Integration by parts. (review).

Let $u(x)$ and $v(x)$ be differentiable function.

$$\boxed{\int u(x)\,\frac{dv}{dx}\,dx = u(x)\cdot v(x) - \int v(x)\,\frac{du}{dx}\,dx.}$$

more concisely:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Integration by parts - definite integrals.

$$\int_a^b u(x)\frac{dv}{dx}\,dx = \underbrace{u(x)\cdot v(x)}_{?} - \int_a^b v(x)\frac{du}{dx}\,dx.$$

Recall product rule:

$$\frac{d}{dx}(u(x)\cdot v(x)) = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$$

$$\Rightarrow \int_a^b \frac{d}{dx}(u(x)\cdot v(x))\,dx = \int_a^b u(x)\frac{dv}{dx}\,dx + \int_a^b v(x)\frac{du}{dx}\,dx.$$

$$\Rightarrow \int_a^b u(x)\frac{dV}{dx}\,dx = u(x)\cdot v(x)\Big|_a^b - \int_a^b v(x)\frac{du}{dx}\,dx$$

$$\int_a^b \frac{d}{dx}(u(x)\cdot v(x))\,dx$$

Example

Evaluate $I_n := \int_0^\infty x^n e^{-x}\,dx$, n is a integer.

$u(x) = x^n$ $\quad v'(x) = e^{-x}$ $\quad \int x^2 e^x\,dx$

$u'(x) = n\,x^{n-1}$ $\quad v(x) = -e^{-x}$

$$I_n = -x^n e^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})\,n x^{n-1}\,dx$$

$$= \left[0 - 0 + \int_0^\infty n e^{-x} x^{n-1}\,dx\right]$$

$$= n\int_0^\infty x^{n-1} e^{-x}\,dx$$

$= n\,I_{n-1}$ $\Rightarrow$ recursive relation!

$x^n e^{-x}\Big|_0^\infty$

$\lim_{x\to\infty} x^n e^{-x} - 0^n \cdot e^{-0}$

$= 0 - 0$

$\lim_{x\to\infty} \frac{x^n}{e^x}$

Example contd.

$$I_0 = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = \left[0-(-1)\right] = 1$$

$$I_1 = 1\cdot I_0 = 1\cdot 1$$

$$I_n = n\, I_{n-1}$$

$$I_2 = 2\cdot I_1 = 2\cdot 1$$

$$I_3 = 3\cdot I_2 = 3\cdot 2\cdot 1.$$

$$f(x)\Big|_a^b = F(b) - F(a)$$

$$\vdots$$

$$I_n = n! \quad (\text{mathematic induction})$$

try: Solve $\underbrace{\int_0^{2\pi} x^2 \cos(x)\,dx}_{I}$.

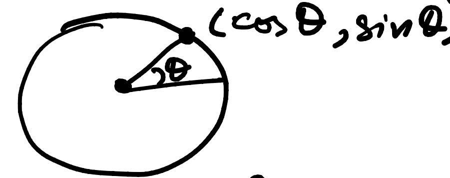
$$I = x^2 \sin x \Big|_0^{2\pi} - \int_0^{2\pi} 2x \sin x\,dx$$

$u(x) = x^2$ $\qquad v'(x) = \cos(x)$

$u'(x) = 2x$ $\qquad v = \sin(x)$

$$= \left((2\pi)^2 \sin 2\pi - 0^2 \sin 0\right) - \int$$

Trigonometric integrals.



$$x^2+y^2=1$$

$$\int \sin^a x \cos^b x \, dx$$

$$\int \tan^a x \sec^b x \, dx.$$

Rewrite higher power of trig functions into lower power

trig identities.

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

sum of angles.

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b.$$

double angle

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin 2x = 2 \sin x \cos x.$$

Example. (Integrating even power of sin)

Simplify $\sin^4 x$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$\Rightarrow 1 - \cos 2x = 2\sin^2 x$$

$$\Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

so, $\sin^4(x) = \frac{1}{4}(1 - \cos 2x)^2$

$$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right)$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x$$

$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.$$

power of sin is even.

$$\int \sin^4 x \, dx$$

$$= \left[\frac{3}{8}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x + \frac{1}{8}\cdot\frac{1}{4}\sin 4x\right] + C$$

Example (Integrating odd power)

Solve $\int \cos^5 x \, dx$

$$= \int \cos x \cos^4 x \, dx$$

$$= \int \cos x \, (1 - \sin^2 x)^2 \, dx$$

let $u(x) = \sin x$

$$\Rightarrow \frac{du}{dx} = \cos x \qquad dx \to \frac{du}{\cos x}$$

Note that power is odd.

so, we get

$$\int (1 - u^2)^2 \, du \Big|_{u = \sin x} = \int 1 - 2u^2 + u^4 \, du \Big|_{\sin x}$$

$$= \left[u - \frac{2u^3}{3} + \frac{u^5}{5} \right]_{u = \sin(x)} + C$$

so,

$$\int \cos^5 x \, dx$$

$$= \sin x - \sin^2 x + \frac{\sin^3 x}{3} + C$$

Algorithm for integrating $\sin^a x \cos^b x$.

if b is odd then there exist a $k \in \mathbb{N}$ s.t. $2k+1=b$

Thm To integrate $\int \sin^a x \cos^b x \, dx$.

- If b is odd, hold 1 power of cosine and turn all other into sine using $\cos^2 x = 1 - \sin^2 x$.

$$\int \sin^a x \cos^{2k+1} x \, dx = \int \sin^a x \left(\cos^2 x\right)^k \cos x \, dx$$

$$= \int \sin^a x \left(1 - \sin^2 x\right)^k \cos x \, dx.$$

- If a is odd, hold 1 power of sine and turn all other into cosine using $\sin^2 x = 1 - \cos^2 x$.

$$\int \sin^{2k+1} x \cos^b x \, dx = \int \left(\sin^2 x\right)^k \sin x \cos^b x \, dx$$

$$= \int \left(1 - \cos^2 x\right)^k \cos^b x \sin x \, dx.$$

Algorithm for integrating $\sin^a x \cos^b x$. (contd)

∘ If a and b are even then use

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) \qquad \sin^2 x = \frac{1}{2}(1-\cos 2x).$$

sine and cosine are derivatives of each other

try: $\int \cos^2 x \sin^5 x \, dx.$

$$= \int \cos^2 x \sin^4 x \sin x \, dx$$

$$= \int \cos^2 x (1-\cos^2 x)^2 \sin x \, dx$$

let $u(x) = \cos x \Rightarrow \frac{du}{dx} = -\sin x$

$$= -\int u^2 (1-u^2)^2 \, du$$

Integrating power of tangent and secant

Recall: $\frac{d}{dx} \tan x = \sec^2 x$ $\qquad$ $\frac{d}{dx} \sec x = \tan x \sec x$.

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \qquad \left(\frac{\sin 2x}{\cos 2x} \right)$$

$$\sec^2 x = 1 - \tan^2 x$$

Ex: $\int \tan^2 x \sec^4 x \, dx$

$\int \tan^2 x \sec^2 x \sec^2 x \, dx$ $\qquad$ $u = \tan x$ $\quad \frac{du}{dx} = \sec^2 x$.

$\int u^2 (1 - u^2) \, du$

power of sec is even

Example: power of tan and sec.

$$\int \tan^3 x \sec^7 x \, dx.$$

$$\int \tan^3 x \sec^5 x \sec^2 x \, dx$$

$$= \int \tan^2 x \sec^6 x \tan x \sec x \, dx$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x.$$

$$= \int (\sec^2 x + 1) \sec^6 x \tan x \sec x \, dx$$

$$\sec^2 x - \tan^2 x = 1$$

$u = \sec x.$

$$\frac{du}{dx} = \tan x \sec x \Rightarrow dx \to \frac{du}{\tan x \sec x}.$$

$$= \int (\sec^2 x + 1) \sec^6 x \, du$$

$$= \int (u^2 + 1) u^6 \, du,$$

power of tan is odd

Algorithm for integrating $\tan^a x \sec^b x$.

$\int \sec x\,dx \quad \int \tan x\,dx$

Thm: To integrate $\int \tan^a x \sec^b x\,dx$

- If b is even, hold onto $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ on remaining $\sec^2 x$.

$$\int \tan^a x \sec^{2k} x\,dx = \int \tan^a x (\sec^2)^{k-1} \sec^2 x\,dx$$
$$= \int \tan^a x (1 + \tan^2 x)^{k-1} \sec^2 x\,dx.$$

- If a is odd, hold onto $\sec x \tan x$ and use $\tan^2 x = 1 - \sec^2 x$ on remaining $\tan^2 x$.

$$\int \tan^{2k+1} \sec^b x\,dx = \int \tan^{2k} \sec^{b-1} x \sec x \tan x\,dx$$
$$= \int (1-\sec^2 x)^k \sec^{b-1} x \sec \tan x\,dx.$$

- All other cases are challenging.

Examples:

try: $\int \tan x \, dx.$

$$= \int \frac{1}{\sec x} \sec x \tan x \, dx.$$

$$= \int \frac{1}{u} \, du \Big|_{u = \sec x}$$

$$= \log|u| + C \Big|_{u = \sec x}$$

$$= \log|\sec x| + C$$

$$\int \frac{\sin x}{\cos x} \, dx.$$

$u = \cos x$

$$-\int \frac{1}{u} \, du \Big|_{\cos x.}$$

$$-\log|\cos x| +$$

$$-\log a$$

$$= \log \frac{1}{a}$$

try: $\int \sec x \, dx = \log|\sec x + \tan x| + C$

Example

$$\int \tan^4 x \, dx.$$

$$= \int \tan^2 x (\sec^2 x - 1) \, dx.$$

$$= \int \tan^2 x \sec^2 x - \tan^2 x \, dx$$

$$= \int u^2 \, du \Big|_{u=\tan x} - \int (\sec^2 x - 1) \, dx$$

$$= \frac{u^3}{3} \Big|_{u=\tan x} - \tan x + x + C$$

Example.

$$\int u\,dv = uv - \int v\,du.$$

$$\int \sec^3 x\,dx.$$

$$\sec^2 x - \tan^2 x = 1$$

$$= \int \sec^2 x \sec x\,dx$$

$v'(x) = \sec^2 x.$

$u(x) = \sec x$

$v(x) = \tan x.$

$$\frac{du}{dx} = \tan x \sec x$$

$$\Rightarrow \int \sec^3 x\,dx = \sec x \tan x - \int \tan x \tan x \sec x\,dx$$

$$= \sec x \tan x - \int \tan^2 x \sec x\,dx.$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)\,dx.$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\Rightarrow \int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \log|\sec x + \tan x|\right) + C$$