11 Infinite sequences and series

11.6 More convergence tests

So the last series we looked at

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is an interesting case — its convergence is quite delicate. If we sum the absolute values of the terms then we get the harmonic series which diverges. On the other hand if we consider

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n} = \frac{1/2}{1-(-1/2)} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = 1$$

they both converge. The convergence is somehow more "robust"? This leads to 2 definitions

Definition. Consider the series $\sum a_n$.

- If the series $\sum |a_n|$ converges then we say that $\sum a_n$ is absoluely convergent.
- If the series $\sum |a_n|$ diverges, but $\sum a_n$ converges then we say that $\sum a_n$ is conditionally convergent.

Being absolutely convergent is a very nice property (for lots of reasons) — perhaps foremost is

Theorem. *If a series is absoluely convergent, then it is convergent.*

Conditionally convergent series are much much more delicate things and have many strange and weird properties — indeed there is a very famous result due to Riemann that says that the sum of a conditionally convergent series depends on the order in which you sum the terms!

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots = \log 2$$
$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots = \frac{1}{2} \log 2$$

This stuff is very strange and way beyond the scope of this course. Suffice to say — absolute convergence = good, conditional convergence = dodgy.

The following is perhaps the most useful convergence test for series — roughly speaking it is an extension what we saw for the sum of geometric series

$\sum r^n$ convergences when $|r| < 1$ and otherwise diverges.

Theorem (ratio test)**.** *Consider the series $\sum a_n$ and let*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \qquad \textit{limit of ratios}$$

Then

- *If $L < 1$ the series is absolutely convergent*
- *If $L > 1$ the series diverges*
- *If $L = 1$ then the test is inconclusive — apply another test.*

So lets apply this to a few things

- $\sum \frac{(-1)^n(n^2+3)}{7^n}$. Ratio is

$$\frac{(-1)^n(n^2+3)7^{n+1}}{(n^2+2n+4)7^n} = \frac{(-1)(n^2+3)}{(n^2+2n+4)7} \to \frac{-1}{7}$$