# Simplex method

- Example
- General bound
- 2-phase simplex

# Example

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

iteration 0: $\mathcal{B} = \{3, 4, 5\}$, $\mathcal{N} = \{1, 2\}$

Which of the following is the new basic set (and non-basic set)?

1. $\mathcal{B} = \{2, 4, 5\}$, $\mathcal{N} = \{1, 3\}$.

2. $\mathcal{B} = \{3, 2, 5\}$, $\mathcal{N} = \{1, 4\}$

3. $\mathcal{B} = \{1, 4, 5\}$, $\mathcal{N} = \{2, 3\}$

# Example

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

iteration 0: $\mathcal{B} = \{3, 4, 5\}$, $\mathcal{N} = \{1, 2\}$

iteration 1:

# Example contd

iteration 2:

- current basis: $\mathcal{B} = \{2,4,5\}$, $\mathcal{N} = \{1,3\}$

$$\boxed{\begin{array}{l} x_j \cdot z_j \geq 0 \\ z \geq 0,\ x \geq 0 \end{array}}$$

- $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- solve $Bx_B = b \rightarrow x_B = (2,3,3)$ $[x_N = 0]$
- simplex multiplier: $B^T y = c_B \longrightarrow y = (-2,0,0)$
- reduced cost: $z_N = c_N - N^T y \rightarrow z_N = (-5,2)$, $[z_B = 0]$
- choose $\eta_1 = 1$ to enter basis.
- search direction: $Bd_B = -a_1 \rightarrow d_B = (2,-3,1)$
- ratio test: $q = \underset{q \mid d_{\beta_q} < 0}{\arg\min} \dfrac{-x_{\beta_q}}{d_{\beta_q}} \longrightarrow q = 2,\ \beta_q = 4$ exits
- $\mathcal{B} = \{2,1,5\}$, $\mathcal{N} = \{4,3\}$

iteration 3:

- current basis $\mathcal{B} = \{2, 1, 5\}$, $\mathcal{N} = \{4, 3\}$
- $B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $B^{-1} = (1/3)\begin{bmatrix} -1 & 2 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$
- solve: $Bx_B = b \rightarrow x_B = (4, 1, 2)$
- simplex multiplier: $B^T y = c_B \rightarrow y = (4/3, -5/3, 0)$
- reduced cost: $z_N = c_N - N^T y \rightarrow z_N = (5/3, -4/3)$
- choose $\eta_2 = 3$ to enter basis
- search direction: $B d_B = -a_3 \rightarrow d_B = (1/3, 2/3, -2/3)$
- ratio test: $q = -3$, $\beta_q = 5$ exits basic
- $\mathcal{B} = \{2, 1, 3\}$, $\mathcal{N} = \{4, 5\}$.

iteration 4:

- current basis : $\mathcal{B} = \{2, 1, 3\}$ , $\mathcal{N} = \{4, 5\}$
- $B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ , $B^{-1} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1 & -1/2 & 3/2 \end{bmatrix}$
- solve $Bx_B = b \rightarrow x_B = (4, 1, 2)$
- simplex multiplier : $B^T y = c_B \longrightarrow y = (0, -1, -2)$
- reduced cost : $z_N = c_N - N^T y \rightarrow z_N = (1, 2)$
- $z_N \geq 0 \rightarrow$ basis is optimal.

# General upper and lower bound

standard form

$$\min_{x} \quad c^T x$$

s.t. $\quad Ax = b \;, \; x \geq 0$

general bound

$$l \leq x \leq u$$

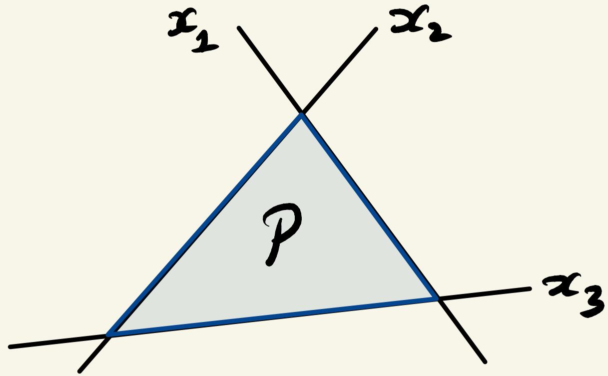
reduction to standard form

$$x - s_1 = l \;, \quad s_1 \geq 0 \quad \Rightarrow \quad l \leq x$$

$$x + s_2 = u \;, \quad s_2 \geq 0 \quad \Longrightarrow \quad x \geq u$$

standard form problem

# General bounds and non basic variable




- nonbasic variable are always at their bounds
- basic variables are uniquely determined by non-basic variables.

$$b = Ax = [B \quad N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = Bx_B + Nx_N$$

thus,

# Simplex with general bounds

effect on objective: need to choose a "good" $d_N$. Solve

$$B^T y = c_B \quad \text{and} \quad z := c - A^T y$$

$$\bar{\phi} = \phi + \alpha z_N^T d_N$$

pricing: only one non-basic variable $x_p$ moves, implying

# Finding an initial Basic Feasible Solution

not always obvious how to choose a linearly independent set of columns of $B$ so that $x \geq 0$

# Two Phase Simplex