## Recall (Ratio test)

The series $\sum_{n=1}^{\infty} a_n$

① converges absolutely if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$.

② diverges if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.

## Recall (absolute/conditional convergence)

① $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

② $\sum_{n=1}^{\infty} a_n$ converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

# Ratio test and examples.

- The ratio test is always inconclusive for ratios of polynomials since. For example, consider $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+3n}{n^3+4n^2}$ with $a_n = (-1)^n \frac{n^2+3n}{n^3+4n}$. Then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

Example: $\sum_{n=1}^{\infty} \frac{n}{5^n}$.

## Example 2

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

## Example

$$\sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$$

## Example

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n^2+1)(n!)^2}.$$

# Power Series.

**Def$^n$:** An expression of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$ is a power series centered at $x=0$. An expression of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x=a$.

# Convergence of power series.

For a general power series, there are three possibilities for convergence of $\sum_{n=0}^{\infty} c_n (x-a)^n$.

**Theorem** (convergence of power series): For $\sum_{n=0}^{\infty} a_n (x-a)^n$, one of the following holds:

1. There is a positive number $R$ such that $\sum_{n=0}^{\infty} a_n (x-a)^n$ diverges for $|x-a| > R$ and converges for $|x-a| < R$. The series may or may not converge for $x = a \pm R$.
2. $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for all $x$ (i.e. $R = \infty$).
3. $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges at $x = a$ (i.e. $R = 0$).

## Radius of convergence

**Def$^n$:** The number $R$ is called radius of convergence of $\sum_{n=0}^{\infty} a_n (x-a)^n$ and the set of all values for which the series converges is called interval of convergence.

- For $\sum_{n=0}^{\infty} x^n$, radius of convergence, $R = 1$
  interval of convergence is $R \in (-1, 1)$.
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## Example 1

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

Sol$^n$:

Example 2

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$$

## Example 3.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots \quad .$$ Let $a_n = \frac{(-1)^{n-1} x^{2n-1}}{2n-1}$

## Example 4

$$\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\, x^2 + \cdots$$