Simplex method

- step length
- Blocking variable & basis change
- Optimality

Reduced cost

$$\boxed{\phi(\tilde{x}) = \phi(x) + \alpha \cdot z_j}$$

reduced cost for any variable x_j, $j = 1, \dots, n$:

$$z_j := c_j + c_B^T d_B = c_j - c_B^T B^{-1} a_j$$

reduced cost for a basic variable, $j \in \mathcal{B}$

$$\begin{aligned} z_j &= c_j - c_B^T B^{-1} a_j \\ &= c_j - c_j \\ &= 0 \end{aligned}$$

thus only non-basic variables need to be considered.

note: if $z \geq 0$, then all feasible directions increase objective.

<u>Thm:</u> consider a basic feasible solution x with reduced cost z.

- if $z \geq 0$ then x is optimal
- if x is optimal and non-degenerate then $z \geq 0$.

Choosing a stepsize

Basis change

A new basis

Simplex without B^{-1}

Example

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

iteration 0: $\mathcal{B} = \{3,4,5\}$, $\mathcal{N} = \{1,2\}$

Which of the following is the new basic set (and non-basic set)?

1. $\mathcal{B} = \{2,4,5\}$, $\mathcal{N} = \{1,3\}$.
2. $\mathcal{B} = \{3,2,5\}$, $\mathcal{N} = \{1,4\}$
3. $\mathcal{B} = \{1,4,5\}$, $\mathcal{N} = \{2,3\}$

Example

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

iteration 0: $\mathcal{B} = \{3, 4, 5\}$, $\mathcal{N} = \{1, 2\}$

Example contd

iteration 2:

- current basis: $B = \{2,4,5\}$, $N = \{1,3\}$

$$\boxed{\begin{array}{c} x_j \cdot z_j \geq 0 \\ z \geq 0,\ x \geq 0 \end{array}}$$

- $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- solve $Bx_B = b \rightarrow x_B = (2,3,3)$ $[x_N = 0]$
- simplex multiplier: $B^T y = c_B \longrightarrow y = (-2,0,0)$
- reduced cost: $z_N = c_N - N^T y \rightarrow z_N = (-5,2)$, $[z_B = 0]$
- choose $\eta_1 = 1$ to enter basis.
- search direction: $Bd_B = -a_1 \rightarrow d_B = (2,-3,1)$
- ratio test: $q = \underset{q \,|\, d_{\beta_q} < 0}{\arg\min} \dfrac{-x_{\beta_q}}{d_{\beta_q}} \longrightarrow q = 2$, $\beta_q = 4$ exits
- $B = \{2,1,5\}$, $N = \{4,3\}$

iteration 3:

- current basis $\mathcal{B} = \{2, 1, 5\}$, $\mathcal{N} = \{4, 3\}$
- $B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $B^{-1} = (1/3) \begin{bmatrix} -1 & 2 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$
- solve : $Bx_B = b \rightarrow x_B = (4, 1, 2)$
- simplex multiplier : $B^T y = c_B \rightarrow y = (4/3, -5/3, 0)$
- reduced cost : $z_N = c_N - N^T y \rightarrow z_N = (5/3, -4/3)$
- choose $\eta_2 = 3$ to enter basis
- search direction : $B d_B = -a_3 \rightarrow d_B = (1/3, 2/3, -2/3)$
- ratio test : $q = -3$, $\beta_q = 5$ exits basic
- $\mathcal{B} = \{2, 1, 3\}$, $\mathcal{N} = \{4, 5\}$.

iteration 4:

- current basis : $\mathcal{B} = \{2, 1, 3\}$, $\mathcal{N} = \{4, 5\}$
- $B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1 & -1/2 & 3/2 \end{bmatrix}$
- solve $B x_B = b \rightarrow x_B = (4, 1, 2)$
- simplex multiplier : $B^T y = c_B \longrightarrow y = (0, -1, -2)$
- reduced cost : $z_N = c_N - N^T y \rightarrow z_N = (1, 2)$
- $z_N \geq 0 \rightarrow$ basis is optimal.