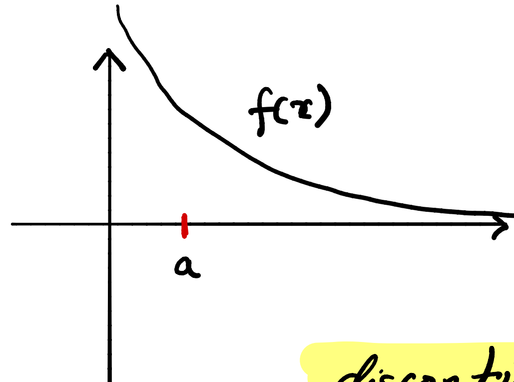
Improper integrals.

Goal: compute definite integrals in the cases where

- limit of integration is ∞ or $-\infty$
- the integrand is discontinuous / unbounded.



terminal at $\pm\infty$

$$\int_a^\infty f(x)\,dx$$

discontinuous / unbounded.

$$\int_0^a f(x)\,dx.$$

Definition (infinite domain)

- If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then we define

$$\int_0^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx.$$

  if the limits exists.

- Similarly for $\int_{-\infty}^b f(x)\,dx$.

- If the limits exist the integrals are called convergent, else divergent.

- If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx. \quad , \quad a \in \mathbb{R}.$$

Key result

Consider $I = \int_1^\infty \frac{1}{x^p}\,dx$. Then I is convergent iff $p > 1$.

proof: Define $I_t = \int_1^t \frac{1}{x^p}\,dx = \begin{cases} \left.\frac{x^{1-p}}{1-p}\right|_1 & p \neq 1 \\ \left.\ln(x)\right|_1 & p = 1 \end{cases}$

$$\text{So, } I_t = \begin{cases} \frac{t^{1-p}}{1-p} - \frac{1}{1-p} & \text{if } p \neq 1 \\ \ln(t) & \text{if } p = 1 \end{cases}$$

Now, let $t \to \infty$, we $t^{1-p} \to 0$ if $p > 1$ and $\ln(t) \to \infty$.

So, I is convergent if and only if $p > 1$.

$$I = \frac{1}{p-1} \quad \text{iff } p > 1.$$

Convergent iff $f(x) = x^{-p}$ decays fast enough.

Definition (discontinuous / unbounded integrand)

- If f is continuous on $[a,b)$ and discontinuous at b then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$

- If f is continuous on $(a,b]$ and discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx.$$

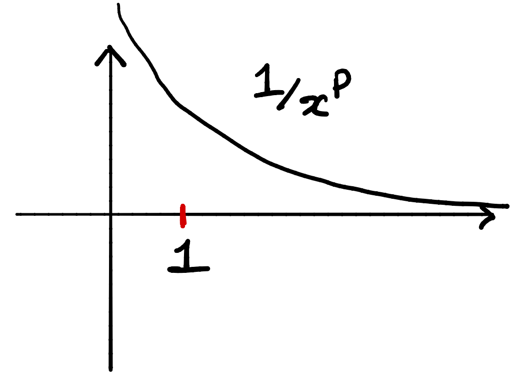
if the limit exists.

- If limit exists then integral is convergent, else divergent.
- If f is discontinuous as $c \in (a,b)$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Key result

Consider an example where $I = \int_0^1 \frac{1}{x^p}\,dx$. Then I is finite iff $p<1$, i.e. x^{-p} blows up slow enough.



proof: Define $I_t = \int_t^1 \frac{1}{x^p}\,dx$. We are interested in $\lim_{t\to 0^+} I_t$.

$$I_t = \begin{cases} \left.\frac{x^{1-p}}{1-p}\right|_t^1 & \text{if } p \neq 1 \\ \left.\ln(x)\right|_t^1 & \text{if } p = 1. \end{cases}$$

$$= \begin{cases} \frac{1}{1-p}\left(1 - t^{1-p}\right) & \text{if } p \neq 1 \\ -\ln(t) & \text{if } p = 1. \end{cases}$$

Key result (contd.)

note that $\lim_{t \to 0^+} t^{1-p} = 0$ if and only if $p < 1$.

and $\lim_{t \to 0^+} \ln(t) = -\infty$.

so, $I = \frac{1}{1-p}$ iff $p < 1$.

Interior singularity

Let $I = \int_{-1}^{1} \frac{1}{x^2}\,dx$. Does I converge?

Note that $f(x) = \frac{1}{x^2}$ has a discontinuity at $x=0$. so, we can't simply find anti derivative:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -x^{-1}\Big|_{-1}^{1} = -2$$ is wrong

Instead:

$$I = \lim_{t\to 0^+} \int_{t}^{1} \frac{1}{x^2}\,dx + \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^2}\,dx.$$

diverges since $p = 2 > 1$

Example (Interior singularity)

Let $I = \int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$. Does I converge?

observe that $f(x) = \frac{1}{(x-1)^{2/3}}$ is discontinuous at $x = 1$.

so, write

$$I = \lim_{t\to 1^+} \int_t^3 \frac{1}{(x-1)^{2/3}}\,dx + \lim_{t\to 1^-} \int_0^t \frac{1}{(x-1)^{2/3}}\,dx.$$

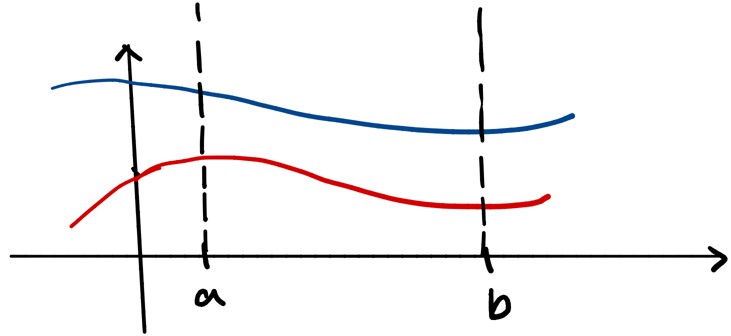
$$= \lim_{t\to 1^+} 3(x-1)^{1/3}\Big|_t^3 + \lim_{t\to 1^-} 3(x-1)^{1/3}\Big|_0^t$$

$$= \lim_{t\to 1^+} \left(3\cdot 2^{1/3} - 3(t-1)^{1/3}\right) + \lim_{t\to 1^-} \left(3(t-1)^{1/3} - 3(-1)^{1/3}\right)$$

$$= 3\cdot 2^{1/3} + 3$$

so, I is convergent

Comparison test:



Theorem (comparison test). Suppose $f(x)$ and $g(x)$ are continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \in [a, \infty)$. Then

- if $\int_a^\infty g(x)\,dx < \infty \Rightarrow \int_a^\infty f(x)\,dx < \infty$
- if $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ diverges.

Example 1

Let $f(x) = \dfrac{\sin^2 x}{x^2}$ for $x \geq 1$. Does $\int_1^\infty f(x)$ converge?

Solⁿ We first find a comparison function. Observe that

$$\sin^2 x < 1 \Rightarrow \frac{\sin^2 x}{x^2} < \frac{1}{x^2}.$$

Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges ($\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge if $p>1$)

we get $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converge.

Example 2

Consider $\int_1^\infty \frac{dx}{\sqrt{x^2-5}}$. Does this integral converge or diverge?

Intuition: For large x $\frac{1}{\sqrt{x^2-5}} \approx \frac{1}{x}$ and $\int_1^\infty \frac{1}{x}\,dx$ diverges.

Observe that $\sqrt{x^2-5} < \sqrt{x^2} \Rightarrow \frac{1}{\sqrt{x^2}} < \frac{1}{\sqrt{x^2-5}}$.

we have $\int_1^\infty \frac{1}{\sqrt{x^2}}\,dx < \int_1^\infty \frac{1}{\sqrt{x^2-5}}\,dx$.

By comparison test $\int_1^\infty \frac{1}{\sqrt{x^2-5}}\,dx$ diverges.

Example 3

Consider $\displaystyle\int_2^{\infty} \frac{x}{x^3+x^2+1}\,dx$. Does this integral converge?

Intuition: $x^3+x^2+1 \simeq x^3$ and $\displaystyle\int_2^{\infty} \frac{1}{x^2}\,dx$ converges.

observe that $x^3+x^2+1 \geq x^3$ if $x \geq 2$.

so, $\displaystyle\frac{1}{x^3} \leq \frac{1}{x^3+x^2+1}$ if $x \geq 2$.

Since $\displaystyle\int_2^{\infty} \frac{x}{x^3}\,dx$ converges $(p = 2 > 1)$,

by comparison test, $\displaystyle\int_2^{\infty} \frac{x}{x^3+x^2+1}\,dx$ converges.