Simplex method

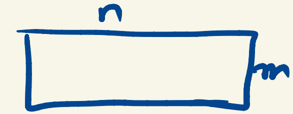
- computing BFS
- computing feasible direction
- reduced cost

Assumptions

we will develop simplex method for an LP in standard form:

$$\min_{x} \quad c^T x \qquad \text{subject to} \quad Ax = b, \; x \geq 0$$

$A \in \mathbb{R}^{m \times n}$,



m - equality

n - inequality

we will assume throughout the section:

- A has full row rank (no redundant rows)
- the LP is feasible. ← might be unbounded. need to check!
- all the basic feasible solution (i.e. extreme points) are non-degenerate.

variable index set: ← incides for column of A.

- $B = \{\beta_1, \dots, \beta_m\}$: basic variables
- $N = \{\eta_1, \eta_2, \dots, \eta_{n-m}\}$: non-basic variables

Constructing basic feasible solution

Basic feasible solution portitions the columns of A:

$$AP = [B \quad N], \quad \text{where } B \text{ is non-singular}$$

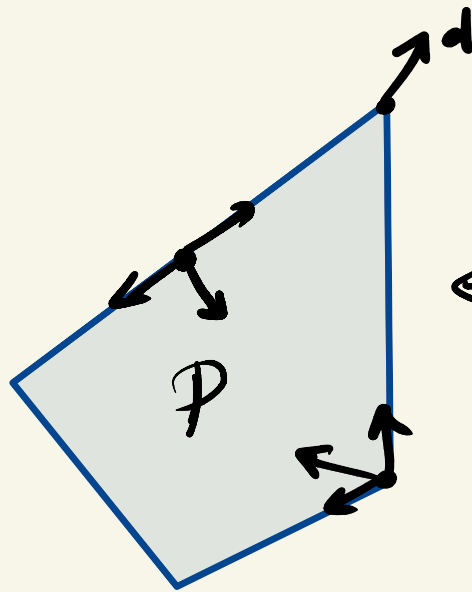
$$x_B = B^{-1}b, \quad x_N = 0$$

Procedure for constructing basic solution

1. choose m linearly independent columns of A indexed in B
2. set $x_i = 0$ for $i \notin B$ ($x_N = 0$)
3. Solve $A_B x_B = b$

Feasible direction.

A direction $d \in \mathbb{R}^n$ is a feasible at $x \in P$ if there exist $\alpha > 0$ such that $\boxed{x + \alpha d \in P}$



$Ax = b$
$x \geq 0$

Constructing a feasible direction

given $x \in P$ with $Ax = b$, $x \geq 0$ — move along d while $x + \alpha d \in P$, $\alpha > 0$.

require for $\alpha > 0$ that

$$b = A(x + \alpha d) = Ax + \alpha Ad = b + \alpha Ad$$

Thus, we require $Ad = 0 \iff$ direction d is in Nullspace of A.

Suppose x is a basic feasible solution so that

$$Ad = [B \;\; N] \begin{bmatrix} d_B \\ d_N \end{bmatrix} = Bd_B + Nd_N = 0$$

basic

non-basic

$$\Rightarrow Bd_B = -Nd_N, \quad N \in \mathbb{R}^{m \times (n-m)}$$

Construct search directions by moving a single non-basic variable $\eta_k \in N$:

$$d_N = e_k \quad \text{and} \quad Bd_B = -Ne_k = a_{\eta_k}$$

$\hookrightarrow \mathbb{R}^{n-m}$

select k^{th} column of N

Example

$$\begin{aligned} \text{minimize} \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\ \text{subject to} \quad & \boxed{\begin{matrix} x_1 & + & x_2 \\ 2x_1 & + & \end{matrix}} \begin{matrix} + \; x_3 + x_4 = 2 \\ + \; 3x_3 + 4x_4 = 2 \end{matrix} \\ & x \geq 0 \end{aligned}$$

Basic

$$[B \quad N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b.$$

$$x_N \equiv 0$$

$$x_B = B^{-1}b$$

Construct basic feasible solution:

$$\mathcal{B} = \{1, 2\} \Rightarrow B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \Rightarrow Bx_B = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow x_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;\&\; x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

increase a non basic variable, say x_3. That is, $d_N = \begin{bmatrix} d_{\eta_1} \\ d_{\eta_2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$Bd_B = -Nd_N = -a_{\eta_1} = -\begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow d_B = \begin{bmatrix} 1/2 \\ -3/2 \end{bmatrix}$$

so, $d = (\underbrace{\tfrac{1}{2}, -\tfrac{3}{2}}_{B}, \underbrace{1, 0}_{N})$ ✓

$$\bar{x} = x + \alpha d = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1/2 \\ -3/2 \\ 1 \\ 0 \end{bmatrix}$$

$A\bar{x} = b$ ✓ $\quad \bar{x} \geq 0$?

Change in objective

objective value at $\bar{x} = x + \alpha d$, where x is basic feasible sol^n.

how does $c^T\bar{x} = c^T(x+\alpha d)$ change as α increases?

Let $\phi(x) = c^T x$, then — current objective value

$$\bar{\phi} = \phi(\bar{x}) = c^T x + \alpha c^T d$$ — new objective value.

$$= \phi + \alpha \left[c_B^T \; c_N^T\right] \begin{bmatrix} d_B \\ d_N \end{bmatrix}$$

$$= \phi + \alpha \left(c_B^T d_B + c_N^T d_N\right)$$

$$= \phi + \alpha \underbrace{\left(c_B^T d_B + c_{n_k}\right)}_{\text{reduced cost}}$$

$d_B = -B^{-1} a_{n_k}$

$B d_B = -N d_N$

where $d_N = e_k$, i.e. only k^{th} non-basic variable increases.

$d_N = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ → k^{th} entry, $d = \begin{bmatrix} d_B \\ d_N \end{bmatrix}$

Reduced cost

$d_B = -B^{-1} a_{\eta_K} = -B^{-1} a_j$

reduced cost for any variable x_j, $j = 1, \dots, n$:

$$z_j := c_j + c_B^T d_B = c_j - c_B^T B^{-1} a_j$$

reduced cost for a basic variable, $j \in \mathcal{B}$

$$\begin{aligned} z_j &= c_j - c_B^T B^{-1} a_j \\ &= c_j - c_j \\ &= 0 \end{aligned}$$

thus only non-basic variables need to be considered.

note: if $z \geq 0$, then all feasible directions increase objective.

<u>Thm:</u> consider a basic feasible solution x with reduced cost z.

- if $z \geq 0$ then x is optimal
- if x is optimal and non-degenerate then $z \geq 0$.

Choosing a stepsize

change in objective value from moving p^{th} non-basic variable $\eta_p \in \mathcal{N}$:

$$\phi(\bar{x}) = \phi + \alpha z_{\eta_p}, \quad z_j = c_j - c_B^T B^{-1} a_j$$

$z_{\eta_p} < 0$, so choose α as large as possible. → depends on η_p

$$\alpha^* = \max\{\alpha \geq 0 \mid x + \alpha d \geq 0\}$$

$$A\bar{x} = Ax + \alpha \cancel{Ad}^{\,0}$$

<u>case 1</u>: if $d \geq 0$ then it is unbounded feasible direction of descent i.e. $x + \alpha d \geq 0$ for all $\alpha \geq 0$.

$$\bar{x} = x + \alpha d$$

$$= \begin{bmatrix} x_B \\ x_N \end{bmatrix} + \alpha \begin{bmatrix} d_B \\ d_N \end{bmatrix}$$

$x_N \equiv 0$ d_N: one non-zero

<u>case 2</u>: if $d_j < 0$ for some j, then $x + \alpha d \geq 0$ if $\alpha \leq -x_j/d_j$ for every $d_j < 0$

<u>ratio test</u>

$$\alpha^* = \min\left\{-\frac{x_j}{d_j} \,\middle|\, j \in B, d_j < 0\right\}$$

Basis change

<u>case 1</u>: no "blocking" basic variable. Therefore d is a direction of unbounded descent.

<u>case 2</u>: the first basic variable to "hit" bound is "blocking".

<u>variable swap</u>:

- entering basic variable $\eta_p \in N$ becomes basic ($x_{\eta_p} \to +$)
- blocking basic variable $\beta_q \in B$ becomes non-basic ($x_{\beta_q} \to 0$)

new basic and non-basic set

- $\bar{B} \leftarrow (B \setminus \{\beta_q\}) \cup \{\eta_p\}$
- $\bar{N} \leftarrow (N \setminus \{\eta_p\}) \cup \{\beta_q\}$