

Numerical integration: short recap

Definite integral approximations: n intervals, $n+1$ points numbered from 0 to n

Mid point rule

$$\int_a^b f(x)dx \approx M_n = \sum f(\bar{x}_i)\Delta x = \Delta x(f(\bar{x}_1) + f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n))$$
$$\Delta x = \frac{b-a}{n} \quad x_i = a + i\Delta x \quad \bar{x}_i = (x_{i-1} + x_i)/2$$

Trapezoidal rule

$$\int_a^b f(x)dx \approx T_n = \sum (f(x_i) + f(x_{i-1}))/2 \cdot \Delta x$$
$$= \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) \cdots + 2f(x_{n-1}) + f(x_n))$$
$$\Delta x = \frac{b-a}{n} \quad x_i = a + i\Delta x$$

Simpson rule

$$\int_a^b f(x)dx \approx S_n$$
$$= \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Where $\Delta x = \frac{b-a}{n}$ and n is an even number. $x_i = a + i\Delta x$

The error associated to the approximation is simply the difference between the actual value and the approximation. For the Midpoint rule, it reads as

$$Error(n) = \left| \int_a^b f(x)dx - M_n \right|$$

For Trapezoidal rule and Simpson rule, we simply replace M_n by T_n and S_n in the error formula above, respectively.

We can bound the error in order to tell how good the approximation is. Below are the formulas to bound the error for each approximation:

Theorem (Error bounds — CLP theorem 1.11.13). *Suppose that $|f''(x)| < K$ on $a \leq x \leq b$ and $|f^{(4)}(x)| < L$ on $a \leq x \leq b$. If $E_M(n)$, $E_T(n)$ and $E_S(n)$ are the errors of the midpoint, trapezoid and Simpson's rules respectively then*

$$\text{Mid point rule : } |E_M(n)| \leq \frac{K(b-a)^3}{24n^2}$$

$$\text{Trapezoidal rule : } |E_T(n)| \leq \frac{K(b-a)^3}{12n^2}$$

$$\text{Simpson rule : } |E_S(n)| \leq \frac{L(b-a)^5}{180n^4}$$

So Simpson rule is better than Mid point rule and Trapezoidal rule as error for Simpson rule varies as $1/n^4$ compared to $1/n^2$ for Mid point rule and Trapezoidal rule.

Mid point rule is slightly better than Trapezoidal rule due to the pre-factor $1/24$ compared to $1/12$ respectively.

Hence we have:

Simpson better than Mid point better than Trapezoidal