2 Applications of integration

2.2 Average value of a function

Integration is a very useful tool in probability and statistics.

- Given a "probability density function" compute the average and variance.
- A well known example would be the "bell curve", $f(x) = e^{-x^2}$.
- Let us look at a simpler version of this problem — compute the average value of a function over a given interval.
- Again we do it by Riemann sum arguments.

First — lets recall what we mean by the average of a set of numbers

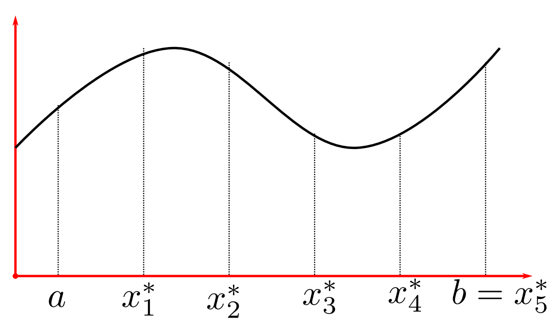
Definition (Average — clp 2.2.1)**.** The average (or mean) of a set of n numbers $\{y_1, y_2, \ldots, y_n\}$ is

$$y_{ave} = \bar{y} = \langle y \rangle = \frac{1}{n}(y_1 + y_2 + \cdots + y_n)$$

These 3 notations are all commonly used to denote the average.

Note (for the very pedantic) — there is a distinction between "mean" and "average" in more general discussions but here we will use them interchangeable.

So now lets compute the average value of the function $f(x)$ over the interval $[a, b]$. Again we follow our same strategy of cutting the problem up into little pieces and then adding them back together with the integral:



(Here I've used the right end-point rule)

- Split the interval up as per Riemann sum x_i and $\Delta x = (b - a)/n$.
- The average value is approximately the average of the values at each point x_i^* $(i = 1, 2, \ldots, n)$

$$\text{average of } f \approx \frac{1}{n}\sum_{i=1}^{n} f(x_i^*)$$

- But this doesn't look like a Riemann sum — no Δx.

- What happens as $n \to \infty$? There are n terms in the sum (presumably bounded?) and the sum is divided by n. So it is plausible that the limit exists.
- The Δx is there — it is hidden in the $\frac{1}{n}$.

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ \frac{\Delta x}{b-a} &= \frac{1}{n}\end{aligned}$$

- So we can rewrite our sum to look like a Riemann sum

$$\begin{aligned}\text{average of } f &\approx \frac{1}{n}\sum_{i=1}^{n} f(x_i^*) \\ &= \frac{\Delta x}{b-a}\sum_{i=1}^{n} f(x_i^*) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x\end{aligned}$$

- So now as $n \to \infty$ our approximation becomes exact (you can prove this rigorously if you have to) and the sum becomes a definite integral.

Definition (Mean of an integrable function — clp 2.2.2.)**.** Let f be an integrable function, then the average value $\bar{f}$ of $f(x)$ for x in the interval $[a, b]$ is

$$f_{ave} = \bar{f} = \langle f \rangle = = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

Notice that by rearranging this we find a very simple link between the signed area under a curve and the mean:

$$(b-a)f_{ave} = \int_a^b f(x)\mathrm{d}x = \text{ signed area}$$

Examples

- Find the average value of the function $y = x^n$ (with $n \geq 0$) on $[0, 1]$

$$\begin{aligned}\bar{y} &= \frac{1}{1-0}\int_0^1 x^n \mathrm{d}x \\ &= \frac{1}{1}\left[\frac{1}{n+1}x^{n+1}\right]_0^1 \\ &= 1/(n+1)\end{aligned}$$

- Find the average value of the function $y = \sin x$ on the interval $[0, \pi/2]$

$$\begin{aligned}\bar{y} &= \frac{1}{\pi/2-0}\int_0^{\pi/2} \sin x \mathrm{d}x \\ &= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi/2} \\ &= \frac{2}{\pi}(-(0)-(-1)) = \frac{2}{\pi}\end{aligned}$$

Now consider the function $f(x) = \sin(x)$ and take its mean on $[0, 2\pi]$

$$\begin{aligned} f_{ave} &= \frac{1}{2\pi}\int_0^{2\pi} \sin(x)\mathrm{d}x \\ &= \frac{1}{2\pi}[-\cos(x)]_0^{2\pi} = 0 \end{aligned}$$

Of course it should be this because we know what its sketch looks like. Now if this were representing alternating current in a circuit, then the "mean" is a poor measure of what is happening since there definitely is voltage there. Another measure might be $|\sin(x)|$, but another standard measure is the "root mean square". (The RMS is also sometimes called a "quadratic mean".)

First we square the function (to make it positive), then take the average, and finally take the square-root to compensate for squaring things. Hence

$$\begin{aligned} f_{rms}^2 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)^2\mathrm{d}x \\ &= \frac{1}{2\pi}\int \sin^2 x\mathrm{d}x = \frac{1}{2\pi}\int \frac{1+\cos 2x}{2}\mathrm{d}x \\ &= \frac{1}{4\pi}[x + \sin(2x)/2]_0^{2\pi} \\ &= \frac{1}{4\pi}[2\pi + 1/2 - 0 - 1/2]_0^{2\pi} \\ &= \frac{1}{2} & \text{and so} f_{rms} = \frac{1}{\sqrt{2}}. \end{aligned}$$

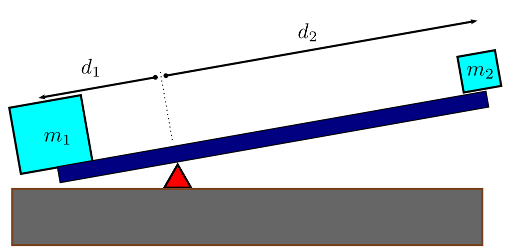
2.3 Centre of mass

When you sit on a see-saw (I think you call them teeter-totters in canadia?) the forces acting on the beam need to balance. These rotational forces are called torques

$$T = r \times F \qquad \text{where } r \text{ is distance from hinge}$$

These rotaional forces need to be balanced in order for the beam to be stable (and not resting on the ground). This will be the case if

$$m_1 d_1 = m_2 d_2.$$



More generally everything will balance if the lever is balanced at the "centre of mass". So assume the first mass is at x_1 and the second is at x_2. If we put the balance at $\bar{x}$ and assume that mass 1 is to the left and mass 2 is to the right then the torques will be

$$\begin{aligned} T_1 &= m_1(\bar{x} - x_1) \\ T_2 &= m_2(x_2 - \bar{x}) \end{aligned}$$

So if these are equal we get balance, and so

$$\begin{aligned} m_1 \underbrace{(\bar{x} - x_1)}_{\text{to the left}} &= m_2 \underbrace{(x_2 - \bar{x})}_{\text{to the right}} \\ (m_1 + m_2)\bar{x} &= m_1 x_1 + m_2 x_2 \\ \bar{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \end{aligned}$$

More generally — if we have a system of discrete masses on a line, the balance point should go at position given by

$$\bar{x} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{m}$$

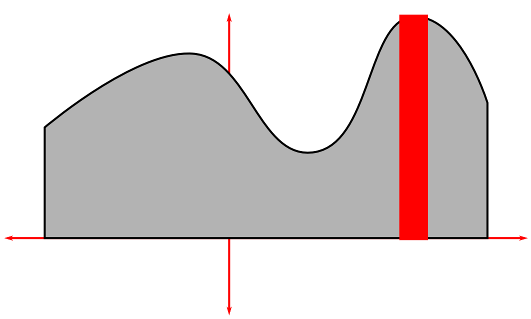
where m is the total mass and $m_i x_i$ is called the moment of mass i. The above is eqn 2.3.1 in the text.

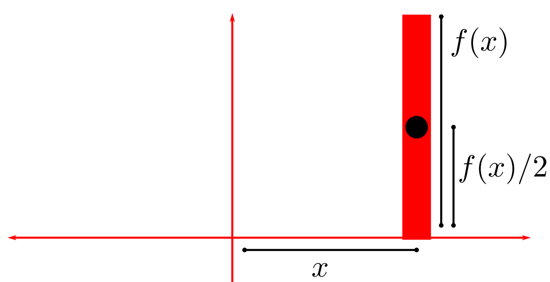
Now we can do very similar Riemann sum arguments to those we have done for means etc to get the centre of mass of a (continuous) object which has density $\rho(x)$:

$$\bar{x} = \frac{1}{m} \int x\rho(x)\mathrm{d}x$$

(this is equation 2.3.4 in the text).

Now consider a flat 2-d plate of constant density ρ whose lower boundary is the x-axis and upper boundary is $y = f(x)$. Let us compute the moment about the y-axis.





A vertical strip at position x has mass $\rho f(x)\Delta x$. This has moment $x\rho f(x)\Delta x$. Adding up these strips gives

$$M_y = \rho \int x f(x)\mathrm{d}x$$

We denote this M_y since it is the moment about the y-axis. However it really describes a horizontal distribution (about the y-axis). Let's come back to that.

What about its moment about the x-axis. This is a bit harder. What does a column at position x contribute now? It is a strip of width Δx and length $f(x)$. Its mass is $\rho f(x)\Delta x$. The average position of mass in this strip (from the x-axis) is $f(x)/2$. Hence it contributes $\frac{1}{2} f(x) \cdot \rho(x) f(x)\Delta x$.

Adding these up (Riemann sum) and taking the limit (converts to an integral) gives

$$M_x = \frac{1}{2} \int \rho f(x)^2 \mathrm{d}x$$

Again this is denoted M_x since it is the moment about the x-axis, but it describes a vertical distribution (about the x-axis).

Dividing both of these by the total mass (density times area)

$$m = \int \rho f(x) \mathrm{d}x$$

gives the coordinates of the centre of mass

$$\begin{aligned}\bar{x} &= \frac{M_y}{m} = \frac{\rho \int x f(x) \mathrm{d}x}{\rho \int f(x) \mathrm{d}x} = \frac{\int x f(x) \mathrm{d}x}{\int f(x) \mathrm{d}x} = \frac{\int x f(x) \mathrm{d}x}{A} \\ \bar{y} &= \frac{M_x}{m} = \frac{1}{A} \int \frac{1}{2} f(x)^2 \mathrm{d}x\end{aligned}$$

This coordinate is called the "centroid" of the area. Notice that it is independent of the density (provided the density is constant). These are given (slightly more generally) as equation 2.3.5 in the text.

Example: Find the centre of mass of a parabolic plate $y = 1 - x^2, y = 0, -1 \leq x \leq 1$.

$$\begin{aligned}A &= \int_{-1}^{1} y \mathrm{d}x = \int_{-1}^{2} (1 - x^2) \mathrm{d}x \\ &= [x - x^3/3]_{-1}^{1} = (1 - 1/3) - (-1 + 1/3) = 4/3\end{aligned}$$

Now use the formulas

$$\begin{aligned}\bar{x} &= \frac{1}{A} \int_{-1}^{1} x f(x) \mathrm{d}x \\ &= \frac{1}{A} \int_{-1}^{1} (x - x^3) \mathrm{d}x \\ &= 0 && \text{function is odd}\end{aligned}$$

and

$$\begin{aligned}\bar{y} &= \frac{1}{2A} \int_{-1}^{1} (1 - x^2)^2 \mathrm{d}x \\ &= \frac{3}{8} \int_{-1}^{1} (1 - 2x^2 + x^4) \mathrm{d}x \\ &= \frac{3}{8} [x - 2x^3/3 + x^5/5]_{-1}^{1} \\ &= \frac{3}{8} ((1 - 2/3 + 1/5) - (-1 + 2/3 - 1/5)) \\ &= \frac{3}{8} \times (\frac{8}{15} + \frac{8}{15}) \\ &= \frac{2}{5}\end{aligned}$$

So centre of mass is at $(0, 2/5)$.

Another example — find the centroid of a quarter disc (example 2.3.8 in the text) of constant density. The equation of the curve is

$$x^2 + y^2 = r^2 \qquad x, y \geq 0$$

So the equation of the function is

$$y = \sqrt{r^2 - x^2} \qquad\qquad 0 \le x \le r$$

So first the area (this one is easy)

$$A = \int_0^r \sqrt{r^2 - x^2}\mathrm{d}x = \frac{\pi r^2}{4} \qquad\qquad \text{a quarter circle}$$

Now (from our equations)

$$\begin{aligned} \bar{x}\frac{1}{A}&\int xy(x)\mathrm{d}x \\ \bar{y} = \frac{1}{2A}&\int y(x)^2\mathrm{d}x \end{aligned}$$

So

$$\begin{aligned} \bar{x} &= \frac{1}{A}\int_0^r x\sqrt{r^2 - x^2}\mathrm{d}x && \text{sub } u = r^2 - x^2, u' = -2x \\ &= \frac{1}{A}\int_0^r \frac{-1}{2}\sqrt{u}u'\mathrm{d}x \\ &= \frac{-1}{2A}\int_{r^2} 0\sqrt{u}\mathrm{d}u \\ &= \frac{1}{2A}[\frac{2}{3}u^{3/2}]_0^{r^2} = \frac{1}{3A}\cdot r^3 && A = \pi r^2/4 \\ &= \frac{4r}{3\pi} \end{aligned}$$

and then

$$\begin{aligned} \bar{y} &= \frac{1}{2A}\int_0^r (r^2 - x^2)\mathrm{d}x \\ &= \frac{1}{2A}[r^2x - x^3/3]_0^r = \frac{1}{2A}\cdot\frac{2r^3}{3} \\ &= \frac{r^3}{3A} = \frac{4r}{3\pi} \end{aligned}$$

This is reassuring — the problem is symmetric in $x \leftrightarrow y$, so we should get the same moments.

2.4 Separable differential equations

First let us talk a bit more generally about DEs.

- A differential equation is an equation that involves a function and one or more of its derivatives.
- The order of the equation is the order of the highest derivative in the equation.
- Consider the equation

  $$y' = xy$$

  It is understood that y is really a function of x.

- A function $f(x)$ is a solution of the differential equation if when you substitute $y = f(x)$ the equation is satisfied.

$$f'(x) = xf(x)$$

- When we solve a differential equation it is important to find *all* the solutions. Just like finding an indefinite integral — you have to find all the anti-derivatives.
- In the above example

$$\begin{aligned} y &= Ce^{x^2/2} && \text{for any } C \\ y' &= Ce^{x^2/2} \cdot x = xy \end{aligned}$$

- Last term you saw some simple differentia equations for population models, radioactive decay etc

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

Here P is the dependent variable and t was the independent variable. You solved this (by inspection?) and got the solution

$$P(t) = A \cdot e^{kt}$$

The constant A needs to then be determined by extra information (like the initial population). At time $t = 0$, the initial population is $P(0) = P_0$, then

$$P(0) = P_0 \qquad \text{initial population} = Ae^{k0} = A$$

Hence $A = P_0$.

- This is an example of an *initial value problem* — a differential equation together with an initial condition.

So — how do we solve a differential equation? For some very simple ones we can (almost) do it by inspection (like we did last term). Like integration there are some families of differential equations that can be solved nicely, but in general it is a very hard problem.

In fact fact most of the integration problems we have done can be considered differential equations:

$$\begin{aligned} F'(x) &= \text{some function of } x && \text{so} \\ F(x) &= \int (\text{some function of } x) \mathrm{d}x \end{aligned}$$

Now while this might seem a bit simplistic, but it in fact will allow us to solve quite a large family of equations — separable equations.

Definition (CLP 2.4.1)**.** A *separable equation* is one of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)g(x)$$

ie, the equation can be written so that the LHS is just $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the RHS can be *factored* into a function of x and a function of y.

To solve such equations there is a simple trick. *Separate* this equation — pull all the y-stuff to the left and leave all the x-stuff on the right. This writes it as

$$\frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$

It is important that the RHS really is of the form $f(y)g(x)$ *not* $f(y) + g(x)$ — that is a very common error.

Now integrate both sides of the equation wrt x:

$$\int \frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int g(x)\mathrm{d}x$$

Hopefully the RHS is an integral we can do. Notice the form of the LHS, it is exactly the form of a substitution integral — so we can rewrite this as

$$\int \frac{1}{f(y)}\mathrm{d}y = \int g(x)\mathrm{d}x$$

If we are lucky then we can do both of these integrals and end up with an equation linking x and y. If we are really lucky then we will be able to isolate y as a function of x.

Let us do an example

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -x/y && \text{separate} \\
y\frac{\mathrm{d}y}{\mathrm{d}x} &= -x && \text{integrate} \\
\int y\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x &= -\int x\mathrm{d}x \\
\int y\mathrm{d}y &= c - \frac{1}{2}x^2 && \text{don't forget the constant} \\
\frac{1}{2}y^2 &= c - \frac{1}{2}x^2 && \text{rearrange — cannot always do this} \\
y^2 &= 2c - x^2 \\
y &= \pm\sqrt{2c - x^2}
\end{aligned}$$

Solution is a circle!

Another slightly harder example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy \qquad \text{separate}$$

Provided $y \neq 0$ — we will come back to $y = 0$ later

$$
\begin{aligned}
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= x && \text{integrate} \\
\int y^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x &= \int x\mathrm{d}x \\
\int y^{-1}\mathrm{d}y &= x^2/2 + c \\
\log|y| &= x^2/2 + c \\
|y| &= e^c e^{x^2/2} \\
y &= \pm e^c e^{x^2/2} \\
y &= Ae^{x^2/2}
\end{aligned}
$$

where $A = \pm e^c$. Now $e^c > 0$, so $\pm e^c$ is any real number *except* 0. So now we check $A = 0$ separately:

$$
y = 0e^{x^2/2} = 0\frac{\mathrm{d}y}{\mathrm{d}x} \qquad = 0 = x \times 0
$$

So $y = 0$ is a solution, so A is any real number. Hence our general solution is

$$
y = Ae^{x^2/2}
$$

A couple more examples. Solve

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{x^2+1} \qquad y(0) = 3
$$

So — we separate:

$$
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{x^2+1}
$$

And then integrate:

$$
\begin{aligned}
\int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x &= \int \frac{x}{x^2+1}\mathrm{d}x \\
\int \frac{\mathrm{d}y}{y} &= \int \frac{1}{2u}\mathrm{d}u && \text{sub } u = x^2+1, u' = 2x \\
\log|y| &= \frac{1}{2}\log|u| + C \\
\log|y| &= \frac{1}{2}\log|x^2+1| + C
\end{aligned}
$$

Now since $x^2 + 1 > 0$ we can get rid of the abs value:

$$
\begin{aligned}
\log|y| &= \log(\sqrt{x^2+1}) + C \\
|y| &= e^C \cdot \sqrt{x^2+1}
\end{aligned}
$$

Now we are told that $y(0) = 3$. Hence

$$3 = e^C \cdot 1$$

Thus we have

$$|y| = 3\sqrt{x^2 + 1}$$

Hmmm — but we want to get rid of the $|blah|$. We can write

$$y = \pm 3\sqrt{x^2 + 1} \qquad \text{but this is not a single function}$$

but recall our initial condition $y(0) = 3$ — hence we take the positive branch

$$y = 3\sqrt{x^2 + 1}$$

Find a solution of

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2 \qquad y(1) = -1.$$

First we need to massage this around a little:

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2 - y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y^2 - y}{x} \end{aligned}$$

Now this is separable and we can separate and integrate:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y^2 - y}{x} \frac{1}{y(y-1)} \frac{\mathrm{d}y}{\mathrm{d}x} && = \frac{1}{x} \\ \int \frac{1}{y(y-1)}\mathrm{d}y &= \int \frac{1}{x}\mathrm{d}x \end{aligned}$$

We need to partial fractionate the LHS (we get):

$$\frac{1}{y(y-1)} = \frac{-1}{y} + \frac{1}{y-1}$$

So we now have

$$\begin{aligned} \int \frac{1}{y-1} - \frac{1}{y}\mathrm{d}y &= \int \frac{1}{x}\mathrm{d}x \\ \log|y-1| - \log|y| &= \log|x| + C \\ \log|\frac{y-1}{xy}| &= C \\ \frac{y-1}{xy} &= e^C = A \end{aligned}$$

Now, we know that when $x = 1, y = -1$, so

$$\frac{-2}{-1} = A$$

Hence

$$\frac{y-1}{xy} = 2$$

Now rearrange to solve for y:

$$\begin{aligned} y - 1 &= 2xy \\ y(1-2x)1 \\ y &= \frac{1}{1-2x}. \end{aligned}$$

Oof!

To check this we can differentiate:

$$\begin{aligned} y' &= \frac{2}{(1-2x)^2} & \text{so} xy' &= \frac{2x}{(1-2x)^2} \quad \text{and} \\ y^2 - y &= \frac{1}{(1-2x)^2} - \frac{1}{1-2x} = \frac{1-(1-2x)}{(1-2x)^2} = \text{ the above.} \end{aligned}$$

and

$$y(1) = \frac{1}{(1-2)} = -1 \qquad \text{as required.}$$

3 Infinite sequences and series

Consider the function e^{-x^2}. It is very easy to differentiate

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{-x^2} = -2xe^{-x^2}$$

But its anti-derivative cannot be written in terms of the functions we know — polynomials, trig functions, exponential and logarithms. That being said, the anti-derivative of this function is extremely important. It turns up quite frequently in different bits of mathematics — perhaps most obviously in probability and statistics as the bell-curve and the normal-distribution. So we need some way of representing this antiderivative and evaluating it. To do this we use taylor series — but to get at taylor series we need to understand power series, and to understand that we need series and to understand that we need sequences. So that is where we start.

3.1 Sequences

The easiest way to think of a sequence is a function from the positive integers to the real numbers.

$$a_n = f(n) \qquad f : \mathbb{N} \to \mathbb{R}$$

Or simply as a list of real numbers

$$\{a_1, a_2, a_3, \dots\}$$

which we also write as

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

We call a_n the n^{th} term of the sequence. Some examples

- $\left\{\frac{1}{n^2}\right\}$ is the sequence $1, 1/4, 1/9, 1/16, \dots$.
- $\left\{\frac{n}{2n+1}\right\}$ is the sequence $1/3, 2/5, 3/7, \dots$.
- $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots$ is the sequence $\left\{\frac{1}{2n}\right\}$
- $1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{8}, \dots$ is the sequence $\{1+2^{-n}\}$
- $1, \frac{3}{5}, \frac{1}{2}, \frac{5}{11}, \frac{3}{7}, \frac{7}{17}, \dots$ is the sequence $\left\{\frac{n+1}{3n-1}\right\}$

Some other famous sequences

$$\{2, 3, 5, 7, 11, 13, 17, \dots\}$$
$$\{1, 1, 2, 3, 5, 8, 13, \dots\}$$

The first is just prime numbers, while the second is the Fibonacci sequence. The second satisfies a "recurrence" $a_n = a_{n-1} + a_{n-2}$.

We aren't so interested in recurrences for this course, the main point is to look at limits. In particular what can we say about the behaviour of a_n as n becomes very large. To introduce this idea let us look at a very simple example

$$a_n = 1 + \frac{1}{n}$$

So now as n becomes larger and larger, it is clear that $\frac{1}{n}$ becomes smaller and smaller. Hence a_n gets closer and closer to 1. In this case we write

$$\lim_{n\to\infty} a_n = 1 \qquad\qquad \text{or } a_n \to 1$$

More generally

Definition. (not so rigorous — CLP 3.1.3) A sequence $\{a_n\}$ has the limit L if we can make the terms a_n as close to L as we like by taking n sufficiently large. In this case we write

$$\lim_{n\to\infty} a_n = L \qquad\qquad \text{or } a_n \to L$$

If this limit exists (and is finite) we say that the sequence is convergent. Otherwise we say it is divergent.

We can make the above definition more rigorous (the interested reader should see section 1.7 of the CLP-1 textbook).

Now we have seen limits before, in particular limits of some functions as $x \to \infty$ or $x \to 0$. The following is a useful theorem that relates the limits of sequences to the limits of functions

Theorem (clp 3.1.6)**.** *If* $a_n = f(n)$ *and* $\lim_{x\to\infty} f(x) = L$ *then* $\lim_{n\to\infty} a_n = L$.

For example

$$\lim_{x\to\infty} e^{-x} = 0 \qquad\qquad a_n = e^{-n} \to 0.$$

And perhaps more usefully for any $r > 0$

$$\lim_{x\to\infty} {}^{1}\!/\!_{x^r} = 0 \qquad\qquad a_n = n^{-r} \to 0$$

Now we want to build up some tools so that we can express the limits of complicated things in terms of the limits of simpler things. Again this is like what you did for derivatives and integrals. Indeed it is a very standard thing to do in mathematics. You define some new object or property (here it is limits) and then you see how it interacts with some standard things (like addition and multiplication etc). Thankfully limits are nice — even easier than derivatives and integrals — they "play nicely" with arithmetic

Theorem (Arithmetic of limits — clp 3.1.8)**.** *Let* $\{a_n\}$ *and* $\{b_n\}$ *be convergent sequences with* $a_n \to A$ *and* $b_n \to B$*. Further let* c *be a constant. Then*

$$\begin{aligned}
\lim_{n\to\infty} (a_n + b_n) &= A + B \\
\lim_{n\to\infty} (a_n - b_n) &= A - B \\
\lim_{n\to\infty} (ca_n) &= cA \\
\lim_{n\to\infty} (a_n \cdot b_n) &= A \cdot B \\
\lim_{n\to\infty} {}^{a_n}\!/\!_{b_n} &= {}^{A}\!/\!_{B} && \textit{provided } B \neq 0 \\
\lim_{n\to\infty} (a_n)^p &= A^p && \textit{provided } p > 0 \textit{ and } a_n > 0
\end{aligned}$$

So lets use this to compute the limit

$$\lim_{n\to\infty} \frac{3n}{2n+7} =$$

We cannot use our limit laws yet because numer and denom are divergent

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{3}{2 + 7/n} && \text{divide top and bottom by } n \\
&= \frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} (2 + 7/n)} \\
&= \frac{3}{\lim 2 + \lim 7n} = \frac{3}{2 + 7 \lim \frac{1}{n}} \\
&= \frac{3}{2+0} = {}^{3}\!/\!_{2}
\end{aligned}$$

Now you lot do

$$\lim_{n\to\infty} \frac{3n^2 + 4}{5n^2 + n + 7}$$

What about a sequence like $a_n = \sin(\pi/n)$. What do we think is going to happen? Well — as $n \to \infty$ we think that $\pi/n \to 0$ and so we'd expect that $\sin(\pi/n) \to \sin(0) = 0$. Now this is true, but it only works because $\sin(x)$ has a very nice property — it is continuous. More generally this reasoning is fine provided we have continuity

Theorem (CLP 3.1.12)**.** *If $a_n \to L$ and $f(x)$ is continuous at $x = L$ then $f(a_n) \to f(L)$.*

Now sometimes the form of the sequence is very messy but we can "squeeze" it between two simpler sequences

Theorem (squeeze theorem — clp 3.1.10)**.** *If $a_n \leq b_n \leq c_n$ and $\lim a_n = \lim c_n = L$ then $b_n \to L$.*

Draw a simple pic. One easy consequence of this is the following

$$\text{if } |a_n| \to 0 \text{ then } a_n \to 0$$

This tells us things like

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

since $|(-1)^n/n| = 1/n$. This is interesting because $a_n = (-1)^n$ does not converge.

Another application — consider $a_n = \frac{n!}{n^n}$ — a little massaging is required to make sense of this.

$$\begin{aligned}\frac{n!}{n^n} &= \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n}\\ &= \frac{1}{n}\left(\frac{2}{n}\frac{3}{n}\cdots\frac{n}{n}\right) \leq \frac{1}{n}\end{aligned}$$

Hence we see that $0 \leq a_n \leq {}^1\!/\!{}_n$. Hence $a_n \to 0$.

A very useful example — one we will need when we get to sequences. Lets call it a lemma:

Lemma. *The sequence $a_n = r^n$ is convergent when $-1 < r \leq 1$ and otherwise is divergent.*

$$\lim_{n\to\infty} r^n = \begin{cases} 1 & r = 1 \\ 0 & -1 < r < 1 \\ \textit{divergent} & \textit{otherwise} \end{cases}$$

Okay — a couple of these are easy $1^n = 1, 0^n = 0$. And we have seen from playing around with exponential functions that

$$\lim_{x\to\infty} r^x = \begin{cases} \infty & r > 1 \\ 0 & 0 < r < 1 \end{cases}$$

Now if $r < 0$ then $|r^n| = |r|^n$ and so the negative cases can just be expressed in terms of the positive cases.

Okay — one last little bit of sequences. In some cases we dont actually care what the limit is, just that the sequence converges. Indeed in some cases this is all we can prove. Our last result gives quite general conditions for a sequence to converge.