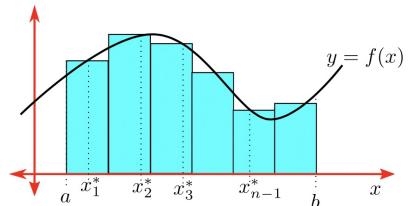


Riemann sum (contd)

Consider n sub-interval $[x_{i-1}, x_i]$

pick $x_i^* \in [x_{i-1}, x_i]$.

$$R_n = \Delta x f(x_1^*) + \Delta x f(x_2^*) + \dots + \Delta x f(x_n^*)$$



$$\text{area} = \lim_{n \rightarrow \infty} R_n.$$

Definition of definite integral (contd)

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

provided limit exists.

- If the limit exists then we say f is integrable on $[a, b]$.
- Integration of even and odd functions.

$F(x) := \int_a^x f(t) dt$ is a function of x . — ①

Thm (Fundamental Theorem of Calculus Part 1, CLP 1.3.1)

Let $F(x)$ be as defined in ①. Then

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Thm (CLP 1.3.1) Let f be any continuous function on $[a, b]$. Let F be antiderivative of f . Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Integration Techniques

- o u -substitution.

$$\int f(u(x)) \frac{du}{dx} dx = \int f(u) du \Big|_{u=u(x)}$$

- o Integration by parts.

$$\boxed{\int u(x) \frac{dv}{dx} dx = u(x) \cdot v(x) - \int v(x) \frac{du}{dx} dx}$$

Integration Techniques

• Trigonometric substitution.

Identity	Expression	Substitution
$1 - \sin^2 \theta = \cos^2 \theta$	$\sqrt{a^2 - x^2}$	$x = a \sin \theta$
$\sec^2 \theta - 1 = \tan^2 \theta$	$\sqrt{x^2 - a^2}$	$x = a \sec \theta$
$1 + \tan^2 \theta = \sec^2 \theta$	$\sqrt{a^2 + x^2}$ or $\frac{1}{a^2 + x^2}$	$x = a \tan \theta$

• Trigonometric integrals.

$$\int \sin^a(x) \cos^b(x) dx$$

$$\int \sec^a(x) \tan^b(x) dx.$$

Integration Techniques

o Partial fractions. $\int \frac{P(x)}{Q(x)} dx$, $P(x)$ and $Q(x)$ are polynomials.

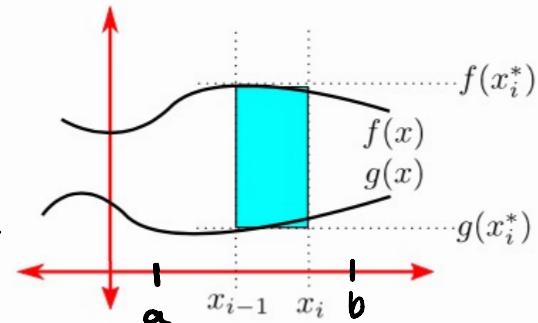
denominator factor	partial fraction expansion
$(x - a)$	$\frac{A}{x-a}$
$(x - a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r}$
$(x^2 + bx + c)$	$\frac{Bx+C}{x^2+bx+c}$
$(x^2 + bx + c)^r$	$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \frac{B_3x+C_3}{(x^2+bx+c)^3} + \cdots$

Area between curves.

Vertical slice

$$\text{Area} = \lim_{n \rightarrow \infty} R_n = \int_a^b (f(x) - g(x)) dx.$$

↑ top function ↓ bottom function.



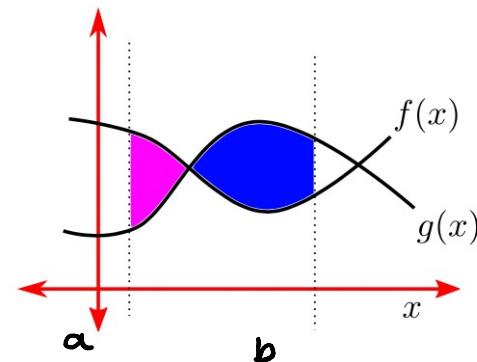
Horizontal slice:

$$\text{Area} = \int_a^b (f(y) - g(y)) dy$$

↑ right func. ↑ left func.

Careful

$f(x) - g(x)$ change sign at intersection point



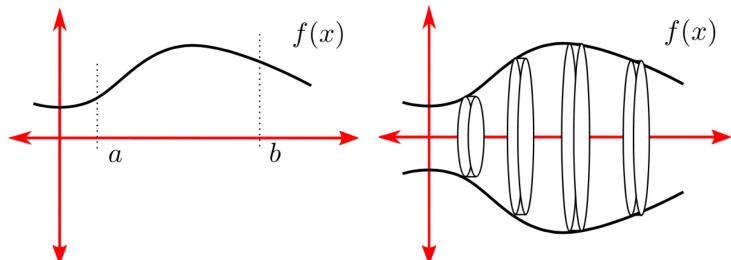
Volume by rotation

$$V = \int_a^b g(x) dx, \quad g(x) = \pi(f(x))^2$$

more generally: $V = \int_a^b A(x) dx$.

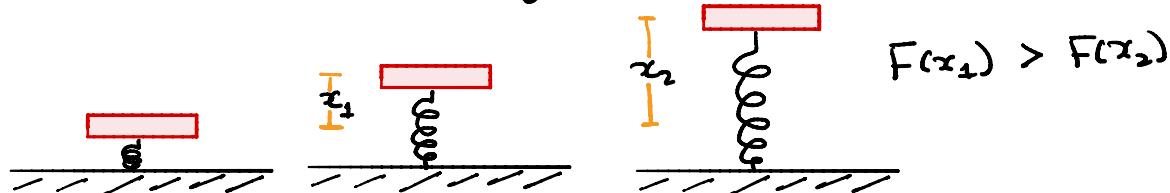
Volume enclosed between two surfaces.

$$V = \pi \int_a^b (f(x)^2 - g(x)^2) dx$$



Work.

$F(x)$ is the force acting at position x .



Use Riemann sum to approx. work done to move from $x=a$ to $x=b$.

- o n subintervals $[x_{i-1}, x_i]$, $x_i = a + i \frac{(b-a)}{n}$

- o Constant force $[x_{i-1}, x_i]$, $F(x_i^*)$

- o Approximate work, $R_n = \sum_{i=1}^n F(x_i^*) \Delta x$.

- o Work = $\int_a^b F(x) dx$

Mean of integrable function.

Definition.

Let f be an integrable function, then

the average value \bar{f} of $f(x)$ for x in $[a,b]$ is.

$$f_{\text{ave}} = \bar{f} = \frac{1}{b-a} \int_a^b f(x) dx. \quad \leftarrow \begin{matrix} \text{we get here from Riemann} \\ \text{sum.} \end{matrix}$$

$$\langle f \rangle = \frac{1}{n} \sum_{n=1}^N f(x_i^*).$$

Separable differential equation.

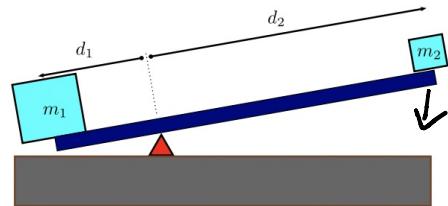
Definition: A separable differential equation is one of the form

$$\frac{dy}{dx} = f(y)g(x).$$

$$\int \frac{1}{f(y)} dy = \int g(x) dx.$$

Center of mass.

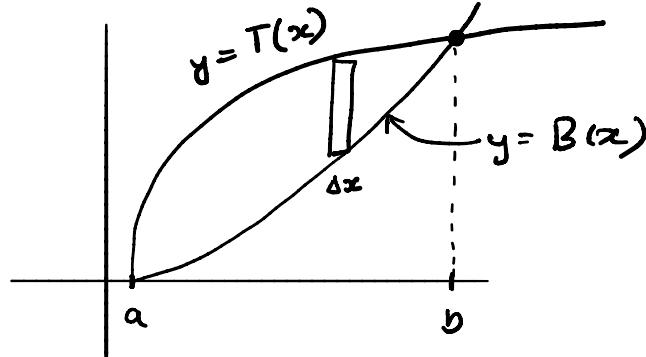
- Torque exerted by m_1 and m_2 is equal if balanced at center of mass.



$$\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{2} \int_a^b (T(x)^2 - B(x)^2) dx}{\int_a^b (T(x) - B(x)) dx}$$

$$\bar{x} = \frac{My}{M} = \frac{\int_a^b x(T(x) - B(x)) dx}{\int_a^b (T(x) - B(x)) dx}$$

$$a \leq x \leq b, \quad B(x) \leq y \leq T(x)$$



Improper integral.

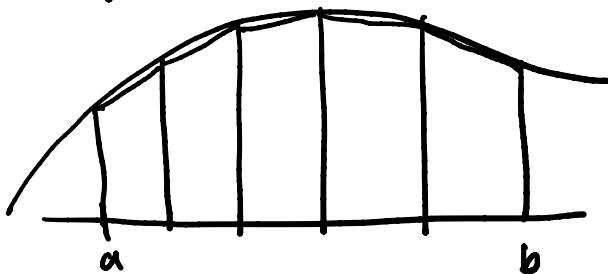
Infinite domain : $\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$

$f(x) \rightarrow \infty$ as $x \rightarrow a$. $\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$

Numerical approximation.

Midpoint rule : Riemann sum with x_i^* as mid point of sub interval.

Trapezoid rule



Simpson's rule

