

## u-substitution

• let  $f(x) = e^x$ . What is  $\int f(x) dx$ ?  $= e^x + C \Big|_a^b$

• let  $g(x) = x^2$ . what is  $\int g(x) dx$ ?  $= \frac{x^3}{3} + C$

• Solve  $\int f(g(x)) dx$ .

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$\int e^{x^2} dx \quad \int x e^{x^2} dx$$

$$\frac{d}{dx} (e^{x^2}) = e^{x^2} \quad \text{not true!}$$

## Chain rule in reverse

$$\frac{d}{dx} F(u(x)) = F'(u(x)) u'(x)$$

Let  $f(x)$  be a continuous function. (and  $F'(x) = f(x)$ )

Let  $u(x)$  be a differentiable function.

Indefinite :

$$\int f(u(x)) u'(x) dx = \int f(u) du \Big|_{u=u(x)}.$$

Definite:

$$\int_a^b f(u(x)) u'(x) dx = \int f(u) du \Big|_{u(a)}^{u(b)}$$

## Strategy for picking $u(x)$ .

- Practice

- look for  $u(x)$  and  $u'(x)$  in the integrand.

For example:  $\int e^{2x} \cos(e^{2x}) dx$

$$\frac{d}{dx} e^{2x} = 2e^{2x}. \text{ so, pick } u(x) = e^{2x}.$$

- Chain rule requires composition of functions.  
Choose  $u(x)$  to be the inner function.
- Choose  $u(x)$  to be the complicated argument.
- Practice

### Example 1

Evaluate:  $\int_0^1 \frac{x^2}{(x^3+1)^2} dx$

$$\frac{d}{du} u^{-1} = -u^{-2}$$

o Recognize that  $\frac{d}{dx} x^3 = 3x^2$ .

o Pick  $u(x) = x^3 + 1$  .  $\frac{du}{dx} = 3x^2$

o  $dx \rightarrow \frac{1}{3x^2} du$

$$\begin{aligned} \int \frac{x^2}{u^2} \frac{1}{3x^2} du &= \frac{1}{3} \int_{u(0)}^{u(1)} \frac{1}{u^2} du = \frac{1}{3} \int_1^2 \frac{1}{u^2} du \\ &= \frac{1}{3} \left[ -u^{-1} \right]_1^2 \\ &\equiv \end{aligned}$$

## Example

Evaluate  $\int \tan(x) \log(\cos x) dx$ . (Q22 CLP Problem book)

- Choose inner function as  $u(x) = \cos(x)$

$$\text{so, } \frac{du}{dx} = -\sin(x)$$

- Replace  $dx \rightarrow \frac{du}{-\sin(x)}$

$$\text{so, } \int \tan(x) \log(\cos(x)) \frac{du}{-\sin(x)}$$

$$= \int \frac{\sin(x)}{\cos(x)} \log(u) \frac{1}{-\sin(x)} du$$

$$= - \int \frac{1}{u} \log(u) du$$

$$= - \int \frac{1}{u} \log(u) du \Big|_{u=u(x)}$$

$$v(u) = \log(u)$$

$$\frac{dv}{du} = \frac{1}{u}$$

### Example

Evaluate  $\int \tan(x) \log(\cos x) dx$ .

$$u(x) = \log(\cos x)$$

$$\frac{du}{dx} = \frac{1}{\cos x} \cdot (-\sin(x)) = -\tan(x)$$

$$dx \rightarrow -\frac{1}{-\tan x} du$$

$$\begin{aligned}\int \tan(x) \log(\cos x) dx &= \int \tan x u \frac{du}{-\tan x} \\&= - \int u du \\&= - \left[ \frac{u^2}{2} \right]_{u=\log(\cos x)} + C \\&= - \frac{\log^2(\cos x)}{2} + C.\end{aligned}$$

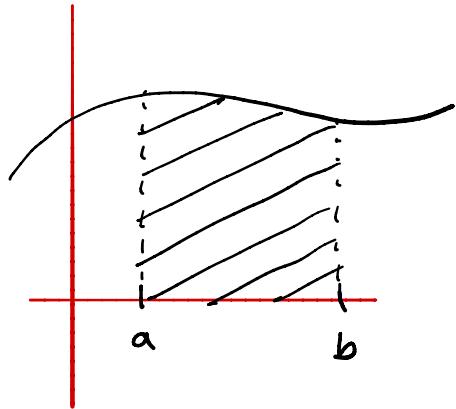
## Area between curves (CLD 1.5)

The definite integral  $\int_a^b f(x) dx$  is the area under the curve.

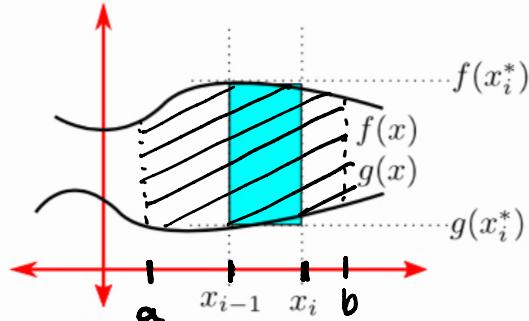
Precisely,

$\int_a^b f(x) dx$  is area bounded by

- o line  $x=a$
- o line  $x=b$
- o line  $y=0$
- o curve  $y=f(x)$



## Area between curves.



find the area between two curves  $y = f(x)$  and  $y = g(x)$ .

- o Generalization of area below the curve.  
think  $g(x) = 0$ .
- o Setup Riemann sum: Height of subrectangle will depend on  $f(x)$  and  $g(x)$ .

## Riemann sum for area between curves.

- Divide  $[a, b]$  into  $n$  sub-intervals.

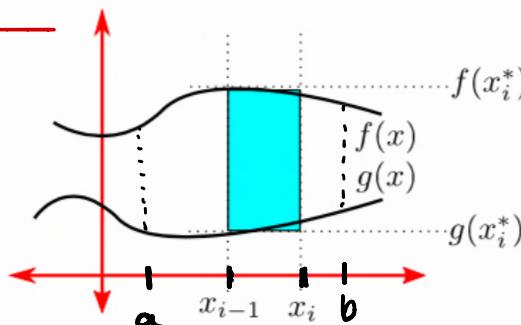
$$\Delta x = (b-a)/n$$

$$x_i = a + i \Delta x$$

- height of  $i^{\text{th}}$  subrectangle

$$\text{height} = \frac{f(x_i^*) - g(x_i^*)}{n}, \quad x_i^* \in [x_{i-1}, x_i].$$

- Riemann sum :  $R_n = \sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x$

- Area :  $A = \lim_{n \rightarrow \infty} R_n = \int_a^b (f(x) - g(x)) dx.$
- 
  
 top function      bottom function

## Example

Compute the finite area between the curves

$$f(x) = x+2 \quad \text{and} \quad g(x) = x^2 - 4.$$

o Step: plot the functions.

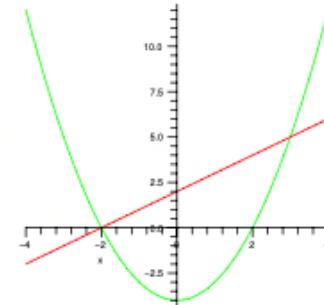
o intersection points:

$$x+2 = x^2 - 4 \Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow x = -2, +3$$

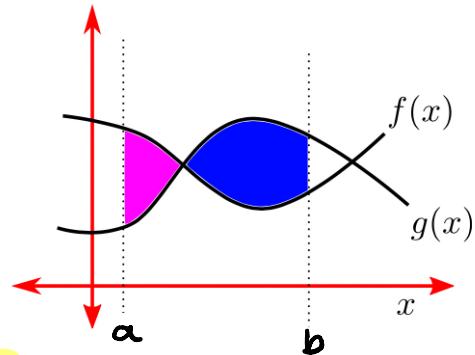
$$o A = \int_{-2}^3 ((x+2) - (x^2 - 4)) dx$$

$$= \int_{-2}^3 (6 + x - x^2) dx = \left[ 6x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^3 \\ = \frac{125}{6}$$



## Separate domain of integration.

- If  $y=f(x)$  and  $y=g(x)$  cross, separate domain of integration.
- $f(x)-g(x)$  change sign at intersection point



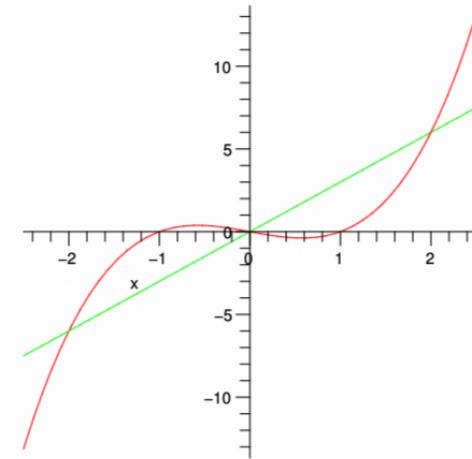
Ex: Find the finite area bounded by the curves  
 $y = x^3 - x$  and  $y = 3x$ .

## Ex. solution

$$y = x^3 - x, \quad y = 3x$$

- Plot.
- Intersection points:

$$\begin{aligned} 3x &= x^3 - x \\ \Rightarrow x^3 - 4x &= 0 \Rightarrow x(x^2 - 4) = 0 \\ \Rightarrow x &= 0, \pm 2. \end{aligned}$$



↑ sign change happens.

$$\begin{aligned} A &= \int_{-2}^0 (\underbrace{x^3 - x - 3x}_{x^3 - 4x}) dx + \int_0^2 (\underbrace{3x - x^3 + x}_{4x - x^3}) dx. \\ &= \left[ \frac{x^4}{4} - 2x^2 \right]_{-2}^0 + \left[ 2x^2 - \frac{x^4}{4} \right]_0^2 = 8 \end{aligned}$$

Interchange  $x$  and  $y$ .

For some problem, it's easier to integrate with respect to  $y$ .

For ex: Find the finite area bounded by  $y^2 = 4x$  and  $4x - 3y = 4$ .  
 $\rightarrow y = \frac{1}{3}(4 - 4x)$        $\rightarrow y = \begin{cases} +\sqrt{4x} \\ -\sqrt{4x} \end{cases}$

Reverse the role of  $x$  and  $y$ .

○ Find intersection points:

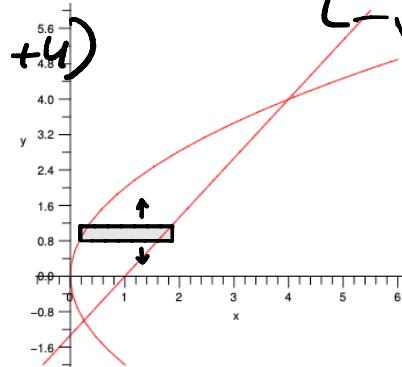
$$y^2 = 4x = 3y + 4$$

$$\Rightarrow y^2 - 3y - 4 = 0$$

$$\Rightarrow y = -1, 4.$$

Check

$$\int_{-1}^4 (x_r - x_l) dy$$



$$x = y^2/4 \quad \text{--- left}$$

$$x = \frac{1}{4}(3y + 4) \quad \text{--- right}$$

## Example contd.

- Riemann sum by splitting interval  $[-1, 4]$  in  $y$ .

$$\Delta y =$$

$$y_i =$$

$$\text{Area} \approx \sum_{i=1}^n (\text{right-left}) \Delta y$$

$$= \sum_{i=1}^{4n} \left( \frac{3}{4}y + 1 - \frac{y^2}{4} \right) \Delta y$$

$$A = \int \left( 1 + \frac{3}{4}y - \frac{y^2}{4} \right) dy$$

$$= \left[ y + \frac{3}{8}y^2 - \frac{y^3}{12} \right]_1^4$$

$$= \frac{125}{24}$$

