/2} \cos(x)\,dx.$$

$$= 2\left(\sin(x)\Big|_0^{\pi/2}\right)$$

$$= 2\left(\sin(\pi/2) - \sin(0)\right)$$

$$= 2.$$

Proof of FTC-2

- $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. by FTC-1

- So, $\int_a^x f(t)\,dt = F(x) + c$ (by definition)

- For $x = a$, $\int_a^a f(t)\,dt = F(a) + c \Rightarrow c = -F(a)$.

- For $x = b$, $\int_a^b f(t)\,dt = F(b) - F(a)$.

Inverse operations.

A function $f : \mathbb{R} \to \mathbb{R}$ is inverse of function $g : \mathbb{R} \to \mathbb{R}$ if they satisfy :

$$f(x) = y \iff g(y) = x \qquad \forall\ x, y \in \mathbb{R}.$$

Remark : $f(g(y)) = y$, $g(f(x)) = x$.

- Differentiating undoes integration : $\dfrac{d}{dx} \displaystyle\int_a^x f(t)\, dt = f(x)$
- Integrating undoes differentiation: $\displaystyle\int_a^x \left(\frac{d}{dt} F(t) \right) dt = F(t) \Big|_a^x$

Examples

- $\int_0^1 3x^2+2x+1 \, dx$
- Antiderivative : x^3+x^2+x

$$\int_0^1 3x^2+2x+1 \, dx = \left[x^3+x^2+x\right]_{x=0}^{1}$$

$$= 3 - 0$$

$$= 3$$

Antiderivative : x^3+x^2+x+C

$$\int_0^1 (3x^3+2x+1)dx$$

$$= \left[x^3+x^2+x+C\right]_{x=0}^{1}$$

$$= (3+C) - (0+C)$$

$$= 3$$

Example (contd)

$$\int_{-1}^{1} \frac{1}{x^4}\, dx$$

- Anti derivative of $\frac{1}{x^4}$ is $\frac{x^{-3}}{-3}$

So, $\displaystyle\int_{-1}^{1} \frac{1}{x^4} = \frac{x^{-3}}{-3} \Bigg|_{-1}^{1} = \frac{1}{-3} - \left(-\frac{1}{-3}\right) = -\frac{2}{3}$?