Integration by parts. (review).

Let $u(x)$ and $v(x)$ be differentiable function.

$$\int u(x) \frac{dv}{dx} dx = u(x) \cdot v(x) - \int v(x) \frac{du}{dx} dx.$$

more concisely:

$$\int u\, dv = uv - \int v\, du.$$

Integration by parts – definite integrals.

$$\int_a^b u(x)\,\frac{dv}{dx}\,dx = u(x)\cdot v(x) - \int v(x)\,\frac{du}{dx}\,dx.$$

Recall product rule:

$$\frac{d}{dx}(u(x)\cdot v(x)) = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$$

Example

Evaluate $I_n := \int_0^\infty x^n e^{-x} dx$, n is a integer.

Example contd.

try: Solve $\int_0^{2\pi} x^2 \cos(x)\, dx$.

Trigonometric integrals.

$$\int \sin^a x \cos^b x \, dx$$

$$\int \tan^a x \sec^b x \, dx.$$

Rewrite higher power of trig functions into lower power

trig identities.

$\sin^2 x + \cos^2 x = 1$

$\sec^2 x - \tan^2 x = 1$

sum of angles.

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\sin(a+b) = \sin a \cos b + \cos a \sin b.$

Example. (Integrating even power of sin)

Simplify $\sin^4 x$.

Example (Integrating odd power)

Solve $\int \cos^5 x \, dx$

$= \int \cos x \cos^4 x \, dx$

$= \int \cos x \, (1 - \sin^2 x)^2 \, dx$

let $u(x) = \sin x$

$\Rightarrow \frac{du}{dx} = \cos x$

so, we get

$$\int (1-u)^2 \, du \Big|_{u=\sin x}$$

$$= \left[u - u^2 + \frac{u^3}{3} \right]_{u=\sin(x)} + c$$

so,

$$\int \cos^5 x \, dx$$

$$= \sin x - \sin^2 x + \frac{\sin^3 x}{3} + C$$

Note that power is odd.

Algorithm for integrating $\sin^a x \cos^b x$.

Thm To integrate $\int \sin^a x \cos^b x \, dx$.

- If b is odd, hold 1 power of cosine and turn all other into sine using $\cos^2 x = 1 - \sin^2 x$.

$$\int \sin^a x \cos^{2k+1} x \, dx = \int \sin^a x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^a x \, (1 - \sin^2 x)^k \cos x \, dx.$$

- If a is odd, hold 1 power of sine and turn all other into cosine using $\sin^2 x = 1 - \cos^2 x$.

$$\int \sin^{2k+1} x \cos^b x \, dx = \int (\sin^2 x)^k \sin x \cos^b x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^b x \sin x \, dx.$$

Algorithm for integrating $\sin^a x \cos^b x$. (contd)

- If a and b are even then use

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) \qquad \sin^2 x = \frac{1}{2}(1-\cos 2x).$$

sine and cosine are derivatives of each other

try: $\int \cos^2 x \sin^5 x \, dx$.

Integrating power of tangent and secant:

Recall: $\frac{d}{dx}\tan x = \sec^2 x$ $\qquad$ $\frac{d}{dx}\sec x = \tan x \sec x$.

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} \qquad \left(\frac{\sin 2x}{\cos 2x}\right)$$

Ex: $\int \tan^2 x \sec^4 x \, dx$

$\int \tan^2 x \sec^2 x \sec^2 x \, dx$ $\qquad$ $u = \tan x$ $\qquad$ $\frac{du}{dx} = \sec^2 x$.

$\int u^2 (1-u^2) \, du$

power of sec is even

Example: power of tan and sec.

$$\int \tan^3 x \sec^7 x \, dx.$$

Algorithm for integrating $\tan^a x \sec^b x$.

Thm: To integrate $\int \tan^a x \sec^b x \, dx$

- If b is even, hold onto $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ on remaining $\sec^2 x$.

$$\int \tan^a x \sec^{2k} x \, dx = \int \tan^a x \left(\sec^2\right)^{k-1} \sec^2 x \, dx$$
$$= \int \tan^a x \left(1 + \tan^2 x\right)^{k-1} \sec^2 x \, dx.$$

- If a is odd, hold onto $\sec x \tan x$ and use $\tan^2 x = 1 + \sec^2 x$ on remaining $\tan^2 x$.

$$\int \tan^{2k+1} x \sec^b x \, dx = \int \tan^{2k} x \sec^{b-1} x \sec x \tan x \, dx$$
$$= \int \left(1 - \sec^2 x\right)^k \sec^{b-1} x \sec x \tan x \, dx.$$

- All other cases are challenging.

Examples:

try: $\int \tan x \, dx$.

try: $\int \sec x \, dx$.

Example

$$\int \tan^4 x \, dx.$$

Example.

$$\int \sec^3 x \, dx.$$