3 Infinite sequences and series

3.3 Convergence tests

3.3.3 Comparison test

Recall from our work using the integral test, that the "p-series"

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges provided $p > 1$. This is just like we saw back in improper integrals — the integral

$$\int_{x=1}^{\infty} \frac{\mathrm{d}x}{x^p}$$

converges provided $p > 1$. In that case, the integrand is decaying fast enough for the integral to converge.

Now consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}$$

Notice that

$$n^2 + 2n + 3 \geq n^2 \qquad \text{for } n \geq 1$$

and hence

$$0 \leq \frac{1}{n^2+2n+3} \leq \frac{1}{n^2} \qquad \text{for } n \geq 1.$$

Now the integral test tells us that $\sum \frac{1}{n^2}$ converges (its summands decay fast enough), so it makes sense that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}$$

also converges —since its summands are smaller.

We can make this reasoning rigorous via the following theorem.

Theorem (The comparison test — CLP 3.3.8)**.** *Let N be a natural number and let $K > 0$. Then*

- *If $|a_n| < Kc_n$ for all $n \geq N$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ also converges.*
- *If $a_n > Kd_n$ for all $n \geq N$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ also diverges. (note — no absolute value signs)*

So notice that we have generalised things a bit — we don't really care about what happens to the summands when n is small, we only care about what happens when they are big $n \geq N$. Similarly, we don't need to compare against c_n or d_n exactly, we can compare against any fixed multiple of those summands.

Another example

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 - 5}$$

Now — this looks like we want to compare it with $3n^2$ and we would expect it to converge. But we cannot do that directly, since $3n^2 - 5 < 3n^2$. But:

$$3n^2 - 5 = 2n^2 + (n - 5) \geq 2n^2 \qquad\qquad \text{provided } n \geq \sqrt{5}$$

Hence

$$0 \leq \frac{1}{3n^2 - 5} \leq \frac{1}{2n^2}$$

for $n \geq 5$. And so, by the comparison test, the series converges.

Notice that we should tell the reader that we are using the comparison test. Just a few words.

Another one

$$\sum_{n=1}^{\infty} \frac{2n + 1}{6n^2 - 5}$$

Again — we should look at this and see that the numerator is "almost" $2n$ and the denominator is "almost" $6n^2$ and so we expect the summands to be "almost" $3/n$. And thus the series should diverge. To make this more rigorous:

$$\frac{2n + 1}{6n^2 - 5} \geq \frac{2n}{6n^2 - 5} > \frac{2n}{6n^2} = \frac{3}{n}$$

So by the comparison test, this series diverges.

Notice that this same problem would be harder if we had

$$\sum_{n=1}^{\infty} \frac{2n + 1}{6n^2 + 5}$$

Because now the denominator is larger than $6n^2$. Here we might need to do something along the lines of

$$\frac{2n + 1}{6n^2 + 5} \geq \frac{2n}{6n^2 + 5} > \frac{2n}{6n^2 + n} = \frac{2n}{7n^2} = \frac{2}{7n}$$

provided $n > 5$. Hence the series diverges.

Now — wouldn't it be easier if we could just look at the "almost" and be done — ie how do the summands look as $n \to \infty$.

The following theorem helps us.

Theorem (Limit comparison theorem — clp 3.3.11)**.** *Let* $\sum a_n$ *and* $\sum b_n$ *be two series with* $b_n > 0$ *for all* n*. Now assume that the limit*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists. Then

- *If* $\sum b_n$ *converges, then* $\sum a_n$ *also converges.*
- *If* $\sum b_n$ *diverges AND* $L \neq 0$ *then* $\sum a_n$ *also diverges.*

Notice that the above is less useful when $L = 0$*.*

This theorem makes our life much easier because we can do away with delicate inequalities and just look at limits (which wash out a lot of the detail and just leave the dominant behaviour of the summands).

Back to our previous example. Examining the summands we see that

$$\frac{2n+1}{6n^2+5} \approx \frac{2n}{6n^2} \text{just looking at the dominant terms}$$

So — it makes sense to compare against either $1/3n$ or just $1/n$. Applying the above theorem.

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n(2n+1)}{6n^2+5} \\ &= \lim_{n\to\infty} \frac{n^2(2+1/n)}{n^2(6+5/n^2)} \\ &= 3 \end{aligned}$$

Since this limit is non-zero and $\sum \frac{1}{n}$ diverges, by the limit comparison theorem, our original series diverges.

One more example. Does the following series converge or diverge?

$$\sum_{n\geq 10} \frac{\sqrt{2n^2+1}}{n^3-7n+13}$$

Notice that the summands are approximately (for very large n)

$$\frac{\sqrt{2n^2+1}}{n^3-7n+13} \approx \frac{n\sqrt{2}}{n^3} = \frac{\sqrt{2}}{n^2}$$

So it makes sense to compare against $\sum \frac{1}{n^2}$ and we expect it to converge. So we look at the limit of the ratio:

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sqrt{2n^2+1}}{n^3-7n+13} \frac{n^2}{1} \\ &= \lim_{n\to\infty} \frac{n^2\sqrt{2n^2+1}}{n^3-7n+13} \\ &= \lim_{n\to\infty} \frac{n^3\sqrt{2+1/n^2}}{n^3(1-7/n^2+13/n^3)} \\ &= \sqrt{2} \end{aligned}$$

Since this limit is non-zero and $\sum n^{-2}$ converges, by the limit comparison test, our original series converges.

3.3.4 Alternating series

So the integral tests we looked at above only work for series that are all positive. Frequently we will come across *alternating* series — the summands alternate positive / negative.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

This is a good example — it is like the harmonic series, but it actually converges. It actually converges to $\log 2$ (but we need to get to Taylor series to see that — coming soon). It is called (not very inventively) the alternating harmonic series.

In fact it is easy to tell when an alternating series converges

Theorem (Alternating series test — CLP 3.3.14)**.** *Consider the alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 \ldots$$

where $b_n > 0$. Then if

- $b_n \geq b_{n+1}$ *for all n (decreasing), and*
- $b_n \to 0$

then the series is convergent.

So if we rexamine the series above, we see that $b_n = \frac{1}{n}$

- $b_n - b_{n+1} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \geq 0$ — decreasing.
- $b_n \to 0$

So by the alternating series test, the alternating harmonic series converges (as noted above, we need to do more work to determine exactly what it converges to).

What about this one

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n^2}{n^2 + 6n + 2}$$

Its alternating — we could test for decreasingness, but its perhaps easier to check the limit first.

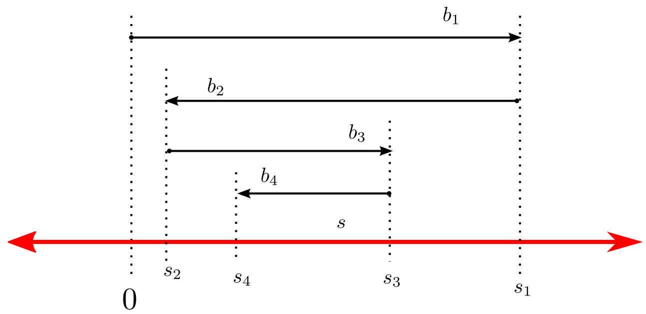
$$\frac{2n^2}{n^2 + 6n + 2} \to 2$$

So the terms do not go to zero — we cannot apply this test. What other test can we apply? We can't apply the integral tests, but we can apply our "test for divergence"

> If a_n does not converge to zero, then the series diverges.

Hence $a_n = (-1)^n \frac{2n^2}{n^2+6n+2}$, so the limit does not exist. Hence the series diverges.

Lets draw a picture of the general case



As we add each term we alternatively jump right and left (since the terms alternate positive / negative) and each jump gets smaller and smaller (since the terms decrease).

But one thing we notice is that the final sum must be somewhere in the middle. ie the final sum is always trapped between s_n and s_{n+1}. So $R_n = |s - s_n| \leq b_{n+1}$. This gives us a nice way of estimating alternating series.

Theorem (Alternating series test (continued) — CLP 3.3.14)**.** *If* $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *is a convergent alternating series with*

$$0 \leq b_{n+1} \leq b_n \qquad\qquad b_n \to 0$$

Then

$$R_n = |s - s_n| \leq b_{n+1}$$

So if we look back at our $\log 2$ example, we see that

$$R_n = |s - s_n| = b_{n+1} = \frac{1}{n+1}$$

So if we want to estimate $\log 2$ to within 10^{-6} decimal places, we need to take the first 10^6 terms of the sum — very slow convergence. This means that simply truncating the series is not an especially efficient way to compute $\log 2$ numerically.

Lets do a couple more:

$$\sum_{n\geq 1}\frac{1}{\sqrt{n}} \qquad\qquad \sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n}}$$

The first diverges (by the integral test), while the second converges. To see this we apply the alternating series test. Rewrite the series as

$$\sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n}} = \sum(-1)^n b_n \qquad\qquad \text{where } b_n = n^{-1/2}$$

Since the series is alternating and $b_n \to 0$ the series converges.

How about

$$\sum_{n\geq 0}\frac{(-1)^n}{n!} \qquad\qquad \text{recall } n! = 1\cdot 2\cdot 3\cdots n$$

This series is alternating, and the summands go to zero, so by the alternating series test, the series converges. In fact it converges to $1/e$ — we'll see this later with Taylor series.

How quickly does it converge — by the alternating series test (again), the difference between the limit L and the partial sums s_N is bounded by the first dropped term $= 1/(N+1)!$ — which shrinks very quickly. This is a very effective way to compute e^{-1}. eg — if we compute

$$1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{10!} = \frac{16481}{44800} \approx 0.36787946428571428571\ldots$$

This differs from $1/e$ by less than $1/11! \approx 2.5 \times 10^{-8}$.