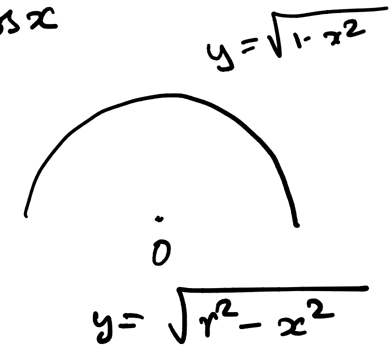
Integration by trigonometric substitution.

Goal: use trigonometric substitution to simplify integrand.

trigonometric substitution : $x \longmapsto \sin x$. $\quad \tan x \quad \cos x \quad \sec x$

Recall $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}$

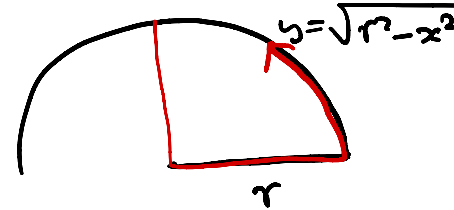


more generally,

Area of circle of radius r $\displaystyle = 4\int_{0}^{r} \sqrt{r^2 - x^2}\, dx$

$= \pi r^2.$

Area using trigonometric substitution.



$$A = 4\int_0^r \sqrt{r^2 - x^2}\, dx.$$

$a^2 \pm x^2$

let $x = r\sin\theta$

$$\frac{dx}{d\theta} = r\cos\theta \rightarrow dx = r\cos\theta\, d\theta$$

$\sin^2\theta + \cos^2\theta = 1$

$r^2\cos^2\theta = r^2(1 - \sin^2\theta)$

$$A = 4\int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2\theta}\; r\cos\theta\, d\theta$$

when $x = 0$, $0 = r\sin\theta$

$\Rightarrow \sin\theta = 0$

$\Rightarrow \theta = 0$

$$= 4\int_0^{\pi/2} r\sqrt{1-\sin^2\theta}\; r\cos\theta\, d\theta$$

$$= 4r^2\int_0^{\pi} \cos^2\theta\, d\theta$$

when $x = r$, $r = r\sin\theta$

$\Rightarrow \sin\theta = 1$

$\Rightarrow \theta = \frac{\pi}{2}$

$$A = 4r^2 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$

$$\Rightarrow \cos^2\theta = \frac{1}{2}\left(\cos 2\theta + 1\right)$$

$$A = \cancel{4}^{2} r^2 \int_0^{\pi/2} \frac{1}{\cancel{2}} \left(\cos 2\theta + 1\right) d\theta$$

$$= 2r^2 \left[\frac{\sin 2\theta}{2} + \theta \right]_0^{\pi/2}$$

$$= 2r^2 \left[\cancel{\frac{\sin \pi}{2}} + \frac{\pi}{2} - \cancel{\frac{\sin 0}{2}} - \cancel{0} \right]$$

$$= \pi r^2$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

$$\sin^2 a + \cos^2 a = 1$$

Connection to u-substitution.

Recall:

$$\int f(u(x))\,\frac{du}{dx}\,dx = \int f(u)\,du \Big|_{u=u(x)}.$$

$a \to b \qquad\qquad u(a) \to u(b)$

In u-substitution, we tried to go from left to right.

Trigonometric substitution goes from right to left.

$$\int_0^r \sqrt{r^2-x^2}\,dx = \int_0^{\pi/2} \sqrt{r^2-r^2\sin^2\theta}\,\frac{dx}{d\theta}\,d\theta.$$

Trigonometric identities:

$$\int \frac{1}{x^2\sqrt{x^2-a^2}}\,dx$$

Identity	Expression	Substitution
$1-\sin^2\theta=\cos^2\theta$	$\sqrt{a^2-x^2}$	$x=a\sin\theta$
$\sec^2\theta-1=\tan^2\theta$	$\sqrt{x^2-a^2}$	$x=a\sec\theta$
$1+\tan^2\theta=\sec^2\theta$	$\sqrt{a^2+x^2}$ or $\dfrac{1}{a^2+x^2}$	$x=a\tan\theta$

$$\int_0^r \sqrt{r^2-x^2}\,dx$$

Q- Solve the indefinite integral $\int\sqrt{a^2-x^2}\,dx$.

using same steps: Let $x = a\sin\theta \Rightarrow \dfrac{dx}{d\theta} = a\cos\theta$;

Example (contd.)

$x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a}$

so, $\int\sqrt{a^2-x^2}\,dx = \int\sqrt{a^2-a^2\sin^2\theta}\; a\cos\theta\,d\theta$

$$= \frac{a^2}{2}\int(\cos 2\theta + 1)\,d\theta$$

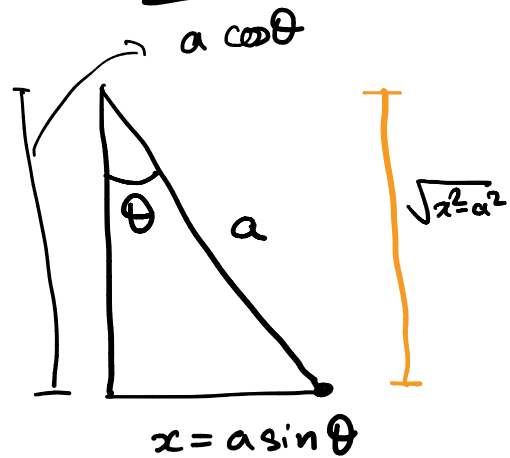
$$= \frac{a^2}{2}\left(\frac{\sin 2\theta}{2} + \theta\right) + C.$$

$$\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}$$

need to express solution in x.

so, $a\cos\theta = \sqrt{x^2-a^2} \Rightarrow \cos\theta = \frac{1}{a}\sqrt{x^2-a^2}$

and $\sin\theta = x/a$



so, $\int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\left(\frac{2\sin\theta\cos\theta}{2} + \theta\right) + c$

$$= \frac{a^2}{2}\left(\frac{x}{a}\,\frac{1}{a}\sqrt{x^2-a^2} + \arcsin\left(x/a\right)\right) + C$$

Example.

$$\int \frac{1}{x^2+4x+7}\,dx \longrightarrow \boxed{(x+a)^2+b}$$

$$(x^2+2ax+a^2)+b$$

$$x^2+4x+7 = x^2+4x+4-4+7 = (x+2)^2+3.$$

$$\frac{1}{\underbrace{a^2+x^2}}$$

$$x = a\tan\theta$$

$$(x+2) = \sqrt{3}\tan\theta$$

$$\frac{dx}{d\theta} = \sqrt{3}\sec^2\theta \Rightarrow dx \to \sqrt{3}\sec^2\theta\, d\theta$$

$$\int \frac{1}{x^2+4x+7}\,dx = \int \frac{1}{(x+2)^2+3}\,dx$$

$$= \int \frac{1}{3\tan^2\theta+3}\sqrt{3}\sec^2\theta\, d\theta$$

$$= \frac{1}{3}\int \frac{1}{\tan^2\theta+1}\sqrt{3}\sec^2\theta\, d\theta$$

$$\sec^2\theta - \tan^2\theta = 1$$

Example contd.

$$\int \frac{1}{x^2+4x+7}\,dx = \sqrt{3}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$= \sqrt{3}\int 1\,d\theta$$

$$= \sqrt{3}\,\theta + C$$

$$(x+2) = \sqrt{3}\tan\theta \Rightarrow \theta = \arctan\left(\frac{x+2}{\sqrt{3}}\right)$$

$$\int \frac{1}{x^2+4x+7}\,dx = \sqrt{3}\arctan\left(\frac{x+2}{\sqrt{3}}\right) + C.$$

Example.

Solve $\int \frac{dx}{x^2\sqrt{x^2-16}}$

Let $x = 4\sec\theta$

$\frac{dx}{d\theta} = 4\sec\theta\tan\theta \Rightarrow dx \to 4\sec\theta\tan\theta\, d\theta$

$$\int \frac{\cancel{4\sec\theta}\tan\theta\, d\theta}{\cancel{16}_{4}\sec^{\cancel{2}}\theta\sqrt{16\sec^2\theta-16}}$$

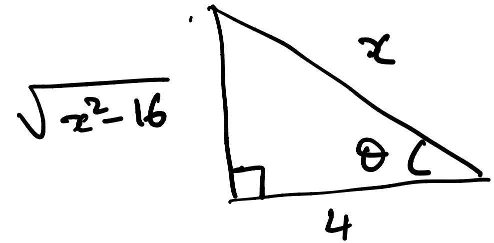
$$= \frac{1}{4}\int \frac{\cancel{\tan\theta}}{\sec\theta\, 4\cancel{\tan\theta}}\, d\theta$$

$$= \frac{1}{16}\int \cos\theta\, d\theta$$

$$= \frac{1}{16}\sin\theta + C$$

Example contd.

$$\int \frac{1}{x^2\sqrt{x^2-16}}\,dx = \frac{1}{16}\sin\theta + C$$



$$x = 4\sec\theta \Rightarrow \cos\theta = \frac{4}{x}$$

$$x = \frac{4}{\cos\theta}$$

$$\Rightarrow \sin\theta = \frac{\sqrt{x^2-16}}{x}$$

$$\int \frac{1}{x^2\sqrt{x^2-16}}\,dx = \frac{1}{16}\,\frac{\sqrt{x^2-16}}{x} + C$$