7 Techniques of integration

7.2 Trigonometric integrals

In this section we will learn to integrate combinations of trig functions.

$$\int \sin^a x \cos^b x \mathrm{d}x$$
$$\int \tan^a x \sec^b x \mathrm{d}x$$

You will need to remember

$$\sin^2 x + \cos^2 x = 1 \qquad \text{divide by cos} \qquad \tan^2 x + 1 = \sec^2 x$$

and some double angle formulas

$$\begin{aligned} \sin 2x &= 2 \sin x \cos x \\ \cos 2x &= 1 - 2\sin^2 x = 2\cos^2 x - 1 = \cos^2 x - \sin^2 x \end{aligned}$$

Examples — not obviously a substitution integral

$$\begin{aligned} \int \sin^5 x \mathrm{d}x &= \int \sin x (1 - \cos^2 x)^2 \mathrm{d}x \\ &= \int \sin x (1 - 2\cos^2 x + \cos^4 x) \mathrm{d}x \end{aligned}$$

now this is clearly a substitution integral with $u = \cos x$ and $u' = -\sin x$

$$\begin{aligned} &= -\int (1 - 2u^2 + u^4) \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x = -\int (1 - 2u^2 + u^4) \mathrm{d}u \\ &= -u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + c \\ &= -\cos x + \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x + c \end{aligned}$$

So even powers of sine became $\cos^2 x$, leaving us with a single sine and a polynomial in cosine.

What about even powers? We need the double angle formulas

$$\begin{aligned}
\int \cos^4 x \mathrm{d}x &= \int (\cos^2 x)^2 \mathrm{d}x && 2\cos^2 x = \cos 2x + 1 \\
&= \int \left(\frac{\cos 2x + 1}{2}\right)^2 \mathrm{d}x \\
&= \int \left(\frac{\cos^2 2x}{4} + \frac{\cos 2x}{2} + \frac{1}{4}\right) \mathrm{d}x && 2\cos^2 2x = \cos 4x + 1 \\
&= \int \left(\frac{\cos 4x + 1}{8} + \frac{\cos 2x}{2} + \frac{1}{4}\right) \mathrm{d}x \\
&= \frac{1}{8} \int (\cos 4x + 4\cos 2x + 3) \, \mathrm{d}x \\
&= \frac{1}{8} \left(\frac{1}{4} \sin 4x + 2 \sin 2x + 3x\right) + c
\end{aligned}$$

The algorithm......

Theorem (page 484 Cpht 7.2)**.** *To integrate* $\int \sin^a x \cos^b x \mathrm{d}x$

- *If power of cosine is odd, then hold onto 1 power of cosine, and turn all the others into sines using* $\cos^2 x = 1 - \sin^2 x$*.*

$$\begin{aligned}
\int \sin^a x \cos^{2k+1} x \mathrm{d}x &= \int \sin^a x (\cos^2 x)^{2k} \cos x \mathrm{d}x \\
&= \int \sin^a x (1 - \sin^2 x)^{2k} \cos x \mathrm{d}x
\end{aligned}$$

- *If power of sine is odd, then hold onto 1 power of sine, and turn all the others into cosines using* $\sin^2 x = 1 - \cos^2 x$*.*

$$\begin{aligned}
\int \sin^{2k+1} x \cos^b x \mathrm{d}x &= \int \sin x (\sin^2 x)^{2k} \cos^b x \mathrm{d}x \\
&= \int \sin x (1 - \cos^2 x)^{2k} \cos^b x \mathrm{d}x
\end{aligned}$$

- *If both powers of sine and cosine are even, then use*

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad\qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Why does this work? Because $\sin x$ and $\cos x$ are derivatives of each other.
We can do similar things with sec and tan.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \tan x &= \sec^2 x \\
\frac{\mathrm{d}}{\mathrm{d}x} \sec x &= \frac{\sin x}{\cos^2 x} = \sec x \tan x
\end{aligned}$$

So we try to massage the integrand so it looks like a subsitution of either $u = \tan x$ or $u = \sec x$

$$\begin{aligned}\int \tan^2 x \sec^4 x \mathrm{d}x &= \int \tan^2 x \sec^2 x (\sec^2 x \mathrm{d}x) \\ &= \int \tan^2 x (1 + \tan^2 x)(\sec^2 x \mathrm{d}x) && u = \tan x, \frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 \\ &= \int u^2 (1 + u^2) \mathrm{d}u && \text{and so on}\end{aligned}$$

$$\int \tan^3 x \sec^7 x \mathrm{d}x = \int \tan^3 x \sec^5 x (\sec^2 x \mathrm{d}x)$$

How do we turn $\sec^{odd}$ into tan? urgh. Instead try $u = \sec x$, so $u' = \tan x \sec x$

$$\begin{aligned}\int \tan^3 x \sec^7 x \mathrm{d}x &= \int \tan^2 x \sec^6 x (\sec x \tan x \mathrm{d}x) \\ &= \int (\sec^2 x - 1) \sec^6 x (\sec x \tan x \mathrm{d}x) \\ &= \int (u^2 - 1) u^6 \mathrm{d}u\end{aligned}$$

Theorem (page 486 Cpht 7.2)**.** *To integrate* $\int \tan^a x \sec^b x \mathrm{d}x$

- *If power of secant is even the hold onto* $\sec^2 x$ *and turn other factors of* $\sec^2 x$ *into* $(1 + \tan^2 x)$

$$\begin{aligned}\int \tan^a x \sec^{2k} x \mathrm{d}x &= \int \tan^a x (\sec^2 x)^{k-1} \sec^2 x \mathrm{d}x \\ &= \int \tan^a x (1 + \tan^2 x)^{k-1} \sec^2 x \mathrm{d}x\end{aligned}$$

  then sub $u = \tan x$*.*

- *If power of tangent is odd, then hold onto 1 factor of* $\sec x \tan x$ *and turn remaining factors of* $\tan^2 x$ *into* $(\sec^2 x - 1)$

$$\begin{aligned}\int \tan^{2k+1} x \sec^b x \mathrm{d}x &= \int \tan^{2k} x \sec^{b-1} x \sec x \tan x \mathrm{d}x \\ &= \int (\sec^2 x - 1)^k \sec^{b-1} x \sec x \tan x \mathrm{d}x\end{aligned}$$

  then sub $u = \sec x$*.*

- *Othercases are harder.*

One can show this using substitution

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x = -\log|\cos x| + c \equiv \log|\sec x| + c$$

This one is a bit harder

$$\int \sec x \mathrm{d}x = \log \left|\frac{1+\sin x}{\cos x}\right| + c = \log|\sec x + \tan x| + c$$

See the textbook p 486.

Examples

$$\begin{aligned}
\int \tan^4 x \mathrm{d}x &= \int \tan^2 x(1+\sec^2 x)\mathrm{d}x \\
&= \int (\tan^2 x \sec^2 x + \tan^2 x)\mathrm{d}x \\
&= \int \tan^2 x \sec^2 x \mathrm{d}x + \int (1+\sec^2 x)\mathrm{d}x \\
&= \frac{1}{3}\tan^3 x + x + \tan x + c
\end{aligned}$$

A harder one

$$\int \sec^3 x \mathrm{d}x = \int \sec^2 x \sec x \mathrm{d}x$$

Integration by parts $f = \sec x$ and $g' = \sec^2 x$, so $g = \tan x$ and $f' = \sec x \tan x$

$$\begin{aligned}
\int \sec^2 x \sec x \mathrm{d}x &= \sec x \tan x - \int \sec x \tan^2 x \mathrm{d}x \\
&= \sec x \tan x - \int \sec x(1+\sec^2 x)\mathrm{d}x \\
&= \sec x \tan x - \log|\sec x + \tan x| - \int \sec^3 x \mathrm{d}x
\end{aligned}$$

So

$$\int \sec^3 x \mathrm{d}x = \frac{1}{2}\left(\sec x \tan x - \log|\sec x + \tan x|\right) + c$$

7.3 Trigonometric substitutions

Consider the problem of computing the area of a circle - or a semi-circle.

- The formula for a circle is $x^2 + y^2 = r^2$, so the top half is

$$y = \sqrt{r^2 - x^2}$$

- So the area is given by

$$\begin{aligned}
A &= 4\int_0^r \sqrt{r^2 - x^2}\mathrm{d}x \\
&= \pi r^2
\end{aligned}$$

- But how is this going to work? Where did the π come from?

In order to do this sort of integral we need to do the substitution rule again.

$$\int f(u)\mathrm{d}u = \int f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

Before we started with something like the right-hand side and tried to find a $\frac{\mathrm{d}u}{\mathrm{d}x}$ so that we could write it as the left-hand side.

But now instead of looking for $u(x)$, we will substitute $x = x(\theta)$.

$$\int f(x)\mathrm{d}x = \int f(x(\theta))\frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta$$

ie we start with the left-hand side and rewrite it as the right-hand side.

Go back to the circle example and substitute $x = r\sin\theta$

$$\int_0^r \sqrt{r^2 - x^2}\mathrm{d}x = \int \sqrt{r^2 - r^2\sin^2\theta}\frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta$$

Now what happens to the terminals?

$$\begin{aligned} x = r &= r\sin\theta & \theta &= \pi/2 \\ x = 0 &= r\sin\theta & \theta &= 0 \end{aligned}$$

So - put this in:

$$\begin{aligned} \int_0^r \sqrt{r^2 - x^2}\mathrm{d}x &= \int_0^{\pi/2} r\sqrt{1-\sin^2\theta}\cdot r\cos\theta\mathrm{d}\theta \\ &= \int_0^{\pi/2} r^2|\cos\theta|\cos\theta\mathrm{d}\theta \\ &= r^2\int_0^{\pi/2}\cos^2\theta\mathrm{d}\theta && \text{cos is positive on range} \\ &= r^2\int_0^{\pi/2}(1+\cos 2\theta)/2\mathrm{d}\theta \\ &= \frac{r^2}{2}[\theta + (\sin 2\theta)/2]_0^{\pi/2} \\ &= \frac{r^2}{2}\frac{\pi}{2} \end{aligned}$$

Why does this work? Because of trig-identities

Identity	Expression	Substitution	range
$1-\sin^2\theta = \cos^2\theta$	$\sqrt{a^2-x^2}$	x $= a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$
$\sec^2\theta - 1 = \tan^2\theta$	$\sqrt{x^2-a^2}$	x $= a\sec\theta$	$0 \le \theta \le \pi/2$
$1+\tan^2\theta = \sec^2\theta$	$\sqrt{a^2+x^2}$ or $\dfrac{1}{a^2+x^2}$	x $= a\tan\theta$	$-\pi/2 \le \theta \le \pi/2$

We saw the first of these when we did $\int \sqrt{r^2 - x^2} \mathrm{d}x$.

$$\begin{aligned} \int \sqrt{a^2 - x^2} \mathrm{d}x &= \int \sqrt{a^2 - a^2 \sin^2 \theta} a \cos \theta \mathrm{d}\theta \\ &= a^2 \int \cos^2 \theta \mathrm{d}\theta \\ &= \frac{a^2}{2} \int 1 + \cos 2\theta \mathrm{d}\theta \\ &= \frac{a^2}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right] + c \end{aligned}$$

Need to re-express this in terms of x.

$$\begin{aligned} x &= a \sin \theta & \theta &= \sin^{-1}(x/a) \\ a \cos^2 \theta &= a^2 - a^2 \sin^2 \theta & a \cos \theta &= \sqrt{a^2 - x^2} \\ \sin 2\theta &= 2 \underbrace{\sin \theta} \underbrace{\cos \theta} \\ &= 2 \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} \end{aligned}$$

So solution is

$$\int \sqrt{a^2 - x^2} \mathrm{d}x = \frac{a^2}{2} \sin^{-1}(x/a) + \frac{x}{2} \sqrt{a^2 - x^2} + c$$

Another example

$$\int \frac{1}{x^2 + 4x + 7} \mathrm{d}x = \int \frac{1}{(x+2)^2 - 4 + 7} \mathrm{d}x = \int \frac{1}{(x+2)^2 + 3} \mathrm{d}x$$

So this is a $\tan \theta$ example — put $(x+2) = \sqrt{3} \tan \theta$, so $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sqrt{3} \sec^2 \theta$

$$\begin{aligned} \int \frac{1}{x^2 + 4x + 7} \mathrm{d}x &= \int \frac{1}{(x+2)^2 - 4 + 7} \mathrm{d}x = \int \frac{1}{(x+2)^2 + 3} \mathrm{d}x \\ e &= \int \frac{1}{3 \tan^2 \theta + 3} \cdot \sqrt{3} \sec^2 \theta \mathrm{d}\theta \\ &= \int \frac{\sqrt{3} \sec^2 \theta}{3(\sec^2 \theta)} \mathrm{d}\theta \\ &= \frac{1}{\sqrt{3}} \int 1 \mathrm{d}\theta \\ &= \frac{1}{\sqrt{3}} \theta + c \end{aligned}$$

Now $\frac{x+2}{\sqrt{3}} = \tan \theta$, so

$$\int \frac{1}{x^2 + 4x + 7} \mathrm{d}x = \frac{1}{\sqrt{3}} \theta + c = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x+2}{\sqrt{3}} \right) + c$$

Another one

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-16}}$$

This contains $\sqrt{x^2-a^2}$ — so it is a $\sec\theta$ one. Substitute $x = 4\sec\theta$, so $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 4\sec\theta\tan\theta$.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-16}} &= \int \frac{4\sec\theta\tan\theta\mathrm{d}\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}} \\
&= \int \frac{\tan\theta\mathrm{d}\theta}{4\sec\theta\sqrt{16\tan^2\theta}} \\
&= \int \frac{\tan\theta\mathrm{d}\theta}{4\sec\theta\cdot 4\tan\theta} \\
&= \int \frac{1}{16\sec\theta}\mathrm{d}\theta \\
&= \int \frac{\cos\theta}{16}\mathrm{d}\theta = \frac{1}{16}\sin\theta + C
\end{aligned}$$

Now we need to convert this back to x. How? Well x is in terms of sec. But tan = sin / cos, so tan / sec = sin.

$$\begin{aligned}
\sin\theta = \frac{\tan\theta}{\sec\theta} &= \frac{\sqrt{\sec^2\theta-1}}{\sec\theta} \\
&= \frac{\sqrt{x^2/16-1}}{x/4} = \frac{4\sqrt{x^2/16-1}}{4x/4} \\
&= \frac{\sqrt{x^2-16}}{x}
\end{aligned}$$

So our integral is

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-16}} = \frac{\sqrt{x^2-16}}{16x} + c$$

7.4 Partial fractions

This technique allows us to integrate any rational function — ie ratio of polynomials.

We have seen a couple of examples

$$\begin{aligned}
\int \frac{1}{x+a}\mathrm{d}x &= \log|x+a| + c \\
\int \frac{1}{x^2+a^2}\mathrm{d}x &= \frac{1}{a}\arctan(x/a) + c
\end{aligned}$$

In general we want to rewrite a general rational function $P(x)/Q(x)$ as a sum of simpler things that we can integrate.

Let us start by adding together simple things we know how to integrate:

$$\begin{aligned}
\frac{1}{x-1} + \frac{2}{x+3} &= \frac{1(x+3)+2(x-1)}{(x-1)(x+3)} \\
&= \frac{3x+1}{x^2+2x-3}
\end{aligned}$$

If we reverse this process then we can integrate things like

$$\int \frac{3x+1}{x^2+2x-3}\mathrm{d}x = \int \left(\frac{1}{x-1} + \frac{2}{x+3} \right)$$
$$= \log|x-1| + 2\log|x+3| + c$$

So how do we reverse this process for a general rational function $f(x) = P(x)/Q(x)$?

1. First, if $\deg(P) \geq \deg(Q)$ then rewrite as a proper fraction $S + R/Q$ with $\deg R < \deg Q$ — ie do polynomial division.

$$\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1} \qquad \text{do the division}$$

2. Split the ratio into simpler pieces — depends on the denominator factors.

$$\frac{A}{(x-a)^n} \qquad\qquad \frac{Bx+C}{(x^2+bx+c)^M}$$

   where $(x-a)$ and (x^2+bx+c) are factors of the denominator.

3. Find the constants

4. Integrate term by term.

How do we know how to split it up?

denominator factor	partial fraction expansion
$(x-a)$	$\frac{A}{x-a}$
$(x-a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r}$
(x^2+bx+c)	$\frac{Bx+C}{x^2+bx+c}$
$(x^2+bx+c)^r$	$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \frac{B_3x+C_3}{(x^2+bx+c)^3} + \cdots$

Go back to our example — first step is okay. Second step, factorise and split:

$$\frac{3x+1}{x^2+2x-3} = \frac{3x+1}{(x+3)(x-1)}$$
$$= \frac{A}{x+3} + \frac{B}{x-1} 2 \cdot \frac{1}{2}$$

How do we now find the constants? Add it all back together:

$$\frac{3x+1}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$
$$= \frac{A(x-1)+B(x+3)}{(x+3)(x-1)} = \frac{x(A+B)+(-A+3B)}{(x+3)(x-1)}$$

So we have 2 equations to solve (coming from the

$$A + B = 3 \qquad -A + 3B = 1$$
$$A = 2 \qquad B = 1$$

Hence

$$\begin{aligned}\int \frac{3x+1}{(x+3)(x-1)} &= \int \frac{2}{x+3} + \frac{1}{x-1} \\ &= 2\log|x+3| + \log|x-1| + c \\ &= \log|(x+3)^2(x-1)| + c\end{aligned}$$

Another example — repeated factor

$$\begin{aligned}\frac{x^2-9x+17}{(x-2)^2(x+1)} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1} \\ &= \frac{A(x-2)(x+1) + B(x+1) + C(x-2)^2}{(x-2)^2(x+1)} \\ &= \frac{x^2(A+C) + x(-A+B-4C) + (-2A+B+4C)}{(x-2)^2(x+1)}\end{aligned}$$

Hence we have the system of equations

$$A + C = 1 \qquad -A + B - 4C = -9 \qquad -2A + B + 4C = 17$$

Solve this to get

$$A = -2 \qquad B = 1 \qquad C = 3$$

And so

$$\begin{aligned}\int \frac{x^2-9x+17}{(x-2)^2(x+1)}\mathrm{d}x &= \int \frac{-2}{x-2}\mathrm{d}x + \int \frac{1}{(x-2)^2}\mathrm{d}x + \int \frac{3}{x+1}\mathrm{d}x \\ &= -2\log|x-2| - \frac{1}{x-2} + 3\log|x+1| + c \\ &= \log\left|\frac{(x+1)^3}{(x-2)^2}\right| - \frac{1}{x-2} + c\end{aligned}$$

Another example — quadratic factor with complex roots

$$\int \frac{x+1}{x^2-2x+5}\mathrm{d}x$$

- Start by completing the square $(x^2 - 2x + 5) = (x-1)^2 + 4$.
- This suggests setting $u = x - 1$, so $\mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}\int \frac{x+1}{x^2-2x+5}\mathrm{d}x &= \int \frac{u+2}{u^2+4}\mathrm{d}u \\ &= \int \frac{u}{u^2+4} + \frac{2}{u^2+4}\mathrm{d}u \\ &= \frac{1}{2}\log|u^2+4| + 2\cdot\frac{1}{2}\arctan(u/2) + c \\ &= \frac{1}{2}\log|(x-1)^2+4| + \arctan((x-1)/2) + c\end{aligned}$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\arctan(x/a)$$

7.5 Strategies for integration

You should read this chapter carefully! It gives a nice 4-stage integration strategy guide

1. Simplify the integrand as much as possible — $x(1-\sqrt{x}) = x - x^{3/2}$.
2. Look for a substitution — $\frac{x^2}{1+x^3}$.
3. Try to classify the integrand
   - Trig functions — $\sin^3 x \cos^2 x$.
   - Rational — $\frac{x}{(1-x)(2-x)}$
   - Integration by parts — $x \sin x$
   - Radicals — $\sqrt{a^2+x^2}$.
4. Try again — at first sight this sounds kinda stupid, but it isnt such bad advice. In the first 3 steps you eliminate several obvious possibilities, but perhaps things need to be forced?

Anyway - please read through this stuff **AND DO PLENTY OF EXAMPLES**.

7.8 Improper integrals

- Up until this point we have looked at integrals of nice functions on nice regions — ie continuous functions $f(x)$ on a bounded region $[a, b]$.
- We now extend this to functions that have a discontinuity and / or on an infinite region $[a, \infty)$ or $(-\infty, b]$ or $(-\infty, \infty)$.
- Such integrals are called improper integrals.
- We handle them using limits.
- We do 2 types — (1) infinite intervals and (2) discontinuous integrands.

  Motivating example

$$\int_0^\infty e^{-x} \mathrm{d}x$$

We would naively like to just put $[-e^{-x}]_0^\infty$ — but we can do it more rigorously using limits.

$$\begin{aligned} \int_0^b e^{-x} \mathrm{d}x &= [-e^{-x}]_0^b \\ &= 1 - e^{-b} \end{aligned}$$

As $b \to \infty$, $e^{-b} \to 0$ and the integral becomes 1. So we can define

$$\begin{aligned} \int_0^\infty e^{-x} \mathrm{d}x &= \lim_{b\to\infty} \int_0^b e^{-x} \mathrm{d}x \\ &= \lim_{b\to\infty} (1 - e^{-b}) = 1. \end{aligned}$$

On the other hand if we take

$$\begin{aligned}\int_0^\infty e^x \mathrm{d}x &= \lim_{b\to\infty}\int_0^b e^x \mathrm{d}x \\ &= \lim_{b\to\infty}(e^b - 1)\end{aligned}$$

The limit is divergent — the integral does not exist. Consider it as the area under the curve — it is infinite.

Definition (Improper integral type 1 — p 531)**.**

- If $\int_a^t f(x)\mathrm{d}x$ exists for every $t \geq a$, then we define

  $$\int_a^\infty f(x)\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\mathrm{d}x$$

  if the limit exists and is finite.
- Similarly for $\int_{-\infty}^b f(x)\mathrm{d}x$.
- If these limits exist the integrals are called convergent, else divergent.
- If $\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^a f(x)\mathrm{d}x$ are convergent then we define

  $$\int_{-\infty}^\infty f(x)\mathrm{d}x = \int_{-\infty}^a f(x)\mathrm{d}x + \int_a^\infty f(x)\mathrm{d}x$$

  for any real number a.

For what values of q is this $\displaystyle\int_1^\infty x^q \mathrm{d}x$ convergent? If $q \neq -1$ then

$$\begin{aligned}\int_1^\infty x^q \mathrm{d}x &= \lim_{b\to\infty}\int_1^b x^q \mathrm{d}x \\ &= \lim_{b\to\infty}[\frac{x^{q+1}}{q+1}]_1^b \\ &= \lim_{b\to\infty}\frac{1}{q+1}(b^{q+1} - 1) \\ &= -\frac{1}{q+1} + \lim_{b\to\infty}\frac{1}{q+1}b^{q+1}\end{aligned}$$

This limit exists if $q + 1 < 0$ or $q < -1$. Now, if $q = -1$ then we have

$$\begin{aligned}\int_1^\infty 1/x \mathrm{d}x &= \lim_{b\to\infty}\int_1^b 1/x \mathrm{d}x \\ &= \lim_{b\to\infty}[\log|x|]_1^b \\ &= \lim_{b\to\infty}(\log b - 0)\end{aligned}$$

which does not exist. Hence the integral is convergent for all real $q < -1$ and divergent for $q \geq -1$.

Another example (over the whole $\mathbb{R}$)

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \int_{0}^{\infty} \frac{1}{1+x^2}\mathrm{d}x + \int_{-\infty}^{0} \frac{1}{1+x^2}\mathrm{d}x$$

Let us look at these individually

$$\begin{aligned}
\int_{0}^{\infty} \frac{1}{1+x^2}\mathrm{d}x &= \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\mathrm{d}x \\
&= \lim_{b\to\infty} [\arctan(x)]_0^b \\
&= \lim_{b\to\infty} \arctan(b) - \arctan(0) \\
&= \pi/2
\end{aligned}$$

And very similarly

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\mathrm{d}x = \pi/2$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \pi$.

The other type of improper integral is one in which the integrand is discontinuous. For example

$$\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x = \left[-\frac{1}{x}\right]_{-1}^{1} = -2 \qquad\qquad \text{WRONG — integrand is positive!}$$

We cannot do this because the integrand is divergent at $x = 0$.

Definition (Improper integral type 2 — p534)**.**

- If f is continous on $[a, b)$ and is discontinous at b then

$$\int_{a}^{b} f(x)\mathrm{d}x = \lim_{t\to b^-} \int_{a}^{t} f(x)\mathrm{d}x$$

  if the limit exists and is finite.

- If f is continous on $(a, b]$ and is discontinous at a then

$$\int_{a}^{b} f(x)\mathrm{d}x = \lim_{t\to a^+} \int_{t}^{b} f(x)\mathrm{d}x$$

  if the limit exists and is finite.

- If these limits exist then the integral is convergent, else divergent.

- If f has a discontinuity at $c \in (a, b)$ and both $\int_a^c f(x)\mathrm{d}x$ and $\int_c^b f(x)\mathrm{d}x$ are convergent then

$$\int_{a}^{b} f(x)\mathrm{d}x = \int_{a}^{c} f(x)\mathrm{d}x + \int_{c}^{b} f(x)\mathrm{d}x$$

Evaluate $\displaystyle\int_2^6 \frac{1}{\sqrt{z-2}}\mathrm{d}z$. Has singularity at $z=2$, so we use limits.

$$\begin{aligned}\int_2^6 \frac{1}{\sqrt{z-2}}\mathrm{d}z &= \lim_{a\to 2^-}\int_a^6 \frac{1}{\sqrt{z-2}}\mathrm{d}z \\ &= \lim_{a\to 2^-}[2\sqrt{z-2}]_a^6 \\ &= \lim_{a\to 2^-}(2\sqrt{4}-2\sqrt{a-2}) \\ &= 4-0=4\end{aligned}$$

Evaluate $\displaystyle\int_{-1}^1 \frac{1}{x^2}\mathrm{d}x$

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\mathrm{d}x &= \lim_{b\to 0^-}\int_{-1}^b \frac{1}{x^2}\mathrm{d}x + \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}\mathrm{d}x \\ &= \underbrace{\lim_{b\to 0^-}[-1-1/b]}_{\infty} + \underbrace{\lim_{a\to 0^+}[-1+1/a]}_{\infty}\end{aligned}$$

Answer is divergent.

Do not do convergence / divergence comparison test.