11 Infinite sequences and series

11.3 Integral test and estimates of sums

In general it is very difficult to evaluate series exactly

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645\ldots$$

Clearly very different techniques are needed for each case. But if we dont care about getting the exact answer, and just want to get an estimate, then integrals can be of big help.

Let us draw a picture of the second case

So we can think of this series as being the sum of the areas of those little rectangles. How can we bound that area?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} x^{-2}\mathrm{d}x$$
$$= 1 + \lim_{b\to\infty} \left[\frac{-1}{x}\right]_1^b = 2$$

Hence the area under the curve is finite, so the sequence of partial sums must be bounded. What else do we need? Since each summand is positive, the sequence of partial sums is an increasing sequence. Thus by the monotone convergence theorem, the series converges.

Can't we do something like this for the harmonic series?

So the harmonic series is the sum of the areas of these little rectangles, but the rectangles are clearly bigger than the area under the curve, so

$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\mathrm{d}x$$
$$= \lim_{b\to\infty} [\log x]_1^b$$

which just gets bigger and bigger. Hence the series just gets bigger and bigger and so does not converge.

But what if we had put the rectangles a bit to the left? Then we'd get

And now we'd see that

$$\sum_{n=1}^{\infty} \leq 1 + \int_{2}^{\infty} x^{-1} \mathrm{d}x$$

But since the RHS blows up to infinity, this inequality doesnt really tell us anything.

More generally we have

Theorem (integral test)**.** *Let f be a positive continuous and decreasing function on $[1, \infty)$, and let $a_n = f(n)$. Then*

- *If $\int_1^\infty f(x)\mathrm{d}x$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.*
- *If $\int_1^\infty f(x)\mathrm{d}x$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.*

Note that if the series converges it need not be equal to the integral (as we saw above).

A very useful example. For what values of p does the sum $\sum \frac{1}{n^p}$ converge? This series is called a p-series.

- If $p < 0$ then the summands do not converge to 0 and so the series diverges.
- If $p = 0$ then we have $\sum 1$ which also diverges.
- If $0 < p$ then we use the integral test

$$\int_1^\infty x^{-p} \mathrm{d}x$$

We did this example before — it diverges if $0 < p \leq 1$ and converges for $p \geq 1$.

Hence the series converges when $p > 1$ and otherwise diverges.

Let us look back at our sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

we know this converges by the above. Further one can prove (not in this course) that it converges to $\pi^2/6$.

We can also use this idea to help us estimate sums — you could try just estimating

$$\sum_{n=1}^{\infty} f(n) \approx \int_{x=1}^{\infty} f(x) \mathrm{d}x$$

But this gives cruddy estimates. Perhaps it is just easier to actually do the