1 Integration

1.12 Improper integrals

- Up until this point we have looked at integrals of nice functions on nice regions — ie continuous functions $f(x)$ on a bounded region $[a, b]$.
- We now extend this to functions that have a discontinuity and / or on an infinite region $[a, \infty)$ or $(-\infty, b]$ or $(-\infty, \infty)$.
- Such integrals are called improper integrals.
- We handle them using limits.
- We do 2 types — (1) infinite intervals and (2) discontinuous integrands.

Definition (Improper integrals — clp 1.12.1)**.** An integral having either a terminal at infinity or an unbounded integrand is called an improper integral.

Motivating example 1

$$\int_0^\infty e^{-x}\mathrm{d}x$$

We would naively like to just put $[-e^{-x}]_0^\infty$ — but we can do it more rigorously using limits.

$$\begin{aligned}\int_0^b e^{-x}\mathrm{d}x &= [-e^{-x}]_0^b \\ &= 1 - e^{-b}\end{aligned}$$

As $b \to \infty$, $e^{-b} \to 0$ and the integral becomes 1. So we can define

$$\begin{aligned}\int_0^\infty e^{-x}\mathrm{d}x &= \lim_{b\to\infty}\int_0^b e^{-x}\mathrm{d}x \\ &= \lim_{b\to\infty}(1 - e^{-b}) = 1.\end{aligned}$$

On the other hand if we take

$$\begin{aligned}\int_0^\infty e^{x}\mathrm{d}x &= \lim_{b\to\infty}\int_0^b e^{x}\mathrm{d}x \\ &= \lim_{b\to\infty}(e^{b} - 1)\end{aligned}$$

The limit is divergent — the integral does not exist. Consider it as the area under the curve — it is infinite.

Another good example is

$$\int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x$$

The integrand blows up at $x = 0$ so we actually compute it (by sneaking up on the discontinuity)

$$\int_a^1 \frac{1}{\sqrt{x}} \mathrm{d}x = [2\sqrt{x}]_a^1 = 2 - 2\sqrt{a}$$

Now taking the limit as $a \to 0$ we get

$$\int_0^1 \frac{1}{\sqrt{x}} \mathrm{d}x = \lim_{a = \to 0} (2 - 2\sqrt{a}) = 2$$

So let us make the two examples of improper integrals above a little more formal:

Definition (Improper integral with infinite domain CLP 1.12.4)**.**

- If $\int_a^t f(x)\mathrm{d}x$ exists for every $t \geq a$, then we define

  $$\int_a^\infty f(x)\mathrm{d}x = \lim_{t \to \infty} \int_a^t f(x)\mathrm{d}x$$

  if the limit exists and is finite.
- Similarly for $\int_{-\infty}^b f(x)\mathrm{d}x$.
- If these limits exist the integrals are called convergent, else divergent.
- If $\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^a f(x)\mathrm{d}x$ are convergent then we define

  $$\int_{-\infty}^\infty f(x)\mathrm{d}x = \int_{-\infty}^a f(x)\mathrm{d}x + \int_a^\infty f(x)\mathrm{d}x$$

  for any real number a.

For what values of q is this $\displaystyle\int_1^\infty x^q \mathrm{d}x$ convergent? If $q \neq -1$ then

$$\begin{aligned}
\int_1^\infty x^q \mathrm{d}x &= \lim_{b \to \infty} \int_1^b x^q \mathrm{d}x \\
&= \lim_{b \to \infty} [\frac{x^{q+1}}{q+1}]_1^b \\
&= \lim_{b \to \infty} \frac{1}{q+1}(b^{q+1} - 1) \\
&= -\frac{1}{q+1} + \lim_{b \to \infty} \frac{1}{q+1} b^{q+1}
\end{aligned}$$

This limit exists if $q + 1 < 0$ or $q < -1$. Now, if $q = -1$ then we have

$$\begin{aligned}
\int_1^\infty 1/x \mathrm{d}x &= \lim_{b \to \infty} \int_1^b 1/x \mathrm{d}x \\
&= \lim_{b \to \infty} [\log |x|]_1^b \\
&= \lim_{b \to \infty} (\log b - 0)
\end{aligned}$$

which does not exist. Hence the integral is convergent for all real $q < -1$ and divergent for $q \geq -1$.

Another example (over the whole $\mathbb{R}$)

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \int_{0}^{\infty} \frac{1}{1+x^2}\mathrm{d}x + \int_{-\infty}^{0} \frac{1}{1+x^2}\mathrm{d}x$$

Let us look at these individually

$$\begin{aligned}
\int_{0}^{\infty} \frac{1}{1+x^2}\mathrm{d}x &= \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\mathrm{d}x \\
&= \lim_{b\to\infty} [\arctan(x)]_0^b \\
&= \lim_{b\to\infty} \arctan(b) - \arctan(0) \\
&= \pi/2
\end{aligned}$$

And very similarly

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\mathrm{d}x = \pi/2$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \pi$.

Now lets look at the other type of improper integral — where the integrand is unbounded. We saw a good example of this a couple of weeks ago:

$$\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x = \left[-\frac{1}{x}\right]_{-1}^{1} = -2 \qquad \text{WRONG — integrand is positive!}$$

We cannot do this because the integrand is divergent at $x = 0$.

Definition (Improper integral with unbounded integrand — clp 1.12.6)**.**

- If f is continous on $[a, b)$ and is discontinous at b then

  $$\int_{a}^{b} f(x)\mathrm{d}x = \lim_{t\to b^-} \int_{a}^{t} f(x)\mathrm{d}x$$

  if the limit exists and is finite.

- If f is continous on $(a, b]$ and is discontinous at a then

  $$\int_{a}^{b} f(x)\mathrm{d}x = \lim_{t\to a^+} \int_{t}^{b} f(x)\mathrm{d}x$$

  if the limit exists and is finite.

- If these limits exist then the integral is convergent, else divergent.

- If f has a discontinuity at $c \in (a, b)$ and both $\int_a^c f(x)\mathrm{d}x$ and $\int_c^b f(x)\mathrm{d}x$ are convergent then

  $$\int_{a}^{b} f(x)\mathrm{d}x = \int_{a}^{c} f(x)\mathrm{d}x + \int_{c}^{b} f(x)\mathrm{d}x$$

Evaluate $\int_2^6 \frac{1}{\sqrt{z-2}}\mathrm{d}z$. Has singularity at $z = 2$, so we use limits.

$$\begin{aligned}\int_2^6 \frac{1}{\sqrt{z-2}}\mathrm{d}z &= \lim_{a\to 2^-}\int_a^6 \frac{1}{\sqrt{z-2}}\mathrm{d}z \\ &= \lim_{a\to 2^-}[2\sqrt{z-2}]_a^6 \\ &= \lim_{a\to 2^-}(2\sqrt{4}-2\sqrt{a-2}) \\ &= 4-0=4\end{aligned}$$

Evaluate $\int_{-1}^1 \frac{1}{x^2}\mathrm{d}x$

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\mathrm{d}x &= \lim_{b\to 0^-}\int_{-1}^b \frac{1}{x^2}\mathrm{d}x + \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}\mathrm{d}x \\ &= \underbrace{\lim_{b\to 0^-}[-1-1/b]}_{\infty} + \underbrace{\lim_{a\to 0^+}[-1+1/a]}_{\infty}\end{aligned}$$

Answer is divergent.

Some more examples (if time)

$$\int_2^\infty \frac{1}{x(\log x)^p}\mathrm{d}x$$

$$\int \frac{1}{x}(\log x)^{-p}\mathrm{d}x \text{sub } u = \log x, u' = 1/x$$

$$\begin{aligned}&= \int u^{-p}\mathrm{d}u \\ &= \frac{1}{1-p}u^{1-p} = \frac{1}{1-p}(\log x)^{1-p} \quad \text{so} \int_2^b \frac{1}{x(\log x)^p}\mathrm{d}x = \frac{1}{1-p}\left((\log b)^{1-p} - (\log 2)^{1-p}\right)\end{aligned}$$

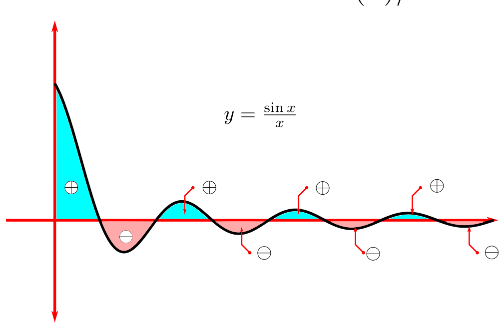
So when we take the limit $b \to \infty$ we get a convergent limit provided $p > 1$, and in that cased we get

$$\int_2^\infty \frac{1}{x(\log x)^p}\mathrm{d}x = \frac{1}{p-1}(\log 2)^{1-p}$$

On can do similarly for the integral

$$\int_0^{1/2} \frac{1}{x(\log x)^p}\mathrm{d}x$$

This is a famous example. A very important function in signal-processing and elsewhere in mathematics and physics is the "sinc" function $\sin(x)/x$.



So if we want to integrate this from 0 to infinity then we are going to get a sequence of plus and minus areas. One can ask if this integral will converge. First lets check the limits of the function: As $x \to 0$

$$\lim_{x\to 0} \operatorname{sinc} x = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

and as $x \to \infty$

$$\lim_{x\to\infty} \operatorname{sinc} x = \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

And — with more advanced maths (complex numbers etc etc) than we have that

$$\int_0^\infty \operatorname{sinc} x = \frac{\pi}{2}$$

So — this infinite sum of plus-areas and minus-areas converges to a finite limit. And we've seen examples where the area is divergent, like

$$\int_1^\infty \frac{1}{x} \mathrm{d}x$$

I want to take this sinc example and tweak it a bit to make it something we can actually integrate:

$$\begin{aligned}
\int \frac{\sin(\log(x))}{x} \mathrm{d}x & & \text{sub } u = \log x,\ u' = 1/x \\
= \int \sin(u) \mathrm{d}u & & \text{easy!}
\end{aligned}$$

So now, integrate this from 1 to infinity:

$$\begin{aligned}
\int_1^\infty \frac{\sin(\log(x))}{x} \mathrm{d}x &= \lim_{b\to\infty} \int_1^b \frac{\sin(\log(x))}{x} \mathrm{d}x \\
&= \lim_{b\to\infty} \int_0^{\log b} \sin u \mathrm{d}u \\
&= \lim_{b\to\infty} \left[-\cos u\right]_0^{\log b} \\
&= \lim_{b\to\infty} (\cos \log b - \cos 0)
\end{aligned}$$

which does not exist — but not because it blows up to infinity, but because it keeps alternating without settling down to a finite answer.

One more example if there is time:

$$\int_1^\infty \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} \right) \mathrm{d}x$$

Easy to see that if we separate the integrand into two integrals then both will be infinite (since the $\sqrt{1+x^2} > x$ on the domain). But together?

Well — we can see (or we should be able to see) that we can integrate both terms separately:

$$\int \frac{1}{x}\mathrm{d}x \log x + C$$

and the first term is tangent-flavoured

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} &\text{ sub } x = \tan\theta,\ x' = \sec^2\theta \\
&= \int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\mathrm{d}\theta \\
&= \int \frac{\sec^2\theta}{\sec\theta}\mathrm{d}\theta \\
&= \int \sec\theta \mathrm{d}\theta \\
&= \log|\sec\theta + \tan\theta| + C \\
&= \log|x + \sqrt{x^1+1}| + c
\end{aligned}$$

Hence

$$\begin{aligned}
\int_1^b \left(\frac{1}{\sqrt{1+x^2}} - \frac{1}{x}\right) &= \left[\log|x+\sqrt{x^2+1}| - \log x\right]_1^b \\
&= \log\left|\frac{b+\sqrt{b^2+1}}{b}\right| - \log(1+\sqrt{2})
\end{aligned}$$

Now in the limit as $b \to \infty$

$$\begin{aligned}
\lim_{b\to\infty}\left(\log\left|\frac{b+\sqrt{b^2+1}}{b}\right| - \log(1+\sqrt{2})\right) &= \lim_{b\to\infty}\left(\log\left|\frac{1+\sqrt{1+1/b^2}}{1}\right| - \log(1+\sqrt{2})\right) \\
&= \log 2 - \log(1+\sqrt{2}) = \log\frac{2}{1+\sqrt{2}}
\end{aligned}$$

Does this answer make sense? Well — $2 < 1+\sqrt{2} \approx 2.4$, so our answer is negative. But lets think about the integrand:

$$\begin{aligned}
\sqrt{x^2+1} &> x & \text{when } x > 0 \\
\frac{1}{x^2+1} &< \frac{1}{x}
\end{aligned}$$

so the integrand is negative. Oof!

So now we have learned plenty of methods of integration to start doing some simple applications.

2 Applications of integration

2.1 Work

In everyday English the word “work” means something like “exertion or effort directed to produce or accomplish something”. In physics (and mathematics) it has a precise meaning (based on this idea) — it is the energy expended acting against a force. eg — the energy expended moving a weight against gravity.

We need some definitions

Definition.

- Time t — measured in seconds
- Position s — measured in metres
- Mass m — measured in grams of kilograms
- Newton's second law

$$\text{Force } = \text{mass} \times \text{acceleration} \qquad\qquad F = m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

  Force is measured in Newtons $= kg.m/s^2$

- Work at constant force measures energy required to act against a force

$$\text{Work } = \text{Force} \times \text{displacement} \qquad\qquad W = Fd$$

  Measured in Newton-metres = Joules

So if the force is constant, then the work is simply the force times the distance moved against the force — eg moving a heavy weight up off the floor. How much work is done moving a 1kg book from the floor to the top of a 2m heigh shelf?

- Acceleration due to gravity $= 9.8m/s^2$. Force due to gravity $= ma = 9.8N$.
- Work done against force is $9.8 \times 2 = 19.6J$.

Very easy. But what happens when the force is not constant? If it varies with distance — eg a spring — then we approximate the work by a Riemann sum.

- Let $f(x)$ be the force acting on an object at position x.
- To compute the work done in order to move the object from $x = a$ to $x = b$ we cut up the interval $[a, b]$ into n segments $[x_{i-1}, x_i]$ each of width $(b - a)/n$.
- We approximate the varying force in the interval $[x_{i-1}, x_i]$ by a constant force $f(x_i^*)$ where $x_i^* \in [x_{i-1}, x_i]$ — just as we approximated the area in a Riemann sum.
- The work done in this interval $[x_{i-1}, x_i]$ is then approximately $f(x_i^*)\Delta x$.
- So the total work is

$$W \approx \sum_{i=1^n} f(x_i^*)\Delta x$$

  which is exactly a Riemann sum.

- Hence as $n \to \infty$ we have

$$W = \int_a^b f(x)\mathrm{d}x$$

Definition (Work — clp 2.1.1)**.** The word done by a force $F(x)$ moving an object from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\mathrm{d}x$$

Note that if $F(x) = c$ is constant then $W = c(b - a)$.

Hooke's law relates the force exterted by a string, F, to the distance it has been stretched x:

$$F = kx \qquad\qquad k = \text{spring constant}$$

Holds for lots of materials provided x isnt too large.

A standard example: A spring has natural length of 20cm. If a 25N force is required to keep it stretched at a length of 30cm how much work is required to stretch it from 20cm to 25cm?

- Careful of units — we need newtons and metres.
- First work out the spring constant:

$$\begin{aligned} F &= kx \\ 25 &= k(0.30 - 0.20) \\ k &= 25/0.1 = 250N/m \end{aligned}$$

- So we can now work out the work

$$\begin{aligned} W &= \int_0^{0.05} F(x)\mathrm{d}x \\ &= \int_0^{0.05} 250x\mathrm{d}x \\ &= [125x^2]_0^{0.05} = 125 \times 0.0025 = 0.3125J \end{aligned}$$

Another slightly less standard example: A chain lying on the ground is 10m long and weighs 80kg. How much work is required to raise one end to a height of 6m?

- Assume that the chain will be "L" shaped when it has been lifted, with 4m left on the ground. Also assume no friction and constant density of the chain = 8kg/m. (A picture might help)
- Let us do this with a Riemann sum — split the chain into segments $[x_{i-1}, x_i]$ and work out how much work is done lifting each segment.
- Let x be the distance (in m) from the top of the chain. The piece of chain at x is lifted $6 - x$ m. Hence the segment $[x_{i-1}, x_i]$ is lifted $6 - x_i^*$ m.
- The segment weighs $8\Delta x$, so experiences $8 \times 9.8\Delta x = 78.4\Delta x$ gravitational force.
- So the Riemann sum is given by

$$W \approx \sum_{i=1}^{n} (6 - x_i^*)78.4\Delta x$$

- So the work is given (limit of $n \to \infty$)

$$\begin{aligned} W &= 78.4 \int_0^6 (6-x)\mathrm{d}x \\ &= 78.4[6x - x^2/2]_0^6 = 78.4 \times 18 = 1411.2J \end{aligned}$$

What if the chain was dangling from the roof and we were to lift the far end?

- Let x be the distance from the middle of the chain.
- The piece of chain at x is lifted a distance of $2x$
- Hence the Riemann sum is

$$W \approx \sum_{i=1}^{n} 2x_i^* 78.4\Delta x$$

- So the work done is

$$\begin{aligned} W &= \int_0^5 156.8x\mathrm{d}x \\ &= 156.8[x^2/2]_0^5 = 156.8 \times 12.5 = 1960 \end{aligned}$$

Similarly if we calculate the work done pumping water from a tank — compute the work done pumping out each "slice" of water.

A very standard example:

- The tank is shaped like an inverted cone — height = 10m, radius = 4m.
- Filled to height of 8m.
- Find work pumping water out of top.
- Density of water is $1000kg/m^3$.

How do we do this?

- Draw a picture.
- How much work done to remove each "slice" of water?
- Let x be distance from bottom of the tank. The slice at x has volume

$$\begin{aligned} V(x) &= \pi r(x)^2 \Delta x \\ &= \pi \left(\frac{2x}{5}\right)^2 \Delta x \end{aligned}$$

- The weight of this slice is $1000V(x)$. So the gravitation force acting on the slice is $1000V(x) \times 9.8$.
- We need to move the slice at x up $10 - x$ metres.

- So the work done is

$$\begin{aligned} W &= \int_0^8 9800 V(x)(10-x)\mathrm{d}x \\ &= 1568\pi \int_0^8 x^2(10-x)\mathrm{d}x \\ &\approx 3.36 \times 10^6 J \end{aligned}$$

Finally — let is look at how work relates to some of newton's laws of motion. To do this assume that we are moving an object against a force, $F(x)$, and that the position of the object is given by $x(t)$. Then the work is given by

$$\begin{aligned} W &= \int_a^b F(x)\mathrm{d}x && \text{sub } x = x(t) \\ &= \int_{t=\alpha}^{t=\beta} F(x(t))\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t \end{aligned}$$

Newton to the rescue with his laws $F = ma$:

$$\begin{aligned} &= \int_\alpha^\beta m\frac{\mathrm{d}^t 2}{\mathrm{d}x^t}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t \\ &= m\int_\alpha^\beta v'(t)v(t)\mathrm{d}t && \text{a little chain rule} \\ &= m\int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v(t)^2\right)\mathrm{d}t \\ &= m\left[\frac{1}{2}v(t)^2\right]_\alpha^\beta \\ &= \frac{1}{2}mv(\beta)^2 - \frac{1}{2}mv(\alpha)^2 \end{aligned}$$

This new function $\frac{m}{2}v^2$ is the *kinetic energy* — so this is telling us that the work done is equal to the difference in kinetic energy. (this is related to the concept of "conservation of energy").