11 Infinite sequences and series

Consider the function e^{-x^2}. It is very easy to differentiate

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{-x^2} = -2xe^{-x^2}$$

But its anti-derivative cannot be written in terms of the functions we know — polynomials, trig functions, exponential and logarithms. That being said, the anti-derivative of this function is extremely important. It turns up quite frequently in different bits of mathematics — perhaps most obviously in probability and statistics as the bell-curve and the normal-distribution. So we need some way of representing this antiderivative and evaluating it. To do this we use taylor series — but to get at taylor series we need to understand power series, and to understand that we need series and to understand that we need sequences. So that is where we start.

11.1 Sequences

The easiest way to think of a sequence is a function from the positive integers to the real numbers.

$$a_n = f(n) \qquad\qquad f : \mathbb{N} \to \mathbb{R}$$

Or simply as a list of real numbers

$$\{a_1, a_2, a_3, \dots\}$$

which we also write as

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

We call a_n the n^{th} term of the sequence. Some examples

- $\left\{\frac{1}{n^2}\right\}$ is the sequence $1, 1/4, 1/9, 1/16, \dots$.
- $\left\{\frac{n}{2n+1}\right\}$ is the sequence $1/3, 2/5, 3/7, \dots$.
- $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots$ is the sequence $\left\{\frac{1}{2n}\right\}$
- $1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{8}, \dots$ is the sequence $\{1+2^{-n}\}$
- $1, \frac{3}{5}, \frac{1}{2}, \frac{5}{11}, \frac{3}{7}, \frac{7}{17}, \dots$ is the sequence $\left\{\frac{n+1}{3n-1}\right\}$

Some other famous sequences

$$\{2, 3, 5, 7, 11, 13, 17, \dots\}$$
$$\{1, 1, 2, 3, 5, 8, 13, \dots\}$$

The first is just prime numbers, while the second is the Fibonacci sequence. The second satisfies a “recurrence” $a_n = a_{n-1} + a_{n-2}$.

We aren't so interested in recurrences for this course, the main point is to look at limits. In particular what can we say about the behaviour of a_n as n becomes very large. To introduce this idea let us look at a very simple example

$$a_n = 1 + \frac{1}{n}$$

So now as n becomes larger and larger, it is clear that $\frac{1}{n}$ becomes smaller and smaller. Hence a_n gets closer and closer to 1. In this case we write

$$\lim_{n\to\infty} a_n = 1 \qquad\qquad \text{or } a_n \to 1$$

More generally

Definition. (not so rigorous) A sequence $\{a_n\}$ has the limit L if we can make the terms a_n as close to L as we like by taking n sufficiently large. In this case we write

$$\lim_{n\to\infty} a_n = L \qquad\qquad \text{or } a_n \to L$$

If this limit exists (and is finite) we say that the sequence is convergent. Otherwise we say it is divergent.

only do if there is time

Now this is not so rigorous what do "as close as we like" and "sufficiently large" actually mean. Well - lets write down the rigorous definition

Definition. A sequence $\{a_n\}$ has limit L if for every $\epsilon > 0$ there is some integer N so that

$$\text{if } n > N \text{ then } |a_n - L| < \epsilon$$

In this case we write

$$\lim_{n\to\infty} a_n = L \qquad\qquad \text{or } a_n \to L$$

So what does this mean. Let us parse this in the context of our example sequence $a_n = 1 + 1/n$. We think that this has limit $L = 1$. Now pick any $\epsilon > 0$ — ask the audience. So we need to be able to make n large enough so that

$$|a_n - 1| = |1/n| = 1/n < \epsilon.$$

Hence we need to have $n > 1/\epsilon$. So let $N = 1/\epsilon$. Then if we take any $n > N$ then $1/n < 1/N = \epsilon$. Hence

$$|a_n - 1| = 1/n < 1/N = \epsilon.$$

That is — no matter what ϵ you pick, you can always pick some point in the sequence N so that all the terms coming after that are no further than ϵ from L.

only do above if there is time

Okay - I don't want to dwell on this any longer — if you want to see this sort of stuff you should take Maths220. We dont do things too rigorously here. Mostly we want to build some tools and intuition for doing things. Worry about strict rigor later.

Do not worry about defn 2 and 5 in the text — we do not do this.

Now we have seen limits before, in particular limits of some functions as $x \to \infty$ or $x \to 0$. The following is a useful theorem that relates the limits of sequences to the limits of functions

Theorem. *If $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.*

For example

$$\lim_{x\to\infty} e^{-x} = 0 \qquad\qquad a_n = e^{-n} \to 0.$$

And perhaps more usefully for any $r > 0$

$$\lim_{x\to\infty} {}^1\!/_{x^r} = 0 \qquad\qquad a_n = n^{-r} \to 0$$

Now we want to build up some tools so that we can express the limits of complicated things in terms of the limits of simpler things. Again this is like what you did for derivatives and integrals. Indeed it is a very standard thing to do in mathematics. You define some new object or property (here it is limits) and then you see how it interacts with some standard things (like addition and multiplication etc). Thankfully limits are nice — even easier than derivatives and integrals — they "play nicely" with arithmetic

Theorem (limit laws)**.** *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences with $a_n \to A$ and $b_n \to B$. Further let c be a constant. Then*

$$\begin{aligned}
\lim_{n\to\infty} (a_n + b_n) &= A + B \\
\lim_{n\to\infty} (a_n - b_n) &= A - B \\
\lim_{n\to\infty} (ca_n) &= cA \\
\lim_{n\to\infty} (a_n \cdot b_n) &= A \cdot B \\
\lim_{n\to\infty} {}^{a_n}\!/_{b_n} &= {}^{A}\!/_{B} && \textit{provided } B \neq 0 \\
\lim_{n\to\infty} (a_n)^p &= A^p && \textit{provided } p > 0 \textit{ and } a_n > 0
\end{aligned}$$

So lets use this to compute the limit

$$\lim_{n\to\infty} \frac{3n}{2n+7} =$$

We cannot use our limit laws yet because numer and denom are divergent

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{3}{2 + 7/n} && \text{divide top and bottom by } n \\
&= \frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} (2 + 7/n)} \\
&= \frac{3}{\lim 2 + \lim 7n} = \frac{3}{2 + 7 \lim \frac{1}{n}} \\
&= \frac{3}{2+0} = {}^3\!/_2
\end{aligned}$$

Now you lot do

$$\lim_{n\to\infty} \frac{3n^2 + 4}{5n^2 + n + 7}$$

What about a sequence like $a_n = \sin(\pi/n)$. What do we think is going to happen? Well — as $n \to \infty$ we think that $\pi/n \to 0$ and so we'd expect that $\sin(\pi/n) \to \sin(0) = 0$. Now this is true, but it only works because $\sin(x)$ has a very nice property — it is continuous. More generally this reasoning is fine provided we have continuity

Theorem. *If $a_n \to L$ and $f(x)$ is continuous at $x = L$ then $f(a_n) \to f(L)$.*

Now sometimes the form of the sequence is very messy but we can "squeeze" it between two simpler sequences

Theorem (squeeze theorem)**.** *If $a_n \leq b_n \leq c_n$ and $\lim a_n = \lim c_n = L$ then $b_n \to L$.*

Draw a simple pic. One easy consequence of this is the following

$$\text{if } |a_n| \to 0 \text{ then } a_n \to 0$$

This tells us things like

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

since $|(-1)^n/n| = 1/n$. This is interesting because $a_n = (-1)^n$ does not converge.

Another application — consider $a_n = \frac{n!}{n^n}$ — a little massaging is required to make sense of this.

$$\begin{aligned} \frac{n!}{n^n} &= \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} \\ &= \frac{1}{n}\left(\frac{2}{n}\frac{3}{n}\cdots\frac{n}{n}\right) \leq \frac{1}{n} \end{aligned}$$

Hence we see that $0 \leq a_n \leq {}^1/_n$. Hence $a_n \to 0$.

A very useful example — one we will need when we get to sequences.

Theorem. *The sequence $a_n = r^n$ is convergent when $-1 < r \leq 1$ and otherwise is divergent.*

$$\lim_{n\to\infty} r^n = \begin{cases} 1 & r = 1 \\ 0 & -1 < r < 1 \\ \textit{divergent} & \textit{otherwise} \end{cases}$$

Okay — a couple of these are easy $1^n = 1, 0^n = 0$. And we have seen from playing around with exponential functions that

$$\lim_{x\to\infty} r^x = \begin{cases} \infty & r > 1 \\ 0 & 0 < r < 1 \end{cases}$$

Now if $r < 0$ then $|r^n| = |r|^n$ and so the negative cases can just be expressed in terms of the positive cases.

Okay — one last little bit of sequences. In some cases we dont actually care what the limit is, just that the sequence converges. Indeed in some cases this is all we can prove. Our last result gives quite general conditions for a sequence to converge.

Consider the sequence $a_n = \frac{1}{n+2}$. We know that this converges to zero. But it also has 2 other very useful properties.

- $a_n > 0$ — it has a lower bound.
- $a_n > a_{n+1}$ — it is a "decreasing sequence"

$$a_n - a_{n+1} = \frac{1}{n+2} - \frac{1}{n+3} = \frac{(n+3)-(n+2)}{(n+3)(n+2)} > 0$$

So we have a sequence that must always decrease, and is bounded below (ie cannot be less than a certain number). This is enough to know that it must converge (without knowing exactly what it converges to).

Some definitions

Definition.
- A sequence $\{a_n\}$ is decreasing when $a_n \geq a_{n+1}$ for all $n \geq 1$.
- A sequence $\{a_n\}$ is increasing when $a_n \leq a_{n+1}$ for all $n \geq 1$.
- A sequence is monotonic if it is either increasing or decreasing.
- A sequence is bounded below if there is some number M so that $a_n \geq M$ for all $n \geq 1$.
- A sequence is bounded above if there is some number M so that $a_n \leq M$ for all $n \geq 1$.
- If a sequence is bounded above and below then it is simply called "bounded".

Theorem (Monotonic sequence theorem)**.** *Every bounded monotonic sequence converges.*

This is a very useful result — even though it does not tell us what the limit is, it does tell us that the limit exists.

11.2 Series

Consider the sequence $\{a_n\}$. Now add up the terms

$$a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

This is called an infinite series. We denote it by

$$\sum_{n=1}^{\infty} a_n$$

But this infinite sum does not always make sense. How can we understand

$$1 + 2 + 3 + \ldots \qquad \text{or } \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \ldots$$

Lets think about the first one — clearly this sum just keeps getting larger and larger as we add more terms. If we consider the partial sums:

$$\begin{aligned} s_1 &= 1 = 1 \\ s_2 &= 1 + 2 = 3 \\ s_3 &= 1 + 2 + 3 = 6 \\ s_4 &= 1 + 2 + 3 + 4 = 10 \\ s_n &= 1 + \cdots + n = \frac{n(n+1)}{2} \end{aligned}$$

So we can think about what happens to these partial sums s_n as $n \to \infty$. Clearly they diverge as $n \to \infty$. Hence the sum $1 + 2 + 3 + \dots$ does not make sense.

What about the other one

$$
\begin{aligned}
s_1 &= \frac{1}{2} \\
s_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
s_4 &= \frac{1}{2} + \cdots + \frac{1}{16} = \frac{15}{16} \\
s_n &= \frac{1}{2} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} = 1 - 2^{-n}
\end{aligned}
$$

Now it is clear that $s_n \to 1$ as $n \to \infty$. In this case we write

$$\sum_{n=1}^{\infty} 2^{-n} = 1$$

In both cases we determine whether or not the infinite series exists by considering its partial sums. In particular whether the sequence of partial sums converges. So we have reduced the convergence of these infinite series to a problem of convergence of sequences.

Anyone want to take a guess at the "harmonic series"

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

Let us make this more general with a definition

Definition. Let $a_1 + a_2 + \dots$ be an infinite series, and let

$$s_n = a_1 + \dots a_n$$

denote its n^{th} partial sum. If $s_n \to s$ for some finite real number s we write

$$a_1 + a_2 + a_3 + \cdots = s \qquad\qquad \sum_{n=1}^{\infty} a_n = s$$

and say that the series is convergent. If the sequence of partial sums does not converge we say that the series is divergent.

Again we have defined this nice mathematical object, we can see how it works together with our standard arithmetic. Not quite as nice as sequences, but still pretty good

Theorem. *Let $\sum a_n$ and $\sum b_n$ be convergent sequences with $\sum a_n = A$ and $\sum b_n = B$. Further let c be some real number. Then*

$$
\begin{aligned}
\sum_{n=1}^{\infty} c a_n &= cA \\
\sum_{n=1}^{\infty} (a_n + b_n) &= A + B \\
\sum_{n=1}^{\infty} (a_n - b_n) &= A - B
\end{aligned}
$$

Unfortunately $\sum(a_n b_n)$ and $\sum a_n/b_n$ are not so simple to evaluate.

There are lots of ways of testing if a series converges or not, but unlike the case of sequences — actually working out what the series converges to is quite difficult and might be impossible in general. There are some nice exceptions to this where it is relatively easy to work things out.

Theorem (geometric series)**.** *The series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is called a geometric series. If $|r| < 1$ then it converges and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| \geq 1$ (and $a \neq 0$) then the series diverges.

Easy proof —

$$\begin{aligned} s_n &= a + ar + ar^2 + \dots ar^{n-1} \\ rs_n &= ar + ar^2 + ar^3 + \dots ar^n \\ s_n - rs_n = (1-r)s_n &= a - ar^n \\ s_n &= a\frac{1-r^n}{1-r} \end{aligned}$$

Now if $|r| < 1$ we know that $r^n \to 0$ and so the result follows. And if $|r| > 1$ then things clearly diverge. What about if $r = 1$, then $s_n = na$ which diverges. And if $r = -1$ then $s_n = a - a + a - a + a \dots$ which is either 0 or a depending on whether n is even or odd.

Please work out

$$9 - \frac{27}{5} + \frac{81}{25} - \frac{243}{125} + \dots$$

Note — it is a geometric series — so you need to work out a and r. How can you do that?

$$\begin{aligned} a &= 9 && \text{first term} \\ r &= \frac{(-27/5)}{9} = \frac{-3}{5} && \text{ratio of terms} \end{aligned}$$

Since $|r| < 1$, the series converges and it sums to

$$\frac{a}{1-r} = \frac{9}{1-(-3/5)} = \frac{9 \cdot 5}{5+3} = \frac{45}{8}$$

There are very few other families of series that are relatively easy to work out (without using calculus) — but one such class is "telescoping" series named after the old fashioned collapsing telescopes. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Lets write out the first few partial sums

$$\begin{aligned}
s_1 &= \frac{1}{2} \\
s_2 &= \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \\
s_3 &= \frac{1}{2} + \cdots + \frac{1}{12} = \frac{9}{12} = \frac{3}{4} \\
s_4 &= \frac{1}{2} + \cdots + \frac{1}{20} = \frac{16}{20} = \frac{4}{5}
\end{aligned}$$

Looks like $s_n = \frac{n}{n+1}$ which is surprisingly nice. Why does this happen. Partial fractions helps

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

So

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Lots of cancellations

$$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

The sequence of partial sums converges to 1 and hence the series equals 1. In general this sort of thing will work when you are summing a_n and $a_n = b_n - b_{n+1}$.

Now we have seen quite a few convergent series. Consider any one of these — so that $\sum a_n$ converges — what can we say about a_n?

Theorem. *If $\sum a_n$ converges then $a_n \to 0$.*

A logically equivalent statement of this (called the contrapositive) says

If a_n does not converge to 0 then $\sum a_n$ is divergent.

This is perhaps a bit easier to see — though both theorems are logically equivalent. This gives a very quick and easy test to see if a theorem diverges.

$$\sum \frac{n+2}{2n+3} \text{ diverges}$$

The sequence of summands converges to $1/2$.

What about the converse of the theorem

if $a_n \to 0$ then $\sum a_n$ converges

This is **very false**. Consider again the harmonic series

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

Clearly $a_n \to 0$, but we can prove (if we are careful) that the sequence of partial sums diverges.

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{2}{4}} \geq 1 + \frac{2}{2} \\
s_8 &= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{2}{4}} + \underbrace{\left(\frac{1}{5} + \cdots + \frac{1}{8}\right)}_{\geq \frac{4}{8}} > 1 + \frac{3}{2} \\
s_{16} &= 1 + \frac{1}{2} + \cdots + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{\geq \frac{8}{16}} > 1 + \frac{4}{2} \\
s_{2^n} &> 1 + \frac{n}{2}
\end{aligned}$$

Hence the sequence of partial sums keeps getting bigger, but very slowly. So it does not converge.