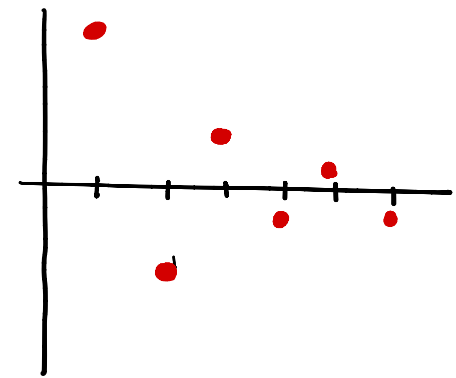
Alternating series.

Last time we looked at integral test.

Recall, integral test only works for series $\sum_{n=1}^{\infty} a_n$

with $a_n > 0$ and a_n decreasing.



Alternating series test:

Theorem (CLP 3.3.14) Consider the alternating series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots \quad \text{with } b_n > 0 \ \forall n.$$

If

- $b_n \geq b_{n+1}$ for all $n \geq N_0$, for some N_0 (i.e tail is decreasing)
- $b_n \to 0$

then the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

Alternating series test

Remarks

i. A proof is given in CLP 3.3.10.

ii.

iii.

Examples 1

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges?

Example 2

Does $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\log n}{n^2}$ converge?

Sol^n:

Example 3

Does $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4}$ converge?

Sol^n:

Example 4.

Does $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+5}$ converge or diverge?

Sol^n

Example 5

Does $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \cos\left(\frac{\pi}{n}\right)$ converge or diverge?

Log 2

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2.$

proof:

Log 2 (cont'd).

Log 2 (contd)

log 2 (contd).

Estimating remainder

Theorem: If $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is a convergent alternating series with $b_n \geq 0$, $b_{n+1} \leq b_n$, and $\lim_{n \to \infty} b_n = 0$, then the remainder R_N defined by

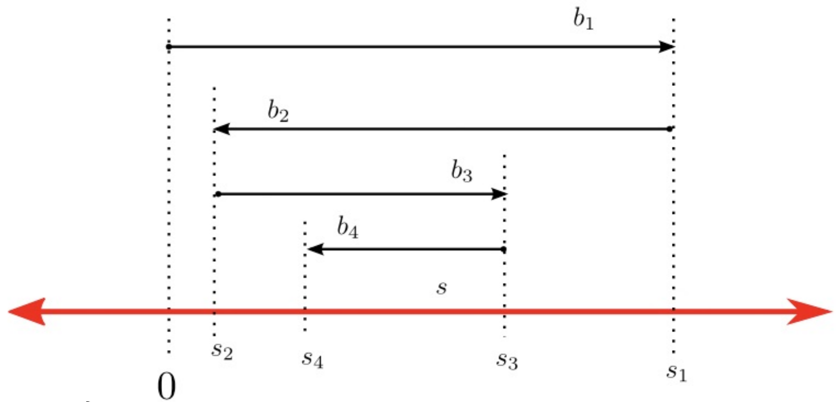
$$R_N = |S - S_N| \quad \text{with} \quad S_N = \sum_{n=1}^{N} (-1)^{n-1} b_n$$

satisfies bound

$$R_N \leq b_{N+1}$$

i.e. Remainder is bounded by first term after the N^{th} partial sum.

N^{th} partial sum of alternating series.



Consider $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$. with

$b_n \to 0$

$b_n > 0$

$b_n > b_{n+1}$

Example 6

Recall from Taylor approximation. (n^{th} order approx.)

$$f(x) \approx \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k.$$

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(x-a)^k. \rightarrow \text{taylor expansion.}$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

How many terms in the infinite sum are needed to get an error of 2.5×10^{-8} for e^{-1}?

Example 6 (cont d.)

Sol^n

Example

How many terms in the infinite series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is needed to approximate $\log 2$ with 10^{-5} accuracy?

Let $b_n = \frac{1}{n}$. Note that $b_n > 0$, $b_n \to 0$, $b_n > b_{n+1}$.

The Remainder R_N satisfy $R_N < \frac{1}{N+1}$.

So, to get an accuracy of 10^{-5} we need

$$\frac{1}{N+1} = 10^{-5} \quad \text{or} \quad N \approx 10^5$$

An enormous number of term since the series converges very slowly.