hw1_soln

January 27, 2020

1 Homwork 1 solution

1.

1.1 Backsolve

1.

$$x_1 = b_1/R_{11}, \quad x_{i+1} = \frac{b_{i+1} - \sum_{k=1}^{i} R_{i+1,k}x_k}{R_{i+1,i+1}}.$$

2.

$$x_n = b_n/R_{nn}, \quad x_i = \frac{b_i - \sum_{k=i+1}^{n} R_{i,k}x_k}{R_{i,i}}.$$

2.

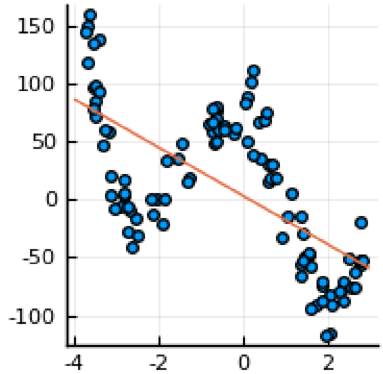
1.2 Linear Data fit

[3]:
```
using JLD
using LinearAlgebra
using Polynomials
using Plots
using BenchmarkTools
using Statistics
```

[22]:
```
D = load("../hw1_p2_data.jld")["data"]
m = size(D)[1]

A = [ones(m,1) D[:,1]]
b = D[:,2]
xls = A\b
p1 = Poly(xls)
xlim = (-4,3)

r = A*xls-b
print("The norm of residual is: ", norm(r,2))

pyplot(size=(200,200), legend=false)
plot1 = scatter(D[:,1],D[:,2])
plot!(t->p1(t), xlim...)
```

```
The norm of residual is: 498.56421135979997
```

[22]:



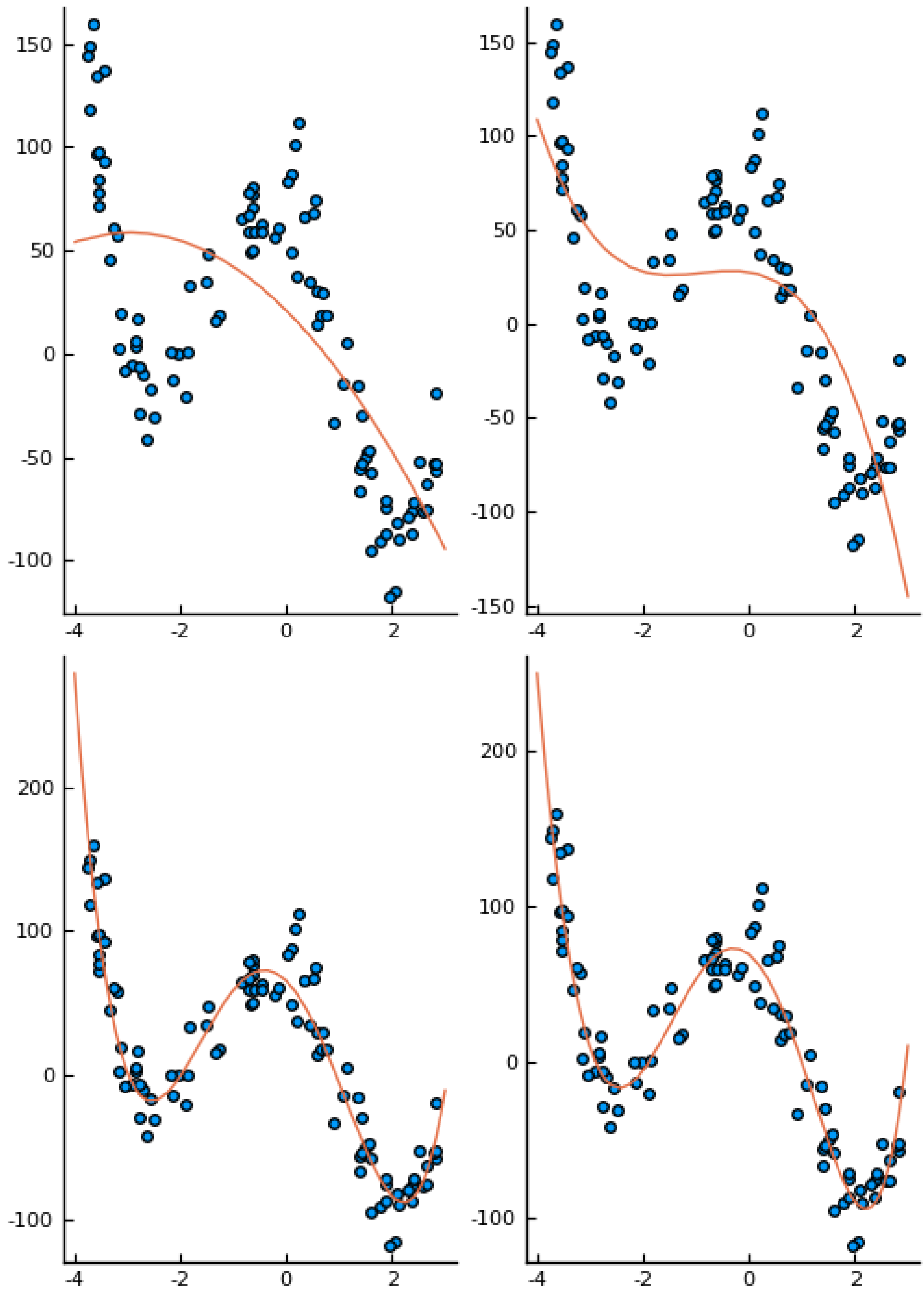
3.

1.3 Polynomial data fit

[44]:
```
plot_list = []
for d in 1:5
    pp = polyfit(D[:,1], D[:,2], d)
    plot_t = scatter(D[:,1],D[:,2])
    plot!(t->pp(t),xlim..., grid = false)
    plot_list = push!(plot_list,plot_t)
    r= pp(D[:,1])-b;
    print("The norm of residual for d = $(d) is: $(norm(r,2))\n")
end
plot(plot_list[2],plot_list[3],plot_list[4],plot_list[5], layout=4)
```

```
The norm of residual for d = 1 is: 498.56421135979997
The norm of residual for d = 2 is: 473.93158732071345
The norm of residual for d = 3 is: 439.1465751343405
The norm of residual for d = 4 is: 194.79584942144604
The norm of residual for d = 5 is: 189.05547456952073
```

[44]:



4.

1.4 Least norm solution

1. There are two ways we can factor A. If we factor as $A = QR$, we get something like

$$\underbrace{\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \end{bmatrix}}_{R}$$

and we can solve

$$x = R^{-1}Q^Tb.$$

The key advantage is that Q is only $m \times m$, and R is the same storage as A, so the only increase in storage is m^2. But, inverting R is tricky, as it is not exactly triangular.

We can also factor $A^T = QR$ to get something like

$$\underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{A^T} = \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}}_{R}.$$

Overall, we will solve this system in two steps:

$$R^Tz = b, \quad Q^Tx = z.$$

The first step is now much easier. When R is wide, it is tricky to figure out how to invert it. But when R is square, it is easy to ``invert" through backsolving.

The second step is tricky, because Q^T is wide, and not easily left invertible. In fact, there are many solutions for x. One possible solution is $x = QQ^Tz$, which is the least squares solution, and perhaps easiest to compute in this context.

In this regime, the solve system $z = R^{-T}b$ takes $O(m^2)$ flops, and $x = Qz$ requires $O(mn)$ flops, for a total of $O(mn + m^2)$ flops for the solve, and an extra $O(nm^2)$ flops for the original QR factorization.

B. Here's a code for timing

[5]:
```
function qrtest(A,b)
    F = qr(A')
    return F.Q*(F.R'\b)
end
```

```julia
m = [10, 100, 100, 100, 100]
n = [20, 200, 2000, 20000, 200000]

bench_times = []

for k in 1:5
    println("Benchmarking for m = $(m[k]) and n = $(n[k])")

    A = randn(m[k],n[k])
    x = randn(n[k])
    b = A*x

    t_qr = @benchmark qrtest($A,$b) samples = 10 seconds = 50

    t_bs = @benchmark $A\$b samples = 10 seconds = 50

    push!(bench_times, (time_qr = t_qr, time_bs = t_bs))
    println(bench_times[k])
end
```

```
Benchmarking for m = 10 and n = 20
(time_qr = Trial(16.043  s), time_bs = Trial(18.324  s))
Benchmarking for m = 100 and n = 200
(time_qr = Trial(839.127  s), time_bs = Trial(1.689 ms))
Benchmarking for m = 100 and n = 2000
(time_qr = Trial(3.882 ms), time_bs = Trial(17.728 ms))
Benchmarking for m = 100 and n = 20000
(time_qr = Trial(39.769 ms), time_bs = Trial(260.832 ms))
Benchmarking for m = 100 and n = 200000
(time_qr = Trial(665.707 ms), time_bs = Trial(5.214 s))
```

C. Let x_{ln} be the least norm solution to

$$\min_{x \in \mathbb{R}^n} \|x\|_x^2 \ \text{s.t.} \ \ Ax = b.$$

The solution x_{ln} can be decomposed as $x_{ln} = x_1 + x_2$, where $x_1 \in \mathcal{R}(A^T)$, $x_2 \in \mathcal{N}(A)$ and $ x_1^Tx_2 = 0$. Since $Ax_2 = 0$, the component of x_{ln} along $\mathcal{N}(A)$ must be zero. So, $x_{ln} \in \mathcal{R}(A^T)$.

Consider the QR decomposition of $A^T = QR$, where

$$Q = \begin{bmatrix} \hat{Q} & \bar{Q} \end{bmatrix}, \text{ and } R = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}.$$

Here $\hat{Q} \in \mathbb{R}^{n \times m}$, $\bar{Q} \in \mathbb{R}^{n \times (n-m)}$ and $R \in \mathbb{R}^{m \times m}$. Additionally, we have $\mathcal{R}(A^T) = \mathcal{R}(\hat{Q})$.

Since, $x_{ln} \in \mathcal{R}(A^T)$, $\mathcal{R}(A^T) = \mathcal{R}(\hat{Q})$, and $\mathcal{N}(A) = \mathcal{R}(\bar{Q})$, we have $\bar{Q}^T x_{ln} = 0$. Also, since x_{ln} satisfies the equation $Ax = b$, we have

$$
\begin{aligned}
&(QR)^T x_{ln} = b \\
\Rightarrow &\hat{R}^T \hat{Q}^T x_{ln} = b
\end{aligned}
$$

.

`Note that` x_{ln} `that satisfies` $\hat{R}^T\hat{Q}^T x_{ln} = b$ `and` $\bar{Q}^T x_{ln} = 0$ `is` $x_{ln} = \hat{Q}\hat{R}^{-T}b$.

[ ]:

```
```