Definite integral.

Goals:

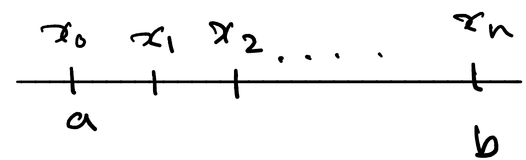
- Definite integrals.
- Even and odd functions.

Definition of definite integral

Recall limit of Riemann sum is the area under curve.

$$\text{Area} = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{Riemann sum.}}$$

- Δx is the width of the sub intervals.
- $x_i = a + i\Delta x$ and $x_i^* \in [x_{i-1}, x_i]$



- $f(x_i^*)$ = height of f at x_i^*.

Definition of definite integral (contd)

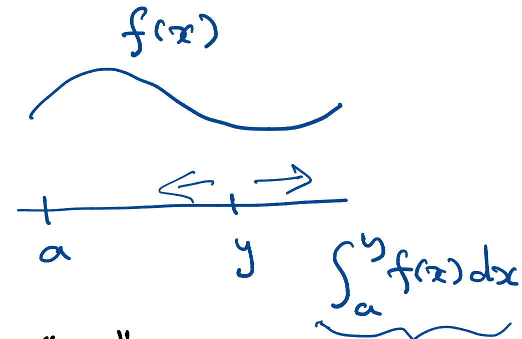
This limit is called definite integral denoted by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

provided limit exists. (see CLP 1.1.8 and 1.1.9)

- If the limit exists then we say f is integrable on $[a,b]$.

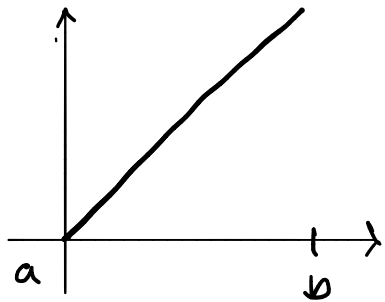
Definition of definite integral (contd)



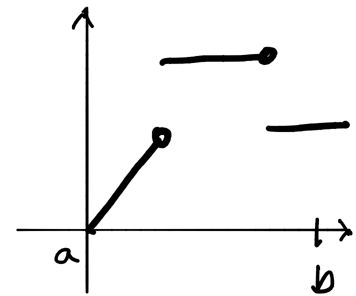
$$\int_a^b f(x)\,dx$$

- $\int$ is the integral sign resembling "S" for sum.
- $f(x)$ is called integrand.
- a and b are called limits of integration.
- $\int_a^b f(x)\,dx$ outputs a number that depends on a, b but does not depend on x.

Integrability of a function.



Continous



finite # of discontinuity

Thm (CLP Thm 1.1.10)

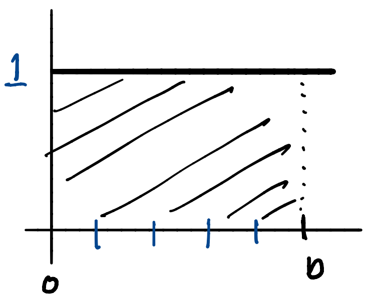
If f is continous on $[a,b]$ or only has a finite # of discontinuities, the f is integrable on $[a,b]$.

Example

- $\int_0^b 1\, dx = ?$

integrand $= 1$

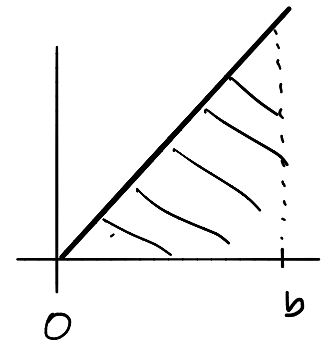
limits of integration $= 0,\ b$



area $= \underline{b}\ b$

- $\int_0^b x\, dx = ?$

integrand $= x$



area $= b \cdot \frac{b}{2} = \frac{b^2}{2}$

$f(x) = \sqrt{1-x^2}$

A more involved example

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\int_0^4 (x^2-3x)\,dx$$

integrand : $x^2 - 3x$

limits of integration : 0 and 4.

- Riemann sum : $a=0$, $b=4$, $\Delta x = (b-a)/n = 4/n$, $x_i = a + i\Delta x$

height (right end point) $= f(x_i)$

$$= f(4i/n)$$

$$= \frac{16i^2}{n^2} - \frac{12i}{n}$$

So,

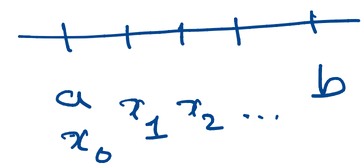
$$R_n = \sum_{i=1}^{n} \underbrace{\left(\frac{16i^2}{n^2} - \frac{12i}{n}\right)}_{f(x_i)} \underbrace{\frac{4}{n}}_{\Delta x}$$

$$= \frac{64}{n^3}\sum_{i=1}^{n} i^2 - \frac{48}{n^2}\sum_{i=1}^{n} i$$

$$= \frac{64}{n^3}\,\frac{n(n+1)(2n+1)}{6} - \frac{48}{n^2}\,\frac{n(n+1)}{2}$$

- $\sum(a_i+b_i)$
$= \sum a_i + \sum b_i$
- $\sum c a_i = c\sum a_i$

$\Delta x = (b-a)/n$



$x_1 = a + \Delta x$

$x_i = a + i \cdot \Delta x$.

A more involved example (contd)

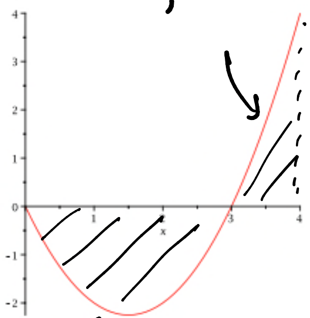
- Area is limit of R_n:

$64 \cdot 2 \cdot n^3 + \dots$ $\qquad$ $48n^2 + \dots$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(\frac{64\, n(n+1)(2n+1)}{6n^3} - \frac{48\, n(n+1)}{2n^2} \right)$$

$$= \frac{64}{3} - \frac{48}{2} = -\frac{8}{3}$$

$f(x)\Delta x > 0$

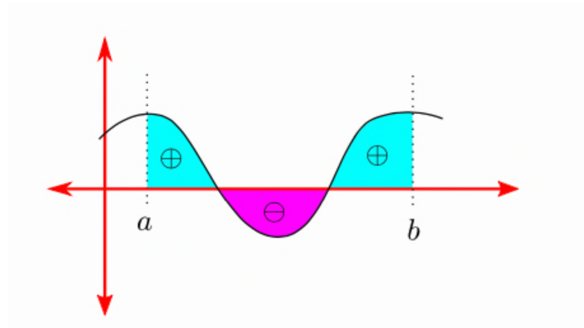


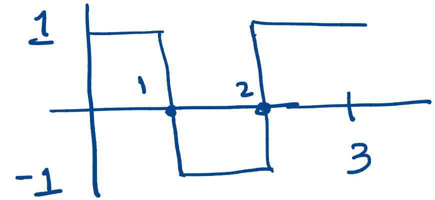
$f(x)\Delta x < 0$

- most of the area is under x-axis.

$f(x) = x^2 - 3x$ on $[0,4]$

Definite integral outputs signed area





$$\int_0^3 f(x)\,dx$$

$$= 1$$

- Adds area above x-axis.
- Substracts area below x-axis.

exercise: Evaluate $\int_0^1 x^3 + 2x \, dx$.

<u>Exercise</u> $\int_0^1 x^3+2x\,dx$ limits of integration, $a=0,\ b=1$

- Riemann sum : $\Delta x = (b-a)/n = 1/n$, $x_i = a + i\,\Delta x = i/n$

$$f(x_i^*) = f(x_i) = \frac{i^3}{n^3} + \frac{2i}{n}$$

$$R_n = \sum_{i=1}^{n}\left(\frac{i^3}{n^3} + \frac{2i}{n}\right)\frac{1}{n}$$

$$= \sum_{i=1}^{n}\frac{i^3}{n^4} + \sum_{i=1}^{n}\frac{2i}{n^2}$$

$$= \frac{1}{n^4}\,\frac{n^2(n+1)^2}{4} + \frac{2}{n^2}\,\frac{n(n+1)}{2}$$

$$= \frac{n^2(n+1)^2}{4n^4} + \frac{n(n+1)}{n^2} \xrightarrow{n\to\infty} \frac{1}{4} + 1 = \underline{\underline{\frac{5}{4}}}$$

Application computing distance

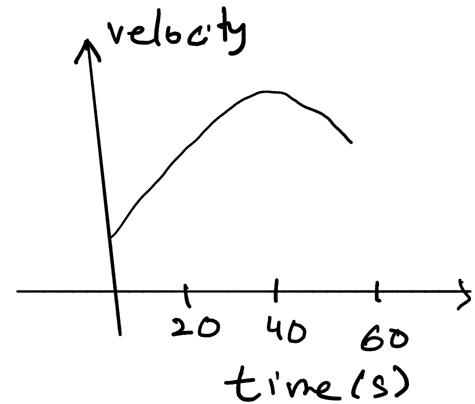
Recall: distance = velocity × time.

compare to: area = height × width.

Integral of velocity as a function of time is distance.

Assume we have velocity reading at different time:

time (s)	0	10	20	30	40	50	60
velocity (m/s)	3	7	8	11	13	12	11



approx distance: (using right end point)

$$R_n = 10 \cdot (7 + 8 + 11 + 13 + 12 + 11) = 620 \text{ m}.$$

Basic properties of definite integral.

Thm (CLP 1.2.1). Let f, g be any integrable functions on $[a, b]$. Let A, B, C be any constants.

- $$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

  $$\left(\sum (a_i + b_i) = \sum a_i + \sum b_i \right)$$

- $$\int_a^b C f(x)\,dx = C \int_a^b f(x)\,dx.$$

$$\sum_{i=1}^{n} (a_i + b_i)$$

Integral is a linear operator.

Basic properties of definite integral. (contd)

- $\int_a^b (f(x) - g(x))\, dx = \int^b f(x)dx - \int_a^b g(x)\, dx.$

  $\int_a^b (f(x) + (-g(x)))\, dx$

- $\int_a^b c\, dx = c\,(b-a)$

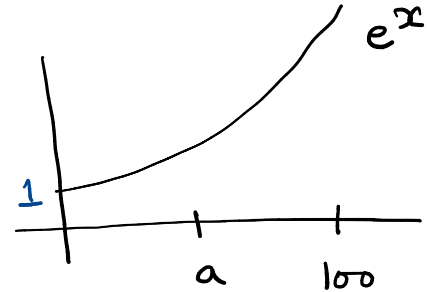
- $\int_a^b (A f(x) + B g(x))\, dx = A \int_a^b f(x)\, dx + B \int_a^b g(x)\, dx.$

follows from first two property!

ex: $\int_0^1 (5x^2 + 3x + 2)\, dx = 5\int_0^1 x^2\, dx + 3\int_0^1 x\, dx + \underbrace{2\int_0^1 dx}_{2.}$

Domain of integration.

ex: ○ $\displaystyle\int_{100}^{100} e^x\,dx = ?$ (0)



○ For any $a \in \mathbb{R}$

$$\int_0^{100} e^x\,dx = \int_0^a e^x\,dx + \int_a^{100} e^x\,dx$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$(b-a)/n$

$(a-b)/n$

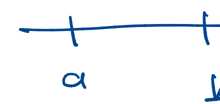
$= -(b-a)/n$

Thm:

○ $\displaystyle\int_a^a f(x)\,dx = 0$

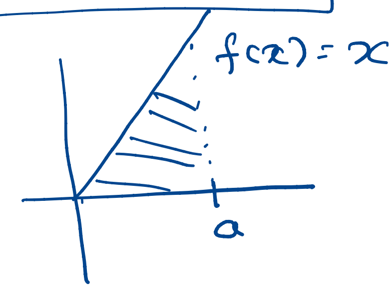
○ $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

○ $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



Example

Compute $\int_a^b x \, dx$ using $\int_0^a x \, dx = \frac{a^2}{2}$.



$$= \int_a^0 x \, dx + \int_0^b x \, dx$$

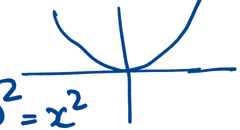
$$= - \int_0^a x \, dx + \int_0^b x \, dx$$

$$= - \frac{a^2}{2} + \frac{b^2}{2}.$$

Even and odd functions.

$f(x) = x^2$

$f(-x) = (-x)^2 = x^2$



$f(x) = x^3$

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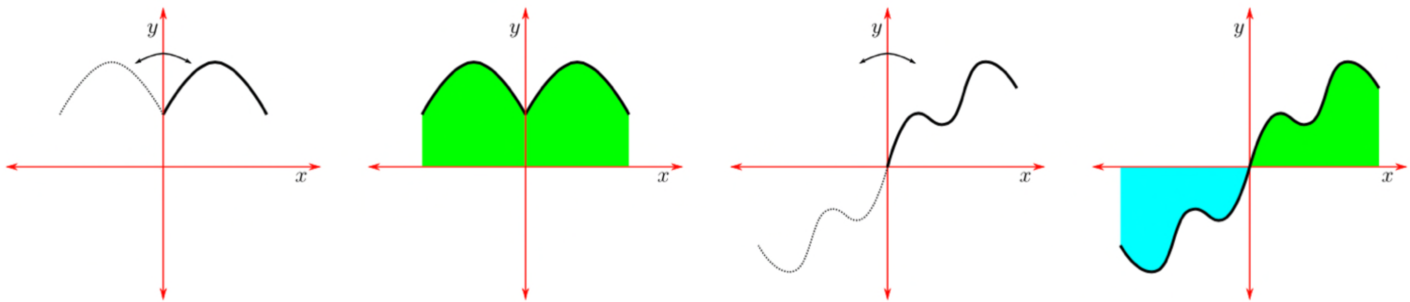


Even function $\qquad\qquad$ odd function.

Def^n (CLP defn 1.2.9) Let $f(x)$ be a function, t

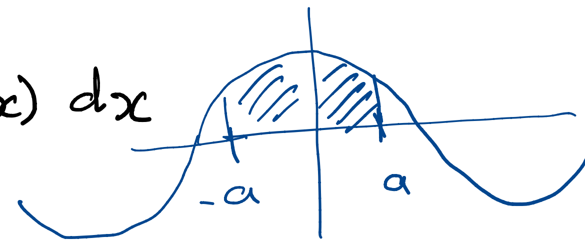
- $f(x)$ is even if $f(x) = f(-x)$ for all x.
- $f(x)$ is odd if $f(x) = -f(-x)$ for all x.

Symmetry in integration.

Thm (CLP 1.2.12) Let $a > 0$ be a real number and f an integrable function.

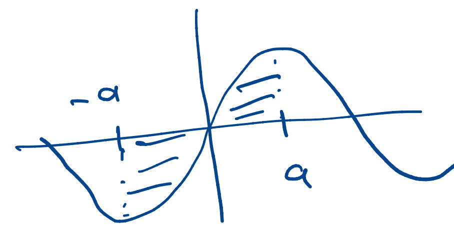
- If f is an even function then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



- If f is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0$$



Example

Compute $\displaystyle\int_{\pi/2}^{\pi/2} \cos x \, dx.$ given $\displaystyle\int_{0}^{\pi/2} \cos x \, dx = 1.$

$$2\int_{0}^{\pi/2} \cos(x)\, dx$$

$$= 2 \cdot 1$$

$$= 2.$$