Squeeze theorem.

Theorem If $a_n \le b_n \le c_n$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$

then $\lim\limits_{b_n\to\infty} = L$.

Squeeze theorem.

- Compute $\lim\limits_{n\to\infty} \dfrac{n!}{n^n}$

- True or False: Let $a_n \leq b_n \leq c_n$ with $a_n \to L$ and $c_n \to L+1$. Then $\{b_n\}$ to a real number M that satisfy $L \leq M \leq L+1$.

Example

- Given that $|a_n| \to 0$, does $\{a_n\}$ converge?

sequences and continuous functions.

Thm: If $a_n \to L$ and $f(x)$ is continuous at $x = L$ then $f(a_n) \to f(L)$.

Example

(1) Calculate $\lim_{n \to \infty} \sin\left(\frac{\pi n}{2n+1}\right)$

notice that $a_n = \frac{\pi n}{2n+1}$ and $a_n \to \frac{\pi}{2}$.

Let $f(x) = \sin(x)$. Since $\sin(x)$ is continuous at $x = \pi/2$, $\sin\left(\frac{\pi n}{2n+1}\right) \to \sin\left(\frac{\pi}{2}\right) = 1$ by the theorem.

Example

11) Calculate $\lim_{n\to\infty} e^{-\left(n^2+2/(n^2+1)\right)}$

let $a_n = \frac{n^2+2}{n^2+1}$. Notice that $a_n \to 1$.

let $f(x) = e^{-x}$. Since e^{-x} is continuous at $x = 1$, we get $e^{-\frac{n^2+2}{n+1}} \to e^{-1}$ by the theorem.

Example.

Let $f(x) = |x|$.

Compute $\lim_{n \to \infty} f'\left(\frac{n^2+1}{(n+1)(n^2+2)}\right)$

Exponential function.

This is a very useful lemma that we will use later in next lecture.

Lemma: The sequence $a_n = r^n$ is convergent when $-1 \leq r \leq 1$ and otherwise is divergent.

$$\lim_{n\to\infty} r^n = \begin{cases} 1 & r = 1 \\ 0 & |r| < 1 \\ \text{divergent} & \text{otherwise.} \end{cases}$$

Recall:

Thm: If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$ for $n = 1, 2, 3, \ldots$

then $\lim_{n\to\infty} a_n = L$.

Proof of Lemma.

First case is easy since $1^n = 1$.

For second case: Consider the function $f(x) = r^x$

Notice that $r^x \to 0$ as $x \to \infty$ if $|r| < 1$.

eg: $\left(\frac{1}{2}\right)^x \to 0$ as $x \to \infty$.

So, $a_n = f(n) = r^n \to 0$ if $|r| < 1$.

Bounded sequences:

Sometimes it is useful to whether a sequence converges without computing the exact value.

Consider the sequence $a_n = \frac{1}{n+2}$. Does $\{a_n\}$ converge?

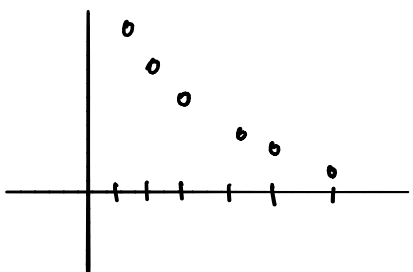
We know that it converges to 0. It also has two other important properties:

- $a_n > 0$ for all n. — lower bounded
- $a_n < a_{n+1}$ for all n. — it is decreasing.

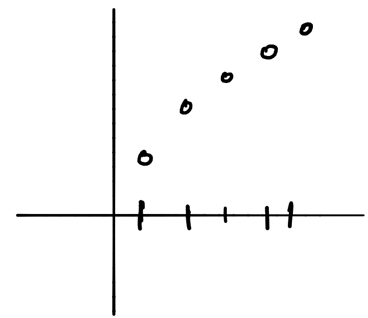
$$a_{n+1} - a_n = \frac{1}{n+3} - \frac{1}{n+2} = \frac{(n+2)-(n+3)}{(n+2)(n+3)} = \frac{-1}{(n+2)(n+3)} < 0$$

Definition.

- A sequence $\{a_n\}$ is decreasing if $a_n \geq a_{n+1}$ for all n.
- A sequence $\{a_n\}$ is increasing if $a_n \leq a_{n+1}$ for all n.
- A sequence is monotonic if it is either decreasing or increasing



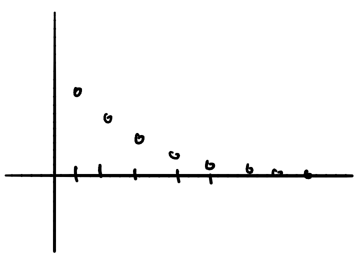
decreasing



increasing.

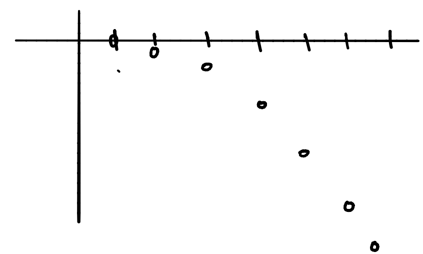
Definition.

- A sequence $\{a_n\}$ is bounded below if there is some number M s.t $a_n \geq M$ for all $n \geq 1$.
- A sequence $\{a_n\}$ is bounded above if there is some number M s.t $a_n \leq M$ for all $n \geq M$.
- If a sequence is bounded from above and below then it is simply called bounded.



$a_n = e^{-n}$

bounded below



$a_n = \ln\left(\frac{1}{n}\right)$.

unbounded below.

Monotonic sequence theorem (Monotonic Convergence Theorem):

Thm Every bound and monotonic sequence converges.

example: $a_n = \dfrac{n}{n+1}$

- $a_{n+1} - a_n = \dfrac{n+1}{n+2} - \dfrac{n}{n+1} = \dfrac{n^2+2n+1-n^2-2n}{(n+2)(n+1)} = \dfrac{1}{(n+2)(n+1)}$

  so, $a_{n+1} \geq a_n$ (increasing).

- $a_n \geq 0$. and $\dfrac{n}{n+1} \leq 1$. so, it's bounded.

Thus, $\left\{ \dfrac{n}{n+1} \right\}$ converges.

Converse of MCT.

True/False: If a sequence is convergent then it is monotonic and bounded.

Ex : $a_n = \frac{\sin(n)}{n}$.