Taylor series.

The general form for a Taylor series of a smooth function $f(x)$ about $x = x_0$ is.

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n.}$$

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Maclaurin series:

Some important Taylor series about $x=0$ are:

$e^x =$

$\sin(x) =$

$\cos(x) =$

-

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Maclaurin series of $\cos(x)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x-0)^n.$$

Maclaurin series of $\cos(x)$.

Radius of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}}{(2(n+1))!}\ \frac{(2n)!}{x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{(2n+1)(2n+2)}\right|$$

$$= 0.$$

so, the Maclaurin series representation of $\cos(x)$ converges for all $x \in \mathbb{R}$.

Geometric series

The geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$, is also a Maclaurin series.

We can derive series representation of other functions like $\log(1-x)$ and $\arctan(x)$ by manipulating geometric series. See previous lecture

Example.

Write the Maclaurin series of $f(x) = (1+x)^p$ for $p \in \mathbb{R}$.

Summary

Our key results that should be memorized are:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Taylor series can be integrated, differentiated, added, multiplied and divided when inside interval of convergence.

Example 1

Let $f(x) = \log(1+2x^2)$ for $|x| < 1/\sqrt{2}$.

1. Find the Maclaurin series of $f(x)$.

Soln:

Example 1 (contd).

ii) compute $\lim_{x \to 0} \dfrac{\log(1+2x^2)}{x^2}$

iii) Compute $f^{(8)}(0)$.

Example 2.

Calculate $\lim_{x \to 0} \dfrac{\cos(x^2) - (1 - x^4/2)}{x^8}$

using Taylor expansion of $\cos(x)$.

Examples

(a) Compute $x^7 e^x$

$$x^7 e^x = \sum_{n=0}^{\infty} \frac{x^7 x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+7}}{n!}$$

(b) Compute $\cos x \, e^x$

$$\cos x \, e^x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= 1 + x - \frac{x^3}{3} + \cdots$$

Integrating e^{-x^2}:

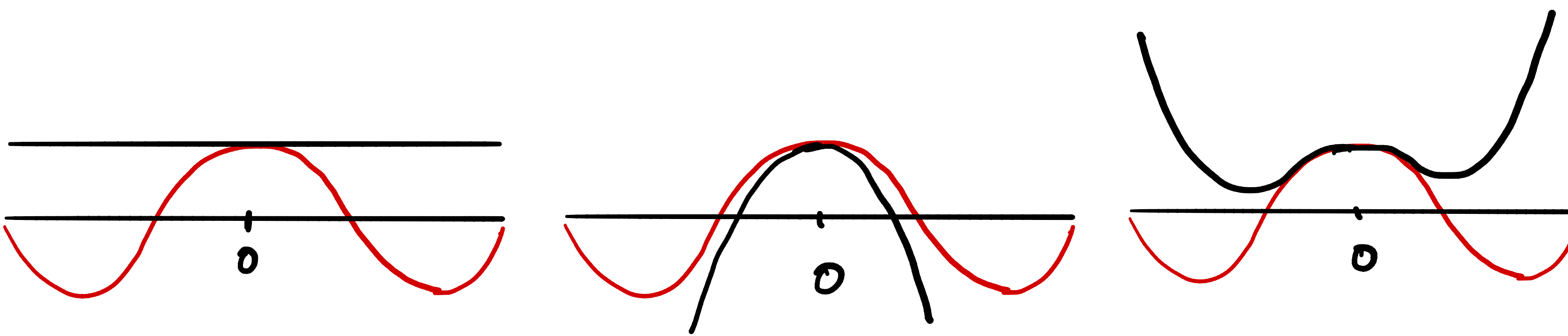
We can use Maclaurin series of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Integrating e^{-x^2} (contd).

So, we can estimate $\int_0^1 e^{-x^2}\,dx$ using N^{th} partial

Remainder Theorem.



Taylor Remainder Theorem.

Thm: If we can bound $|f^{(n+1)}(x)| \leq M$ for all $|x-a| \leq d$ then the remainder $R_n(x)$ of Taylor series satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1},$$

for all $|x-a|<d$.

Ex: