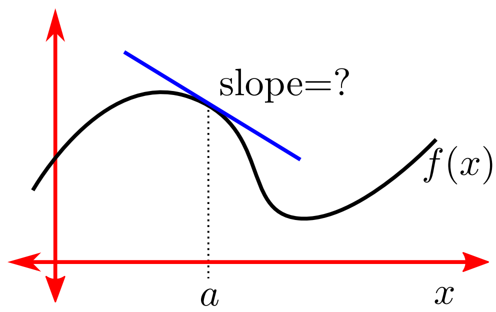
1 Integrals — CLP Chapter 1

1.1 The definite integral — CLP 1.1

Slopes and areas

There are 2 basic problems in Calculus

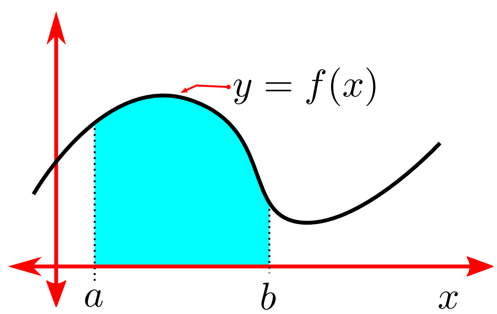
1. Tangent problem — Define the tangent to a curve and calculate its slope.



$$\text{slope} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{\mathrm{d}f}{\mathrm{d}x} = f'(x)$$

- This is Differential Calculus and is covered in Maths100.
- This is useful, not to compute the tangent line, but because it gives you the rate of change — allows you to compute maxima, minima and optimise things.

2. Area problem — Calculate the area between the x-axis and the curve $y = f(x)$.

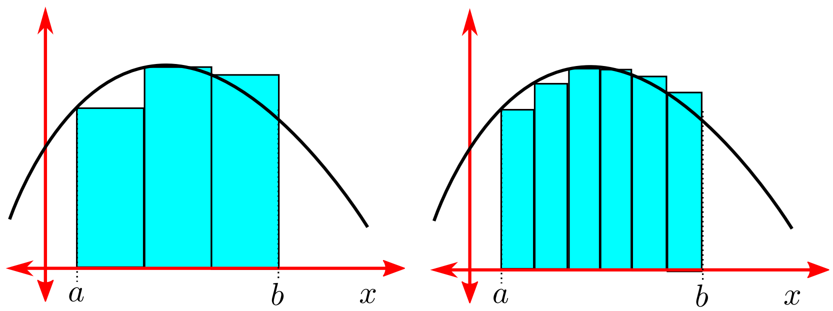


- This is Integral Calculus and is covered in this course.
- Integration is useful for far more things than computing areas. In particular it helps us solve differential equations which turn up everywhere!
- One of the main reasons that mathematics is so useful (both of itself and to other subjects).

Calculus (both differential and integral) was discovered independently by Newton and Leibniz around the same time. They spent the rest of their lives fighting over who really invented it. It caused much resentment between english-speaking mathematicians on Newton's side and European mathematicians on Leibniz's side — because it is so useful and fundamental the fight became so big.

It is not immediately obvious that these two problems are related, however the Fundamental Theorem of Calculus shows that they linked. Before we get to this theorem we need some more machinery.

In particular we will write the area under a curve as a sum of areas of rectangles under the curve. Then we let the widths of the rectangles get smaller and smaller.



Now here we have made sure that the top-left corner of the rectangle is touching the curve. We could have done the same but with the top-right corner and gotten a different approximation. However, you can see that as we make the width of the rectangles smaller, we get more rectangles, but we get a better approximation of the area. Indeed we will have a theorem that tells us that (provided $f(x)$ is a nice function) as the width of the rectangles goes to zero, the sum of those rectangles will be exactly the area under the curve.

Now I want to introduce some useful notation that many of you have seen before — this will make it easier to write down the sums of areas of rectangles that we need.

Summation notation — CLP section 1.1.3

We need to learn how to use the $\sum$ symbol — it is a capital "sigma". It is a very useful shorthand for writing sums of many terms. Some examples

- The sum of the first 20 integers is

$$1 + 2 + 3 + \cdots + 19 + 20 = \sum_{i=1}^{20} i = \sum_{k=1}^{20} k$$

  i and k are "dummy variables" we could even write

$$\sum_{\spadesuit=1}^{20} \spadesuit$$

  At each stage we replace $\spadesuit$ by the integers from 1 to 20 and sum the result.

- The sums of cubes

$$\sum_{k=4}^{7} k^3 = 4^3 + 5^3 + 6^3 + 7^3$$

- A sum of a function evaluated at integer points

$$\sum_{k=3}^{6} f(k) = f(3) + f(4) + f(5) + f(6)$$

- A sum of constants
$$\sum_{i=1}^{n} C = \underbrace{C + C + \cdots + C}_{n \text{ terms}} = nC$$
- A more formal sum
$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

The sum symbol is a "linear operator" (see Theorem 1.1.5 in CLP)

$$\begin{aligned}
\sum_{i=1}^{n} ca_i &= c\sum_{i=1}^{n} a_i && \text{move the constant out the front} \\
\sum_{i=1}^{n}(a_i + b_i) &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i && \text{split the sum into two sums} \\
\sum_{i=1}^{n}(a_i - b_i) &= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i
\end{aligned}$$

You can prove this by writing out both sides of each equation.

So the sum-symbol interacts with these very basic bits of arithmetic very nicely. It does not, in general, interact with multiplication and division as nicely — there are no formulas similar to the above.

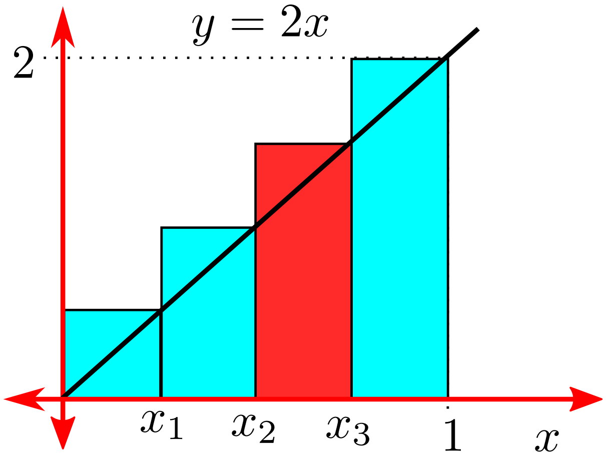
There are also some special sums that you should know (see Theorem 1.1.6 in CLP):

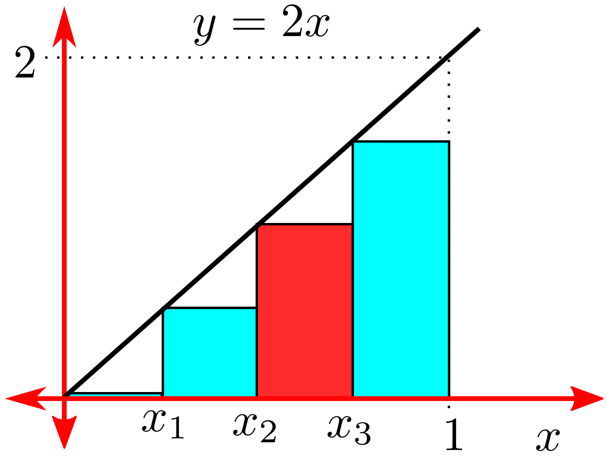
- First n integers: $\displaystyle\sum_{k=1}^{n} k = \frac{k(k+1)}{2}$
$$\begin{aligned}
S &= 1 + 2 + \cdots + (n-1) + n \\
S &= n + (n-1) + \cdots + 2 + 1
\end{aligned}$$
  Now add these together…
$$\begin{aligned}
2S &= (n+1) + (n+1) + \cdots + (n+1) + (n+1) \\
&= n(n+1)
\end{aligned}$$
- Sum of first n squares $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- Sum of first n cubes $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

The easiest way to prove these last two formulas is using "Mathematical induction" — but there are some nice (and sneaky) proofs relying on geometry. These are in

Back to areas and an example

The text does a nice example of the area under the curve $y = e^x$ between $x = 0, x = 1$. Now, rather than doing this example in class (you can read the text instead), let us do something a little simpler that gives us the same idea. Find the area between the curve $y = 2x$ and the x-axis between $x = 0$ and $x = 1$ — of course the area should come out to be 1.





- Let us use n rectangles determined by the right-endpoints. This will overestimate the area. To make things much easier, make them all have the same width $\Delta x = 1/n$.
- So the x-ordinates are $1/n, 2/n, 3/n, \ldots, n/n = 1$.
- The heights of the i^{th} rectangle is therefore $f(x_i) = 2x_i = 2i/n$.
- The area of the i^{th} rectangle is $f(x_i)\Delta x = \frac{2i}{n} \cdot \frac{1}{n}$.
- So the total area of the n rectangles is

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \frac{2i}{n^2} \\ &= \frac{2}{n^2}\sum_{i=1}^{n} i && \text{take out constants} \\ &= \frac{2}{n^2} \cdot \frac{n(n+1)}{2} \end{aligned}$$

- Now, in the limit as $n \to \infty$ we get

$$A = \lim_{n\to\infty} R_n = \frac{2}{2} \cdot \lim_{n\to\infty} \frac{n(n+1)}{n^2} = 1$$

- Now if we repeat using the underestimate (left-endpoint) — rectangles of height $f(x_{i-1})$ we obtain the same answer $A = 1$ — you can check it at home.

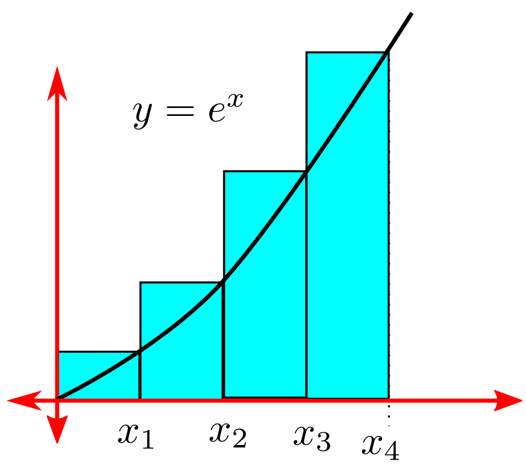
$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} \frac{2}{n^2} \cdot (i-1) \cdot \frac{1}{n} \qquad = \frac{2}{n^2} \underbrace{\sum_{i=1}^{n}(i-1)}_{=\sum_{j=0}^{n-1} j}$$

$$= \frac{2}{n^2} \cdot \frac{(n-1)(n+1-1)}{2} = \frac{n(n-1)}{n^2}$$

- If we let $n \to \infty$ we obtain the same answer $A = 1$.

NOTE — left-endpoints will not always give an underestimate of the area. Nor will right-endpoints always give an overestimate.

We can also use these sums to find a numerical approximation of the area. For example — lets approximate the area under the curve $y = e^x$ between $x = 0$ and $x = 1$ using 4 rectangles defined by their right-endpoints.



$$\Delta x = \frac{1}{4}$$

$x_1 = 1/4 \qquad x_2 = 1/2 \qquad x_3 = 3/4 \qquad x_4 = 1$

So our approximation of the area is just

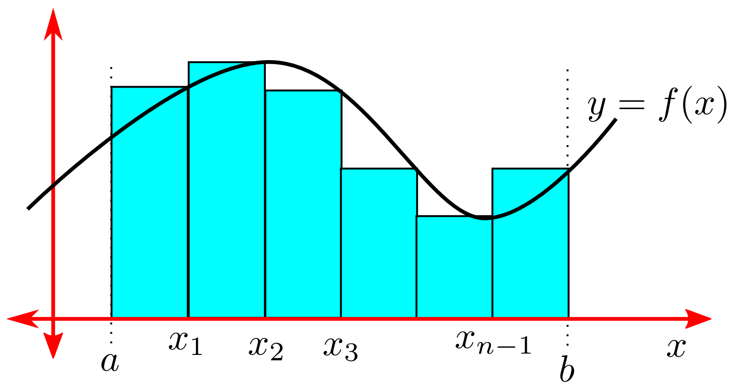
$$\begin{aligned} R_4 &= f(1/4)\Delta x + f(3/2)\Delta x + f(3/4)\Delta x + f(1)\Delta x \\ &= \frac{1}{4}\left(e^{1/4} + e^{1/2} + e^{3/4} + e^1\right) \qquad \approx 1.94 \end{aligned}$$

The actual area is $e - 1 \approx 1.72$ — so we are not so far off. (of course we have just used our knowledge of e to estimate e which is rather circular. Better computations of e are later in the text (and text)).

More generally

Now everything worked out very very nicely in our first example because things ended in such a simple sum — and we knew the sum exactly. More generally (as happened in the second example) we are going to be dealing with much more general functions and sums that we won't know. That being said we can still do pretty much the same thing.

If we want the area under a curve from $x = a$ to $x = b$, we can approximate things by cutting that area up into n rectangles and summing them. Such sums are called Riemann sums (more in a moment and also see CLP Definition 1.1.11).



Here I used right-endpoints, but I could have used left-endpoints too. It is easy to work out the little rectangles.

- All have width $\Delta x = (b-a)/n$.
- The ordinate $x_i = a + i\Delta x$.
- The height of the i^{th} rectangle is just $f(x_i)$

So our approximation of the area is just

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

And if we let $n \to \infty$ we get the area.

More generally, the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is the limit of the sum of areas of the approximating rectangles

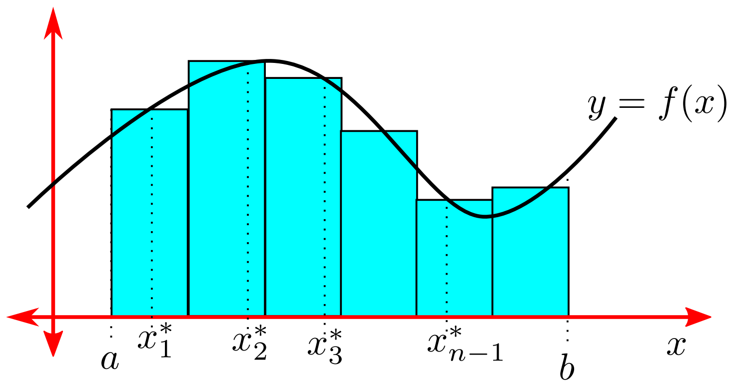
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

Now provided $f(x)$ is a continuous function one can prove that the above limit always exists — but this doesnt tell us what it actually is. We need some more calculus to do that.

Notice that we could have also defined things in terms of the left-endpoint rectangles L_n.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x$$

And we can go a bit further still. Keep the x_i as we have defined them — very nicely spaced and the rectangle widths all the same Δx. Now let x_j^* be *any* point in $[x_{j-1}, x_j]$ and make the height of the j^{th} rectangle be $f(x_j^*)$ — so we have a pic something like



$$A = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_j^*)\Delta x$$

As I noted above, these sums are called Riemann sums (after Bernhard Riemann — a 19th century German mathematician who did a lot of very important mathematics including Riemann surfaces and the Riemann hypothesis — the latter being arguably the most important unsolved problem in pure mathematics).

Definition (CLP Defn 1.1.11)**.** Let a, b be two real numbers, let n be a positive integer, and let $f(x)$ be a function defined on $[a, b]$. Set $\Delta x = \frac{b-a}{n}$ and then (as above) divide the interval $[a, b]$ up into n even subintervals $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$. Then the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x$$

is called a Riemann sum.

Above we have been computing areas using Riemann sums — more generally and more importantly, we use Riemann sums to define the "definite integral".

The definition of the definite integral

We have now seen how we can find the area under the curve by summing lots of small rectangles and using a Riemann sum

$$A = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_j^*)\Delta x$$

The above form is useful not just for finding areas — indeed it still works when $f(x)$ is negative (why is this a problem for areas?) Because it is so useful we give this limit a special name — the definite integral.

Definition (CLP Defn 1.1.8 and 1.1.9)**.** Let f be a function defined on $[a, b]$. We divide the interval $[a, b]$ into n subintervals each of width $\Delta x = (b - a)/n$. Hence the endpoints of the intervals are $x_i = a + i\Delta x$. Further let $x_i^* \in [x_{i-1}, x_i]$ be a *sample point* from the i^{th} subinterval. Then the *definite integral of f from a to b* is

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided the limit exists. If it does exist then we say that f is integrable on $[a, b]$.

Note that

- $\int$ was introduced by Leibniz (it is a big "S") called "integral sign".
- $f(x)$ is called the "integrand"
- a and b are called the limits of integration
- $\mathrm{d}x$ does not have meaning by itself — it only means something as part of the whole $\int_a^b f(x)\mathrm{d}x$.
- $\int_a^b f(x)\mathrm{d}x$ is a number — it does not depend on x.

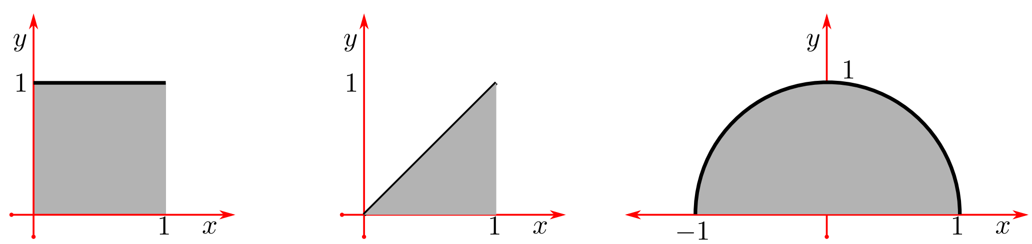
How do we know when the definite integral exists — there is a nice simple theorem?

Theorem (CLP Theorem 1.1.10)**.** *If f is continuous on $[a, b]$ or has only a finite number of jump discontinuities, then f is integrable on $[a, b]$.*

Okay — so with what we know presently we can compute some definite integrals, but not many. Since we already know things like the areas of rectangles and triangles and circles, we can compute integrals which involve those. So integrals like

$$\int_0^1 1\mathrm{d}x \qquad \int_0^1 x\mathrm{d}x \qquad \int_{-1}^1 \sqrt{1-x^2}\mathrm{d}x$$

since these correspond to the following simple geometric shapes.



Let us make a couple of these into a lemma

Lemma. *Let b be a real number, then*

$$\int_0^b 1\mathrm{d}x = b \qquad \int_0^b x\mathrm{d}x = \frac{b^2}{2}$$

Notice that we have a nice geometric picture when $b > 0$ — more care is required when $b < 0$ (see the text). Similar examples are done in CLP 1.1.5.

We can also (using the summation formulas we saw earlier) compute definite integrals of low-degree polynomials. The first step is to express the integral as a Riemann sum. Then we have to take a limit. So lets do this for a simple polynomial function.

Let us compute the definite integral $\int_0^4 (x^2 - 3x)\mathrm{d}x$.

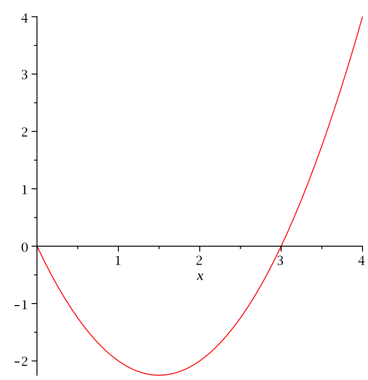
- Turn it into a Riemann sum
  - $a = 0, b = 4$ so $\Delta x = 4/n$ and $x_i = \frac{4i}{n}$.
  - The Riemann sum (right-endpoints) is then

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{16i^2}{n^2} - \frac{12i}{n} \right) \frac{4}{n} \\
&= \sum_{i=1}^{n} \frac{64i^2}{n^3} - \sum_{i=1}^{n} \frac{48i}{n^2} \\
&= \frac{64}{n^3} \sum_{i=1}^{n} i^2 - \frac{48}{n^2} \sum_{i=1}^{n} i \\
&= \frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{48}{n^2} \frac{n(n+1)}{2}
\end{aligned}$$

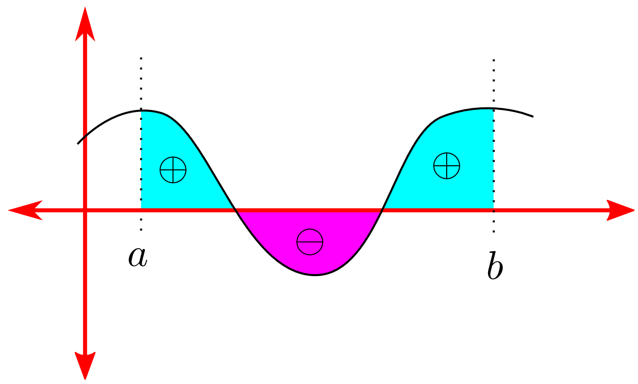
- So in the limit as $n \to \infty$ we have

$$\begin{aligned}
\int_0^3 f(x)\mathrm{d}x &= \lim_{n\to\infty} R_n \\
&= \lim_{n\to\infty} \frac{64}{6} \cdot \frac{n(n+1)(2n+1)}{n^3} - \lim_{n\to\infty} \frac{48}{2} \frac{n(n+1)}{n^2} \\
&= \frac{64}{3} - 24 = -\frac{8}{3}
\end{aligned}$$

So there are 2 things to notice here — firstly this is not such an easy way of doing things and we would really like a nicer way of doing it. Secondly — the answer is negative! What is going on here. Plot the graph



Since the function is negative between 0 and 3, those rectangles, $f(x)\Delta x$ are negative. Thus the definite integral computes what is called a “signed area”.



So the definite integral computes the area of all the pieces above the axis and then subtracts the area of all the pieces below the axis.

(If there is time) — Why don't you try $\int_0^1 (x^3 + 2x) \mathrm{d}x$.

- $a = 0, b = 1, \Delta x = 1/n, x_i = i/n$.
- Riemann sum is

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i) \Delta x \\ &= \sum_{i=1}^{n} \left(\frac{i^3}{n^3} + \frac{2i}{n} \right) \frac{1}{n} \\ &= \frac{1}{n^4} \sum_{i=1}^{n} i^3 + \frac{2}{n^2} \sum_{i=1}^{n} i \\ &= \frac{n^2(n+1)^2}{4n^4} + \frac{2n(n+1)}{2n^2} \end{aligned}$$

- Take limits as $n \to \infty$ to get

$$\begin{aligned} A &= \lim_{n\to\infty} R_n \\ &= \frac{1}{4} + 1 = \frac{5}{4} \end{aligned}$$

Distances — CLP 1.1.6

Integrals are not just used to compute areas under curves. Here is another simple interpretation of the integral.

Back in differential calculus we discussed distance and velocity. If velocity is constant then

$$\text{distance} = \text{velocity} \times \text{time}$$

Notice that this relationship is very similar to

$$\text{area} = \text{height} \times \text{width}$$

Of course the area under a curve is not as simple as this because the height varies as we move along the x-axis, but we approximated the area using lots of little rectangles. We can do the same thing with distances — the velocity is not constant as time increases, but we can approximate the total distance in the same way we approximated area.

Say we know the velocity of a car at regular times

time (s)	0	10	20	30	40	50	60
velocity (m/s)	3	7	8	11	13	12	11

Then during the first 10 seconds it travelled (using right-endpoints) $7 \times 10 = 70m$. Similarly for the next 10 seconds and the next etc etc.

$$\begin{aligned} \text{distance} &= 7 \times 10 + 8 \times 10 + 11 \times 10 + 13 \times 10 + 12 \times 10 + 11 \times 10 \\ &= (7 + 8 + 11 + 13 + 12 + 11) \times 10 = 620m \end{aligned}$$

1.2 Basic properties of the definite integral — CLP 1.2

Now when we looked at derivatives of functions we saw some useful properties that helped us compute things — eg

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) + g(x)\right) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}$$

Indeed — some of the most important things we learned about derivatives was how the derivative interacts with the basic operations of arithmetic. Good news and bad news — integrals interact with addition and multiplication by a constant. However they do not interact as nicely with multiplication, division or composition — the equivalents of the product and chain rules exist, but are more involved. We will get to that later.

Theorem (Arithmetic of integration — CLP theorem 1.2.1)**.** *Let f and g be integrable on $[a, b]$ and let A, B, C be constants. Then*

$$\begin{aligned} \int_a^b \left(f(x) + g(x)\right) \mathrm{d}x &= \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x \\ \int_a^b \left(f(x) - g(x)\right) \mathrm{d}x &= \int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x \\ \int_a^b Cf(x)\mathrm{d}x &= C \cdot \int_a^b f(x)\mathrm{d}x \end{aligned}$$

Combining these gives

$$\int_a^b C\mathrm{d}x = C \cdot (b - a)$$

$$\int_a^b \left(Af(x) + Bg(x)\right) \mathrm{d}x = A \int_a^b f(x)\mathrm{d}x + B \int_a^b g(x)\mathrm{d}x$$

These all follow quite directly from the Riemann sum definition.

Theorem (Domain of integration — CLP Theorem 1.2.3)**.** *Let a, b, c be real numbers and let f be integrable on a domain that contains a, b, c. Then*

$$\int_a^a f(x)\mathrm{d}x = 0$$
$$\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$$
$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

The proofs of these are no too difficult from the Riemann sum definition. We get the first by noting $\Delta x = \frac{a-a}{n} = 0$, and the second by noting that $\frac{b-a}{n} = \frac{a-b}{n}$. See the text for more details.

Using what we now know, we can compute

$$\int_a^b x\mathrm{d}x$$

First up — we know that $\int_0^b x\mathrm{d}x = b^2/2$. So write the integral as

$$\int_a^b x\mathrm{d}x = \int_a^0 x\mathrm{d}x + \int_0^b x\mathrm{d}x$$

Now turn the terminals around

$$= -\int_0^a x\mathrm{d}x + \int_0^b x\mathrm{d}x$$

Use our know integral

$$= -\frac{a^2}{2} + \frac{b^2}{2}.$$

Of course, this is still quite involved for such simple functions — we really need a better way.

Even and odd functions

Symmetries can sometimes make integrating easier. Recall

Definition (CLP defn 1.2.9)**.** Let $f(x)$ be a function, then

- $f(x)$ is even, when $f(x) = f(-x)$ for all x, and
- $f(x)$ is odd, when $f(x) = -f(-x)$ for all x.

Why "even" and "odd"? because of monomials

$$(-x)^{even} = (-1)^{even} x^{even} = x^{even}$$
$$(-x)^{odd} = (-1)^{odd} x^{odd} = -x^{odd}$$

Also $\sin(x)$ is odd and $\cos(x)$ is even. Hence $\tan(x)$ is odd.

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

Integrating even and odd functions over a symmetric domain $[-a, a]$ gives

Theorem (CLP theorem 1.2.12)**.** *Let $a > 0$ be a real number and f an integrable function.*

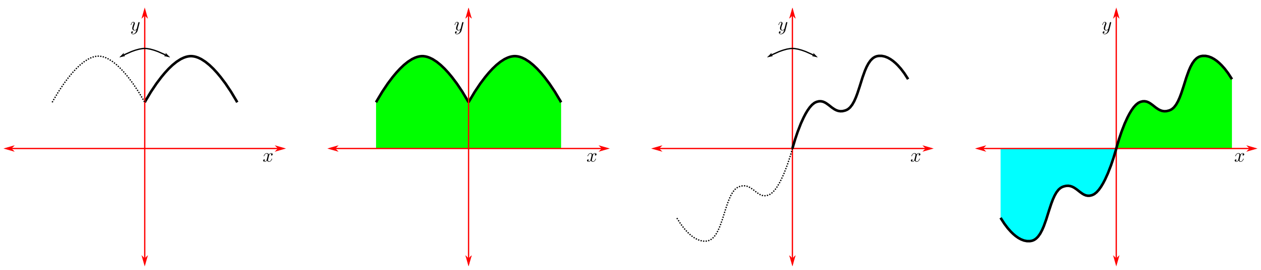
- *If $f(x)$ is an even function then*

$$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x.$$

- *If $f(x)$ is an odd function then*

$$\int_{-a}^{a} f(x)\mathrm{d}x = 0$$

To understand this simply draw some pictures



This theorem tells us that

$$\int_{-\pi}^{\pi} \sin(x)\mathrm{d}x = 0 \qquad \int_{-\pi}^{\pi} \cos(x)\mathrm{d}x = 2\int_{0}^{\pi} \cos(x)\mathrm{d}x$$

even though we don't know

$$\int_{0}^{\pi} \sin(x)\mathrm{d}x = ??? \qquad \int_{0}^{\pi} \cos(x)\mathrm{d}x = ???$$

1.3 Fundamental Theorem of Calculus — CLP 1.3

So I think I have now convinced you that computing definite integrals using Riemann sums is quite painful. There is a better way, and that is through a link to the calculus you did last term. The fundamental theorem of calculus links integral calculus to differential calculus. It is a remarkable result.

It says (approximately) that "differentiating undoes integrating." This is the first part of the theorem — the second part soon.

Theorem (Fundamental Theorem of Calculus Part 1 — CLP theorem 1.3.1)**.** *Let f be continuous on $[a, b]$. Then define the function*

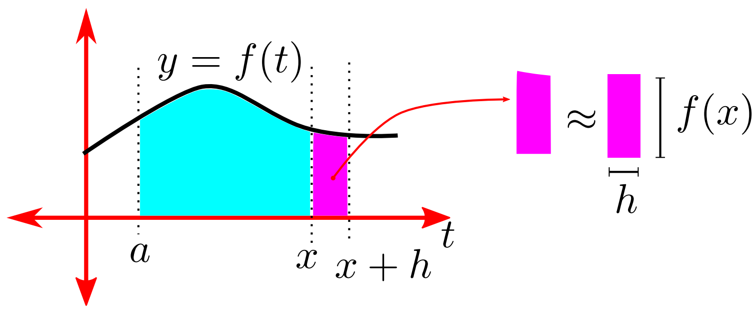
$$F(x) = \int_a^x f(t)\mathrm{d}t \qquad \textit{for } x \in [a, b]$$

then

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$$

Notice here that we are doing something slightly sneaky with the definite integral. We are setting one of its terminals to be x — by changing x we change the terminal of the definite integral. It is still a perfectly fine function — choose x and it returns a number.

We can get the idea of the proof of this theorem via the following picture



So what happens to the area when we move x to $x + h$? Go back to the definition of the derivative.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}F(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \left(\int_a^{x+h} f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t \right) / h \\
&= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\mathrm{d}t \qquad \text{splitting the interval}
\end{aligned}$$

So now think of what this definite integral is — it is a tiny area that is very nearly a rectangle of height $f(x)$ and width h. So we expect the definite integral to be approximately $f(x) \cdot h$. Indeed we can show (if we are careful — see proof in text) that as h gets closer and closer to zero, the definite integral $\int_x^{x+h} f(t)\mathrm{d}t$ gets closer and closer to $f(x) \cdot h$. Hence as $h \to 0$ we get

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$$

Some examples:

- Let $F(x) = \int_0^x \cos(t)\mathrm{d}t$. Then $F'(x) = \cos(x)$.
- Let $F(x) = \int_1^x (t^2 + \sqrt{t+1})\mathrm{d}t$. Then $F'(x) = x^2 + \sqrt{x+1}$.
- Let $F(x) = \int_1^{x^2} \cos(t)\mathrm{d}t$ Then find $F'(x)$.

- We need to use the chain rule.
- Write $G(u) = \int_1^u \cos(t)\mathrm{d}t$, with $u = x^2$.
- Then using the chain rule we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}G(u(x)) &= \underbrace{\frac{\mathrm{d}G}{\mathrm{d}u}}_{FTC1}\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \cos u \cdot 2x = 2x\cos(x^2)\end{aligned}$$

While this is cute, it isnt so useful. What we really want to do is given $f(x)$ compute the integral of f. The FTC tells us that it is undone by differentiating. Let us make this more precise

Definition (CLP defn 1.3.7)**.** Given a function f, then any function F with $F'(x) = f(x)$ is called an anti-derivative of f.

For example, $F(x) = x^2$ is an anti-derivative of $f(x) = 2x$. But so is $F(x) = x^2 + 3$ and $F(x) = x^2 - 7.35$. We can check by differentiating. What other anti-derivatives are out there? How are they related to each other? — thankfully it is not hard.

Lemma (CLP lemma 1.3.8)**.** *If F is an anti-derivative of f, then any other anti-derivative of f is of the form $F(x) + c$, where c is a constant.*

(If we have time, then do this proof)

Proof. A common proof trick when trying to show that 2 functions are the same (or nearly so) is to show that their difference is zero (or nearly so).

- Suppose F and G are anti-derivatives of f on the interval $[a, b]$.
- Let $H = F - G$.
- By the FTC (part 1) we know that $H' = F' - G' = 0$ for all $x \in [a, b]$.
- Hence $H(x)$ is a constant.
- So $F(x) = G(x) + c$.

Based on the above we define the indefinite integral

Definition (CLP defn 1.3.9)**.** The indefinite integral of $f(x)$ is denoted $\int f(x)\mathrm{d}x$ (without terminals) and should be regarded as the general antiderivative of $f(x)$. In particular, if $F(x)$ is an antiderivative of $f(x)$ then

$$\int f(x)\mathrm{d}x = F(x) + C \tag{1}$$

where C is an arbitary constant — also called the constant of integration.

Last term we learned some simple anti-derivatives — basically by taking a table of derivatives and switching the order of the columns. Here are some of the ones we learned (see CLP theorem 1.3.17), which we write as indefinite integrals

$f(x)$	$F(x) = \int f(x)\mathrm{d}x$
1	$x + C$
x^n	$\frac{1}{n+1}x^{n+1} + C$ provided $n \neq 1$
$\frac{1}{x}$	$\log\lvert x\rvert + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\frac{1}{1+x^2}$	$\arctan x + C$

Armed with all of these lets finish off the FTC. The second part of the theorem links anti-derivatives to the area under the curve $\int_a^b f(x)\mathrm{d}x$. It seems very surprising that these two things are linked — somehow differentiating (and its inverse) are very symbolic operations — we juggle the symbols around — and yet they are intimately linked with the very tangible geometric ideas of the area under a curve and the slope of a tangent line.

Theorem (Fundamental Theorem of Calculus Part 2 — CLP theorem 1.3.1)**.** *Let f be continuous on $[a,b]$ and let F be any anti-derivative of f. Then*

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a)$$

Note that we might also write the RHS as (see CLP defn 1.3.10)

$$\begin{aligned}\int_a^b f(t)\mathrm{d}t &= F(b) - F(a)\\ &\equiv \int f(x)\mathrm{d}x\bigg|_a^b\\ &\equiv F(x)\Big|_a^b\\ &\equiv \Big[F(x)\Big]_a^b\end{aligned}$$

Lets put this to work. We saw (and can easily check) that an anti-derivative of $f(x) = 2x$ is $F(x) = x^2$. So we can write

$$\int 2x\mathrm{d}x = x^2 + C.$$

And if we want to compute

$$\int_0^1 2x\mathrm{d}x = F(1) - F(0) = 1^2 - 0^2 = 1$$

just as we got by the much more difficult method of Riemann sums before.

The proof of the second part of the FTC is actually quite straight forward.

Proof.

- By FTC1 we know that $\int_a^x f(t)\mathrm{d}t$ is an anti-derivative of $f(x)$.
- Hence we write $\int_a^x f(t)\mathrm{d}t = F(x) + c$ for some constant c.
- What is c? what extra information do we have?
- When $x = a$ we have

$$\int_a^a f(t)\mathrm{d}t = F(a) + c = 0$$

- So $c = -F(a)$.
- Now when $x = b$ we get

$$\int_a^b f(t)\mathrm{d}t = F(b) + c = F(b) - F(a)$$

- Note, it doesnt matter which anti-derivative we choose, because the constant "$+C$" cancels out.

□

So by FTC 1 and 2, we see that integration and differentiation are inverse operations — they undo each other.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\mathrm{d}t\right) = f(x)$$
$$\int_a^x \left(\frac{\mathrm{d}}{\mathrm{d}t}F(t)\right)\mathrm{d}t = F(x) - F(a)$$

A few more examples:

$$\begin{aligned}\int_0^1 (x^2 - 3x)\mathrm{d}x &= \int_0^1 x^2\mathrm{d}x - \int_0^1 3x\mathrm{d}x \\ &= \int_0^1 x^2\mathrm{d}x - 3\int_0^1 x\mathrm{d}x \\ &= [x^3/3]_0^1 - 3[x^2/2]_0^1 \\ &= \frac{1}{3} - 3 = -\frac{8}{3}\end{aligned}$$

Much much easier. Why don't you redo the other example

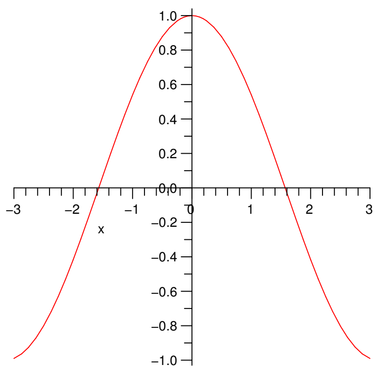
$$\int_0^1 x^3 + 2x\mathrm{d}x = \left[\frac{x^4}{4} + x^2\right]_0^1 \qquad = 1/4 + 1 = 5/4$$

What happens if we change the terminals —

$$\begin{aligned}\int_1^3 x^3 + 2x\mathrm{d}x &= \left[\frac{x^4}{4} + x^2\right]_1^3 \\ &= \left(\frac{81}{4} + 9\right) - \left(\frac{1}{4} + 1\right) \\ &= \frac{117}{4} - \frac{5}{4} = 112/4 = 28\end{aligned}$$

Another example for you to do.

- Area under the curve $y = \cos x$ between $x = \pm\pi/2$.



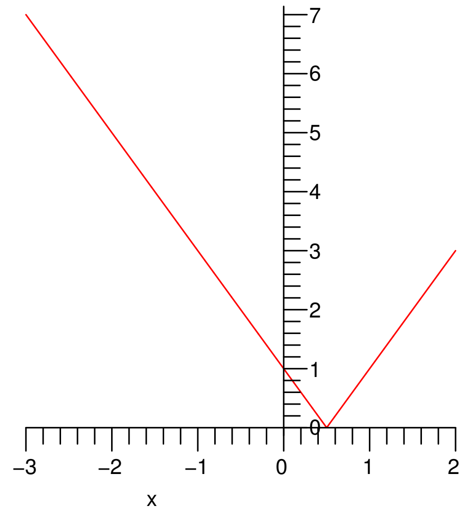
- Area is given by

$$\begin{aligned}A &= \int_{-\pi/2}^{\pi/2} \cos(x)\mathrm{d}x \\ &= [\sin(x)]_{-\pi/2}^{\pi/2} \\ &= \sin(\pi/2) - \sin(-\pi/2) = 1 - (-1) = 2.\end{aligned}$$

Another example in which we have to be a bit more careful.

$$\int_{-2}^{1} |2x - 1|\mathrm{d}x$$

The integrand is continuous, but the $|.|$ is a bit worrying. So first step is to plot things.



We could just compute areas of triangles, but let us do it using integration.

- Split the integral to take into account

$$|q| = \begin{cases} q & q \geq 0 \\ -q & q < 0 \end{cases}$$

- We need to split things where $2z - 1 = 0$, namely at $z = 1/2$.

$$|2z-1| = \begin{cases} 2z-1 & z \geq 1/2 \\ 1-2z & z < 1/2 \end{cases}$$

- Hence the integral becomes

$$\begin{aligned} \int_{-2}^{1} |2z-1| \mathrm{d}z &= \int_{-2}^{1/2} (1-2z)\mathrm{d}z - \int_{1/2}^{1} (2z-1)\mathrm{d}z \\ &= [z - z^2]_{-2}^{1/2} - [z^2 - z]_{1/2}^{1} \\ &= ((1/2 - 1/4) - (-2 - 4)) - ((1/4 - 1/2) - (1 - 1)) \\ &= 1/4 + 6 + 1/4 - 0 = 6 + 1/2 = 13/2 \end{aligned}$$

Not too bad?

$$\begin{aligned} \int_{-1}^{1} \arctan x \mathrm{d}x &= \left[\frac{1}{1+x^2}\right]_{-1}^{1} \\ &= \frac{1}{1+1} - \frac{1}{1+(-1)^2} = 0 \end{aligned}$$

(of course we could also have noted that the integrand was an odd function).

So we are on our way to being able to integrate families of functions

- we know how the integral interacts with addition and multiplication by a constant (nicely)
- we have a nice table of building blocks (anti-derivatives of easy functions)
- we have the FTC to express integrals in terms of anti-derivatives

but we don't yet know how to integrate things like e^{x^2} or $x \sin x$ — this requires the integration equivalent of the chain rule and the product rule. That being said, we can do some simple compositions (which you can check by differentiating). Let $f(x)$ be a function with an antiderivative $F(x)$, then by the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x} F(ax) = af(ax) \qquad \text{where } a \text{ is a constant}$$

Hence

$$\int f(ax)\mathrm{d}x = \frac{1}{a}F(ax) + C$$

This allows us to integrate things like $\cos(3x)$ and e^{2x} — do it!

This works precisely because a is a constant and its derivative is zero. Composing with non-constant functions requires a bit more work and a careful look at the chain rule. That is our next topic.