

## Partial fractions.

Goal: Integrate rational functions.

Definition: A function  $f(x)$  is called a rational function if and only if it can be written in the form:

$$f(x) = \frac{P(x)}{Q(x)}, \quad P(x) - \text{polynomial}$$
$$Q(x) - \text{polynomial}.$$

Rational function is ratio of polynomials.

ex:

$$\frac{1}{x+a}$$

$$\frac{1}{x^2 + a^2}$$

Rational functions. (Why is it easy to integrate).

Rational function = sum of simpler functions.

Consider  $\frac{1}{x-1} + \frac{2}{x+3} = \frac{1(x+3) + 2(x-1)}{(x-1)(x+3)}$

$$= \frac{3x+2}{x^2+2x-3}$$

So,  $\int \frac{3x+2}{x^2+2x-3} dx = \int \frac{1}{x-1} dx + \int \frac{2}{x+3} dx$

$$= \ln|x-1| + 2\ln|x+3| + C$$

Easier to integrate:

$$\frac{A}{x+a}, \quad \frac{A}{x^2+ax+b}, \quad \frac{Ax}{x^2+ax+b}$$

## Structure of the decomposition.

$$\frac{A}{x-a} + \frac{B}{x-b} = \frac{A(x-b) + B(x-a)}{(x-a)(x-b)} = \frac{x(A+B) - (Ab + Ba)}{(x-a)(x-b)}.$$

- Decomposition depends on the structure of denominator.
- "Easier to integrate" functions has numerator with lower degree than denominator.

in  $\frac{P(x)}{Q(x)}$ , we have  $\deg(P) < \deg(Q)$ .

## Steps of partial fraction.

Let  $f(x) = P(x)/Q(x)$ .

1. If  $\deg(P) \geq \deg(Q)$ , then rewrite as a proper fraction:  $\frac{P}{Q} = S + \frac{R}{Q}$  with  $\deg(R) < \deg(Q)$ .  
that is do polynomial division.

e.g.:  $\frac{x^3+x}{x+1} = x^2 + x + 2 + \frac{2}{x-1}$

2. Factor the denominator into linear  $(x-a)$  and irreducible quadratic  $(x^2+bx+c)$  factors.

Fundamental theorem of Algebra.

## Steps of partial fraction.

3. Split the ratio into simpler pieces.

$$\frac{A}{(x-a)^n} \quad \frac{Bx + C}{(x^2 + bx + c)^m}$$

where  $(x-a)$  and  $(x^2 + bx + c)$  are factors of the denominator.

4. Find constants

5. Integrate term by term.

## Polynomial division.

Practice polynomial division.

e.g.:  $\frac{x^3 + x}{x - 1}$

$$\begin{array}{r} x^2 + x + 2 \\ \hline x-1 ) x^3 + x \end{array}$$

$$\begin{array}{r} - x^3 + x^2 \\ \hline x^2 + x \end{array}$$

$$\begin{array}{r} - x^2 + x \\ \hline \end{array}$$

$$\begin{array}{r} 2x \\ - 2x + 2 \\ \hline 2 \end{array}$$

$$\begin{aligned} (x^3 + x) &= (x-1)(ax^2 + bx + c) + d \\ &= ax^3 + x^2(b-a) + x(c-b) \\ &\quad + d - c \end{aligned}$$

$$\text{so, } a = 1, b = 1, c = 2, d = 2$$

$$\text{so, } \frac{x^3 + x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$$

$$\text{so, } \text{ans} = x^2 + x + 2 + \frac{2}{x-1}$$

## Step 3: How to split up the fraction?

denominator factor	partial fraction expansion	covered in this course
$(x - a)$	$\frac{A}{x-a}$	✓
$(x - a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r}$	✓
$(x^2 + bx + c)$	$\frac{Bx+C}{x^2+bx+c}$	✓
$(x^2 + bx + c)^r$	$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \frac{B_3x+C_3}{(x^2+bx+c)^3} + \cdots$	✗ (phew)

For example:

$$\begin{aligned} & \frac{P(x)}{(x-a)(x-b)^2(x^2-cx+d)} \\ = & \frac{A}{x-a} + \frac{B}{(x-b)} + \frac{C}{(x-b)^2} + \frac{Dx+E}{(x^2-cx+d)} \end{aligned}$$

### Example 1

Solve  $\int \frac{3x+1}{x^2+2x-3} dx.$

- No need for step 1.
- Factorization:  $x^2+2x-3 = (x+3)(x-1)$ . (Read A.16)  
If  $p(a) = 0$  then  $p(x) = Q(x)(x-a)$ .
- Split:  $\frac{3x+1}{x^2+2x-3} = \frac{A}{x+3} + \frac{B}{x-1}$

## Example 1 (contd.)

• Find constants:

$$\begin{aligned} 1. \frac{3x+1}{(x+3)(x-1)} &= \frac{A(x-1) + B(x+3)}{(x+3)(x-1)} \\ &= \frac{x(A+B) + (-A+3B)}{(x+3)(x-1)} \end{aligned}$$

$$\text{so, } A+B = 3 \quad \text{and} \quad -A+3B = 1$$

$$\Rightarrow A = 2, \quad B = 1.$$

$$2. \frac{3x+1}{(x+3)(x-1)} = \frac{A(x-1) + B(x+3)}{(x+3)(x-1)}$$

Comparing numerators: if  $x = 1, 4B = 4 \Rightarrow B = 1$   
if  $x = -3, -4A = -8 \Rightarrow A = 2$ .

### Example 1 (Contd.)

o Integrate terms:

$$\begin{aligned} \text{so, } \int \frac{3x+1}{x^2+2x-3} dx &= \int \left( \frac{2}{x+3} + \frac{1}{x-1} \right) dx \\ &= \int \frac{2}{x+3} dx + \int \frac{1}{x-1} dx \\ &= 2 \ln|x+3| + \ln|x-1| + C \end{aligned}$$

## Example 2.

$$\int \frac{x^2 - 9x + 17}{(x-2)^2(x+1)} dx.$$

- Polynomial division step done.
- Factorization step done.
- Split:

$$\begin{aligned}\frac{x^2 - 9x + 17}{(x-2)^2(x+1)} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1} \\ &= \frac{A(x-2)(x+1) + B(x+1) + C(x-2)^2}{(x-2)^2(x+1)} \\ &= \frac{x^2(A+C) + x(-A+B-4C) + (-2A+B+4C)}{(x-2)^2(x+1)}\end{aligned}$$

## Example 2 (contd)

$$\text{So, } A+C=1, \quad -A+B-4C=-9, \quad -2A+B+4C=17$$
$$\Rightarrow A=-2, \quad B=1, \quad C=3.$$

Alternatively:

$$\frac{x^2-9x+17}{(x-2)^2(x+1)} = \frac{A(x-2)(x+1) + B(x+1) + C(x-2)^2}{(x-2)^2(x+1)}$$

$$\text{pick } x=2, \quad 3B=3 \Rightarrow B=1$$

$$\text{pick } x=-1, \quad 9C=27 \Rightarrow C=3$$

$$\text{pick } x=0, \quad 17=-2A+B+4C \Rightarrow A=-2$$

### Example 3 (contd.)

◦ Integrations.

$$\begin{aligned}\int \frac{x^2 - 9x + 17}{(x-2)^2(x+1)} dx &= \int \frac{-2}{x-2} dx + \int \frac{1}{(x-2)^2} dx + \int \frac{3}{x+1} dx \\&= -2 \ln|x-2| - (x-2)^{-1} + 3 \ln|x+1| + C \\&= \ln \left| \frac{(x+1)^3}{(x-2)^2} \right| - (x-2)^{-1} + C\end{aligned}$$

≡

### Example 3.

$$\int \frac{x+1}{x^2-2x+5} dx$$

- complete square denominator:

$$x^2 - 2x + 5 = (x-1)^2 + 4$$

- let  $u = x-1 \Rightarrow du = dx$ .

$$\begin{aligned} \text{so, } \int \frac{x+1}{x^2-2x+5} dx &= \int \frac{u+2}{u^2+4} du \\ &= \int \frac{u}{u^2+4} du + \int \frac{2}{u^2+4} du \\ &= \frac{1}{2} \ln|u^2+4| + 2 \frac{1}{2} \arctan\left(\frac{1}{2}u\right) + C \\ &= \frac{1}{2} \ln|(x-1)^2+4| + \arctan((x-1)/2) + C \end{aligned}$$

