Example

Lets approximate $\int_1^2 \frac{1}{x}\,dx$ with the trapezoidal rule. We know the exact value $\int_1^2 \frac{1}{x}\,dx = \left[\log|x|\right]_1^2 = \log 2$ and would like to calculate T_n its trapezoidal approximation and bound the error for $n=5$.

- Applying trapezoidal rule for $n=5$, we get

$$T_5 = \frac{0.2}{2}\left(f(1) + 2\cdot f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\right)$$

$$= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

$$\simeq 0.6956$$

while $\log 2 \simeq 0.6931$

- To bound the error, we first need to bound $|f''(x)|$ over $1 \leq x \leq 2$

$$f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2} \Rightarrow f''(x) = \frac{2}{x^3}$$

so over $1 \leq x \leq 2$, we have $|f''(x)| \leq 2$

The general formula for the error is

$$|E_T(n)| \leq \frac{K(b-a)^3}{12n^2}$$

Here we have : $a = 1$, $b = 2$, $n = 5$ and $K = 2$ (2)

so $|E_T(5)| \leq \frac{2 \cdot (2-1)^3}{12 \cdot 5^2} = \frac{1}{150} \simeq 0.00666$

Now we can determine the minimum number of intervals to make the error less than ε.

In other words, how should we choose n to have

$|E_T(n)| \leq \varepsilon$

Let's assume we want the error to get less than 10^{-6}

so we want to determine the minimum n to have

$|E_T(n)| \leq 10^{-6}$

$\Rightarrow \frac{K(b-a)^3}{12 n^2} \leq 10^{-6}$

In our example we get $\frac{2 \cdot (2-1)^3}{12 n^2} \leq 10^{-6}$

$\Rightarrow n \geq \sqrt{\frac{10^6}{6}} = \frac{10^3}{\sqrt{6}} \simeq 408$

So we need at least 408 intervals

Note: this is a conservative bound, you can show that the actual minimum n is $\simeq 260$.

• Finally, let's compare the trapezoidal rule and the Simpson rule for the same error 10^{-6} ③

For the Simpson rule, we need first to bound $f^{(4)}(x)$ over $1 \leq x \leq 2$

$$f^{(3)}(x) = -\frac{6}{x^4} \quad \text{and} \quad f^{(4)}(x) = \frac{24}{x^5}$$

so we have $|f^{(4)}(x)| \leq 24$ over $1 \leq x \leq 2$

Applying the Simpson rule error formula, we have with $K = 24$:

$$|E_S(n)| \leq \underbrace{\frac{K(b-a)^5}{180\, n^4} \leq 10^{-6}}$$

$$\Rightarrow \frac{24 \cdot (2-1)^5}{180\, n^4} \leq 10^{-6}$$

$$\Rightarrow n \geq \left(\frac{2}{15} \cdot 10^6\right)^{1/4} \sim 19.1$$

Hence we need $n \geq 20$ with Simpson rule compared to $n \geq 408$ with trapezoidal rule to have the same quality of approximation, i.e., about 20 times less intervals! a significant reduction of the computing effort!